Solutions to Practice Exam

MA1521 Calculus for Computing

Time allowed: 2 hours

Answer all 10 questions. Each question carries 10 marks.

Show your steps clearly.

1. The plane that passes through the point $(1,-1,1)$ and contains the line with parametric equations $x = t, y = t/2, z = t/3$ has an equation of the form $9z = ax + by$. Determine the value of $a+b$.

   Answer. 1.

   Solution. Since the plane contains the line $x = t, y = t/2, z = t/3$ which passes through the origin when the parameter $t = 0$, the plane also contains the origin. Therefore, a normal vector to the plane is given by

   $\langle 1,-1,1\rangle \times \langle 1, \frac{1}{2}, \frac{1}{3}\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} \end{vmatrix} = \langle -5, 4, 9\rangle$. Therefore, an equation of the plane is $-5x + 4y + 9z = 0 \Leftrightarrow 9z = 5x - 4y$. Thus $a + b = 5 - 4 = 1$.

2. Determine whether the following series converges or diverges. Justify your answers.

   (a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n - \ln n}$

   Solution. (a) $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1$. Thus by root test, the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$ converges.

   (b) By L'Hôpital's rule, $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{x} = \lim_{x\to\infty} \frac{1}{x^2} = 0$.

   Thus $\displaystyle\lim_{n\to\infty} \frac{\ln n}{n - \ln n} = \lim_{x\to\infty} \frac{\ln x}{x - \ln x} = \lim_{x\to\infty} \frac{\frac{\ln x}{x}}{1 - \frac{\ln x}{x}} = \lim_{n\to\infty} \frac{0}{1-0} = 0$.

   Next $\frac{d}{dx}\frac{\ln x}{x - \ln x} = \frac{\frac{1}{x}(x - \ln x) - \ln x(1 - \frac{1}{x})}{(x-\ln x)^2} = \frac{1 - \ln x}{(x - \ln x)^2} < 0$ for $x \geq 3$.

   Thus the sequence $\{\frac{\ln n}{n - \ln n}\}$ is decreasing for $n \geq 3$.

By alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n - \ln n}$ converges.

3. Find the number of saddle points of the function $f(x,y) = 3x^2y + x^2 - 6x - 3y - 2$.

   Answer. 2.

   Solution. $f_x = 6xy + 2x - 6, f_y = 3x^2 - 3$. We need to solve the system of equations $6xy + 2x - 6 = 0, 3x^2 - 3 = 0$. The second equation gives $x = -1, 1$. When $x = -1$, we have by the first equation, $-6y - 2 - 6 = 0$ so that $y = -\frac{4}{3}$. When $x = 1$, we have by the first equation, $6y + 2 - 6 = 0$ so that $y = \frac{2}{3}$. Thus we have two critical points $(-1, -\frac{4}{3})$ and $(1, \frac{2}{3})$.

   To apply the second derivative test, we compute $f_{xx} = 6y + 2, f_{yy} = 0, f_{xy} = 6x$, and $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -36x^2$. We have $D(-1, -\frac{4}{3}) = -36 < 0$ and $D(1, \frac{2}{3}) = -36 < 0$. Thus f has a saddle point at each of the points $(-1, -\frac{4}{3})$ and $(1, \frac{2}{3})$.

4. Let $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$. It is known that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{k}{f}$, where k is a positive integer. Determine the value of k.

   Answer. 2.

   Solution. $f_x = x/f$ and $f_{xx} = (f - xf_x)/f^2 = (f - x^2/f)/f^2 = (y^2 + z^2)/f^3$. Similarly, $f_{yy} = (z^2 + x^2)/f^3$ and $f_{zz} = (y^2 + z^2)/f^3$. Therefore, $f_{xx} + f_{yy} + f_{zz} = 2/f$. Thus $k = 2$.

5. Given $\int_{-2}^{0} \int_{0}^{x^2} e^{y - \frac{1}{3}y^{\frac{3}{2}}}\, dy dx = ae^{\frac{4}{3}} + b$, where a and b are integers, determine the value of $a + b$.

   Answer. 0.

   Solution. The region of integration is

$$R = \{(x,y) \mid 0 \le y \le x^2,\ -2 \le x \le 0\} = \{(x,y) \mid -2 \le x \le -\sqrt{y},\ 0 \le y \le 4\}.$$

   Thus $\int_{-2}^{0} \int_{0}^{x^2} e^{y - \frac{1}{3}y^{\frac{3}{2}}}\, dy dx = \int_{0}^{4} \int_{-2}^{-\sqrt{y}} e^{y - \frac{1}{3}y^{\frac{3}{2}}}\, dx dy = \int_{0}^{4} \left[x e^{y - \frac{1}{3}y^{\frac{3}{2}}} \right]_{x=-2}^{x=-\sqrt{y}} dy$

   $= \int_{0}^{4} (2 - y^{\frac{1}{2}}) e^{y - \frac{1}{3}y^{\frac{3}{2}}}\, dy = \int_{0}^{4} 2e^{y - \frac{1}{3}y^{\frac{3}{2}}}\, d\left(y - \frac{1}{3}y^{\frac{3}{2}}\right) = 2\left[e^{y - \frac{1}{3}y^{\frac{3}{2}}} \right]_{0}^{4} = 2e^{\frac{4}{3}} - 2.$

   Thus $a = 2, b = -2$ and $a + b = 0$.

6. Let R be the circular region bounded by the circle $x^2 + (y-1)^2 = 1$. It is known that

$$\iint_R \frac{dA}{(1 + 2x^2 + 2y^2)^2} = \frac{\pi}{a},$$

where a is a positive integer. Determine the value of a.

[Hint: Use polar coordinates and evaluate the resulting integral by means of the substitution $t = \tan\theta$].

Answer. 6.

Solution. First note that the region R is above the x-axis. That means all points inside R have their y-coordinates nonnegative. Using polar coordinates, $x^2+(y-1)^2 = 1 \Leftrightarrow x^2+y^2-2y = 0 \Leftrightarrow r^2-2r\sin\theta = 0 \Leftrightarrow r = 2\sin\theta$, $0 \le \theta \le \pi$. Thus we may describe R in polar coordinates as

$$R = \{(r,\theta) \mid 0 \le r \le 2\sin\theta, 0 \le \theta \le \pi\}.$$

The integrand $\frac{1}{(1+2x^2+2y^2)^2}$ and the region of integration R are both symmetric with respect to the y axis. We can simply find the integral over the right semicircular disk and multiply the result by 2.

Hence, $\iint_R \frac{dA}{(1+2x^2+2y^2)^2} = 2\int_0^{\frac{\pi}{2}}\int_0^{2\sin\theta} \frac{1}{(1+2r^2)^2}\, r dr d\theta = 2\int_0^{\frac{\pi}{2}} \left[-\frac{1}{4(1+2r^2)}\right]_0^{2\sin\theta} d\theta$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1+8\sin^2\theta}\right) d\theta = \frac{\pi}{4} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+8\sin^2\theta}\, d\theta. \qquad (*)$$

Let's evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+8\sin^2\theta} d\theta$. Let $t = \tan\theta$. Then $dt = \sec^2\theta d\theta = (1+\tan^2\theta)d\theta = (1+t^2)d\theta$. Thus $d\theta = \frac{1}{1+t^2}dt$. Also $\sin^2\theta = 1-\cos^2\theta = 1 - \frac{1}{\sec^2\theta} = 1 - \frac{1}{1+\tan^2\theta} = \frac{\tan^2\theta}{1+\tan^2\theta} = \frac{t^2}{1+t^2}$.

Thus $\int_0^{\frac{\pi}{2}} \frac{1}{1+8\sin^2\theta}\, d\theta = \int_0^{\infty} \frac{1}{1+\frac{8t^2}{1+t^2}} \frac{1}{1+t^2}\, dt = \int_0^{\infty} \frac{1}{1+9t^2}\, dt = \lim_{b\to\infty}\left[\frac{1}{3}\tan^{-1}(3t)\right]_0^b = \frac{\pi}{6}$.

Finally by (*), the value of the integral is $\frac{\pi}{4} - \frac{1}{2}\times\frac{\pi}{6} = \frac{\pi}{6}$. Therefore, $a = 6$.

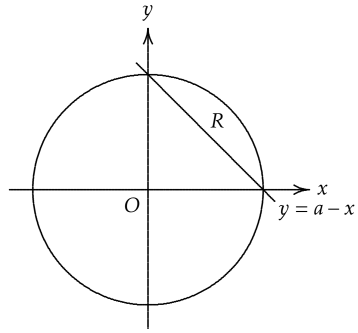
7. Let a be a positive number. Let R be the smaller cap region of the circular disk $x^2+y^2 \le a^2$ cut off by the line $x+y = a$. Suppose $\displaystyle\iint_R xy^2\, dA = 5000$. Determine the value of a.

   Answer. 10.

   Solution. $\displaystyle\iint\limits_R xy^2\, dydx = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} xy^2\, dydx = \int_0^a \left[\frac{1}{3}xy^3\right]_{y=a-x}^{y=\sqrt{a^2-x^2}} dx$

   $\displaystyle= \int_0^a \frac{1}{3}x(a^2-x^2)^{3/2} - \frac{1}{3}x(a-x)^3\, dx$

   $\displaystyle= \left[-\frac{1}{15}(a^2-x^2)^{5/2} - \frac{1}{6}a^3x^2 + \frac{1}{3}a^2x^3 - \frac{1}{4}ax^4 + \frac{1}{15}x^5\right]_0^a = \frac{a^5}{20}.$



Thus $a^5 = 20 \times 5000 = 100000$ so that $a = 10$.

8. Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^2}, \text{ with } x > 0, y > 0 \text{ and } y(1) = \sqrt{\frac{5}{7}}.$$

Find the value of $y(\frac{1}{10})$. Give your answer correct to two decimal places.

Answer. 0.50.

Solution. This is a Bernoulli equation with $n = 3$. Let $u = y^{-2}$. The equation becomes

$$u' - \frac{4}{x}u = -\frac{2}{x^2}.$$

This is a first order linear differential equation. An integrating factor is $e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4}$. Multiplying through the above equation by x^{-4}, we have

$$(x^{-4}u)' = -\frac{2}{x^6}.$$

Integrating, $x^{-4}u = \frac{2}{5x^5} + C$. That is $u = Cx^4 + \frac{2}{5x}$. Therefore, $y = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{Cx^4 + \frac{2}{5x}}}$. Since $y(1) = \sqrt{\frac{5}{7}}$, we have $\sqrt{\frac{5}{7}} = \frac{1}{\sqrt{C + \frac{2}{5}}}$ so that $C = 1$. Consequently, $y = \frac{1}{\sqrt{x^4 + \frac{2}{5x}}}$. Therefore, $y(\frac{1}{10}) = \frac{1}{\sqrt{(\frac{1}{10})^4 + \frac{2}{5(\frac{1}{10})}}} = \frac{1}{\sqrt{10^{-4} + 4}} = 0.50$.

9. You started an experiment with 100 mg of a radioactive substance X which has a half life of 30 minutes. After 0.86 hour, you had m mg of X left. Find the value of m. Give your answer correct to the nearest integer.

Answer. 30.

Solution. Let y in mg be the amount of the substance X at time t in minutes. We have $y = 100e^{-\frac{\ln 2}{T}t}$, where T is the half-life. That is $y = 100e^{-\frac{\ln 2}{30}t}$. Therefore, $m = 100e^{-\frac{\ln 2}{30}0.86\times 60} = 100e^{-1.72\ln 2} = 100(2)^{-1.72} = 30.35 \approx 30$ to the nearest integer.

10. The growth of the sandhill crane population follows a logistic model (the modified Malthus model) with a birth rate per capita of 10% per year. Initially at time $t = 0$ there were 1521 sandhill cranes. It is known that at time $t = 10$ years there were 2019 sandhill cranes. How many sandhill cranes will there be after a very long time? Give your answer correct to the nearest integer.

Answer. 2494.

Solution. Let $N(t)$ be the number of sandhill crane at time t in years. The logistic model gives $\frac{1}{N} = \frac{s}{B} + Ce^{-Bt}$, where s, C are constants and $B = 10\% = 0.1$ is the birth rate per capita. Thus $\frac{1}{N} = 10s + Ce^{-t/10}$, $t \geq 0$.

$N(0) = 1521 \Rightarrow \frac{1}{1521} = 10s + C$, $N(10) = 2019 \Rightarrow \frac{1}{2019} = 10s + Ce^{-1}$. Subtracting the two equations, we obtain

$\frac{1}{1521} - \frac{1}{2019} = C(1 - e^{-1}) \Rightarrow C = \frac{498}{1521 \times 2019 \times (1-e^{-1})} = \frac{166}{1023633(1-e^{-1})} = 0.000256545.$

Therefore, $10s = \frac{1}{1521} - C$. As $t \to \infty$, $N \to N_\infty = \frac{1}{10s} = \frac{1}{\frac{1}{1521} - C} = \frac{1}{\frac{1}{1521} - \frac{166}{1023633(1-e^{-1})}}$ $= 2494.28 = 2494$ to the nearest integer.