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Math 309

Homework 1

7. Less $S = 0, 1$ and $F = R$. In $\mathcal{F}(S, R$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1, g(t) = 1 + 4t + 2t^2$, and $h(t) = 5^t + 1$.

Solution:
For 0:

$$f0 = 2(0) + 1 = g(0) = 1 + 4(0) - 2(0)^2 = 1$$
$$h(0) = 5^0 + 1 = f(0) + g(0) = 1 + 1 = 2$$

For 1:

$$f(1) = 2(1) + 1 = g(1) = 1 + 4(1) - 2(1)^2 = 3$$
$$h(1) = 5^1 + 1 = f(1) + g(1) = 3 + 3 = 6$$

Thus, we have shown what was to be shown.

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in V$ and any $a, b \in F$.

Solution:
By (VS 8) we know that we can distribute the vectors in a vector-scalar multiplication, so $(a + b)(x + y) = a(x + y) + b(x + y)$. By (VS 7) we know that we can distribute the scalar in a vector-scalar multiplacation, so $a(x + y) + b(x + y) = ax + ay + bx + by$. And so we have shown that $(a + b)(x + y) = ax + ay + bx + by$.

12. A real-valued function f defined on the real line is called an even function if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

Solution:

- Suppose E is the set of even functions. Take the function $f \in E$, . Since $f(-t) = f(t)$, we know that $af(-t) = af(t)$ for all $a \in \mathbf{R}$, so $(af)(t) \in E$, i.e. closed under scalar multiplication.
- Take $f, g \in E$. $(f + g)(t) = f(t) + g(t) = f(-t) + g(-t) = (f + g)(-t)$. In short $(f + g)(t) = (f + g)(-t)$ so $(f + g)(t) \in E$, i.e. closed under addition.
- Since $f(t) = f(-t)$ for $f \in E$ where $f(t) = 0$, $f(t) = 0 \in E$, i.e. has a 0 function.

18. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbf{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ adn $c \in \mathbf{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2)$$

Is V a vector space over $\mathbf{R}$ wit these operations?

Solution:
No, V is not a vector space over $\mathbf{R}$.
Take $a_1, a_2 \neq 0 \in \mathbf{R}$. Since $(a_1, a_2) + (0, 0) = (a_1, a_2)$ and $(0, 0) + (a_1, a_2) = (2a_1, 3a_2)$, i.e. $(a_1, a_2) + 0 \neq 0 + (a_1, a_2)$. The set does not have commutative addition so it is not a vector space.

21. Let V adn W be vector spaces over a field F. Let

$$Z = \{(v, w) : v \in V, w \in W\}$$

Prove that Z is a vector space over F with operations:

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_1, w_1 + w_2) \text{ and } c(v_1, w_1) = (cv_1, cw_1)$$

Solution:

- Take $v_1, v_2 \in V$, $w_1, w_2 \in W$. Since V and W are vector spaces, $v_1 + v_2 \in V$ and $w_1 + w_2 \in W$, so it follows that $(v_1, w_1) + (v_2, w_2) \in Z$, i.e. closed under addition.
- Take $v \in V, w \in W, c \in F$. Since V and W are vector spaces, $cv \in V$ and $cw \in W$. So, $c(v, w) = (cv, cw) \in Z$, i.e. closed under scalar multiplication.
- Since V and W are vector spaces, $\vec{0} \in V, W$. So $(\vec{0}_V, \vec{0}_W) \in Z$, i.e. Z has a 0 vector.

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution:

- Proof (1): $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$ implies $W_1 \cup W_2$ is a subspace.

  Assume, without loss of generality, that $W_1 \subseteq W_2$. Then, $W_1 \cup W_2 = W_2$ which we know is a subspace (the same logic applies when $W_2 \subseteq W_1$).

- Proof (2): $W_1 \cup W_2$ is a subspace implies $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

  We will use prooof by contradiction. Assume $W_1 \cup W_2$ is a subspace and $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. Take $w_1 \in W_1, \notin W_2, w_2 \in W_2, \notin W_1$. We know what $w_1 + w_2 \in W_1 \cup W_2$, which means $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_1$. We assume, without loss of generality, that $w_1 + w_2 \in W_1$. Now, we add the additive inverse of w_1, $-w_1 \in W_1$ to $w_1 + w_2$ to get $(w_1 + w_2) + (-w_1) = w_2 \in W_1$ which contradicts our initial condition that $w_2 \notin W_1$

20. Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$ for any scalar $a_1, a_2, ..., a_n$.

Solution:
Proof by induction:

- Base case: From their definition we know that vector spaces are closed under scalar multplication and addition. So it follows that $a_1w_1 + a_2w_2 \in W$.
- Assume: $a_1w_1 + \cdots + a_{n-1}w_{n-1} \in W$.
- Need to show: $(a_1w_1 + \cdots + a_{n-1}w_{n-1}) + a_nw_n \in W$
- Proof: We assumed $a_1w_1 + \cdots + a_{n-1}w_{n-1} \in W$. Since $a_nw_n \in W$ (W is closed under scalar multiplication) and since we also know that W is closed under addition, it follows that $(a_1w_1 + \cdots + a_{n-1}w_{n-1}) + a_nw_n \in W$. **PMI**.