NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST ANSWERS
AY2021/22 Semester 1

CS1231S — DISCRETE STRUCTURES

7 October 2021 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **SIXTEEN (16)** questions in **THREE (3)** parts and comprises **SIX (6)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions and write your answers only on the **ANSWER SHEETS**. You may write in pen or pencil.
5. The maximum mark of this assessment is 50.
6. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

1 **C**	2 **B**	3 **C**	4 **B**	5 **D**	6 **C**

7 **AD**	8 **ABD**	9 **ABC**	10 **AD**	11 **BD**	12 **C**	13 **ABC**

Part A: Multiple Choice Questions (Total: 12 marks)

Each multiple choice question (MCQ) is worth <u>two marks</u> and has exactly **one** correct answer.

1. Given this statement:

   "If Aiken or Dueet can do it, then all CS1231S students can do it."

   Suppose the above is true, which of the following is always true?

   A. "Aiken or Dueet are CS1231S students."
   B. "If Aiken can do it, then Dueet can do it."
   C. "If Aiken can do it, then all CS1231S students can do it."
   D. "If all CS1231S students can do it, then Aiken or Dueet can do it."
   E. None of (A), (B), (C), (D) is correct.

   Answer: C

   $\big((p \vee q) \rightarrow r\big) \rightarrow (p \rightarrow r)$ is a tautology.

2. Consider the predicate $P(x, y, z) \equiv$ "$xyz = 1$" for $x, y, z \in \mathbb{Q}^+$.

   Which of the following statements is/are true?

   (I) $\forall x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \forall z \in \mathbb{Q}^+\ P(x, y, z)$.
   (II) $\forall x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \exists z \in \mathbb{Q}^+\ P(x, y, z)$.
   (III) $\exists x \in \mathbb{Q}^+\ \forall y \in \mathbb{Q}^+\ \forall z \in \mathbb{Q}^+\ P(x, y, z)$.

   A. (I) only.
   B. (II) only.
   C. (III) only.
   D. (II) and (III) only.
   E. None of (A), (B), (C), (D) is correct.

   Answer: B

   (I) Counterexample: $x = 1, y = 1, z = 2$.

   (II) For any $x, y \in \mathbb{R}^+$, let $z = \frac{1}{xy}$, then $xyz = 1$.

   (III) Counterexample: For any x, let $y = \frac{1}{x}, z = 2$.

3. Which of the following statements is/are true?

   (I) $\mathcal{P}(\{\emptyset\}) = \mathcal{P}\big(\{\{\emptyset\}\}\big)$.
   (II) $|\mathcal{P}(\{\emptyset\})| = \big|\mathcal{P}\big(\{\{\emptyset\}\}\big)\big|$.

   A. Both (I) and (II) are true.
   B. (I) is true but (II) is not.
   C. (II) is true but (I) is not.
   D. Both (I) and (II) are not true.

   Answer: C

   $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$, while $\mathcal{P}\big(\{\{\emptyset\}\}\big) = \big\{\emptyset, \{\{\emptyset\}\}\big\}$. So $|\mathcal{P}(\{\emptyset\})| = 2 = \big|\mathcal{P}\big(\{\{\emptyset\}\}\big)\big|$.

As $\{\emptyset\} \in \mathcal{P}(\{\emptyset\})$ but $\{\emptyset\} \notin \mathcal{P}(\{\{\emptyset\}\})$, we see that $\mathcal{P}(\{\emptyset\}) \neq \mathcal{P}(\{\{\emptyset\}\})$.

4. Consider the congruence-mod-5 relation as an equivalence relation on $\mathbb{Z}$. Of which of the following sets is 1231 an element?

   A. $[0]$.
   B. $[1]$.
   C. $[2]$.
   D. $[3]$.
   E. $[4]$.

   Answer: B
   Note that $1231 = 5 \times 246 + 1$.

5. Define $f\colon \mathbb{Z} \to \mathbb{Z}_{\geqslant 0}$ and $g\colon \mathbb{Q} \to \mathbb{Q}_{\geqslant 0}$ by setting, for all $a \in \mathbb{Z}$ and all $x \in \mathbb{Q}$,

$$f(a) = \{a^2n^2 : n \in \mathbb{Z}\} \quad \text{and} \quad g(x) = x^2\sqrt{2}.$$

   Which of the following is true?

   A. f and g are both well defined.
   B. f is well defined but g is not.
   C. g is well defined but f is not.
   D. f and g are both not well defined.

   Answer: D
   $f(0) = \{0^2n^2 : n \in \mathbb{Z}\} = \{0\} \notin \mathbb{Z}_{\geqslant 0}$.
   $g(1) = 1^2\sqrt{2} = \sqrt{2} \notin \mathbb{Q}_{\geqslant 0}$.

6. Consider the equivalence relation $\sim$ on $\mathbb{Z}$ satisfying, for all $x, y \in \mathbb{Z}$,

$$x \sim y \quad \Leftrightarrow \quad x = y \text{ or } x = -y.$$

   Define two functions $f, g\colon \mathbb{Z}/{\sim} \to \mathbb{Z}/{\sim}$ by setting, for all $x \in \mathbb{Z}$,

$$f([x]) = [3x+1] \quad \text{and} \quad g([x]) = [x^4].$$

   Which of the following is true?

   A. f and g are both well defined.
   B. f is well defined but g is not.
   C. g is well defined but f is not.
   D. f and g are both not well defined.

   Answer: C
   $[1] = [-1]$ but $[3 \times 1 + 1] = [4] \neq [-2] = [3 \times (-1) + 1]$ as $4 \nsim -2$. So f is not well defined.
   Let us show that g is well defined.
   1. Let $x, y \in \mathbb{Z}$ such that $[x] = [y]$.
   2. Then $x \sim y$ by Lemma 6.4.4.
   3. So $x = y$ or $x = -y$ by the definition of $\sim$.
   4. If $x = y$, then $[x^4] = [y^4]$.
   5. If $x = -y$, then $[x^4] = [(-y)^4] = [y^4]$ too.
   6. So $[x^4] = [y^4]$ in all cases.

Part B: Multiple Response Questions [Total: 21 marks]

Each multiple response question (MRQ) is worth three marks and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C. Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

7. The floor and the ceiling of a real number x, denoted as $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively, are defined as follows:

   $\lfloor x \rfloor$ = the largest integer n such that $n \leq x$.

   $\lceil x \rceil$ = the smallest integer n such that $n \geq x$.

   Which of the following statements is/are true?

   A. $\forall x \in \mathbb{R}, \lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.
   B. $\forall x \in \mathbb{R}, \lceil x \rceil = \lfloor x \rfloor + 1$.
   C. $\forall x \in \mathbb{R}, \lfloor 2x \rfloor = 2\lfloor x \rfloor$.
   D. $\forall x \in \mathbb{R}, x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$.
   E. $\forall x, y \in \mathbb{R}, \lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$.

Answer: A, D

A. True. Suppose $x \in \mathbb{R}$. $\lfloor x \rfloor \in \mathbb{Z}$ (by definition of floor function).
   $\therefore$ $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ (by definition of ceiling, $\lceil a \rceil = a, \forall a \in \mathbb{Z}$).

B. False. Counterexample: $x = 0$. RHS = $\lfloor 0 \rfloor + 1 = 1 \neq 0 = \lceil 0 \rceil$ = LHS.

C. False. Counterexample: $x = 0.5$. LHS = $\lfloor 2 \times 0.5 \rfloor = 1 \neq 0 = 2\lfloor 0.5 \rfloor$ = RHS.

D. True.
   1. Suppose $x \in \mathbb{R}$.
   2. Case 1: $x \in \mathbb{Z}$.
      2.1 Then $\lfloor x \rfloor = \lceil x \rceil = x$ (by definitions of the ceiling and the floor).
      2.2 Note that $x - 1 < x < x + 1$ (by basic algebra).
      2.3 Then $x - 1 < \lfloor x \rfloor = \lceil x \rceil < x + 1$.
   3. Case 2: $x \notin \mathbb{Z}$.
      3.1 Then $n < x < n + 1$ for some integer n.
      3.2 Then $\lfloor x \rfloor = n$ and $\lceil x \rceil = n + 1$ (by definitions of the ceiling and the floor).
      3.3 Note that $n + 1 < x + 1$ and $x - 1 < n$ (by step 3.1).
      3.4 Then $x - 1 < \lfloor x \rfloor = n < n + 1 = \lceil x \rceil < x + 1$.
   4. In both cases, $x - 1 < \lfloor x \rfloor \leq \lceil x \rceil < x + 1$.

E. False. Counterexample: $x = y = 0.3$. LHS = $\lceil 0.3 + 0.3 \rceil = 1 \neq 2 = \lceil 0.3 \rceil + \lceil 0.3 \rceil$ = RHS.

8. Which of the following statements is/are equivalent to $(p \wedge q) \rightarrow q$?

   A. $p \rightarrow p$
   B. $(p \wedge q) \rightarrow p$
   C. $(p \vee q) \rightarrow q$
   D. $p \rightarrow (p \vee q)$
   E. $p \rightarrow (p \wedge q)$

   Answer: A, B, D (Given statement is a tautology)

9. To prove the statement $\forall x \in D\ \big(P(x) \rightarrow Q(x)\big)$, it is enough to prove that ______

   A. $\exists x \in D\ \big(P(x) \wedge {\sim}Q(x)\big) \rightarrow \exists y \in D\ \big(P(y) \wedge {\sim}P(y)\big)$
   B. $\forall x \in D\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$
   C. $\forall x \in D\ \Big(\big(P(x) \wedge {\sim}Q(x)\big) \rightarrow \big(P(x) \wedge {\sim}P(x)\big)\Big)$
   D. $\exists\, x \in D\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$

   Answer: A,B,C

   A : Proof by contradiction

$$\exists x \in D\ \big(P(x) \wedge {\sim}Q(x)\big) \rightarrow \exists y \in D\ \big(P(y) \wedge {\sim}P(y)\big)$$
$$\exists x \in D\ \big(P(x) \wedge {\sim}Q(x)\big) \rightarrow F \because P(y) \wedge {\sim}P(y)\ is\ a\ contradiction$$
$${\sim}\Big(\exists x \in D\ \big(P(x) \wedge {\sim}Q(x)\big)\Big)$$
$$\forall\, x \in D\ {\sim}P(x) \vee Q(x)$$
$$\forall\, x \in D\ P(x) \rightarrow Q(x)$$

   B : $\forall x \in D\ \big({\sim}Q(x) \rightarrow {\sim}P(x)\big)$ Proof by contraposition

   C : $\forall\, x \in D\ \Big(\big(P(x) \wedge {\sim}Q(x)\big) \rightarrow \big(P(x) \wedge {\sim}P(x)\big)\Big)$

$$\forall\, x \in D\ \big(P(x) \wedge {\sim}Q(x)\big) \rightarrow F \qquad \because P(x) \wedge {\sim}P(x)\ is\ a\ contradiction$$
$$\forall\, x \in D\ {\sim}\big(P(x) \wedge {\sim}Q(x)\big) \vee F$$
$$\forall\, x \in D\ {\sim}P(x) \vee Q(x)$$
$$\forall\, x \in D\ P(x) \rightarrow Q(x)$$

10. Let $A = \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$ and $B = \{x \in \mathbb{Q} : 1 \leqslant x \leqslant 2\}$ and $C = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 3\}$. Which of the following is a partition (or are partitions) of $\mathbb{Q}$?

   A. $\{B, \mathbb{Q} \setminus B\}$.
   B. $\{A \cap C, \mathbb{Q} \setminus (A \cap C)\}$.
   C. $\{A, \mathbb{Q} \setminus A, B, \mathbb{Q} \setminus B\}$.
   D. $\{A, C, (\mathbb{Q} \setminus A) \cap (\mathbb{Q} \setminus C)\}$.
   E. $\{A, B, C\}$.

Answer: A, D

   B: $A \cap C = \emptyset$.

   C: $A \cap B \neq \emptyset$.

   D: Note that $(\mathbb{Q} \setminus A) \cap (\mathbb{Q} \setminus C) = \mathbb{Q} \setminus (A \cup C)$, where $A \cap C = \emptyset$.

   E: $1231 \in \mathbb{Q}$ but $1231 \notin A \cup B \cup C$.

11. Let $A = \{3,4,5,6,7,8\}$. Which of the following is/are equal to $A/{\sim}$ for some equivalence relation $\sim$ on A?

    A. $\{\{1,2,3\}, \{4,5,6\}\}$.

    B. $\{\{3,4,5\}, \{6,7,8\}\}$.

    C. $\big\{\{\{3,4\}, \{5\}\}, \{6,7,8\}\big\}$.

    D. $\{\{3,4,5\}, \{6\}, \{7,8\}\}$.

    E. $\{3,4,5,6,7,8\}$.

Answer: B, D

In view of Theorem 6.4.9 and Tutorial 4 Question 7, this is equivalent to checking whether the given set is a partition of A.

A: $\{1,2,3\} \nsubseteq A$.

C: $\big\{\{3,4\}, \{5\}\big\} \nsubseteq A$ as $\{3,4\} \in \{\{3,4\}, \{5\}\}$ but $\{3,4\} \notin A$.

E: $3 \in \{3,4,5,6,7,8\}$ but $3 \nsubseteq A$.

12. Let $A = \{3,4,5,6,7,8\}$. Partially order A by the divisibility relation, i.e., consider the partial order $\preccurlyeq$ on A defined by setting, for all $a, b \in A$,

$$a \preccurlyeq b \quad \Leftrightarrow \quad \exists k \in \mathbb{Z}\ (b = ka).$$

Which of the following is/are equal to the set of all minimal elements in this partially ordered set?

    A. $\{x \in A : \exists k \in \mathbb{Z}\ (x = 2k + 1)\}$.

    B. $\{3\}$.

    C. $A \setminus \{x + x : x \in A\}$.

    D. $\{x \in \mathbb{Z} : \exists k \in \mathbb{Z}\ (420 = kx)\}$.

    E. $\{x \in A : x + x \in A\}$.

Answer: C

The set of all minimal elements is $\{3,4,5,7\}$, as one can see from the Hasse diagram below.

$$\begin{tikzcd} 6 \arrow[d, dash] & 8 \arrow[d, dash] & & \\ 3 & 4 & 5 & 7 \end{tikzcd}$$

A: This set is $\{3,5,7\}$, which does not contain the minimal element 4.

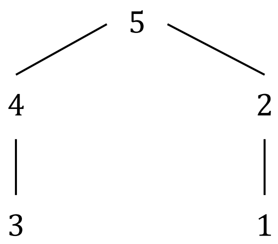
B: The set $\{3\}$ does not contain the minimal element 4.

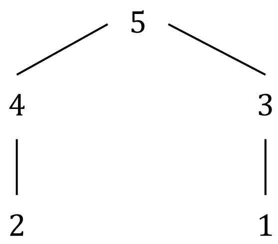
C: This set is $A \setminus \{6,8\} = \{3,4,5,7\}$.

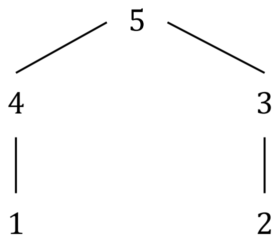
D: As $420 = 2 \times 210$, this set contains 2, but 2 is not an element of A.

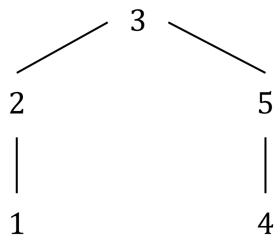
E: This set is $\{6,8\}$, which does not contain the minimal element 4.

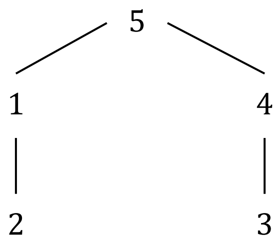
13. Which of the following is a Hasse diagram (or are Hasse diagrams) for a partial order of which the usual non-strict order $\leqslant$ on $\{1,2,3,4,5\}$ is a linearization?

A. 

B. 

C. 

D. 

E. 

Answer: A, B, C

D: 5 is below 3 in the Hasse diagram, but $\sim(5 \leqslant 3)$.

E: 2 is below 1 in the Hasse diagram, but $\sim(2 \leqslant 1)$.

Part C: There are 3 questions in this part [Total: 17 marks]

14. Theorem 2.1.1 is given as follows:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r$ $\equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r$ $\equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
5	Negation laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	${\sim}$**true** $\equiv$ **false**	${\sim}$**false** $\equiv$ **true**

Simplify the following expression using the laws above, justifying every step. You may combine consecutive steps using the same law in one step. [3 marks]

$$(p \wedge q) \vee (q \wedge r) \vee ({\sim}p \wedge r)$$

Answer:

$(p \wedge q) \vee (q \wedge r) \vee ({\sim}p \wedge r)$

$\equiv (p \wedge q) \vee \big((q \wedge r) \wedge \mathbf{true}\big) \vee ({\sim}p \wedge r)$ by identity law

$\equiv (p \wedge q) \vee \big((q \wedge r) \wedge (p \vee {\sim}p)\big) \vee ({\sim}p \wedge r)$ by negation law

$\equiv (p \wedge q) \vee \Big(\big((q \wedge r) \wedge p\big) \vee \big((q \wedge r) \wedge {\sim}p\big)\Big) \vee ({\sim}p \wedge r)$ by distributive law

$\equiv (p \wedge q) \vee \Big(\big(p \wedge (q \wedge r)\big) \vee \big({\sim}p \wedge (r \wedge q)\big)\Big) \vee ({\sim}p \wedge r)$ by commutative law (2 times)

$\equiv \Big((p \wedge q) \vee \big((p \wedge q) \wedge r\big)\Big) \vee \Big(\big(({\sim}p \wedge r) \wedge q\big) \vee ({\sim}p \wedge r)\Big)$ by associative law (5 times)

$\equiv (p \wedge q) \vee \Big(\big(({\sim}p \wedge r) \wedge q\big) \vee ({\sim}p \wedge r)\Big)$ by absorption law

$\equiv (p \wedge q) \vee \Big(({\sim}p \wedge r) \vee \big(({\sim}p \wedge r) \wedge q\big)\Big)$ by commutative law

$\equiv (p \wedge q) \vee ({\sim}p \wedge r)$ by absorption law

Alternative answer:

$(p \wedge q) \vee (q \wedge r) \vee (\sim p \wedge r)$

$\equiv \big((p \wedge q) \vee (q \wedge r) \vee (\sim p \wedge r)\big) \vee \text{false}$ by identity law

$\equiv \big((p \wedge q) \vee (q \wedge r) \vee (\sim p \wedge r)\big) \vee (p \wedge \sim p)$ by negation law

$\equiv \big((q \wedge p) \vee (q \wedge r) \vee (\sim p \wedge r)\big) \vee (\sim p \wedge p)$ by commutative law (2 times)

$\equiv \big((q \wedge p) \vee (q \wedge r)\big) \vee \big((\sim p \wedge r) \vee (\sim p \wedge p)\big)$ by associative law (3 times)

$\equiv \big(q \wedge (p \vee r)\big) \vee \big(\sim p \wedge (r \vee p)\big)$ by distributive law (2 times)

$\equiv \big((p \vee r) \wedge q\big) \vee \big((p \vee r) \wedge \sim p\big)$ by commutative law (3 times)

$\equiv (p \vee r) \wedge (q \vee \sim p)$ by distributive law

15. Prove that $(n^3 - n^2)$ is even for any positive integer n. [4 marks]

(You may quote the claim without proof that an integer is either odd or even but not both.)

Proof:

1. Take any positive integer n.
2. Case 1: n is even.
   2.1. Then $n = 2k$ for some integer k. (by definition of an even integer)
   2.2. Hence $n^3 - n^2 = (2k)^3 - (2k)^2 = 8k^3 - 4k^2 = 2(4k^3 - 2k^2) = 2m$, where $m = 4k^3 - 2k^2$. (by basic algebra)
   2.3. Since $m \in \mathbb{Z}$ (by closure of integers under $\times$ and $+$), $n^3 - n^2$ is even. (by definition of an even integer)
3. Case 2: n is odd.
   3.1. Then $n = 2k + 1$ for some integer k. (by definition of an odd integer)
   3.2. Hence $n^3 - n^2 = (2k+1)^3 - (2k+1)^2 = (8k^3 + 12k^2 + 6k + 1) - (4k^2 + 4k + 1) = 8k^3 + 8k^2 + 2k = 2(4k^3 + 4k^2 + k) = 2p$, where $p = 4k^3 + 4k^2 + k$. (by basic algebra)
   3.3. Since $p \in \mathbb{Z}$ (by closure of integers under $\times$ and $+$), $n^3 - n^2$ is even. (by definition of an even integer)
4. In all cases, $n^3 - n^2$ is even.

16. Let $A = \{1,2,3,4,5,6\}^2$. Define a relation R on A by setting, for all $(a_1, a_2), (b_1, b_2) \in A$,

$$(a_1, a_2)\ R\ (b_1, b_2) \quad \Leftrightarrow \quad \left|\left\{(i,j) \in \{1,2\}^2 : a_i \leqslant b_j\right\}\right| \geqslant 2.$$

(Hint: the number 2 on the right-hand side of the inequality above is equal to $|\{1,2\}^2|/2$.)

(a) Is R reflexive? [3 marks]

(b) Is R symmetric? [2 marks]

(c) Is R antisymmetric? [2 marks]

(d) Is R transitive? [3 marks]

For each part, if your answer is yes, then give a proof; if your answer is no, then give a counterexample.

Answer

(a) Yes, as shown below.

1. Take any $(a_1, a_2) \in A$.
2. As $a_1 \leqslant a_1$ and $a_2 \leqslant a_2$, we know $(1,1)$ and $(2,2)$ are 2 distinct elements of $\{(i,j) \in \{1,2\}^2 : a_i \leqslant a_j\}$.
3. So $(a_1, a_2)\ R\ (a_1, a_2)$.

(b) No. One counterexample: $(1,2)\ R\ (3,4)$ but $\sim\big((3,4)\ R\ (1,2)\big)$.

(c) No. One counterexample: $(1,2)\ R\ (2,1)$ and $(2,1)\ R\ (1,2)$, but $(1,2) \neq (2,1)$.

(d) No. One counterexample: $(2,6)\ R\ (3,4)$ and $(3,4)\ R\ (1,5)$, but $\sim\big((2,6)\ R\ (1,5)\big)$.

Generalizations: search for "intransitive dice"

=== END OF PAPER ===