

*Problem 4.* Suppose  $A$  is an  $m \times n$  matrix such that the  $\text{rank}(A) = n$ . Show that  $A^T A \succ 0$ .

*Proof.* Let the column vectors of  $A$  be denoted  $A_i$  for all  $i \in \{1, 2, \dots, n\}$ , which is

$$A = \begin{bmatrix} | & | & & | \\ A_1 & A_2 & \dots & A_n \\ | & | & & | \end{bmatrix}.$$

Since the  $\text{rank}(A) = n$ , this means that the collection of the vectors  $A_i$  are linearly independent. By this linear independence we have

$$0 < \|Ax\| = (Ax)^T Ax = x^T A^T Ax,$$

which gives  $x^T A^T Ax > 0$ . ■