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Problem 2. Show that the product of upper (lower) triangular matrices is upper (lower) triangular.

Proof. Without loss of generality, let A and B be upper triangular matrices, where A and B have dimensions $m \times p$ and $p \times n$, respectively. The product of A and B is

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^{j} a_{ik} b_{kj} + \sum_{k=j+1}^{p} a_{ik} b_{kj}.$$

Notice that for all $k < i$ we have

$$\sum_{k=1}^{i-1} a_{ik} b_{kj} = 0$$

because A is upper triangular. Similarly, for all $k > j$ we have

$$\sum_{k=j+1}^{n} a_{ik} b_{kj} = 0$$

because B is upper triangular. This gives

$$c_{ij} = \sum_{k=i}^{j} a_{ik} b_{jk},$$

which is zero for $j < i$. So, by definition C must also be an upper triangular matrix.

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Proof. Without loss of generality, let A and B be lower triangular matrices, where A and B have dimensions $m \times p$ and $p \times n$, respectively. The product of A and B is

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = \sum_{k=1}^{j-1} a_{ik} b_{kj} + \sum_{k=j}^{i} a_{ik} b_{kj} + \sum_{k=i+1}^{p} a_{ik} b_{kj}.$$

Notice that for all $i < k$ we have

$$\sum_{k=i+1}^{p} a_{ik} b_{kj} = 0$$

because A is lower triangular. Similarly, for all $k < j$ we have

$$\sum_{k=1}^{j-1} a_{ik} b_{kj} = 0$$

because B is lower triangular. This gives

$$c_{ij} = \sum_{k=j}^{i} a_{ik} b_{jk},$$

which is zero for $i < j$. So, by definition C must also be a lower triangular matrix.

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