Math 211 Homework 5 Kelvin Abrokwa-Johnson

1. (a) If A and B are invertible, then we can pre-multiply both sides by A^{-1} to get $A^{-1}ABC = A^{-1}I$ thus, $BC = A^{-1}$. We continue by pre-multiplying both sides by B^{-1} to get $B^{-1}BC = B^{-1}A^{-1}$, thus, $C = B^{-1}A^{-1}$. Now we can undo those operation in a slightly different order. We can post-multiply both sides by A to get $CA = B^{-1}A^{-1} = BI$. And now we pre-multiply both sides by B to get $BCA = BB^{-1} = I$. So, we have shown that $BCA = I$. In general, matrix multiplication of this sort is only commutativein a cyclical manner.

(b)

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

2. Computing the product of two partitioned matrices, we have

$$\begin{bmatrix} I & 0 \\ -E & I \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} W & X \\ -EW + Y & -EX + Z \end{bmatrix}$$

3. When we multiple the first row by the first column we find that $AX = I$ and so we can multiply the left side of both sides by A^{-1} to get $X = A^{-1}$. Using the first row and second column we see that $AY + B0 = 0$ which is simply $AY = 0$. Since we know from the first calculation that $A \neq 0$, Y must be 0. Finally, using the first row and third column we find that $AZ + BI = 0$. If we subtract B from both sides and then multiply on the left by A^{-1} we get $Z = -A^{-1}B$.

So we have:

$$X = A^{-1}$$

$$Y = 0$$

$$Z = -A^{-1}B$$

4.

$$M^2 = \begin{bmatrix} A & O_n \\ I_n & -A \end{bmatrix} \begin{bmatrix} A & O_n \\ I_n & -A \end{bmatrix} = \begin{bmatrix} A^2 & O_n \\ 0_n & A^2 \end{bmatrix}$$

We are given $A^2 = I_n$, so we substistute to get:

$$\begin{bmatrix} I_n & O_n \\ 0_n & I_n \end{bmatrix} = I_{2n}$$

5.

(a) In order to show that $A^{-1} = \begin{bmatrix} A_{11}^{-1} & O & O \\ O & A_{22}^{-1} & O \\ O & O & A_{33}^{-1} \end{bmatrix}$ it is sufficient to show that $AA^{-1} = I_3$. So we have the following matrix multiplication:

$$\begin{bmatrix} A_{11} & O & O \\ O & A_{22} & O \\ O & O & A_{33} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & O & O \\ O & A_{22}^{-1} & O \\ O & O & A_{33}^{-1} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} & O & O \\ O & A_{22}A_{22}^{-1} & O \\ O & O & A_{33}A_{33}^{-1} \end{bmatrix} = \begin{bmatrix} I & O & O \\ O & I & O \\ O & O & I \end{bmatrix}$$

(b) To find A^{-1} we can simply invert each block:

$$A_{11}^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}, A_{22}^{-1} = \begin{bmatrix} 3 & -4 \\ -\frac{5}{2} & \frac{7}{2} \end{bmatrix}, A_{33}^{-1} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$$

So,

$$A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & -\frac{5}{2} & \frac{7}{2} & \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

6. We have the following LU factorization of the matrix

$$A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 8 & 0 \\ 0 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix}.$$

Solve the equation $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ by first solving (a) $L\mathbf{y} = \mathbf{b}$, and then solving (b) $U\mathbf{x} = \mathbf{y}$.

(a)

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & -4 \\ 0 & 1 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & -4 \\ 2 & 0 & 1 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -6 & 4 & 2 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & -2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 6 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$x = \begin{bmatrix} -11 \\ -6 \\ -3 \end{bmatrix}$$

7. Find the LU factorization of

$$A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix},$$

by applying elementary operations to yield $[A|I_n]$ to $[U|L^{-1}]$.

$$\begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix} \to \begin{bmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 10 & 24 \end{bmatrix} \to \begin{bmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

$$U = \begin{bmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix}$$

We apply the appropriate column divions for L to get:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}$$

8. If B is invertible, then we can find elementary matrices $E_1, ..., E_k$ such that $E_k...E_1B = I$. If we apply these elementary matrices to the augmented matrix $[\,B \quad I\,]$, which would yield $E_k...E_1\,[\,B \quad I\,] = [\,I \quad B^{-1}\,]$. Likewise, we can apply that same series of elementary matrix manipulations to the augmented matrix $[\,B \quad A\,]$ yielding $[\,I \quad E_k...E_1A\,]$. We are given that B is invertible and that $A = BC$, so we know $B^{-1}A = IC$ and since $E_k...E_1A = B^{-1}A$ we have shown that the reduced row echelon form of $[B\ A]$ is $[I\ C]$.