Appendix A

Properties of the Real Numbers*

In this text we take the real numbers and their basic properties as our starting point. We give a core set of properties, called axioms, which the real numbers are assumed to satisfy, and we state some useful properties that can be deduced from these axioms.

We assume that there are two binary operations defined on the set of real numbers, called **addition** and **multiplication**, such that if a and b are any two real numbers, the **sum** of a and b, denoted $\boldsymbol{a+b}$, and the **product** of a and b, denoted $\boldsymbol{a \cdot b}$ or $\boldsymbol{ab}$, are also real numbers. These operations satisfy properties F1, F2, F3, F4, F5, and F6, which are called the **field axioms**.

F1. *Commutative Laws* For all real numbers a and b,

$$a+b=b+a \quad \text{and} \quad ab=ba.$$

F2. *Associative Laws* For all real numbers a, b, and c,

$$(a+b)+c=a+(b+c) \quad \text{and} \quad (ab)c=a(bc).$$

F3. *Distributive Laws* For all real numbers a, b, and c,

$$a(b+c)=ab+ac \quad \text{and} \quad (b+c)a=ba+ca.$$

F4. *Existence of Identity Elements* There exist two distinct real numbers, denoted 0 and 1, such that for every real number a,

$$0+a=a+0=a \quad \text{and} \quad 1\cdot a=a\cdot 1=a.$$

F5. *Existence of Additive Inverses* For every real number a, there is a real number, denoted $-a$ and called the **additive inverse** of a, such that

$$a+(-a)=(-a)+a=0.$$

F6. *Existence of Reciprocals* For every real number $a \neq 0$, there is a real number, denoted $1/a$ or a^{-1}, called the **reciprocal** of a, such that

$$a\cdot\left(\frac{1}{a}\right)=\left(\frac{1}{a}\right)\cdot a=1.$$

All the usual algebraic properties of the real numbers that do not involve order can be derived from the field axioms. The most important are collected as theorems T1, T2, T3, T4, T5, T6, T7, T8, T9, T10, T11, T12, T13, T14, T15, and T16 as follows. In all these theorems the symbols a, b, c, and d represent arbitrary real numbers.

T1. *Cancellation Law for Addition* If $a + b = a + c$, then $b = c$. (In particular, this shows that the number 0 of Axiom F4 is unique.)

T2. *Possibility of Subtraction* Given a and b, there is exactly one x such that $a + x = b$. This x is denoted by $b - a$. In particular, $0 - a$ is the additive inverse of a, $-a$.

T3. $b - a = b + (-a)$.

T4. $-(-a) = a$.

T5. $a(b - c) = ab - ac$.

T6. $0 \cdot a = a \cdot 0 = 0$.

T7. *Cancellation Law for Multiplication* If $ab = ac$ and $a \neq 0$, then $b = c$. (In particular, this shows that the number 1 of Axiom F4 is unique.)

T8. *Possibility of Division* Given a and b with $a \neq 0$, there is exactly one x such that $ax = b$. This x is denoted by b/a and is called the **quotient** of b and a. In particular, $1/a$ is the reciprocal of a.

T9. If $a \neq 0$, then $b/a = b \cdot a^{-1}$.

T10. If $a \neq 0$, then $(a^{-1})^{-1} = a$.

T11. *Zero Product Property* If $ab = 0$, then $a = 0$ or $b = 0$.

T12. *Rule for Multiplication with Negative Signs*

$$(-a)b = a(-b) = -(ab), \quad (-a)(-b) = ab,$$

and

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

T13. *Equivalent Fractions Property*

$$\frac{a}{b} = \frac{ac}{bc}, \quad \text{if } b \neq 0 \text{ and } c \neq 0.$$

T14. *Rule for Addition of Fractions*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}, \quad \text{if } b \neq 0 \text{ and } d \neq 0.$$

T15. *Rule for Multiplication of Fractions*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \text{if } b \neq 0 \text{ and } d \neq 0.$$

T16. *Rule for Division of Fractions*

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ad}{bc}, \quad \text{if } b \neq 0, c \neq 0, \text{and } d \neq 0.$$

The real numbers also satisfy the following axioms, called the **order axioms.** It is assumed that among all real numbers there are certain ones, called the **positive real numbers**, that satisfy properties Ord1, Ord2, and Ord3.

Ord1. For any real numbers a and b, if a and b are positive, so are $a+b$ and ab.

Ord2. For every real number $a \neq 0$, either a is positive or $-a$ is positive but not both.

Ord3. The number 0 is not positive.

The symbols $<$, $>$, $\leq$, and $\geq$, and negative numbers are defined in terms of positive numbers.

Definition

Given real numbers a and b,

$a < b$ means $b + (-a)$ is positive.	$b > a$ means $a < b$.
$a \leq b$ means $a < b$ or $a = b$.	$b \geq a$ means $a \leq b$.
If $a < 0$, we say that a is **negative**.	If $a \geq 0$, we say that a is **nonnegative**.

From the order axioms Ord1, Ord2, and Ord3 and the above definition, all the usual rules for calculating with inequalities can be derived. The most important are collected as theorems T17, T18, T19, T20, T21, T22, T23, T24, T25, T26, and T27 as follows. In all these theorems the symbols a, b, c, and d represent arbitrary real numbers.

T17. *Trichotomy Law* For arbitrary real numbers a and b, exactly one of the three relations $a < b$, $b < a$, or $a = b$ holds.

T18. *Transitive Law* If $a < b$ and $b < c$, then $a < c$.

T19. If $a < b$, then $a + c < b + c$.

T20. If $a < b$ and $c > 0$, then $ac < bc$.

T21. If $a \neq 0$, then $a^2 > 0$.

T22. $1 > 0$.

T23. If $a < b$ and $c < 0$, then $ac > bc$.

T24. If $a < b$, then $-a > -b$. In particular, if $a < 0$, then $-a > 0$.

T25. If $ab > 0$, then both a and b are positive or both are negative.

T26. If $a < c$ and $b < d$, then $a + b < c + d$.

T27. If $0 < a < c$ and $0 < b < d$, then $0 < ab < cd$.