Revision
Past Years' Exams

13-14 April 2022

Aaron Tan

(AY2021/22 Sem1)

20. Define sequence $a_1, a_2, a_3, \cdots$ by setting, for each $n \in \mathbb{Z}_{\geq 1}$,

$$a_1 = 1 \text{ and } a_{n+1} = a_n + \frac{(n+1)(n+2)(n+3)(n+4)}{4!}$$

Prove by induction: for every $n \in \mathbb{Z}_{\geq 1}$, $a_n = \frac{(n+4)!}{5!(n-1)!}$

1. For each $n \in \mathbb{Z}_{\geq 1}$, let $P(n) \equiv a_n = \frac{(n+4)!}{5!(n-1)!}$
2. (Base step) $P(1)$ is true because $\frac{(1+4)!}{5!(1-1)!} = \frac{5!}{5!0!} = 1 = a_1$.
3. (Induction step)

   3.1. Let $k \in \mathbb{Z}_{\geq 1}$ such that $P(k)$ is true, i.e. $a_k = \frac{(k+4)!}{5!(k-1)!}$.

   3.2. Then $a_{k+1} = a_k + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the defn of $a_1, a_2, a_3, \cdots$;

   3.3. $= \frac{(k+4)!}{5!(k-1)!} + \frac{(k+1)(k+2)(k+3)(k+4)}{4!}$ by the induction hypothesis;

   3.4. $= \frac{(k+4)!}{5!(k-1)!}\left(1 + \frac{5}{k}\right) = \frac{(k+4)!}{5!(k-1)!}\left(\frac{k+5}{k}\right) = \frac{(k+5)!}{5!k!} = \frac{((k+1)+4)!}{5!((k+1)-1)!}$

   3.5. So $P(k+1)$ is true.
4. Hence $n \in \mathbb{Z}_{\geq 1}, P(n)$ is true by MI.

21. Let $f: A \to A$ such that $X \subseteq f(X)$ for all $X \subseteq A$.
Prove that $f = id_A$, the identity function on A.

1. The domain and codomain of f and id_A are A.
2.
   2.1. Let $x \in A$ and $X = \{x\}$.
   2.2. Then $X \subseteq A$.
   2.3. Given $X \subseteq f(X)$, this implies $x \in f(X)$.
   2.4. But $f(X) = f(\{x\}) = \{f(x)\}$ by the definition of $f(X)$.
   2.5. So $f(x) = x = id_A(x)$ by the definition of id_A.
3. Therefore, $f = id_A$, .

22(a). Must $|A_1 \cup A_2| \neq |B_1 \cup B_2|$ if A_1, A_2, B_1 are uncountable sets and B_2 is a countable set? Prove your answer is correct.

Answer: **No.**

1. Let $A_1 = A_2 = B_1 = \mathbb{R}$ and $B_2 = \emptyset$.
2. Then A_1, A_2, B_1 are uncountable — by Lecture #9 Example #5.
3. Also, as B_2 is finite, it is countable.
4. Note that $A_1 \cup A_2 = \mathbb{R} \cup \mathbb{R}$ — by the definition of A_1 and A_2;
5. $= \mathbb{R}$ — by the idempotent law;
6. $= \mathbb{R} \cup \emptyset$ — by the identity law;
7. $= B_1 \cup B_2$ — by the definition of B_1 and B_2;
8. So $|A_1 \cup A_2| = |B_1 \cup B_2|$ — by Theorem 7.4.1(a).

Theorem 7.4.1 The cardinality relation is an equivalence relation

For all sets A, B and C:

a. **Reflexive**: $|A| = |A|$.
b. **Symmetric**: $|A| = |B| \to |B| = |A|$.
c. **Transitive**: $(|A| = |B|) \wedge (|B| = |C|) \to |A| = |C|$.

22(b). Must $|C_1 \times C_2| = |D_1 \times D_2|$ if C_1, C_2, D_1, D_2 are all countable sets? Prove your answer is correct.

Answer: **No.**

1. Let $C_1 = C_2 = \{1\}$ and $D_1 = D_2 = \{1,2\}$.
2. As C_1, C_2, D_1, D_2 are finite, they are countable.
3. Note that $|C_1 \times C_2| = |C_1| \times |C_2|$ by the product rule;
4. $= 1 \times 1 = 1$
5. $\neq 4 = 2 \times 2$
6. $= |D_1| \times |D_2|$
7. $= |D_1 \times D_2|$ by the product rule.

23(a). You have \$50,000 that you can use for investment. You are recommended 4 properties to invest in . Each investment must be in multiples of \$1000. How many different investment strategies are possible

(i) if you invest \$50,000 in total?

(ii) If you need not invest the entire amount of \$50,000?

(i) Number of possible solutions to $x_1 + x_2 + x_3 + x_4 = 50$ such that $x_i \geq 0$ is

$$\binom{50+4-1}{50} = \binom{53}{50} = \mathbf{23426}.$$

(ii) Number of possible solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ such that $x_i \geq 0$ is

$$\binom{50+5-1}{50} = \binom{54}{50} = \mathbf{316251}.$$

23(b). A car manufacturer has 3 factories A, B and C producing 25%, 35% and 40% of all cars respectively. The percentages of defective cars produced by factories A, B and C are 5%, 4% and 2% respectively.

(i) What is the probability that a car produced by the manufacturer is defective?

(ii) You, as a customer, received a defective car. What is the probability that this car was manufactured in factory B?

(i) $P(defective)$

$$= P(A) \cdot P(defective|A) + P(B) \cdot P(defective|B) + P(C) \cdot P(defective|C)$$

$$= (0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)$$

$$= \mathbf{0.0345}$$

(ii) $P(B|defective)$

$$= \frac{P(B \cap defective)}{P(defective)} = \; = \frac{P(B) \cdot P(defective|B)}{P(defective)} = \frac{0.35 \times 0.04}{0.0345} = \mathbf{0.406}$$

23(c). On returning home after today's exam, you realized that you had lost your student card. The probability that you left it in the exam hall is 60%, while the probability that you left it at the bus stop is 20%. Having not found your card in the exam hall, what is the probability of finding it at the bus stop? You may assume in your calculations that no one move any lost properties at the exam hall and at the bus stop.

$$P(busstop \mid \sim examhall)$$

$$= \frac{P(busstop \cap \sim examhall)}{P(\sim examhall)}$$

$$= \frac{P(busstop)}{1 - P(examhall)}$$

$$= \frac{0.2}{0.4}$$

$$= \mathbf{0.5}$$

23(d). Each student of CS1231S submitted a 2-page report in Discrete Math. We have observed the following pattern of errors. The probability of finding an error on the first page is 9%. If we find errors on page 1, the probability of finding an error on page 2 jumps to 25%; but the probability of finding an error on page 2 is 5% if page 1 has no errors. On average, how many pages of a report will have errors?

Let X = number of pages with errors.

	Probability	X
1st page only	$0.09 \times (1 - 0.25) = 0.0675$	1
1st and 2nd	$0.09 \times 0.25 = 0.0225$	2
2nd page only	$(1 - 0.09) \times (0.05) = 0.0455$	1
None	$(1 - 0.09) \times (1 - 0.05) = 0.8645$	0

$$E[X] = 0.0675 + (2 \times 0.0225) + 0.0455 = \mathbf{0.158}$$

23(e). Let A be an event with probability p. We say that I is an indicator variable for the event A if

$$I = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{if } \bar{A} \text{ occurs} \end{cases}$$

Find $E[I]$, the expected value of I. Show your working.

$$P(1) = P(A) = p.$$

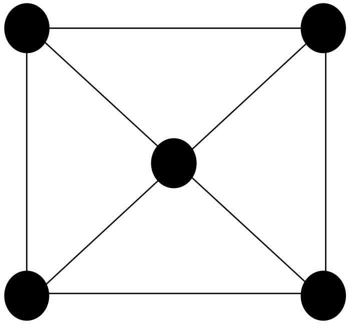
$$P(0) = P(\bar{A}) = 1 - P(A) = 1 - p.$$

$$E[I] = 1(p) + 0(1-p) = \boldsymbol{p}.$$

23(f). Show that given any 5 distinct non-negative integers, two of them have a difference that is divisible by 4. Explain your answer clearly.

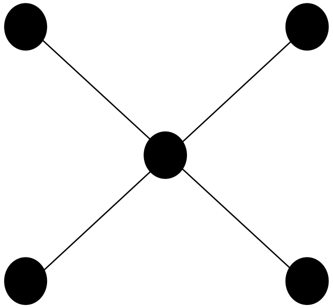
1. Let the 4 pigeonholes be the equivalence classes $[0], [1], [2]$ and $[3]$ of the congruence-mod-4 relation $\sim_4$.
2. Given any 5 distinct non-negative integers, two of them will be in $[i]$, where $i = 0,1,2,3$, by the PHP.
3. Let these 2 numbers be x and y. Hence $x = 4k + i$ and $y = 4l + i$, for some integers k and l, by the definition of congruence-mod-n.
4. Then $x - y = (4k + i) - (4l + i) = 4(k - l)$.
5. Therefore, x and y have a difference that is divisible by 4.

24(a). Draw all non-isomorphic spanning trees of the following graph.

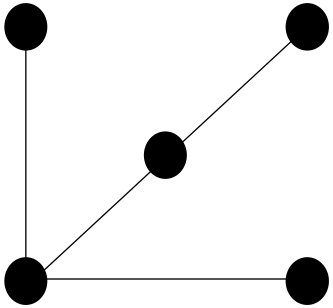


A spanning tree of 5 vertices has 4 edges (Theorem 10.5.2) and has total degree 8 (Handshake Theorem).

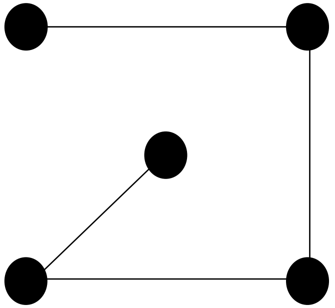
Answer



Degrees: (1,1,1,1,4)



(1,1,1,2,3)



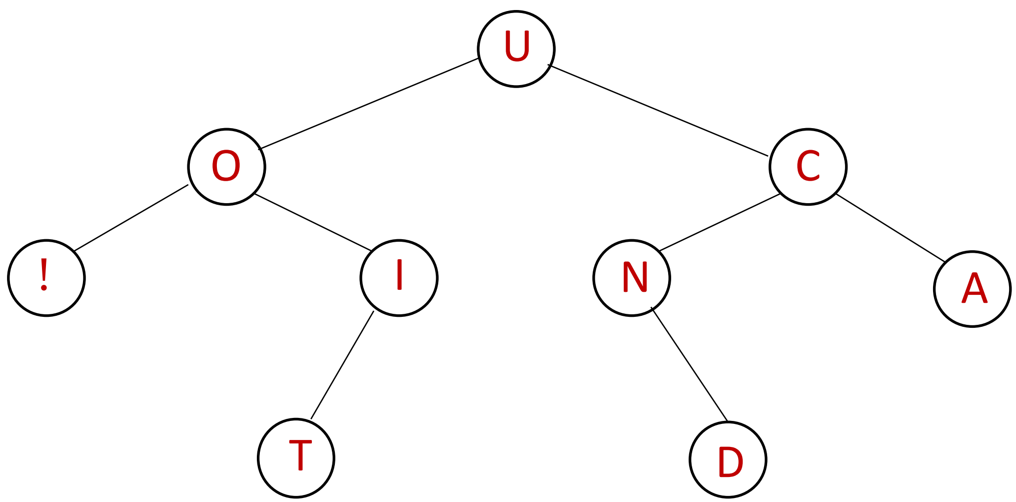
(1,1,2,2,2)

24(b). Aiken defines his own versions of the binary tree traversals. In his versions, one visits the right subtree before traversing the left subtree. Therefore, his pre-order traversal is root, right subtree, left subtree, and his post-order traversal is right subtree, left subtree, root. A particular binary tree has the following pre-order traversal and post-order traversal according to Aiken's versions.

Pre-order: **U C A N D O I T !**

Post-order: **A D N C T I ! O U**

Draw one example of such a binary tree.



24(c). A function $f: A \to A$ can be represented by a directed graph $G(A, E)$ where $E = \{(u, v) \in A \times A : f(u) = v\}$.
Let $deg^-(v)$ and $deg^+(v)$ denote the indegree and outdegree of vertex v respectively.

(i) Using the notation of indegree and outdegree above, define the following predicates:

$IsFunction(G)$: the graph G represents a function;

$IsSurjective(G)$: the graph G represents a surjective function;

$IsInjective(G)$: the graph G represents an injective function.

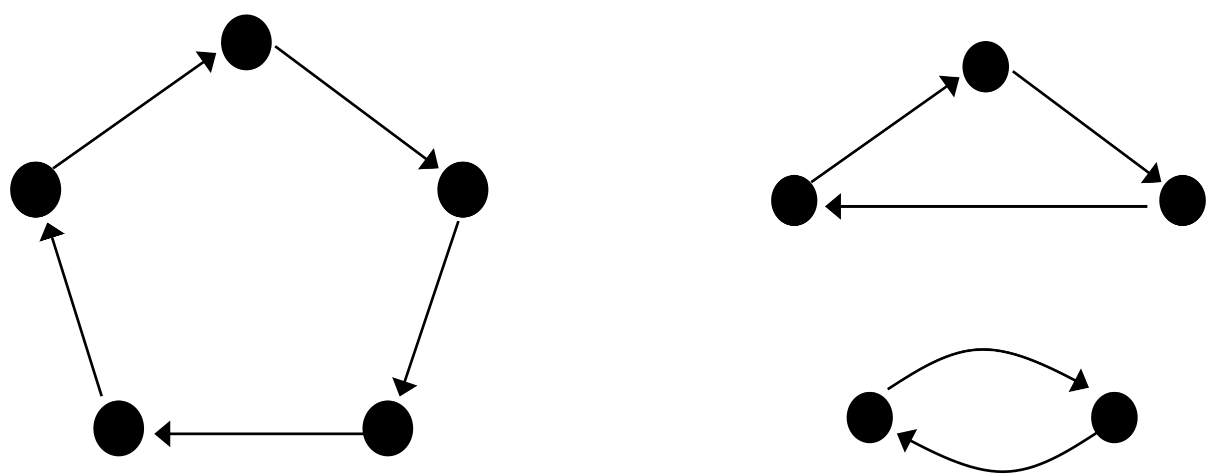
$$IsFunction(G) \equiv \forall v \in A\ deg^+(v) = 1.$$

$$IsSurjective(G) \equiv \forall v \in A\ (deg^+(v) = 1 \wedge deg^-(v) \geq 1).$$

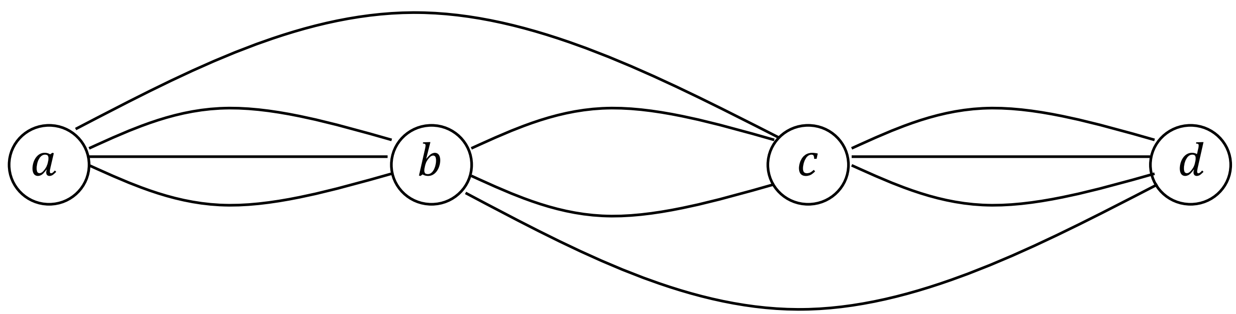
$$IsInjective(G) \equiv \forall v \in A\ (deg^+(v) = 1 \wedge deg^-(v) \leq 1).$$

24(c). A function $f: A \to A$ can be represented by a directed graph $G(A, E)$ where $E = \{(u, v) \in A \times A : f(u) = v\}$.
Let $deg^{-}(v)$ and $deg^{+}(v)$ denote the indegree and outdegree of vertex v respectively.

(ii) Draw all non-isomorphic loopless directed graphs that represent bijections $f: A \to A$ where $|A| = 5$. A loopless graph is a graph with no loops.



24(d). Given the undirected graph below, how many walks of length 3 are there from vertex a to vertex d?



Method 1:

$$M = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \qquad M^2 = \begin{bmatrix} 10 & 2 & 6 & 6 \\ 2 & 14 & 6 & 6 \\ 6 & 6 & 14 & 2 \\ 6 & 6 & 2 & 10 \end{bmatrix}$$

#walks of length 3 from a to d = $\begin{bmatrix} 0 & 3 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 2 \\ 10 \end{bmatrix} = 18 + 2 = \mathbf{20}$

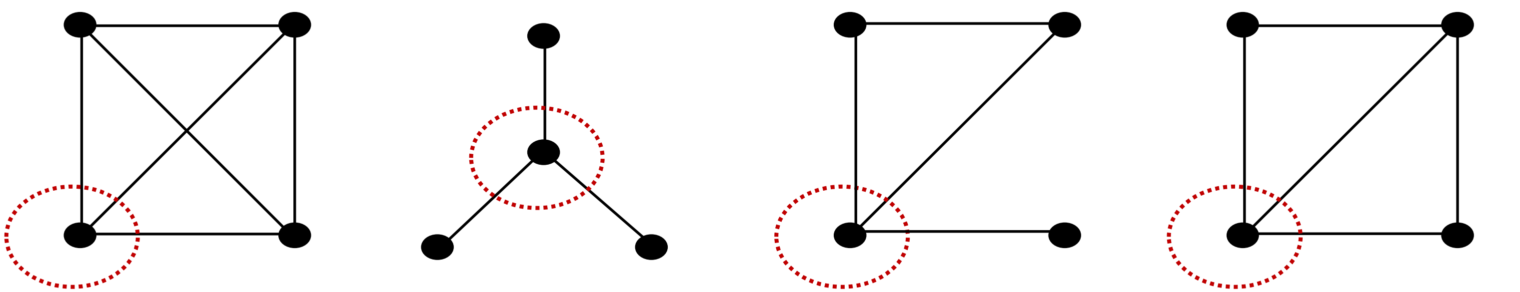
Method 2:

$a \to b$ = 3 ways; $b \to c$ = 2 ways; $c \to d$ = 3 ways; therefore $3 \times 2 \times 3 = 18$ ways.
$a \to c \to b \to d$ = 2 ways.
Hence, total = $18 + 2 = \mathbf{20}$ ways.

24(e). A set of vertices, D, in an undirected graph is said to be a ***dominating set*** if every vertex not in D is adjacent to at least one vertex in D. A ***minimal dominating set*** is a dominating set such that none of its proper subsets are dominating sets.

(i) Draw two non-isomorphic simple graphs with four vertices that have minimal dominating sets of **size one**. Highlight the vertices in the minimal dominating set in your graphs.

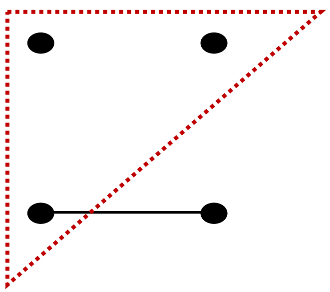
A number of possible answers (a complete graph, a star graph, and others). 4 are shown below. Dominating set shown in dotted circle. We accept any 2 correct graphs.



24(e). A set of vertices, D, in an undirected graph is said to be a ***dominating set*** if every vertex not in D is adjacent to at least one vertex in D. A ***minimal dominating set*** is a dominating set such that none of its proper subsets are dominating sets.

(ii) Draw a simple graphs with four vertices that have minimal dominating sets of **size three**. Highlight the vertices in the minimal dominating set in your graph.

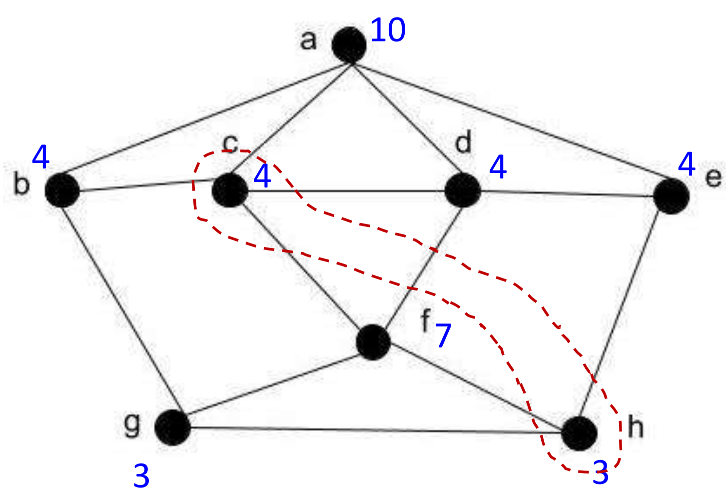
Dominating set shown in dotted triangle.



24(e). A set of vertices, D, in an undirected graph is said to be a ***dominating set*** if every vertex not in D is adjacent to at least one vertex in D. A ***minimal dominating set*** is a dominating set such that none of its proper subsets are dominating sets.

(iii) The graph below shows different villages (vertices) and their connectivity in a certain district. A telephone company wants to set up cellular towers such that all villages are covered by the network. Each cellular tower can cover the village in which it is set up, together with all the villages adjacent to it.

The cost of setting up a tower in the village *a* is 10 units, in the villages *b, c, d,* or *e* is 4 units, in the village *f* is 7 units and in the villages *g* or *h* is 3 units. Describe one way in which the company can set up its towers such that the total cost is minimized (that is, list out the villages and the total cost).



$\{c, h\}$ or $\{d, g\}$.

Total cost is $4 + 3 = \mathbf{7}$ units.

AY2020/21 Semester 2

(AY2020/21 Sem2)

17. Define sequence $a_1, a_2, a_3, \cdots$ by setting, for each $n \in \mathbb{Z}^+$,

$$a_1 = \frac{1}{10} \text{ and } a_{n+1} = a_n + \frac{1}{(3n+2)(3n+5)}$$

Prove by induction: for every $n \in \mathbb{Z}^+$, $a_n = \frac{n}{2(3n+2)}$

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv a_n = \frac{n}{2(3n+2)}$
2. (Base step) $P(1)$ is true because $\frac{1}{2(3\times 1+2)} = \frac{1}{10} = a_1$.
3. (Induction step)

   3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $a_k = \frac{k}{2(3k+2)}$.

   3.2. Then $a_{k+1} = a_k + \frac{1}{(3k+2)(3k+5)}$ by the defn of a_{k+1};

   3.3. $= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$ by the induction hypothesis;

   3.4. $= \frac{1}{3k+2}\left(\frac{k}{2} + \frac{1}{3k+5}\right) = \frac{1}{(3k+2)} \cdot \frac{3k^2+5k+2}{2(3k+5)} = \frac{1}{(3k+2)} \cdot \frac{(3k+2)(k+1)}{2(3k+5)} = \frac{k+1}{2(3(k+1)+2)}$

   3.5. So $P(k+1)$ is true.
4. Hence $n \in \mathbb{Z}^+, P(n)$ is true by MI.

18. Let A be a set and $\mathcal{C}$ be a partition of A. Prove that there exists a function $f: A \to A$ such that

$$f \circ f = f \text{ and } \mathcal{C} = \{f^{-1}(\{y\}): y \in A \text{ and } f(y) = y\}.$$

1. For each $S \in \mathcal{C}$, pick an element $y_S \in S$ that will be fixed throughout.
2. Define $f: A \to A$ by setting, for all $x \in A$ and all $S \in \mathcal{C}$,

$$f(x) = y_S \Leftrightarrow x \in S.$$

3. Then f is well defined because every element of A is in exactly one component of $\mathcal{C}$.
4. Note that for all $S \in \mathcal{C}$, $y_S \in S$ (by the choice of y_S on line 1), $\therefore f(y_S) = y_S$.
5. (To verify $f \circ f = f$)
   - 5.1. Let $x \in A$.
   - 5.2. Use the fact that $\mathcal{C}$ is a partition of A to find $S \in \mathcal{C}$ such that $x \in S$.
   - 5.3. Then $f(x) = y_S$ by the definition of f.
   - 5.4. So $(f \circ f)(x) = f\big(f(x)\big) = f(y_S) = y_S = f(x)$ by line 4.
6. (To verify $\mathcal{C} = \{f^{-1}(\{y\}): y \in A \text{ and } f(y) = y\}$)

(AY2020/21 Sem2)

18. Let A be a set and $\mathcal{C}$ be a partition of A. Prove that there exists a function $f: A \to A$ such that

$$f \circ f = f \text{ and } \mathcal{C} = \{f^{-1}(\{y\}): y \in A \text{ and } f(y) = y\}.$$

2. Define $f: A \to A$ by setting, for all $x \in A$ and all $S \in \mathcal{C}$,

$$f(x) = y_S \Leftrightarrow x \in S.$$

:

6. (To verify $\mathcal{C} = \{f^{-1}(\{y\}): y \in A \text{ and } f(y) = y\}$)

6.1. ($\subseteq$) 6.1.1. Let $S \in \mathcal{C}$.

6.1.2. From line 4, we know $f(y_S) = y_S$.

6.1.3. For all x, $x \in S \Leftrightarrow f(x) = y_S$ by the definition of f;

$\Leftrightarrow x \in f^{-1}(\{y_S\})$ by the definition of $f^{-1}(\{y_S\})$.

6.1.4 So $S = f^{-1}(\{y_S\})$.

6.2. ($\supseteq$) 6.2.1. Let $y \in A$ such that $f(y) = y$.

6.2.2. Use that fact that $\mathcal{C}$ is a partition of A to find $S \in \mathcal{C}$ such that $y \in S$.

6.2.3. Then the definition of f implies $y_S = f(y) = y$.

6.2.4. For all x, $x \in f^{-1}(\{y_S\}) \Leftrightarrow f(x) = y$

$\Leftrightarrow f(x) = y_S$ by line 6.2.3.

$\Leftrightarrow x \in S$ by the definition of f.

6.2.5. So $f^{-1}(\{y_S\}) = S$.

(AY2020/21 Sem2)

19. Let A be a set. Let S be the set of all functions $\{0,1\} \to A$, i.e.,

$$S = \{\alpha \mid \alpha : \{0,1\} \to A\}.$$

Prove that $|S| = |A^2|$ according to Cantor's definition of same-cardinality.

1. Define $f: S \to A^2$ by setting $f(\alpha) = \big(\alpha(0), \alpha(1)\big)$ for all $\alpha \in S$.
2. Define $g: A^2 \to S$ by setting $g(a, b)$ to be the function $\alpha: \{0,1\} \to A$ satisfying

   $\alpha(0) = a$ and $\alpha(1) = b$, for all $a, b \in A$.
3. For all $(a, b) \in A^2$ and all $\alpha \in S$,

   3.1. $f(\alpha) = (a, b) \Leftrightarrow \alpha(0) = a \wedge \alpha(1) = b$ by the definition of f;

   3.2. $\Leftrightarrow g(a, b) = \alpha$ by the definition of g.
4. So g is an inverse of f.
5. Thus f is bijective.
6. Therefore, $|S| = |A^2|$.

20(c). There are 3 urns U_1, U_2 and U_3. Urn $U_k (1 \le k \le 3)$ contains k red balls and $k+1$ blue balls.

(i) If you draw 2 balls at random from U_2 without replacement, what is the probability of drawing at least one red ball?

- 2 red balls and 3 blue balls in U_2.
- $\binom{5}{2} = 10$ ways to draw 2 balls.
- $\binom{2}{1} \times \binom{3}{1} = 6$ ways to draw 1 red ball and 1 blue ball.
- $\binom{2}{2} \times \binom{3}{0} = 1$ way to draw 2 red balls.
- Therefore, P(≥1 red ball) = $\frac{6+1}{10} = \mathbf{\frac{7}{10}}$.

- Alternatively: $\binom{3}{2} \times \binom{2}{0} = 3$ ways to draw 2 blue balls.
- Therefore, P(≥1 red ball) = $\frac{10-3}{10} = \mathbf{\frac{7}{10}}$.

20(c). There are 3 urns U_1, U_2 and U_3. Urn $U_k (1 \leq k \leq 3)$ contains k red balls and $k + 1$ blue balls.

(ii) Four words "I", "CAN", "DO" and "IT" are separately written on 4 slips of paper and concealed, each having an equal chance of being selected. You select a slip of paper at random and reveal the word written on it. The length of the word, k, directs you to urn U_k to pick a ball. If the ball picked is blue, what is the probability that it comes from U_2?

- $P(U_1) = \frac{1}{4}; P(U_2) = \frac{1}{2}; P(U_3) = \frac{1}{4}.$
- $P(U_1 \cap Blue) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}.$
- $P(U_2 \cap Blue) = \frac{1}{2} \times \frac{3}{5} = \frac{3}{10}.$
- $P(U_3 \cap Blue) = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}.$
- Therefore $P(U_2|Blue) = \frac{\frac{3}{10}}{\frac{1}{6}+\frac{3}{10}+\frac{1}{7}} = \frac{\frac{3}{10}}{\frac{256}{420}} = \mathbf{\frac{63}{128}}.$

20(e). You are to pick 14 numbers from 1 through 20. Is it true that no matter how you pick the 14 numbers, there is always a pair of numbers such that one is three times the other? Explain.

False

Counterexamples:

- 1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19;
- 7 through 20;
- others...

Or, partition the 20 numbers into these 14 sets:

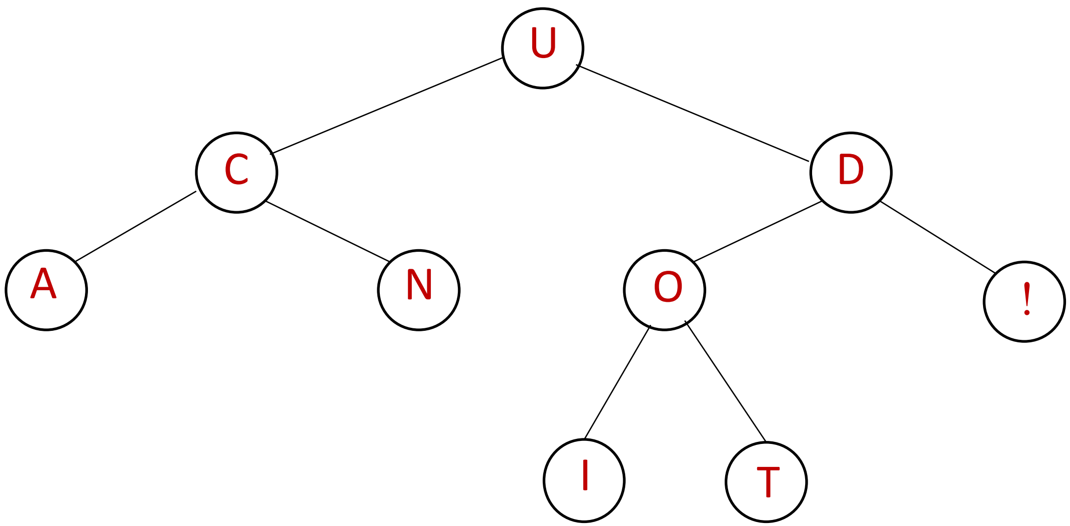
{1,3,9}, {2,6,18}, {4,12}, {5,15}, {7}, {8}, {10}, {11}, {13}, {14}, {16}, {17}, {19}, {20}.

You can pick one number from each of the 14 sets and do not get any pair of numbers such that one is three times the other.

21(a). The pre-order traversal and post-order traversal of a full binary tree with 9 vertices are given below. Draw the full binary tree.

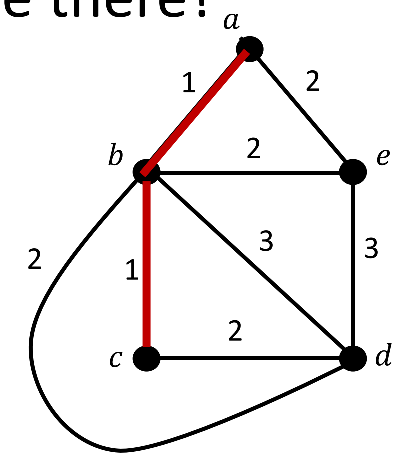
Pre-oder: U C A N D O I T !

Post-order: A N C I T O ! D U



21(b). In the following weighted graph, how many minimum spanning trees are there?

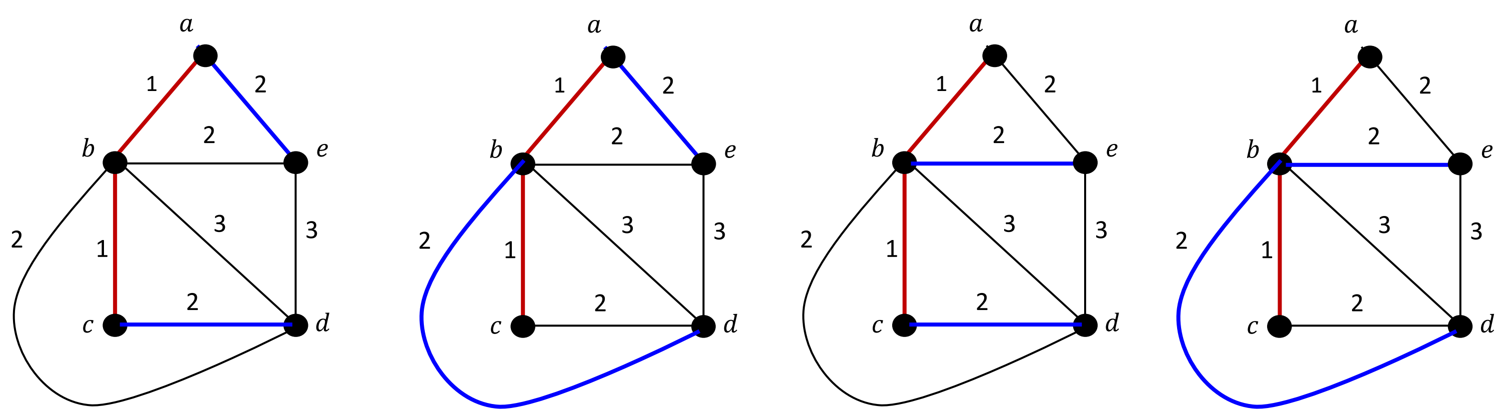
The MST consists of 4 edges, 2 of which have weight 1.



The remaining 2 edges have weight 2.

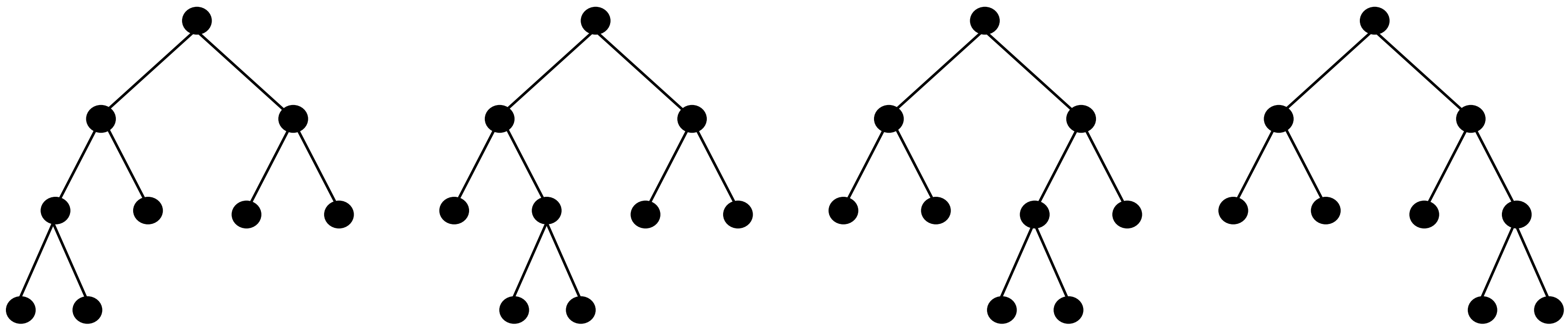
There are $\binom{4}{2} = 6$ choices, but 2 of them result in a circuit.

Therefore, there are **4** MSTs.



21(c). A *height-balanced binary tree* (or simply *balanced binary tree*) is a binary tree in which the heights of the left and right subtrees under any vertex differ by not more than one. Draw all balanced full binary trees with 9 vertices.

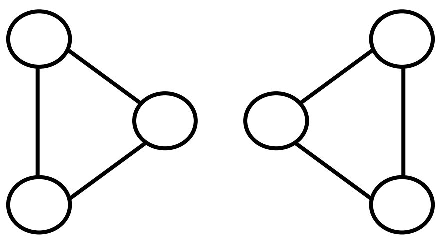
4



21(d). A *regular graph* is a simple undirected graph where every vertex has the same degree. A *2-regular graph* is a regular graph where every vertex has degree 2. Prove or disprove:

All 2-regular graphs are connected graphs.

False

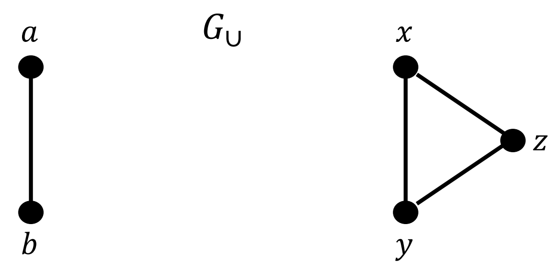


(AY2020/21 Sem2)

21(e). Given 2 simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the following are defined.

> **Graph union:**
> The union $G_u = G_1 \cup G_2$ has vertex set $V_u = V_1 \cup V_2$ and edge set $E_u = E_1 \cup E_2$.

(i) Given the following graphs G_1 and G_2, draw their union graph.

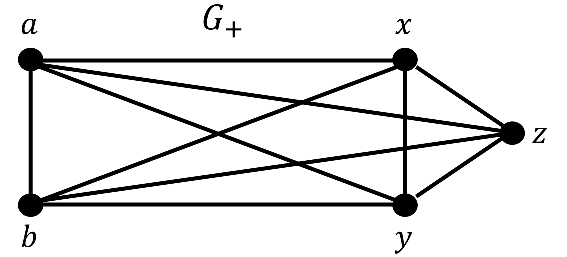


21(e). Given 2 simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the following are defined.

Graph join:

The join $G_+ = G_1 + G_2$ has vertex set $V_+ = V_1 \cup V_2$ and edge set $E_+ = E_1 \cup E_2 \cup \{$ all edges connecting every vertex in V_1 with every vertex in $V_2\}$.

(ii) Given the following graphs G_1 and G_2, draw their join graph.



(AY2020/21 Sem2)

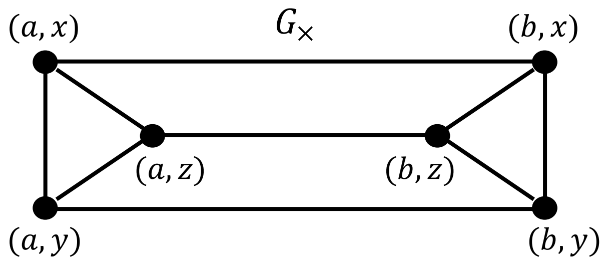
21(e). Given 2 simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $V_1 \cap V_2 = \emptyset$ and $E_1 \cap E_2 = \emptyset$, the following are defined.

Graph product:

The product $G_\times = G_1 \times G_2$ has vertex set $V_\times = V_1 \times V_2$ and two vertices $(\alpha, \beta), (\gamma, \delta) \in V_\times$ are connected by an edge iff the vertices $\alpha, \beta, \gamma, \delta$ satisfy the following (with $\sim$ denoting "is adjacent to"):

$$(\alpha = \gamma \wedge \beta \sim \delta) \vee (\beta = \delta \wedge \alpha \sim \gamma)$$

(iii) Given the following graphs G_1 and G_2, draw their product graph.



Additional question:

Given a fair coin, what is the expected number of coin tosses to get 5 consecutive heads?

Solution 1:

1. Let E be the expected number of tosses.
2. If we get a T (with probability ½), then the expected value is $E + 1$.
3. If we get HT (with probability ¼), then the expected value is $E + 2$.
4. If we get HHT (probability 1/8), then the expected value is $E + 3$.
5. If we get HHHT (probability 1/16), then the expected value is $E + 4$.
6. If we get HHHHT (probability 1/32), then the expected value is $E + 5$.
7. If we get HHHHH (probability 1/32), then the expected value is 5.
8. Putting these together, we have

$$E = \frac{1}{2}(E+1) + \frac{1}{4}(E+2) + \frac{1}{8}(E+3) + \frac{1}{16}(E+4) + \frac{1}{32}(E+5) + \frac{1}{32}(5)$$

9. Solving the above equation, we get $E = \mathbf{62}$.

Generalizing, we may get $E_n = 2(2^n - 1)$, where n is the number of consecutive heads.

Additional question:

Given a fair coin, what is the expected number of coin tosses to get 5 consecutive heads?

Solution 2:

Find recurrence relation for E_n, the expected number of tosses to get n consecutive heads. After getting $n-1$ consecutive heads, there are two scenarios:

- We get a head (with probability ½), in which case we have got n consecutive heads and the expected value is $E_{n-1}+1$.
- We get a tail (with probability ½), in which case we have to start all over, and hence the expected value is $E_{n-1}+1+E_n$.

$$E_0 = 0$$

$$E_n = \frac{1}{2}(E_{n-1}+1) + \frac{1}{2}(E_{n-1}+1+E_n)$$

or $$E_n = 2\,E_{n-1} + 2$$

Therefore, $E_5 = \mathbf{62}$.

Solving the recurrence relation, we get the closed-form formula $E_n = 2(2^n - 1)$.