Math 481 Final Paper - Non-Standard Analysis and Loeb Measure

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Introduction

Loeb Measure is a way to convert between nonstandard and standard measure spaces, and it has many applications, including in the world of probability theory and stochastic calculus. One can also formalize intuitive definitions of Lebesgue Measure and Brownian motion using this construction.

First, we need to rigorously define a non-standard measure space, which uses principles from non-standard analysis.

In practice, we will mostly use the real numbers as our set we would like to extend. In order to do this, we will take sequences in our real numbers and quotient them by a free (non-principal) ultrafilter. In the colloquial way one can think about a filter, a free ultrafilter "cares about" all cofinite sets. In this way, two of our hyperreal numbers are equivalent modulo the ultrafilter iff the set of points at which they disagree is finite.

However, this construction lacks tangibility if we can't relate it back to the underlying structure. We first make use of the Transfer Principle to demonstrate that the truth values of all statements in first order logic correspond, and then show how Loeb Measure provides another way to transfer results in our constructed space back to the base space.

We will then go on to show a few examples of how one can use Loeb Measure to do so, including constructions of Lebesgue Measure and Brownian Motion. This has far-reaching applications into the world of differential equations and stochastic calculus, as we can model these infinitesmials as hyperfinite steps and use non-standard techniques to approach certain problems in those fields.

Non-Standard Analysis

We define our hyperreal numbers ${}^*\mathbb{R}$ as ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$ where $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$.
Rigorously, an ultrafilter $\mathcal{U}$ on a set X is defined as follows:

$$\mathcal{U} \subseteq \mathcal{P}(X)$$
$$\text{If } A, B \in \mathcal{U}, A \cap B \in \mathcal{U}.$$
$$\text{If } A \in \mathcal{U}, \text{ and } A \subseteq B, \text{ then } B \in \mathcal{U}.$$

Furthermore, we say $\mathcal{U}$ is free, or non-principal, if $\mathcal{U}$ contains no finite sets.

We can then consider elements $a = (a_n)$ and $b = (b_n)$ of ${}^*\mathbb{R}$ as being equivalent modulo the ultrafilter $\mathcal{U}$

$$(a_n \equiv_{\mathcal{U}} b_n) \iff \{n \in \mathbb{N} : a_n = b_n\} \in \mathcal{U}$$

If so, we write $a \approx b$, and say $x \in {}^*\mathbb{R}$ is infinitesimal if $x \approx 0$.

For $r \in \mathbb{R}$, the canonical representation of r in ${}^*\mathbb{R}$ is the constant sequence $(r_n) \in \mathbb{R}^{\mathbb{N}}$. Via this construction, ${}^*\mathbb{R}$ extends $\mathbb{R}$. In general, given any set V, we can perform this extension modulo an ultrafilter to produce a nonstandard set ${}^*V$.

We can then check that our construction ${}^*\mathbb{R}$ has all the properties of an ordered field that we desire:
Addition, subtraction, and multiplication are (well)-defined termwise, and multiplicative inverses are defined termwise as follows:
If $(a_n) \neq 0$ then there are only zeroes in finitely many entries. Define $(a_n)^{-1}$ as

$$a_n{}^{-1} = \begin{cases} a_n{}^{-1} & a_n \neq 0 \\ 0 & a_n = 0 \end{cases}$$

We can also define an order on our set ${}^*\mathbb{R}$: define ${}^* \leq$ on ${}^*\mathbb{R}$ by $(a_n) \; {}^* \leq (b_n) \iff \{n \in \mathbb{N} : a_n \leq b_n \in \mathcal{U}\}$
A few more important notes:

Theorem. *Transfer Principle*
Let φ be any first-order logical statement. Then φ holds in $\mathbb{R} \iff {}^*\varphi$ holds in our construction $({}^*\mathbb{R}, {}^*f, {}^*R)$, where ${}^*f$ and ${}^*R$ are the pushthroughs of our standard functions and relations to ${}^*\mathbb{R}$.

Since ${}^*\mathbb{R}$ is an ultrapower construction, the proof of the Transfer Principle follows from Łoś' Theorem.
In that vein,

Prop. *For any finite $x \in {}^*\mathbb{R}$, we can decompose x uniquely as $x = r + \varepsilon$ where $r \in \mathbb{R}$ and $\varepsilon \equiv_{\mathcal{U}} 0$ in ${}^*\mathbb{R}$.*

Proof. Since x is finite, the set $X = \{y \in \mathbb{R} : y \leq x\}$ has a supremum, $\sup(X) = r \in \mathbb{R}$. So, $|x - r|$ must be infinitesimal, as if it was not, $|x - r| \geq s$ for some $s \in \mathbb{R}$, and then $r + s$ would be a greater upper bound for the set X. □

Definition. *Define the standard part of $x \in {}^*\mathbb{R}$ as $st(x) = {}^\circ x = r$ for $x = r + \varepsilon$. Since this r is unique by proposition, the standard part function is well-defined on all finite $x \in {}^*\mathbb{R}$.*

Now, we need some concept of internal sets – crucially, the Transfer Principle as stated above only applies to internal sets. Loosely, an internal set is a set we could define using only our local elements.

Definition. *Define $\mathbb{V}$ recursively as follows:*
Let $V_0(\mathbb{R}) = \mathbb{R}$
For $n < \omega$, let $V_{n+1} = V_n(\mathbb{R}) \cup \mathcal{P}(V_n(\mathbb{R}))$
And finally let $\mathbb{V} = \bigcup_{n<\omega} V_n(\mathbb{R})$.
Let A be a set in our non-standard universe $V({}^*\mathbb{R})$. Importantly, ${}^*\mathbb{V} \subset \mathbb{V}({}^*\mathbb{R})$. We say A is an internal set if A is an element of ${}^*\mathbb{V}$, and that A is external if not.

We will also state the propositions of overflow, underflow, and $\aleph_1$-saturation here, but not rigorously prove them in the interest of space. Proofs to each of these statements can be found in [2].

Overflow/Underflow:

Let $A \subseteq {}^*\mathbb{R}$ be an internal set
(i) (Overflow) If A has the property that for any $n < \omega$, A contains some finite element a s.t. $a > n$, then A contains an infinite number r.

(ii) (Underflow) if A has the property that for any $N > \omega$, A contains some infinite element b s.t. $\omega < b < N$, then A contains a finite element.

Definition. *Fix an internal set Ω. A non-standard universe ${}^*V$ is said to be $\aleph_1$-saturated if, for any countable decreasing sequence of internal sets $(A_n)_{n<w}$, with each $A_n \neq \Omega$, $\bigcap_{n<\omega} A_n \neq \Omega$.*

Notably, this implies that for any sequence of internal subsets of that internal set Ω, there is some other internal infinite sequence that extends that sequence. This property is called countable comprehension, and is formalized as follows:

Definition. *Countable Comprehension*
Given any internal set Ω and a sequence of internal sets $(A_n)_{n<w}$ with each $A_n \subset \Omega$, there is an extension to that internal sequence, $(A_n)_{n\in {}^*\mathbb{N}}$, which is also internal.

Now that we have our toolbox of definitions with which we can work with our non-standard universe, we go on to actually define Loeb Measure.

Loeb Measure

Our setup for the rest of the paper is as follows:
Given an internal set Ω and an internal algebra $\mathcal{A}$ of subsets of Ω, and an internal finitely additive measure $\mu : \mathcal{A} \to {}^*[0, \infty)$, we can define the standard-part premeasure as follows:

Definition. ${}^\circ\mu : \mathcal{A} \to [0, \infty)$ *as expected:* ${}^\circ\mu(A) = {}^\circ(\mu(A))$ *for* $A \in \mathcal{A}$.

$(\Omega, \mathcal{A}, {}^\circ\mu)$ is a standard finitely additive measure space, but not necessarily countably additive unless, of course, $\mathcal{A}$ is finite.
In search of a complete measure, we turn to Loeb's construction.

Theorem. *(Loeb, 1975)*
There is a unique σ-additive extension of ${}^\circ\mu$ to the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$.

For context, the completion of ${}^\circ\mu$, μ_L, is Loeb measure, and $\sigma(\mathcal{A}) = L(\mathcal{A})$ is the Loeb σ-algebra.
This can be proved as a consequence of Caratheodory's extension theorem, but a direct proof is much more informative of what Loeb's construction actually does.
In order to prove that Loeb's construction produces an actual σ-algebra, we first need a definition and a lemma.

Definition. *Suppose $U \subseteq \Omega$. Note that U is not necessarily internal. We say that U is a Loeb Null Set if $\forall \varepsilon \in \mathbb{R}_{>0}, \exists A \in \mathcal{A}$ such that $U \subseteq A$, and $\mu(A) < \varepsilon$.*

Then, we will prove the below lemma, which we will use to approximate subsets B of Ω by internal sets in our σ-algebra $\mathcal{A}$.

Lemma. *Let $(A_n)_{n<\omega}$ be an increasing family of sets with each $A_n \in \mathcal{A}$, and let $B = \bigcup_{n<\omega} A_n$. Then, $\exists A \in \mathcal{A}$ such that:*

(i) $B \subseteq A$

(ii) $^\circ\mu(A) = \lim_{n\to\infty} {}^\circ\mu(A_n)$, and

(iii) $A \backslash B$ is null.

Proof. Let

$$\alpha = \lim_{n\to\infty} {}^\circ\mu(A_n)$$

First, notice that for all $n < w$,

$$\mu(A_n) \leq ({}^\circ\mu(A_n) + \frac{1}{n}) \leq (\alpha + \frac{1}{n})$$

Since Ω is $\aleph_1$-saturated, it has the property of countable comprehension, so we can take an extension of our sequence $(A_n)_{n<\omega}$. This extension will consist of internal sets in $\mathcal{A}$.
The overflow principle shows that there must be some infinite $N \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that $\mu(A_N) \leq \alpha + \frac{1}{N}$.
We will set $A = A_N$, and check that (i), (ii), and (iii) hold.

(i) Since our sequence (A_n) was increasing, and our $A = A_N$ extends that sequence, $A_n \subseteq A$ for all $n < \omega$. So, $B = \bigcup_{n<\omega} A_n \subseteq A$ as desired.

(ii) We show $^\circ\mu(A) = \lim_{n\to\infty} {}^\circ\mu(A_n) = \alpha$.
But by our unique standard part decomposition, $^\circ\mu(A) \leq \alpha + \frac{1}{N}$ and $^\circ\mu(A) \geq {}^\circ\mu(A_n)$ for all $n < w$, so $\alpha \leq {}^\circ\mu(A) \leq \alpha + \frac{1}{N}$ and $\frac{1}{N} \approx 0$, so $^\circ\mu(A) = \alpha$.

(iii) Now notice that $\lim_{n\to\infty} {}^\circ\mu(A\backslash A_n) = \lim_{n\to\infty}({}^\circ\mu(A) - {}^\circ\mu(A_n)) = 0$, and $A\backslash B \subseteq A\backslash A_n$ for all $n < \omega$. Therefore, for any $\varepsilon > 0$, $A\backslash B$ can be covered by $A\backslash A_m$ for some $m < \omega$ such that $^\circ\mu(A\backslash A_m) < \varepsilon$. Therefore $^\circ\mu(A\backslash B) < \varepsilon$, so $(A\backslash B)$ is Loeb Null.

□

Now we can actually concretely define our Loeb extension. We do so as follows:

Definition.

(i) Let $B \subseteq \Omega$. We say that B is Loeb Measurable if $\exists A \in \mathcal{A}$ such that $A \triangle B$ is Loeb Null. If so, $B \in L(\mathcal{A})$.

(ii) For $B \in L(\mathcal{A})$, define $\mu_L(B) = {}^\circ\mu(A)$ for any $A \in \mathcal{A}$ with $A \triangle B$ Loeb Null.

Now, we can prove

Theorem. *(Loeb, 1975)*
$L(\mathcal{A})$, constructed in this way, is a σ-algebra, and μ_L is a complete measure on $L(\mathcal{A})$.

Proof. Obviously, as $L(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$, it is a σ-algebra. Furthermore, since μ is an internal finitely additive measure on $\mathcal{A}$, we only need to check countable additivity of μ_L on $L(\mathcal{A})$.

In that vein, let $(A_n)_{n<\omega}$ be a countable collection of sets in $\mathcal{A}$. We will construct an increasing sequence of sets in $\mathcal{A}$ and use our previous lemma to approximate $A = \bigcup A_n$.

Define a new sequence of sets B_n recursively by

$$B_1 = A_1, B_2 = B_1 \cup A_2, \text{ and } B_i = B_{i-1} \cup A_i \text{ for all } i < \omega.$$

Since each B_i is a finite union of $A_i \in \mathcal{A}$, each $B_i \in \mathcal{A}$. Now we have an increasing sequence (B_n) with

$$\mu(B_i) = \sum_{n=1}^{i} {}^\circ\mu(A_n) \text{ for all } i < \omega \text{ and } \bigcup_{n<k} B_n = \bigcup_{n<k} A_n \text{ for all } k < \omega.$$

By our lemma, there exists some $B \in \mathcal{A}$ such that $A \subseteq B$, $^\circ\mu(B) = \lim_{n\to\infty} {}^\circ\mu(B_n) = \sum_{n=1}^{\infty} {}^\circ\mu(A_n)$, and $B\backslash A$ is null, so $\mu_L(A) =^\circ \mu(B)$.

□

Applications

As an example, we will construct alternate definitions of both Lebesgue measure and Brownian motion using non-standard tools and Loeb's construction. We will then show that our constructions agree with the pre-existing definitions.

Lebesgue Measure:

First, fix an infinite hypernatural number $N \in^* \mathbb{N}\backslash\mathbb{N}$. Construct a hyperfinite set on $[0,1]$ which looks like $^*H = \{0, \frac{1}{N}, \frac{2}{N}, ..., 1\}$. If we let $t = N^{-1}$,

$$^*H = \{0, \delta, 2\delta, ..., 1-\delta, 1\}.$$

Let μ_L be the Loeb counting measure on $^*H$:

$$\mu_L(A) = \frac{|A|}{N}$$

$M \subseteq [0,1]$ is Lebesgue measurable if and only if $st^{-1}(M)$ is Loeb measurable. Let $\mathcal{M}$ represent the family of Lebesgue measurable sets defined in this way.
And we define Lebesgue measure, m, on the Lebesgue measurable sets as

$$m(M) = \mu_L(st^{-1}(M)).$$

Theorem. *The triple* $([0,1], \mathcal{M}, m)$ *is exactly Lebesgue measure on [0,1].*

Proof. Let $[a,b] \in [0,1]$. First, we show $[a,b]$ is Lebesgue measurable by our definition. We have

$$st^{-1}([a,b]) = \bigcap_{n<\omega} {}^*[a - \frac{1}{n}, b + \frac{1}{n}]$$

Which is Loeb measurable as it is a countable intersection of internal (Loeb Measurable) sets. Hence, the interval $[a,b]$ is Lebesgue measurable by our definition.
Furthermore,

$$m([a,b]) = \mu_L(\bigcap_{n<\omega} {}^*[a - \frac{1}{n}, b + \frac{1}{n}]) = \mu_L({}^*[a,b]) =^\circ \mu[a,b] = \frac{|[a,b]|}{N} = \frac{bN - aN}{N} = b - a$$

as desired. m is also clearly translation invariant on intervals $[a,b]$.
We also already have that m is complete by our construction. Since this completion is unique, m is exactly Lebesgue measure on $[0,1]$.

□

Now, we will use our Loeb Measure to construct an intuitive definition of Brownian Motion as a hyperfinite random walk. First, we need a bit of setup:

Definition. *We define a Stochastic Process on a probability space* Ω *as a collection of random variables* $\{W_t\}$ *on* Ω.

Definition. *Further, the classical definition of Brownian Motion is that Brownian motion is a stochastic process* $\{W_t\}_{t\geq 0}$ *indexed by time* t *such that:*

(i) $W_0 = 0$
(ii) The function $t \to W_t$ *is continuous with probability 1*
(iii) The process $\{W_t\}_{t\geq 0}$ *has stationary increments: For any* $s > 0$, $W_{t+s} - W_s$ *has the same distribution as* $W_t - W_0$.
(iv) The process has independent increments: If $W_{t_1} - W_{t_2}$ *and* $W_{s_1} - W_{s_2}$ *are non-overlapping increments (meaning* $t_1 < t_2 \leq s_1 < s_2$*), then their distributions are independent of each other.*
(v) Each interval $W_t \sim N(0,t)$.

We will now construct a one-dimensional Brownian Motion on the interval $[0,1]$ using a similar technique to our Lebesgue measure construction. While the proof is far too long to include in this paper, we will also state a theorem (Anderson, 1976) that states that it does indeed satisfy our definition above.

Fix $N \in {}^*\mathbb{N}\backslash\mathbb{N}$, and let $\Delta t = N^{-1}$ as before. Consider our hyperfinite timeline from above, $*H = \{0, \delta, 2\delta, 3\delta, ..., 1-\delta, 1\}$.

We let $\Omega = \{-1,1\}^H$ be our underlying set, and let our internal measure on Ω be $\mu(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{|2^N|}$.
Note that since $\mu(\Omega) = 1$, μ is an internal probability measure on Ω.
For each $t \in H$ and $\omega \in \Omega$, let

$$B_t(\omega) = \sum_{s}^{H_{<t}} (\omega(s)\sqrt{\delta})$$

Essentially, $B_t(\omega)$ says: pick a time t and a subset of Ω, ω. This set ω will tell us what to do at each point of time $s < t$. Then, we move $\pm\sqrt{\delta}$ depending on the value of $\omega(s)$. Do this t times. Ranging over all subsets ω of Ω, B_t will give us a distribution of where we might expect to end up at time t.

However, this all lives within our nonstandard set Ω. We need to transfer back to the standard interval [0,1]. We will do so as follows:
If t = 0 or 1, $b_t =^\circ B_t$ for each $t \in (0,1)$, $n\delta \leq t \leq (n+1)\delta$ for some $n < N$. Define $k = n\delta$ with n satisfying the above property. Then let $b_t = {}^\circ B_k$.
Then we claim the collection $\{b_t\}_{t\in[0,1]}$ is Brownian motion on the interval $[0,1]$.
In fact, we state without proof:

Theorem 1. *(Anderson, 1976) The process $b : [0,1] \times \Omega \to \mathbb{R}$ defined, as above, by*

$$b_t(\omega) = \begin{cases} {}^\circ B_t(\omega) & t = 0,1 \\ {}^\circ B_k(\omega) & t \in (0,1) \end{cases}$$

is Brownian Motion on the probability space Ω.

Another notable conclusion we can draw from this construction is that

$$B_{(t+\Delta t)} - B_{(t)} = B_{\Delta t} = \Delta B = \pm\sqrt{\Delta t}\ ,$$

So in some sense, we have $(\Delta B)^2 = \Delta t$, as in Itô's lemma (essentially stating that for Itô processes, $dB^2 = dt$, or that Brownian motion has quadratic variation with respect to time). This is a final example of how intuitive (and simple) non-standard analysis can be as a way of constructing these tools. One can in fact go on to define a concept of an integral under Loeb measure, and represent Itô integrals in this way.

This concludes my introductory presentation of non-standard analysis and Loeb measure, and a few applications to general results in measure and probability theory. This construction hinges on a couple concepts we discussed in class, notably the transfer principle for internal sets and our ultrapower construction. Once one understands those technical details, this construction provides a much more intuitive picture of many analytical concepts.

References

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