Math 211 Homework 11 Kelvin Abrokwa-Johnson

1. When $\lambda = 2$ our solution set to solving the system set equal to 0 is $Span \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and when $\lambda = 3$ we have $Span \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. So we have:

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. (a) We know that the dimension of $\text{Nul}(A - \lambda_3 I) = 2$ because, by Theorem 6, an nxn matrix with n distinct eigenvalues is diagonalizable. Since, $A - \lambda_1 I$ and $A - \lambda_2 I$ produce 5 eigenvalues, $A - \lambda_3 I$ must produce 2.

(b) If A is not diagonalizable then $\dim \text{Nul}(A - \lambda_3 I) = 1$ I cannot be 2 becuase of the reason above and it cannot be more because there cannot be a set of more than n linearly independent vectors in an nxn space.

3. We see immediately that $[T(\mathbf{d}_1)] = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and $[T(\mathbf{d}_2)] = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. So the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$ is $\begin{bmatrix} 3 & -2 \\ -3 & 5 \end{bmatrix}$.

4. (a)

$$T(3 - 2t + t^2) = (3 - 2t + t^2) + 2t^2(3 - 2y + t^2) = 3 - 2t + 7t^2 - 4t^3 + 2t^4$$

(b) First we show that the transformation is closed under addition. Take $p(t) = a_1 + b_1 t + c_1 t^2$ and $q(t) = a_2 + b_2 t + c_2 t^2$

$$T(p(t) + q(t)) = T((a_1 + a_2) + (b_1 + b_2)t + (c_1 + c_2)t^2)$$

$$= ((a_1 + a_2) + (b_1 + b_2)t + (c_1 + c_2)t^2) + 2t^2((a_1 + a_2) + (b_1 + b_2)t + (c_1 + c_2)t^2)$$
$$= (a_1 + b_1 t + c_1 t^2 + 2t^2 a_1 + 2t^3 b_1 + 2t^4 c_1) + (a_2 + b_2 t + c_2 t^2 + 2t^2 a_2 + 2t^3 b_2 + 2t^4 c_2)$$
$$= T(p(t)) + T(q(t))$$

Now we show that it is closed under scalar multiplication. Take $p(t) = a + bt + ct^2$ and some scalar k:

$$T(kp(t)) = T(ka + kbt + kct^2)$$
$$= ka + kbt + kct^2 + 2kat^2 + 2kbt^3 + 2kct^4$$
$$= k(a + bt + ct^2 + 2at^2 + 2bt^3 + 2ct^4)$$
$$= kT(p(t))$$

(c) We simply define the transformation on the standard polynomials to get:

$$T(1) = 1 + 2t^2$$

$$T(t) = t + 2t^3$$
$$T(t^2) = t^2 + 2t^3$$

so the matrix T relative to the bases is:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5.

$$T(4b_1 - 3b_2) = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$$

Which is

$$5b_2 - 5b_3$$

6. In essence, we must find a basis $\mathcal{B}$ such that the transformation T in that basis is a diagonal matrix: $T([x]_{\mathcal{B}}) = D[x]_{\mathcal{B}}$ where D is a diagonal matrix. The $\mathcal{B}$ coordinates of a vector x, $[x]_{\mathcal{B}}$, can be transformed if we first convert it back to the standard basis (by multiplying by the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$), multiply by A and then convert back to the $\mathcal{B}$ basis (by multiplying it by $P_{\mathcal{B}\leftarrow\mathcal{C}}$). So $T([x]_{\mathcal{B}}) = D[x]_{\mathcal{B}}$ becomes $T([x]_{\mathcal{B}}) = P_{\mathcal{B}\leftarrow\mathcal{C}} A P_{\mathcal{C}\leftarrow\mathcal{B}} = D[x]_{\mathcal{B}}$.

Now it is clear that we are looking for a diagonal matrix similar to A. We know that the diagonal matrix D similar to A is composed of the eigenvalues. So we have $A = PDP^{-1}$ where the columns of P are the eigenvectors of A. So $D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = [T]_{\mathcal{B}}$ and $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

7. Similar to the problem above, we can rephrase the problem as finding a diagonal vector such that the matrix of transformation $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$. This matrix has eigenvalues $\lambda = 5, -1$ so the eigenvectors are $Span \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $Span \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. So a basis for said polynomial space is $\mathcal{B} = \{1 + t, -1 + t\}$.