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Chapter 1 - Solving Single Variable Equations
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Problem 4. Show that the sequence defined by the recursive statement $d_{k+1} = \frac{d_k}{2}$, with $d_1 = a$ converges with order 1 for any constant a. What does this say about the bisection method, especially if compared to Newton's method?

Proof. Notice that the sequence

$$d_{k+1} = \frac{d_k}{2},$$

where $d_1 = a$ has a closed-form solution

$$d_{k+1} = \frac{a}{2^k}.$$

Linear convergence of a sequence guarantees that

$$\lim_{k\to\infty} \frac{\left|x^{k+1} - x^*\right|}{\left|x^k - x^*\right|^p} < \infty,$$

where $p = 1$. Without loss of generality, consider a nonzero. From the closed-form solution we have

$$\lim_{k\to\infty} \frac{\left|x^{k+1} - x^*\right|}{\left|x^k - x^*\right|} = \lim_{k\to\infty} \frac{\left|a2^{-k} - 0\right|}{\left|a2^{1-k} - 0\right|} = \lim_{k\to\infty} \frac{\left|a2^{-k}\right|}{\left|a2^{1-k}\right|} = \lim_{k\to\infty} \frac{\left|2^{-k}\right|}{\left|2^{1-k}\right|} = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2} < \infty,$$

which gives us order 1 convergence for when a is nonzero.

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This shows the bisection method has a linear convergence. This comes from the iteration limit based on the bracketed interval we have

$$\varepsilon \geq \frac{b - a}{2^k},$$

where k is the number of iterations. The same analysis can be done since the form of the sequence and the error are equivalent to a constant. However, Newton's method has a quadratic convergence, which means that the bisection and Newton's method for this function would have the same order of convergence.