NATIONAL UNIVERSITY OF SINGAPORE

CS1231S – DISCRETE STRUCTURES

(Semester 2: AY2021/22)

(ANSWERS)

Time Allowed: 2 Hours

INSTRUCTIONS

1. This assessment paper contains **TWENTY ONE (21)** questions in **THREE (3)** parts and comprises **TWELVE (12)** printed pages. The last two pages are intentionally left blank.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS** provided.
5. Do **not** write your name on the ANSWER SHEETS.
6. The maximum mark of this assessment is 100.

Question	Max. mark
Part A: Q1 – 10	20
Part B: Q11 – 16	18
Part C: Q17	5
Part C: Q18	10
Part C: Q19	20
Part C: Q20	20
Part C: Q21	7
Total	**100**

——— END OF INSTRUCTIONS ———

1	E	2	A	3	D	4	B	5	E
6	D	7	C	8	B	9	C	10	D

11	A,B,D	12	B,C,D	13	C,E	14	A,B,C	15	A,C,E	16	A,B,C

Part A: Multiple Choice Questions [Total: 10×2 = 20 marks]

Each multiple choice question (MCQ) is worth **TWO marks** and has exactly **one** correct answer.

1. Which answer in this list is the correct answer to *this* question?

   A. All of the below.
   B. None of the below.
   C. All of the above.
   D. One of the above.
   E. None of the above.
   F. None of the above.

Answer: E

2. Aiken is taking the CS1231S final exam which consists of the dreadful MRQs (multiple-response questions). If each MRQ has 5 options A, B, C, D and E, in how many ways can Aiken write his answer for each MRQ, assuming that he writes at least one option in his answer. Note that the order of the options in his answer does not matter, for example, ABC, ACB and BCA are considered the same answer.

   A. 31.
   B. 32.
   C. 325.
   D. 3125.
   E. None of the above.

$\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}=31$
or
$2^5-\binom{5}{0}=32-1=31.$

Answer: A

3. Which of the following statements is true?

   A. $(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$
   B. $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$
   C. $p \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow (p \wedge r)$
   D. $p \vee (q \rightarrow r) \equiv (p \vee q) \rightarrow (p \vee r)$
   E. None of the above.

Counterexample:
A. $p \equiv true; q \equiv false; r \equiv false.$
B. $p \equiv true; q \equiv false; r \equiv false.$
C. $p \equiv false; q \equiv true; r \equiv true.$

Answer: D

4. Given that $\forall x\, \exists y\, P(x,y)$ is true on a non-empty domain of discourse, which of the following statements is true?

   A. $\exists y\, \forall x\, P(x,y)$
   B. $\exists x\, \exists y\, P(x,y)$
   C. $\forall x\, \forall y\, P(x,y)$
   D. $\exists x\, \forall y \sim P(x,y)$
   E. None of the above.

Answer: B

5. Define a set S recursively as follows.

   (1) $5 \in S$. (base clause)

   (2) If $x \in S$, then $x + 3 \in S$ and $x + 5 \in S$. (recursion clause)

   (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

   What is the smallest integer k such that all integers $n \geq k$ are in S?

   A. 7.
   B. 8.
   C. 9.
   D. 10.
   E. None of the above.

   $5,8 \in S$

   $6,7,9,12 \notin S$

   $k = 13$

Answer: E

6. Given the set $A = \{1,2,3,4,5\}$ and bijection $f: A \to A$ as follows:

$$f = \{(1,3), (2,4), (3,5), (4,1), (5,2)\}.$$

   What is the order of $f \circ f$?

   A. 2
   B. 3
   C. 4
   D. 5
   E. None of the above.

   $f \circ f = \{(1,5), (2,1), (3,2), (4,3), (5,4)\}$.
   Let $h = f \circ f$.
   $h \circ h = \{(1,4), (2,5), (3,1), (4,2), (5,3)\}$.
   $h \circ h \circ h = \{(1,3), (2,4), (3,5), (4,1), (5,2)\}$.
   $h \circ h \circ h \circ h = \{(1,2), (2,3), (3,4), (4,5), (5,1)\}$.
   $h \circ h \circ h \circ h \circ h = \{(1,1), (2,2), (3,3), (4,4), (5,5)\} = id_A$.

Answer: D

7. Let $S = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$, $V = \{\mathrm{A}, 2,3,4,5,6,7,8,9,10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$, and $B = \{(\spadesuit, \mathrm{A}), (\heartsuit, 7), (\clubsuit, 9), (\spadesuit, 6), (\diamondsuit, \mathrm{J})\}$.

   Which of the following sets contains B as an element? (Note: $\mathcal{P}(X)$ denotes the power set of X).

   A. $S \times V$
   B. $S \cup V$
   C. $\mathcal{P}(S \times V)$
   D. $\mathcal{P}(S \cup V)$
   E. None of the above.

   (A) $S \times V = \{(\diamondsuit, \mathrm{A}), (\diamondsuit, 2), \cdots, (\diamondsuit, \mathrm{K}), (\clubsuit, \mathrm{A}), \cdots, (\spadesuit, \mathrm{K})\}$.
   (B) $S \cup V = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit, \mathrm{A}, 2,3,4,5,6,7,8,9,10, \mathrm{J}, \mathrm{Q}, \mathrm{K}\}$.
   (C) B is a subset of $S \times V$, hence it is an element of $\mathcal{P}(S \times V)$.
   (D) B is not a subset of $S \cup V$, hence it is not an element of $\mathcal{P}(S \cup V)$.

Answer: C

8. Given $A = \{1,2,3,4,5\}$ and the partial order R on A as follows:

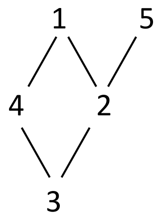
$$R = \{(x,x) : x \in A\} \cup \{(2,1),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1)\}$$

How many distinct linearizations of R are there?

A. 3
B. 5
C. 7
D. 12
E. None of the above.

Answer: B

Hasse diagram:



Linearizations:

5	1	5	1	1
1	5	1	5	4
2	2	4	4	5
4	4	2	2	2
3	3	3	3	3

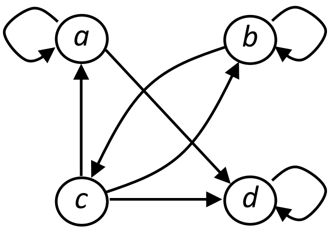
9. Aiken working on a problem from circuit design came up with a graph G, while Dueet working on a problem from computational biology came up with a graph G*. When they met for dinner, they were surprised to find that that G and G* are isomorphic. Aiken's graph G has 6 connected components.

How many connected components are there in G*?

A. G* has at most 6 connected components.
B. G* has at least 6 connected components.
C. G* has at exactly 6 connected components.
D. G* does not have exactly 6 connected components.
E. There is insufficient information to determine.

Answer: C
Since G and G* are isomorphic, they have the same number of connected components.

10. Given the following directed graph $G = (V, E)$, how many walks of length 3 in total are there between u and w, $\forall u, w \in V$?



$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 3 & 2 & 3 \\ 2 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A. 8.
B. 20.
C. 22.
D. 24.
E. 25.

Answer: D

Part B: Multiple Response Questions [Total: 6×3 = 18 marks]

Each multiple response question (MRQ) is worth **THREE marks** and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C.

Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

11. Which of the following have the same answer?

    A. Number of subsets of $\{1,2,3,4,5,6,7,8,9,10\}$ with cardinality 4.

    B. Number of ways to choose 6 out of 10 persons.

    C. Number of ways to arrange 6 persons around a round table.

    D. Number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 6$ where $x_i \in \mathbb{N}$.

    E. Number of permutations of ICANDOIT.

Answer: A, B, D (with solution = $\binom{10}{6} = 210$)
(C) has solution $5! = 120$. (E) has solution $\frac{8!}{2!} = 20160$.

12. Given $A = \{1,2,3,4,5\}$ and the partial order R on A as follows:

    $$R = \{(x,x) : x \in A\} \cup \{(2,1),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1)\}$$

    (This is the same partial order in Question 8.)

    Which of the following statements is/are true with respect to this partial order?

    A. 1 is a minimal element.

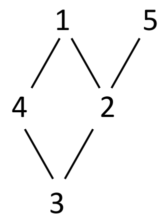
    B. 1 is a maximal element.

    C. 4 and 5 are non-comparable.

    D. 3 is a smallest element.

    E. 5 is a largest element.

Hasse diagram:



Answer: B, C, D

13. Given the equivalence relation $\sim$ on $\mathbb{Z}^2$ defined by

    $$(a,b) \sim (c,d) \Leftrightarrow 3 \mid (a-c) \wedge 2 \mid (b-d).$$

    Which of the following is/are equivalence classes under this relation?

    A. $\{(x,y) \in \mathbb{Z}^2 : x, y \in \mathbb{Z}\}$

    B. $\{(x,y) \in \mathbb{Z}^2 : 3x, 2y \in \mathbb{Z}\}$

    C. $\{(x,y) \in \mathbb{Z}^2 : x = 3k-1, y = 2m \text{ where } k, m \in \mathbb{Z}\}$

    D. $\{(x,y) \in \mathbb{Z}^2 : x = 2k, y = 3m \text{ where } k, m \in \mathbb{Z}\}$

    E. $\{(x,y) \in \mathbb{Z}^2 : x = 3k+4, y = 2m+3 \text{ where } k, m \in \mathbb{Z}\}$

Answer: C, E
Two points in $\mathbb{Z}^2$ are equivalent if their first coordinates are congruent modulo 3 and their second coordinates are congruent modulo 2.

14. Suppose f and g are functions, which of the following statements is/are true?

   A. If f and g are injective, then $g \circ f$ is injective.
   B. If f and g are surjective, then $g \circ f$ is surjective.
   C. If f and g are bijective, then $g \circ f$ is bijective.
   D. If $g \circ f$ is bijective, then f and g are bijective.
   E. If $g \circ f$ is not bijective, then f and g are not bijective.

Answer: A, B, C

Counterexample for (D): Let $A = \{0\}, B = \{1,2\}, C = \{3\}$. Define $f: A \to B$ by $f(0) = 1$; define $g: B \to C$ by $g(1) = g(2) = 3$. Then $g \circ f$ is bijective, but neither f nor g are bijective.

Counterexample for (E): Let $X = \{a, b\}, Y = \{c, d\}, Z = \{e\}$. Define $f: X \to Y$ by $f(a) = c$ and $(b) = d$; define $g: Y \to Z$ by $g(c) = g(d) = e$. Then $g \circ f$ is not bijective, but f is bijective.

15. Which of the following sets is/are countable?

   A. The set A of all points in the plane with rational coordinates.
   B. The set B of all infinite sequences of integers.
   C. The set C of all functions $f: \{0,1\} \to \mathbb{N}$.
   D. The set D of all functions $f: \mathbb{N} \to \{0,1\}$.
   E. The set E of all 2-element subsets of $\mathbb{N}$.

Answer: A, C, E

(A) $\mathbb{Q}$ is countable and the cartesian product of finitely many countable sets is countable. Hence $A = \mathbb{Q} \times \mathbb{Q}$ is countable.
(B) The set of all infinite sequences of integers is a superset of the set of infinite binary sequences. The latter can be interpreted as the binary expansion of a real number in the interval (0,1), which is uncountable.
(C) The function $f: \{0,1\} \to \mathbb{N}$ is a bijection with $\mathbb{N} \times \mathbb{N}$.
(D) This is the set of all infinite binary sequences. Cantor's diagonal argument shows that this set is uncountable.
(E) Let S be the set of all subsets of $\mathbb{N}$ with at most 2 elements. Observe that S is countable since C (in part C) is countable and the function $C \to S$ that maps each $f \in C$ to its range is surjective. Since $E \subseteq S$, it follows that E is countable.

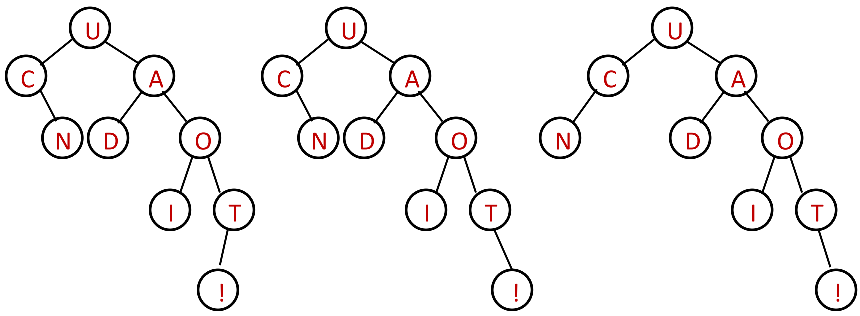
16. The following are the pre-order traversal and post-order traversal of a binary tree:

Pre-order: **U C N A D O I T !**

Post-order: **N C D I ! T O A U**

Which of the following is/are possible in-order traversals of this tree?

   A. C N U D A I O ! T
   B. C N U D A I O T !
   C. N C U D A I O T !
   D. N C U D A I ! O T
   E. None of the above.



Answer: A, B, C

Part C: There are 5 questions in this part [Total: 62 marks]

17. Prove by mathematical induction: $\mathbf{7 \mid (5^{2n+1} + 2^{2n+1})}$ for all $n \in \mathbb{N}$. **[Total: 5 marks]**

Answer.

1. For each $n \in \mathbb{N}$, let $P(n) \equiv 7 \mid (5^{2n+1} + 2^{2n+1})$.
2. (Base step) $P(0)$ is true as $5^{2(0)+1} + 2^{2(n)+1} = 5 + 2 = 7$ is divisible by 7.
3. (Induction step)
   - 3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $7 \mid (5^{2k+1} + 2^{2k+1})$ or $5^{2k+1} + 2^{2k+1} = 7m$ for some integer m.
   - 3.2 Then $5^{2(k+1)+1} + 2^{2(k+1)+1}$
   - 3.3. $= 5^{2k+1+2} + 2^{2k+1+2}$
   - 3.4. $= 5^2(5^{2k+1}) + 2^2(2^{2k+1})$
   - 3.5. $= 25(5^{2k+1}) + 4(7m - 5^{2k+1})$ from line 3.1
   - 3.6. $= 25(5^{2k+1}) + 4(7m) - 4(5^{2k+1})$
   - 3.7. $= 4(7m) + 21(5^{2k+1})$
   - 3.8. $= 7(4m) + 7(3 \cdot 5^{2k+1})$
   - 3.9. $= 7(4m + 3 \cdot 5^{2k+1})$
   - 3.10. Since $(4m + 3 \cdot 5^{2k+1}) \in \mathbb{Z}$ (by closure of integers), $7 \mid (5^{2(k+1)+1} + 2^{2(k+1)+1})$ or $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{N}\ P(n)$ is true by MI.

18. **[Total: 10 marks]**

(a) For each of the following functions, indicate whether it is injective and whether it is surjective. Write **True** or **False** on the Answer Sheets. You do not need to prove or disprove it. [6 marks]

(i) $f: \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(n) = n + 1$.

(ii) $f: [3, \infty) \rightarrow [7, \infty)$ defined by $f(x) = (x - 3)^2 + 7$.

(iii) $f: \mathbb{Z}^+ \rightarrow \mathbb{Q}$ defined by $f(n) = 1/n$.

(b) Let $\preccurlyeq$ be a partial order on a set S. A subset C of S is called a **chain** if and only if each pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \lor b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain.

Given a finite set S with cardinality n, what is the maximum number of maximal chains that a partial order on S can have? Explain your answer. (Answer without explanation will not receive any credit.) [2 marks]

(c) Let $\preccurlyeq$ be a partial order on a set S. An **antichain** is a subset of S such that no two elements in it are comparable under $\preccurlyeq$. A **maximal antichain** is an antichain that is not a proper subset of any other antichain.

Let D_n be the set of positive divisors of integer n. Given a partial order on D_{30} under divisibility, write out all the maximal antichains in this partial order. [2 marks]

Answers

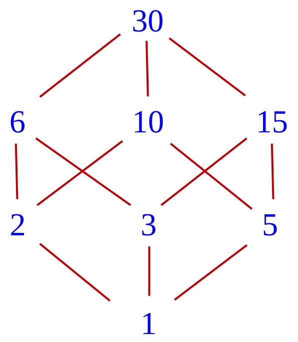
(a) (i) Injective and not surjective. No $n \in \mathbb{N}$ such that $f(n) = 0$. (2 marks)

(ii) Injective and surjective. (2 marks)

(iii) Injective and not surjective. No $n \in \mathbb{Z}^+$ such that $f(n) = 2/3$. (2 marks)

(b) It turns out that I made a mistake in this question as it is a very hard question which I myself have no answer to! I will void this question and award 2 marks to everybody, and additional mark(s) to student who gave a reasonable answer. (2 marks)

(c) {1}, {30}, {2,15}, {3,10}, {5,6}, {2, 3, 5} and {6, 10, 15}. (2 marks)



19. Counting and Probability **[Total: 20 marks]**

Note that you need **not** show your working for parts (a) to (c).

(a) [Subtotal: 4 marks]

Aiken, Dueet and Surely went to the famous Gluttons Gourmet Hawker Centre where four of the city's best hawker stalls are housed: DiDi Chicken Rice, Singing Char Kway Teow, Alamak Nasi Lemak and Hurry Hurry Curry Fish Head. If each of them is to order from one of these four stalls,

(i) in how many ways can they order their food? [1 mark]

(ii) in how many ways can they order their food from the same stall? [1 mark]

(ii) in how many ways can they order their food from two of the four stalls? [2 marks]

Answers

(i) $n = 4, r = 3$: $\binom{r+n-1}{r} = \binom{\mathbf{3+4-1}}{\mathbf{3}} = \binom{6}{3} = \mathbf{20.}$

(ii) Choose a stall from 4 stalls: $\binom{4}{1} = \mathbf{4.}$

(iii) Let $N(x)$ be the number of ways to order from x stalls, $x = 1,2,3$.
$N(2) = 20 - N(1) - N(3) = 20 - 4 - 4 = \mathbf{12.}$

Alternatively, choose 2 stalls: $\binom{4}{2} = 6$. For each choice, there are 2 ways: 1 person orders from one of the two stalls and the other 2 persons order from the other stall. Therefore, $6 \times 2 =$ **12**.

(b) [Subtotal: 5 marks]

A rare disease broke out in a city with a prevalence of 0.1%, that is, it affects 1 out of every 1000 persons. A quick test kit has been developed that has a sensitivity of 85%, which is the probability that a person with the rare disease is tested positive. Among those who took the test, 10% of the time it came out positive.

Write your answers correct to 3 significant figures.

(i) Divoc has shown symptoms of the disease. Should he be tested positive, what is the probability that he actually has the disease? [2 marks]

(ii) What is the probability of a false positive result, that is, a person does not have the disease but is tested positive? [3 marks]

Answers

P(Disease) = 0.001; P(+|Disease) = 0.85; P(+) = 0.1.

(i) $P(Disease \mid +) = \frac{P(+\mid Disease)\cdot P(Disease)}{P(+)} = \frac{0.85\times 0.001}{0.1} = \mathbf{0.00850}$ or $\mathbf{0.850\%}$.

(iii) $P(+ \mid \overline{Disease}) = \frac{P(\overline{Disease} \mid +)\cdot P(+)}{P(\overline{Disease})} = \frac{(1-0.00850)\times 0.1}{0.999} = \mathbf{0.0992}$ or $\mathbf{9.92\%}$

(c) [Subtotal: 6 marks]
How many equivalence relations are there on a set with n elements, where
(i) $n = 2$? [1 mark]
(ii) $n = 3$? [2 marks]
(iii) $n = 4$? [3 marks]

Answers

In general, the answers follow the Bell numbers sequence: 1, 1, 2, 5, 15, 52, 203, 877, … (https://en.wikipedia.org/wiki/Bell_number), which counts the number of possible partitions of a set.

For small values of n, we can list out all possible cases.

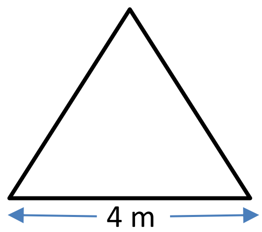
(i) Let $A = \{1,2\}$. There are two possible partitions {{1},{2}} and {{1,2}}. Hence answer = $\mathbf{2}$.

(ii) Let $A = \{1,2,3\}$. There is 1 partition with 3 components: {{1},{2},{3}}; 3 partitions with 2 components: {{1}, {2,3}}, {{2}, {1,3}} and {{3}, {1,2}}; and 1 partition with 1 component: {{1,2,3}}. Hence answer = $\mathbf{5}$.

(iii) Let $A = \{1,2,3,4\}$. There is 1 partition with 4 components: {{1},{2},{3},{4}}; $\binom{4}{1} = 4$ partitions with 2 components where one component is a singleton and the other is a tripleton: {{1}, {2,3,4}}, {{2}, {1,3,4}}, {{3}, {1,2,4}} and {{4}, {1,2,3}}; $\binom{4}{2}/2 = 3$ partitions with 2 components where each component is a doubleton: {{1,2},{3,4}}, {{1,3},{2,4}} and {{1,4},{2,3}}; $\binom{4}{2} = 6$ partitions with 3 components: {{1,2},{3},{4}}, {{1,3},{2},{4}}, {{1,4},{2},{3}}, {{2,3},{1},{4}}, {{2,4},{1},{3}} and {{3,4},{1},{2}}; and 1 partition with 1 component: {{1,2,3,4}}. Hence answer = 1 + 4 + 3 + 6 + 1 = $\mathbf{15}$.

(d) [Subtotal: 5 marks]
(i) Dueet owns a beautiful cat called Meow who loves to wear socks on its four feet. Dueet has a collection of many cat socks in three colours: white, red and yellow. What is the least number of socks Dueet must pull out from the drawer to guarantee getting four socks of matching colour? You do not need to explain your answer. [2 marks]

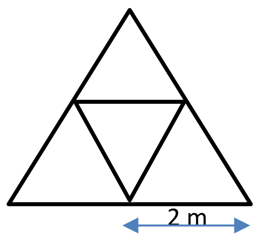
(ii) Aiken has a mini garden which is an equilateral triangle with side of 4 metres. He wants to plant five seeds in his mini garden. Explain that there will be two seeds within 2 metres of each other. [3 marks]



Answers

(i) $\mathbf{10}$ socks. $k < \frac{10}{3}$, so there are at least $k + 1 = 4$ socks of matching colour.

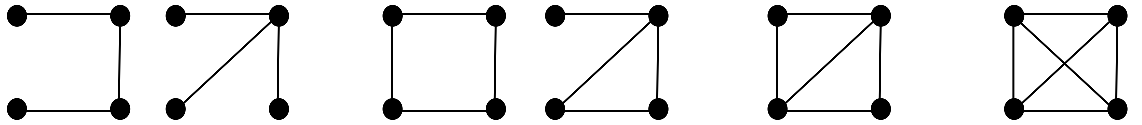
(ii) Divide the mini garden into 4 smaller equilateral triangles each with side 2 metres. From PHP, there will be at least two seeds (pigeons) in a small triangle (pigeonhole), and hence are within 2 metres of each other.



20. Graphs and Trees [Total: 20 marks]

(a) Draw all non-isomorphic, connected simple graphs on four vertices. [3 marks]

Answer: 6 graphs in total.



with 3 edges with 4 edges with 5 edges with 6 edges

(b) Prove or disprove the following statements: [8 marks]

(i) The graph $K_{2,4}$ is a planar graph.

(ii) The graph $K_{3,4}$ is a planar graph.

(iii) There exists a simple graph that is connected and planar with 5 vertices and 8 faces.

(iv) The graph $K_5 - \{e\}$ where e is any edge in the graph is 4-colourable.

Answers

(i) True: Simple to show – one way is to place 1 vertex on each side of the 4-vertices subset.

(ii) False: $K_{3,4}$ contains $K_{3,3}$, which is non-planar (Kuratowski's theorem).

(iii) False. Using Euler's formula ($f = e - v + 2$), the graph has $e = f + v - 2 = 11$ edges. But a simple graph on 5 vertices has at most $\binom{5}{2} = 10$ edges.

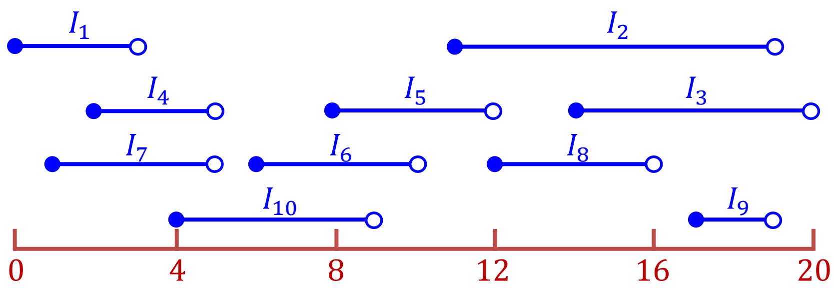
(iv) True: Let $e = \{x, y\}$. Colour both x and y with the same colour, then the other vertices can be coloured with 3 other colours.

(c) [Subtotal: 9 marks] You are given a set of n tasks $T = \{T_1, T_2, T_3, \dots, T_n\}$. Each task T_k is represented by the interval $I_k = [s_k, e_k)$ where s_k is the *start time* and e_k (where $s_k < e_k$) is the *end time* of the task, for $k = 1,2, \dots, n$. An instance of this problem with $n = 10$ is shown below.

Instance: $n = 10$, and $T = \{T_1, T_2, \dots, T_{10}\}$, and

$I_1 = [0, 3)$, $I_2 = [11, 19)$, $I_3 = [14, 20)$, $I_4 = [2, 5)$, $I_5 = [8, 12)$,

$I_6 = [6, 10)$, $I_7 = [1, 5)$, $I_8 = [12, 16)$, $I_9 = [17, 19)$, $I_{10} = [4, 9)$.



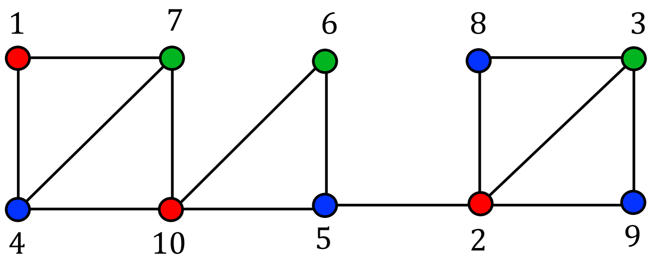
Define a graph $G = (T, E)$ where

$$E = \{ \{T_x, T_y\} : (I_x \cap I_y \neq \emptyset) \}$$

Namely, there is an edge $\{T_x, T_y\} \in E$ if and only if the intervals $[s_x, e_x)$ of task T_x *overlaps* with the interval $[s_y, e_y)$ of task T_y. We call G an **interval graph**.

(i) Draw the interval graph G for this instance. Ten labelled vertices have been drawn on the Answer Sheets. [3 marks]

Answer



(ii) Give a minimum colouring for the graph G from (i) above. [2 marks]

Answer

Graph is 3 colourable. There are many possible ways to colour G with 3 colours. All are acceptable. (Grader needs to check that this.)

(**Note:** In the answer book, students are told that if they do not have colour pens/pencils, you can label the vertices with colour labels C1, C2, C3, C4, etc.)

(iii) You want to assign all the tasks to machines so as to *minimize* the number of machines used. Each machine can only work on one task at a time. Each machine can do any number of tasks as long as the tasks do not overlap in time. (You can assume that a machine can (if necessary) start on a new task immediately after finishing a previous task. In the instance above, tasks T_5 and T_8 can be assigned to the same machine, if necessary.)

Give a short argument (in one or two sentences) that the tasks assigned the same colour in the coloured graph G (obtained in part (ii) above) can be done by one machine. [2 marks]

Answer

Vertices with the same colour are mutually not connected in G. Hence, they do not overlap in time and can be assigned to one machine.

(iv) Give an assignment of tasks in T to machines that minimizes the number of machines needed. You can use the one from part (c)(ii) above or a different assignment that minimizes the number of machines. [2 marks]

Answer

Minimum number of machines needed is 3. Task assignment is given below:

M1: $\{T_1, T_{10}, T_2\}$

M2: $\{T_7, T_6, T_3\}$

M3: $\{T_4, T_5, T_8, T_9\}$

(Note: Like (ii), there are many possible answers with 3 machines.)

21. **[Total: 7 marks]**

(a) Define a partition of $\mathbb{Z}$ that divides $\mathbb{Z}$ into 1231 countably infinite subsets. You do not need to explain your answer [2 marks]

(b) Define a bijection $f: (0,1) \to (0,1]$ so that you may conclude that $(0,1)$ and $(0,1]$ have the same cardinality. Prove that your function f is a bijection. [5 marks]

Answer.

(a) The partition induced by the **congruence-mod-1231** relation on $\mathbb{Z}$ divides $\mathbb{Z}$ into 1231 components, each of which is countably infinite.

Alternatively, $\{\{1231k : k \in \mathbb{Z}\}, \{1231k + 1 : k \in \mathbb{Z}\}, \cdots, \{1231k + 1230 : k \in \mathbb{Z}\}\}$ is a partition where each component is countably infinite.

(b) Define a bijection $f: (0,1) \to (0,1]$ as follows

$$f(x) = \begin{cases} 2x, & \text{if } x = \frac{1}{2^n} \text{ for some } n \in \mathbb{Z}^+ \\ x, & \text{otherwise} \end{cases}$$

To prove that f is a bijection:

1. (Injective)
   1.1. Let $f(x_1) = f(x_2)$ for some $x_1, x_2 \in (0,1)$.
   1.2. Case 1: $f(x_1) = f(x_2) = \frac{1}{2^p}$ for some $p \in \mathbb{Z}_{\geq 0}$
      1.2.1. Then $x_1 = x_2 = \frac{1}{2^{p+1}}$.
   1.3. Case 2: $f(x_1) = f(x_2) \neq \frac{1}{2^p}$ for all $p \in \mathbb{Z}_{\geq 0}$
      1.3.1. Then $x_1 = x_2$.
   1.4. In all cases $x_1 = x_2$, hence f is injective.
2. (Surjective)
   2.1. Let $\in (0,1]$.
   2.2. Case 1: If $y = \frac{1}{2^p}$ for some $p \in \mathbb{Z}_{\geq 0}$
      2.2.1. Then $f\left(\frac{1}{2^{p+1}}\right) = \frac{2}{2^{p+1}} = \frac{1}{2^p} = y$.
   2.3. Case 2: If $y \neq \frac{1}{2^p}$
      2.3.1. Then $f(y) = y$.
   2.4. In all cases there exists an x such that $f(x) = y$, hence f is surjective.
3. Therefore f is a bijection.

=== END OF PAPER ===

CS1231S Exam

AY2021/22 Semester 2

Some quick comments on your performance in this exam while they are still fresh on my mind. I will do a more complete report when all the grading is done.

I am midway grading Q18 (relations and functions) Q19 (counting and probability), and Q21 (cardinality). Prof Leong is grading his Q20 (graphs) and Ben is grading Q17 (mathematical induction).

General comments

The performance is disappointing and honestly speaking it really makes me feel like crying. After the poor showing in the midterm test, I actually have adjusted the exam questions to make them a notch easier than I had originally planned, that is, I have made the questions more direct and less deep (as compared to previous semesters').

I put quite a bit of effort in designing exam questions and spent several weeks setting and revising them over and over again to ensure a mixed level of difficulty. I tried to relate the questions to what you have seen or done in lectures, tutorials, midterm test and the online quizzes. However, the relation may not be obvious; you need to observe, analyze and think.

When we anticipate that a question might appear too novel or complex in an exam, but a good question nonetheless, we prepare you in advance. For instance, for the exam question on graphs, Prof Leong has deliberately set a similar question in tutorial #10 to give you some head start. Likewise, in my explanation below, you can see that I have tried to relate some of the exam questions to situations you have encountered in class.

One thing for sure, at this level of learning, you wouldn't expect assessment to be entirely on content recall or pattern recognition. Our objective is problem solving – applying what you have learned to solve novel problems. This is an essential skill if you aspire to be a competent computer scientist. Computer scientists are not only very good at programming, they are – I would even say more importantly so – very good at logical and computational thinking, in order to create useful solutions for complex problems.

I observed some students kept referring to their notes during the exam. In an open book exam, if you find yourself having to refer to the notes very often, then you are not really ready for the exam.

After a test or an exam, invariably I will have students emailing me saying that they didn't do well and asking for resources for them to practice more during the vacation.

My answer is, it is NOT so much about discrete structures or whatever the subject matter is. Instead of reading up more about discrete structures (or whatever subject), I would rather advise students to find out more about how to approach problem solving, how to think logically ("The Art of Logical Thinking") and computationally ("The Power of Computational Thinking"), and to engage in more activities that give you the opportunity to hone these skills. Also, do self-reflection on the way you learn and how to make your learning more effective (meta-learning – learning about how to learn).

To arouse your interest, you may also read interesting books like "The New Turing Omnibus – 66 Excursions in Computer Science".

Now, on to the specific exam questions.

Q18a

These are simple questions. I don't even bother to provide the proofs or explanations in my answers file. Part (ii) might take a little more effort to check that it is both injective and surjective.

Most students got the injective part right for all 3 functions. But I am quite surprised and disappointed that a number of students didn't detect that (i) and (iii) are not surjective. For (i), there is no pre-image for 0; for (iii), there is no pre-image for, say, 2/3, as shown in the answers file.

Many students left parts (b) and (c) blank!

Q18b

It turns out that I made a mistake in this question as it is a very hard question which I myself have no answer to! I will void this question and award 2 marks to everybody, and additional mark(s) to student who gave a reasonable answer.

But I encountered strange answers that are contradictory. For example, "the maximum number of maximal chains is n when all elements are comparable to one another". When all elements are comparable to one another, then we have one big chain with n elements; we don't have n chains!

Q18c

More students skipped this question.

You see from the answers file that all you need is to draw the Hasse diagram, and you will find the maximal antichains. Some students have some invalid antichains in the list, in which case I didn't accept it.

Q19

I was expecting students to do well for Q19 but so far the performance has proved me wrong. Only a handful of students received double-digit marks for this question. (I am midway in my grading, so I hope to see better results for the rest.)

Q19a

Standard multiset problem. Interestingly, more students got part (ii) correct than part (i).

What concerns me is the lack of common logic in some answers. Some students have the answer for part (ii) or part (iii) larger than their answer for part (i)! This is illogical because part (i) is the most general of all (how many ways without any restriction) while parts (ii) and (iii) come with restrictions (must get from one stall; must get from two stalls), so the answers for (i) must be the largest among the three parts.

Q19b

This is a standard question on conditional probability but is very poorly attempted. Please refer to the answers.

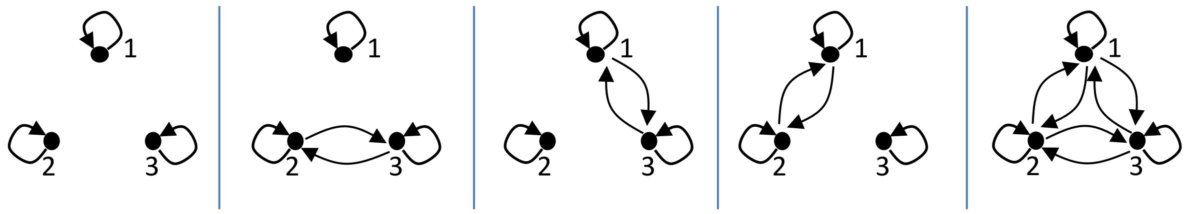
Q19c

This is the Bell numbers sequence, not covered in class, but I am asking for small values of n (n=2,3,4), so I was hoping students could use some exhaustive technique for such small values, which should work for n=2 and n=3 easily, albeit a little challenging for n=4.

In the lecture, we have shown equivalence relation and partition are related – one induces the other. Here we apply this connection – the number of possible equivalence relations is simply the number of possible partitions.

Even if you don't see this connection, you can try other means. For example, draw all possible directed graphs that represent an equivalence relation on a set of 2 elements. You'll realize that there are only 2 possible ways:

Likewise for n=3, there are 5 ways:



Q19d(i)

I thought this is a give-away question but I was wrong. When you draw out 9 socks, you may still not get 4 socks of the same matching colour as you may have 3 white, 3 red and 3 yellow. This is the worst-case scenario. The moment you pick the 10th sock, you get 4 matching socks.

Q19d(ii)

In tutorial 9, we discussed a PHP question on 51 points inside a unit square. We solved the problem by dividing the unit square into smaller squares.

Here, you are given an equilateral triangle with side 4m, and 5 seeds. Do you see the similarity with the tutorial question?

I was expecting students to divide the equilateral triangles into 4 smaller equilateral triangles (pigeonholes) with side 2m, and with 5 seeds (pigeons), we solved this with PHP the same way we did in the tutorial.

Some students answered this using geometry or by computing the area. We have discussed in tutorial why this doesn't work.

Some students answered this by putting 3 seeds at the corners of the triangle, one in the middle, and argued that the 5th seed must be within 2m of another seed, because the "best" place to put the 5th seed is on the edge of the triangle, in the midpoint of two corners. The problem with this answer is that it relies on visual inspection more than a rigorous proof. Why does the 4th seed in the centre of the triangle matter? What if it is not in the middle of the triangle but a bit off? Could the 5th seed then be able to place slightly off the edge but outside of 2m range from any of the other seeds? Or, why is the middle of the edge the "best" position to place the 5th

seed? Wouldn't the 5th seed be too close to the seed at the centre. And there are more unanswered questions arising from this argument. Of course, from the diagram we know it is the case, all thanks to visual inspection, but there isn't rigour in this "proof".

The proper solution is to divide the equilateral triangle into 4 smaller ones and then simply apply PHP.

Q21

Part (a) is easy but part (b) is very difficult (intentional). That's why I put this as the last question.

Most students skipped this question, or at most attempted only part (a).

There are examples of partitions of $\mathbb{Z}$ by using congruence-mod-2, congruence-mod-3, congruence-mod-4 (lecture 6 slide 52), so this is a direct application of that in part (a). Most students who attempted part (a) got this right.

So far, only one student attempted part (b) correctly.

Prepared by Aaron Tan
28 April 2022

CS1231S Final Assessment Report

AY2021/22 Semester 2

This is the report for the CS1231S final assessment held on 27 April 2022.

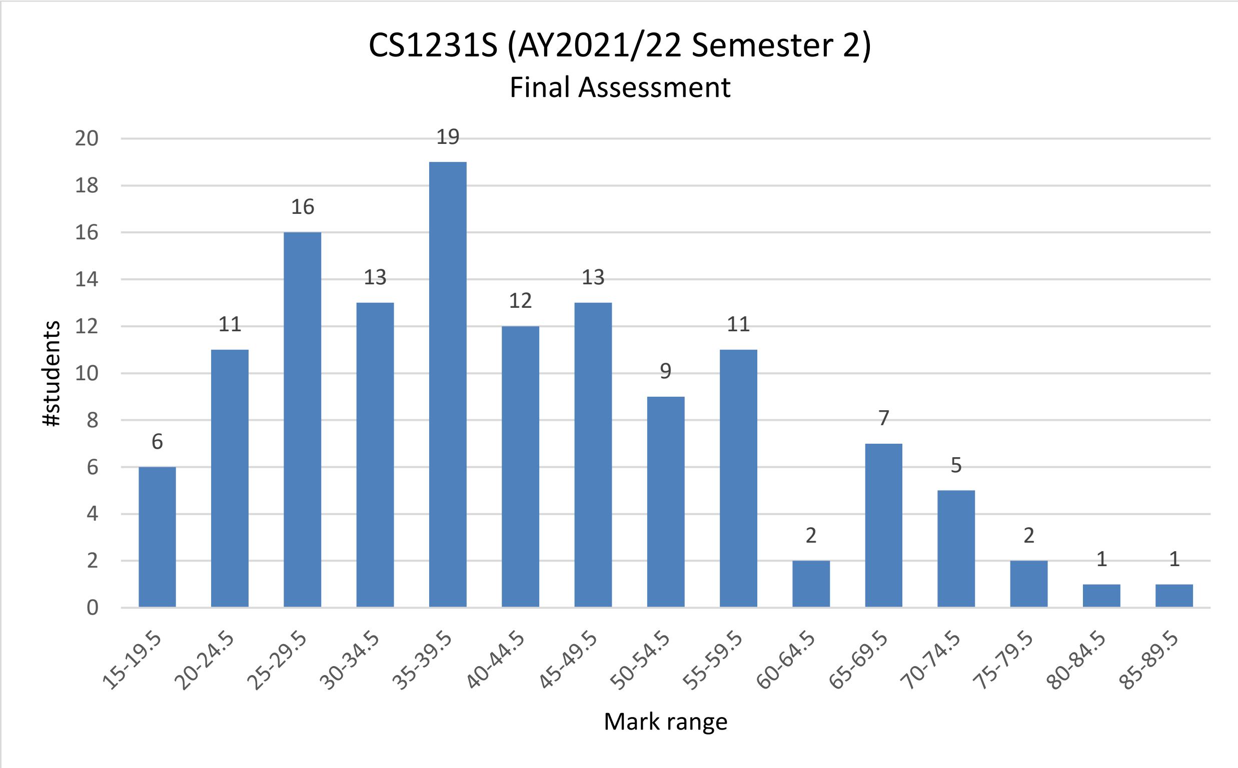
128 (out of 130) students sat for the final assessment.

Below are the statistics:

1. General statistics

	Part A (Q1-10) (20 marks)	Part B (Q11-16) (18 marks)	Q17 Induction (5 marks)	Q18 Functions/ Relations (10 marks)	Q19 Counting (20 marks)	Q20 Graphs (20 marks)	Q21 Cardinality (7 marks)	**Total (100 marks)**
Average	11.52	4.13	3.02	5.34	5.90	9.75	0.76	**42.02**
Average (normalized)	57.6%	22.9%	60.5%	53.4%	29.5%	48.8%	10.8%	42.0%
Median	12	3	3	5	5	10	0	38.75
Standard deviation	4.15	3.70	1.13	1.89	3.99	5.22	1.16	16.44

Below is the chart for the overall results.



2. Part A: MCQs 1 - 10

The table below shows the percentage of students who chose the correct answers, and of those who chose the most popular wrong answers:

	Q1	Q2	Q3	Q4	Q5
			Hardest		
%students who chose the correct answer	E (32.8%)	A (80.5%)	D (17.2%)	B (69.5%)	E (67.2%)
%students who chose the most popular wrong answer	F (42.2%)	E (8.6%)	B (33.6%)	D (22.7%)	B (22.7%)

	Q6	Q7	Q8	Q9	Q10
				Easiest	
%students who chose the correct answer	D (61.7%)	C (79.7%)	B (40.6%)	C (78.1%)	D (57.8%)
%students who chose the most popular wrong answer	A (20.3%)	A (12.5%)	A and C (17.2% each)	C (8.6%)	C (14.1%)

3. Part B: MRQs 11 - 16

The table below shows the answers and percentages of students who got the questions right:

	Q11	Q12	Q13	Q14	Q15	Q16
Answer	A,B,D	B,C,D	C,E	A,B,C	A,C,E	A,B,C
%students who got it right	48.4%	35.9%	11.7%	21.1%	4.7%	18.0%
	Easiest				Hardest	

Prepared by Aaron Tan

Updated: 3 May 2022