CS1231S TUTORIAL #11

Graphs and Trees

Learning objectives of this tutorial

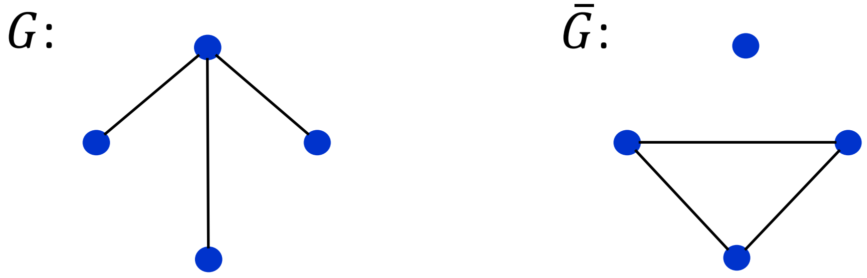
Graphs and Trees

- Complement graphs and self-complementary graphs.
- Counting spanning trees in a graph
- Isomorphic and non-isomorphic trees,
- Theorem on trees
- Counting Binary Trees
- Understanding pre-order, in-order and post-order traversals.
- Finding MST using Kruskal's, Prim-Dijkstra's algorithms
  (and for fun, with Guan's algorithm)

Definitions and Theorems

If G is a simple graph, the complement of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge iff v and w are not connected by an edge in G.

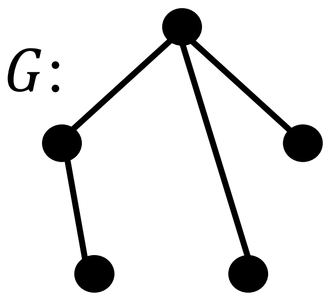
A self-complementary graph is isomorphic with its complement.



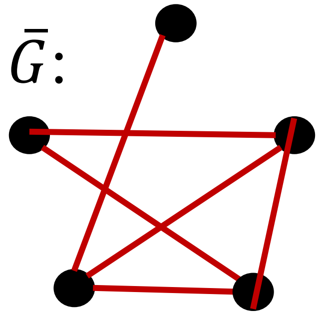
A graph G and its complement $\bar{G}$.

Lemma 10.5.5. Let G be a simple, undirected graph. Then if there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

Q1. (a) For graph G, draw its complement graph $\bar{G}$.



Complement graph



Not Self Complementary

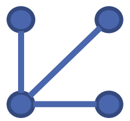
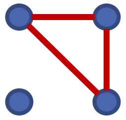
Q1. (b) Draw all self-complementary graphs with

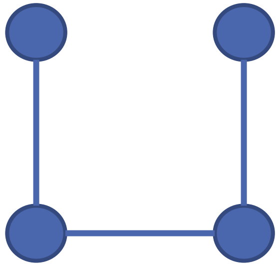
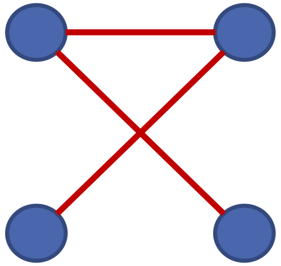
Complement graph

K_4 has 6 edges.
So 3 each for G and $\bar{G}$

4 vertices

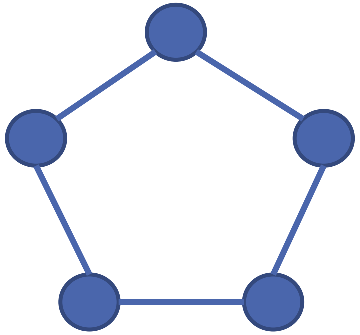
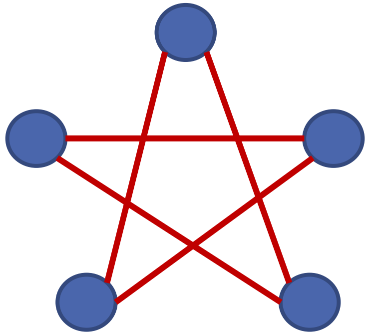
How about this?

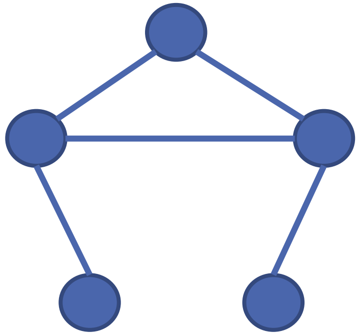
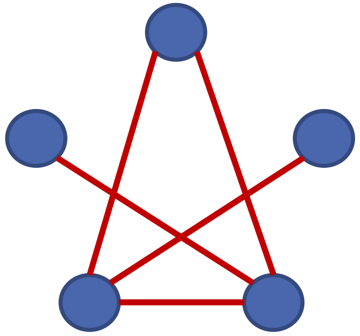
 $\ncong$ 

 $\cong$ 

K_5 has 10 edges.
So 5 each for G and $\bar{G}$

5 vertices

 $\cong$ 

 $\cong$ 

3 & 6 vertices

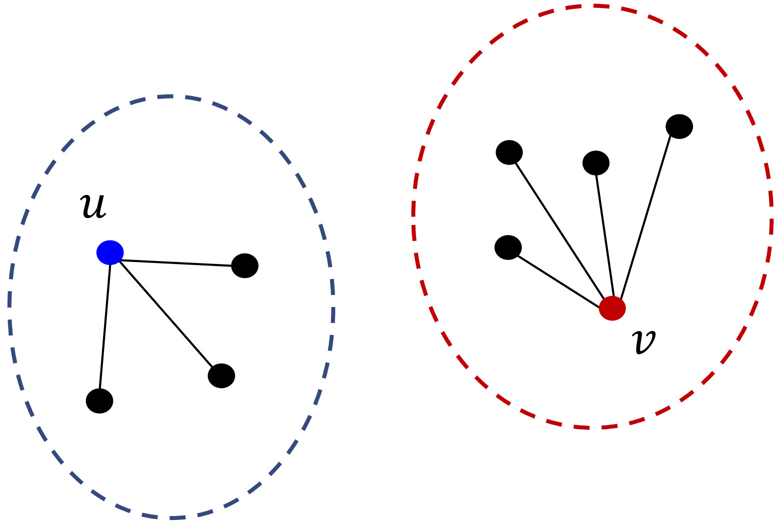
For $n = 3$, K_3 has 3 edges.
For $n = 6$, K_6 has 15 edges.

Cannot be evenly divided into 2 equal halves for G and $\bar{G}$.

Q2. Let G be a simple graph with n vertices where every vertex has degree at least $\left\lfloor\frac{n}{2}\right\rfloor$. Prove that G is connected.

Proof by contradiction

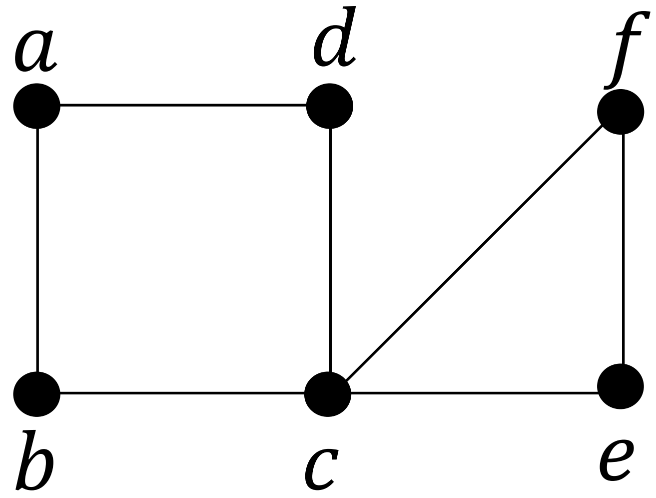
1. Suppose G is not connected.
2. Let u and v be the vertices in two separate connected components.
3. Then the number of vertices in the union of their neighbourhood, including u and v, is at least $\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2 \geq n + 1$.
4. This contradicts that there are n vertices.
5. Hence G is connected.



3.1 Case: n is even, then $n = 2k$ for some integer k, so $\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2 = \left\lfloor\frac{2k}{2}\right\rfloor + \left\lfloor\frac{2k}{2}\right\rfloor + 2 = k + k + 2 = n + 2.$

3.2 Case: n is odd, then $n = 2k + 1$ for some integer k, so $\left\lfloor\frac{n}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor + 2 = \left\lfloor\frac{2k+1}{2}\right\rfloor + \left\lfloor\frac{2k+1}{2}\right\rfloor + 2 = k + k + 2 = 2k + 2 = n + 1.$

Q3. Consider the graph G given below. How many spanning trees of G are there?



There are 2 edge-disjoint cycles
$C_1 = \{a, b, c, d\}$ and
$C_2 = \{c, e, f\}$.

We need to remove 1 edge from each cycle:
4 choices for C_1 and
3 choices for C_2.

Product rule: Total #ways = $4 \times 3 = 12$.

DIY: Draw these 12 spanning trees.

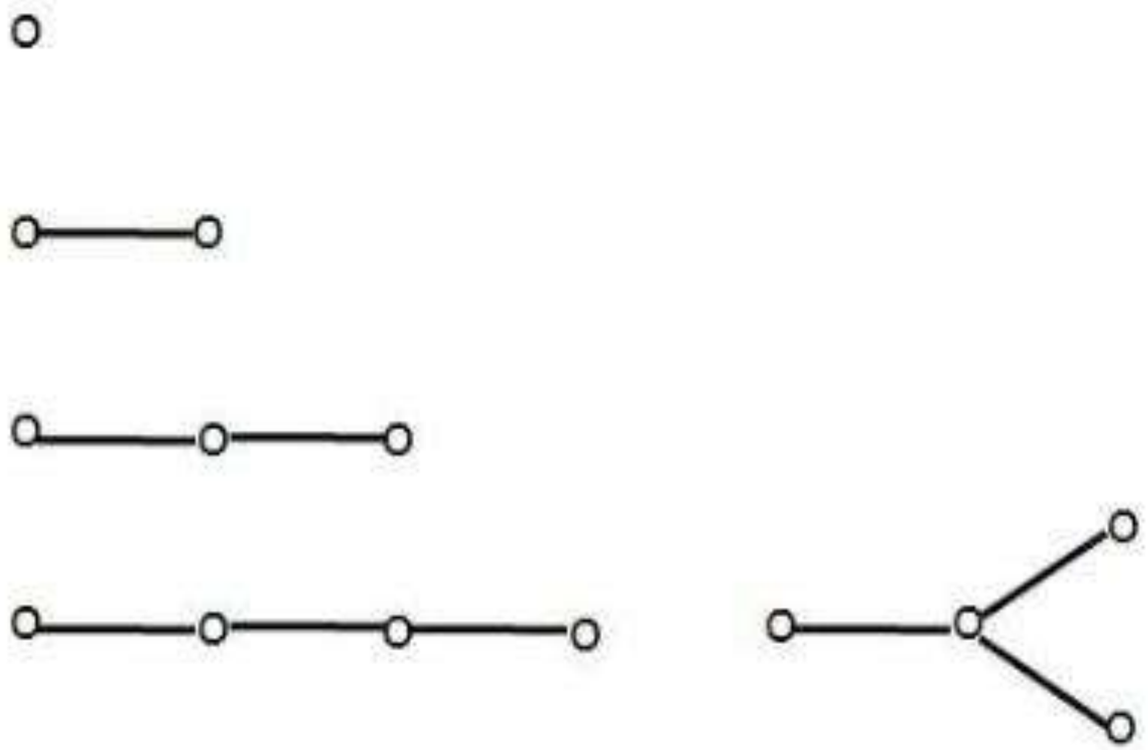
Q4. (a) Draw all non-isomorphic trees with n nodes, $n = 1,2,3,4$.

$n = 1$ — Total = 1

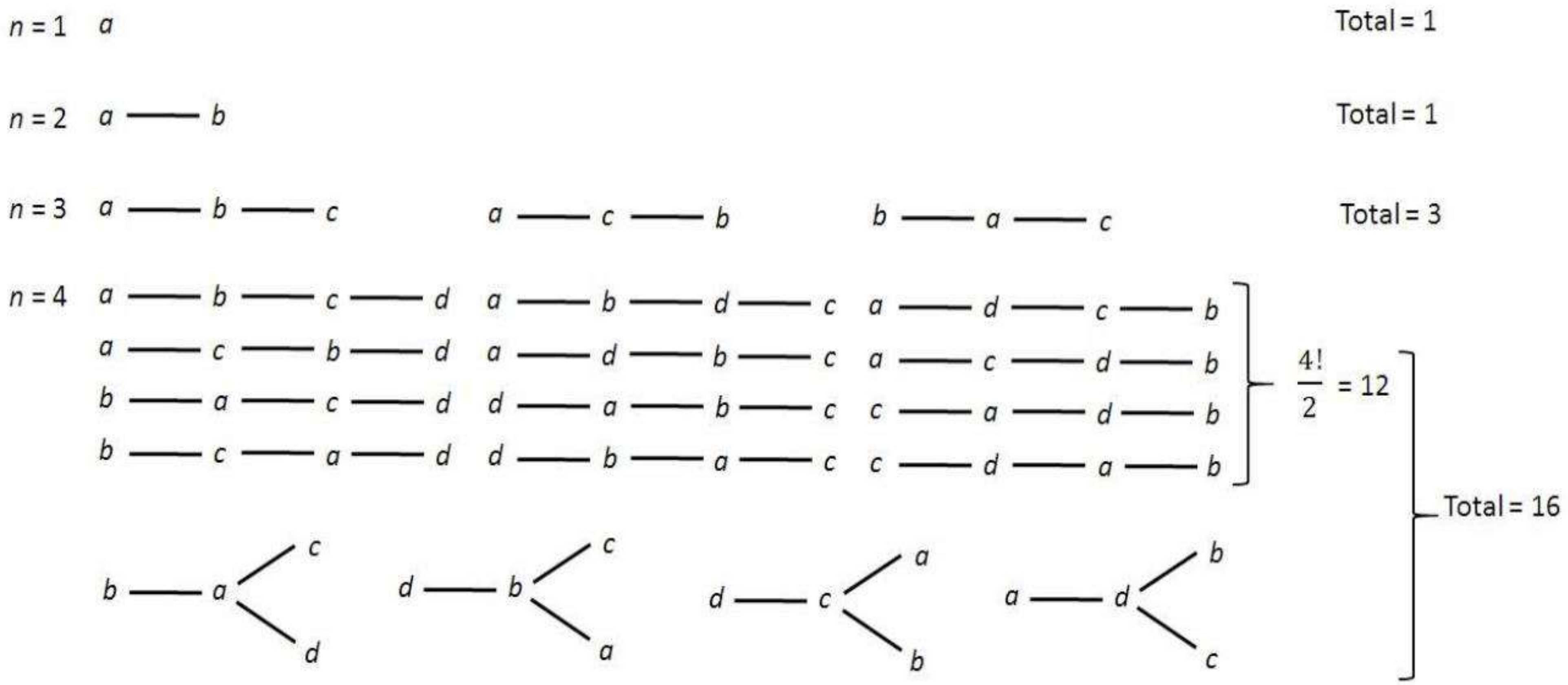
$n = 2$ — Total = 1

$n = 3$ — Total = 1

$n = 4$ — Total = 2



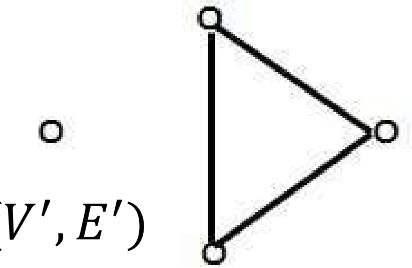
Q4. (b) For each non-isomorphic tree in part (a), how many isomorphic copies are there?



Q5. (a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is connected, then $|E| \geq |V| - 1$.

1. Suppose that $G = (V, E)$ is connected.
2. Then G has a spanning tree $T = (V, F)$, where $F \subseteq E$, (by Theorem 10.7.1)
3. Then $|F| = |V| - 1$ (by Theorem 10.5.2)
4. Thus, $|E| \geq |F| = |V| - 1$.

(b) Is the converse true?



$G' = (V', E')$

Converse is NOT true.
This graph $G' = (V', E')$ has $(|V'| - 1)$ edges, but the graph is not connected.

Q6.

(a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

1. Suppose that $G = (V, E)$ is acyclic.

2. Decompose G into its **connected** component(s)
$H_1 = (V_1, E_1), H_2 = (V_2, E_2), \ldots, H_k = (V_k, E_k)$, where $k \geq 1$.

2.1 Each $H_j = \left(V_j, E_j\right)$ is connected (defn of connected component)
2.2 … and is acyclic (since G is acyclic)
2.3 … is therefore, a tree. (definition of tree)
2.4 Therefore, $\left| E_j \right| = \left|V_j\right| - 1$, for each $j = 1,2, \ldots, k$. (by Thm 10.5.2))

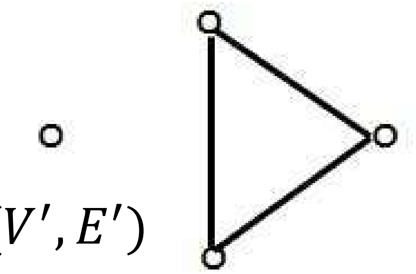
3. Hence, $|E| = |E_1| + |E_2| + \cdots + |E_k|$
$= (|V_1| - 1) + (|V_2| - 1) + \cdots + (|V_k| - 1) = |V| - k$
4. Thus, $|E| = |V| - k \leq |V| - 1$ (since $k \geq 1$).

Q6.

(a) Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is acyclic, then $|E| \leq |V| - 1$.

(b) Is the converse true?

$G' = (V', E')$



Converse is NOT true.

This graph $G' = (V', E')$ has $(|V'| - 1)$ edges,
but the graph is not acyclic (and not connected).

Q7. Let $G = (V, E)$ be a simple, undirected graph. Prove that if G is a tree if and only if there is exactly one path between every pair of vertices.

Lemma 10.5.5. Let G be a simple, undirected graph. Then if there are two distinct paths from a vertex v to a different vertex w, then G contains a cycle (and hence G is cyclic).

($\Rightarrow$)

1. Let G be a tree.
2. As G is connected, there is a path between every pair of nodes.
3. If some pair has more than one path, then G is cyclic (lemma above).
4. Therefore, every pair of nodes has exactly one path between them.

($\Leftarrow$)

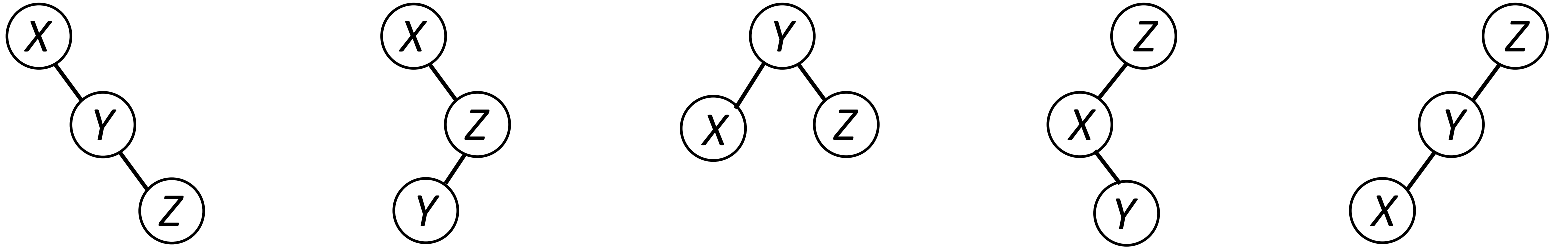
1. Let G be a simple, undirected graph.
2. Suppose there is exactly one path between every pair of vertices in G, then G is connected. (defn of connected)
3. Suppose G is cyclic, then there is a cycle C in G.
4. Let x and y be two distinct vertices in the cycle C. Then there are *two* different paths between x and y, which contradicts the assumption in (2).
5. Therefore, G is acyclic and hence G is a tree.

Q8. (a) Draw all possible binary trees with 3 vertices X, Y and Z with in-order traversal: $X\ Y\ Z$.

STRATEGY:

- Fix the ROOT of the binary tree;
- Then, know # nodes in Left-Subtree,
- #nodes in Right-Subtree

Tutors: May be good to do Q9 before Q8. Then they will be very familiar with left, right subtrees in in-order.



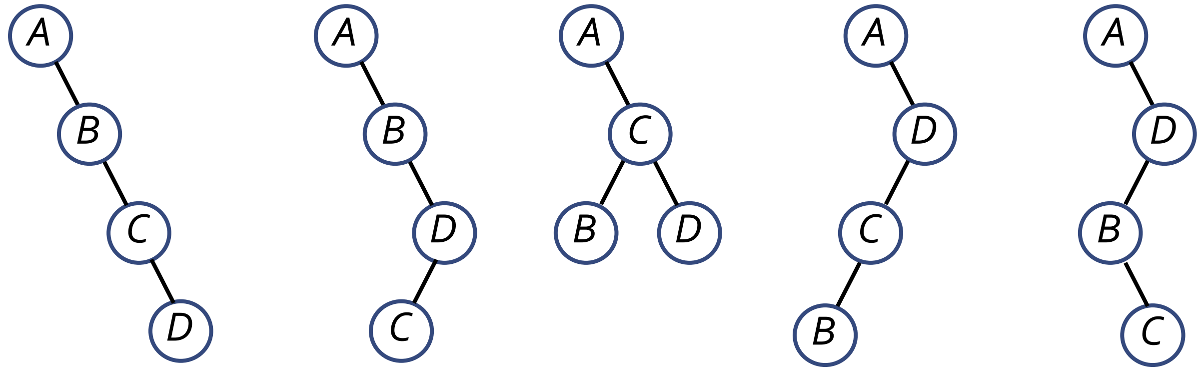
Total: 5

Q8.

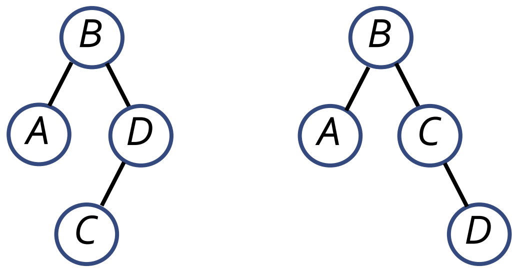
(b) Draw all possible binary trees with 4 vertices A, B, C and D with in-order traversal: $A\ B\ C\ D$.

Total: 14

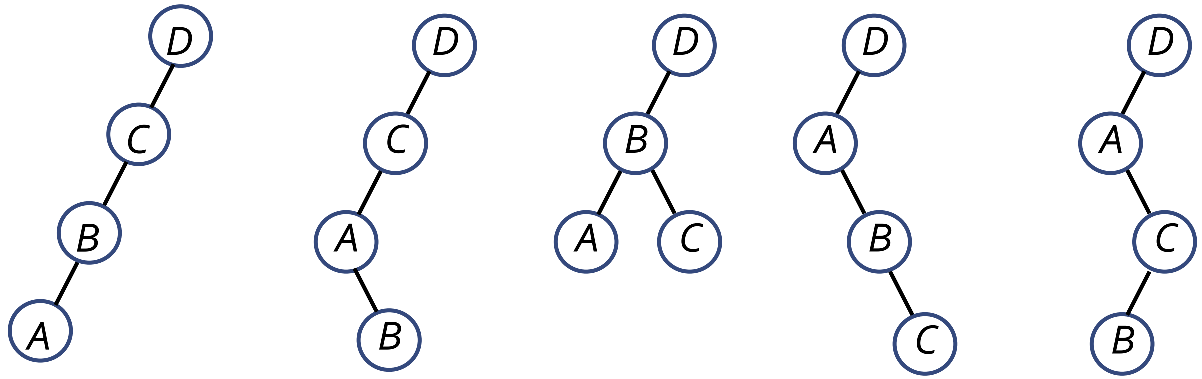
Rooted at A: 5



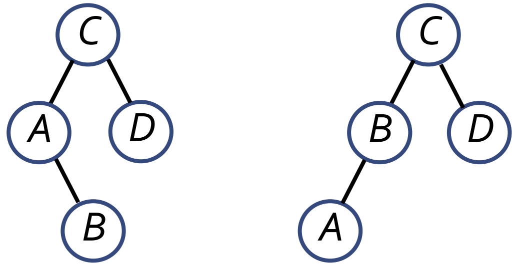
Rooted at B: 2



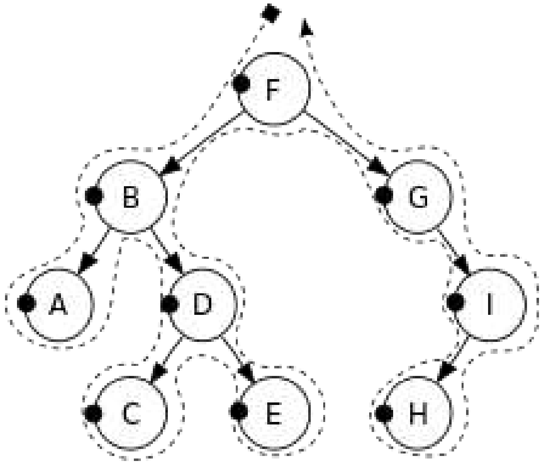
Rooted at D: 5



Rooted at C: 2



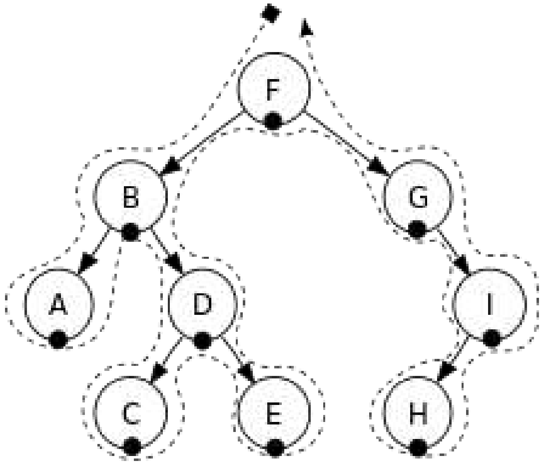
Depth-First Search



Pre-order:

F, B, A, D, C, E, G, I, H

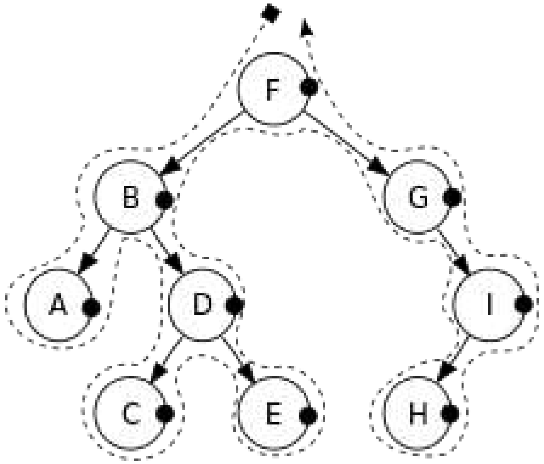
R	T_l	T_r
F	B A D C E	G I H



In-order:

A, B, C, D, E, F, G, H, I

T_l	R	T_r
A B C D E	F	G H I



Post-order:

A, C, E, D, B, H, I, G, F

T_l	T_r	R
A C E D B	H I G	F

Acknowledgement: Wikipedia
https://en.wikipedia.org/wiki/Tree_traversal

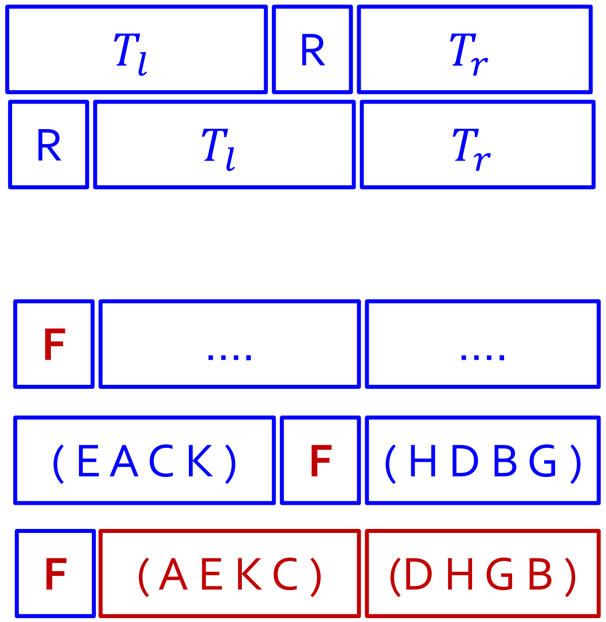
Q9. (a) Construct a binary tree T_1 given in-order and pre-order traversals. Draw tree. Give its post-order traversal.

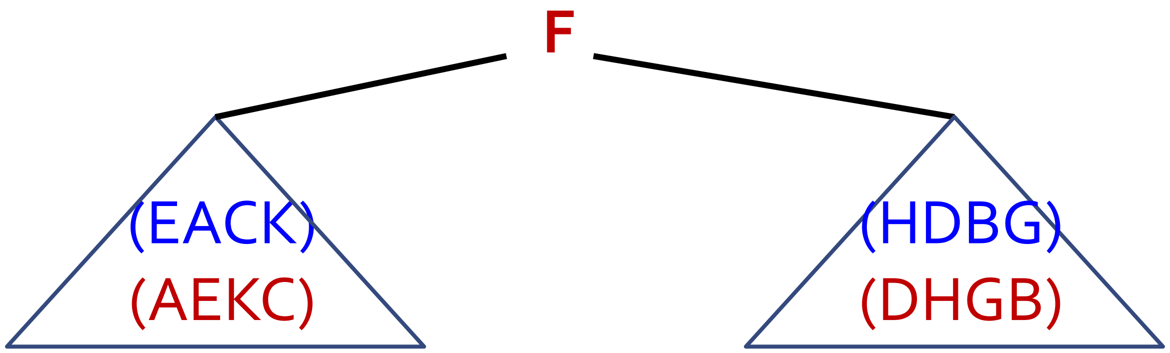
In-order: E A C K F H D B G

Pre-order: F A E K C D H G B

STRATEGY:

- Identify ROOT
- Identify Left-Subtree, Right-Subtree
- Recursively Solve Left-Subtree,
  Recursively Solve Right-Subtree;

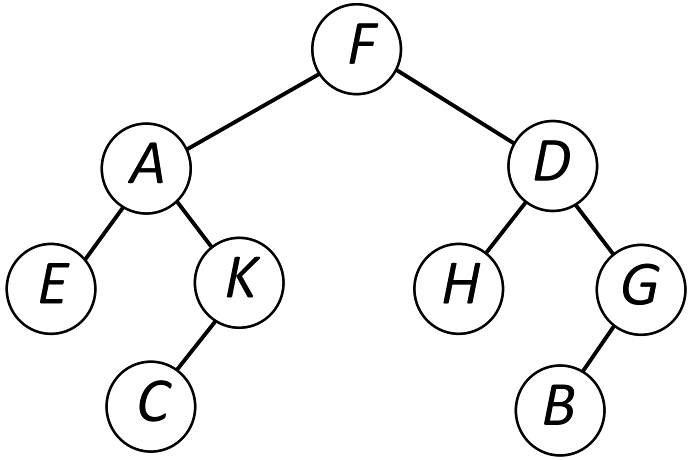




Q9. (a) Construct a binary tree T_1 given in-order and pre-order traversals. Draw tree. Give its post-order traversal.

In-order: E A C K F H D B G

Pre-order: F A E K C D H G B



Post-order: E C K A H B G D F

Q9. (b) Construct a binary tree T_2 given in-order and post-order traversals. Draw tree. Give its pre-order traversal.

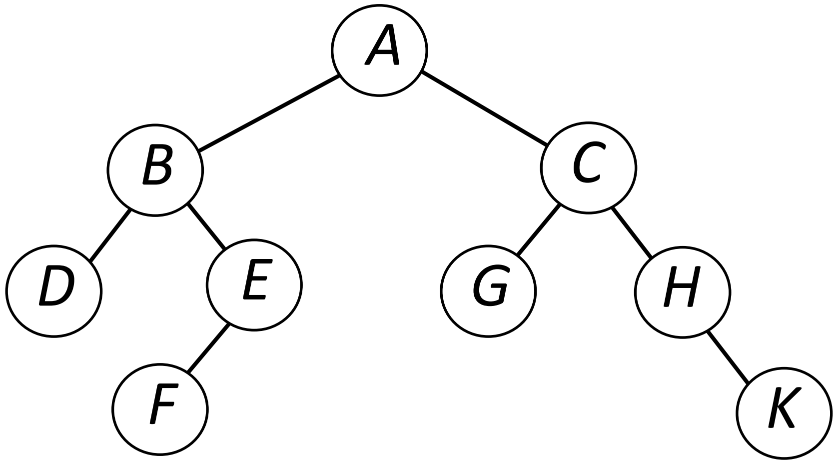
In-order: D B F E A G C H K

Post-order: D F E B G K H C A

T_l	R	T_r
T_l	T_r	R

SAME STRATEGY:

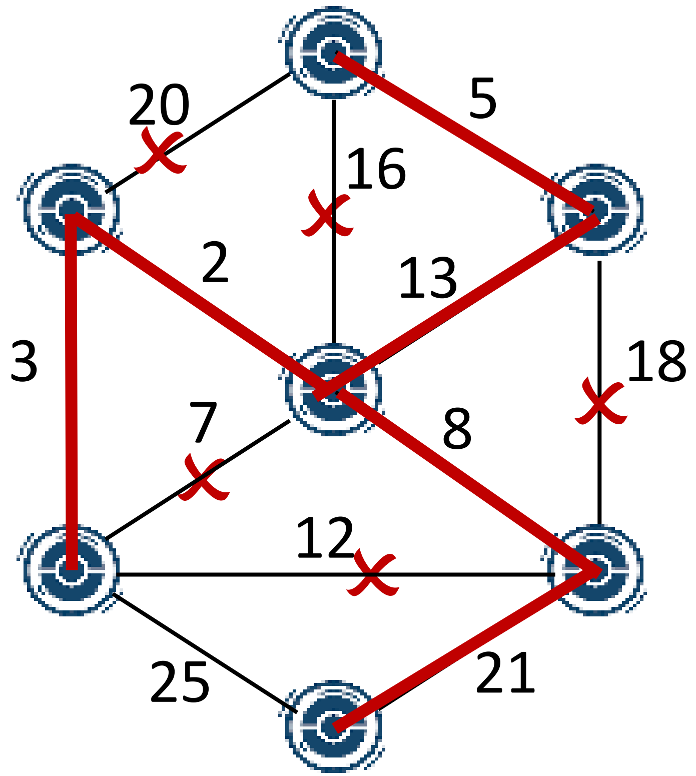
- In post-order traversal, ROOT is last node.



Pre-order: A B D E F C G H K

Q10. Find the MST of the graph below.

Kruskal's algorithm



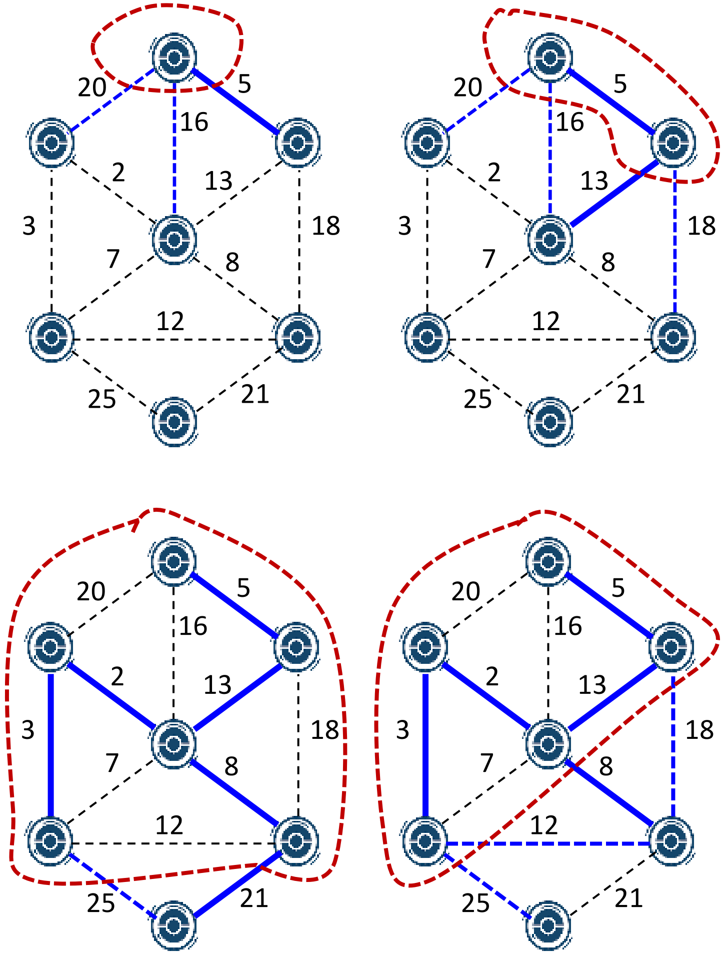
Edges in non-decreasing order:

2
3
5
7
8
12
13
16
18
20
21
25

Weight of MST = $2 + 3 + 5 + 8 + 13 + 21 = 52.$

Q10. Find the MST of the graph below.

Prim-Dijkstra's algorithm



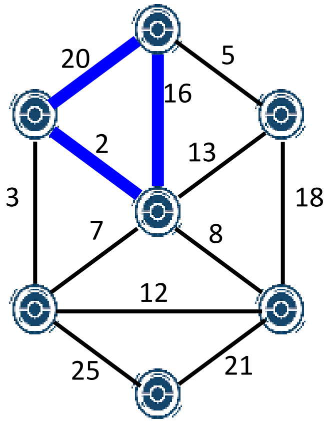
Start from top vertex.

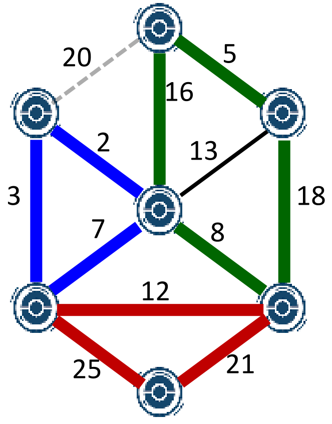
1. $\{20, 16, 5\} \rightarrow 5$
2. $\{20, 16, 13, 18\} \rightarrow 13$
3. $\{20, 2, 7, 8, 18\} \rightarrow 2$
4. $\{3, 7, 8, 18\} \rightarrow 3$
5. $\{25, 12, 8, 18\} \rightarrow 8$
6. $\{25, 21\} \rightarrow 21$

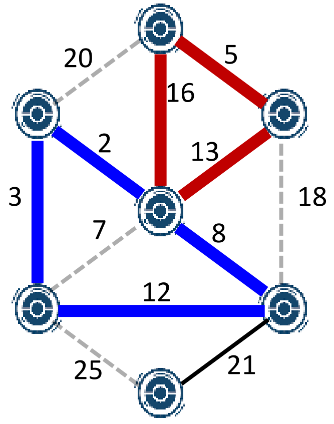
Weight of MST = $2 + 3 + 5 + 8 + 13 + 21 = 52$.

Q10. Find the MST of the graph below.

Guan's algorithm







Cycles considered:

C1: $\{\mathbf{20}, 2, 16\} \rightarrow \boldsymbol{D20}$

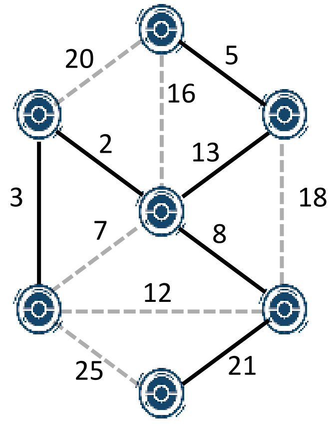
C2: $\{3, \mathbf{7}, 2\} \rightarrow \boldsymbol{D7}$

C3: $\{8, \mathbf{18}, 5, 16\} \rightarrow \boldsymbol{D18}$

C4: $\{\mathbf{25}, 21, 12\} \rightarrow \boldsymbol{D25}$

C5: $\{3, \mathbf{12}, 8, 2\} \rightarrow \boldsymbol{D12}$

C6: $\{\mathbf{16}, 13, 5\} \rightarrow \boldsymbol{D16}$



Weight of MST = $2 + 3 + 5 + 8 + 13 + 21 = 52$.

THE END (of Tut #11)