Math 211 Homework 4 Name: Kelvin Abrokwa-Johnson

Five points for each question unless specified otherwise.

1.

$$AB = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. (a)

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}, B = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 7 & 3 & 1 & 0 \\ -6 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ -6 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -3 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -\frac{7}{3} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix}$$

(c)

$$A^{-1}b = \begin{bmatrix} 1 & 1 \\ -2 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix}$$

$$Ax_0 = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix} = b$$

4. For the Augmented matrix $[\, A \quad I_3 \,]$ we can say use the the variables S and T for each section such that $[\, A \quad I_3 \,] = [\, S \quad T \,]$.

(a) The matrix is not invertible

$$S = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, T = \left\{ \begin{bmatrix} 0 & 0 & * \\ 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(b) The matrix is not invertible

$$S = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}, T = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

(c) The matrix is invertible

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

5. Using regular matrix algebra we see:

$$\begin{aligned} AB &= AC \\ AB - AC &= 0 \\ A(B-C) &= 0 \\ A^{-1}A(B-C) &= A^{-1}0 \\ B - C &= 0 \\ B &= C \end{aligned}$$

6. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -3 \end{bmatrix}$. Apply row reduction to the matrix $[\,A \quad I_3\,]$ until we have the reduced row echelon form $[\,I_3 \quad B\,]$, and verify that $AB = I_3$.

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & -6 & 1 & -1 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 6 & -1 & 1 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 6 & -1 & 1 \\ 10 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7. (a) We can augment the matrix with both e_1 and $e_2 \in \mathbf{R}^2$ to find the columns of the matrix B.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

In both cases we have two free variables, x_3 and x_4, which we can set equal to s and t, respectively. If we set $s = 0$ and $t = 1$, then we have $b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. and so $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Indeed

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Assume there is a $4x2$ matrix C such that $CA = I_4$. In order for $I_{1,1} = 1$, $C_{1,1}$ must equal 1. Then in order for $I_{1,2} = 0$, $C_{1,2}$ must equal -1. So we have the first row of Matrix C being $[1 \quad -1]$. However we find that if this is the case, then $I_{1,3} = -1$ and this is not the identity matrix. So there is no such matrix C.

8. Assume A has linearly dependent columns. That means there exists some vector x, where $x \neq 0$, such that $Ax = 0$. This entails that A^2 is also linearly dependent because $A^2 = AA$, and we know there is some x such that $Ax = 0$, so we have $A^2x = A(Ax)$ which becomes $A^2 = A0$ and finally $A^2x = 0$. However, we are given that A^2 spans $\mathbf{R}^n$, which, by the IVT, entails that its columns are linearly independent. Therefore, we have a contradiction and A must have linearly independent columns.

9. (10 points)

(a)

$$A = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}$$

$$A^{-1} = detA \begin{bmatrix} 7 & 8 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & -4 \\ -1 & -1 \end{bmatrix}$$

(b)

$$R(x_1, x_2) = (-\frac{7}{2}x_1 - x_2, -4x_1 - x_2)$$

(c)

$$R \circ T(x_1, x_2)$$
$$= R(2x_1 - 8x_2, -2x_1 + 7x_2)$$
$$= (-\frac{7}{2}(2x_1 - 8x_2) - (-2x_1 + 7x_2), -4(2x_1 - 8x_2) - (-2x_1 + 7x_2))$$
$$= (x_1, x_2)$$

(d)

$$T \circ R(x_1, x_2)$$
$$= T(-\frac{7}{2}x_1 - x_2, -4x_1 - x_2)$$
$$= (2(-\frac{7}{2}x_1 - x_2) - 8(-4x_1 - x_2), -2(-\frac{7}{2}x_1 - x_2) + 7(-4x_1 - x_2))$$
$$= (x_1, x_2)$$