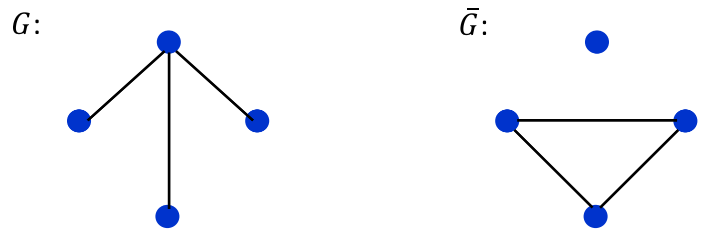
CS1231S: Discrete Structures

Counting and Probability II and Graph I

Definition. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition. A *self-complementary* graph is isomorphic with its complement.

Definition. A simple circuit (cycle) of length three is called a *triangle*.

CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II and Graphs I

(Week 12: 4 – 8 April 2022)

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

II. Tutorial Questions

1. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

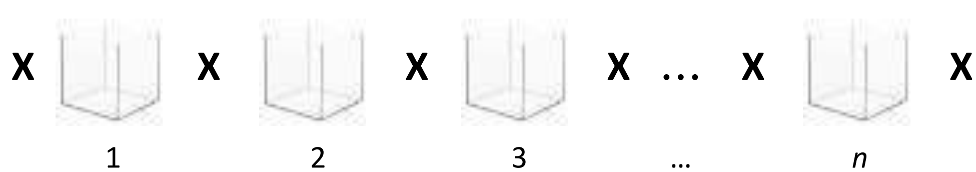
2. Let's revisit Tutorial #9 Question 5:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



CS1231S: Discrete Structures

Counting and Probability II and Graph I

Answers

1. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

Answer:

Recall the Binomial Theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b^1 + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

We have $a = 2x^2, b = \frac{1}{x}, n = 9$.

The general term in the expansion of $(a+b)^n$ is given by:

$$\binom{n}{r} a^{n-r}b^r = \binom{9}{r}(2x^2)^{9-r}\left(\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} \cdot x^{18-2r} \cdot x^{-r} = \binom{9}{r} 2^{9-r} \cdot x^{18-3r}$$

For this term to be independent of x, we must have $18 - 3r = 0$, or $r = 6$.

Therefore, the term independent of x is

$$\binom{9}{6} 2^{9-6} = 84 \times 2^3 = \mathbf{672}$$

2. Let's revisit Tutorial #9 Question 5:

   Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   Last week, the answer given was

   For k $(1 \le k \le n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

   Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?

1 2 3 … *n*

Answer:

The task is similar to choosing two out of the $n+1$ crosses to mark the start and end of the consecutively numbered boxes that contain white balls.

This is $\binom{n+1}{2}$, which is also equal to $n(n+1)/2$.

3. [AY2020/21 Semester 2 Exam Question]
On a die there are 6 numbers. We call 4, 5, 6 the big numbers and 1, 2, 3 the small numbers. Given a loaded die in which the probability of rolling any fixed big number is twice the probability of rolling any fixed small number, answer the following questions.
   (i) What is the probability of rolling a 6? Write your answer as a single fraction.
   (ii) If two such loaded dice are rolled, what is the expected value of the maximum of the two dice? Write your answer as a single fraction.

Answer:

Probability of $\frac{1}{9}$ to roll a 1, 2 or 3; probability of $\frac{2}{9}$ to roll a 4, 5 or 6.

Maximum of 1: 1 way → (1,1) with probability $\frac{1}{81}$.

Maximum of 2: 3 ways → (1,2), (2,1), (2,2) with probability $\frac{3}{81}$.

Maximum of 3: 5 ways → (1,3)x2, (2,3)x2, (3,3) with probability $\frac{5}{81}$.

Maximum of 4: 7 ways → (1,4)x2, (2,4)x2, (3,4)x2, (4,4) with probability $\frac{12}{81}+\frac{4}{81}=\frac{16}{81}$.

Maximum of 5: 9 ways → (1,5)x2, (2,5)x2, (3,5)x2, (4,5)x2, (5,5) with probability $\frac{12}{81}+\frac{12}{81}=\frac{24}{81}$.

Maximum of 6: 11 ways → (1,6)x2, (2,6)x2, (3,6)x2, (4,6)x2, (5,6)x2, (6,6) with probability $\frac{12}{81}+\frac{20}{81}=\frac{32}{81}$.

Expected value = $\left(\frac{1}{81}\times 1\right)+\left(\frac{3}{81}\times 2\right)+\left(\frac{5}{81}\times 3\right)+\left(\frac{16}{81}\times 4\right)+\left(\frac{24}{81}\times 5\right)+\left(\frac{32}{81}\times 6\right)=\mathbf{\frac{398}{81}}$.

4. An urn contains five balls numbered 1, 2, 2, 8 and 8. If a person selects a set of two balls at random, what is the expected value of the sum of the numbers on the balls?

Answer:

Let $2_a, 2_b$ denote the two balls with the number 2, and $8_a, 8_b$ the two balls with the number 8.

The list below shows the sums of the numbers on the balls in each set:

- Sum of 3: $\{1, 2_a\}, \{1, 2_b\}$.
- Sum of 4: $\{2_a, 2_b\}$.
- Sum of 9: $\{1, 8_a\}, \{1, 8_b\}$.
- Sum of 10: $\{2_a, 8_a\}, \{2_a, 8_b\}, \{2_b, 8_a\}, \{2_b, 8_b\}$.
- Sum of 16: $\{8_a, 8_b\}$.

Expected value = $3\times\left(\frac{2}{10}\right)+4\times\left(\frac{1}{10}\right)+9\times\left(\frac{2}{10}\right)+10\times\left(\frac{4}{10}\right)+16\times\left(\frac{1}{10}\right)=\mathbf{8.4}$.

Or, we could compute the expected value of the number when one ball is picked:
$1\times\left(\frac{1}{5}\right)+2\times\left(\frac{2}{5}\right)+8\times\left(\frac{2}{5}\right)=4.2$.
Then apply linearity of expected value to get $2\times 4.2=\mathbf{8.4}$.

5. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

Write your answers correct to three significant figures.

(a) What is the probability that the chosen ball is green?

(b) If the chosen ball is green, what is the probability that it was picked from the first urn?

Answers:

(a) $\frac{4}{10} \times \frac{25}{35} + \frac{6}{10} \times \frac{15}{37} = \frac{137}{259} = 52.9\%.$

Probability that the chosen ball is green is **52.9%**.

(b) Let G be the event that the chosen ball is green, U_1 the event that the ball came from the first urn, and U_2 the event that the ball came from the second urn.

$P(U_1) = 0.4, P(U_2) = 0.6, P(G|U_1) = \frac{25}{35}, P(G|U_2) = \frac{15}{37}$.

By Bayes' Theorem,

$$P(U_1|G) = \frac{P(G|U_1) \cdot P(U_1)}{P(G|U_1) \cdot P(U_1) + (G|U_2) \cdot P(U_2)}$$

$$= \frac{\left(\frac{25}{35}\right) \times 0.4}{\left(\frac{25}{35}\right) \times 0.4 + \left(\frac{15}{37}\right) \times 0.6} = \frac{74}{137} = 54.0\%$$

Therefore, the probability that the green ball is chosen from the first urn is **54.0%**.

Alternatively, using the result from part (a),

$$P(U_1|G) = \frac{P(U_1 \cap G)}{P(G)} = \frac{0.4 \times \frac{25}{35}}{\frac{137}{259}} = \frac{74}{137} = \mathbf{54.0}\%$$

6. [AY2015/16 Semester 1 Exam Question]

Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

Answers:

(a) 1/16

(b) 1/64

For the general case, let $|A| = n$. In general, we need to consider all possible n^2 pairs. Thus the total number of possible combinations, i.e. the number of elements in $P(A \times A)$, is 2^{n^2}.

(a) To count the number of reflexive relations, note that all the pairs (a, a) for all $a \in A$ must be in the relation, so these n pairs are fixed. We are then free to choose to include or not any of the $(n^2 - n)$ remaining pairs. This gives us:

$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

(b) For symmetric relations, if some pair (a, b) is in the relation, then (b, a) must also be in the relation. Hence, either (a, b) and (b, a) are both included in the relation, or both are not. This gives us $\frac{n^2+n}{2}$ pairs to choose to include or not, giving us:

$$\frac{2^{\frac{n^2+n}{2}}}{2^{n^2}} = \frac{1}{2^{\frac{n^2-n}{2}}}$$

(Refer to the slides uploaded for more details.)

7. Let's revisit Tutorial #9 Question 1:

   In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

   Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a,b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b-1) + W(a-1, b), & \text{if a} > 0 \text{ and } b > 0. \end{cases}$$

   We may express $W(a, b)$ as a simple combination formula as follows:

$$W(a,b) = \binom{a+b}{a}.$$

   Verify the above.

   Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \leq n$) games.

   Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in Tutorial #9 Question 1.

   Answer:

   To verify that $W(a,b) = \binom{a+b}{a}$, it suffices to verify that the function $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ such that $(a, b) \mapsto \binom{a+b}{b}$ satisfies the recursive definition of $W(a, b)$. We use mathematical induction (though the solution below is not written formally).

   Pick any $a, b \in \mathbb{Z}_{\geqslant 0}$.

   To verify that $W(a,b) = \binom{a+b}{a}$:

   Case 1: $a = 0$, then $\binom{a+b}{a} = \binom{b}{0} = 1 = W(a,b)$.

   Case 2: $b = 0$, then $\binom{a+b}{a} = \binom{a}{a} = 1 = W(a,b)$.

   Case 3: $a > 0$ and $b > 0$, then

$$\begin{aligned} \binom{a+b}{a} &= \binom{a+b-1}{a-1} + \binom{a+b-1}{a} \text{ (by Pascal's formula: } \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}\text{)} \\ &= \binom{a+(b-1)}{a} + \binom{(a-1)+b}{a-1} \text{ (by commutativity)} \\ &= W(a, b-1) + W(a-1, b) = W(a,b) \end{aligned}$$

   Hence, we have verified that $W(a,b) = \binom{a+b}{a}$ for all cases.

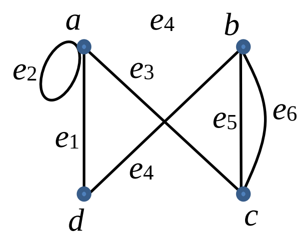
Now, we can see that

$$T(n,k) = W(n-k,n) = \binom{2n-k}{n-k} = \binom{2n-k}{n}.$$

(by Lecture #12 example 8: $\binom{n}{r} = \binom{n}{n-r}$)

Hence, $T(4,2) = \binom{6}{4} = \mathbf{15}$.

8. Given the graph shown below:



(a) Give the adjacency matrix A for the graph, with vertices in the order a, b, c, d.

(b) Compute A^0, A^2 and A^3.

(c) How many walks of length 2 are there from a to b? From c to itself? List out all the walks.

(d) How many walks of length 3 are there from a to c? List out all the walks.

Answers:

(a)
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

(b) $A^0 = I_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $\quad A^2 = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 3 & 5 & 0 & 0 \\ 1 & 0 & 5 & 3 \\ 1 & 0 & 3 & 2 \end{bmatrix}$; $\quad A^3 = \begin{bmatrix} 5 & 3 & 9 & 6 \\ 3 & 0 & 13 & 8 \\ 9 & 13 & 1 & 1 \\ 6 & 8 & 1 & 1 \end{bmatrix}$

(c) There are 3 walks of length 2 from a to b: $< ae_1de_4b >, < ae_3ce_5b >$ and $< ae_3ce_6b >$.

There are 5 walks of length 2 from c to itself:
$< ce_3ae_3c >, < ce_5be_5c >, < ce_5be_6c >, < ce_6be_6c >, < ce_6be_5c >$.

(d) There are 9 walks of length 3 from a to c:

1 way via a first: $< ae_2ae_2ae_3c >$.

0 way via b first:

5 ways via c first: $< ae_3ce_3ae_3c >, < ae_3ce_5be_5c >, < ae_3ce_6be_6c >,$
$< ae_3ce_5be_6c >, < ae_3ce_6be_5c >$.

3 ways via d first: $< ae_1de_1ae_3c >, < ae_1de_4be_5c >, < ae_1de_4be_6c >$.

9. A lady hosted a party of n ($n \geq 2$) people (including herself). At the party, various friends met and some of them shook hands with each other. The thoughtful host made sure that she shook hands with everyone in the party.

Prove that there are at least two people who have shaken hands the same number of times.

Answer:

Model this with a graph $G = (V, E)$ where each person is a vertex in V and there is an undirected edge $\{x, y\} \in E$ whenever x and y shook hand in the party.

1. Let h be the host vertex. Then $deg(h) = n - 1$ (given by the problem)
2. For other vertices v, $deg(v) > 0$ (shook hand with host *h*)
3. So all n vertices have $deg(v)$ in the range $[1..(n-1)]$.
4. Hence, at least two vertices have the same degree (by the Pigeonhole Principle)

Hence at least two people have the *same* number of handshakes.

10. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.

(Note: This problem is equivalent to the following: Show that in any group of 6 people, there must be either 3 mutual friends or 3 mutual strangers. We can assume that for every pair of people, they are either friends or strangers. **BUT for your answers, please use the graph formulation.**)

Answer:

1. Consider any simple graph G with 6 vertices. (eg: Fig 10a)
2. In K_6 the complete graph on 6 vertices, colour the edges in G **black**, and those not in G red. Namely, the red edges are those in the complement graph $\bar{G}$. (Figs 10a and 10b illustrate this process.)

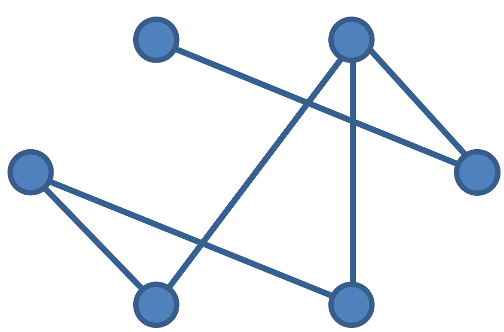


Figure 10a

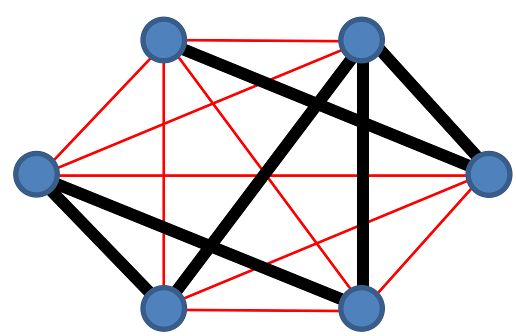


Figure 10b

3. Call this graph G'. Now, G' is a complete graph with the edge coloured black or red, and we want to prove that G' contains a black triangle or a red triangle.

4. Let v be an arbitrary vertex of G'.

4.1 There are 5 edges incident to v (since G' is a complete graph), coloured black or red.

4.2 Therefore, (at least) 3 them have the same colour C, by the Generalized PHP. (In Figure 10c, 3 edges incident to v are red.)

4.3 WLOG, let C be red. Let the red edges be $\{v, u_1\}$, $\{v, u_2\}$, $\{v, u_3\}$. (eg: Fig 10c).

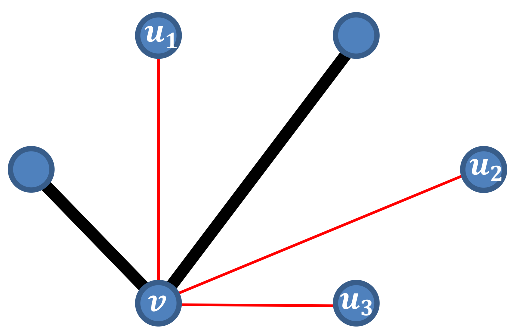


Figure 10c

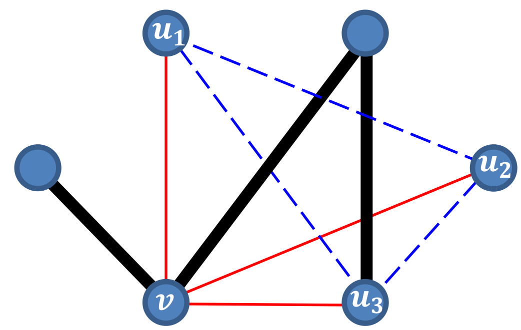


Figure 10d

4.4 **Case 1:** If at least one of the edges $\{u_1, u_2\}$, $\{u_2, u_3\}$ or $\{u_1, u_3\}$ is red.
If $\{u_x, u_y\}$ is the edge that is red, then we have a red triangle $\{v, u_x, u_y\}$.
This is a triangle in $\bar{G}$.

4.5 **Case 2:** If all the three edges $\{u_1, u_2\}$, $\{u_2, u_3\}$ or $\{u_1, u_3\}$ are black.
Then, we have a black triangle $\{u_1, u_2, u_3\}$. This is a triangle in G.

5. Therefore, G or $\bar{G}$ contains a triangle.

[Note: For practice, students may want to also write out the case for "WLOG, let C be black". And also modify Figures 10c and 10d, accordingly.]

(For those interested, look up Ramsey's theory for the *generalization* of this problem, where this result is expressed as "$R(3,3) = 6$". And google the interesting story of Fan Chung's PhD research work on this problem.)

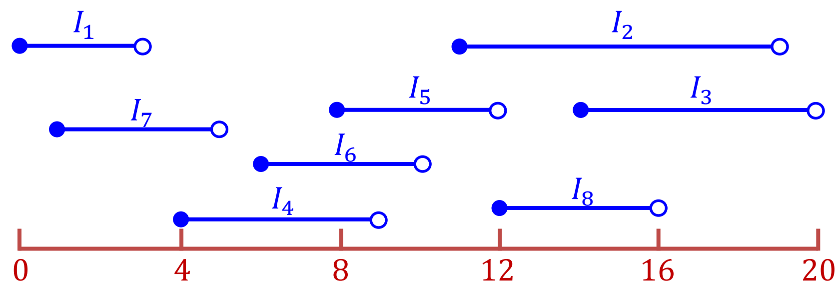
The following definition is used in Question 11.

Definition:
A relation $\prec$ on a set A is said to be ***irreflexive,*** if and only if, $\forall a \in A, (a \nprec a)$.
Note: irreflexive is not that same as (*not reflexive*).

11. You are given a set of n jobs $J = \{J_1, J_2, J_3, \dots, J_n\}$. Each job J_k is represented by the interval $I_k = [s_k, e_k)$ where s_k is the *start time* and e_k (where $s_k < e_k$) is the *end time* of the job, for $k = 1,2,\dots,n$. An instance of this problem with $n = 8$ is shown below.

Instance: $n = 8$, and $\{J_1, J_2, J_3, \dots, J_8\}$, and

$$I_1 = [0,3), \quad I_2 = [11,19), \quad I_3 = [14,20), \quad I_4 = [4,9) \quad ,$$
$$I_5 = [8,12), \quad I_6 = [6,10), \quad I_7 = [1,5), \quad I_8 = [12,16),$$



Define a relation $\prec$ on J by

$$(J_x \prec J_y) \Leftrightarrow (e_x \le s_y)$$

Namely, $J_x \prec J_y$ means that the job J_x is *to the left of* the job J_y.

(a) Show that the relation $\prec$ is irreflexive, is anti-symmetric, and is transitive.
(This kind of relation is also called **strict** partial order.)

Answer

To show $\prec$ is irreflexive:

1. Consider any job $J_k \in J$ with interval $I_k = [s_k, e_k)$.
2. Then $e_k > s_k$ (by definition of the jobs J_k)
3. Hence $(J_k \nprec J_k)$. (by definition of $\prec$).
4. Hence, $\prec$ is irreflexive. (by definition of irreflexivity)

To show $\prec$ is anti-symmetric: (Note that the proof below doesn't use the definition of anti-symmetry in the lecture, but an alternative definition that is equivalent.)

2 distinct

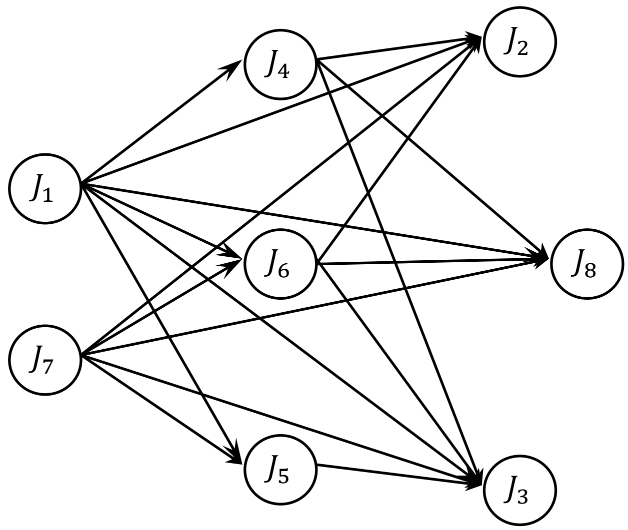
1. For any jobs J_x, J_y with interval $I_x = [s_x, e_x)$ and $I_y = [s_y, e_y)$, respectively.
2. Suppose $(J_x \prec J_y)$. Then $(e_x \le s_y)$. (by definition of $\prec$)
3. We have $(e_y > s_y)$ and $(e_x > s_x)$. (by definition of the jobs J_x J_y)
4. Then, $e_y > s_y \ge e_x > s_x \Rightarrow e_y > s_x \Rightarrow e_y \nleq s_x$. (by 2 and 3)
5. Hence $(J_y \nprec J_x)$. And the relation $\prec$ is anti-symmetric.

To show that $\prec$ is transitive:

1. For any jobs $J_x, J_y, J_z \in J$, suppose $(J_x \prec J_y)$ and $(J_y \prec J_z)$.
2. Then $(e_x \le s_y)$ and $(e_y \le s_z)$. (by definition of $\prec$)
3. We have $(s_y < e_y)$. (by definition of the job J_y)
4. Then, $e_x \le s_y < e_y \le s_z \Rightarrow e_x \le s_z$. (by 2 and 3)
5. Hence $(J_x \prec J_z)$. And the relation $\prec$ is transitive.

(b) Draw the graph $G = (J, \prec)$ of the relation $\prec$ for this instance given above.

Answer



You want to assign all the jobs to workers so as to *minimize* the number of workers deployed. Each worker can only work on one job at a time. Each worker can do any number of jobs as long as the jobs do not overlap in time. (You can assume that a worker can (if necessary) start on a new job immediately after finishing a previous job. In the instance above, jobs T_5 and T_8 can be assigned to the same worker, if necessary.) This problem can be solved in parts (c) and (d) as follows:

(c) Give a maximal chain C in the graph G. Argue that the jobs in C can be assigned to a single worker.

Answer

Many answers. For example, $C = \{J_1, J_4, J_2\}$, $C = \{J_1, J_4, J_3\}$, etc.

Every chain consists of jobs that do not overlap in time, and can be assigned to a single worker.

(d) Now, partition the vertices in G into a *minimum* number of vertex-disjoint chains. For each chain, we assign a worker to do all the jobs in that chain. What is the minimum number of workers needed? Show the jobs assigned to each worker (W1, W2, etc.).

Answer

Partition: $J = \{\{J_1, J_4, J_2\}, \{J_7, J_6, J_8\}, \{J_5, J_3\}\}$.

Minimum number of workers needed is 3. Job assignment is given below:

W1: $\{J_1, J_4, J_2\}$

W2: $\{J_7, J_6, J_8\}$

W3: $\{J_5, J_3\}$

(Note: There are many possible answers with 3 workers.)

CS1231S: Discrete Structures

Supplementary discussion on Solution to Tut #10, Q11(a)

This short note is about the solution to Tut #10, Q11(a) where we prove that the relation $\prec$ is anti-symmetric.

The published solution had used an alternative definition of anti-symmetry. The original and the alternative (equivalent) definitions are given below.

Two equivalent definitions of anti-symmetry:

(a) from the lecture notes

$$\forall x, y, \big((x, y) \in R\big) \wedge \big((y, x) \in R\big) \Rightarrow (x = y)$$

(b) alternative definition

$$\forall x, y, (x \neq y) \wedge \big((x, y) \in R\big) \Rightarrow \big((y, x) \notin R\big)$$

They are equivalent since they both reduces to

$$\equiv \forall x, y, \big((x, y) \notin R\big) \vee \big((y, x) \notin R\big) \vee (x = y)$$

In my (LeongHW's) proof (earlier version), I had missed out the condition that $(x \neq y)$. This is now corrected and I also include the proof using the original definition (from the lectures).

To show $\prec$ is anti-symmetric: (using alternative definition)

1. For any ***2 distinct*** jobs J_x, J_y with interval $I_x = [s_x, e_x)$ and $I_y = \big[s_y, e_y\big)$, respectively.
2. Suppose $\big(J_x \prec J_y\big)$. Then $(e_x \leq s_y)$. (by definition of $\prec$)
3. We have $(e_y > s_y)$ and $(e_x > s_x)$. (by definition of the jobs J_x J_y)
4. Then, $(e_y > s_y \geq e_x > s_x) \Rightarrow \big(e_y > s_x\big) \Rightarrow (e_y \nleq s_x)$. (by 2 and 3)
5. Hence $\big(J_y \nprec J_x\big)$. And the relation $\prec$ is anti-symmetric.

To show $\prec$ is anti-symmetric: (using definition from the lecture notes)

1. For any jobs J_x, J_y with interval $I_x = [s_x, e_x)$ and $I_y = \big[s_y, e_y\big)$, respectively.
2. Suppose $\big(J_x \prec J_y\big)$ and $\big(J_y \prec J_x\big)$. Then $(e_x \leq s_y)$ and $(e_y \leq s_x)$. (by definition of $\prec$)
3. We have $(s_x < e_x)$ and $(s_y < e_y)$. (by definition of the jobs J_x J_y)
4. Then, $\big(s_x < e_x \leq s_y < e_y \leq s_x\big) \Rightarrow (s_x < s_x)$ (by 2 and 3)
5. This is a contradiction.
6. Hence, $\big(J_x \prec J_y\big) \wedge \big(J_y \prec J_x\big)$ is False, and $\Big(\big(J_x \prec J_y\big) \wedge \big(J_y \prec J_x\big) \Rightarrow (x = y)\Big)$ is vacuously true.

(**Note for those who want to explore more:** In this problem, there is not enough information to get the conclusion that $(x = y)$. In many real applications, this is also true. Hence the alternative definition can often be more convenient to use.)