Math 211 Homework 10 Kelvin Abrokwa-Johnson

1. (10 points) Show that

$$\mathbb{B} = \{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$$

is a basis for $\mathbb{P}_3$, and find the change of basis matrix from $\mathbb{B}$ to $\mathbb{C} = \{1, t, t^2, t^3\}$ and the change of the basis matrix from $\mathbb{C}$ to $\mathbb{B}$ Find $[u]_{\mathbb{B}}$ for $u = 1 - 3t + 9t^2 - t^3$.

We can represent the polynomial as a set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -18 \\ 9 \\ 1 \end{bmatrix} \right\}$$

In this form it is clear that the set is linearly independent, therefore the set is linearly independent and is a basis for $\mathbb{P}_3$.

The change of basis matrix from $\mathbb{C}$ to $\mathbb{B}$ is composed of the vectors above:

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[u]_{\mathbb{B}} = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} 22 \\ -51 \\ 18 \\ -1 \end{bmatrix}$$

2. (5 points) Find the rank, the nullity (dimension of null space), a basis for the column space, a basis for the row space of the following matrix

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -13 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

rank $A = 5$

dim Nul $A = n - rank A = 1$

$$\text{Basis Col } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -13 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{Basis for row space} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -13 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3. Find the characteristic polynomial for the matrix and two linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 4 \\ 1 & 4 \end{bmatrix}.$$

4. Show that $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ has at most one linear independent eigenvector.

$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 0 & 1-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{bmatrix}$ so $detA = (1-\lambda)^3$, and the matrix is singular when $\lambda = 1$.

We substitute this eigenvalue back into $A - \lambda I$ to get $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. We set this matrix to 0 and get the solution $Span \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and this is the only eigenvector.

5. Find the characteristic polynomial for the matrix and three linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

6. Find the values h so that there are two linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 4$ for the matrix

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

[Hint: Find h such that the null space of $A - 4I_4$ has nullity 2.]

7. Show that A and A^T have the same characteristic polynomial.
[Hint: Show that $\det(A - \lambda I) = \det(A^T - \lambda I)$.]