CS1231S: Discrete Structures

Tutorial #5: Relations & Partial Orders

(Week 7: 28 February – 4 March 2022)

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1. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. Let R be a binary relation on a non-empty set A. If $R = \emptyset$, then is R reflexive? Symmetric? Transitive?

D2. Suppose a binary relation R on a non-empty set A is reflexive, transitive, symmetric and antisymmetric. What can you conclude about R? Explain.

D3. Asymmetry is defined in question 6 as follows: A binary relation R on a set A is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

Are there any binary relations that are both symmetric and asymmetric?

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Answers

1. **Lexicographic order:** To compare two English words ("strings"), you compare their letters one by one from left to right. If they have been the same to a certain point, and one word runs out of letters, then the shorter word comes first. For example, "run" comes before "runner". If all letters up to a certain point are the same, but the next letter differs, then the word with the "smaller" differing letter (in the usual alphabetic sense) comes first. For example, "run<u>n</u>er" comes before "run<u>w</u>ay"; also, "<u>c</u>hicken" comes before "<u>d</u>og". This is formally defined in Theorem 8.5.1 (Epp).

   A **string** over a set A is a finite sequence of elements from A. The null string is denoted by ε. For example, if $A = \{a, b, c\}$, then $\varepsilon, a, ac, bca, bbbb$ and $cbbbbabcc$ are some strings over A.

   > **Theorem 8.5.1**
   >
   > Let A be a set with a partial order relation R, and let S be a set of strings over A.
   > Define a relation $\preccurlyeq$ on S as follows:
   >
   > For any two strings in S, $a_1a_2 \cdots a_m$ and $b_1b_2 \cdots b_n$, where m and n are positive integers,
   >
   > 1. If $m \le n$ and $a_i = b_i$ for all $i = 1,2,\cdots,m$, then
   > $$a_1a_2 \cdots a_m \preccurlyeq b_1b_2 \cdots b_n.$$
   > 2. If for some integer k with $1 \le k \le m$ and $1 \le k \le n$, $a_i = b_i$ for all $i = 1,2,\cdots,k-1$, and $a_k \ne b_k$, but $a_k \; R \; b_k$, then
   > $$a_1a_2 \cdots a_m \preccurlyeq b_1b_2 \cdots b_n.$$
   > 3. If ε is the null string and s is any string in S, then $\varepsilon \preccurlyeq s$.
   >
   > If no strings are related other than by these three conditions, then $\preccurlyeq$ is a partial order relation.

   Let $A = \{a, b\}$ be our alphabet (instead of the usual 26-letter alphabet of the English language), and the partial order R defined on A: $x \; R \; y$ iff $x = y$ or letter x comes before y for all $x, y \in A$. Now define S to be the set of all strings over alphabet A. Define the partial order $\preccurlyeq$ on S to be the lexicographic order specified in Theorem 8.5.1 (Epp). Determine which of the following statements are true, and for each true statement, state which of the three rules in the theorem makes the statement true.

   (a) $aab \preccurlyeq aaba$
   (b) $bbab \preccurlyeq bba$
   (c) $\varepsilon \preccurlyeq aba$
   (d) $ababb \preccurlyeq abb$
   (e) $bbab \preccurlyeq bbaa$
   (f) $ababa \preccurlyeq ababaa$
   (g) $bbaba \preccurlyeq bbabb$

   Answers
   (a) True by (1).
   (b) False.
   (c) True by (3).
   (d) True by (2).
   (e) False.
   (f) True by (1).
   (g) True by (2).

2. Let R be a binary relation on a non-empty set A. Let $x, y \in A$. Define a relation S on A by

$$x\ S\ y \Leftrightarrow x = y \lor x\ R\ y \text{ for all } x, y \in A.$$

Show that:

(a) S is reflexive;

(b) $R \subseteq S$; and

(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.

What is this relation S called? (Hint: Refer to Transitive Closure in Lecture 6).

Answers:

> Let R be a relation on a set A.
> R is **reflexive** iff $\forall x \in A\ (xRx)$.

(a)
1. Let $x \in A$.
2. $x = x$, so $x\ S\ x$ by the definition of S.
3. Therefore, S is reflexive.

(b)
1. Suppose $(x, y) \in R$, that is, $x\ R\ y$.
2. So $x\ S\ y$ by the definition of S.
3. So $(x, y) \in S$.
4. Therefore, $R \subseteq S$ by the definition of $\subseteq$.

(c)
1. Suppose $(x, y) \in S$.
2. Then $x\ S\ y$, which means $x = y \lor x\ R\ y$ by the definition of S.
3. Case 1: $x = y$
   - 3.1. Then $x\ S'\ y$ since S' is reflexive.
   - 3.2. So $(x, y) \in S'$.
4. Case 2: $x\ R\ y$
   - 4.1. Then $(x, y) \in R \subseteq S'$.
   - 4.2. Then $(x, y) \in S'$.
5. In all cases, $(x, y) \in S'$.
6. Therefore, $S \subseteq S'$.

The relation S is called the reflexive closure of R. It is the smallest relation on A that is reflexive and contains R as a subset.

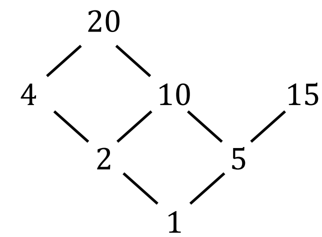
3. Consider the "divides" relation on each of the following sets of integers. For each of them, draw a Hasse diagram, and find all minimal, maximal, smallest and largest elements.

(a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

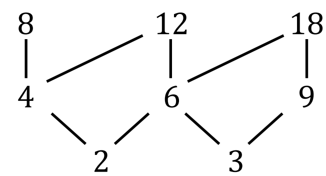
(b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

Answers:

(a) Minimal: 1.
Maximal: 15 and 20.
Smallest: 1.
Largest: None.



(b) Minimal: 2 and 3.
Maximal: 8, 12 and 18.
Smallest: None.
Largest: None.



Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

- c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$.
- c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
- c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
- c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

4. Let A be a set and $\wp(A)$ the power set of A. Prove that the binary relation $\subseteq$ on $\wp(A)$ is a partial order.

Answer:

Proof:

1. (Reflexivity) Take any $S \in \wp(A)$,
   1.1. $S \subseteq S$ by the definition of subset.
   1.2. Hence $\subseteq$ is reflexive.
2. (Antisymmetry) Take any $S, T \in \wp(A)$,
   2.1. Suppose $S \subseteq T$ and $T \subseteq S$.
   2.2. Then $S = T$ by the definition of set equality.
   2.3. Hence $\subseteq$ is antisymmetric.
3. (Transitivity)
   3.1. $\subseteq$ is transitive by Theorem 6.2.1.
4. Therefore $\subseteq$ on $\wp(A)$ is a partial order.

Theorem 6.2.1
For all sets A, B, C,
$A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$.

5. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B\ \big((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \wedge (b \le d)\big).$$

   (a) Prove that R is a partial order.

   (b) Draw the Hasse diagram for R.

   (c) Find the maximal, largest, minimal and smallest elements.

   (d) Is $(B \times B, R)$ well-ordered?

Let R be a relation on a set A.

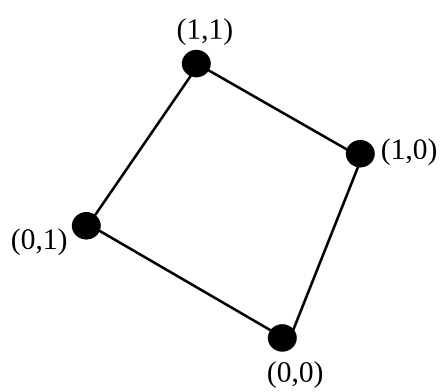
- R is **reflexive** iff $\forall x \in A\ (xRx)$.
- R is **antisymmetric** iff $\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$.
- R is **transitive** iff $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

Answers:

(a) Proof:

1. (Reflexivity) Take any $(a,b) \in B \times B$,
   1.1. $a \le a$ and $b \le b$.
   1.2. So $(a,b)\ R\ (a,b)$ by the definition of R.
   1.3. Hence R is reflexive.
2. (Antisymmetry) Take any $(a,b),(c,d) \in B \times B$,
   2.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (a,b)$.
   2.2. Then $a \le c, b \le d, c \le a$ and $d \le b$ by the definition of R.
   2.3. Then $a = c$ and $b = d$ by the antisymmetry of $\le$.
   2.4. So $(a,b) = (c,d)$ by equality of ordered pairs.
   2.5. Hence R is antisymmetric.
3. (Transitivity) Take any $(a,b),(c,d),(e,f) \in B \times B$,
   3.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (e,f)$.
   3.2. Then $a \le c, b \le d, c \le e$ and $d \le f$ by the definition of R.
   3.3. Then $a \le e$ and $b \le f$ by the transitivity of $\le$.
   3.4. So $(a,b)\ R\ (e,f)$ by the definition of R.
   3.5. Hence R is transitive.
4. Therefore R on $B \times B$ is a partial order.

(b)



(c) Maximal and largest: (1,1); minimal and smallest: (0,0).

(d) No. It is not even a total order, because (0,1) and (1,0) are not comparable.

6. Let R be a binary relation on a set A.
We have defined antisymmetry in class: R is **antisymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y).$$

We define asymmetry here. R is **asymmetric** iff

$$\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x).$$

(a) Find a binary relation on A that is both asymmetric and antisymmetric.

(b) Find a binary relation on A that is not asymmetric but antisymmetric.

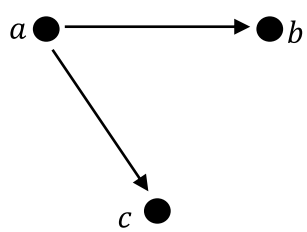
(c) Find a binary relation on A that is asymmetric but not antisymmetric.

(d) Find a binary relation on A that is neither asymmetric nor antisymmetric.

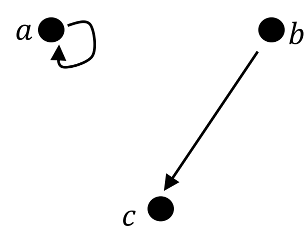
Answers:

Let $A = \{a, b, c\}$.

(a) $R = \{(a, b), (a, c)\}$.

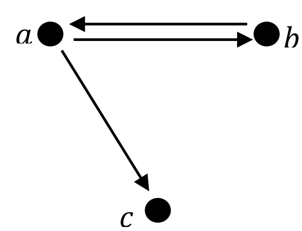


(b) $R = \{(a, a), (b, c)\}$



(c) No solution.
Every asymmetric relation is antisymmetric (see proof below).

(d) $R = \{(a, b), (b, a), (a, c)\}$



Proof: (Every asymmetric relation is antisymmetric.)

1. Take any binary relation R on a set A.
2. Suppose R is asymmetric.
   2.1. Then $\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x)$ by the definition of asymmetry.
   2.2. $\equiv \forall x, y \in A\ (x \not R\ y \vee y \not R\ x)$ by the implication law.
   2.3. $\Rightarrow \forall x, y \in A\ ((x \not R\ y \vee y \not R\ x) \vee x = y)$ by generalization.
   2.4. $\equiv \forall x, y \in A\ ({\sim}(x\ R\ y \wedge y\ R\ x) \vee x = y)$ by the De Morgan's law.
   2.5 $\equiv \forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$ by the implication law.
3. Line 2.5 is the definition of antisymmetry, hence R is antisymmetric.

We see that asymmetry property forces the antisymmetry property to be vacuously true.

7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

Answer:

1. Let $c \in A$ that is minimal with respect to $\preccurlyeq$.
2. Pick any $x \in A$.
3. As $\preccurlyeq$ is a total order, either $x \preccurlyeq c$ or $c \preccurlyeq x$.
4. Case 1: Suppose $x \preccurlyeq c$.
    4.1. Then $x = c$ by the minimality of c.
    4.2. So $c \preccurlyeq x$ by the reflexibility of $\preccurlyeq$.
5. Case 2: Suppose $c \preccurlyeq x$.
    5.1. Then $c \preccurlyeq x$.
6. In all cases, $c \preccurlyeq x$, i.e. c is the smallest element.

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

- c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
- c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

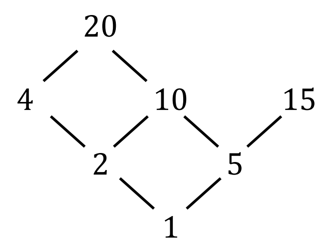
8. **Definitions.** Consider a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.

- We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
- We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

In question 3, you are given the "divides" relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$. List out the pairs of distinct elements in A that are (a) comparable; (b) compatible. Use the notation $\{x, y\}$ to represent the pair of elements x and y.

Answers:

Hasse diagram



(a) Comparable:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$,

$\{2,4\}$, $\{2,10\}$, $\{2,20\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$,

$\{4,20\}$ and $\{10,20\}$.

Not comparable: $\{2,5\}$, $\{2,15\}$, $\{5,4\}$, $\{4,10\}$, $\{4,15\}$, $\{10,15\}$ and $\{15,20\}$.

(b) Compatible:

$\{1,2\}$, $\{1,5\}$, $\{1,4\}$, $\{1,10\}$, $\{1,15\}$, $\{1,20\}$,

$\{2,4\}$, $\{2,5\}$, $\{2,10\}$, $\{2,20\}$, $\{5,4\}$, $\{5,10\}$, $\{5,15\}$, $\{5,20\}$,

$\{4,10\}$, $\{4,20\}$ and $\{10,20\}$.

Not compatible: $\{2,15\}$, $\{4,15\}$, $\{10,15\}$ and $\{15,20\}$.

9. For each of the following statements, state whether it is true or false and justify your answer.

   (a) In all partially ordered sets, any two comparable elements are compatible.

   (b) In all partially ordered sets, any two compatible elements are comparable.

> Let $\preccurlyeq$ be a partial order $\preccurlyeq$ on a set A and let $a, b \in A$.
> - We say a, b are **comparable** iff $a \preccurlyeq b$ or $b \preccurlyeq a$.
> - We say a, b are **compatible** iff there exists $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

Answers:

(a) Yes. Proof:

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. Case 1: $a \preccurlyeq b$
   - 3.1. Let $c = b$.
   - 3.2. Then $a \preccurlyeq b = c$ by assumption and
$b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   - 3.3. Hence a and b are compatible by the definition of compatibility.
4. Case 2: $b \preccurlyeq a$
   - 4.1. Let $c = a$.
   - 4.2. Then $b \preccurlyeq a = c$ by assumption and
$a \preccurlyeq a = c$ by the reflexivity of $\preccurlyeq$.
   - 4.3. Hence a and b are compatible by the definition of compatibility.
5. In all cases, a and b are compatible.

Note that the 2 cases are symmetrical and hence very similar (just switch the roles of a and b). We may use WLOG (without loss of generality) to cover one of the cases. (Use WLOG with care! Make sure that the cases are indeed similar before you use WLOG.)

1. Let $a, b \in A$ such that a and b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. WLOG, let $a \preccurlyeq b$.
   - 3.1. Let $c = b$.
   - 3.2. Then $a \preccurlyeq b = c$ by assumption and
$b \preccurlyeq b = c$ by the reflexivity of $\preccurlyeq$.
   - 3.3. Hence a and b are compatible by the definition of compatibility.
4. Hence a and b are compatible.

(b) No. Consider the poset $(\mathbb{Z}^+, |)$ where $|$ is the "divides" relation. 2 and 3 are compatible as $2 \mid 6$ and $3 \mid 6$. However, 2 and 3 are not comparable as $2 \nmid 3$ and $3 \nmid 2$.

10. Let $A = \{a, b, c, d\}$. Consider the following partial order on A:

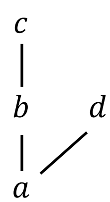
$$R = \{(a,a),\ (a,b),\ (a,c),\ (a,d),\ (b,b),\ (b,c),\ (c,c),\ (d,d)\}.$$

(a) Draw a Hasse diagram of R..

(b) Draw Hasse diagrams of all the linearizations of R.

Answers:

(a)



(b)