Problem 1 – Source Coding (20 Points)

(1) **(14 Points)** Consider symbol-wise source coding on a finite alphabet $\mathcal{X} = \{a, b, \dots\}$ (of unspecified size), with symbol distribution P_X, code lengths $\ell(a), \ell(b), \dots$, and an average code length of $L(C)$.

(i) Does there exist a source P_X for which the value of $\ell(a)$ for Shannon-Fano coding exceeds that of Huffman coding by at least 5? Explain.

(ii) Does there exist a source P_X for which the value of $L(C)$ for Shannon-Fano coding exceeds that of Huffman coding by at least 5? Explain.

(iii) Give an example of a source P_X such that Huffman coding applied to pairs (i.e., applied to the alphabet $\mathcal{X}^2$ and distribution $P_{X_1X_2}(x_1, x_2) = P_X(x_1)P_X(x_2)$) gives a strictly smaller average length per $\mathcal{X}$-symbol compared to Huffman coding applied to P_X alone. Explain.

(b) **(6 Points)** Consider a discrete memoryless source with per-symbol distribution P_X and block length n. Suppose that someone claims to come up with a "better" notion of a typical set (different from the typical set definition in the lecture), denoted $\mathcal{T}_n^*$, such that $\mathbb{P}[\mathbf{X} \in \mathcal{T}_n^*] \to 1$ as $n \to \infty$ and $|\mathcal{T}_n^*| \leq 2^{nA}$ for some constant value of $A < H(X)$ (where A and P_X do not vary as n increases, and where $\mathbf{X}$ is distributed according to $\prod_{i=1}^n P_X(x_i)$ as usual). Is this possible? Explain briefly.

Problem 2 – Discrete and Continuous Information Measures (25 Points)

(a) **(16 Points)** Answer the following:

(i) Write down a chain of inequalities between $H(X)$, $H(X,Y)$, and $I(X;Y)$, ordering them from smallest to largest. Briefly explain each inequality (2 in total).

(ii) Consider comparing $I(X;Y)$ to $H(X|Y)$ for an arbitrary joint distribution P_{XY}. Can we say that $I(X;Y) \geq H(X|Y)$ always, that $I(X;Y) \leq H(X|Y)$ always, or that both $I(X;Y) > H(X|Y)$ and $I(X;Y) < H(X|Y)$ are possible? Explain.

(iii) Prove that $I(X,Z;Y,Z) = H(Z) + I(X;Y|Z)$ for any random variables (X,Y,Z).

(iv) Describe a joint distribution on (X,Y,Z) such that X, Y, and Z are binary-valued (0 or 1), $H(X) = H(Y) = H(Z) = 1$, $I(X;Y) = I(X;Z) = I(Y;Z) = 0$, and $H(X,Y,Z) = 2$. Briefly explain why all these entropy and mutual information values are attained under your choice of joint distribution.

(b) **(9 Points)** Suppose that someone has already computed the differential entropy $h(\tilde{U}+\tilde{V})$ in the case that $\tilde{U}$ and $\tilde{V}$ are independent random variables distributed uniformly in the interval $[0,1]$; let ξ denote this value of $h(\tilde{U}+\tilde{V})$. (Its precise value is not needed for the purpose of answering this question.)

Let U and V be independent random variables distributed uniformly in the interval $[0,4]$, and define $Z = U + V$. Compute the differential entropies $h(U)$, $h(Z)$, and $h(Z|U)$, and the mutual information $I(U;Z)$, leaving your answers in terms of the quantity ξ introduced above as appropriate.

Problem 3 – Practical Codes (25 Points)

(a) **(10 Points)** Consider the following code with 4 codewords: $\{000000, 001111, 111100, 111111\}$.

(i) Explain why this is not a linear code.

(ii) What is the minimum distance of this code? Explain briefly.

(iii) Replace one of the 4 codewords by a new codeword such that the resulting code (different from the one above) is a linear code.

(b) **(8 Points)** Write down (i) the generator matrix $\mathbf{G}$, (ii) the parity check matrix $\mathbf{H}$, and (iii) the code rate, for a systematic code with five information bits $(u_1, u_2, u_3, u_4, u_5)$, and three parity check bits taking the form

$$u_1 \oplus u_2 \oplus u_3$$
$$u_1 \oplus u_3 \oplus u_5$$
$$u_2 \oplus u_4.$$

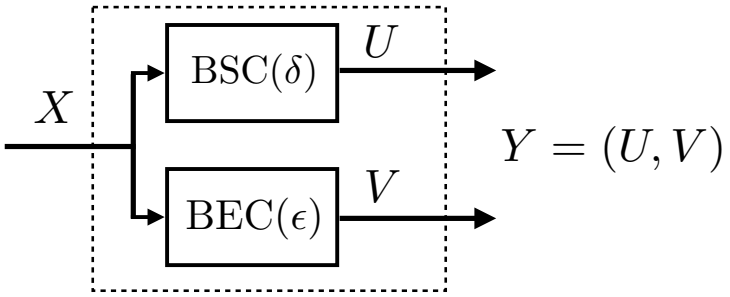
Explanations are not required.

(c) **(7 Points)** In the lecture, we introduced the Hamming code with 16 binary codewords of length 7, and showed that the code is guaranteed to correct up to one error (i.e., given a received string containing no bit flips or a single bit flip compared to some codeword, it's guaranteed that this codeword can be uniquely recovered).

Prove that it is *impossible* to find a code with 17 or more binary codewords of length 7 while being guaranteed to correct up to one error, even if the code may be non-linear.

Problem 4 – Channel Coding (30 Points)

(a) **(20 Points)** Consider the following setup in which $X \in \{0,1\}$ is passed through a binary symmetric channel (BSC) to get $U \in \{0,1\}$, and X is also passed through a binary erasure channel (BEC) to get $V \in \{0,1,e\}$, where e is the erasure symbol. (Assume that the BSC and BEC outputs are conditionally independent given X.) Then, the overall output is a pair consisting of both resulting values, i.e., $Y = (U,V)$.



Overall Channel

Consider choosing $P_X(1) = p$ and $P_X(0) = 1-p$ for some $p \in (0,1)$, and answer the following with your answers depending on p and/or δ and/or ϵ as needed:

(i) Explain why $I(X;U|V=0)$ and $I(X;U|V=1)$ are both zero.

(ii) Prove that the joint conditional distribution of (X,U) given $V=e$ is the same as the unconditional joint distribution of (X,U)

(iii) Using the previous parts or otherwise, show that $I(X;U|V) = \epsilon \cdot I(X;U)$.

(iv) Using the previous parts or otherwise, find $I(X;V)$ and $I(X;U|V)$ when $p = \frac{1}{2}$.

(v) Using the previous parts or otherwise, find the capacity of the overall channel $P_{Y|X}$ (and explain why this is the capacity).

(Note: You may make use of the binary entropy function $H_2(\cdot)$ as usual. You may also use any known facts/properties from the lecture regarding the BSC and/or the BEC and/or their associated mutual information terms.)

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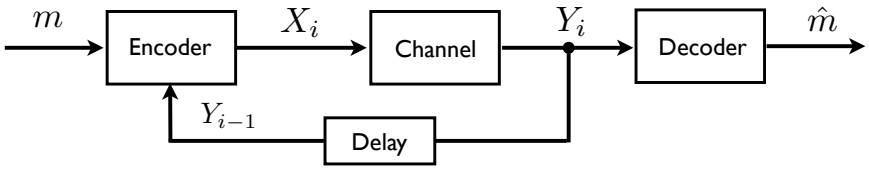
(b) **(10 Points – Advanced)** Consider the following setup of communication with feedback:



Formally, the setup described is as follows:

- As usual, the message m is uniformly distributed over the set $\{1, \ldots, M\}$.
- Different from the lecture, the encoder may choose the next input as a function of all previous outputs: $X_i = f_i(Y_1, \ldots, Y_{i-1})$ for some deterministic function f_i.
- The channel is assumed to be a discrete memoryless channel, with the memorylessness assumption now being that Y_i is conditionally independent of the message and previous inputs/outputs (i.e., $(m, X_1, \ldots, X_{i-1}, Y_1, \ldots, Y_{i-1})$) given the input X_i. The corresponding conditional distribution is written as $P_{Y|X}$, as usual.
- The decoder outputs the estimate $\hat{m}$ after all outputs $Y_1, \ldots, Y_n$ have been received.

We also define $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ as usual.

Defining the channel capacity with feedback, C_{F}, in an analogous manner to the lecture, it turns out that $C_{\mathrm{F}} = \max_{P_X} I(X;Y)$, i.e., feedback does not increase the capacity. The converse proof is mostly the same as the non-feedback case, but we no longer necessarily have the Markov chain relation $m \to \mathbf{X} \to \mathbf{Y} \to \hat{m}$, so it is now more difficult to show that $I(m;\hat{m}) \leq \sum_{i=1}^{n} I(X_i;Y_i)$. This question works through filling in this missing step:

(i) Explain why $I(m;\hat{m}) \leq I(m;\mathbf{Y})$.

(ii) Prove that $I(m;\mathbf{Y}) \leq \sum_{i=1}^{n} I(X_i;Y_i)$, carefully explaining all steps.

(Note: Marks will <u>not</u> be awarded for copying the same steps as the non-feedback case.)

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