

Homework 3

1.3.12

Solution:

Let D be the set of diagonal matrices.

- The 0 matrix is in D because $A_{ij} = 0$ for all entries where $i > j$ (moreover, all entries are zero).
- Take some scalar $c \in F$ and matrix $A \in D$. When we perform the multiplication cA , for every entry A_{ij} where $i > j$ and thus $A_{ij} = 0$, $cA_{ij} = 0$. Thus $cA \in D$, i.e. D is closed under scalar multiplication.
- Take matrices $A, B \in D$. When we add A and B , since $A_{ij} = B_{ij} = 0$ whenever $i > j$, $A_{ij} + B_{ij} = 0$ when those entries are below the diagonal. So $A + B \in D$, i.e. D is closed under addition.

1.6.16

Solution:

A basis for W is the set of E^{ij} where the only non-zero entry is a 1 in the i, i position and $1 \leq i \leq j \leq n$.

$$\dim W = n + (n - 1) + \cdots + 1 = \frac{1}{2}n(n + 1)$$

1.6.22

Solution:

A necessary and sufficient condition is that $W_1 \subseteq W_2$.

- $W_1 \subseteq W_2 \rightarrow \dim(W_1 \cap W_2) = \dim(W_1)$
If $W_1 \subseteq W_2$ then $W_1 \cap W_2 = W_1$. So, since we assumed $W_1 \subseteq W_2$, it follows that $\dim(W_1 \cap W_2) = \dim(W_1)$.
- $\dim(W_1 \cap W_2) = \dim(W_1) \rightarrow W_1 \subseteq W_2$
Let $\dim(W_1 \cap W_2) = n$, $\dim(W_1) = n$, and $\dim(W_2) = m$. Now we define some arbitrary basis for $W_1 \cap W_2$, $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$. Since \mathcal{B} is a basis it is linearly independent. We also know that $W_1 \cap W_2 \in W_1$, so $\mathcal{B} \in W_1$. Since \mathcal{B} has n linearly independent elements and W_1 had a dimension of n it forms a maximal linearly independent subset of W_1 , so \mathcal{B} is a basis for W_1 . Now, we also know that $W_1 \cap W_2 \subseteq W_2$, so its basis, \mathcal{B} can be extended into a basis for W_2 . Since the basis for W_2 contains the basis for W_1 we can conclude that $W_1 \subseteq W_2$.

1.6.29.a

Solution:

Let $\dim(W_1 \cap W_2) = k$, $\dim(W_1) = n$, and $\dim(W_2) = m$. Then we may express a basis for $W_1 \cap W_2$ as some linearly independent subset $\mathcal{B}_3 = \{u_1, u_2, \dots, u_k\}$. Since, $W_1 \cap W_2 \subseteq W_1, W_2$ we may extend \mathcal{B}_3 to bases for W_1 and W_2 . Let $\mathcal{B}_1 = \mathcal{B}_3 \cup \{v_1, \dots, v_{n-k}\}$ be a basis for W_1 and $\mathcal{B}_2 = \mathcal{B}_3 \cup \{w_1, \dots, w_{m-k}\}$ be a basis for W_2 . Now we can show that the set $\mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ is a basis for $W_1 + W_2$. First let's show that the set is linearly independent. First we know that each part of the union is linearly independent because they were each subsets of bases. Now, take $v \in \text{span}(\mathcal{B}_1 \setminus \mathcal{B}_3)$. We know that $v \notin \text{span}(\mathcal{B}_2 \setminus \mathcal{B}_3)$ because those portions of the bases didn't share any elements in common. Likewise, for any $v \in \text{span}(\mathcal{B}_2 \setminus \mathcal{B}_3)$, $v \notin \text{span}(\mathcal{B}_1 \setminus \mathcal{B}_3)$. Now take $v \in \text{span}(\mathcal{B}_3)$, we immediately see that $v \notin \mathcal{B}_1 \setminus \mathcal{B}_3$ and $v \notin \mathcal{B}_2 \setminus \mathcal{B}_3$. So the set $\mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ is linearly independent. Now we can show that the set $\mathcal{B} = \mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ spans $W_1 + W_2$. We can show this by contradiction. Take $v \in W_1 + W_2$. If $v \notin \text{span}(\mathcal{B})$ then $v \notin \text{span}(W_1)$ and $v \notin \text{span}(W_2)$, so $v \notin W_1 + W_2$. Which is a contradiction. So \mathcal{B} spans $W_1 + W_2$.

1.6.31

Solution:

- a. Let $\dim(W_1 \cap W_2) = k$. Then we can define some arbitrary basis $\mathcal{B} = \{u_1, u_2, \dots, u_k\}$ for $W_1 \cap W_2$. Since $W_1 \cap W_2$ is a subspace of W_2 , $\mathcal{B} \subseteq W_2$. Since \mathcal{B} is linearly independent, a maximal linearly independent subset of W_2 must have at least k elements. So $k = \dim(W_1 \cap W_2) \leq n$.
- b. We have shown that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. Since $\dim(W_1 \cap W_2) \geq 0$, $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2) = m + n$.

1.6.32

Solution:

- a. We let $W_1 = R^3$ and $W_2 = \text{span}((1, 0, 0))$. So $m = 3$ and $n = 1$. $W_1 \cap W_2 = W_2$ since $W_2 \subset W_1$. So, $\dim(W_1 \cap W_2) = \dim(W_2) = 1 = n$.
- b.
- c.