Math 211 Homework 2 **Name: Kelvin Abrokwa-Johnson**

1 (5 points) Write the augmented matrix of a system of linear equations for the following vector equation

$$x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and solve the system.

$$\begin{bmatrix} 3 & 7 & -2 & 0 \\ -2 & 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 7 & -2 & 0 \\ 0 & 23 & -1 & 0 \end{bmatrix}$$

The solution:

$$x_1 = \frac{13}{23}t$$

$$x_2 = \frac{1}{23}t$$

$$x_3 = t$$

2 (5 points) Write the following linear system as vector equation,

$$3x_1 - 2x_2 + 4x_3 = 3,$$

$$-2x_1 - 7x_2 + 5x_3 = 1,$$

$$5x_1 + 4x_2 - 3x_3 = 2,$$

and solve the system.

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

To solve:

$$\begin{bmatrix} 3 & -2 & 4 & 3 \\ -2 & -7 & 5 & 1 \\ 5 & 4 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 4 & 3 \\ 0 & -25 & 23 & 9 \\ 0 & 22 & -29 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 4 & 3 \\ 0 & -25 & 23 & 9 \\ 0 & 0 & -73 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{2}{3} & \frac{4}{3} & 1 \\ 0 & 1 & -\frac{23}{25} & -\frac{9}{25} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{2}{3} & 0 & \frac{61}{73} \\ 0 & 1 & 0 & -\frac{18}{73} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{61}{73} \\ 0 & 1 & 0 & -\frac{18}{73} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix}$$

$$x_1 = \frac{61}{73}$$

$$x_2 = -\frac{18}{73}$$

$$x_3 = \frac{9}{73}$$

3 (5 points) Find all the values of h such that the following vector equation has solution(s),

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}.$$

Solution:
$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{bmatrix}$$
We see here that the system is only consistent when $2h + 4 = 0$. When we solve we find that the system has a solution when:
$$h = -2$$

4 (10 points) Given the following matrix A and a vector $\mathbf{b}$.
$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}.$$

(a) Determine if $\mathbf{b}$ is a linear combination of the columns of A.

(b) Show that the second column of A is a linear combination of the columns of A.

Solution:

(a)
$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$
We can see from the final matrix above that the matrix is inconsistent since the last equation has all 0 coefficients resulting in a non-zero answer.

(b)
$$x_1 = 0$$
$$x_2 = 1$$
$$x_3 = 0$$

5 (5 points) Compute the matrix product $A\mathbf{b}$ for
$$A = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
$$Ab = 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

6 (5 points) Let $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$. Show that $A\mathbf{x} = \mathbf{b}$ is not always solvable for all $\mathbf{b} \in \mathbb{R}^3$, and describe the set of $\mathbf{b} \in \mathbb{R}^3$ (as a plane or a line) such that the system is solvable. Solution:
$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 0 & 1 & 1 & -\frac{4}{7}b_1 + \frac{1}{7}b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -1 & -1 & b_1 + \frac{b_2}{2} \\ 0 & 0 & 0 & \frac{3}{7}b_1 + \frac{1}{2}b_2 + \frac{1}{7}b_3 \end{bmatrix}$$

We see that in order for the system to be solvable when:

$$\left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : \frac{3}{7}b_1 + \frac{1}{2}b_2 + \frac{1}{7}b_3 = 0, b_1, b_2, b_3 \in \mathbb{R} \right\}$$

7 (5 points) Given three vectors

$$\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix},$$

determine if these vectors generate, or span, $\mathbb{R}^3$.

Solution:

$$\begin{bmatrix} 0 & 0 & 4 & b_1 \\ 2 & 1 & 1 & b_2 \\ 5 & 3 & 1 & b_3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 4 & b_1 \\ 2 & 1 & 1 & b_2 \\ 0 & \frac{1}{2} & -\frac{3}{2} & b_3 - b_2\frac{5}{2} \end{bmatrix}$$

Since there are 3 pivoting columns we can conclude that these vectors span $\mathbb{R}^3$.

8 (5 points) Given that

$$\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad 2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = 0.$$

Find x_1 and x_2 that satisfy

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:
We can quickly manipulate the information given above like so:

$$2u - 3v - w = 0$$

$$2u - 3v = w$$

So we have:

$$x_1 = 2$$

$$x_2 = -3$$

9 (5 points) Suppose the solution set of a linear system are described as:

$$x_1 = 5x_4, x_2 = 3 - 2x_4, x_3 = 2 + 5x_4, \qquad x_4 \in \mathbb{R}.$$

Describe the solution set as a line in $\mathbb{R}^4$ in the format of $\{\mathbf{p} + t\mathbf{v} : t \in \mathbb{R}\}$.
Solution:
Since x_4 is a free variable we can set it equal to a variable t, $x_4 = t$. We can thens substitute to the get the following vector:

$$\begin{bmatrix} 5t \\ 3 - 2t \\ 2 + 5t \\ t \end{bmatrix}$$

So our final solution is:

$$\left\{\begin{bmatrix}0\\3\\2\\0\end{bmatrix}+t\begin{bmatrix}5\\-2\\5\\1\end{bmatrix}:t\in\mathbb{R}\right\}$$

10 (5 points) Describe geometrically (a line, a plane, etc.) the solution set of $x_1 - 2x_2 + 3x_3 = 0$, and the solution set of $x_1 - 2x_2 + 3x_3 = 4$, as subsets in $\mathbb{R}^3$.
Solution:
For the first equation, we have the matrix $\begin{bmatrix}1 & -2 & 3 & 0\end{bmatrix}$. When we row reduce we find that we have 2 free variables, $x_3 = t$ and $x_2 = s$. We solve for x_1 to get the following solution set:

$$x_1 = 2s - 3t$$

$$x_2 = s$$

$$x_3 = t$$

Which can be expressed as the plane:

$$\left\{t\begin{bmatrix}-3\\0\\1\end{bmatrix}+s\begin{bmatrix}2\\1\\0\end{bmatrix}:s,t\in\mathbb{R}\right\}$$

For the second equation we can set $x_2 = s$ and $x_3 = t$ to get the following solution set:

$$x_1 = 4 + 2s - 3t$$

$$x_2 = s$$

$$x_3 = t$$

Which can be expressed as:

$$\left\{\begin{bmatrix}4\\0\\0\end{bmatrix}+s\begin{bmatrix}2\\1\\0\end{bmatrix}+t\begin{bmatrix}-3\\0\\1\end{bmatrix}:s,t\in\mathbb{R}\right\}$$

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11 (5 points) **Extra credits** Show that the reduced row echelon form of an augmented matrix is unique.
Solution:
Assume A_1, A_2 are different reduced row echelon forms of an augmented matrix A. Say:

$$A_1 = \begin{bmatrix}a_{11} & * & * & c_1\\0 & a22 & * & c_2\\0 & 0 & a_{33} & c_3\end{bmatrix}$$

$$A_2 = \begin{bmatrix}b_{11} & * & * & c_1\\0 & b_{22} & * & c_2\\0 & 0 & b_{33} & c_3\end{bmatrix}$$

If these matrices are indeed different, then $a_{33} \neq b_{33}$. So, in A_1, $x_3 = \frac{c_3}{a_{33}}$ and in A_2, $x_3 = \frac{c_3}{b_{33}}$. Since $\frac{c_3}{a_{33}} \neq \frac{c_3}{b_{33}}$ when $a_{33} \neq b_{33}$, the solutions are different, so we have shown by contrapositive that A_1 and A_2 must be different.