Math 211 Homework 3 Kelvin Abrokwa-Johnson

1. Determine whether the following three vectors are linearly independent.

$$\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 3 & 0 \\ 3 & -8 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & -8 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

After row reduction, we see that since there are 3 pivoting columns we know that there is only one solution to the system, the trivial solution. So, the system is **linearly independent**.

2. Determine the value(s) h such that the following three vectors are linearly independent.

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & 3 & 3 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -6 & 9 & 0 \\ 0 & -8 & 18+h & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & -6 & 9 & 0 \\ 0 & 0 & 18+h & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We can see that the vectors are linearly independent when $h \neq -18$ because you will then have 3 pivoting columns which means one solution to the system.

3. Determine (with explanation) the reduced echelon form of a 3×3 matrix A if it has linearly independent columns.
Solution:
The reduced echelon form of a linearly independent matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We see that in this case, we hae 3 pivoting columns which means there is only one solution when we equate it to the 0 matrix, the trivial solution. Hence, the matrix is linearly independent.

4. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Determine $T(\mathbf{u})$ and $T(\mathbf{v})$ if $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$.
Solution: By performing matrix multiplication, we get the following:

$$T(u) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 12 \\ -21 \\ -54 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 + 2b + 3c \\ 4b + 5c \\ 6c \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}.$$

Find a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.
Solution:

$$\begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -7 \end{bmatrix}$$

And so from that row reduction we can see that $x_1 = -5$, $x_2 = -7$, and $x_3 = -1$. So,

$$x = \begin{bmatrix} -5 \\ -1 \\ -7 \end{bmatrix}$$

6. Let

$$A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}.$$

Find all vectors that are mapped into the zero vector by the transformation $\mathbf{x} \rightarrow A\mathbf{x}$.
Solution:

$$\begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -10 & -20 & -30 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We find that we have to free variables:

$$x_4 = t, x_3 = s, x_2 = -3t - 2s, x_1 = -6s$$

And so vectors that are mapped into the zero vector are of the form:

$$\left\{ s \begin{bmatrix} -6 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

7. Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 7 \\ -2 \end{bmatrix},$$

and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(\mathbf{x}) = x_1\mathbf{v_1} + x_2\mathbf{v_2}$. Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^2$.
Solution:
The question virtually defines matrix multiplication of x and a matrix with rows x_1 and x_2. So our solution is:

$$\begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix}$$

8. Find the standard matrix of a transformation T that reflects points through the horizontal x_1 axis and then reflects the point through the line $x_2 = x_1$.
Solution:

$$T(e_1) = T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T(e_2) = T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

And so the solution is:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

9. Suppose $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$. Find a matrix A such that

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_3 \\ x_1 \end{bmatrix}.$$

Solution: We can rewrite like so:

$$T(x_1, x_2) = \begin{bmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_2 \\ x_1 \end{bmatrix}$$

And so we see that:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix}$$