Theory

## Proportional-hazards models and intra-class correlation

Cox proportional-hazards models (or Cox regression) estimate the hazard (rate) of an event occurring in relation to predictor variables with time-to-event data (i.e., time taken till the event or censoring; Cox 1972). The hazard is a rate (or risk) of which an event occurring at time . The hazard rate is defined in a Cox model as:

where is the hazard rate at time for the *i*th subject (individual), is the baseline hazard rate, are the regression coefficients, and are the predictor variables. Note take the place of the intercept as or where (‘ln’ is a natural logarithm). Equation 1 can be rearranged to:

for the right hand side to take a linear form, which is more familiar for many readers although it does not have the intercept (i.e., ). To fit such a model using, for example, R, one needs to provide the time-to-event data in the form of a Surv object (e.g., Surv(time, event) where time is time taken till an event or censoring and event is usually 0 or 1, indicating whether the event occurred or not). The Cox model can be fitted using the coxph in the survival package (Therneau et al. 2015).

Now let us assume that we have a single predictor variable, sex, for a time-to-event data set (e.g. a latency to solve a task), and we have a single random effect (or cluster) (e.g. individuals or populations). The Cox proportional-hazards model can be extended to include a random effect (individual identify), which is often referred to as the ‘frailty’ term (Cox regressions with a single random effect is known as the frailty model):

where is the hazard rate at time for the *i*th individual for the *j*th occasion (observation). This model (frailty model) can be fitted using the coxme function in the R package coxme (Therneau 2015) as well as coxph.

Now we have defined the Cox model so let us define repeatably or intra-class correlations (ICC) in its simplest form when the trait of interest (the response variable) is a Gaussian variable (i.e. having normally distributed residuals):

where is the variance of the random effect (identity for a cluster: e.g. individuals so the between-cluster variance) and is the variance of the residuals (or within-cluster variance; Nakagawa & Schielzeth 2010). The ICC can be interpreted as the proportion of the total variance that is due to the between-cluster variance. The ICC can be calculated for (generalized) linear mixed-effect models (LMMS or GLMMs). For example, the R pacakge, rptR can be used to calculate the ICC for a GLMM, via lmer and glmer function in the lme4 package.

Nakagawa & Schielzeth (2010) suggest that for non-Guassian data (e.g. binomial or Poisson), the within-cluster variance can be determined by what distributional assumptions GLMM makes (e.g. binomial or Poisson). For example, to obtain ICC for binary GLMMs on the latest (link/transformed) scale, can be assumed to be (they call as the distributional specific variance; for more details, see Nakagawa & Schielzeth 2010. Nakagawa et al. 2017). However, Cox models make no distributional assumptions about the hazard rate (i.e., non-parametric; Equation 1 & 2) although frailty models (Equation 3) has a random effect of a Gaussian distribution (this is why, this model is known as semi-parametric. Therefore, we could not calculate the ICC for Cox models.

Yet, in the statistical literature, a formula for parametric version of ICC () for the frailty model is known when we make the random effect is disbursed as a Gamma distribution on the exponential scale. If we denote the variance from a Gamma distribution as under Equation 3, can be written as:

where the first and the second are , the shape and the rate parameter of the Gamma distribution, respectively (such parameterization results in the mean, and variance, ).

The estimate, is Kendall’s (the rank correlation or concordance for within-cluster observations) for the frailty model (Hougaard 2000). Unfortunately, there is no closed formula when assuming a Gaussian distribution for the random effect as in Equation 3. Yet, can be obtained numerically and we provide an R function based on the tau function from the R package, parfm (Munda et al. 2012). We note that and are unlikely to be the same but values, which we show in the simulation below.

An issue with the is that it is not a parametric version of ICC and more importantly, it is not clear whether this method can be extended where a Cox model has more than one random effect (at least, piratically speaking). Therefore, we need to use a trick to turn a time-to-event data for Cox models into a data set where we could fit a GLMM so that we can obtain parametric versions of ICCs.

## Cox proportional-hazards models and generalized linear mixed models

In the statistical literature, it seems to be well known that the frailty model (Equation 3) can be fitted as a Poisson GLMM (known as, the piece-wise exponential model; e.g., Hirsh et al. 2016) or a binomial GLMM (the discrete-time model; Finkelstein 1986; for an accessible non-technical account, see Austin 2017). Here, we show the discrete-time model, more specifically, the binomial GLMM with the complementary log-log (cloglog) link can be used to fit a comparable model as Equation 3 by “exploding” the time-to-even data set by defining arbitrary discrete time intervals (Fig XXX shows an example of such a exploded data set compared to the original). If we assume we have three (arbitrary discrete) time intervals (*t1*, *t2* & *t3*) This binomial GLMM (without the intercept) is defined as:

where is the hazard rate at time for the *i*th subject for the *j*the occasions for the *k*th time interval (), is the baseline hazard rate for is the time interval, are the indicator variables for the time intervals, and are the regression coefficients for the time intervals. Note that the cloglog link is where is the probability of the event occurring so that the left hand side of Equation 6 consistes of the cloglog-transformed hazard rate () and base hazard rate ().

Rather remarkably, and are “the same” as in Equation 3 although the data structures used for two models are very different (i.e., time-to-event data vs. exploded data; Fig XXX). Using a simple simulation, we show the equivalence of and between the Cox (frailty) model (fitted with coxph and coxme) and the binomial GLMM (fitted with glmer;see below). Note that the number of intervals do not affect these estimates from the binomial GLMM.

Therefore, we can use variance components obtained from Cox models to estimate ICC under a binomial GLMM with the complementary log-log link where (or the distributional-specific variance; Equation 4) is on the latent scale. This means we can define ICC for Equation 3 and 6 as:

In the simulation below, we show the parametric version of ICC and the non-parametric version () are well correlated but not equivalent (analogous to the relationship between Pearson’s *r* and Kendall’s . We prefer the use of ICC as in Equation 7 over because the parametric version is more comparable to other ICC estimates (on the latent/link scale) derived from GLMMs (e.g. Poisson and binomial data), which are now commonly used in ecology and evolution.

Furthermore, the advantage of this approach is for us to add more than one random effect, which was eluded above. For example, imagine we have another cluster (a random effect), adding it to Equation 3 and then:

where is the random effect for the th level of the second cluster, which is normally distributed with the mean of zero and the variane of . It is interesting to notice that the two random effects can be ‘nested’ or ‘crossed’ (Schielzeth and Nakagawa, 2013).

An example of the nested random effects are individual () and population () where individuals are nested within populations. In this case, the ICC for individuals can be defined as:

The reason we need both variance components are required for ICC for individuals is that some of similarities of a pair of individuals come from belonging to the same population. An example of the crossed random effects are individual () and year () where individuals are not nested within years. This time, the ICC for individuals can be written as:

If one wants to remove or adjusting the effect of year, and then, ICC for individuals simplifes to Equation 7. Speaking of adjusting, all the ICC formulas presented above represents ‘adjusted’ repeatablity (ICC; *sensu* Nakagawa and Schielzeth, 2010) because the effect of sex is accounted for in these models above. We can obtain ‘unadjusted’ repeatablity (ICC) by fitting the model without the fixed effect (sex), for example by changing Equation 3:

Importantly, and so ICC values should be larger obtained from this model (Equation 11) than those obtained from Equation 3 given the fixed effect explains non-zero variance. On the online page (www.github….xxxx), we show how to fit models and ICC estimates, described above.

## Methods (for simulaiton)

Note - I do not think we need a section for simulation???

## Case studies