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Chapter 2 - Solving Systems of Equations
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Problem 3. Let $Q \succ 0$ be an $n \times n$ matrix and let $v_1, v_2, \ldots, v_n$ be a collection of Q-conjugate n-vectors. Show that the collection is linearly independent.

Proof. Consider the linear combination of the Q-conjugate n-vectors such that

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n = 0,$$

where $\alpha_i \in \mathbb{R}$ for all $i \in \{1, 2, \ldots, n\}$. Since $Q \succ 0$

$$\alpha_1 v^1 + \alpha_2 v^2 + \ldots + \alpha_n v^n = 0 \implies \alpha_1 Q v^1 + \alpha_2 Q v^2 + \ldots + \alpha_n Q v^n = 0.$$

Consider an arbitrary $v^j \in \left\{v^1, v^2, \ldots, v^n\right\}$. We now have

$$\alpha_1 Q v^1 + \alpha_2 Q v^2 + \ldots + \alpha_n Q v^n = 0 \implies \alpha_1 \left(v^i\right)^T Q v^1 + \alpha_2, \left(v^i\right)^T Q v^2 + \ldots + \alpha_n \left(v^i\right)^T Q v^n = 0,$$

which gives for $i \neq j$

$$\alpha_j \left(v^i\right)^T Q v^j = 0$$

because every vector $v^i \in \left\{v^1, v^2, \ldots, v^n\right\}$ is Q-conjugate. Notice for $i = j$ we have

$$\alpha_i \left(v^i\right)^T Q v^i = 0 \iff \alpha_i ||v^i||_Q = 0 \implies \alpha_i = 0.$$

Since α_i is arbitrary, then this is applied to every vector in the collection. Thus, $\alpha_i = 0$ for all $i \in \{1, 2, \ldots, n\}$, which means the collection must be linear independent.

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