## Problem 1 – Information Measures and Source Coding (30 Points)

(a) **(10 Points)** Consider a source $X$ on the alphabet $\mathcal{X} = \{1, 2, 3\}$ having probabilities

$$P_X(1) = \frac{1}{2}, \quad P_X(2) = p, \qquad P_X(3) = \frac{1}{2} - p$$

for some $p \in \left(0, \frac{1}{2}\right)$. For which value(s) of $p \in \left(0, \frac{1}{2}\right)$ will the Shannon-Fano code give the same average codeword length as the Huffman code? Explain.

(b) **(10 Points)** Consider a length-$n$ source sequence $\mathbf{X} \in \mathcal{X}^n$ whose probability mass function (PMF) is given by

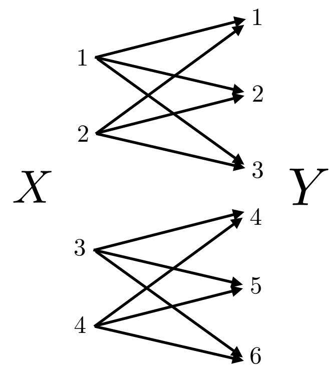
$$P_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2}\biggl(\prod_{i=1}^{n} P_1(x_i)\biggr) + \frac{1}{2}\biggl(\prod_{i=1}^{n} P_2(x_i)\biggr),$$

for some PMFs $P_1$ and $P_2$ on the alphabet $\mathcal{X}$, where $\mathbf{x} = (x_1, \dots, x_n)$. Describe a fixed-length source encoding and decoding method for $\mathbf{X}$ based on typical sets, and briefly explain why it can reconstruct $\mathbf{X}$ with probability arbitrarily close to one for any rate $R > \max\{H(P_1), H(P_2)\}$, where $H(P)$ denotes the entropy of a random variable with PMF given by $P$. *(Note: You may use any properties of typical sets that we proved in class.)*

(c) **(10 Points)** Consider the discrete random variable $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ (i.e., $P_X(0) = P_X(1) = \frac{1}{2}$) and the continuous random variable $Z \sim \text{Uniform}[0, 1]$, and let $Y = \alpha X + Z$ for some $\alpha > 0$. Show that $h(Y) = \min\{1, \alpha\}$, where $h(\cdot)$ denotes the differential entropy.

## Problem 2 – Channel Coding (30 Points)

(a) **(8 Points)** Consider a discrete memoryless channel of the following form with input alphabet $\mathcal{X} = \{1, 2, 3, 4\}$ and output alphabet $\mathcal{Y} = \{1, 2, 3, 4, 5, 6\}$:

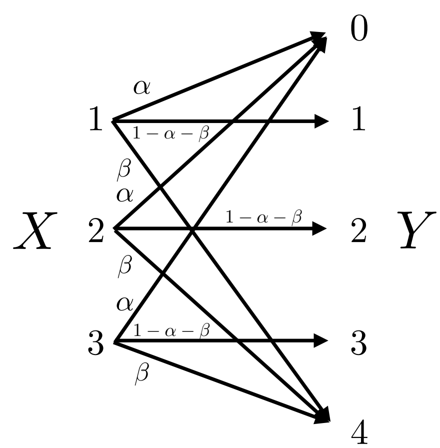



The precise transition probabilities are not shown, but any absent transitions correspond to a transition probability of zero.

(i) In at most two sentences, explain why the channel capacity is at least 1 bit/use.

(ii) In at most two sentences, explain why the channel capacity cannot exceed 2 bits/use.

(b) **(12 Points)** Consider the following erasure-type channel with 3 inputs, 5 outputs, and parameters $\alpha \in \left(0, \frac{1}{2}\right)$ and $\beta \in \left(0, \frac{1}{2}\right)$:




Using the formula $C = \max_{P_X} I(X;Y)$, prove that the capacity is $(1-\alpha-\beta)\log_2 3$ bits/use.

(c) **(10 Points)** Consider a continuous memoryless channel described by $Y = X + Z$, where $X$ is subject to a power constraint $\mathbb{E}[X^2] \le P$, but in contrast with the AWGN channel in the lecture, the noise is correlated with the input: $Z = \alpha X + V$ for some $\alpha > 0$, where $V \sim N(0, \sigma^2)$ is independent of $X$. Starting with the formula

$$C = \max_{f_X : \mathbb{E}[X^2] \le P} I(X;Y),$$

find the channel capacity in terms of $P$, $\sigma^2$, and $\alpha$.

*(Note: Recall that the differential entropy of an $N(0,\sigma^2)$ random variable is $\frac{1}{2}\log_2(2\pi e \sigma^2)$)*

## Problem 3 – Linear Codes (20 Points)

1. **(8 Points)** Consider the repetition code that maps $0 \to 00000$ and $1 \to 11111$. Interpreting this as a linear code, write down the generator matrix $\mathbf{G}$, the parity check matrix $\mathbf{H}$, and the minimum distance $d_{\min}$. Explanations are not required.

2. **(12 Points)** Let $\mathbf{G}_1$ (respectively, $\mathbf{G}_2$) be a generator matrix whose code has rate $R_1$ (respectively, $R_2$) and minimum distance $d_{\min,1}$ (respectively, $d_{\min,2}$). Suppose that $\mathbf{G}_1$ and $\mathbf{G}_2$ have the same number of rows, and consider the generator matrix

$$\mathbf{G} = \begin{bmatrix}\mathbf{G}_1 & \mathbf{G}_2\end{bmatrix}.$$

   Consider the linear code with generator matrix $\mathbf{G}$, and answer the following:

   (i) Show that the rate $R$ of the code satisfies $R = \frac{1}{(1/R_1)+(1/R_2)}$.

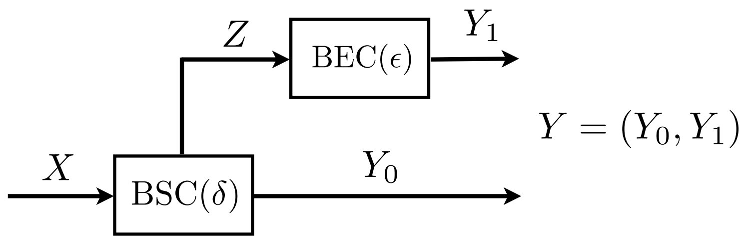
   (ii) Show that the minimum distance $d_{\min}$ of the code satisfies $d_{\min} \geq d_{\min,1} + d_{\min,2}$.

   (iii) Do there exist any $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $d_{\min} > d_{\min,1} + d_{\min,2}$? Explain.

## Problem 4 – Advanced Channel Coding (20 Points)

(a) **(10 Points)** Recall from the lecture that the binary symmetric channel (BSC) has $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and flips the input with probability $\delta$, and that the binary erasure channel (BEC) has $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1, e\}$, and given any input, the output is $e$ with probability $\epsilon$ and equals the input otherwise.

Consider the following channel with input $X$ and output $Y = (Y_0, Y_1)$:




Here BEC($\epsilon$) is a binary erasure channel with parameter $\epsilon \in (0, 1)$, and BSC($\delta$) is a binary symmetric channel with parameter $\delta \in (0, 1)$ that not only provides the usual output (written as $Y_0$ here), but also outputs the value $Z = \mathbf{1}\{X \neq Y_0\}$ equaling 1 if a flip occurred, and 0 otherwise. We can view this channel $P_{Y|X}$ as being a BSC in which we get to "peek" at the noise a fraction $1 - \epsilon$ of the time.

Assuming that the "flip event" in the BSC and the "erasure event" in the BEC occur independently of each other, show that the overall channel has capacity $C = 1 - \epsilon H_2(\delta)$, where $H_2(\delta) = \delta \log_2 \frac{1}{\delta} + (1 - \delta) \log_2 \frac{1}{1-\delta}$ is the binary entropy function.

(b) **(10 Points)** Consider a channel with input $X = (X_A, X_B, X_C)$ and output $Y = (Y_A, Y_B, Y_C)$, where $X_A$, $X_B$, $X_C$, $Y_A$, $Y_B$, and $Y_C$ are all binary-valued, with values $\{0, 1\}$. The channel transition probabilities are described as follows:

$$(Y_A, Y_B, Y_C) = \begin{cases} (X_A, X_B, X_C) & \text{with probability } 1 - \delta \\ (1 - X_A, 1 - X_B, 1 - X_C) & \text{with probability } \delta. \end{cases}$$

That is, either all 3 bits are flipped or none of them are.

Let $\mathcal{C}_1$ be a codebook for a binary symmetric channel with rate $R_1$ and error probability $\epsilon_1$. Describe how this codebook can be used to communicate over the above channel at rate $2 + R_1$ with error probability $\epsilon_1$.

**END OF PAPER**