Q6.

Let $A = \{1,2,3,4\}$. Since each element of $\mathcal{P}(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($\mathcal{P}(S)$ denotes the power set of S.)

Assuming each relation in $\mathcal{P}(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

We will solve the general case. Let $A = \{a_1, a_2, \cdots, a_n\}$ and so $|A| = n$.
A relation R on A can be represented by an $n \times n$ matrix where the entry $a_{i,j} = 1$ if $a_i \; R \; a_j$, or $a_{i,j} = 0$ if $a_i \not R \; a_j$.

Example:

$$\begin{array}{cc} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

This matrix represents this relation R on A:

$$R = \{(a_1, a_2), (a_1, a_4), (a_2, a_2), (a_3, a_1), (a_4, a_3), (a_4, a_4)\}$$

Q6. (a) Reflexive relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A. (why?)
2. For a relation to be reflexive, $a_i \; R \; a_i \; \forall a_i \in A$. Hence, the main diagonal entries $a_{i,i}$ must be filled with 1, as shown below.

$$\begin{bmatrix} 1 & ? & ? & \cdots & ? \\ ? & 1 & ? & \cdots & ? \\ ? & ? & 1 & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & ? \\ ? & ? & ? & ? & 1 \end{bmatrix}$$

3. The remaining $n^2 - n$ entries may be filled with 0 or 1 (two choices).
4. Therefore, there are 2^{n^2-n} reflexive relations on A with n elements.
5. Hence, the probability that a randomly chosen relation on set A is reflexive is:

$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

In particular, when $n = 4$, the probability is $\frac{1}{2^4}$ or $\frac{1}{16}$.

Q6. (b) Symmetric relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A.
2. For a relation to be symmetric, for every entry $a_{i,j}$ (where $i < j$), i.e. in the upper triangular region (red triangle), its corresponding mirror image along the main diagonal, $a_{j,i}$ in the lower triangular region (blue triangle) must follow with the same value.

$$\begin{bmatrix} ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & ? \\ ? & ? & ? & ? & ? \end{bmatrix}$$

3. There are $\frac{n(n-1)}{2}$ entries in the upper triangle. There are n entries along the main diagonal. Therefore, there are $\frac{n(n-1)}{2} + n$, or $\frac{n(n+1)}{2}$ entries to be filled with 0 or 1.
4. Therefore, there are $2^{\frac{n(n+1)}{2}}$ symmetric relations on A with n elements.
5. Hence, the probability that a randomly chosen relation on set A is symmetric is: $\dfrac{2^{\frac{n(n+1)}{2}}}{2^{n^2}} = \dfrac{1}{2^{\frac{n^2-n}{2}}}$

In particular, when $n = 4$, the probability is $\frac{1}{2^6}$ or $\frac{1}{64}$.