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Chapter 2 - Solving Systems of Equations
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Problem 1. Show that the inverse of an elementary matrix is another elementary matrix. Give 3×3 examples.

Proof. Notice that there are three elementary matrix operations, multiplication of a row, addition of two rows, and swapping of two rows. Let A be an $m \times n$ matrix. Then, let E_1 be the matrix that multiplies the ith row of A by a nonzero constant $c \in \mathbb{R}$, which can be represented as $A'_{ik} = cA_{ik}$ for all $k \in \{1, 2, \ldots, n\}$. The A' matrix is the resulting matrix after E_1 is applied to A. Let E'_1 be the reverse operation, which is multiplying the ith row of A' by $\frac{1}{c}$. Equivalently, this becomes $A''_{ik} = \frac{1}{c}A'_{ik}$ for all $k \in \{1, 2, \ldots, n\}$, where A'' is the matrix after applying E'_1 to A'. We know this is a valid operation since $\frac{1}{c} \in R$, which means

$$A''_{ik} = \frac{1}{c}A'_{ik} = \frac{1}{c} \cdot cA_{ik} = A_{ik}.$$

Thus, $E_1 E'_1 = I$ and for our example we multiply the second row by c and in matrix form

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E'_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$E_1 E'_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c \cdot \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

For the addition of two rows we let E_2 be the elementary matrix representing the addition of row j to row i, which is $A'_{ik} = A_{ik} + A_{jk}$ for all $k \in \{1, 2, \ldots, n\}$. Let E'_2 be the reverse operation, which is subtracting row j from row i. Similarly written as $A''_{ik} = A'_{ik} - A'_{jk}$ for all $k \in \{1, 2, \ldots, n\}$. Notice that because A_{jk} is unchanged then we have

$$A''_{ik} = A'_{ik} - A'_{jk} = A_{ik} + A_{jk} - A_{jk} = A_{ik}$$

for all $k \in \{1, 2, \ldots, n\}$. We know that the second operation must be an elementary matrix by definition because subtraction is just addition of the additive identity. Our example for row addition is adding row one and three to row two and in matrix form

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E'_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives

$$E_2 E'_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1-1 & 1 & -1+1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Lastly, we have a row swap operation. So, we let E_3 be the elementary matrix representing the swap of rows i and j. In matrix form we have $A'_{jk} = A_{ik}$ and $A'_{ik} = A_{jk}$, where A' represents the matrix after the multiplication of E_3. Let E'_3 be the reverse operation of swapping rows i and j again, which gives $E'_3 = E_3$. So, we have $A''_{ik} = A'_{jk} = A_{ik}$ and $A''_{jk} = A'_{ik} = A_{jk}$, where A'' is the matrix after the inverse operation. Thus, we have that $E_3 E'_3 = I$. For our example we swap the first and second rows, which is

$$E_3 = E'_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice we have

$$E_3 E_3' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

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