Special Matrices

Diagonally Dominant
A matrix is **diagonally dominant** when

$$|a_{ii}| \geq \sum_{j=1,j\neq i}^{n} |a_{ij}| \text{ holds for each } i = 1, 2, ..., n.$$

That is, the absolute value of the element on the diagonal is greater than the sum of the absolute values of all other elements in the row.
It is **strictly diagonally dominant** when,

$$|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|$$

Symmetric Positive Definite Matrices (SPD)
A matrix A is **symmertric positive definite** if it is symmetric and if $x^t A x > 0$ for every n-dimensional vector $x \neq 0$.
If A is an $n \times n$ SPD matrix, then

- A has an inverse
- $a_{ii} > 0$ for each $i = 1, 2, ..., n$
- $max_{1\leq k,j\leq n}|a_{kj}| \leq max_{1\leq i\leq n}|a_{ii}|$
- $(a_{ij})^2 < a_{ii}a_{jj}$, for each $i \leq j$

The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries.
The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries.

Cholesky (LL^T)

Tridiagonal
$O(n)$ time multiplication

Vector Norms
A vector norm on R^n is a function, $||\cdot||$, from R^n into R with the following properties:

- $||x|| \geq 0$ for all $x \in R^n$
- $||x|| = 0$ if and only if $x = 0$
- $||\alpha x|| = |\alpha|||x||$ for all $\alpha \in R$, $x \in R^n$

- $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in R^n$

The l_2 norm is defined as such:

$$||x||_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}$$

The l_∞ norm is defined as such:

$$||x||_\infty = max_{1 \leq i \leq n} |x_i|$$

The distance between two vectors is defined as the norm of the difference of the vectors.

A sequence $\{x^{(k)}\}_{k=1}^{\infty}$ of vectors in R^n is said to converge to x with respect to the norm $|| \cdot ||$ if, given any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$||x^{(\epsilon)} - x|| < \epsilon, \text{ for all } k \geq N(\epsilon)$$

The sequence of vectors $\{x^{(k)}\}$ converges to x in R^n with respect to the l_∞ norm if and only if $lim_{k \to \infty} x^{(k)} = x_i$, for each $i = 1, 2, ..., n$.

Matrix Norms

A matrix norm on the set of all $n \times n$ matrices is a real-valued function, $|| \cdot ||$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

- $||A|| \geq 0$
- $||A|| = 0$ if and only if A is 0, the matrix with all 0 entries
- $||\alpha A|| = |\alpha| ||A||$
- $||A + B|| \leq ||A|| + ||B||$
- $||AB|| \leq ||A|| ||B||$

The distance between $n \times n$ matrices A and B with respect to this matrix norm is $||AB||$.
If $|| \cdot ||$ is a vector norm on R^n, then

$$||A|| = max_{||x||=1} ||Ax||$$

is a matrix norm.

For any vector $z \neq 0$, matrix A, and any natural norm $|| \cdot ||$, we have

$$||Az|| \leq ||A|| \cdot ||z||$$

If $A = (aij)$ is an $n \times n$ matrix, then

$$||A||_\infty = max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

(max row sum).

Spectral Radius
The spectral radius $\rho(A)$ of a matrix A is defined by

$$\rho(A) = max|\lambda|, \text{ where } \lambda \text{ is an eigenvalue of } A$$

Jacobi Method
The Jacobi method can be written in the form $x^{(k)} = Tx^{(k-1)} + c$ by splitting A into its diagonal and off-diagonal parts.

$$A = D - L - U$$

So we have,

$$x^{(k)} = D^{-1}(L+U)x^{(k-1)} + D^{-1}b,\ k = 1, 2, ...$$

To shorten,

$$T_j = D^{-1}(L+U) \text{ and } c_j = D^{-1}b$$

So we have

$$x^{(k)} = T_j x^{(k-1)} + c_j$$

Gauss-Siedel Method
We have

$$x^{(k)} = (D-L)^{-1}Ux^{(k-1)} + (D-L)^{-1}b$$

Letting

$$T_g = (D-L)^{-1}U \text{ and } c_g = (D-L)^{-1}b$$

We get

$$x^{(k)} = T_g x_{(k-1)} + c_g$$

Note: for $D - L$ to be nonsingular, it is necessary and sufficient that $a_{ii} \neq 0$.

Conjugate Gradient Method
Matrix must be positive definite.