Lecture 6: Relations

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Part of the contents here is taken from Dr Wong Tin Lok's lecture notes.

6.1 Relations on Sets

- Definition of relation; arrow diagram, inverse of a relation.
- Relation on a set; directed graph of a relation.
- Composition of relations.
- *N*-ary relations and relational databases.

6.2 Reflexivity, Symmetry and Transitivity

- Definitions of reflexivity, symmetry and transitivity.
- Transitive closure of a relation.

6.3 Equivalence Relations

- Partition of a set; the relation induced by a partition.
- Equivalence relation; equivalence classes.
- Congruence.
- Dividing a set by an equivalence relation.
- Summary

Reference: Epp's Chapter 8 Properties of Relations

6.4 Partial Order Relations

- Antisymmetry.
- Partial order relations.
- Hasse diagrams.
- Comparability.
- Maximal/minimal/largest/smallest element.
- Linearization.
- Total order relations; well ordered sets.

Reference: Epp's Chapter 8 Properties of Relations

6.1 Relations on Sets

6.1.1 Definitions

As the topic of relations is built on sets, definitions on sets, such as ordered pair, ordered n-tuple, Cartesian product, etc. (see Lecture 5 Set Theory) will be referred to here.

Recall: The Cartesian product of sets A and B, denoted $A \times B$, consists of all ordered pairs whose first element is in A and second element in B: $A \times B = \{(x, y) : x \in A \land y \in B\}$.

Example #1: Let $A = \{0,1,2\}$ and $B = \{a, b, c\}$.

Then $A \times B =$
$\{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Definition: Relation

Let A and B be sets. A (binary) **relation from $\boldsymbol{A}$ to $\boldsymbol{B}$** is a subset of $\boldsymbol{A} \times \boldsymbol{B}$.

Given an ordered pair (x, y) in $A \times B$, **$\boldsymbol{x}$ is related to $\boldsymbol{y}$ by $\boldsymbol{R}$**, or **$\boldsymbol{x}$ is $\boldsymbol{R}$-related to $\boldsymbol{y}$**, written $\boldsymbol{x\ R\ y}$, iff $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{R}$.

Symbolically, $x\ R\ y$ means $(x, y) \in R$
$x \not R\ y$ means $(x, y) \notin R$

We read $x\ R\ y$ as "x is R-related to y" or, if there is no risk of confusion, simply "x is related to y".

Example #2: Let $A = \{0,1,2\}$ and $B = \{1,2,3\}$. Suppose we define the relation R s.t. xRy iff $x < y$.

Then $0R1, 0R2, 0R3, 1R2, 1R3$ and $2R3$, but $1\not R1, 2\not R1$ and $2\not R2$.

Example #3: Let $A = \{1,2\}$ and $B = \{1,2,3\}$. Define a relation R from A to B as follows:

$$\forall (x,y) \in A \times B \left((x,y) \in R \Leftrightarrow \frac{x-y}{2} \in \mathbb{Z} \right).$$

State explicitly which ordered pairs are in $A \times B$ and which are in R.

$A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$.

$R = \{(1,1),(1,3),(2,2)\}$.

An application: A simple database

Let S be the set of students, M the set of modules, and R the relation "is enrolled in" from S to M.

Student	Module
Ali	CS1101S
Aiken	CS1231S
Dueet	CS1231S
Bam Boo	MA1101R
Lian Eng	CS1101S
Manimaran	CS1231S
James Tan	MA1101R
:	:

R = { (Ali, CS1101S),
(Aiken, CS1231S),
(Dueet, CS1231S)
(Bam Boo, MA1101R),
(Lian Eng, CS1101S),
(Manimaran, CS1231S),
(James Tan, MA1101R),
… }

Definitions: Domain, Co-domain, Range

Let A and B be sets and R be a relation from A to B.

The **domain** of R, $Dom(R)$, is the set $\{a \in A : aRb \text{ for some } b \in B\}$.

The **co-domain** of R, $coDom(R)$, is the set B.

The **range** of R, $Range(R)$, is the set $\{b \in B : aRb \text{ for some } a \in A\}$.

Example #4: Let $A = \{1,2,3\}$ and $B = \{2,4,9\}$, and define a relation R from A to B as follows:

$$\forall (x, y) \in A \times B, (x, y) \in R \Leftrightarrow x^2 = y.$$

$Dom(R) = \{2,3\}$

$coDom(R) = \{2,4,9\}$

$Range(R) = \{4,9\}$

Example #5: Define a relation R from $\mathbb{Z}$ to $\mathbb{Z}$ as follows:

$\forall (x, y) \in \mathbb{Z} \times \mathbb{Z}\ (xRy \Leftrightarrow x - y$ is even$)$.

a. Is $4R0$? Is $2R6$? Is $3R(-3)$? Is $5R2$?

Yes Yes Yes No

b. List five integers that are related by R to 1.

Infinitely many possible answers.
One answer: 1, 57, 12345, -203, -999.

c. Prove that if a is any odd integer, then $aR1$.

1. Let a be an odd integer.
2. Then $a = 2k + 1$ for some integer k (by the definition of odd).
3. Therefore, $a - 1 = 2k$ which is even (by the definition of even).
4. Hence $aR1$ (by the definition of R).

Arrow Diagram

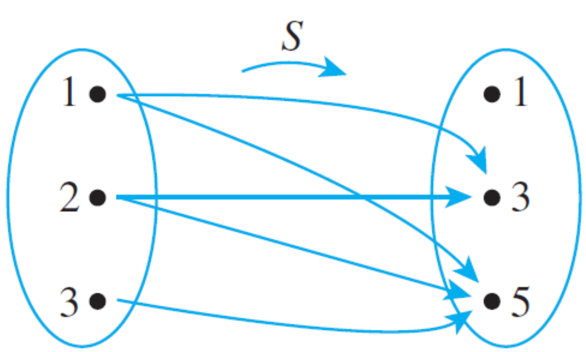
A relation R from set A to set B can be depicted as an **arrow diagram**:

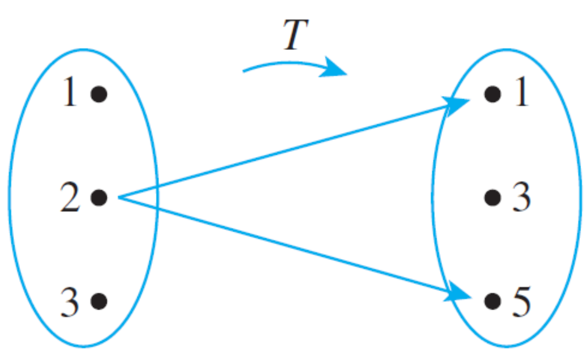
1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each $x \in A$ and $y \in B$, draw an arrow from x to y iff xRy.

Example #6: Let $A = \{1,2,3\}$ and $B = \{1,3,5\}$. Define relations S and T from A to B as follows: $\forall (x, y) \in A \times B$,

$$(x, y) \in S \Leftrightarrow x < y$$

$$T = \{(2,1), (2,5)\}.$$





6.1.2 The Inverse of a Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

Definition: Inverse of a Relation

Let R be a relation from A to B. Define the **inverse relation** R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\boxed{\forall x \in A, \forall y \in B \left((y, x) \in R^{-1} \Leftrightarrow (x, y) \in R\right).}$$

If $n, d \in \mathbb{Z}$:

$d \mid n \Leftrightarrow \exists k \in \mathbb{Z}$ such that $n = dk$.

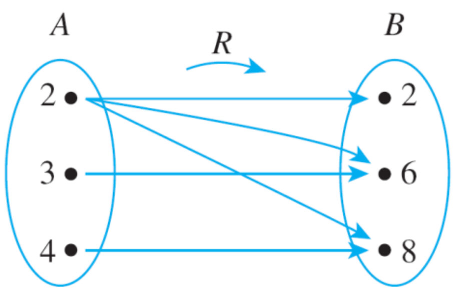
Example #7: Let $A = \{2, 3, 4\}$ and $B = \{2, 6, 8\}$ and let R be the "divides" relation from A to B:

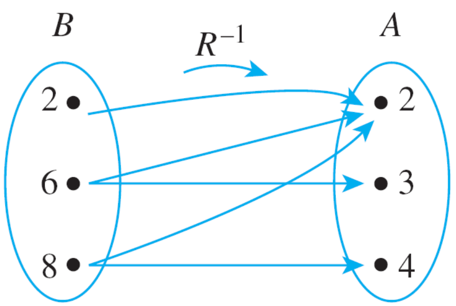
$$\forall (x, y) \in A \times B \; (xRy \Leftrightarrow x \mid y)$$

a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

$R = \{(2,2), (2,6), (2,8), (3,6), (4,8)\}$

$R^{-1} = \{(2,2), (6,2), (8,2), (6,3), (8,4)\}$





b. Describe R^{-1}.

$\forall (y, x) \in B \times A$
$(yR^{-1}x \;\Leftrightarrow\; y = kx)$ for some integer k.

6.1.3 Directed Graph of a Relation

Definition: Relation on a Set

A **relation on a set A** is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

We may write A^2 for $A \times A$.
In general, we may write A^n for $A \times \cdots \times A$ (n times).

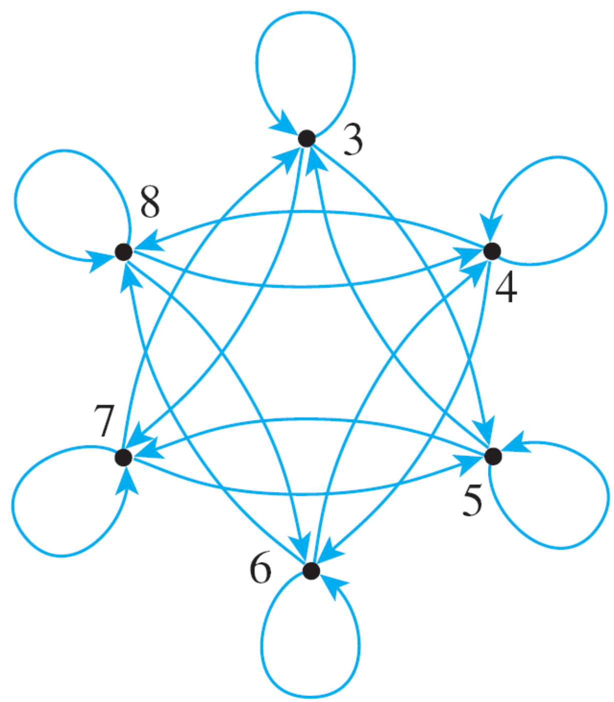
The arrow diagram of such a relation can be modified so that it becomes a **directed graph**. Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to its related point.

If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.

Example #8: Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: $\forall x, y \in A$,

$$xRy \Leftrightarrow 2 \mid (x - y).$$

Draw the directed graph of R.



6.1.4 Composition of Relations

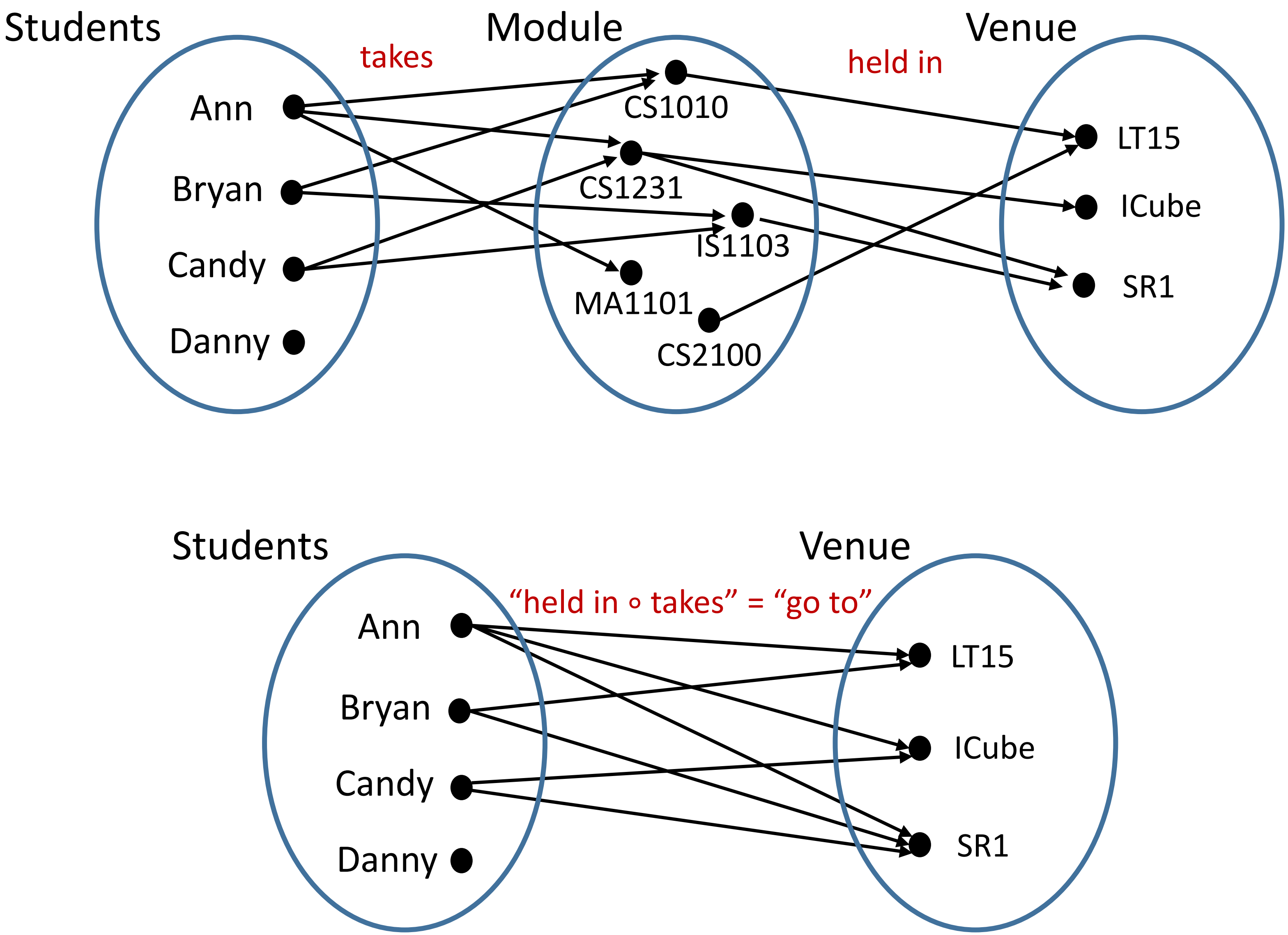
Definition: Composition of Relations

Let A, B and C be sets. Let $R \subseteq A \times B$ be a relation. Let $S \subseteq B \times C$ be a relation. The **composition of R with S**, denoted $S \circ R$, is the relation from A to C such that:

$$\forall x \in A, \forall z \in C \left(x\, S \circ R\, z \Leftrightarrow \left(\exists y \in B\ (xRy \wedge ySz) \right) \right).$$

In other words, $x \in A$ and $z \in C$ are "$S \circ R$"-related iff there is a "path" from x to z via some intermediate element $y \in B$ in the arrow diagram.

Composition of Relations



Proposition: Composition is Associative

Let A, B, C, D be sets. Let $R \subseteq A \times B, S \subseteq B \times C$ and $T \subseteq C \times D$ be relations.

$$T \circ (S \circ R) = (T \circ S) \circ R = T \circ S \circ R$$

Proposition: Inverse of Composition

Let A, B and C be sets. Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations.

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

6.1.5 *N*-ary Relations and Relational Databases

A relation involving two sets is called binary relation. We can generalize a relation to involve more than two sets.

Definition: n-ary Relation

Given n sets $A_1, A_2, \cdots, A_n$, an **$\boldsymbol{n}$-ary relation $\boldsymbol{R}$** on $A_1 \times A_2 \times \cdots \times A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

The special cases of 2-ary, 3-ary and 4-ary relations are called **binary**, **ternary** and **quaternary relations** respectively.

Example #9: (The following is a radically simplified version of a database that might be used in a hospital.)

Let A_1 be a set of positive integers, A_2 and A_4 be sets of alphabetic character strings, and A_3 be a set of numeric character strings.

Define a quaternary relation R on $A_1 \times A_2 \times A_3 \times A_4$ as follows:

$(a_1, a_2, a_3, a_4) \in R \Leftrightarrow$ a patient with ID number a_1, named a_2, was admitted on date a_3 with primary diagnosis a_4.

At a particular hospital, this relation might contain these 4-tuples:

(011985, John Schmidt, 020710, asthma)
(574329, Tak Kurosawa, 011410, pneumonia)
(466581, Mary Lazars, 010310, appendicitis)
(008352, Joan Kaplan, 112409, gastritis)
(011985, John Schmidt, 021710, pneumonia)
(244388, Sarah Wu, 010310, broken leg)
(778400, Jamal Baskers, 122709, appendicitis)

For example, in the database language SQL, if the above database is denoted as S, the result of the query

```
SELECT Patient_ID#, Name FROM S WHERE Admission_Date=010310
```

would yield a list of the ID numbers and names of all patients admitted on 01-03-10:

466581	Mary Lazars
244388	Sarah Wu

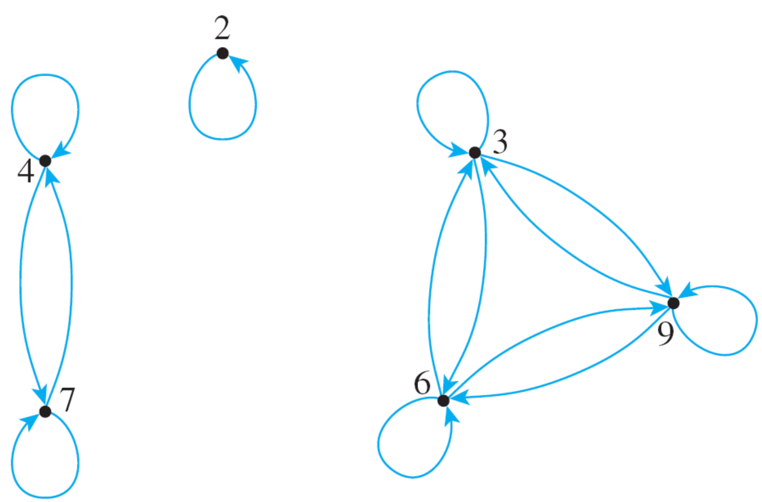
This is obtained by taking the intersection of the set $A_1 \times A_2 \times \{010310\} \times A_4$ with the database and then projecting onto the first two fields.

6.2 Reflexivity, Symmetry and Transitivity

6.2.1 Definitions of Reflexivity, Symmetry and Transitivity

Example #10: Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation R on A as follows: $\forall x, y \in A\ (xRy \Leftrightarrow 3|(x-y))$.

The directed graph for R is shown below:



1. Each point of the graph has an arrow looping around from it back to itself.
2. Wherever there is an arrow going from one point to another, there is also an arrow going from the second point back to the first.
3. Wherever there is an arrow going from one point to a second and from the second point to a third, there is also an arrow going from the first point to the third.

If $n, d \in \mathbb{Z}$:

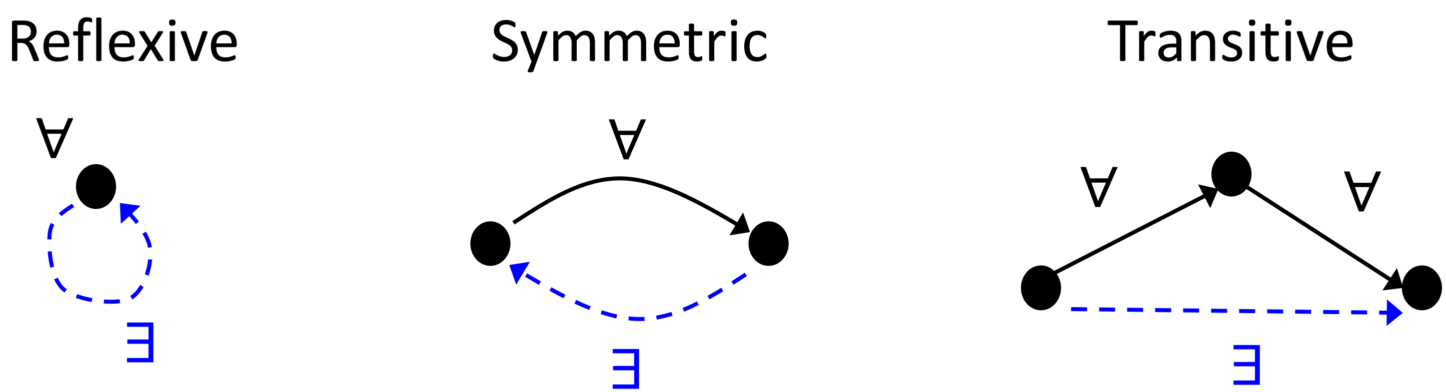
$d \mid n \Leftrightarrow\ \exists k \in \mathbb{Z}$ such that $n = dk$.

Properties (1), (2), and (3) correspond to properties of general relations called *reflexivity*, *symmetry*, and *transitivity*.

Definitions: Reflexivity, Symmetry, Transitivity

Let R be a relation on a set A.

1. R is **reflexive** iff $\forall x \in A\ (xRx)$.
2. R is **symmetric** iff $\forall x, y \in A\ (xRy \Rightarrow yRx)$.
3. R is **transitive** iff $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

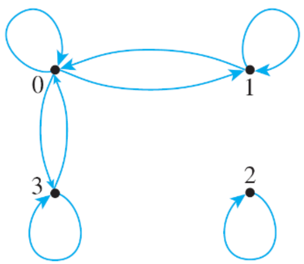


Example #11: Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\},$$
$$S = \{(0,0),(0,2),(0,3),(2,3)\},$$
$$T = \{(0,1),(2,3)\}.$$

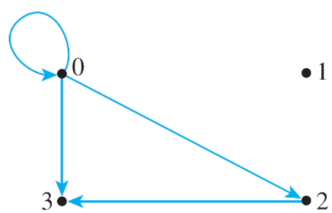
a. Is R reflexive? symmetric? transitive?

Yes Yes No



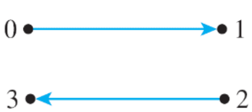
b. Is S reflexive? symmetric? transitive?

No No Yes



c. Is T reflexive? symmetric? transitive?

No No Yes



Common mistake: Talking about reflexivity.

“The element 0 is reflexive.”

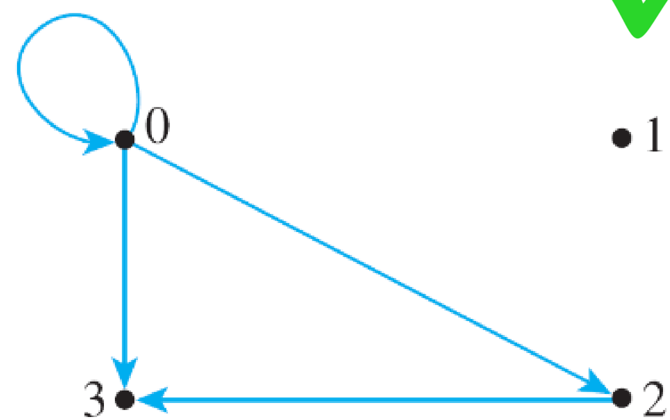
“The element 1 is not reflexive.”

“The element 2 is not reflexive.”

“The element 0 is related to itself.”

“The element 1 is not related to itself.”

“The element 2 is not related to itself.”



Reflexivity, symmetry and transitivity are **properties of a relation**, not properties of members of the set.

We say a relation is reflexive or not reflexive.

Example #12: Define a relation R on $\mathbb{Z}$ as follows:

$$\forall x, y \in \mathbb{Z}\,(x\,R\,y \iff 3 \mid (x-y)).$$

This relation is called congruence modulo 3.

Is R reflexive, symmetric, transitive?

R is reflexive.

Proof of Reflexivity:

1. Let a be an arbitrarily chosen integer.
2. Now $a - a = 0$.
3. But $3 \mid 0$ (since $0 = 3 \cdot 0$), hence, $3 \mid (a-a)$.
4. Therefore $a\,R\,a$ (by the definition of R).

Example #12: Define a relation R on $\mathbb{Z}$ as follows:

$$\forall x, y \in \mathbb{Z}\,\big(x\,R\,y \;\Leftrightarrow 3 \mid (x - y)\big).$$

This relation is called congruence modulo 3.

Is R reflexive, symmetric, transitive? R is symmetric.

Proof of Symmetry:

1. Let a and b be arbitrarily chosen integers that satisfy $a\,R\,b$.
2. Then $3|(a - b)$ (by the definition of R), hence $a - b = 3k$ for some integer k (by the definition of divisibility).
3. Multiplying both sides by -1 gives $b - a = 3(-k)$.
4. Since $-k$ is an integer, $3|(b - a)$ (by definition of divisibility).
5. Therefore $b\,R\,a$ (by the definition of R).

Example #12: Define a relation R on $\mathbb{Z}$ as follows:

$$\forall x, y \in \mathbb{Z}\left(x\,R\,y \iff 3 \mid (x-y)\right).$$

This relation is called congruence modulo 3.

Is R reflexive, symmetric, transitive?

R is transitive.

Proof of Transitivity:

1. Let a, b and c be arbitrarily chosen integers that satisfy $a\,R\,b$ and $b\,R\,c$.
2. Then $3|(a-b)$ and $3|(b-c)$ (by the definition of R), hence $a-b=3r$ and $b-c=3s$ for some integers r and s (by the definition of divisibility).
3. Adding both equations gives $a-c=3(r+s)$.
4. Since $r+s$ is an integer, $3|(a-c)$ (by definition of divisibility).
5. Therefore $a\,R\,c$ (by the definition of R).

6.2.2 The Transitive Closure of a Relation

Generally speaking, a relation fails to be transitive because it fails to contain certain ordered pairs.

For example, if (1, 3) and (3, 4) are in a relation R, then the pair (1, 4) *must* be in R for R to be transitive.

To obtain a transitive relation from one that is not transitive, it is necessary to add ordered pairs.

Roughly speaking, the relation obtained by adding the least number of ordered pairs to ensure transitivity is called the *transitive closure* of the relation.

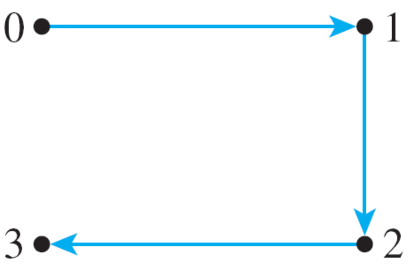
In a sense made precise by the formal definition, the transitive closure of a relation is the smallest transitive relation that contains the relation.

Definition: Transitive Closure

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

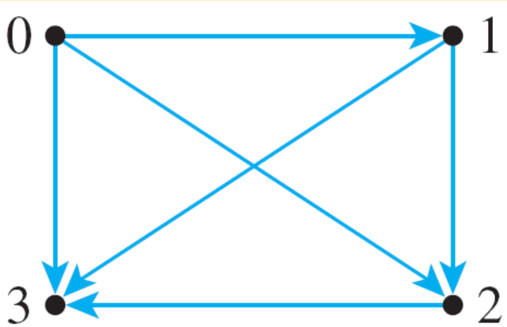
1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

Example #13: Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows: $R = \{(0, 1), (1, 2), (2, 3)\}$. Find the transitive closure of R.



Since there are arrows from 0 to 1 and from 1 to 2, R^t must have an arrow from 0 to 2. Hence $(0,2) \in R^t$. Then $(0,2) \in R^t$ and $(2,3) \in R^t$, so $(0,3) \in R^t$. Also, since $(1,2) \in R^t$ and $(2,3) \in R^t$, so $(1,3) \in R^t$.

Directed graph of R^t:

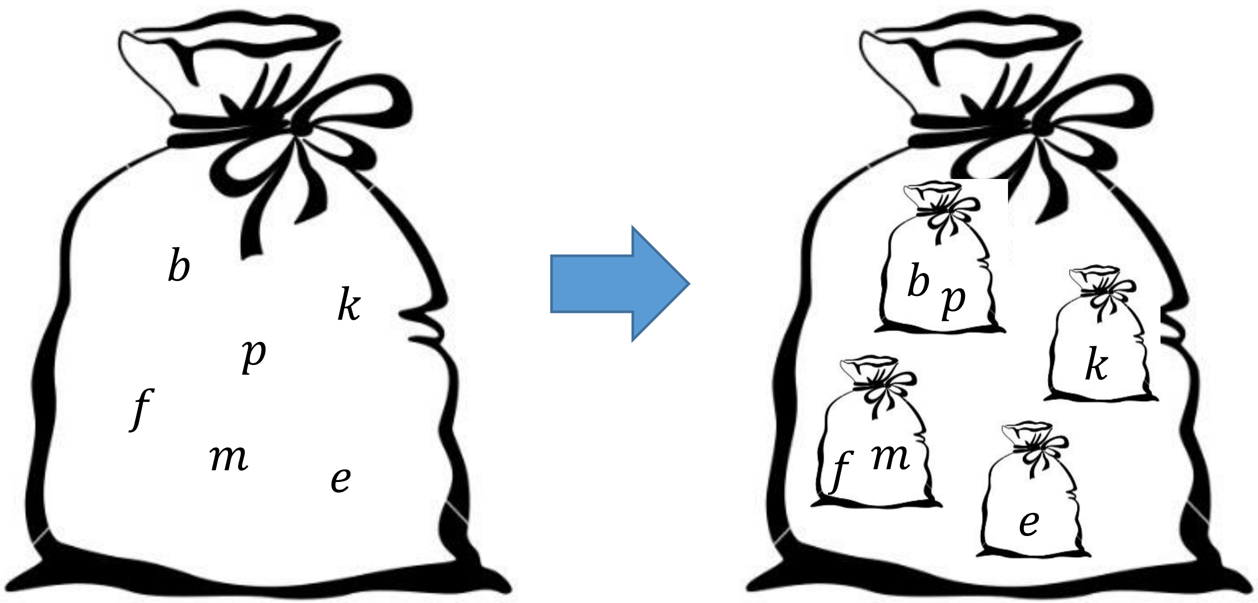


6.3 Equivalence Relations

6.3.1 The Relation Induced by a Partition

A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A.

The diagram below illustrates a partition of a set $A = \{b, e, f, k, m, p\}$ by subsets $\{b, p\}, \{f, m\}, \{k\}, \{e\}$.



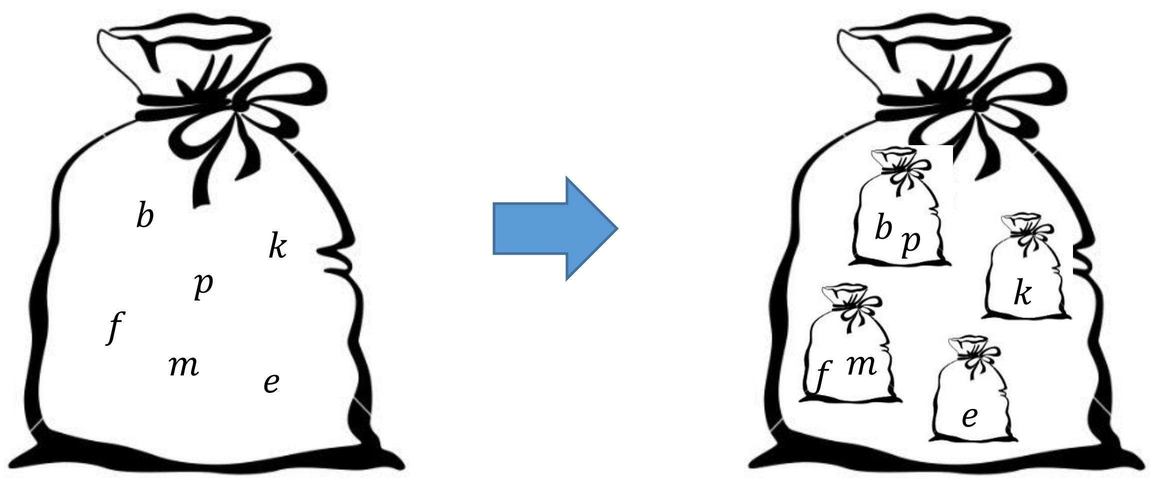
Definition: Partition

$\mathfrak{C}$ is a **partition** of a set A if the following hold:

(1) $\mathfrak{C}$ is a set of which all elements are non-empty subsets of A, i.e., $\emptyset \neq S \subseteq A$ for all $S \in \mathfrak{C}$.

(2) Every element of A is in exactly one element of $\mathfrak{C}$, i.e.,
$\forall x \in A\ \exists S \in \mathfrak{C}\ (x \in S)$ and
$\forall x \in A\ \exists S_1, S_2, \in \mathfrak{C}\ (x \in S_1 \land x \in S_2 \Rightarrow S_1 = S_2)$.

Elements of a partition are called **components** of the partition.



$A = \{b, e, f, k, m, p\}$ $\qquad$ $\mathfrak{C} = \{\{b, p\}, \{f, m\}, \{k\}, \{e\}\}$

Definition: Partition

$\mathfrak{C}$ is a **partition** of a set A if the following hold:

(1) $\mathfrak{C}$ is a set of which all elements are non-empty subsets of A, i.e., $\emptyset \neq S \subseteq A$ for all $S \in \mathfrak{C}$.

(2) Every element of A is in exactly one element of $\mathfrak{C}$, i.e., $\forall x \in A\ \exists S \in \mathfrak{C}\,(x \in S)$ and $\forall x \in A\ \exists S_1, S_2, \in \mathfrak{C}\,(x \in S_1 \wedge x \in S_2 \Rightarrow S_1 = S_2)$.

Elements of a partition are called **components** of the partition.

Definition (shorter): Partition

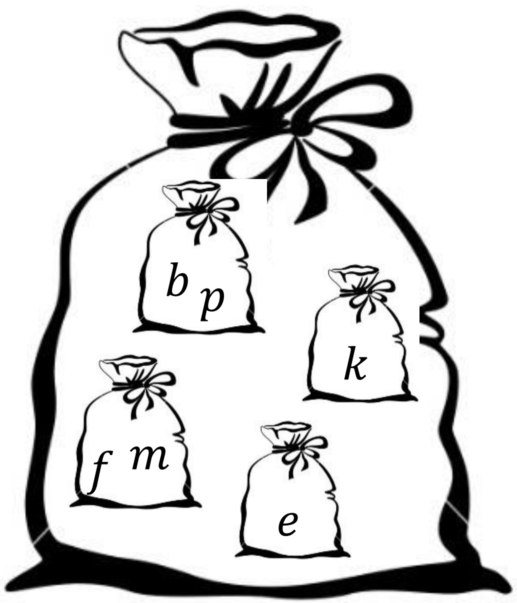
A **partition** of set A is a set $\mathfrak{C}$ of non-empty subsets of A such that

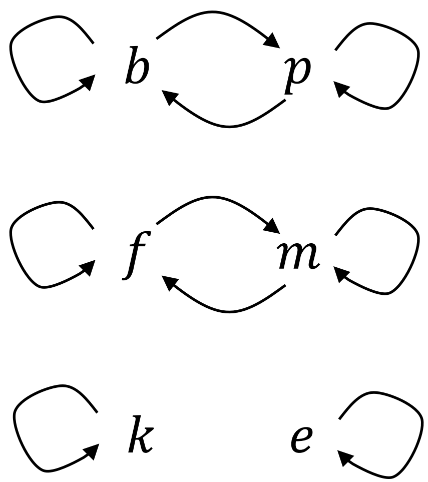
$$\forall x \in A\ \exists!\, S \in \mathfrak{C}\,(x \in S).$$

(Recall: ∃! means “there exists a unique”.)

Partitions as relations

We may view a partition as a “is in the same component as” relation.





Let R be “in the same component as” relation.

$b\ R\ b$ $\quad$ $f\ R\ f$
$p\ R\ p$ $\quad$ $m\ R\ m$
$b\ R\ p$ $\quad$ $f\ R\ m$
$p\ R\ b$ $\quad$ $m\ R\ f$

$k\ R\ k$ $\quad$ $e\ R\ e$

Definition: Relation Induced by a Partition

Given a partition $\mathcal{C}$ of a set A, the relation R **induced by the partition** is defined on A as follows: $\forall x, y \in A$,

$$xRy \Leftrightarrow \exists \text{ a component } S \text{ of } \mathcal{C} \text{ s.t. } x, y \in S.$$

Example #14: Let $A = \{0, 1, 2, 3, 4\}$ and consider this partition of A: $\{\{0, 3, 4\}, \{1\}, \{2\}\}$.

Find the relation R induced by this partition.

{0,3,4} is a component of the partition → 0*R*0, 0*R*3, 0*R*4, 3*R*0, 3*R*3, 3*R*4, 4*R*0, 4*R*3 and 4*R*4.

{1} is a component of the partition → 1*R*1.

{2} is a component of the partition → 2*R*2.

Therefore, *R* = {(0,0), (0,3), (0,4), (1,1), (2,2), (3,0), (3,3), (3,4), (4,0), (4,3), (4,4)}.

The fact is that a relation induced by a partition of a set satisfies all three properties: reflexivity, symmetry, and transitivity.

Theorem 8.3.1 Relation Induced by a Partition

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

6.3.2 Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an *equivalence relation*.

Definition: Equivalence Relation

Let A be a set and R a relation on A. R is an **equivalence relation** iff R is reflexive, symmetric and transitive.

Note: The symbol $\sim$ is commonly used to denote an equivalence relation.

Example #15: Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then $X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Define a relation **R** on X as follows: For all $A, B \in X$,

A **R** $B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

R is reflexive:

Suppose A is a nonempty subset of {1, 2, 3}. It is true to say that the least element of A equals the least element of A. Thus, A **R** A.

R is symmetric:

Suppose A and B are nonempty subsets of {1, 2, 3} and A **R** B. Since A **R** B, the least element of A equals the least element of B. But this means that the least element of B equals the least element of A, and so by definition of **R**, B **R** A.

Example #15: Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then $X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Define a relation **R** on X as follows: For all $A, B \in X$,

A **R** $B \Leftrightarrow$ the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

R is transitive:

Suppose A, B and C are nonempty subsets of {1, 2, 3}, A **R** B and B **R** C.

Since A **R** B, the least element of A equals the least element of B and since B **R** C, the least element of B equals the least element of C.

Thus the least element of A equals the least element of C, and so, by definition of **R**, A **R** C.

6.3.3 Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation on a certain set. If a is any particular element of the set, then one can ask, “What are the elements that are related to a?” This set of elements is called the *equivalence class* of a.

Definition: Equivalence Class

Suppose A is a set and $\sim$ is an equivalence relation on A. For each $a \in A$, the **equivalence class** of a, denoted $[\boldsymbol{a}]$ and called the **class of $\boldsymbol{a}$** for short, is the set of all elements $x \in A$ s.t. a is $\sim$-related to x.

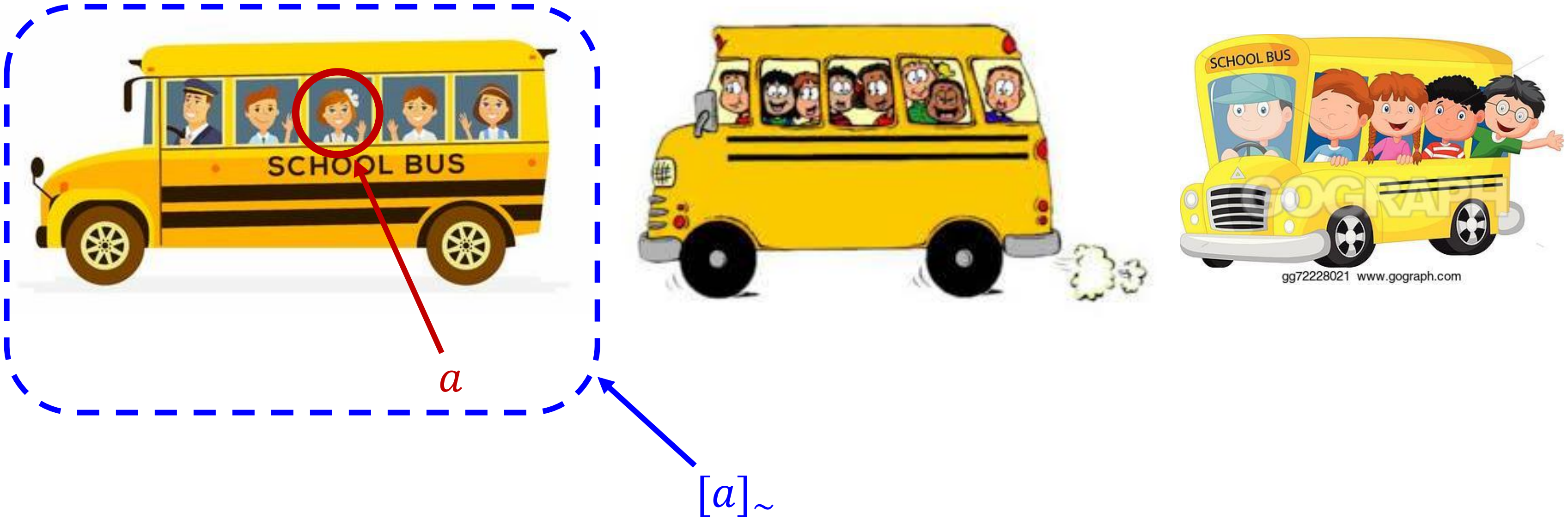
Symbolically,

$$[a]_{\sim} = \{x \in A : a \sim x\}$$

Equivalence Classes of an Equivalence Relation

The procedural version of this definition is

$$\boxed{\forall x \in A\ (x \in [a]_\sim \Leftrightarrow a \sim x).}$$

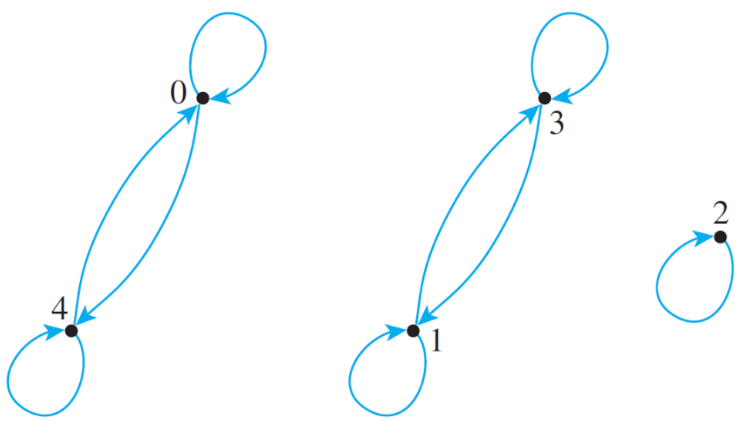


(When there is no risk of confusion, we may drop the subscript $\sim$ and write $[a]$.)

Example #16: Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

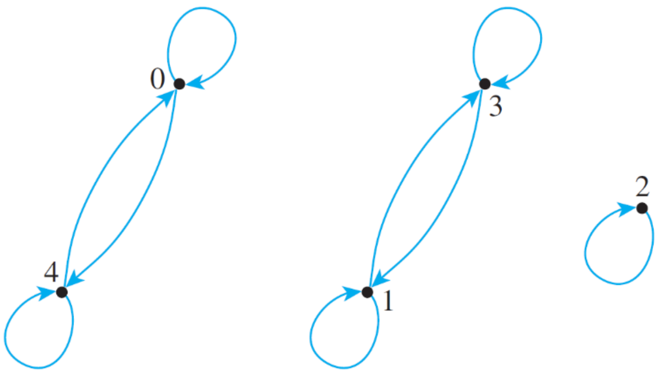
$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}.$$

The directed graph for R is as shown below. As can be seen by inspection, R is an equivalence relation on A. Find the distinct equivalence classes of R.



Equivalence Classes of an Equivalence Relation

First find the equivalence class of every element of A.



$$[0] = \{x \in A : 0\ R\ x\} = \{0,4\}$$
$$[1] = \{x \in A : 1\ R\ x\} = \{1,3\}$$
$$[2] = \{x \in A : 2\ R\ x\} = \{2\}$$
$$[3] = \{x \in A : 3\ R\ x\} = \{1,3\}$$
$$[4] = \{x \in A : 4\ R\ x\} = \{0,4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are {0, 4}, {1, 3}, and {2}.

Lemma Rel.1 Equivalence Classes

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \emptyset$.

We prove this by proving:

$$\begin{array}{ccc} & \text{(i)} & \\ \Nearrow & & \Searrow \\ \text{(iii)} & \Leftarrow & \text{(ii)} \end{array}$$

Definition:
$[a]_\sim = \{x \in A : a \sim x\}$

Proof

1. ((i) $\Rightarrow$ (ii))
   1.1. Suppose $x \sim y$.
   1.2. Then $y \sim x$ by symmetry.
   1.3. For every $z \in [x]$,
      1.3.1. $x \sim z$ by the definition of $[x]$;
      1.3.2. $\therefore y \sim z$ by transitivity, as $y \sim x$;
      1.3.3. $\therefore z \in [y]$ by the definition of $[y]$.
   1.4. This shows $[x] \subseteq [y]$.
   1.5. Switching the roles of x and y, we see also that $[y] \subseteq [x]$.
   1.6. Therefore, $[x] = [y]$.

Lemma Rel.1 Equivalence Classes

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \emptyset$.

We prove this by proving:

$$\begin{array}{ccc} & \text{(i)} & \\ \nearrow & & \searrow \\ \text{(iii)} & \Leftarrow & \text{(ii)} \end{array}$$

Definition:
$[a]_{\sim} = \{x \in A : a \sim x\}$

Proof

2. ((ii) $\Rightarrow$ (iii))
   - 2.1. Suppose $[x] = [y]$.
   - 2.2. Then $[x] \cap [y] = [x]$ by the Idempotent Law for ∩.
   - 2.3. However, we know $x \sim x$ by the reflexivity of ~.
   - 2.4. This shows $x \in [x] = [x] \cap [y]$ by the definition of [x] and line 2.2.
   - 2.5. Therefore, $[x] \cap [y] \neq \emptyset$.

Lemma Rel.1 Equivalence Classes

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$.

(i) $x \sim y$. (ii) $[x] = [y]$. (iii) $[x] \cap [y] \neq \emptyset$.

We prove this by proving:

$$\begin{array}{ccc} & \text{(i)} & \\ \Nearrow & & \Searrow \\ \text{(iii)} & \Leftarrow & \text{(ii)} \end{array}$$

Definition:
$[a]_\sim = \{x \in A : a \sim x\}$

Proof

3. ((iii) $\Rightarrow$ (i))
   - 3.1. Suppose $[x] \cap [y] \neq \emptyset$.
   - 3.2. Take $z \in [x] \cap [y]$.
   - 3.3. Then $z \in [x]$ and $z \in [y]$ — by the definition of $\cap$.
   - 3.4. Then $x \sim z$ and $y \sim z$. — by the definition of $[x]$ and $[y]$.
   - 3.5. $y \sim z$ implies $z \sim y$ — by symmetry.
   - 3.6. Therefore, $x \sim y$ — by transitivity.

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Revisit Example #12:

Define a relation R on $\mathbb{Z}$ as follows:

$$\forall x, y \in \mathbb{Z}\left(x\, R\, y \;\Leftrightarrow 3 \mid (x-y)\right).$$

This relation is called congruence modulo 3.

It has been shown that R is an equivalence relation.

What are the distinct equivalence classes of R?

The distinct equivalent classes of R are:

- $\{3k : k \in \mathbb{Z}\}$, {...,-9,-6,-3,0,3,6,9,...}
- $\{3k+1 : k \in \mathbb{Z}\}$, and {...,-8,-5,-2,1,4,7,10,...}
- $\{3k+2 : k \in \mathbb{Z}\}$. {...,-7,-4,-1,2,5,8,11,...}

Observe that {{...,-9,-6,-3,0,3,6,9,...}, {...,-8,-5,-2,1,4,7,10,...}, {...,-7,-4,-1,2,5,8,11,...}} is a partition of $\mathbb{Z}$.

Equivalence Classes of an Equivalence Relation

Congruence modulo n (congruence-mod-n) relation:

$$\forall x, y \in \mathbb{Z}\left(x\ R\ y \ \Leftrightarrow n \mid (x - y)\right).$$

Congruence modulo 2 ($\mathbb{Z}$)

:	:
-4	-3
-2	-1
0	1
2	3
4	5
:	:

Partition of $\mathbb{Z}$:

$$\left\{\begin{matrix}\{2k : k \in \mathbb{Z}\}, \\ \{2k+1 : k \in \mathbb{Z}\}\end{matrix}\right\}$$

Congruence modulo 3 ($\mathbb{Z}$)

:	:	:
-6	-5	-4
-3	-2	-1
0	1	2
3	4	5
6	7	8
:	:	:

Partition of $\mathbb{Z}$:

$$\left\{\begin{matrix}\{3k : k \in \mathbb{Z}\}, \\ \{3k+1 : k \in \mathbb{Z}\}, \\ \{3k+2 : k \in \mathbb{Z}\}\end{matrix}\right\}$$

Congruence modulo 4 ($\mathbb{Z}$)

:	:	:	:
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
:	:	:	:

Partition of $\mathbb{Z}$:

$$\left\{\begin{matrix}\{4k : k \in \mathbb{Z}\}, \\ \{4k+1 : k \in \mathbb{Z}\}, \\ \{4k+2 : k \in \mathbb{Z}\}, \\ \{4k+3 : k \in \mathbb{Z}\}\end{matrix}\right\}$$

6.3.4 Congruence

Definition: Divisibility

Let $n, d \in \mathbb{Z}$. Then $d \mid n \Leftrightarrow n = dk$ for some $k \in \mathbb{Z}$.

Definition: Congruence

Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then a is congruent to b modulo n iff $a - b = nk$ for some $k \in \mathbb{Z}$. In other words, $n \mid (a - b)$.

In this case, we write $a \equiv b \pmod{n}$.

Example #17: Are the following true?

(a) $7 \equiv 1 \pmod{2}$ Yes, because $7 - 1 = 6 = 2 \times 3$. $k = 3$.

(b) $-3 \equiv 12 \pmod{5}$ Yes, because $-3 - 12 = -15 = 5 \times (-3)$. $k = -3$.

(c) $-4 \equiv 5 \pmod{7}$ No, because $-4 - 5 = -9 \neq 7k$ for any $k \in \mathbb{Z}$.

Proposition

Congruence-mod n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$.

Proof:

1. (Reflexivity) For all $a \in \mathbb{Z}$,
   1.1. $a - a = 0 = n \times 0$.
   1.2. So $a \equiv a \pmod{n}$ by the defn of congruence.
2. (Symmetry)
   2.1. Let $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$.
   2.2. Then there is a $k \in \mathbb{Z}$ such that $a - b = nk$.
   2.3. Then $b - a = -(a - b) = -nk = n(-k)$.
   2.4. $-k \in \mathbb{Z}$ (by closure of integers under $\times$), so $b \equiv a \pmod{n}$ by the definition of congruence.
3. (Transitivity)
   3.1. Let $a, b, c \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
   3.2. Then there are $k, l \in \mathbb{Z}$ such that $a - b = nk$ and $b - c = nl$.
   3.3. Then $a - c = (a - b) + (b - c) = nk + nl = n(k + l)$.
   3.4. $k + l \in \mathbb{Z}$ (by closure of integers under $+$), so $a \equiv c \pmod{n}$ by the definition of congruence.

A relation R on a set A is
reflexive: $\forall x \in A\ (x\ R\ x)$;
symmetric:
$\forall x, y \in A\ (x\ R\ y \Rightarrow y\ R\ x)$;
transitive:
$\forall x, y, z \in A$
$(x\ R\ y \wedge y\ R\ z \Rightarrow x\ R\ z)$.

Definition: Equivalence Class

Suppose A is a set and $\sim$ is an equivalence relation on A. The **equivalence class** of $a \in A$, is $[a]_{\sim} = \{x \in A : a \sim x\}$.

Congruence: Equivalence classes

Let $n \in \mathbb{Z}^+$. The equivalence classes w.r.t. the congruence-mod-n relation on are of the form:

$$\begin{aligned}[x] &= \{y \in \mathbb{Z} : x \equiv y \pmod{n}\} \\ &= \{y \in \mathbb{Z} : x - y = nk \text{ for some } k \in \mathbb{Z}\} \\ &= \{x + nk : k \in \mathbb{Z}\} \\ &= \{\dots, x - 2n, x - n, x, x + n, x + 2n, \dots\} \text{ where } x \in \mathbb{Z}.\end{aligned}$$

Congruence modulo 4

$\mathbb{Z}$

:	:	:	:
-8	-7	-6	-5
-4	-3	-2	-1
0	1	2	3
4	5	6	7
8	9	10	11
:	:	:	:

Note that for all $x \in \mathbb{Z}$, $[x + n] = \{\dots, x - n, x, x + n, x + 2n, x + 3n, \dots\} = [x]$.

For example, if $n = 4$, then

$$\cdots = [-8] = [-4] = [0] = [4] = \cdots \text{ and } \cdots = [-7] = [-3] = [1] = [5] = \cdots$$

and so on.

6.3.5 Dividing a Set by an Equivalence Relation

Definition: Set of equivalence classes

Let A be a set and $\sim$ be an equivalence relation on A. Denote by $A/{\sim}$ the set of all equivalence classes with respect to $\sim$, i.e.,

$$A/{\sim} = \{[x]_\sim : x \in A\}.$$

We may read $A/{\sim}$ as "the quotient of A by $\sim$".

Example #18: Let $n \in \mathbb{Z}^+$. If $\sim_n$ denotes the congruence-mod-n relation on $\mathbb{Z}$, then

$$\begin{aligned}\mathbb{Z}/{\sim_n} &= \{[x] : x \in \mathbb{Z}\} \\ &= \big\{\{nk : k \in \mathbb{Z}\}, \{nk+1 : k \in \mathbb{Z}\}, \cdots, \{nk+(n-1) : k \in \mathbb{Z}\}\big\}.\end{aligned}$$

Theorem Rel.2 Equivalence classes form a partition

Let $\sim$ be an equivalence relation on a set A.
Then $A/\sim$ is a partition of A.

$A/\sim\ = \{[x]_\sim : x \in A\}$.

$\mathfrak{C}$ is a **partition** of a set A if:
(1) $\mathfrak{C}$ is a set of which all elements are nonempty subsets of A.
(2) Every element of A is in exactly one element of $\mathfrak{C}$.

Proof:

1. $A/\sim$ is by definition a set.
2. We show that every element of $A/\sim$ is a nonempty subset of A.
   - 2.1. Let $S \in A/\sim$.
   - 2.2. Use the definition of $A/\sim$ to find $x \in A$ such that $S = [x]$.
   - 2.3. Then $S = [x] \subseteq A$ in view of the definition of equivalence classes.
   - 2.4. $x \sim x$ by the reflexivity of $\sim$.
   - 2.5. Hence $x \in [x] = S$ by the definition of [x].
   - 2.6. In particular, we know S is nonempty.
3. We show that every element of A is in at least one element of $A/\sim$.
4. We show that every element of A is in at most one element of $A/\sim$.

Dividing a Set by an Equivalence Relation

Theorem Rel.2 Equivalence classes form a partition

Let $\sim$ be an equivalence relation on a set A.
Then $A/\sim$ is a partition of A.

$A/\sim \; = \{[x]_\sim : x \in A\}$.

$\mathfrak{C}$ is a **partition** of a set A if:

(1) $\mathfrak{C}$ is a set of which all elements are nonempty subsets of A.

(2) Every element of A is in exactly one element of $\mathfrak{C}$.

Proof:

1. $A/\sim$ is by definition a set.
2. We show that every element of $A/\sim$ is a nonempty subset of A.
3. We show that every element of A is in at least one element of $A/\sim$.
   - 3.1. Let $x \in A$.
   - 3.2. $x \sim x$ by the reflexivity of $\sim$.
   - 3.3. So $x \in [x] \in A/\sim$.
4. We show that every element of A is in at most one element of $A/\sim$.

Theorem Rel.2 Equivalence classes form a partition

Let $\sim$ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

$A/{\sim} = \{[x]_{\sim} : x \in A\}.$

$\mathscr{C}$ is a **partition** of a set A if:

(1) $\mathscr{C}$ is a set of which all elements are nonempty subsets of A.

(2) Every element of A is in exactly one element of $\mathscr{C}$.

Proof:

1. $A/{\sim}$ is by definition a set.
2. We show that every element of $A/{\sim}$ is a nonempty subset of A.
3. We show that every element of A is in at least one element of $A/{\sim}$.
4. We show that every element of A is in at most one element of $A/{\sim}$.
   - 4.1. Let $x \in A$ that is in two elements of $A/{\sim}$, say S_1 and S_2.
   - 4.2. Use the definition of $A/{\sim}$ to find $y_1, y_2 \in A$ such that $S_1 = [y_1]$ and $S_2 = [y_2]$.
   - 4.3. $x \in [y_1] \cap [y_2]$ by lines 4.1 and 4.2.
   - 4.4. So $[y_1] \cap [y_2] \neq \emptyset$.
   - 4.5. Therefore $S_1 = [y_1] = [y_2] = S_2$ by lemma: equivalence classes.

Lemma Rel.1 Equivalence Classes

Let $\sim$ be an equivalence relation on a set A. The following are equivalent for all $x, y \in A$. (i) $x \sim y$; (ii) $[x] = [y]$; (iii) $[x] \cap [y] \neq \emptyset$.

6.3.6 Summary

Definition: A **relation** on set A is a subset of A^2.

Definition: If R is a **relation** on a set A, then we write $x\ R\ y$ for $(x, y) \in R$.

Definition: A **partition** of a set A is a set $\mathfrak{C}$ of non-empty subsets of A such that

$$\forall x \in A\ \exists!\, S \in \mathfrak{C}\, (x \in S).$$

Definition: A relation R on A is an **equivalence relation** if

- (reflexivity) $\forall x \in A\ (x\ R\ x)$;
- (symmetry) $\forall x, y \in A\ (x\ R\ y \Rightarrow y\ R\ x)$; and
- (transitivity) $\forall x, y, z \in A\ (x\ R\ y \wedge yRz \Rightarrow y\ R\ z)$.

Definition: Let $\sim$ be an equivalence relation on A. Then the set of equivalence classes is denoted by $A/\!\sim\ = \{[x]_\sim : x \in A\}$, where $[x]_\sim = \{y \in A : x \sim y\}$.

Proposition: The same-component relation w.r.t. a partition is an equivalence relation.

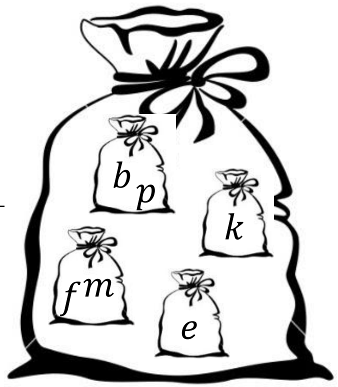
Theorem Rel.2: If $\sim$ is an equivalence relation on A, then $A/\!\sim$ is a partition of A.

Informal descriptions of the terms

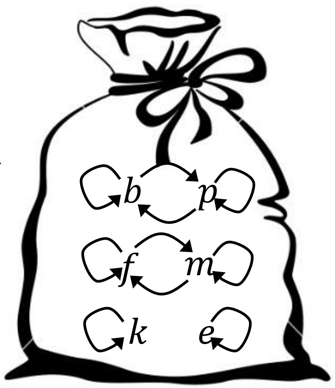
1.	underlying set	A	the set to be "partitioned"
2.	components	S	subsets of A, mutually disjoint, together union to A
3.	partition	$\mathfrak{T}$	the set of all components
4.	same-component relation	$\sim$	equivalence relation

1.	underlying set	A	the set of all vertices
2.	relation	R	the set of all arrows
3.	equivalence relation	$\sim$	if ignoring directions of arrows one can walk from x to y, then there is an arrow from x to y
4.	equivalence classes	$[x]$	connected components
5.	quotient	$A/\sim$	the set of all connected components

$A = \{b, e, f, k, m, p\}$



$\mathfrak{T} = \{\{b,p\},\{f,m\},\{k\},\{e\}\}$



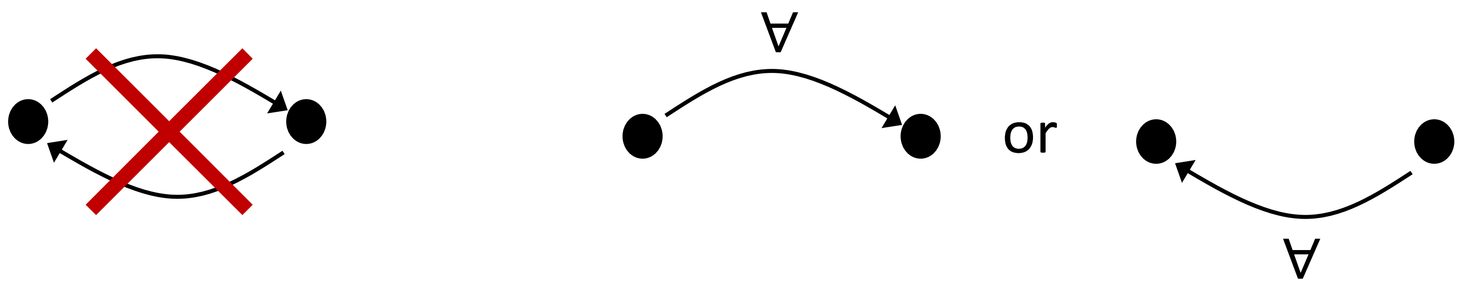
6.4 Partial Order Relations

6.4.1 Antisymmetry

Definition: Antisymmetry

Let R be a relation on a set A. R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y).$$



By taking the negation of the definition, you can see that a relation R is ***not* antisymmetric** iff

$$\exists x, y \in A\ (x\ R\ y \wedge y\ R\ x \wedge x \neq y).$$

The big question: Is antisymmetry $\equiv$ $\sim$(symmetry)?

Let R be a relation on a set A.

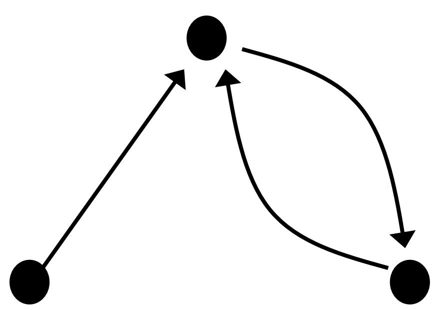
R is **symmetric**
iff $\forall x, y \in A\ (xRy \Rightarrow yRx)$.

R is **not symmetric**
iff $\exists x, y \in A\ (xRy \wedge y \not R x)$.

$\cancel{\equiv}$

R is **antisymmetric**
iff $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.

?



Example #19: Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\forall a, b \in \mathbb{Z}^+, \qquad aR_1b \Leftrightarrow a|b.$$
$$\forall a, b \in \mathbb{Z}, \qquad aR_2b \Leftrightarrow a|b.$$

a. Is R_1 antisymmetric? Prove or give a counterexample.

R_1 is antisymmetric:

1. Suppose $a, b \in \mathbb{Z}^+$ such that aR_1b and bR_1a.
2. Then $b = ra$ and $a = sb$ for some integers r and s (by definition of "divides"). It follows that $b = ra = r(sb)$.
3. Dividing both sides by b gives $1 = rs$.
4. The only product of two positive integers that equals 1 is $1 \cdot 1$.
5. Thus $r = s = 1$, and so $a = sb = 1 \cdot b = b$.

Example #19: Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\forall a, b \in \mathbb{Z}^+, \qquad aR_1b \Leftrightarrow a|b.$$
$$\forall a, b \in \mathbb{Z}, \qquad aR_2b \Leftrightarrow a|b.$$

a. Is R_1 antisymmetric? Prove or give a counterexample.

Alternatively, we may use Theorem 4.3.1 (see lecture #4):

For all $a, b \in \mathbb{Z}^+$, if $a \mid b$ then $a \leq b$.

R_1 is antisymmetric:

1. Suppose $a, b \in \mathbb{Z}^+$ such that aR_1b and bR_1a.
2. Then $a \leq b$ and $b \leq a$ by Theorem 4.3.1.
3. So $a = b$.

Example #19: Let R_1 be the "divides" relation on the set of all positive integers, and let R_2 be the "divides" relation on the set of all integers.

$$\forall a, b \in \mathbb{Z}^+, \qquad aR_1b \Leftrightarrow a|b.$$
$$\forall a, b \in \mathbb{Z}, \qquad aR_2b \Leftrightarrow a|b.$$

b. Is R_2 antisymmetric? Prove or give a counterexample.

R_2 is not antisymmetric.

Counterexample:

Let $a = 2$ and $b = -2$. Then $a|b$ and $b|a$.

Hence aR_2b and bR_2a but $a \neq b$.

6.4.2 Partial Order Relations

Definition: Partial Order Relation

Let R be a relation on a set A. Then R is a **partial order relation** (or simply **partial order**) iff R is reflexive, antisymmetric and transitive.

Two fundamental partial order relations are the "less than or equal to ($\leq$)" relation on a set of real numbers and the "subset ($\subseteq$)" relation on a set of sets.

Definition: Partially Ordered Set

A set A is called a **partially ordered set** (or **poset**) with respect to a partial order relation R on A, denoted by (A, R).

Example #20: Let $|$ be the "divides" relation on a set A of positive integers. That is, $\forall a, b \in A$,

$$\boxed{a|b \Leftrightarrow b = ka \text{ for some integer } k.}$$

Prove that $|$ is a partial order relation on A.

$|$ is **reflexive**: Suppose $a \in A$. Then $a = 1 \cdot a$, so $a|a$ (by the definition of divisibility).

$|$ is **antisymmetric**: To show that $\forall a, b \in A, a|b \ \land b|a \rightarrow a = b$. Proof identical to Exercise #19a.

$|$ is **transitive**: To show that for $\forall a, b, c \in A, a|b \ \land b|c \rightarrow a|c$. This is Theorem 4.3.3 (5^{th}: 4.4.3).

Theorem 4.3.3 (5^{th}: 4.4.3) Transitivity of Divisibility

For all integers a, b and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Theorem 4.3.3 (5th: 4.4.3) Transitivity of Divisibility

For all integers a, b and c, if $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof:

1. Suppose a, b, c are integers such that $a \mid b$ and $b \mid c$.
2. Then $b = ar$ and $c = bs$ for some integers r and s by the definition of divisibility.
3. $c = bs = (ar)s = a(rs)$ by basic algebra.
4. Therefore $a \mid c$ since rs is an integer (by closure of integers under $\times$).

Exercise: Let $\leq$ be the "less than or equal to" relation on $\mathbb{Q}$. Show that $\leq$ is a partial order.

Notation

Because of the special paradigmatic role played by the $\leq$ relation in the study of partial order relations, the symbol $\preccurlyeq$ is often used to refer to a general partial order, and the notation $x \preccurlyeq y$ is read "x is curly less than or equal to y".

Question: The "less than" relation on $\mathbb{Q}$ is denoted as $<$. Is $<$ a partial order? Why?

No. $<$ is not reflexive.

Question: Fix an $n \in \mathbb{Z}^+$. Let R denote the congruence-mod-n relation on $\mathbb{Z}$. Is R a partial order? Why?

No. R is not antisymmetric.

One way of viewing partial orders.

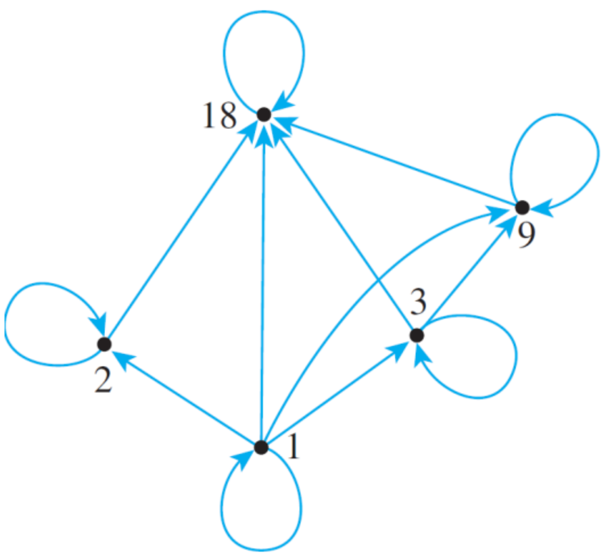
- We may view the set A as a set of tasks.
- Suppose $x, y \in A$. We write $x \preccurlyeq y$ iff task x must be done before or at the same time as task y.
- For some elements x and y, it could be that neither $x \preccurlyeq y$ nor $y \preccurlyeq x$ (that is, x and y are not “comparable” – to be defined later).
- Hence, the order is “partial”, that is, there may not be an order between certain elements.
- This is clearer when we get to the Hasse Diagram.

6.4.3 Hasse Diagrams

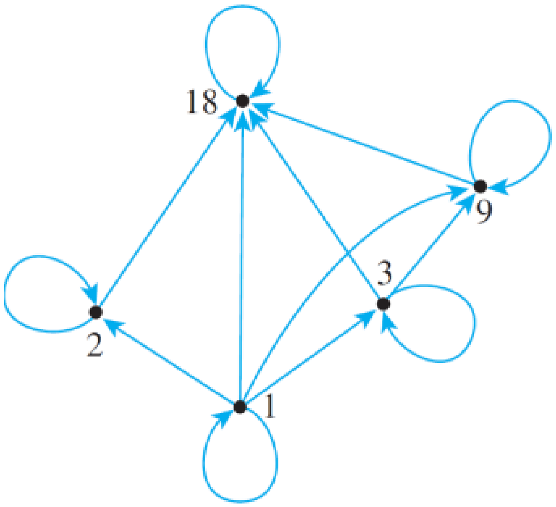
Example #21: Let $A = \{1, 2, 3, 9, 18\}$ and consider the "divides" relation on A: $\forall a, b \in A$,

$$\boxed{a \mid b \Leftrightarrow b = ka \text{ for some integer } k.}$$

The directed graph of this relation, which is a partial order, is as follows:



Note that there is a loop at every vertex, all other arrows point in the same direction (upward), and any time there is an arrow from one point to a second and from the second point to a third, there is an arrow from the first point to the third.



Given any partial order relation defined on a finite set, it is possible to draw the directed graph in such a way that all of these properties are satisfied.

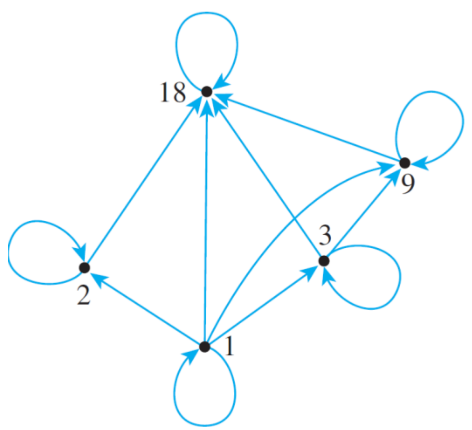
This makes it possible to associate a somewhat simpler graph, called a **Hasse diagram** (after Helmut Hasse, a twentieth-century German number theorist), with a partial order relation defined on a finite set.

Hasse Diagrams

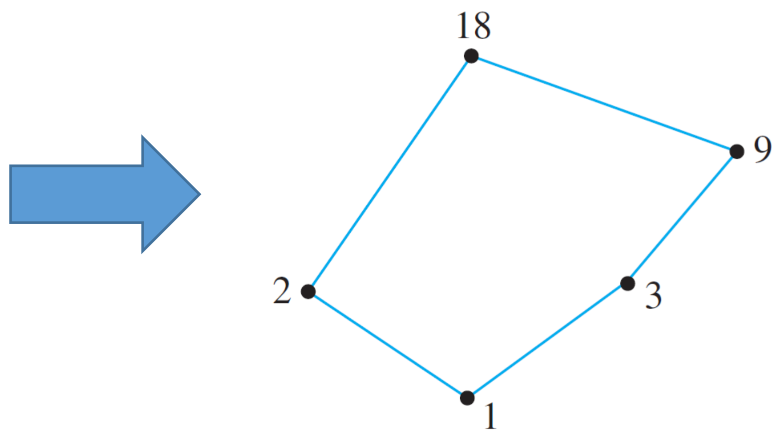
To obtain a Hasse diagram, proceed as follows:

Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property,
3. the direction indicators on the arrows.



Directed graph of the "divides" relation on {1,2,3,9,18}

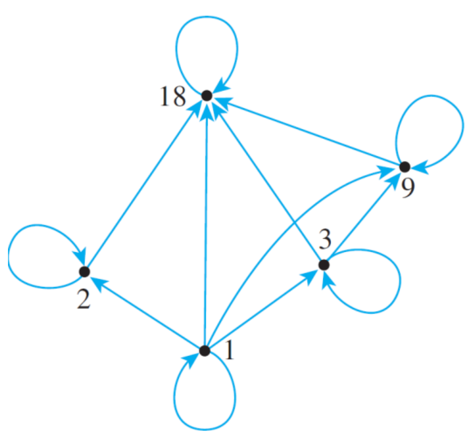


Hasse diagram of the "divides" relation on {1,2,3,9,18}

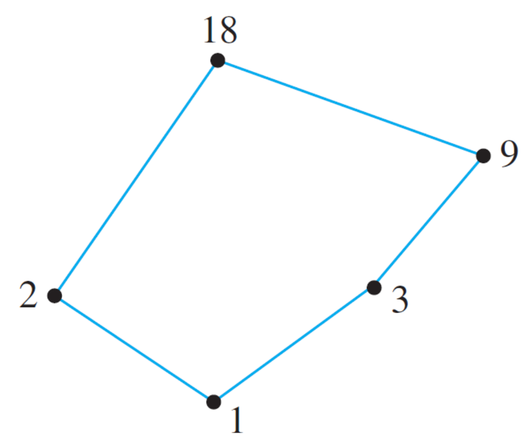
Definition: Hasse Diagram

Let $\preccurlyeq$ be a partial order on a set A. A **Hasse diagram** of $\preccurlyeq$ satisfies the following condition for all <u>distinct</u> $x, y, m \in A$:

> If $x \preccurlyeq y$ and no $m \in A$ is such that $x \preccurlyeq m \preccurlyeq y$,
> then x is placed below y with a line joining them,
> else no line joins x and y.



Directed graph of the "divide" relation on {1,2,3,9,18}

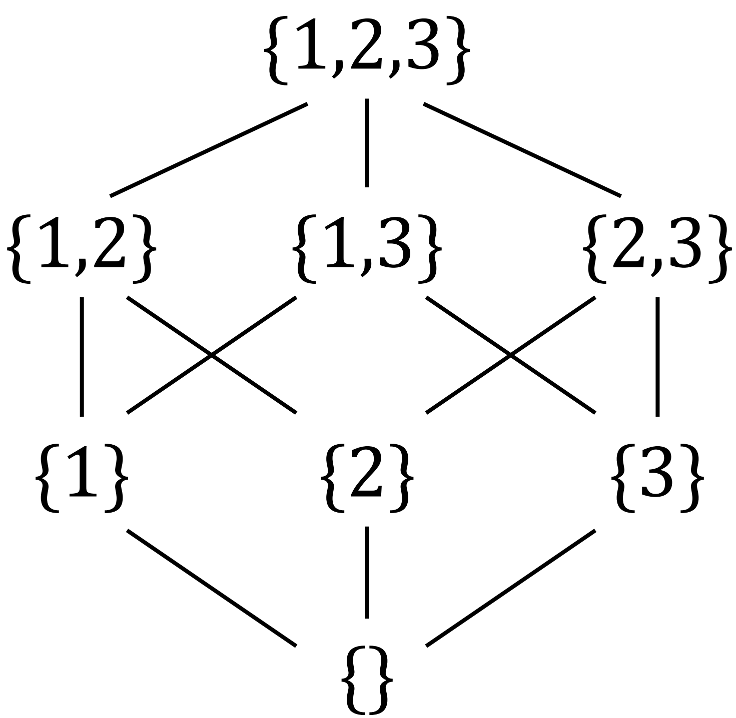


Hasse diagram of the "divide" relation on {1,2,3,9,18}

Example #22: Consider $\wp(\{1,2,3\})$ partially ordered by the subset relation $\subseteq$. Draw its Hasse diagram.

(The directed graph would be too complex.
The Hasse diagram carries the same information.)

> Recall that $\wp(A)$ denotes the power set of A.



Note that certain elements, say x and y, of $\wp(\{1,2,3\})$ are not comparable, that is, neither $x \preccurlyeq y$ nor $y \preccurlyeq x$.

$\{1,2\}$ and $\{1,3\}$ are two such elements, as neither $\{1,2\} \subseteq \{1,3\}$ nor $\{1,3\} \subseteq \{1,2\}$.

6.4.4 Comparability

Given any two real numbers x and y, either $x \leq y$ or $y \leq x$. In a situation like this, the elements x and y are said to be *comparable*.

On the other hand, given two subsets A and B of $\{a, b, c\}$, it may be the case that neither $A \subseteq B$ nor $B \subseteq A$.

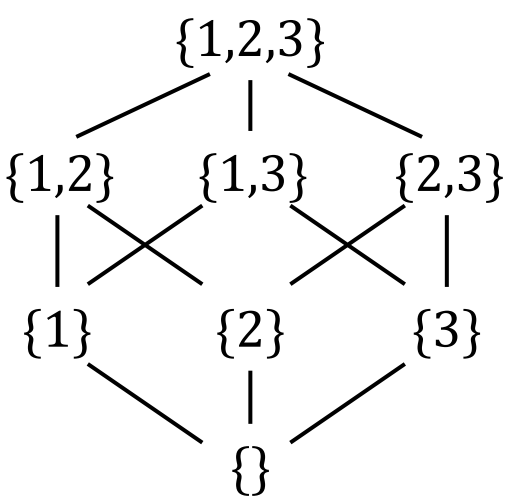
For instance, let $A = \{a, b\}$ and $B = \{b, c\}$. Then $A \nsubseteq B$ and $B \nsubseteq A$.

In such a case, A and B are said to be *noncomparable*.

Definition: Comparability

Suppose $\preccurlyeq$ is a partial order relation on a set A. Elements a and b of A are said to be **comparable** iff either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are **noncomparable**.

$\subseteq$ on $\wp(\{1,2,3\})$



Which of the following pairs of elements are comparable?

(a)	$\{1\}$ and $\{1,3\}$	Yes
(b)	$\{2,3\}$ and $\{2\}$	Yes
(c)	$\{1\}$ and $\{3\}$	No
(d)	$\{1,2\}$ and $\{3\}$	No
(e)	$\{3\}$ and $\{1,2,3\}$	Yes

6.4.5 Maximal/Minimal/Largest/Smallest Element

Definitions

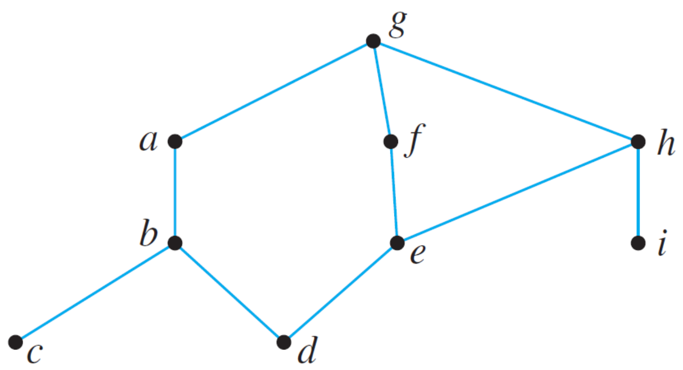
Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A$, either $x \preccurlyeq c$, or x and c are not comparable. Alternatively, c is a maximal element of A iff
$$\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$$
2. c is a **minimal element** of A iff $\forall x \in A$, either $c \preccurlyeq x$, or x and c are not comparable. Alternatively, c is a minimal element of A iff
$$\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x).$$
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Note: Alternative terms

- *largest element = greatest element = maximum;*
- *smallest element = least element = minimum.*

Example #23: Let $A = \{a, b, c, d, e, f, g, h, i\}$ have the partial ordering defined by the following Hasse diagram. Find all maximal, minimal, largest, and smallest elements of A.

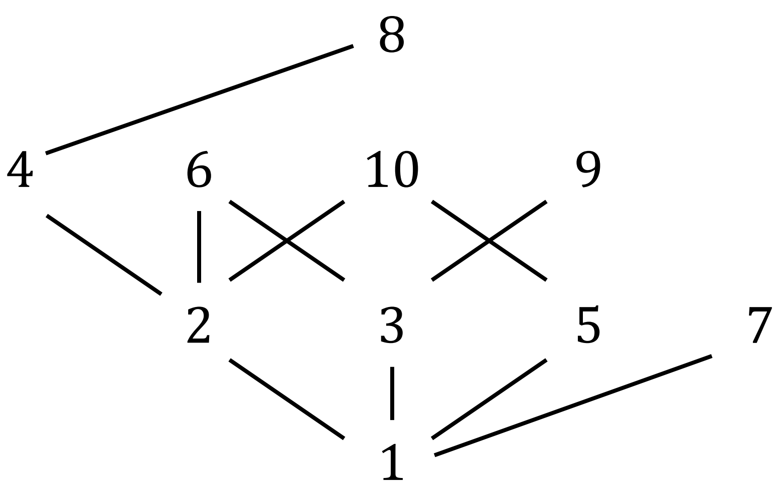


There is just one maximal element, g, which is also the largest element. The minimal elements are c, d, and i, and there is no smallest element.

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$
2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Example #24: The divisibility relation $|$ on $A = \{1, 2, \cdots, 10\}$ is has the following Hasse diagram. Find all maximal, minimal, largest, and smallest elements of A.



Maximal: 6,7,8,9,10

Minimal: 1

Largest: No largest element

Smallest: 1

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$
2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Proposition: A smallest element is minimal.

Consider a partial order $\preccurlyeq$ on a set A. Any smallest element is minimal. (Likewise, any largest element is maximal.)

smallest $\Leftrightarrow$ everything is above

$\Downarrow$

minimal $\Leftrightarrow$ nothing is below

Being above and below implies equality by antisymmetry

$$\forall x, y \in A$$
$$(xRy \land yRx \Rightarrow x = y)$$

Proof:

1. Let c be a smallest element.
2. Take any $x \in A$ such that $x \preccurlyeq c$.
3. By smallestness, we know $c \preccurlyeq x$ too.
4. So $c = x$ by antisymmetry.

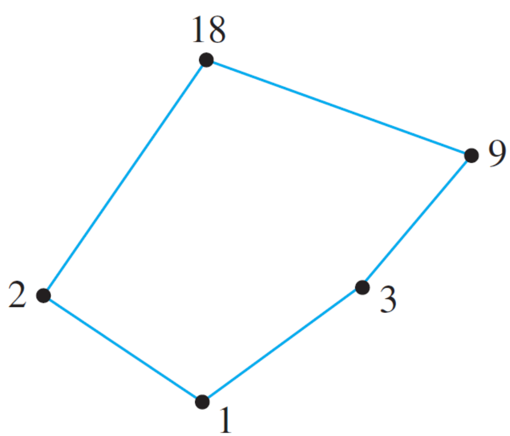
Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.

4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

6.4.6 Linearization

Let $\preccurlyeq$ be a partial order relation on a set A. Suppose we view A as a set of tasks and $x \preccurlyeq y$ to mean x must be performed before or at the same time as y. How can we "line up" the tasks in A assuming that we cannot perform two tasks simultaneously?



Hasse diagram of the "divides" relation on {1,2,3,9,18}

Possible "line-ups":

18		18		18	
|		|		|	
9		9		2	
|	or	|	or	|	or …
3		2		9	
|		|		|	
2		3		3	
|		|		|	
1		1		1	

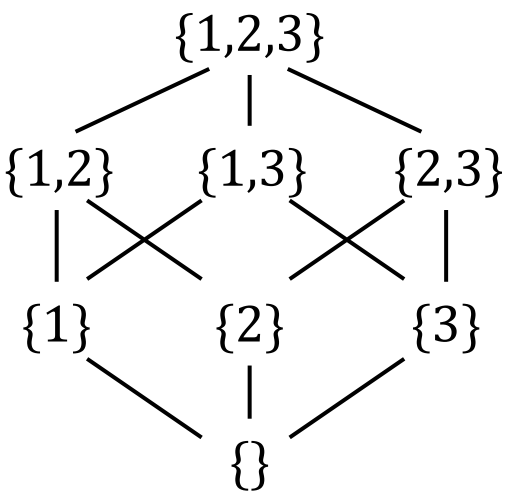
A "line-up" is called a linearization of $\preccurlyeq$.

Such a "line-up" (linearization) happens to be a total order (to be defined later).

Example #25: Consider the subset relation $\subseteq$ on $\wp(\{1,2,3\})$. The Hasse Diagram is shown below.

Which of the following is a linearization of $(\wp(\{1,2,3\}), \subseteq)$?

$\subseteq$ on $\wp(\{1,2,3\})$



$\{1,2,3\} - \{2,3\} - \{1,2\} - \{1,3\} - \{2\} - \{1\} - \{3\} - \{\}$

$\{1,2,3\} - \{2,3\} - \{1\} - \{1,2\} - \{2\} - \{1,3\} - \{3\} - \{\}$

$\{1,2,3\} - \{2,3\} - \{1,2\} - \{1\} - \{2\} - \{1,3\} - \{3\} - \{\}$

6.4.7 Total Order Relations

When all the elements of a partial order relation are comparable, the relation is called a *total order*. (Some call it a *linear order*.)

Definition: Total Order Relations

If R is a partial order relation on a set A, and for any two elements x and y in A, either $x\ R\ y$ or $y\ R\ x$, then R is a **total order relation** (or simply **total order**) on A.

In other words, R is a total order iff

$$R \text{ is a partial order and } \forall x, y \in A\ (x\ R\ y\ \vee y\ R\ x).$$

Example #25: The divisibility relation $|$ on $\mathbb{Z}^+$ is a partial order but not a total order because 3 and 5 are not comparable (i.e., $3 \nmid 5$ and $5 \nmid 3$).

Example #26: The $\leq$ relation on $\mathbb{Q}$ is a total order because for every $x, y \in \mathbb{Q}\ (x \leq y \vee y \leq x)$.

It follows that the Hasse diagram of a total order is one single line (chain). Hence, the linearization of a total order is the total order itself.

A linearization of a partial order can be seen as deriving one total order (among many possible total orders) from that partial order.

Definition: Linearization of a partial order

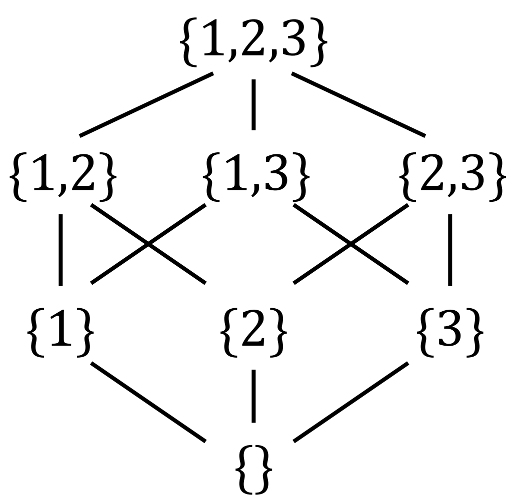
Let $\preccurlyeq$ be a partial order on a set A. A **linearization** of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that

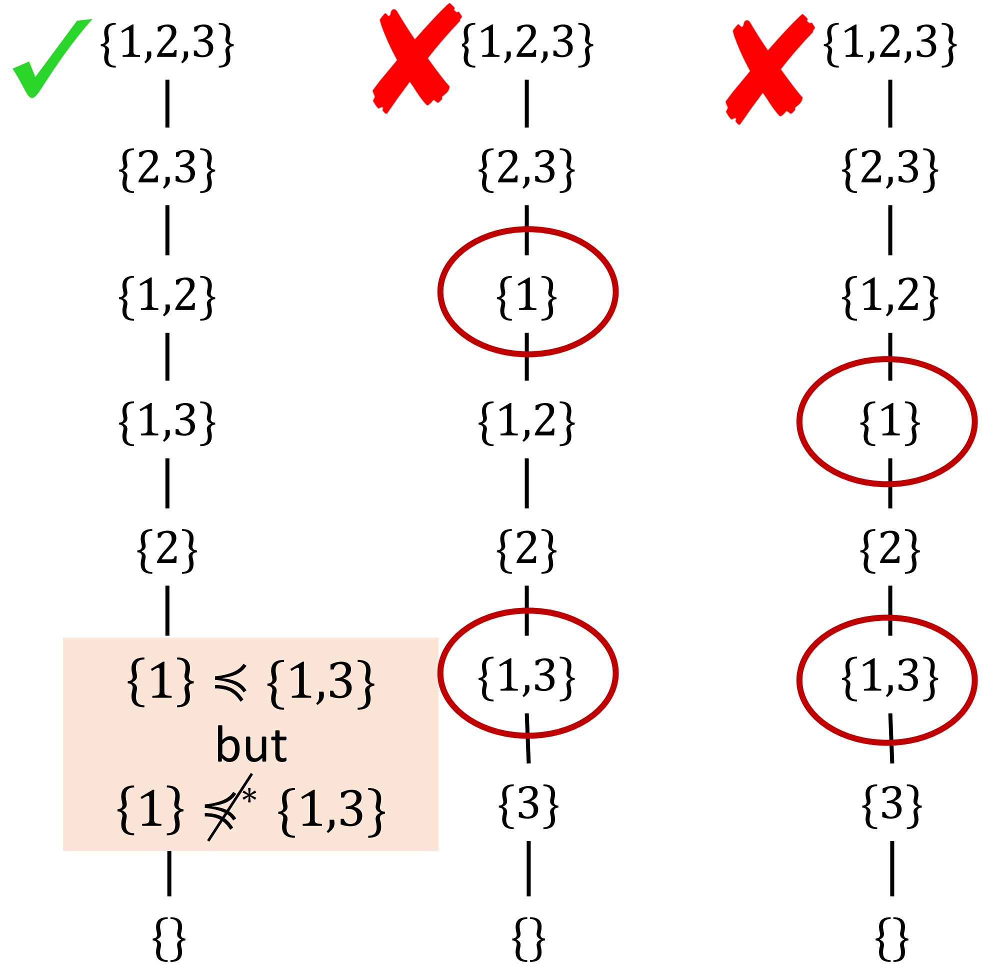
$$\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y).$$

Example #25 (revisit): Consider the subset relation $\subseteq$ on $\wp(\{1,2,3\})$.

Which of the following is a linearization of $(\wp(\{1,2,3\}), \subseteq)$?

$\subseteq$ on $\wp(\{1,2,3\})$





Kahn's Algorithm (1962)

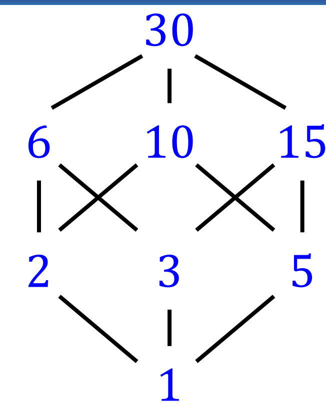
Input: A finite set A and a partial order $\preccurlyeq$ on A.

1. Set $A_0 := A$ and $i := 0$.
2. Repeat until $A_i = \emptyset$
   2.1. find a minimal element c_i of A_i wrt $\preccurlyeq$
   2.2. set $A_{i+1} = A_i \setminus \{c_i\}$
   2.3. set $i := i + 1$

Output: A linearization $\preccurlyeq^*$ of $\preccurlyeq$ defined by setting, for all indices i, j,

$$c_i \preccurlyeq^* c_j \Leftrightarrow i \leq j.$$

Example #26: A run of the Kahn's algorithm on the set $\{d \in \mathbb{Z}^+ : d|30\}$ partially ordered by the divisibility relation $|$.



- Set $A_0 := \{d \in \mathbb{Z}^+ : d|30\}$.
- 1 is the only minimal element of A_0. Set $c_0 := 1$ and $A_1 := A_0 \setminus \{1\}$.
- 2,3,5 are the minimal elements of A_1. Set $c_1 := 3$ and $A_2 := A_1 \setminus \{3\}$.
- 2,5 are the minimal elements of A_2. Set $c_2 := 2$ and $A_3 := A_2 \setminus \{2\}$.
- 5,6 are the minimal elements of A_3. Set $c_3 := 6$ and $A_4 := A_3 \setminus \{6\}$.
- 5 is the only minimal element of A_4. Set $c_4 := 5$ and $A_5 := A_4 \setminus \{5\}$.
- 10,15 are the minimal elements ofA_5. Set $c_5 := 15$ and $A_6 := A_5 \setminus \{15\}$.
- 10 is the only minimal element of A_6. Set $c_6 := 10$ and $A_7 := A_6 \setminus \{10\}$.
- 30 is the only minimal element of A_7. Set $c_7 := 30$ and $A_8 := A_7 \setminus \{30\}$.
- $A_8 = \emptyset$ and so we stop.
- A linearization is obtained: $1 \preccurlyeq^* 3 \preccurlyeq^* 2 \preccurlyeq^* 6 \preccurlyeq^* 5 \preccurlyeq^* 15 \preccurlyeq^* 10 \preccurlyeq^* 30$.

Definition: Well-Ordered Set

Let $\preccurlyeq$ be a total order on a set A. A is **well-ordered** iff every non-empty subset of A contains a smallest element. Symbolically,

$$\forall S \in \wp(A), S \neq \emptyset \Rightarrow \big(\exists x \in S\ \forall y \in S\ (x \preccurlyeq y)\big).$$

Example #27:

- $(\mathbb{N}, \leq)$ is well-ordered.
- $(\mathbb{Z}, \leq)$ is not well-ordered.

Summary

- Relations allow us to model and study many real-world relationships.
- Relations may be inverted and composited.
- Important properties are: reflexivity, symmetry, transitivity, anti-symmetry.
- An Equivalent Relation is the generalization of the notion of "equality".
- A partition of a set and an equivalence relation are two sides of the same coin.
- A Partial Order is the generalization of the notion of "less than or equal to".
- Maximal and minimal elements are generalizations of upper and lower bounds.

END OF FILE