Solutions to Practice Midterm

MA1521 Calculus for Computing

Time allowed: 1 hour 15 mins.

Answer all 7 questions. Each question carries 10 marks.

1. Let a be a positive integer. Given $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = \frac{1}{e^a}$, determine the value of a.

   Answer. 8.

   Solution. The given limit is an indeterminate form of type 1^∞.

   $$\lim_{x\to\infty}\ln\left(\frac{2x-3}{2x+5}\right)^{2x+1} = \lim_{x\to\infty}(2x+1)\ln\left(\frac{2x-3}{2x+5}\right) = \lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{(2x+1)^{-1}}.$$

   By L'Hôpital's Rule, the above limit is equal to

   $\lim_{x\to\infty}\frac{2(2x-3)^{-1}-2(2x+5)^{-1}}{-2(2x+1)^{-2}} = \lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)} = \lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8$. Consequently, $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$. Thus $a = 8$.

2. Let k be a positive number. The area of the region bounded by the curves $y = x^3 - x$ and $y = kx$ is 72. Determine the value of k.

   Answer. 11.

   Solution. First we find the points of intersection of the curve and the line. Setting $x^3 - x = kx$ and solving for x, we obtain $x = 0, \pm\sqrt{k+1}$. Since both are odd functions on $[-\sqrt{k+1}, \sqrt{k+1}]$, we have

   $$\begin{aligned}\text{Area} &= 2\int_0^{\sqrt{k+1}}(kx-(x^3-x))\,dx = 2\left[(k+1)\frac{x^2}{2}-\frac{x^4}{4}\right]_0^{\sqrt{k+1}}\\ &= \frac{1}{2}(k+1)^2 = 72.\end{aligned}$$

   Solving for k, we obtain $k = 11$.

3. An athlete leaves a given point O and runs north at 15 km per hour. One hour later, a car leaves the point O and travels east at 40 km per hour. At what rate in km per hour is the distance between the athlete and the car changing at the instant the car has been traveling for 1 hour?

Answer. 41.

Solution. Denote the athlete and the car by A and C respectively. At time t in hours, the distance of the athlete from the point O is $OA = 15t$ km, and the distance of the car from the point O is $OC = 40(t-1)$ km as the car starts one hour later. Note that AOC forms a right-angled triangle. By Pythagoras' theorem, the distance D between the athlete and the car at time t is $D = AC = \sqrt{OA^2+OC^2} = \sqrt{(15t)^2+(40(t-1))^2}$. Thus

$$\frac{dD}{dt} = \frac{2(15)^2t+2(40)^2(t-1)}{2\sqrt{(15t)^2+(40(t-1))^2}} = \frac{15^2t+40^2(t-1)}{\sqrt{(15t)^2+(40(t-1))^2}}.$$

Therefore, $\left.\dfrac{dD}{dt}\right|_{t=2} = \dfrac{15^2\times 2+40^2}{\sqrt{30^2+40^2}} = 41$ km per hour.

4. Durian trees grown in Malaysia produce 120 durians per tree per year if no more than 20 tree are planted per acre. For each additional tree planted per acre, the yield per tree per year decreases by 4 durians. Assuming that the maximum capacity per acre is 40 durian trees, how many trees per acre should be planted to produce the greatest number of durians per acre per year? Your answer should be a positive integer.

   Answer. 25.

   Solution. Let $F(x)$ be the number of durians produced by planting x durian trees per acre per year. We may assume $x \in [20,40]$. Each tree can produce $120-4(x-20)$ durians. Thus $F(x) = x(120-4(x-20)) = 200x-4x^2$, for $20 \le x \le 40$.

   We have $F'(x) = 200-8x$, and $F'(x) = 0 \Leftrightarrow 200-8x = 0 \Leftrightarrow x = 25$. Thus F has only 1 critical point at $x = 25$ inside the interval $[20,40]$. Then $F(20) = 2400, F(25) = 2500, F(40) = 1600$. Therefore, the maximum value of F is 2500 and is attained at $x = 25$.

5. The curve $\pi^2x^2 = y(2-y)^3$ has a shape like a teardrop . Find the area of the region enclosed by the curve.

   Answer. 1.

   Solution. The curve is symmetric about the y-axis since for any point (x,y) on the curve, the point $(-x,y)$ is also on the curve. It suffices to consider the curve with $x \ge 0$. Solving x in terms of y, we have $x = \frac{1}{\pi}\sqrt{y(2-y)^3}$. This is the equation of the curve for $x \ge 0$. Note that we need to have the condition $y(2-y)^3 \ge 0$ so that the square root is defined. That is $0 \le y \le 2$. Therefore, the area of the region bounded by the curve and the y-axis is

$$\int_0^2 \tfrac{1}{\pi}\sqrt{y(2-y)^3}\,dy = \tfrac{1}{\pi}\int_0^2 (2-y)\sqrt{y(2-y)}\,dy = \tfrac{1}{\pi}\int_0^2 (1-y+1)\sqrt{2y-y^2}\,dy$$
$$= \tfrac{1}{\pi}\int_0^2 (1-y)\sqrt{2y-y^2}\,dy + \tfrac{1}{\pi}\int_0^2 \sqrt{2y-y^2}\,dy = \tfrac{1}{3\pi}[(2y-y^2)^{\frac{3}{2}}]_0^2 + \tfrac{1}{\pi}\int_0^2 \sqrt{1-(y-1)^2}\,dy$$

$= 0 + \frac{1}{\pi}\int_0^2 \sqrt{1-(y-1)^2}\,dy = \frac{1}{\pi}\left[\frac{y-1}{2}\sqrt{1-(y-1)^2} + \frac{1}{2}\sin^{-1}(y-1)\right]_0^2$
$= \frac{1}{\pi}(0 + \frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(-1)) = \frac{1}{2}.$

Therefore the required area is $2 \times \frac{1}{2} = 1$.

6. Let $f(x) = \displaystyle\int_e^x \frac{1}{\ln t}\,dt$. Show that f^{-1} exists by proving that f is increasing on $(1, \infty)$. Find also the value of $(f^{-1})'(0)$.

   Answer. 1.

   Solution. By the fundamental theorem of calculus, $f'(x) = \frac{1}{\ln x} > 0$ for all $x > 1$. Hence, f is increasing on $(1, \infty)$. Since f is increasing, it is injective and f^{-1} exists. Moreover, $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$. Note that $f(e) = 0$. Consequently, $(f^{-1})'(0) = \frac{1}{f'(e)} = \frac{1}{\frac{1}{\ln e}} = 1$.

7. Let $p > 1$ be an integer. Suppose $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^p} = \frac{1}{6}$. Determine p.

   Answer. 7.

   Solution.

$$\int_e^\infty \frac{dx}{x(\ln x)^p} = \lim_{b\to\infty}\int_e^b \frac{d(\ln x)}{(\ln x)^p} = \lim_{b\to\infty}\left[\frac{1}{(1-p)(\ln x)^{p-1}}\right]_e^b$$
$$= \lim_{b\to\infty}\frac{1}{1-p}\left(\frac{1}{(\ln b)^{p-1}} - \frac{1}{(\ln e)^{p-1}}\right) = \frac{1}{p-1} = \frac{1}{6}.$$

   Thus $p = 7$.