

CS1231S: Discrete Structures
Tutorial #7: Mathematical Induction and Recursion
(Week 9: 14 – 18 March 2022)

1. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

Definition:

An integer d is said to be a **linear combination** of integers a and b , if and only if, there exist integers s and t such that $as + bt = d$.

D1. Prove the following proposition:

$$\forall a, b, c \in \mathbb{Z}, \text{ if } a \mid b \text{ and } a \mid c, \text{ then } \forall x, y \in \mathbb{Z} (a \mid bx + cy).$$

The proposition states that if a divides both b and c , then a divides their linear combination.

D2. Aiken attempts to prove the following by Mathematical Induction:

$$\forall n \in \mathbb{N} (3 \mid (n^3 + 44n)).$$

However, his proof below is incorrect. Point out the mistakes.

Proof:

1. Note that $1^3 + 44(1) = 45$ which is divisible by 3.
2. So the statement is true for $n = 1$.
3. Now suppose the statement is true for some natural number k .
4. Then $k^3 + 44k$ is divisible by 3.
5. Therefore $(k + 1)^3 + 44(k + 1)$ is divisible by 3.
6. So by Mathematical Induction, the statement is true for all numbers.

D3. Dueet attempts to prove the following by Mathematical Induction.

Consider a group of n people, each of whom shakes hands exactly once with everybody else in the group. No one shakes his/her own hand. Let $S(n)$ be the total number of handshakes in any group of n people. Prove that

$$\forall n \in \mathbb{Z}^+ \left(S(n) = \frac{n(n-1)}{2} \right).$$

However, her proof below is incorrect. Point out the mistakes.

Proof:

1. Let $P(n) \equiv (S(n) = \frac{n(n-1)}{2})$, for any $n \in \mathbb{Z}^+$.
2. Basis step: $n = 1$
 - 2.1. $S(1) = 0$ because nobody shakes his/her own hand.
 - 2.2. Also, $\frac{1(1-1)}{2} = 0 = S(1)$.
 - 2.3. Thus $P(1)$ is true.
3. Inductive step: Assume $P(k)$, i.e. $S(k) = \frac{k(k-1)}{2}$.
 - 3.1. For any $k \in \mathbb{Z}^+$, consider any group of k people.
 - 3.2. This group makes $S(k)$ handshakes, by the induction hypothesis.
 - 3.3. Now consider one new person joining the group. Since all the original k people have already shaken hands, they just need to shake the newcomer's hand, giving k additional handshakes in total.
 - 3.4. Thus $S(k+1) = S(k) + k = \frac{k(k-1)}{2} + k = \frac{(k+1)k}{2}$ by basic algebra.
 - 3.5. Hence $P(k+1)$ is true.
4. Therefore $P(n)$ is true for any $n \in \mathbb{Z}^+$, by mathematical induction.

2. Tutorial Questions

1. Prove by induction that for all $n \in \mathbb{Z}^+$,
$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$
2. Let $x \in \mathbb{R}_{\geq -1}$. Prove by induction that $1 + nx \leq (1+x)^n$ for all $n \in \mathbb{Z}^+$.
3. In Lecture #5, we claim that given any set A , $|\wp(A)| = 2^n$, where $\wp(A)$ denotes the power set of A and $|A| = n$. Prove by induction on n that this claim is true by using the argument in Lecture #5.
4. Let a be an odd integer. Prove by induction that $2^{n+2} \mid a^{2^n} - 1$ for all $n \in \mathbb{Z}^+$.
Here you may use without proof the fact that the product of any two consecutive integers is even.
(Note that $a^{b^c} = a^{(b^c)}$ by convention.)
5. Prove by induction that
$$\forall n \in \mathbb{Z}_{\geq 8} \exists x, y \in \mathbb{N} (n = 3x + 5y).$$

(In other words, any integer-valued transaction of at least \$8 can be carried out using only \$3 and \$5 notes.)

CS1231S: Discrete Structures
Tutorial #7: Mathematical Induction and Recursion
Answers

1. Prove by induction that for all $n \in \mathbb{Z}^+$,

$$1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. (Basis step) $P(1)$ is true because $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1)$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1^2 + 2^2 + \dots + k^2 = \frac{1}{6}k(k+1)(2k+1)$
 - 3.2. Then $1^2 + 2^2 + \dots + k^2 + (k+1)^2$
 - 3.3. $= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$ by the induction hypothesis
 - 3.4. $= \frac{1}{6}(k+1)(k(2k+1) + 6(k+1))$
 - 3.5. $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$
 - 3.6. $= \frac{1}{6}(k+1)(k+2)(2k+3)$
 - 3.7. $= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$ by basic algebra
 - 3.8. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ P(n)$ is true by MI.

2. Let $x \in \mathbb{R}_{\geq -1}$. Prove by induction that $1 + nx \leq (1+x)^n$ for all $n \in \mathbb{Z}^+$.

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv (1 + nx \leq (1+x)^n)$.
2. (Basis step) $P(1)$ is true because $1 + 1x = 1 + x = (1+x)^1$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e. $1 + kx \leq (1+x)^k$.
 - 3.2. Then $(1+x)^{k+1}$
 - 3.3. $= (1+x)^k(1+x)$
 - 3.4. $\geq (1+kx)(1+x)$ by the induction hypothesis
 - 3.5. $= 1 + (k+1)x + kx^2$
 - 3.6. $\geq 1 + (k+1)x$ as $k \geq 1$ and $x^2 \geq 0$
 - 3.7. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ P(n)$ is true by MI.

3. In Lecture #5, we claim that given any set A , $|\wp(A)| = 2^n$, where $\wp(A)$ denotes the power set of A and $|A| = n$. Prove by induction on n that this claim is true by using the argument in Lecture #5.

Answer:

1. For each $n \in \mathbb{N}$, let $P(n) \equiv (|\wp(A)| = 2^n)$ where A is any n -element set.
2. (Basis step) $P(0)$ is true because $|\wp(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$ as $\wp(\emptyset) = \{\emptyset\}$ and $|\emptyset| = 0$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $|\wp(X)| = 2^k$ where X is any k -element set.
 - 3.2. Let A be a $k + 1$ -element set.
 - 3.3. Since $k \geq 0$, there is at least one element in A . Pick $z \in A$.
 - 3.4. The subsets of A can be split into 2 groups: those that contain z and those that don't.
 - 3.5. Those subsets that do not contain z are the same as the subsets of $A \setminus \{z\}$, which has a cardinality of k , and hence $|\wp(A \setminus \{z\})| = 2^k$ by the induction hypothesis.
 - 3.6. Those subsets that contain z can be matched up one for one with those subsets that do not contain z by unioning $\{z\}$ to the latter.
 - 3.7. Hence there is an equal number of subsets that contain z and subsets that don't.
 - 3.8. Hence $|\wp(A)| = 2^k + 2^k = 2^{k+1}$.
 - 3.9. Thus $P(k + 1)$ is true.
4. Therefore, $\forall n \in \mathbb{N} P(n)$ is true by MI.

4. Let a be an odd integer. Prove by induction that $2^{n+2} \mid a^{2^n} - 1$ for all $n \in \mathbb{Z}^+$. Here you may use without proof the fact that the product of any two consecutive integers is even. (Note that $a^{b^c} = a^{(b^c)}$ by convention.)

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n) \equiv 2^{n+2} \mid a^{2^n} - 1$.
2. (Basis step)
 - 2.1. $a = 2p + 1$ for some integer p by the definition of odd integers.
 - 2.2. Then $a^{2^1} - 1 = a^2 - 1 = (a - 1)(a + 1) = (2p + 1 - 1)(2p + 1 + 1) = 4p(p + 1)$.
 - 2.3. Now $p(p + 1)$ is even (given by the question), so $p(p + 1) = 2m$ for some integer m by the definition of even integers.
 - 2.4. Hence, $a^{2^1} - 1 = 4(2m) = 8m = 2^3m$.
 - 2.5. So $2^{1+2} \mid a^{2^1} - 1$ and hence $P(1)$ is true.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true, i.e., $2^{k+2} \mid a^{2^k} - 1$.
 - 3.2. So $a^{2^k} - 1 = 2^{k+2}m$ for some integer m by the definition of divisibility.
 - 3.3. Then $a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$
 - 3.4.
$$= (a^{2^k})^2 - 1$$
 - 3.5.
$$= (a^{2^k} - 1)(a^{2^k} + 1)$$
 - 3.6.
$$= (2^{k+2}m)((2^{k+2}m + 1) + 1)$$
 by line 3.2
 - 3.7.
$$= 2^{k+3}m(2^{k+1}m + 1)$$
 - 3.8. Thus $2^{k+3}m \mid a^{2^{k+1}} - 1$ and so $P(k + 1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ P(n)$ is true by MI.

5. Prove by induction that

$$\forall n \in \mathbb{Z}_{\geq 8} \exists x, y \in \mathbb{N} (n = 3x + 5y).$$

(In other words, any integer-valued transaction of at least \$8 can be carried out using only \$3 and \$5 notes.)

Answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N} (n = 3x + 5y)$.
2. (Basis step) $P(8)$ is true as $8 = 3(1) + 5(1)$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(k)$ is true.
 - 3.2. Find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
 - 3.3. Case 1: $y > 0$.
 - 3.3.1. Then $k + 1 = (3x + 5y) + 1$ by the choice of x, y .
 - 3.3.2. $= 3(x + 2) + 5(y - 1)$.
 - 3.3.3. $y - 1 \in \mathbb{N}$ as $y > 0$.
 - 3.3.4. As $x + 2 \in \mathbb{N}$ and $y - 1 \in \mathbb{N}$, so $P(k + 1)$ is true.
 - 3.4. Case 2: $y = 0$.
 - 3.4.1. Then $k = 3x + 5(0) = 3x$.
 - 3.4.2. $\therefore x = k/3 \geq 8/3$ as $k \geq 8$.
 - 3.4.3. $\therefore x \geq 3$ as $x \in \mathbb{N}$.
 - 3.4.4. Thus $k + 1 = 3x + 1 = 3(x - 3) + 5(2)$.
 - 3.4.5. As $x - 3 \in \mathbb{N}$ and $2 \in \mathbb{N}$, So $P(k + 1)$ is true.
 - 3.5. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{N} P(n)$ is true by MI.

Alternative answer:

1. For each $n \in \mathbb{Z}_{\geq 8}$, let $P(n) \equiv \exists x, y \in \mathbb{N} (n = 3x + 5y)$.
2. (Basis step)
 - 2.1. $P(8)$ is true because $8 = 3(1) + 5(1)$.
 - 2.2. $P(9)$ is true because $9 = 3(3) + 5(0)$.
 - 2.3. $P(10)$ is true because $10 = 3(0) + 5(2)$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}_{\geq 8}$ such that $P(8), P(9), \dots, P(k + 2)$ are true.
 - 3.2. Apply $P(k)$ to find $x, y \in \mathbb{N}$ such that $k = 3x + 5y$.
 - 3.3. Then $k + 3 = (3x + 5y) + 3$ by the choice of x, y .
 - 3.4. $= 3(x + 1) + 5y$ where $x + 1, y \in \mathbb{N}$.
 - 3.5. Hence $P(k + 3)$ is true.
4. Therefore, $\forall n \in \mathbb{N} P(n)$ is true by Strong MI.

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}^+ \exists l \in \mathbb{Z}^+ \exists i_1, i_2, \dots, i_l \in \mathbb{N} (i_1 < i_2 < \dots < i_l \wedge n = 2^{i_1} + 2^{i_2} + \dots + 2^{i_l}).$$

Answer:

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition
 $"\exists l \in \mathbb{Z}^+ \exists i_1, i_2, \dots, i_l \in \mathbb{N} (i_1 < i_2 < \dots < i_l \wedge n = 2^{i_1} + 2^{i_2} + \dots + 2^{i_l})"$
2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(1), P(2), \dots, P(k)$ are true.
 - 3.2. Find $m \in \mathbb{Z}$ such that $k + 1 = 2m$ or $k + 1 = 2m + 1$. (This is possible because $k + 1$ is either even or odd, by lecture #1 assumption 1.)
 - 3.3. Note that $2m \leq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
 - 3.4. $\leq k + k$ as $k \geq 1$;
 - 3.5. $= 2k$.
 - 3.6. So $m \leq k$.
 - 3.7. Also, $2m + 1 \geq k + 1$ as $k + 1 = 2m$ or $k + 1 = 2m + 1$;
 - 3.8. So $2m \geq k \geq 1$ or $m \geq \frac{1}{2}$ or $m \geq 1$ as $m \in \mathbb{Z}$ and 1 is the smallest integer $\geq \frac{1}{2}$.
 - 3.9. By lines 3.6 and 3.8, $1 \leq m \leq k$, and so $P(m)$ is true by the induction hypothesis.
 - 3.10. Apply $P(m)$ to find $l \in \mathbb{Z}^+$ and $i_1, i_2, \dots, i_l \in \mathbb{N}$ such that
 $i_1 < i_2 < \dots < i_l$ and $m = 2^{i_1} + 2^{i_2} + \dots + 2^{i_l}$.
 - 3.11. Case 1: $k + 1 = 2m$.
 - 3.11.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \dots + 2^{i_l})$
 - 3.11.2. $= 2^{i_1+1} + 2^{i_2+1} + \dots + 2^{i_l+1}$.
 - 3.11.3. Also, $i_1 + 1 < i_2 + 1 < \dots < i_l + 1$ as $i_1 < i_2 < \dots < i_l$.
 - 3.11.4. So $P(k + 1)$ is true.
 - 3.12. Case 1: $k + 1 = 2m + 1$.
 - 3.12.1. Then $k + 1 = 2(2^{i_1} + 2^{i_2} + \dots + 2^{i_l}) + 1$
 - 3.12.2. $= 2^0 + 2^{i_1+1} + 2^{i_2+1} + \dots + 2^{i_l+1}$.
 - 3.12.3. Also, $0 < i_1 + 1 < i_2 + 1 < \dots < i_l + 1$ as $0 \leq i_1 < i_2 < \dots < i_l$.
 - 3.12.4. So $P(k + 1)$ is true.
 - 3.13. Hence $P(k + 1)$ is true for all cases.
4. Therefore, $\forall n \in \mathbb{Z}^+ P(n)$ is true by Strong MI.

Alternative answer (not for the faint-hearted):

1. For each $n \in \mathbb{Z}^+$, let $P(n)$ be the proposition
“ $\exists l \in \mathbb{Z}^+ \exists i_1, i_2, \dots, i_l \in \mathbb{N} (i_1 < i_2 < \dots < i_l \wedge n = 2^{i_1} + 2^{i_2} + \dots + 2^{i_l})$ ”
2. (Basis step) $P(1)$ is true as $1 = 2^0$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{Z}^+$ such that $P(k)$ is true.
 - 3.2. Apply this assumption to obtain $l \in \mathbb{Z}^+$ and $i_1, i_2, \dots, i_l \in \mathbb{N}$ such that
 $i_1 < i_2 < \dots < i_l$ and $k = 2^{i_1} + 2^{i_2} + \dots + 2^{i_l}$.
 - 3.3. Since $k < 2^k$, we know k is an element of the set $\mathbb{N} \setminus \{i_1, i_2, \dots, i_l\}$.
 - 3.4. By the **Well-Ordering Principle**, this set has a minimum, say m .
 - 3.5. Find $j \in \{0, 1, \dots, l\}$ such that $i_1 < i_2 < \dots < i_j < m < i_{j+1} < \dots < i_l$.
 - 3.6. The minimality of m tells us $0, 1, 2, \dots, m-1 \in \{i_1, i_2, \dots, i_l\}$.
 - 3.7. Thus $0, 1, 2, \dots, m-1 \in \{i_1, i_2, \dots, i_j\}$ by the choice of j .
 - 3.8. The choice of j also tell us $i_1, i_2, \dots, i_j \in \{0, 1, 2, \dots, m-1\}$.
 - 3.9. From these, we deduce that $\{i_1, i_2, \dots, i_j\} = \{0, 1, 2, \dots, m-1\}$.
 - 3.10. Now, $k + 1 = 1 + 2^{i_1} + 2^{i_2} + \dots + 2^{i_l}$ by line 3.2;
 - 3.11. $= 2^0 + 2^0 + 2^1 + 2^2 + \dots + 2^{m-1} + 2^{i_{j+1}} + \dots + 2^{i_l}$ by line 3.9;
 - 3.12. $= 2^1 + 2^1 + 2^2 + \dots + 2^{m-1} + 2^{i_{j+1}} + \dots + 2^{i_l}$
 - 3.13. $= 2^2 + 2^2 + \dots + 2^{m-1} + 2^{i_{j+1}} + \dots + 2^{i_l}$
 - 3.14. $= \dots = 2^{m-1} + 2^{m-1} + 2^{i_{j+1}} + \dots + 2^{i_l}$
 - 3.15. $= 2^m + 2^{i_{j+1}} + \dots + 2^{i_l}$
 - 3.16. So $P(k + 1)$ is true.
4. Therefore, $\forall n \in \mathbb{Z}^+ P(n)$ is true **by MI**.

7. Let F_0, F_1, F_2, \dots be the Fibonacci sequence. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{N}$.

Answer:

1. $F_{n+4} = F_{n+3} + F_{n+2}$ by the definition of F_{n+4} ;
2. $= (F_{n+2} + F_{n+1}) + F_{n+2}$ by the definition of F_{n+3} ;
3. $= 2F_{n+2} + F_{n+1}$
4. $= 3F_{n+2} - F_{n+2} + F_{n+1}$
5. $= 3F_{n+2} - (F_{n+1} + F_n) + F_{n+1}$ by the definition of F_{n+2} ;
6. $= 3F_{n+2} - F_n$

Note: Not all questions need to be solved with mathematical induction, unless the question explicitly states so. For this question, a simple direct proof like this suffices.

8. Let F_0, F_1, F_2, \dots be the Fibonacci sequence. Show by induction that for all $n \in \mathbb{N}$,

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n.$$

Answer:

1. For each $n \in \mathbb{N}$, let $P(n)$ be the proposition
 $"F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n"$
2. (Basis step)
 - 2.1. Since $F_0 = 0$ and $F_1 = 1$,
 $F_{0+1}^2 - F_{0+1}F_0 - F_0^2 = 1^2 - (1 \times 0) - 0^2 = 1 = (-1)^0$.
 - 2.2. So $P(0)$ is true.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{N}$ such that $P(k)$ is true, i.e., $F_{k+1}^2 - F_{k+1}F_k - F_k^2 = (-1)^k$.
 - 3.2. Then $F_{(k+1)+1}^2 - F_{(k+1)+1}F_{k+1} - F_{k+1}^2$
 - 3.3. $= F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2$
 - 3.4. $= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} - F_{k+1}^2$ by the definition of F_{k+2} ;
 - 3.5. $= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_kF_{k+1} - F_{k+1}^2$
 - 3.6. $= -(F_{k+1}^2 - F_kF_{k+1} - F_k^2)$
 - 3.7. $= -(-1)^k$ by the induction hypothesis;
 - 3.8. $= (-1)^{k+1}$.
 - 3.9. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{N} P(n)$ is true by MI.

9. Let a_0, a_1, a_2, \dots be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{N}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{N}$.

Answer:

1. For each $n \in \mathbb{N}$, let $P(n) \equiv a_n < 3^n$.
2. (Basis step)
 $P(0), P(1), P(2)$ are true as $a_0 = 0 < 1 = 3^0$, $a_1 = 2 < 3 = 3^1$, and $a_2 = 7 < 9 = 3^2$.
3. (Induction step)
 - 3.1. Let $k \in \mathbb{N}$ such that $P(0), P(1), \dots, P(k+2)$ are true.
 - 3.2. $P(k), P(k+1), P(k+2)$ are true means $a_k < 3^k, a_{k+1} < 3^{k+1}, a_{k+2} < 3^{k+2}$.
 - 3.3. $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ by the definition of a_{k+3} ;
 - 3.4. $< 3^{k+2} + 3^{k+1} + 3^k$ by the induction hypothesis;
 - 3.5. $< 3^{k+2} + 3^{k+2} + 3^{k+2}$
 - 3.6. $= 3(3^{k+2}) = 3^{k+3}$.
 - 3.7. Thus $P(k+1)$ is true.
4. Therefore, $\forall n \in \mathbb{N} P(n)$ is true by Strong MI.

10. Define a set S recursively as follows.

- (1) $2 \in S$. (base clause)
- (2) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)
- (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers 0, 6, 15, 16, 36 are in S ? Which are not?

Answer:

Structure induction over S . To prove that $\forall n \in S P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

(base step) show that $P(2)$ is true; and

(induction step) show that $\forall n \in S (P(n) \Rightarrow P(3n) \wedge P(n^2))$ is true.

- We know $0 \notin S$ because all $n \in S$ satisfy $n \geq 2$, as one can show by structural induction over S as follows.
 1. For each $n \in S$, let $P(n) \equiv n \geq 2$.
 2. (Basis step) $P(2)$ is true because $2 \geq 2$.
 3. (Induction step)
 - 3.1. Let $x \in S$ such that $P(x)$ is true, i.e., that $x \geq 2$.
 - 3.2. Then $3x \geq 3 \times 2 = 6 \geq 2$ and $x^2 \geq 2^2 = 4 \geq 2$.
 - 3.3. So $P(3x)$ and $P(x^2)$ are both true.
 4. Hence $\forall n \in S P(n)$ is true by structural induction over S .
- - $2 \in S$ by the base clause.
 - $\therefore 6 \in S$ by the recursion clause with $n = 2$ and the previous line.
 - $\therefore 36 \in S$ by the recursion clause with $n = 6$ and the previous line.
- - $2 \in S$ by the base clause.
 - $\therefore 4 \in S$ by the recursion clause with $n = 2$ and the previous line.
 - $\therefore 16 \in S$ by the recursion clause with $n = 4$ and the previous line.
- We know $15 \notin S$ because no $n \in S$ is odd, as one can show by structural induction over S .

11. Let $A = \{1,2,3,4,5\}$ and $B = \{1,3,5,7,9\}$. Define a set S recursively as follows.

- (1) $A, B \in S$. (base clause)
- (2) If $X, Y \in S$, then $X \cap Y \in S$ and $X \cup Y \in S$ and $X \setminus Y \in S$ (recursion clause)
- (3) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

For each of the following sets, determine whether it is in S , and use one sentence to explain your answer.

- (a) $C = \{2,4,7,9\}$.
- (b) $D = \{2,3,4,5\}$.

Answer:

Structure induction over S . To prove that $\forall X \in S P(X)$ is true, where each $P(X)$ is a proposition, it suffices to:

(base step) show that $P(A)$ and $P(B)$ are true; and

(induction step) show that

$$\forall X, Y \in S (P(X) \wedge P(Y) \Rightarrow P(X \cap Y) \wedge P(X \cup Y) \wedge P(X \setminus Y))$$

is true.

(a) $A \setminus B = \{2,4\}$ and $B \setminus A = \{7,9\}$.

$C \in S$ because $C = \{2,4,7,9\} = (A \setminus B) \cup (B \setminus A)$.

(b) $D \neq S$ because $1 \notin D$ and $3 \in D$, but one can show by structural induction that

$$\forall X \in S (1 \in X \Leftrightarrow 3 \in X).$$

(Proof shown below.)

1. For each $X \in S$, let $P(X) \equiv 1 \in X \Leftrightarrow 3 \in X$.
2. (Basis step)
 - 2.1. As $1,3 \in A$ and $1,3 \in B$, we know $1 \in A \Leftrightarrow 3 \in A$ and $1 \in B \Leftrightarrow 3 \in B$.
 - 2.2. So $P(A)$ and $P(B)$ are true.
3. (Induction step)
 - 3.1. Let $X, Y \in S$ such that $P(X)$ and $P(Y)$ are true, i.e.,
 $1 \in X \Leftrightarrow 3 \in X$ and $1 \in Y \Leftrightarrow 3 \in Y$.
 - 3.2. Case 1: If $1,3 \in X$ and $1,3 \in Y$, then
 - 3.2.1. $1,3 \in X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
 - 3.2.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
 - 3.2.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
 - 3.3. Case 2: If $1,3 \in X$ and $1,3 \notin Y$, then
 - 3.3.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \in X \setminus Y$;
 - 3.3.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
 - 3.3.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
 - 3.4. Case 3: If $1,3 \notin X$ and $1,3 \in Y$, then
 - 3.4.1. $1,3 \notin X \cap Y$ and $1,3 \in X \cup Y$ and $1,3 \notin X \setminus Y$;
 - 3.4.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
 - 3.4.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
 - 3.5. Case 4: If $1,3 \notin X$ and $1,3 \notin Y$, then
 - 3.5.1. $1,3 \notin X \cap Y$ and $1,3 \notin X \cup Y$ and $1,3 \notin X \setminus Y$;
 - 3.5.2. So $1 \in X \cap Y \Leftrightarrow 3 \in X \cap Y$ and $1 \in X \cup Y \Leftrightarrow 3 \in X \cup Y$ and $1 \in X \setminus Y \Leftrightarrow 3 \in X \setminus Y$.
 - 3.5.3. Thus $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true.
 - 3.6. Hence $P(X \cap Y)$ and $P(X \cup Y)$ and $P(X \setminus Y)$ are all true in all cases.
4. It follows that $\forall X \in S P(X)$ is true by structural induction over S .