CS1231S: Discrete Structures

Tutorial #11: Graph II and Tree

(Week 13: 11 – 15 April 2022)

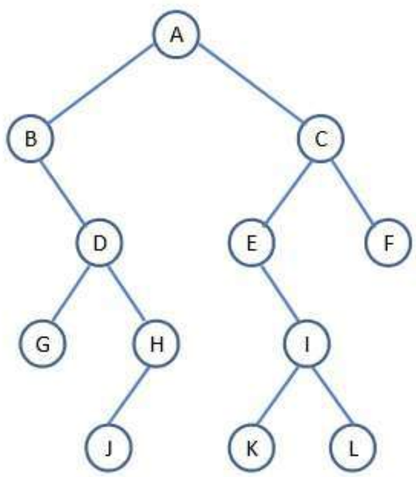
I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

D2. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there?
In general, how many simple graphs on n $(n > 1)$ vertices are there?

D3. Given the following binary tree, write the pre-order, in-order, and post-order traversals of its vertices.

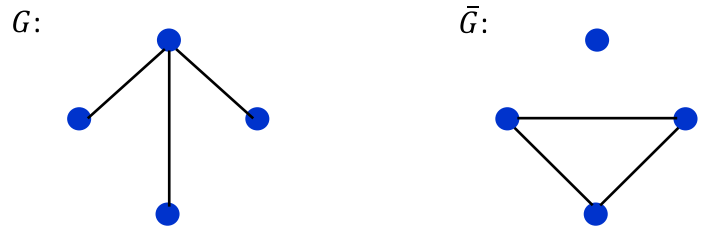


CS1231S: Discrete Structures
Tutorial #11: Graph II and Tree
Answers

II. Definitions

Definition 1. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A *self-complementary* graph is isomorphic with its complement.

Definition 3. A simple circuit (cycle) of length three is called a *triangle*.

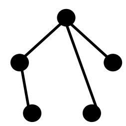
You can freely use the following lemma (without proof). The proof is left as an optional exercise.

Lemma 10.5.5. Let G be a simple, undirected graph. Then if there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

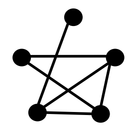
III. Tutorial Questions

1. (a) For the following graph G, draw its complement graph $\bar{G}$.

G:



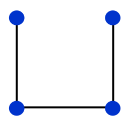
Answer: $\bar{G}$:



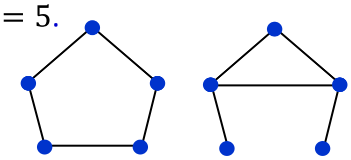
(b) Consider simple graphs on n vertices. Draw all self-complementary graphs with n vertices (for $n = 3, 4, 5, 6$), or justify why there are none.

Answers

For $n = 4$.



For $n = 5$.



For $n = 3$, K_3 has 3 edges. Cannot be evenly divided into 2 equal halves.

For $n = 6$, K_6 has 15 edges. Cannot be evenly divided into 2 equal halves.

2. (AY2016/17 Semester 1 Exam Question)
Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor \frac{n}{2} \right\rfloor$. Prove that G is connected.

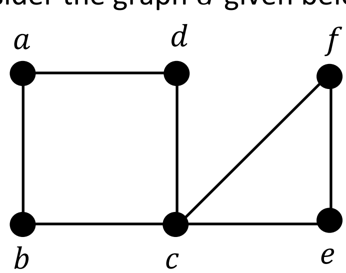
Answer:

Proof by contradiction: Suppose G is not connected. Let u and v be the vertices in two separate connected components. Then the number of vertices in the union of their neighborhood, including u and v, is at least $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor + 2 \geq n + 1$.

But this exceeds the number of vertices in the graph. Hence, G must be connected.

3. Consider the graph G given below. How many spanning trees of G are there?

G:

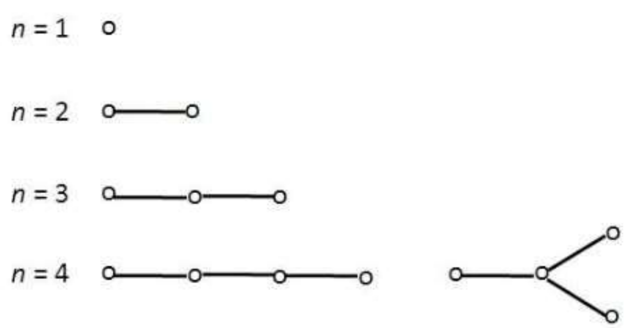


There are 2 cycles $C_1 = \{a, b, c, d\}$ and $C_2 = \{c, e, f\}$.
They are edge-disjoint (no common edge).
We need to remove 1 edge from each cycle:
4 choice for C_1 and 3 choice for C_{12}.
Product rule: Total ways = $4 \times 3 = 12$.

4. (a) Draw all non-isomorphic trees with n nodes, $n = 1,2,3,4$.

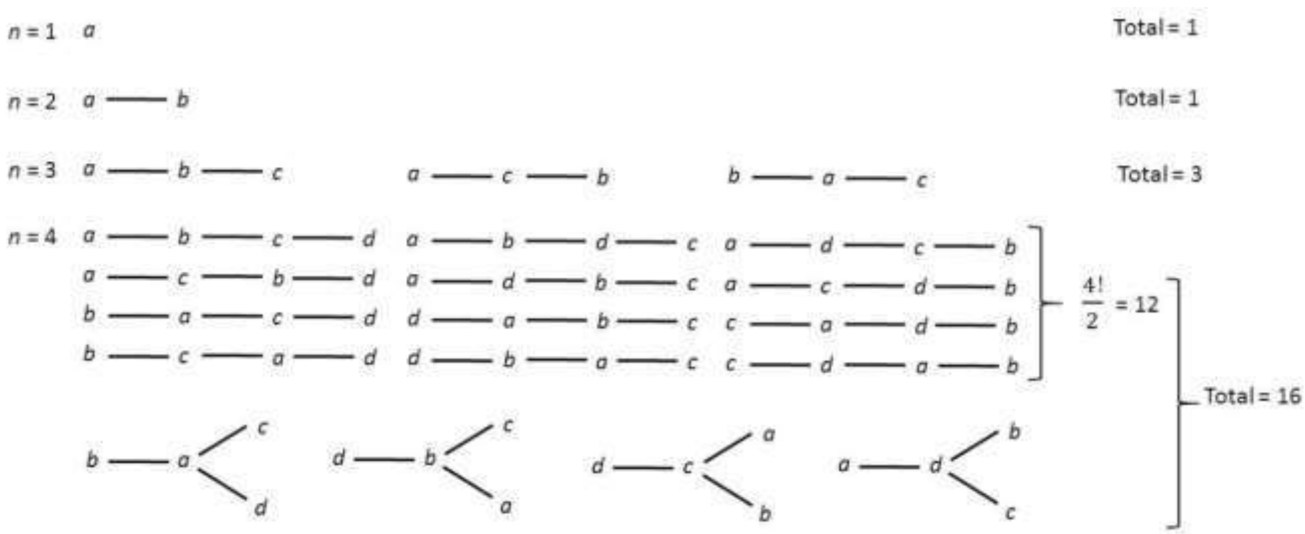
(b) For each non-isomorphic tree in part (a), how many isomorphic copies are there?

Answer:



		# of isomorphic graphs
$n = 1$	Total = 1	1
$n = 2$	Total = 1	2
$n = 3$	Total = 1	3
$n = 4$	Total = 2	12, 4

(b) For each non-isomorphic tree above, we label the vertices and determine how many different ways to permute the labels.



For $n = 3$, there are $3!/2 = 3$ different ways to permute the labels of the graph.
For $n = 4$, there are $4!/2 = 12$ different ways to permute the labels of the a-b-c-d path, and 4 ways to select the middle vertex for the other non-isomorphic graph.

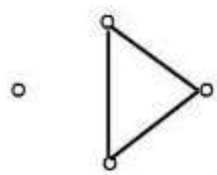
5. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$.

(b) Is the converse true?

Answer: (a)

1. Suppose that $G = (V, E)$ is connected.
2. Then G has a spanning tree $T = (V, F)$, where $F \subseteq E$, (by Theorem 10.7.1)
3. Then $|F| = |V| - 1$ (by Theorem 10.5.2)
4. Thus, $|E| \geq |F| = |V| - 1$.

(b)



$G' = (V', E')$

Converse is NOT true.
This graph $G' = (V', E')$ has $(|V'| - 1)$ edges,
but the graph is not connected.

6. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

(b) Is the converse true?

Answer:

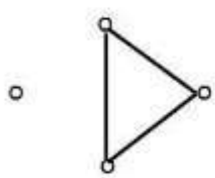
(a)

1. Suppose that $G = (V, E)$ is acyclic.
2. Let the connected components in $G = (V, E)$ be $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \dots, H_k = (V_k, E_k)$, where $k \geq 1$.
   - 2.1. where each $H_i = (V_i, E_i)$, is connected. (definition of connected components)
   - 2.2. Each $H_i = (V_i, E_i)$, is connected *and acyclic*. (since G is acylic)
   - 2.3. Hence, each $H_i = (V_i, E_i)$, is a tree. (definition of tree)
   - 2.4. Hence, $|E_i| = |V_i| - 1$, for $i = 1,2, \dots, k$. (by Theorem 10.5.2)
3. So, $|E| = |E_1| + |E_2| + \cdots |E_k|$ (by Addition Rule)
   $= (|V_1| - 1) + (|V_2| - 1) + \cdots + (|V_k| - 1) = |V| - k.$ (by 2.4)
4. Hence, $||V| - |E| = k \geq 1$, and so $|E| \leq |V| - 1$.

[**Note to students**: Given that G is any simple, undirected graph that is acyclic, we do not have any good leverage/property to use in our proof.

But, we have many theorems (magic wands) dealing with connected graphs.
So, one way is to consider Case 1: G is connected and Case 2: G is not connected.
When G is not connected, we can consider the k *connected* components of G.
In this problem, Case 1 just happen to be a special case where $k = 1$. Hence, Case 1 is just "absorbed" into Case 2.]

(b)



$G' = (V', E')$

Converse is NOT true.
This graph $G' = (V', E')$ has $(|V'| - 1)$ edges, but cyclic. (Also not connected.)

7. Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is a tree if and only if there is exactly one path between every pair of vertices.

Answer:

(⇒)

1. Let G be a tree.
2. Then G is connected. (by definition of a tree)
3. Hence, there is a path between any pair of vertices x and y. (since G is connected)
4. If some pair of vertices x and y has two or more paths connecting them, then by Lemma 10.5.5, the graph G is cyclic.
5. This contradiction 1 above.
6. Therefore, every pair of vertices has exactly one path between them.

(⇐)

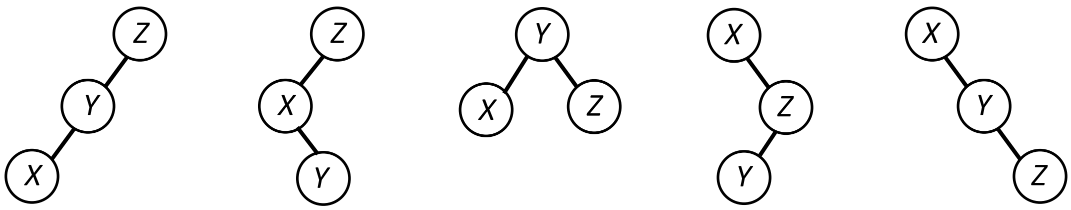
1. Suppose the is exactly one path between every pair of vertices.
2. Then G is connected.
3. Suppose that G is cyclic, then there is a cycle C in G. (definition of cyclic)
4. Let x and y be two distinct vertices in the cycle C.
5. Then there are two paths connecting x and y in the cycle C.
6. This contradicts 1 above.
7. Hence G is acyclic.
8. And therefore G is tree. (by 2 and 7, and definition of a tree.

8. (a) Draw all possible binary trees with 3 vertices X, Y and Z with in-order traversal: $X\ Y\ Z$.

   (b) Draw all possible binary trees with 4 vertices A, B, C and D with in-order traversal: $A\ B\ C\ D$.

Answer:

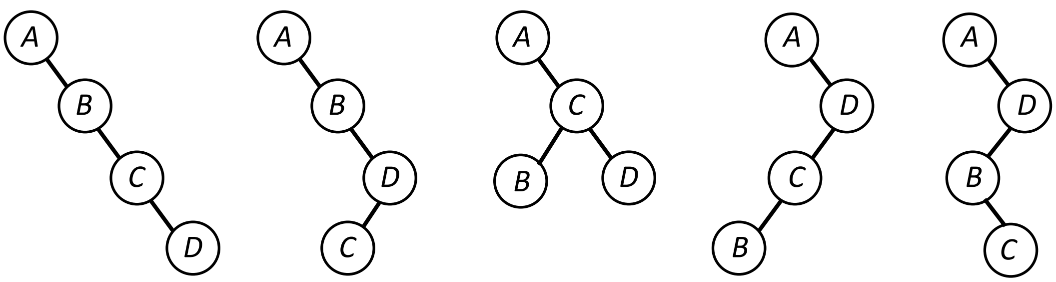
(a) 5 possible binary trees (3 vertices: X, Y, Z)



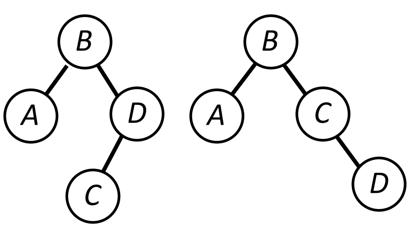
(b) 14 possible binary trees (4 vertices A, B, C, D)

(Strategy: Fix the root of the binary tree; then we know #nodes in left and right subtrees.)

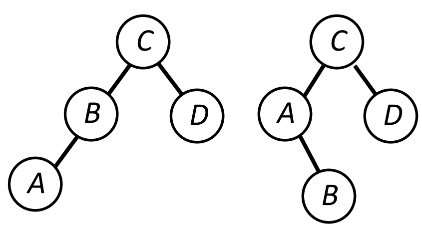
Rooted at *A*: 5



Rooted at *B*: 2



Rooted at *C*: 2



Rooted at *D*: 5
(not shown)

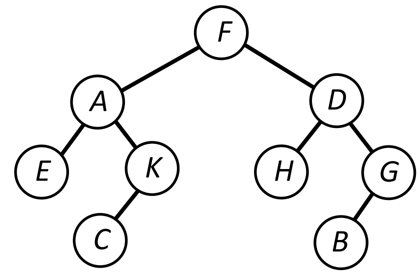
[Optional, for the FUN of it] The above strategy also gives hint to a recurrence relation, that when solved gives the general solution for larger n. It is an example of a *convolution recurrence*.
This sequence is called the Catalan's number sequence: 1, 2, 5, 14, 42, 132,…
The general form for the Catalan's number is $C_n = \frac{1}{(n+1)}\binom{2n}{n}$.]

9. (a) A binary tree T_1 has 9 nodes. The in-order and pre-order traversals of T_1 are given below. Draw the tree T_1 and give its post-order traversal.

In-order: E A C K F H D B G
Pre-order: F A E K C D H G B

Answer:

Post-order: E C K A H B G D F



Strategy:

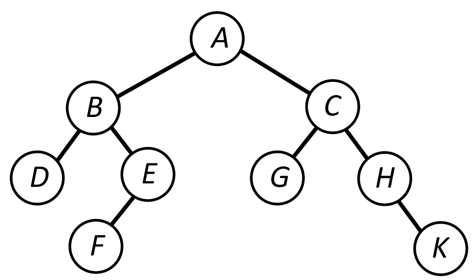
- The first node in pre-order traversal is root of tree.
(In our example, this is node F.
- Find node F in in-order traversal.
All nodes appearing before F (in-order) belong to left subtree;
All nodes appearing after F (in-order) belong to right subtree;
- Recursively apply procedure to left subtree and right subtree;

(b) A binary tree T_2 has 9 nodes. The in-order and post-order traversals of T_2 are given below. Draw the tree T_2 and give its pre-order traversal.

In-order: D B F E A G C H K
Post-order: D F E B G K H C A

Answer:

Pre-order: A B D E F C G H K.



Strategy:

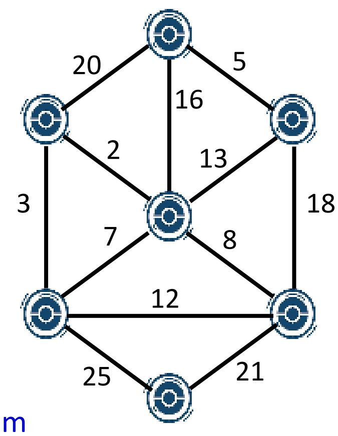
Now last node in post-order traversal is root of tree.
(In our example, this is node A.)

Apply a similar method.

10. (*Modified from* AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

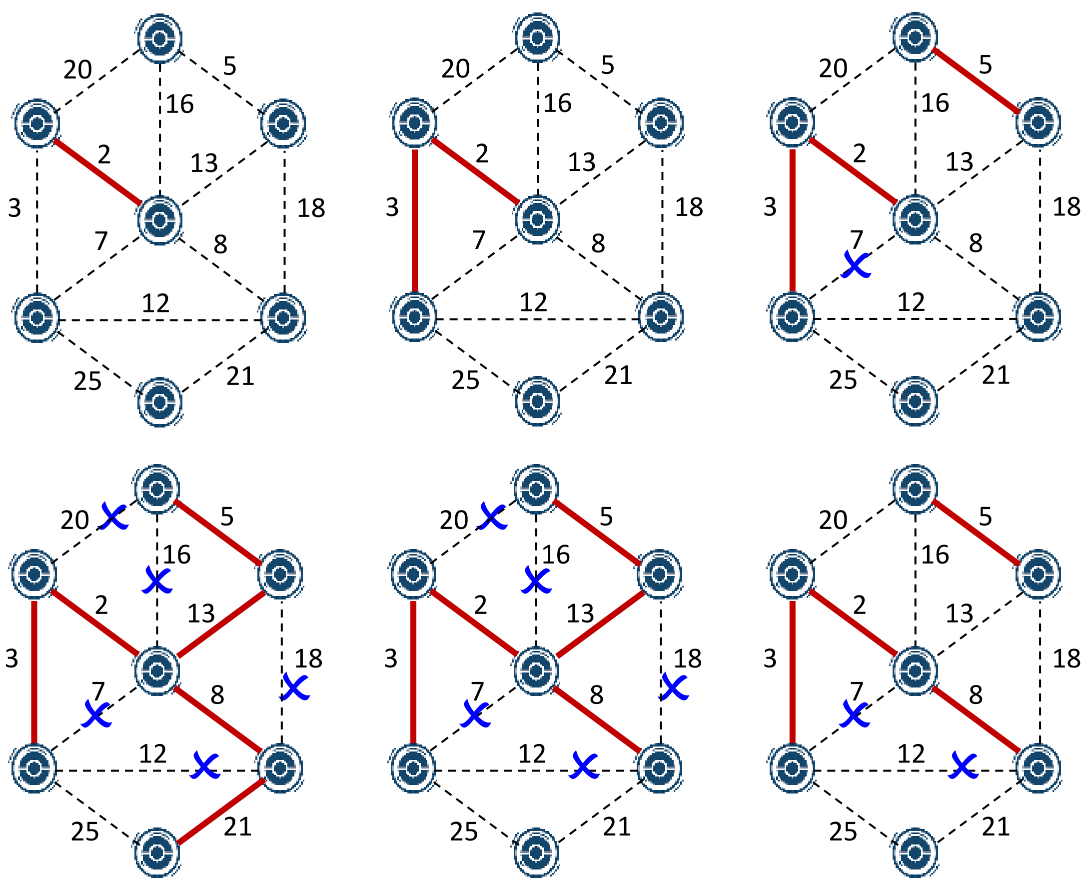
Indicate the order of the edges inserted into the MST in your answer.

[OPTIONAL, for the FUN of it] In addition to (but not in place of), you can also use Guan's algorithm from the optional notes. The one that repeatedly removes the longest edge in *any* cycle.

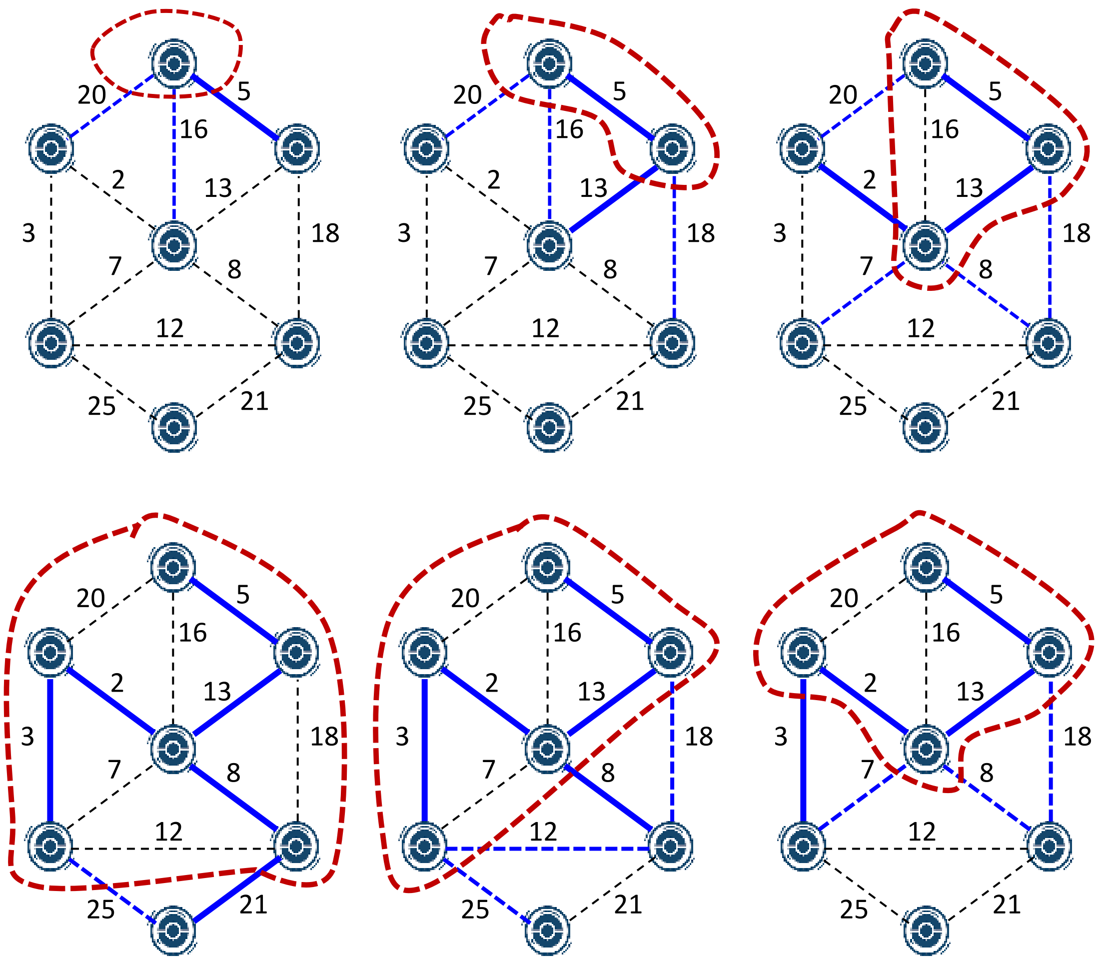


Answer: Kruskal's algorithm

Edges:
2
3
5
7 ×
8
12 ×
13
16 ×
18 ×
20 ×
21
25

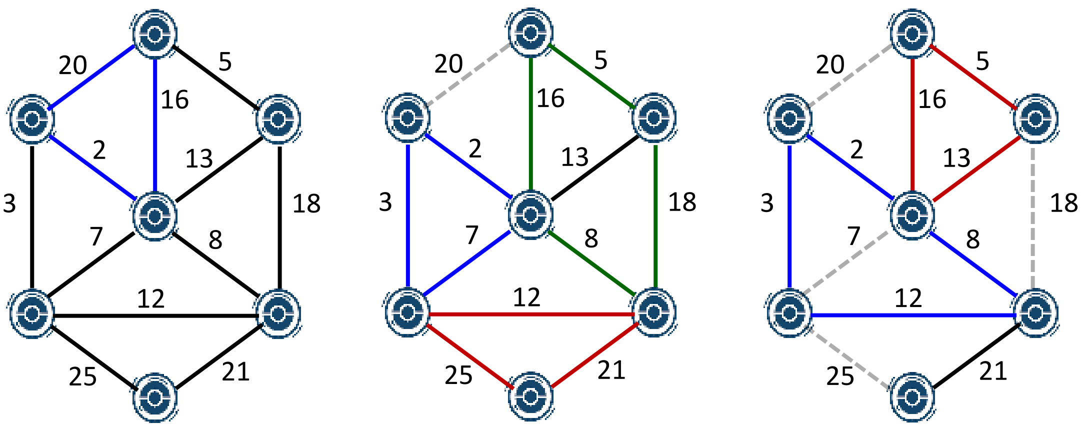


Prim-Dijkstra's algorithm.



[OPTIONAL: Just for the FUN of it.]

Guan's algorithm.



Cycles considered:

C1: {**20**, 2, 16}

C2: {3, **7**, 2)

C3: {8, **18**, 5, 16}

C4: {**25**, 21, 12}

C5: {3, **12**, 8, 2}

C6: {**16**, 13 ,5}