

Homework 1

7. Let  $S = 0, 1$  and  $F = R$ . In  $\mathcal{F}(S, R)$ , show that  $f = g$  and  $f + g = h$ , where  $f(t) = 2t + 1$ ,  $g(t) = 1 + 4t + 2t^2$ , and  $h(t) = 5^t + 1$ .

**Solution:**

For 0:

$$f(0) = 2(0) + 1 = g(0) = 1 + 4(0) + 2(0)^2 = 1$$

$$h(0) = 5^0 + 1 = f(0) + g(0) = 1 + 1 = 2$$

For 1:

$$f(1) = 2(1) + 1 = g(1) = 1 + 4(1) + 2(1)^2 = 3$$

$$h(1) = 5^1 + 1 = f(1) + g(1) = 3 + 3 = 6$$

Thus, we have shown what was to be shown.

8. In any vector space  $V$ , show that  $(a + b)(x + y) = ax + ay + bx + by$  for any  $x, y \in V$  and any  $a, b \in F$ .

**Solution:**

12. A real-valued function  $f$  defined on the real line is called an even function if  $f(-t) = f(t)$  for each real number  $t$ . Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

**Solution:**

- Suppose  $E$  is the set of even functions. Take the function  $f \in E$ . Since  $f(-t) = f(t)$ , we know that  $af(-t) = af(t)$  for all  $a \in \mathbf{R}$ , so  $(af)(t) \in E$ , i.e. closed under scalar multiplication.
- Take  $f, g \in E$ .  $(f + g)(t) = f(t) + g(t) = f(-t) + g(-t) = (f + g)(-t)$ . In short  $(f + g)(t) = (f + g)(-t)$  so  $(f + g)(t) \in E$ , i.e. closed under addition.
- Since  $f(t) = f(-t)$  for  $f \in E$  where  $f(t) = 0$ ,  $f(t) = 0 \in E$ , i.e. has a 0 function.

**18.** Let  $V = \{(a_1, a_2) : a_1, a_2 \in \mathbf{R}\}$ . For  $(a_1, a_2), (b_1, b_2) \in V$  and  $c \in \mathbf{R}$ , define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2)$$

Is  $V$  a vector space over  $\mathbf{R}$  with these operations?

**Solution:**

No,  $V$  is not a vector space over  $\mathbf{R}$ .

Take  $a_1, a_2 \neq 0 \in \mathbf{R}$ . Since  $(a_1, a_2) + (0, 0) = (a_1, a_2)$  and  $(0, 0) + (a_1, a_2) = (2a_1, 3a_2)$ , i.e.  $(a_1, a_2) + 0 \neq 0 + (a_1, a_2)$ . The set does not have commutative addition so it is not a vector space.

**21.** Let  $V$  and  $W$  be vector spaces over a field  $F$ . Let

$$Z = \{(v, w) : v \in V, w \in W\}$$

Prove that  $Z$  is a vector space over  $F$  with operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1)$$

**Solution:**

- Take  $v_1, v_2 \in V, w_1, w_2 \in W$ . Since  $V$  and  $W$  are vector spaces,  $v_1 + v_2 \in V$  and  $w_1 + w_2 \in W$ , so it follows that  $(v_1, w_1) + (v_2, w_2) \in Z$ , i.e. closed under addition.
- Take  $v \in V, w \in W, c \in F$ . Since  $V$  and  $W$  are vector spaces,  $cv \in V$  and  $cw \in W$ . So,  $c(v, w) = (cv, cw) \in Z$ , i.e. closed under scalar multiplication.
- Since  $V$  and  $W$  are vector spaces,  $\vec{0} \in V, W$ . So  $(\vec{0}_V, \vec{0}_W) \in Z$ , i.e.  $Z$  has a 0 vector.

**19.** Let  $W_1$  and  $W_2$  be subspaces of a vector space  $V$ . Prove that  $W_1 \cup W_2$  is a subspace of  $V$  if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

**Solution:**

- Proof (1):  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$  implies  $W_1 \cup W_2$  is a subspace.  
Assume, without loss of generality, that  $W_1 \subseteq W_2$ . Then,  $W_1 \cup W_2 = W_2$  which we know is a subspace (the same logic applies when  $W_2 \subseteq W_1$ ).
- Proof(2):  $W_1 \cup W_2$  is a subspace implies  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .  
We will use proof by contradiction. Assume  $W_1 \cup W_2$  is a subspace and  $W_1 \not\subseteq W_2$  and  $W_2 \not\subseteq W_1$ . Take  $w_1 \in W_1, \notin W_2, w_2 \in W_2, \notin W_1$ . We know what  $w_1 + w_2 \in W_1 \cup W_2$ , which means  $w_1 + w_2 \in W_1$  or  $w_1 + w_2 \in W_2$ . We assume, without loss of generality, that  $w_1 + w_2 \in W_1$ . Now, we add the additive inverse of  $w_1$ ,  $-w_1 \in W_1$  to  $w_1 + w_2$  to get  $(w_1 + w_2) + (-w_1) = w_2 \in W_1$  which contradicts our initial condition that  $w_2 \notin W_1$ .

**20.** Prove that if  $W$  is a subspace of a vector space  $V$  and  $w_1, w_2, \dots, w_n$  are in  $W$ , then  $a_1w_1 + a_2w_2 + \dots + a_nw_n \in W$  for any scalar  $a_1, a_2, \dots, a_n$ .