Brandon M. Keltz An Introduction to Computational Science by Allen Holder and Joseph Eichholz Chapter 2 - Solving Systems of Equations November  $18,\,2019$ 

Problem 4. Suppose A is an  $m \times n$  matrix such that the rank(A) = n. Show that  $A^T A \succ 0$ .

*Proof.* Let the column vectors of A be denoted  $A_i$  for all  $i \in \{1, 2, ..., n\}$ , which is

$$A = \begin{bmatrix} | & | & | \\ A_1 & A_2 & \dots & A_n \\ | & | & | \end{bmatrix}.$$

Since the rank(A) = n, this means that the collection of the vectors  $A_i$  are linearly independent. By this linear independence we have

$$0 < ||Ax|| = (Ax)^T Ax = x^T A^T Ax,$$

which gives  $x^T A^T A x > 0$ .

1