

**1** (5 points) Write the augmented matrix of a system of linear equations for the following vector equation

$$x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and solve the system.

$$\begin{bmatrix} 3 & 7 & -2 & 0 \\ -2 & 3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 7 & -2 & 0 \\ 0 & 23 & -1 & 0 \end{bmatrix}$$

The solution:

$$x_1 = \frac{13}{23}t$$

$$x_2 = \frac{1}{23}t$$

$$x_3 = t$$

**2** (5 points) Write the following linear system as vector equation,

$$3x_1 - 2x_2 + 4x_3 = 3,$$

$$-2x_1 - 7x_2 + 5x_3 = 1,$$

$$5x_1 + 4x_2 - 3x_3 = 2,$$

and solve the system.

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

To solve:

$$\begin{bmatrix} 3 & -2 & 4 & 3 \\ -2 & -7 & 5 & 1 \\ 5 & 4 & -3 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 4 & 3 \\ 0 & -25 & 23 & 9 \\ 0 & 22 & -29 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -2 & 4 & 3 \\ 0 & -25 & 23 & 9 \\ 0 & 0 & -73 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{2}{3} & \frac{4}{3} & 1 \\ 0 & 1 & -\frac{23}{25} & -\frac{9}{25} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{2}{3} & 0 & \frac{61}{73} \\ 0 & 1 & 0 & -\frac{18}{73} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{61}{73} \\ 0 & 1 & 0 & -\frac{18}{73} \\ 0 & 0 & 1 & \frac{9}{73} \end{bmatrix}$$

$$x_1 = \frac{61}{73}$$

$$x_2 = -\frac{18}{73}$$

$$x_3 = \frac{9}{73}$$

**3** (5 points) Find all the values of  $h$  such that the following vector equation has solution(s),

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{bmatrix}$$

We see here that the system is only consistent when  $2h + 4 = 0$ . When we solve we find that the system has a solution when:

$$h = -2$$

**4** (10 points) Given the following matrix  $A$  and a vector  $\mathbf{b}$ .

$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}.$$

- (a) Determine if  $\mathbf{b}$  is a linear combination of the columns of  $A$ .
- (b) Show that the second column of  $A$  is a linear combination of the columns of  $A$ .

Solution:

(a)

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

We can see from the final matrix above that the matrix is inconsistent since the last equation has all 0 coefficients resulting in a non-zero answer.

(b)

$$x_1 = 0$$

$$x_2 = 1$$

$$x_3 = 0$$

**5** (5 points) Compute the matrix product  $A\mathbf{b}$  for

$$A = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

$$A\mathbf{b} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

**6** (5 points) Let  $A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}$ . Show that  $A\mathbf{x} = \mathbf{b}$  is not always solvable for all  $\mathbf{b} \in \mathbb{R}^3$ , and describe the set of  $\mathbf{b} \in \mathbb{R}^3$  (as a plane or a line) such that the system is solvable.

7 (5 points) Given three vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix},$$

determine if these vectors generate, or span,  $\mathbb{R}^3$ .

8 (5 points) Given that

$$\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad 2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = 0.$$

Find  $x_1$  and  $x_2$  that satisfy

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

9 (5 points) Suppose the solution set of a linear system are described as:

$$x_1 = 5x_4, x_2 = 3 - 2x_4, x_3 = 2 + 5x_4, \quad x_4 \in \mathbb{R}.$$

Describe the solution set as a line in  $\mathbb{R}^4$  in the format of  $\{\mathbf{p} + t\mathbf{v} : t \in \mathbb{R}\}$ .

10 (5 points) Describe geometrically (a line, a plane, etc.) the solution set of  $x_1 - 2x_2 + 3x_3 = 0$ , and the solution set of  $x_1 - 2x_2 + 3x_3 = 4$ , as subsets in  $\mathbb{R}^3$ .

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11 (5 points) **Extra credits** Show that the reduced row echelon form of an augmented matrix is unique.

[Hint: Suppose  $A_1, A_2$  are two different reduced row echelon forms of an augmented matrix  $A$ . Consider the first row from the bottom that the two matrices  $A_1, A_2$  are different. Argue that they will yield different solutions....]