

### 3. The Logic of Quantified Statements (aka Predicate Logic) Summary

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# Summary

## 3. The Logic of Quantified Statements

### 3.1 Predicates and Quantified Statements I

- Predicate; domain; truth set
- Universal quantifier  $\forall$ , existential quantifier  $\exists$
- Universal conditional statements; Implicit quantification

### 3.2 Predicates and Quantified Statements II

- Negation of quantified statements; negation of universal conditional statements
- Vacuous truth of universal statements
- Variants of universal conditional statements (contrapositive, converse, inverse)
- Necessary and sufficient conditions, only if

### 3.3 Statements with Multiple Quantifiers

- Negations of multiply-quantified statements; order of quantifiers
- Prolog

### 3.4 Arguments with Quantified Statements

- Universal instantiation; universal modus ponens; universal modus tollens

# Summary

## 3.1 Predicates and Quantified Statements I

### Definition 3.1.1 (Predicate)

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.

The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

### Definition 3.1.2 (Truth set)

If  $P(x)$  is a predicate and  $x$  has domain  $D$ , the **truth set** is the set of all elements of  $D$  that make  $P(x)$  true when they are substituted for  $x$ .

The truth set of  $P(x)$  is denoted  $\{x \in D \mid P(x)\}$ .

# Summary

## 3.1 Predicates and Quantified Statements I

### Definition 3.1.3 (Universal Statement)

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ .

A **universal statement** is a statement of the form “ $\forall x \in D, Q(x)$ ”.

- It is defined to be true iff  $Q(x)$  is true for every  $x$  in  $D$ .
- It is defined to be false iff  $Q(x)$  is false for at least one  $x$  in  $D$ .

A value for  $x$  for which  $Q(x)$  is false is called a **counterexample**.

### Definition 3.1.4 (Existential Statement)

Let  $Q(x)$  be a predicate and  $D$  the domain of  $x$ .

An **existential statement** is a statement of the form “ $\exists x \in D$  such that  $Q(x)$ ”.

- It is defined to be true iff  $Q(x)$  is true for at least one  $x$  in  $D$ .
- It is defined to be false iff  $Q(x)$  is false for all  $x$  in  $D$ .

$\exists!$  is the **uniqueness quantifier symbol**. It means “there exists a unique” or “there is one and only one”.

### Theorem 3.2.1 Negation of a Universal Statement

The **negation** of a statement of the form

$$\forall x \in D, P(x)$$

is logically equivalent to a statement of the form

$$\exists x \in D \text{ such that } \sim P(x)$$

Symbolically,

$$\sim(\forall x \in D, P(x)) \equiv \exists x \in D \text{ such that } \sim P(x)$$

### Theorem 3.2.2 Negation of an Existential Statement

The **negation** of a statement of the form

$$\exists x \in D \text{ such that } P(x)$$

is logically equivalent to a statement of the form

$$\forall x \in D, \sim P(x)$$

Symbolically,

$$\sim(\exists x \in D \text{ such that } P(x)) \equiv \forall x \in D, \sim P(x)$$

# Summary

## 3.2 Predicates and Quantified Statements II

### Definition 3.2.1 (Contrapositive, converse, inverse)

Consider a statement of the form:  $\forall x \in D (P(x) \rightarrow Q(x))$ .

1. Its **contrapositive** is:  $\forall x \in D (\sim Q(x) \rightarrow \sim P(x))$ .
2. Its **converse** is:  $\forall x \in D (Q(x) \rightarrow P(x))$ .
3. Its **inverse** is:  $\forall x \in D (\sim P(x) \rightarrow \sim Q(x))$ .

### Definition 3.2.2 (Necessary and Sufficient conditions, Only if)

- “ $\forall x, r(x)$  is a **sufficient condition** for  $s(x)$ ” means “ $\forall x (r(x) \rightarrow s(x))$ ”.
- “ $\forall x, r(x)$  is a **necessary condition** for  $s(x)$ ” means “ $\forall x (\sim r(x) \rightarrow \sim s(x))$ ” or, equivalently, “ $\forall x (s(x) \rightarrow r(x))$ ”.
- “ $\forall x, r(x)$  **only if**  $s(x)$ ” means “ $\forall x (\sim s(x) \rightarrow \sim r(x))$ ” or, equivalently, “ $\forall x (r(x) \rightarrow s(x))$ ” .

# Summary

## 3.4 Arguments with Quantified Statements

### Universal Modus Ponens

#### *Formal version*

$\forall x (P(x) \rightarrow Q(x)).$

$P(a)$  for a particular  $a$ .

- $Q(a)$ .

#### *Informal version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  makes  $P(x)$  true.

- $a$  makes  $Q(x)$  true.

### Universal Modus Tollens

#### *Formal version*

$\forall x (P(x) \rightarrow Q(x)).$

$\sim Q(a)$  for a particular  $a$ .

- $\sim P(a)$ .

#### *Informal version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  does not make  $Q(x)$  true.

- $a$  does not make  $P(x)$  true.

### Definition 3.4.1 (Valid Argument Form)

To say that **an argument form is valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An **argument is called valid** if, and only if, its form is valid.

# Summary

## 3.4 Arguments with Quantified Statements

### Converse Error (Quantified Form)

#### *Formal version*

$\forall x (P(x) \rightarrow Q(x)).$

$Q(a)$  for a particular  $a$ .

- $P(a)$ .

#### *Informal version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  makes  $Q(x)$  true.

- $a$  makes  $P(x)$  true.

### Inverse Error (Quantified Form)

#### *Formal version*

$\forall x (P(x) \rightarrow Q(x)).$

$\sim P(a)$  for a particular  $a$ .

- $\sim Q(a)$ .

#### *Informal version*

If  $x$  makes  $P(x)$  true, then  $x$  makes  $Q(x)$  true.

$a$  does not make  $P(x)$  true.

- $a$  does not make  $Q(x)$  true.

### Universal Transitivity

#### *Formal version*

$\forall x (P(x) \rightarrow Q(x)).$

$\forall x (Q(x) \rightarrow R(x)).$

- $\forall x (P(x) \rightarrow R(x)).$

#### *Informal version*

Any  $x$  that makes  $P(x)$  true makes  $Q(x)$  true.

Any  $x$  that makes  $Q(x)$  true makes  $R(x)$  true.

- Any  $x$  that makes  $P(x)$  true makes  $R(x)$  true.



# Summary

## 3.4 Arguments with Quantified Statements

Rule of Inference for quantified statements	Name
$\forall x \in D P(x)$ $\therefore P(a)$ if $a \in D$	Universal instantiation
$P(a)$ for every $a \in D$ $\therefore \forall x \in D P(x)$	Universal generalization
$\exists x \in D P(x)$ $\therefore P(a)$ for some $a \in D$	Existential instantiation
$P(a)$ for some $a \in D$ $\therefore \exists x \in D P(x)$	Existential generalization

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