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Math 309

Homework 3

1.3.12

Solution:
Let D be the set of diagonal matrices.

- The 0 matrix is in D because $A_{ij} = 0$ for all entries where $i > j$ (moreover, all entries are zero).
- Take some scalar $c \in F$ and matrix $A \in D$. When we perform the multiplication cA, for every entry A_{ij} where $i > j$, $A_{ij} = 0$, so $cA_{ij} = 0$. Thus $cA \in D$, i.e. D is closed under scalar multiplication.
- Take matrices $A, B \in D$. When we add A and B, since $A_{ij} = B_{ij} = 0$ whenever $i > j$, $A_{ij} + B_{ij} = 0$ when those entries are below the diagonal. So $A + B \in D$, i.e. D is closed under addition.

1.6.16

Solution:
A basis for W is the set of E^{ij} where the only non-zero entry is a 1 in the i, j position for every E^{ij} where $i \leq j$.
$dim(W) = n + (n-1) + \cdots + 1 = \frac{1}{2}n(n+1)$

1.6.22

Solution:
A neccessary and sufficient condition is that $W_1 \subseteq W_2$.

- Showing $W_1 \subseteq W_2 \rightarrow dim(W_1 \cap W_2) = dim(W_1)$:

  If $W_1 \subseteq W_2$ then $W_1 \cap W_2 = W_1$. So, since we assumed $W_1 \subseteq W_2$, it follows that $dim(W_1 \cap W_2) = dim(W_1)$.

- Showing $dim(W_1 \cap W_2) = dim(W_1) \rightarrow W_1 \subseteq W_2$:

  Let $dim(W_1 \cap W_2) = n$, $dim(W_1) = n$, and $dim(W_2) = m$. Now we define some arbitrary basis for $W_1 \cap W_2$, $\mathcal{B} = \{v_1, v_2, ..., v_n\}$. Since $\mathcal{B}$ is a basis it is linearly independent. We also know that $W_1 \subseteq W_2 \in W_1$, so $\mathcal{B} \in W_1$. Since $\mathcal{B}$ has n linearly independent elements and W_1 had a dimension of n it forms a maximal linearly independent subset of W_1, so $\mathcal{B}$ is a basis for W_1. Now, we also know that $W_1 \cap W_2 \subseteq W_2$, so its basis, $\mathcal{B}$ can be extended into a basis W_2. Since the basis for W_2 contains the basis for W_1 we can conclude that $W_1 \subseteq W_2$.

1.6.29.a

Solution:
Let $dim(W_1 \cap W_2) = k$, $dim(W_1) = n$, and $dim(W_2) = m$. Then we may express a basis for $W_1 \cap W_2$ as some linearly independent subset $\mathcal{B}_3 = \{u_1, u_2, ..., u_k\}$. Since, $W_1 \cap W_2 \subseteq W_1, W_2$ we may extend $\mathcal{B}_3$ to bases for W_1 and W_2. Let $\mathcal{B}_1 = \mathcal{B}_3 \cup \{v_1, ..., v_{m-k}\}$ be a basis for W_1 and $\mathcal{B}_2 = \mathcal{B}_3 \cup \{w_1, ...w_{n-k}\}$ be a basis for W_2. Now we can show that the set $\mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ is a basis for $W_1 + W_2$. First lets show that the set is linearly independent. We know that each part of the union is linearly independent because they were each subsets of bases. Now, take $v \in span(\mathcal{B}_1 \setminus \mathcal{B}_3)$. We know that $v \notin span(\mathcal{B}_2 \setminus \mathcal{B}_3)$ because those portions of the bases didn't share any elements in common. Likewise, for any $v \in span(\mathcal{B}_2 \setminus \mathcal{B}_3)$, $v \notin span(\mathcal{B}_1 \setminus \mathcal{B}_3)$. Now take $v \in span(\mathcal{B}_3)$, we immediately see that $v \notin \mathcal{B}_1 \setminus \mathcal{B}_3$ and $v \notin \mathcal{B}_2 \setminus \mathcal{B}_3$. So the set $\mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ is linearly independent. Now we can show that the set $\mathcal{B} = \mathcal{B}_3 \cup \mathcal{B}_1 \setminus \mathcal{B}_3 \cup \mathcal{B}_2 \setminus \mathcal{B}_3$ spans $W_1 + W_2$. We do this by contradiction. Take $v \in W_1 + W_2$. If $v \notin span(\mathcal{B})$ then $v \notin span(W_1)$ and $v \notin span(W_2)$, so $v \notin W_1 + W_2$. Which is a contradiction. So $\mathcal{B}$ spans $W_1 + W_2$.

1.6.31

Solution:
a. Let $dim(W_1 \cap W_2) = k$. Then we can define some arbitrary basis $\mathcal{B} = \{u_1, u_2, ..., u_k\}$ for $W_1 \cap W_2$. Since $W_1 \cap W_2$ is a subspace of W_2, $\mathcal{B} \subseteq W_2$. Since $\mathcal{B}$ is linearly independent, a maximal linearly independent subset of W_2 must have at least k elements. So $k = dim(W_1 \cap W_2) \leq n$.
b. We have shown that $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$. Since $dim(W_1 \cap W_2) \geq 0$, $dim(W_1 + W_2) \leq dim(W_1) + dim(W_2) = m + n$.

1.6.32

Solution:
a. We let $W_1 = R^3$ and $W_2 = span(\{(1,0,0)\})$. So $m = 3$ and $n = 1$. $W_1 \cap W_2 = W_2$ since $W_2 \subset W_1$. So, $dim(W_1 \cap W_2) = dim(W_2) = 1 = n$.
b. Let $W_1 = span(\{(1,0,0),(0,1,0)\})$ and $W_2 = span(\{(0,0,1)\})$ $(m = 2, n = 1)$. So $dim(W_1 + W_2) = dim(span(\{(1,0,0),(0,1,0),(0,0,1)\})) = dim(R^3) = m + n = 3$.
c. Let $W_1 = span(\{(1,0,0),(0,1,0)\})$ and $W_2 = span(\{(0,1,0),(0,0,1)\})$. So $m = dim(W_1) = 2$, $n = dim(W_2)$, and $dim(W_1 \cap W_2) = 1$. So $dim(W_1 + W_2) = m + n - 1 < m + n$.

1.7.4

Solution:
Let $\mathcal{B} = \{u_1, u_2, ..., u_n\}$ be any basis for W. By Theorem 1.13, there exists a maximal linearly independent subset of V, lets call it $\mathcal{S}$, that contains $\mathcal{B}$. Since $\mathcal{S}$ is a basis for V (see Theorem 1.12), and $\mathcal{B} \subseteq \mathcal{S}$, then $\mathcal{B}$ is a subset of a basis for V.