

Five points for each question unless specified otherwise.

1.

$$AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (a)

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}, B = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 7 & 3 & 1 & 0 \\ -6 & -3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ -6 & -3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -3 & 6 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -\frac{7}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$$

(c)

$$A^{-1}b = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix} = b$$

4. Let A be a 3×3 matrix. Determine all possible reduced row echelon form of the augmented matrix $[A \ I_3]$ and decide whether A is invertible in each of the following cases.

- (a) The matrix A has 1 pivoting column.
- (b) The matrix A has 2 pivoting columns.
- (c) The matrix A has 3 pivoting columns.

5. Using regular matrix algebra we see:

$$AB = AC$$

$$AB - AC = 0$$

$$A(B - C) = 0$$

$$A^{-1}A(B - C) = A^{-1}0$$

$$B - C = 0$$

$$B = C$$

6. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -3 \end{bmatrix}$. Apply row reduction to the matrix $[A \ I_3]$ until we have the reduced row echelon form $[I_3 \ B]$, and verify that $AB = I_3$.

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -3 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & -6 & 1 & -1 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 6 & -1 & 1 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 6 & -1 & 1 \\ 10 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

(a) Find a 4×2 matrix B such that $AB = I_2$.

[Hint: Solve $\mathbf{b}_1, \mathbf{b}_2 \in \mathbf{R}^4$ such that $A\mathbf{b}_1 = \mathbf{e}_1 \in \mathbf{R}^2$ and $A\mathbf{b}_2 = \mathbf{e}_2 \in \mathbf{R}^2$.

(b) Explain why there is no 4×2 matrix C such that $CA = I_4$.

(a) We can augment the matrix with both \mathbf{e}_1 and $\mathbf{e}_2 \in \mathbf{R}^2$ to find the columns of the matrix B .

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In both cases we have two free variables, x_3 and x_4 , which we can set equal to s and t , respectively.

If we set $s = 0$ and $t = 1$, then we have $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. and so $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Indeed

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) There is no 4×2 matrix C such that $CA = I_4$ because

8. Suppose A is an $n \times n$ matrix such that the columns of A^2 spans \mathbf{R}^n .

Show that A has linearly independent columns.

9. (10 points)

(a)

$$A = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}$$

$$A^{-1} = \det A \begin{bmatrix} 7 & 8 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & -4 \\ -1 & -1 \end{bmatrix}$$

(b)

$$R(x_1, x_2) = \left(-\frac{7}{2}x_1 - x_2, -4x_1 - x_2\right)$$

(c)

$$\begin{aligned} & R \circ T(x_1, x_2) \\ &= R(2x_1 - 8x_2, -2x_1 + 7x_2) \\ &= \left(-\frac{7}{2}(2x_1 - 8x_2) - (-2x_1 + 7x_2), -4(2x_1 - 8x_2) - (-2x_1 + 7x_2)\right) \\ &= (x_1, x_2) \end{aligned}$$

(d)

$$\begin{aligned} & T \circ R(x_1, x_2) \\ &= T\left(-\frac{7}{2}x_1 - x_2, -4x_1 - x_2\right) \\ &= \left(2\left(-\frac{7}{2}x_1 - x_2\right) - 8(-4x_1 - x_2), -2\left(-\frac{7}{2}x_1 - x_2\right) + 7(-4x_1 - x_2)\right) \\ &= (x_1, x_2) \end{aligned}$$