

**CS1231S: Discrete Structures**  
**Tutorial #6: Functions**  
**(Week 8: 7 – 11 March 2022)**

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**1. Discussion Questions**

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. Which of the following is a function? If it is not a function, explain.

- (a) Define  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 3 \mid z. \end{cases}$
- (b) Define  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\forall z \in \mathbb{Z}, f(z) = \begin{cases} 1, & \text{if } 2 \mid z, \\ 2, & \text{if } 2 \nmid z. \end{cases}$
- (c) Define  $f: \mathbb{R} \rightarrow \mathbb{Z}$  by  $\forall x \in \mathbb{R}, f(x) = 2x$ .
- (d) Define  $f: \mathbb{Z} \rightarrow \mathbb{R}$  by  $\forall x \in \mathbb{Z}, f(x) = 2x$ .

D2. Let function  $f: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$  be defined by setting,  $\forall x, y \in \mathbb{Z}, f(x, y) = \frac{x+y}{3}$ .  
Find three distinct pre-images of 2.

D3. Definitions: Given any real number  $x$ ,

- (1) the **floor** of  $x$ , denoted  $\lfloor x \rfloor$ , is the unique integer  $n$  such that  $n \leq x < n + 1$ ;
- (2) the **ceiling** of  $x$ , denoted  $\lceil x \rceil$ , is the unique integer  $n$  such that  $n - 1 < x \leq n$ .

Let  $f, g: \mathbb{Q} \rightarrow \mathbb{Q}$  be defined by setting, for each  $x \in \mathbb{Q}$ ,

$$f(x) = \lfloor x \rfloor + 1 \quad \text{and} \quad g(x) = \lceil x \rceil.$$

What is the range of  $f$ ? What is the range of  $g$ ? Is  $f = g$ ? Why?

D4. To prove that a composition of two surjections is a surjection, Aiken wrote:

- 1. Suppose  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are surjections.
- 2. Then  $\forall y \in Y \exists x \in X$  such that  $f(x) = y$  as  $f$  is surjective,
- 3. and  $\forall z \in Z \exists y \in Y$  such that  $g(y) = z$  as  $g$  is surjective.
- 4. So  $(g \circ f)(x) = g(f(x)) = g(y) = z$ .
- 5. Hence  $g \circ f$  is a surjection.

Explain the mistakes in this “proof”.

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**Answers**

1. Define  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $g: \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = x + 3, \quad \text{and} \quad g(x) = -x, \quad \text{for all } x \in \mathbb{R}.$$

Prove that (a)  $f$  is a bijection, (b)  $g$  is a bijection, and (c)  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

**Answers:**

(a) To prove  $f$  is a bijection.

1. (To show  $f$  is injective.)

1.1. Let  $x_1, x_2 \in \mathbb{R}$  such that  $f(x_1) = f(x_2)$ .

1.2. Then  $x_1 + 3 = x_2 + 3$  by the definition of  $f$ .

1.3. Then  $x_1 = x_2$ .

1.4. Therefore,  $f$  is injective.

2. (To show  $f$  is surjective.)

2.1. Take any  $y \in \mathbb{R}$ .

2.2. Let  $x = y - 3$ .

2.3. Then  $f(x) = f(y - 3) = (y - 3) + 3 = y$ .

2.4. Therefore,  $f$  is surjective.

3. Therefore  $f$  is bijective (by 1.4 and 2.4).

(b) To prove  $g$  is a bijection.

1. (To show  $g$  is injective.)

1.1. Let  $x_1, x_2 \in \mathbb{R}$  such that  $g(x_1) = g(x_2)$ .

1.2. Then  $-x_1 = -x_2$  by the definition of  $g$ .

1.3. Then  $x_1 = x_2$ .

1.4. Therefore,  $g$  is injective.

2. (To show  $g$  is surjective.)

2.1. Take any  $y \in \mathbb{R}$ .

2.2. Let  $x = -y$ .

2.3. Then  $g(x) = g(-y) = -(-y) = y$ .

2.4. Therefore,  $g$  is surjective.

3. Therefore  $g$  is bijective (by 1.4 and 2.4).

(c)  $f^{-1}(x) = x - 3$ .

$$g^{-1}(x) = -x.$$

$$(g \circ f)(x) = g(f(x)) = g(x + 3) = -(x + 3) = -x - 3.$$

$$(g \circ f)^{-1}(x) = -x - 3.$$

$$(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(-x) = -x - 3 = (g \circ f)^{-1}(x).$$

2. Let  $A = \{a, b\}$  and  $S$  be the set of all strings over  $A$ . (See Lecture 7 for the definition of string.) Define a concatenate-by- $a$ -on-the-left function  $C: S \rightarrow S$  by  $C(s) = as$  for all  $s \in S$ .

- (a) Is  $C$  an injection? Prove or give a counterexample.  
(b) Is  $C$  a surjection? Prove or give a counterexample.

**Answers:**

- (a) Yes,  $C$  is an injection.

Proof:

1. Take any two strings  $s_1, s_2 \in S$  and suppose  $C(s_1) = C(s_2)$ .
2. Then  $as_1 = as_2$  by the definition of  $C$ .
  - 2.1. Let the length of  $as_1$  and  $as_2$  be  $n$ .
  - 2.2. Then the length of  $s_1$  and  $s_2$  is  $n - 1$ .
  - 2.3. Write  $s_1 = x_1x_2 \cdots x_{n-1}$  and  $s_2 = y_1y_2 \cdots y_{n-1}$  for  $x_i, y_i \in A$ , for all  $i \in \{1, 2, \dots, n - 1\}$ .
  - 2.4. Thus  $ax_1x_2 \cdots x_{n-1} = ay_1y_2 \cdots y_{n-1}$  by substitution.
  - 2.5. Thus  $x_i = y_i$  for all  $i \in \{1, 2, \dots, n - 1\}$  by string equality.
  - 2.6. Hence  $s_1 = s_2$ .
3. Therefore  $C$  is an injection.

- (b) No,  $C$  is not a surjection.

Proof:

1. Take the string  $b$  from  $S$ .
2. (Claim: There is no  $s \in S$  such that  $C(s) = b$ .)
3. Suppose not, i.e., suppose  $\exists s \in S$  such that  $C(s) = b$ .
  - 3.1. Then  $C(s) = as = b$  by the definition of  $C$ .
  - 3.2. Then  $as$  and  $b$  have the same length.
  - 3.3. Since  $a$  and  $b$  have length 1, this means  $s = \varepsilon$  (the empty string, which has a length of 0).
  - 3.4. But this means  $b = as = a\varepsilon = a$ .
  - 3.5. This is a contradiction since  $a \neq b$ .
4. Hence, for  $b$ , there is no  $s \in S$  such that  $C(s) = b$ .
5. Therefore  $C$  is not a surjection.

3. Let  $A = \{s, u\}$ . Define a function  $\text{len}: A^* \rightarrow \mathbb{Z}_{\geq 0}$  by setting  $\text{len}(\sigma)$  to be the length of  $\sigma$  for each  $\sigma \in A^*$ .
- (a) What is  $\text{len}(suu)$ ?
  - (b) What is  $\text{len}(\{\epsilon, ss, uu, ssss\})$ ?
  - (c) What is  $\text{len}^{-1}(\{3\})$ ?
  - (d) Does  $\text{len}^{-1}$  exist? Explain your answer.

**Answers:**

- (a)  $\text{len}(suu) = 3$ .
- (b)  $\text{len}(\{\epsilon, ss, uu, ssss\}) = \{0, 2, 4\}$ .
- (c)  $\text{len}^{-1}(\{3\}) = \{sss, ssu, sus, suu, uss, usu, uus, uuu\}$ .
- (d)  $\text{len}(s) = 1 = \text{len}(u)$  but  $s \neq u$ . So  $\text{len}$  is not injective and so it is not bijective. Thus  $\text{len}^{-1}$  does not exist by Theorem 7.2.3. (Recall that  $\text{len}^{-1}$  refers to the inverse function of  $\text{len}$ .)

4. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by  $\text{Bool}$  the set  $\{\text{true}, \text{false}\}$ .
- (a)  $f: \mathbb{Q} \rightarrow \mathbb{Q};$   
 $x \mapsto 12x + 31.$
  - (b)  $g: \text{Bool}^2 \rightarrow \text{Bool};$   
 $(p, q) \mapsto p \wedge \sim q.$
  - (c)  $h: \text{Bool}^2 \rightarrow \text{Bool}^2;$   
 $(p, q) \mapsto (p \wedge q, p \vee q).$
  - (d)  $k: \mathbb{Z} \rightarrow \mathbb{Z};$   
 $x \mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x - 1, & \text{if } x \text{ is odd.} \end{cases}$

**Answers:**

- (a) We could prove that  $f$  is injective and surjective, and hence bijective. Here, we try another approach: if we manage to find an inverse function for  $f$ , then by Theorem 7.2.3,  $f$  is bijective.
  1. Note that for all  $x, y \in \mathbb{Q}, y = 12x + 31 \Leftrightarrow x = (y - 31)/12$ .
  2. Define  $f^*: \mathbb{Q} \rightarrow \mathbb{Q}$  by setting, for all  $y \in \mathbb{Q}, f^*(y) = (y - 31)/12$ .
  3. Then whenever  $x, y \in \mathbb{Q}, y = f(x) \Leftrightarrow x = f^*(y)$ .
  4. Thus  $f^*$  is the inverse of  $f$ .
  5. Hence  $f$  is bijective (i.e. both injective and surjective) by Theorem 7.2.3.

(b)

1.  $g(\text{false}, \text{true}) = \text{false} = g(\text{false}, \text{false})$ , where  $(\text{false}, \text{true}) \neq (\text{false}, \text{false})$ .
2. So  $g$  is not injective.
3.  $g(\text{true}, \text{false}) = \text{true}$ .
4. Lines 1 and 3 show that every element of the codomain  $Bool$  is in the range of  $g$ .
5. Hence  $g$  is surjective.

(c)

1.  $h(\text{true}, \text{false}) = (\text{false}, \text{true}) = g(\text{false}, \text{true})$ , where  $(\text{true}, \text{false}) \neq (\text{false}, \text{true})$ .
2. So  $h$  is not injective.
3. If  $p, q, r \in Bool$  such that  $h(p, q) = (\text{true}, r)$ , then
  - 3.1.  $p \wedge q = \text{true}$  by the definition of  $h$ ;
  - 3.2.  $\therefore p = \text{true}$
  - 3.3.  $\therefore r = p \vee q = \text{true}$  by the definition of  $h$ .
4. So  $(\text{true}, \text{false})$  in the codomain is not in the range of  $h$ .
5. Hence  $h$  is not surjective.

(d)

1. We first show that if  $x$  is an even integer, then  $k(x)$  is even.
  - 1.1. Let  $x$  be an even integer.
  - 1.2. Then  $k(x) = x$  by the definition of  $k$ .
  - 1.3. So  $k(x)$  is even.
2. Next we show that if  $x$  is an odd integer, then  $k(x)$  is odd.
  - 2.1. Let  $x$  be an odd integer.
  - 2.2. Then  $k(x) = 2x - 1 = 2(x - 1) + 1$ , where  $x - 1$  is an integer.
  - 2.3. So  $k(x)$  is odd.
3. Since every integer is either even or odd but not both, lines 1 and 2 tell us that, for every  $x \in \mathbb{Z}$ ,
  - 3.1.  $x$  is even if and only if  $k(x)$  is even; and
  - 3.2.  $x$  is odd if and only if  $k(x)$  is odd.
4. Now we show that  $k$  is injective.
  - 4.1. Let  $x_1, x_2 \in \mathbb{Z}$  such that  $k(x_1) = k(x_2)$ .
  - 4.2. Case 1:  $k(x_1)$  is even.
    - 4.2.1. Then both  $x_1$  and  $x_2$  are even by line 3.1.
    - 4.2.2. So  $x_1 = k(x_1) = k(x_2) = x_2$  by the definition of  $k$ .
  - 4.3. Case 2:  $k(x_1)$  is odd.
    - 4.3.1. Then both  $x_1$  and  $x_2$  are odd by line 3.2.
    - 4.3.2. So  $2x_1 - 1 = k(x_1) = k(x_2) = 2x_2 - 1$  by the definition of  $k$ .
    - 4.3.3. So  $x_1 = x_2$ .
  - 4.4. Since  $k(x_1)$  is either even or odd, we conclude that  $x_1 = x_2$  in any case.
  - 4.5. Therefore  $k$  is injective.

5. Finally, we prove by contradiction that  $k$  is not surjective.
  - 5.1. Suppose  $k$  is surjective.
  - 5.2. Note that 3 is in the codomain  $\mathbb{Z}$ .
  - 5.3. Use the surjectivity of  $k$  to find  $x \in \mathbb{Z}$  such that  $k(x) = 3$ .
  - 5.4. Note that  $k(x) = 3$  is odd and so  $x$  is odd by line 3.2.
  - 5.5. Thus  $3 = k(x) = 2x - 1$  by the choice of  $x$  and the definition of  $k$ .
  - 5.6. Solving gives  $x = \frac{3+1}{2} = 2$  which is even.
  - 5.7. This contradicts line 5.4 that  $x$  is odd.
  - 5.8. Therefore  $k$  is not surjective.

5. We have shown in Theorem 7.3.3 that if  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are both injective, then  $g \circ f$  is injective.

Now, let  $f: B \rightarrow C$ . Suppose we have a function  $g$  with domain  $C$  such that  $g \circ f$  is injective. Show that  $f$  is injective.

**Answer:**

1. Suppose  $g$  is a function with domain  $C$  such that  $g \circ f$  is injective.
2. Let  $x_1, x_2 \in B$  such that  $f(x_1) = f(x_2)$ .
3. Then  $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$  by the definition of  $g \circ f$ .
4. So  $x_1 = x_2$  as  $g \circ f$  is injective by the choice of  $g$ .
5. Therefore  $f$  is injective.

6. We have shown in Theorem 7.3.4 that if  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are both surjective, then  $g \circ f$  is surjective.

Now, let  $f: B \rightarrow C$ . Suppose we have a function  $e$  with codomain  $B$  such that  $f \circ e$  is surjective. Show that  $f$  is surjective.

**Answer:**

1. Suppose  $e$  is a function with codomain  $B$  such that  $f \circ e$  is surjective.
2. Take any  $y \in C$ .
3. Apply the surjectivity of  $f \circ e$  to find  $w$  in the domain of  $e$  such that  $y = (f \circ e)(w)$ .
4. Let  $x = e(w)$ .
5. Then  $x \in B$  and  $y = (f \circ e)(w) = f(e(w)) = f(x)$  by the definition of  $f \circ e$ .
6. Therefore  $f \circ e$  is surjective.

7. Let  $A = \{1,2,3\}$ . The **order** of a bijection  $f: A \rightarrow A$  is defined to be the smallest  $n \in \mathbb{Z}^+$  such that

$$\underbrace{f \circ f \circ \cdots \circ f}_{n\text{-many } f\text{'s}} = id_A.$$

Define functions  $g, h: A \rightarrow A$  by setting, for all  $x \in A$ ,

$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \quad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of  $g, h, g \circ h$ , and  $h \circ g$ .

**Answers:** The orders are respectively 2, 2, 3 and 3.

8. Let  $f: A \rightarrow B$  be a function. Let  $X \subseteq A$  and  $Y \subseteq B$ .

- Is it always the case that  $X \subseteq f^{-1}(f(X))$ ? Is it always the case that  $f^{-1}(f(X)) \subseteq X$ ? Justify your answer.
- Is it always the case that  $Y \subseteq f(f^{-1}(Y))$ ? Is it always the case that  $f(f^{-1}(Y)) \subseteq Y$ ? Justify your answer.

**Answers:**

- First we show it is always that case that  $X \subseteq f^{-1}(f(X))$ .

- Let  $x \in X$ .
- Then  $f(x) \in f(X)$  by the definition of  $f(X)$ .
- So  $x \in f^{-1}(f(X))$  by the definition of  $f^{-1}(f(X))$ .

Next we show it is possible that  $f^{-1}(f(X)) \not\subseteq X$ .

- Consider  $f: \{-1,1\} \rightarrow \{0\}$  where  $f(-1) = 0 = f(1)$ , and  $X = \{1\}$ .
- Note that  $f(X) = \{f(1)\} = \{0\}$ .
- Since  $f(-1) = 0$ , we know  $-1 \in f^{-1}(\{0\}) = f^{-1}(f(X))$ .
- As  $-1 \notin \{1\} = X$ , we deduce that  $f^{-1}(f(X)) \not\subseteq X$ .

(Other counterexamples possible.)

- First we show it is always that case that  $f(f^{-1}(Y)) \subseteq Y$ .

- Take any  $y \in f(f^{-1}(Y))$ .
- Then we have some  $x \in f^{-1}(Y)$  such that  $y = f(x)$ , by the definition of  $f(f^{-1}(Y))$ .
- Now, as  $x \in f^{-1}(Y)$ , we get  $y' \in Y$  which makes  $y' = f(x)$ .
- Since  $f$  is a function, this implies  $y = f(x) = y' \in Y$  as required.

Next we show it is possible that  $Y \not\subseteq f(f^{-1}(Y))$ .

- Consider  $f: \{0\} \rightarrow \{-1,1\}$  where  $f(0) = 1$ , and  $Y = \{-1\}$ .
- Note that no  $x \in \{0\}$  makes  $f(x) = -1$ .
- So  $f^{-1}(Y) = \emptyset$  by the definition of  $f^{-1}(Y)$ .
- This entails  $f(f^{-1}(Y)) = \emptyset \not\subseteq \{-1\} = Y$ .

(Other counterexamples possible.)

9. [Optional question]

Consider the equivalence relation  $\sim$  on  $\mathbb{Q}$  defined by setting, for all  $x, y \in \mathbb{Q}$ ,

$$x \sim y \Leftrightarrow x - y \in \mathbb{Z}.$$

Define addition and multiplication on  $\mathbb{Q}/\sim$  as follows: whenever  $[x], [y] \in \mathbb{Q}/\sim$ ,

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] \cdot [y] = [x \cdot y].$$

(a) Is  $+$  well defined on  $\mathbb{Q}/\sim$ ?

(b) Is  $\cdot$  well defined on  $\mathbb{Q}/\sim$ ?

Prove that your answers are correct.

**Answers:**

(a) We claim that  $+$  is well defined on  $\mathbb{Q}/\sim$ .

1. Let  $[x_1], [y_1], [x_2], [y_2] \in \mathbb{Q}/\sim$  such that  $[x_1] = [x_2]$  and  $[y_1] = [y_2]$ .
2. So  $x_1 \sim x_2$  and  $y_1 \sim y_2$  by Lemma Rel.1.
3. Use the definition of  $\sim$  to find  $k, l \in \mathbb{Z}$  such that  $x_1 - x_2 = k$  and  $y_1 - y_2 = l$ .
4. Note that  $(x_1 + y_1) - (x_2 + y_2) = (x_1 - x_2) + (y_1 - y_2) = k + l \in \mathbb{Z}$ .
5. So  $x_1 + y_1 \sim x_2 + y_2$  by the definition of  $\sim$ .
6. Hence  $[x_1] + [y_1] = [x_1 + y_1] = [x_2 + y_2] = [x_2] + [y_2]$  by Lemma Rel.1.

(b) We claim that  $\cdot$  is not well defined on  $\mathbb{Q}/\sim$ .

1. Note that  $\frac{1}{2} - \frac{-1}{2} = 1 \in \mathbb{Z}$ .
2. This implies  $\frac{1}{2} \sim \frac{-1}{2}$  and so  $\left[\frac{1}{2}\right] = \left[\frac{-1}{2}\right]$  by Lemma Rel.1.
3. Note that  $\frac{1}{4} - \frac{-1}{4} = \frac{1}{2} \notin \mathbb{Z}$ .
4. This implies  $\frac{1}{4} \not\sim \frac{-1}{4}$  and so  $\left[\frac{1}{4}\right] \neq \left[\frac{-1}{4}\right]$  by Lemma Rel.1.
5. Therefore, according to the definition of  $\cdot$  on  $\mathbb{Q}/\sim$ ,

$$\left[\frac{1}{2}\right] \cdot \left[\frac{1}{2}\right] = \left[\frac{1}{2} \cdot \frac{1}{2}\right] = \left[\frac{1}{4}\right] \neq \left[\frac{-1}{4}\right] = \left[\frac{1}{2} \cdot \frac{-1}{2}\right] = \left[\frac{1}{2}\right] \cdot \left[\frac{-1}{2}\right].$$