MA1301 INTRODUCTORY MATHEMATICS

TEST REVISION

September/October 2021 – Time allowed: 1 hour

Question 1 [10 marks]

(a) Given that $100x^2 - 90xy + 36y^2 = 1575$, —————— (1)

(i) find $\frac{dy}{dx}$ in terms of x and y, [2]

$$200x - 90y - 90x\frac{dy}{dx} + 72y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{90y - 200x}{72y - 90x}$$

$$= \frac{45y - 100x}{36y - 45x}$$

(ii) find the equations of the tangents parallel to the x − axis. [3]

For tangent ∥ to x-axis, $\frac{dy}{dx} = 0$

$$\therefore \frac{45y - 100x}{36y - 45x} = 0$$

$$45y - 100x = 0$$

$$9y - 20x = 0$$

$$x = \frac{9}{20}y \quad — (2)$$

Solving (1) and (2),

$$y = \pm 10$$

Question 1 [continued]

(b) A curve is defined by the parametric equations

$$x = \frac{6k}{t^2}, \quad y = 12kt^2, \qquad x = 6kt^{-2}$$

where $t > 0$ and k is a non-zero constant.

(i) Find $\frac{dy}{dx}$. [2]

$$\frac{dx}{dt} = 6k(-2)t^{-3} \qquad \frac{dy}{dt} = 24kt$$

$$= \frac{-12k}{t^3}$$

$$\frac{dy}{dx} = 24kt \div \frac{-12k}{t^3}$$

$$= -24kt \times \frac{t^3}{12k} = -2t^4$$

(ii) Find the value of k for which $\frac{d^2y}{dx^2} = 2017$ when $t = 1$. [3]

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}(-2t^4) \times \frac{t^3}{-12k}$$

$$= \frac{-8t^3 \times t^3}{-12k}$$

$$= \frac{2t^6}{3k}$$

put $t=1$, $\frac{2(1)^6}{3k} = 2017$

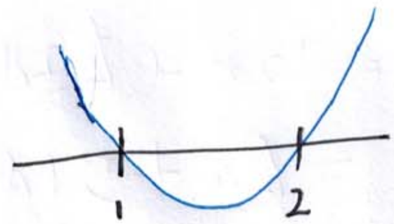
$$k = \frac{2}{6051}$$

Question 2 [10 marks]

Let $f(x) = 8x^3 - 36x^2 + 48x - 7$, where $x \in \mathbf{R}$.

(i) Find the intervals on which f is

(a) increasing (b) decreasing . [2]



$$f'(x) = 24x^2 - 72x + 48$$
$$= 24(x^2 - 3x + 2)$$
$$= 24(x-1)(x-2)$$

$f'(x) = 0$

$x = 1, 2$

(a) $x < 1$ or $x > 2$ (b) $1 < x < 2$

(ii) Find the coordinates and nature (local maximum or local minimum or saddle point) of the stationary points of the curve $y = f(x)$. [4]

$$f''(x) = 48x - 72$$

$f(1) = 13$ $f''(1) = 48 - 72 < 0 \Rightarrow$ max point

$f(2) = 9$ $f''(2) = 96 - 72 > 0 \Rightarrow$ min point

$(1, 13)$ max point

$(2, 9)$ min point

Question 2 [continued]

(iii) Find the interval on which the graph of f is

(a) concave down (b) concave up. [2]

$f''(x) = 48x - 72$. $f''(x) = 0$ $x = \frac{3}{2}$

(a) $x < \frac{3}{2}$

(b) $x > \frac{3}{2}$

(iv) Write down the coordinates of the inflexion point of the curve $y = f(x)$. [2]

$\left(\frac{3}{2}, 11\right)$

Question 3 [16 marks]

(a)(i) Show that for any positive integer n,

$$n(n+1)(n+2)-(n-1)(n)(n+1)=3n(n+1). \quad [2]$$

$L.H.S = n(n+1)(n+2)-(n-1)(n)(n+1)$

$= n(n+1)\{n+2-(n-1)\}$

$= n(n+1)(n+2-n+1) = 3n(n+1) = R.H.S$

(a)(ii) Use the result in (a)(i) to find the sum:

$$1017\times1018+1018\times1019+1019\times1020+\ldots+2016\times2017. \quad [4]$$

$=(1017)(1017+1)+(1018)(1018+1)+\ldots+(2016)(2016+1)$

$=\frac{1}{3}\sum_{n=1017}^{2016} 3n(n+1)$

$=\frac{1}{3}\sum_{n=1017}^{2016}\{n(n+1)(n+2)-(n-1)(n)(n+1)\}$

left

$=\frac{1}{3}\{(1017)\cancel{(1018)(1019)}-(1016)(1017)(1018)$

$+\cancel{(1018)(1019)(1020)}-\cancel{(1017)}(1018)(1019)$

$+ \vdots$

left

$+(2016)(2017)(2018)-\cancel{(2015)}(2016)(2017)\}$

$=\frac{1}{3}\{(2016)(2017)(2018)-(1016)(1017)(1018)\}$

Question 3 [continued]

(b) The first three terms of a geometric progression are $x+5, x+1$ and x. Calculate

(i) the value of x, [2]

$$r = \frac{x+1}{x+5} = \frac{x}{x+1}$$

$$(x+1)^2 = x(x+5)$$

$$x^2+2x+1 = x^2+5x$$

$$1 = 3x$$

$$x = \frac{1}{3}$$

(ii) the common ratio, [1]

$$r = \frac{x}{x+1} = \frac{\frac{1}{3}}{\frac{1}{3}+1} = \frac{1}{4}$$

(iii) the sum to infinity of the geometric progression. [2]

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{16}{3}}{1-\frac{1}{4}}$$

$$= \frac{\frac{16}{3}}{\frac{3}{4}}$$

$$= \frac{64}{9}$$

$$a = T_1$$

$$= x+5$$

$$= \frac{1}{3}+5$$

$$= \frac{16}{3}$$

Question 3 [continued]

(c) An arithmetic progression has 12 terms. The sum of the last six terms is three times the sum of the first 5 terms. Find the ratio of the sixth term to the fourth term. [5]

$$S_{12} - S_6 = \frac{12}{2}\{2a+(12-1)d\} - \frac{6}{2}\{2a+(6-1)d\}$$

$$= 6(2a+11d) - 3(2a+5d)$$

$$= 6a+51d$$

$$S_5 = \frac{5}{2}\{2a+(5-1)d\} = 5a+10d$$

$$S_{12} - S_6 = 3S_5$$

$$6a+51d = 3(5a+10d)$$

$$21d = 9a$$

$$a = \frac{7d}{3}$$

$$\frac{T_6}{T_4} = \frac{a+5d}{a+3d} = \frac{\frac{7d}{3}+\frac{15d}{3}}{\frac{7d}{3}+\frac{9d}{3}} = \frac{22d}{16d} = \frac{11}{8}$$

Question 4 [14 marks]

(a) Given that the term which is independent of x in the binomial expansion of $\left(x^2 + \frac{k}{x}\right)^6$ is 240, calculate the possible values of k. [6]

$$\binom{6}{r}(x^2)^{6-r}\left(\frac{k}{x}\right)^r$$

$$= \binom{6}{r} x^{12-2r} k^r \frac{1}{x^r}$$

$$= \binom{6}{r} x^{12-2r} k^r x^{-r}$$

$$= \binom{6}{r} x^{12-3r} k^r$$

$$\therefore x^{12-3r} = x^0$$

$$12 - 3r = 0$$

$$r = 4$$

$$\binom{6}{4} k^4 = 240$$

$$15k^4 = 240$$

$$k^4 = 16$$

$$k = \pm 2$$

Question 4 [continued]

(b) Find the values of a, b, c and d if the binomial expansion, in ascending powers of x, up to x^4 term, of $\sqrt{\frac{1+x}{1-x}}$ is $1+ax+bx^2+cx^3+dx^4$. [8]

$$\sqrt{\frac{1+x}{1-x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots$$

$$(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\frac{35}{128}x^4+\ldots$$

$$\therefore (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{3}{8}x^4+\ldots$$

Method 2

$$\frac{\sqrt{1+x}}{\sqrt{1-x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}} = \frac{\sqrt{(1-x^2)}}{1-x} = (1-x^2)^{\frac{1}{2}}\left(\frac{1}{1-x}\right)$$

$$(1-x^2)^{\frac{1}{2}} = 1+\frac{1}{2}(-x^2)+\frac{\frac{1}{2}(-\frac{1}{2})(-x^2)^2}{2}+\ldots$$

$$= 1-\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots$$

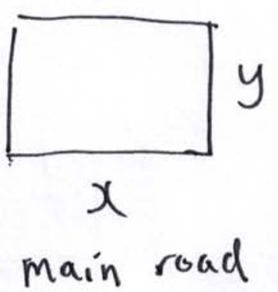
$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\ldots$$

$$\therefore (1-x^2)^{\frac{1}{2}}\left(\frac{1}{1-x}\right) = \left(1-\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots\right)\left(1+x+x^2+x^3+x^4+\ldots\right)$$

$$= 1+x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{3}{8}x^4+\ldots$$

Question 5 [10 marks]

An event organizer needs to build a fence to enclose a rectangular region of area 2400 square meters. As one side of the region is facing a main road, the organizer decides to make that side more attractive by using higher quality fencing that costs \$6 per meter. For the other three sides, he intends to use fencing that costs \$3 per meter. What dimensions of the rectangular region will minimize the cost of the fence?



$$xy = 2400$$

$$y = \frac{2400}{x}$$

$$\text{Cost } C = 6x + 3y + 3x + 3y$$

$$= 9x + 6y$$

$$= 9x + 6\left(\frac{2400}{x}\right)$$

$$= 9x + \frac{14400}{x}$$

$$\frac{dC}{dx} = \frac{d}{dx}\left(9x + \frac{14400}{x}\right)$$

$$= 9 - \frac{14400}{x^2}$$

$$\frac{dC}{dx} = 0 \qquad 9 = \frac{14400}{x^2}$$

$$x^2 = 1600$$

$$x = 40$$

$$y = 60$$