NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2021/22 Semester 2

CS1231S — DISCRETE STRUCTURES

8 March 2022 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **SEVENTEEN (17)** questions in **THREE (3)** parts and comprises **SEVEN (7)** printed pages.
2. This is an **OPEN BOOK** assessment.
3. Printed/written materials are allowed. Apart from calculators, electronic devices are not allowed.
4. Answer **ALL** questions.
5. Write your answers only on the **ANSWER SHEETS**. You may write in pen or pencil. You are to write within the space provided. No extra pages should be submitted.
6. The maximum mark of this assessment is 50.
7. Do not start writing or flip over this page until you are told to do so.

——— END OF INSTRUCTIONS ———

1	2	3	4	5	6
E	B	C	D	D	C

7	8	9	10	11	12	13
AD	ABDE	ACD	BCDE	ACDE	ABD	BC

Part A: Multiple Choice Questions (Total: 12 marks)

Each multiple choice question (MCQ) is worth two marks and has exactly **one** correct answer.

1. Given this statement:

   "If Aiken can do it, then Dueet can do it."

   Which of the following is the negation of the above statement?

   A. "If Dueet cannot do it, then Aiken cannot do it."
   B. "If Aiken cannot do it, then Dueet cannot do it."
   C. "Aiken cannot do it or Dueet can do it."
   D. "Aiken can do it or Dueet cannot do it."
   E. None of (A), (B), (C), (D) is correct.

   Answer: E

   The negation of "$p \rightarrow q$" is "$p \wedge \sim q$". So the negation of the given statement is "Aiken can do it and Dueet cannot do it."

2. The binary relation $R = \{(0,0), (1,1)\}$ on $A = \{0,1,2\}$ has the following properties.

   A. R is reflexive, is not symmetric, is transitive.
   B. R is not reflexive, is symmetric, is transitive.
   C. R is not reflexive, is symmetric, is not transitive.
   D. R is reflexive, is not symmetric, is not transitive.
   E. R is not reflexive, is not symmetric, is not transitive.

   Answer: B

3. Given $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$. Consider the following quantified statements:

   (I) $\exists(x, y) \in A \times B \ \big((x - y), (y - x)\big) \in A \times B$
   (II) $\exists(x, y) \in A \times B \ \big((x + y), (x - y)\big) \in B \times A$
   (III) $\forall(x, y) \in A \times B \ \exists z \in B \ (xz, yz) \in A \times B$
   (IV) $\forall(x, y) \in A \times B \ \exists z \in A \ \big((x - z), (y - z)\big) \in B \times A$

   Which of the above statements are true?

   A. (I) and (III).
   B. (I) and (IV).
   C. (II) and (III).
   D. (II) and (IV).
   E. None of the above 4 options are correct.

   Answer: C

   (I) False: No $(x, y) \in A \times B$ such that $(x - y)$ and $(y - x)$ are positive.
   (II) True: $(2,1) \in A \times B$ and $(2 + 1, 2 - 1) = (3,1) \in A \times B$.
   (III) True: Pick $z = 1 \in B$.
   (IV) False: Negation is $\exists(x, y) \in A \times B \ \forall z \in A \ \big((x - z), (y - z)\big) \notin B \times A$. Pick $(1,1) \in A \times B$.

4. Let $A_n = \{(x, y) \in \mathbb{Z}^2 : xy = n\}$. What is the cardinality of $A_6 \cup A_8$ and $A_6 \cap A_8$?

   A. $|A_6 \cup A_8| = 8;\quad |A_6 \cap A_8| = 4.$
   B. $|A_6 \cup A_8| = 16;\quad |A_6 \cap A_8| = 8.$
   C. $|A_6 \cup A_8| = 8;\quad |A_6 \cap A_8| = 0.$
   D. $|A_6 \cup A_8| = 16;\quad |A_6 \cap A_8| = 0.$
   E. None of the above.

Answer: D

$A_6 = \{(-1,-6), (-2,-3), (-3,-2), (-6,-1), (1,6), (2,3), (3,2), (6,1)\};$
$A_8 = \{(-1,-8), (-2,-4), (-4,-2), (-8,-1), (1,8), (2,4), (4,2), (8,1)\}.$

5. Let R be a binary relation on a set A. Suppose you are given the following statement S:

$$R \text{ is reflexive} \Leftrightarrow \forall a \in A\ \exists b \in A\ (a\ R\ b).$$

Which of the following needs to be true in order for statement S to be true?

   (I) R is symmetric.
   (II) R is antisymmetric.
   (III) R is transitive.

   A. Only (I).
   B. Only (II).
   C. Only (III).
   D. Only (I) and (III).
   E. Only (II) and (III).
   F. None of (I), (II) or (III) needs to be true.

Answer: D

6. Which of the following sets is not well-ordered?

   A. $\{x \in \mathbb{Z} : x = 2^n \text{ for some } n \in \mathbb{N}\}$
   B. $\{x \in \mathbb{Z} : x \geq 0\}$
   C. $\{x \in \mathbb{Q} : x \geq 0\}$
   D. $\{n \in \mathbb{N} : n = 4s + 9t \text{ for some integers } s \text{ and } t\}$
   E. None of the above.

Answer: C

$\{x \in \mathbb{Q} : x \geq 0\}$ contains the subset $S = \{x \in \mathbb{Q} : x > 0\}$ which is non-empty and does not have a smallest element.

Proof:

1. Suppose not, that is, S has a smallest element, say l.
2. Since l is a positive rational number, $\frac{l}{2}$ is also a positive rational number and hence $\frac{l}{2} \in S$.
3. But $\frac{l}{2} < l$, contradicting that l is the smallest element.

Part B: Multiple Response Questions [Total: 21 marks]

Each multiple response question (MRQ) is worth three marks and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, C, E are the correct answers, write A, C, E. Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

7. Given the statement "all cats are mammals", how do you write a quantified statement for it? Assuming that the universe of discourse is the set of animals, the predicates $Cat(x)$ means x is a cat and $Mammal(x)$ means x is a mammal.

   A. $\forall x \left(Cat(x) \rightarrow Mammal(x)\right)$
   B. $\forall x \left(Cat(x) \wedge Mammal(x)\right)$
   C. $\forall x \left(Mammal(x) \rightarrow Cat(x)\right)$
   D. $\forall x \left(Mammal(x) \vee {\sim}Cat(x)\right)$
   E. None of the above.

   Answer: A, D

8. Which of the following statements is/are tautologies?

   A. $\left((p \rightarrow q) \wedge p\right) \rightarrow q$
   B. $\left((p \rightarrow q) \wedge {\sim}q\right) \rightarrow {\sim}p$
   C. $\left((p \vee q) \wedge p\right) \rightarrow {\sim}q$
   D. $({\sim}p \rightarrow (q \wedge {\sim}q)) \rightarrow p$
   E. $(p \wedge {\sim}q) \rightarrow ({\sim}q \vee p)$

   Answer: A, B, D, E

9. Which of the following statements is/are logically equivalent to the following statement?

$$\forall x\, \forall y\, \exists z \left(P(x,z) \wedge P(y,z)\right)$$

   A. $\forall y\, \forall z\, \exists x \left(P(z,x) \wedge P(y,x)\right)$
   B. $\forall x\, \forall y\, \exists z \left(P(z,x) \wedge P(z,y)\right)$
   C. $\forall y\, \forall x\, \exists z \left(P(x,z) \wedge P(y,z)\right)$
   D. $\forall x\, \forall y\, \exists z \left(P(y,z) \wedge P(x,z)\right)$
   E. $\forall z\, \forall y\, \exists x \left(P(z,x) \wedge P(y,z)\right)$

   Answer: A, C, D

10. Consider the predicate $P(x,y,z) \equiv$ "$(x \mid y) \vee (y \mid z)$" for $x,y,z \in \mathbb{Z}^+$.

    Which of the following statements is/are true?

    A. $\forall x \in \mathbb{Z}^+\ \forall y \in \mathbb{Z}^+\ \forall z \in \mathbb{Z}^+\ P(x,y,z)$.
    B. $\forall x \in \mathbb{Z}^+\ \forall y \in \mathbb{Z}^+\ \exists z \in \mathbb{Z}^+\ P(x,y,z)$.
    C. $\exists x \in \mathbb{Z}^+\ \forall y \in \mathbb{Z}^+\ \forall z \in \mathbb{Z}^+\ P(x,y,z)$.
    D. $\forall x \in \mathbb{Z}^+\ \exists y \in \mathbb{Z}^+\ \forall z \in \mathbb{Z}^+\ P(x,y,z)$.
    E. $\exists x \in \mathbb{Z}^+\ \exists y \in \mathbb{Z}^+\ \exists z \in \mathbb{Z}^+\ P(x,y,z)$.

    Answer: B, C, D, E

(B) Let $z = y$.
(C) Let $x = 1$.
(D) Let $y = 1$.
(E) Let $x = y = z = 1$.

11. Suppose $\mathcal{C}$ is a partition of a non-empty set A. Which of the following statements are true?

A. $\forall X \in \mathcal{C}\ (X \neq \emptyset)$.
B. $\forall X, Y \in \mathcal{C}\ (X \cap Y = \emptyset)$.
C. $\bigcup_{X \in \mathcal{C}} X = A$. (Note: $\bigcup_{X \in \mathcal{C}} X$ means "union of all the members of $\mathcal{C}$".)
D. $\mathcal{C} \subseteq \wp(A)$. (Note: $\wp(A)$ is the power set of A.)
E. $\mathcal{C} \neq \emptyset$.

Answer: A, C, D, E

(B) False because if $X = Y$, then $X \cap Y = X \cap X = X \neq \emptyset$.

12. For each $k \in \mathbb{Z}^+$, let $A_k = \{n \in \mathbb{Z}_{\geqslant 2} : k = mn \text{ for some } m \in \mathbb{Z}_{\geqslant 2}\}$.

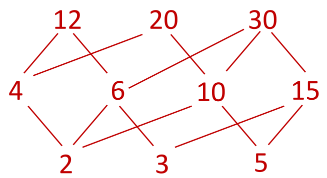
Let $\preccurlyeq$ be the divisibility relation on A_{60}.

Which of the following is/are linearization $\preccurlyeq^*$ of $\preccurlyeq$?

A. $2 \preccurlyeq^* 3 \preccurlyeq^* 4 \preccurlyeq^* 5 \preccurlyeq^* 6 \preccurlyeq^* 10 \preccurlyeq^* 12 \preccurlyeq^* 15 \preccurlyeq^* 20 \preccurlyeq^* 30$.
B. $5 \preccurlyeq^* 3 \preccurlyeq^* 2 \preccurlyeq^* 4 \preccurlyeq^* 6 \preccurlyeq^* 10 \preccurlyeq^* 15 \preccurlyeq^* 30 \preccurlyeq^* 20 \preccurlyeq^* 12$.
C. $2 \preccurlyeq^* 3 \preccurlyeq^* 5 \preccurlyeq^* 4 \preccurlyeq^* 6 \preccurlyeq^* 10 \preccurlyeq^* 15 \preccurlyeq^* 12 \preccurlyeq^* 20 \preccurlyeq^* 30 \preccurlyeq^* 60$.
D. $2 \preccurlyeq^* 4 \preccurlyeq^* 3 \preccurlyeq^* 6 \preccurlyeq^* 12 \preccurlyeq^* 5 \preccurlyeq^* 10 \preccurlyeq^* 20 \preccurlyeq^* 15 \preccurlyeq^* 30$.
E. None of the above is a linearization of $\preccurlyeq$.

Answer: A, B, D

Hasse diagram:



13. Let R and S be binary relations on $A = \{a, b, c\}$ as follows:

$$R = \{(a,c), (b,b), (b,c), (c,a)\};$$
$$S = \{(a,a), (a,b), (b,c), (c,a)\}.$$

Which of the following statements is/are true?

A. $S^{-1} = S$.
B. $|R^{-1}| = |R|$.
C. $|R^2| = |S \circ R|$.
D. $|R^2| = |R \circ S|$.
E. $|S \circ R| = |R \circ S|$.

Answer: B, C

$R^2 = \{(a,a), (b,a), (b,b), (b,c), (c,c)\}$;
$S \circ R = \{(a,a), (b,a), (b,c), (c,a), (c,b)\}$;
$R \circ S = \{(a,b), (a,c), (b,a), (c,c)\}$.

Part C: There are 3 questions in this part [Total: 17 marks]

14. Theorem 2.1.1 is given as follows:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r \equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r \equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
5	Negation laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	${\sim}$**true** $\equiv$ **false**	${\sim}$**false** $\equiv$ **true**

Simplify the following expression using the laws above, justifying every step. You may combine consecutive steps using the same law in one step. [3 marks]

$$\big(({\sim}p \wedge q) \vee {\sim}(q \vee {\sim}p)\big) \wedge q$$

Answer:

$\big(({\sim}p \wedge q) \vee {\sim}(q \vee {\sim}p)\big) \wedge q$

$\equiv \Big(({\sim}p \wedge q) \vee \big({\sim}q \wedge {\sim}({\sim}p)\big)\Big) \wedge q$ by De Morgan's law

$\equiv \big(({\sim}p \wedge q) \vee ({\sim}q \wedge p)\big) \wedge q$ by the double negative law

$\equiv q \wedge \big((q \wedge {\sim}p) \vee ({\sim}q \wedge p)\big)$ by the commutative law (2 times)

$\equiv \big(q \wedge (q \wedge {\sim}p)\big) \vee \big(q \wedge ({\sim}q \wedge p)\big)$ by the distributive law

$\equiv \big((q \wedge q) \wedge {\sim}p\big) \vee \big((q \wedge {\sim}q) \wedge p\big)$ by the associative law

$\equiv (q \wedge {\sim}p) \vee \big((q \wedge {\sim}q) \wedge p\big)$ by the idempotent law

$\equiv (q \wedge {\sim}p) \vee (\mathbf{false} \wedge p)$ by the negation law

$\equiv (q \wedge {\sim}p) \vee (p \wedge \mathbf{false})$ by the commutative law

$\equiv (q \wedge {\sim}p) \vee \mathbf{false}$ by the universal bound law

$\equiv (q \wedge {\sim}p)$ by the identity law

15. Prove the following. [Total: 4 marks]

(a) Suppose $x, y \in \mathbb{Z}$. Prove that if x is odd and xy is even, then y is even. [1 mark]

(b) Prove that $\{x \in \mathbb{Z} : 2 \mid x\} \cap \{x \in \mathbb{Z} : 9 \mid x\} \subseteq \{x \in \mathbb{Z} : 6 \mid x\}$. [3 marks]

Answer:

(a) Proof by contradiction:

1. Suppose not, that is, x is odd and xy is even and y is odd.
2. Since x and y are odd, then xy is odd by tutorial 1 question 9.
3. This contradicts xy is even.
4. Hence if x is odd and xy is even, then y is even.

Or: Proof by contraposition: If y is odd then x is even or xy is odd.

1. Suppose y is odd.
2. Case 1: x is even.
   2.1. Then the conclusion is true.
3. Case 2: x is odd.
   3.1. Then xy is odd by tutorial 1 question 9.
   3.2. Then the conclusion is true.
4. In both cases, the conclusion is true.
5. Hence if x is odd and xy is even, then y is even.

(b) Proof:

1. Let $a \in \{x \in \mathbb{Z} : 2 \mid x\} \cap \{x \in \mathbb{Z} : 9 \mid x\}$.
2. So $a \in \{x \in \mathbb{Z} : 2 \mid x\}$ and $a \in \{x \in \mathbb{Z} : 9 \mid x\}$ by the definition of $\cap$.
3. Hence $2 \mid a$ and $9 \mid a$.
4. Hence $a = 2n$ and $a = 9m$ for some $n, m \in \mathbb{Z}$ by the definition of divisibility.
5. Since $a = 2n$ is even, $9m$ must be even by the definition of even integer.
6. Since 9 is odd and $9m$ is even, then m must be even by part (a).
7. So $m = 2k$ for some $k \in \mathbb{Z}$ by the definition of even integer.
8. Hence $a = 9m = 9(2k) = 6(3k)$.
9. So $6 \mid$ a by the definition of divisibility.
10. Hence $a \in \{x \in \mathbb{Z} : 6 \mid x\}$.
11. Therefore $\{x \in \mathbb{Z} : 2 \mid x\} \cap \{x \in \mathbb{Z} : 9 \mid x\} \subseteq \{x \in \mathbb{Z} : 6 \mid x\}$.

Note that it is <u>not</u> correct to write:

$2 \mid a \ \wedge 9 \mid a \Rightarrow 18 \mid a$ without justification, as in general, $p \mid a \ \wedge q \mid a \nRightarrow pq \mid a$.

Counterexample: $6|18 \ \wedge 9|18 \nRightarrow 54|18$.

16. Let R be a relation $\{(a, b) : a \neq b\}$ on $\mathbb{Z}$. Let S be the reflexive closure of R. [Total: 4 marks]

(a) Is R transitive? Prove or disprove.

(b) Is S transitive? Prove or disprove.

(Recall in tutorial #5 that the reflexive closure, S, of a relation R on a set A, is the smallest relation on A that is reflexive and contains R as a subset.)

Answer:

(a) R is not transitive.
Counterexample: $(1,2) \in R$ and $(2,1) \in R$ but $(1,1) \notin R$.

(b) S is transitive.

1. S includes all elements $(a, a)\ \forall a \in \mathbb{Z}$ as S is the reflexive closure of R.
2. For any $a, b, c \in \mathbb{Z}$, suppose $(a, b) \in S$ and $(b, c) \in S$.
3. There are two cases: $a \neq c$ or $a = c$.
4. Case 1: $a \neq c$
   4.1. Then $(a, c) \in R$ by definition of R.
   4.2. Hence $(a, c) \in S$ as $R \subseteq S$.
5. Case 2: $a = c$
   5.1. Then $(a, c) = (a, a) \in S$ by line 1.
6. In all cases $(a, c) \in S$ and hence S is transitive.

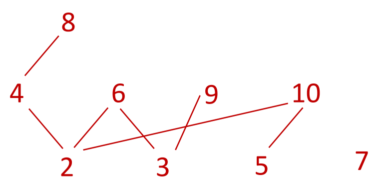
17. You do not need to explain or prove your answers for this question. [Total: 6 marks]

The divisibility relation | is defined by: $a \mid b$ if and only if there exists an integer k such that $b = ak$.

(a) What are the minimal and maximal elements of the set $A = \{2,3,4,5,6,7,8,9,10\}$ under the divisibility relation? [2 marks]

Answer:

Hasse diagram:



The minimal elements are 2,3,5,7. The maximal elements are 6,7,8,9,10.

(b) What are the minimal and maximal elements of the set $\mathbb{N}$ under the divisibility relation? [2 marks]

Answer: The only minimal element (also the smallest element) is 1 since 1 divides all natural numbers. The only maximal element (also the largest element) is 0 since all natural numbers divide 0.

(c) Let $\preccurlyeq$ be a partial order on a set A. A subset C of A is called **chain** if and only if each pair of elements in C is comparable, that is, $\forall a, b \in C\ (a \preccurlyeq b \vee b \preccurlyeq a)$. A **maximal chain** is a chain M such that $t \notin M \Rightarrow M \cup \{t\}$ is not a chain.

Given that the set $\wp(\{a, b, c\})$ is partially ordered with respect to the subset ($\subseteq$) relation, write out two maximal chains. [2 marks]

Answer: Since $\emptyset \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$, the set $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ is a maximal chain. The other 5 maximal chains are:
$\{\emptyset, \{a\}, \{a, c\}, \{a, b, c\}\}, \{\emptyset, \{b\}, \{a, b\}, \{a, b, c\}\}, \{\emptyset, \{b\}, \{b, c\}, \{a, b, c\}\}, \{\emptyset, \{c\}, \{a, c\}, \{a, b, c\}\}$ and $\{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\}$.

=== END OF PAPER ===

CS1231S Midterm Test
AY2021/22 Semester 2

This is the report for the CS1231S midterm test held on 8 March 2022.

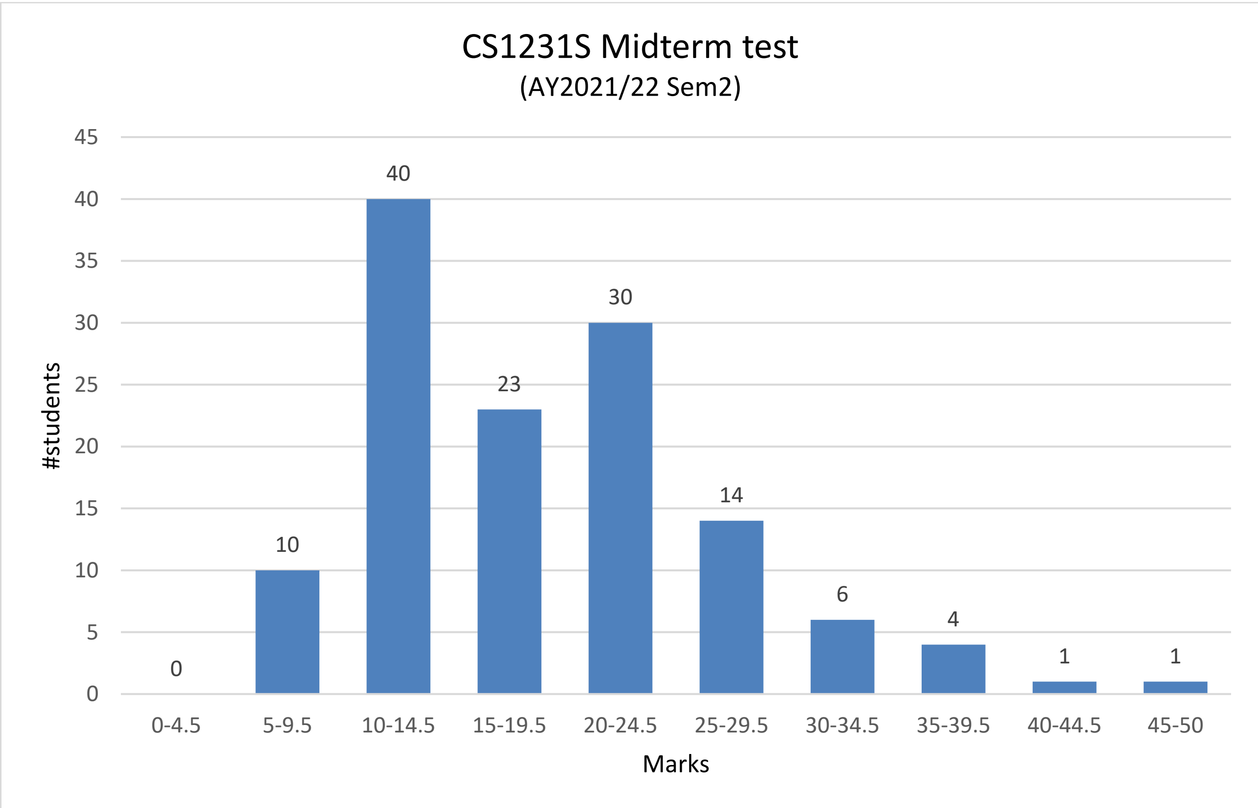
129 (out of 132) students sat for the midterm test.

1. General statistics

	Part A (Q1-6) (12 marks)	Part B (Q7-13) (21 marks)	Q14 (3 marks)	Q15 (4 marks)	Q16 (4 marks)	Q17 (6 marks)	**Total (50 marks)**
Average mark	5.60 (46.6%)	7.21 (34.3%)	1.94 (64.7%)	0.72 (17.9%)	1.17 (29.2%)	1.95 (32.6%)	18.6 (37.2%)
Median mark	6	6	2.5	0	0.5	2	17.5

The average is 18.6 (or 37.2%) as compared against the average of 58.4% in AY2021/22 Semester 1 and 57.0% in AY2021/22 semester 2. This is the worst mid-term result in the past 5 years.

Below is the chart for the overall results.



2. Part A: MCQs 1 - 7

The table below shows the percentage of students who chose the correct answers, and of those who chose the most popular wrong answers:

	Q1	Q2	Q3	Q4	Q5	Q6
			Easiest		Hardest	
%students who chose the correct answer	E (72.1%)	B (65.9%)	C (83.0%)	D (20.2%)	D (17.8%)	C (20.9%)
%students who chose the most popular wrong answer	B (10.1%)	E (14.0%)	E (7.8%)	C (62.8%)	F (31.0%)	D (32.6%)

The easiest question is Q1 with 72.1% of students getting it right. However, as this is a give-away question, so it is worrisome that 27.9% chose the wrong answer. The most popular wrong answer chosen is (B), which is the inverse of the given statement, not the negation.

By now, you should know the basics: negation, contrapositive, converse and inverse.

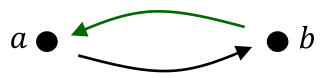
Q4, though not the hardest, is surprising because majority picked answer (C). I think the reason students picked that could be they considered only positive integers and not both positive and negative integers. Could it be that they took the symbol $\mathbb{Z}^2$, which means $\mathbb{Z} \times \mathbb{Z}$ (Cartesian product of $\mathbb{Z}$) to be (wrongfully) the square of integers (and therefore not negative)?

Understanding symbols is very important in mathematics.

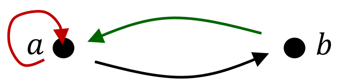
The hardest MCQ seems to be Q5. Many students picked the wrong answer (F). For a relation to be reflexive, it is not sufficient that every element has an element it relates to ($\forall a \in A\ \exists b \in A\ (a\ R\ b)$). The relation must also be symmetric and transitive. Why?

A reflexive relation has a loop around every element in the directed graph. Given that every element, say a, has an element, say b, it is related to (as given in the right hand side of the given statement S), i.e. $a\ R\ b$, so we have this:

To get a loop around a, we must also have $b\ R\ a$ (which means R has to be symmetric):



And then R must also be transitive ($a\ R\ b\ \wedge b\ R\ a\ \Rightarrow a\ R\ a$) to get the loop around a:



(A loop will also appear around b, but here we are focusing on a.)

As you can see, a diagram helps a lot.

3. Part B: MRQs 7 - 13

The table below shows the answer and average mark for each question:

	Q7	Q8	Q9	Q10	Q11	Q12	Q13
Answer	AD	ABDE	ACD	BCDE	ACDE	ABD	BC
%correct	70.5%	47.3%	30.2%	29.5%	12.4%	38.8%	11.6%
	Easiest				2nd hardest		Hardest

Q7 was intended to be a give-away so I am not surprised it has the highest percentage of strike. However, still, 29.5% didn't get it right.

For Q13, I went to check and recheck my answer as so many students didn't get it right. Many students picked BE as their answer. I do not know how they got $|S \circ R| = |R \circ S|$. Do let me know so that I can tell future batches of students of such common mistake.

I deliberately set Q11 to test whether students are careful in reading symbols. But still, I didn't expect it to end up as the hardest question. Though I had expected some would write ABCDE as their answers, didn't expect that many did.

Do note that when you see $\forall x, y$, it doesn't mean that x and y must be distinct, unless you add $x \neq y$.

4. Comments on Q14 (graded by Prof Leong)

I am surprised by the number of elementary mistakes by the students.
So, let me start with an advice and best practice.

Use Spaces and Parenthesis to separate out different logical pieces
First, once an expression gets complicated, try to use *spaces and parenthesis* to separate out the different logical pieces. For example, consider this: with parenthesis and no-spaces

((~p^q)v~q)^((~p^q)^q)

With this same expression, type using the Equation format in Word, and with extra spaces between big-logical pieces:

$$\Big(\big((\sim p \wedge q) \vee \sim q\big) \;\wedge\; \big((\sim p \wedge q) \vee p\big) \;\wedge\; q \;\Big)$$

It is easy to see the logical pieces and the "extend" of the different ∨, ∧, ~ operators. The equation tool in Word (and in TeX or LaTeX) will also automagically enlarge the parentheses for you as well. It greatly helps you (and of course, it helps the graders too).

Missing Parenthesis:
Many of you missed out parenthesis around logical pieces. It is ONLY OK, when all the operators are the same (all ∨ or all ∧), but not when there is a mixture of ∨ and ∧ . The above expression, missing some parenthesis, will look like this.

$$(\sim p \wedge q \;\; \vee \sim q) \wedge \sim p \wedge q \vee p \wedge q$$

This is very confusing and **WRONG** since it is now ambiguous.
For some, this occur very early in your answers, and so the rest, from then on, are **all WRONG**. (This often leads to the next two errors.)

Wrong application of commutative law:
When there are different operators (∨ and ∧) in an expression, if you miss out the parenthesis, then you can easily commit this error.

WRONG: $\sim p \wedge q \vee p \equiv \sim p \wedge p \vee q$ **WRONG:** $\sim p \wedge q \vee p \equiv \sim p \vee p \wedge q$

Wrong application of associative law:
Associative law can ONLY be applied when all the operators are the same (all ∨ or all ∧). You <u>don't</u> apply it when there are mixed operators.

RIGHT: $a \vee (b \vee c) \equiv (a \vee b) \vee c$ **RIGHT:** $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$
WRONG: $a \vee (b \wedge c) \equiv (a \vee b) \wedge c$ **WRONG:** $a \wedge (b \vee c) \equiv (a \wedge b) \vee c$

Wrong application of distributive law:
Another common mistake by students. You mix up the operators.

RIGHT: $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$
WRONG: $a \vee (b \wedge c) \equiv (a \vee b) \vee (a \wedge c)$ **WRONG:** $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \wedge c)$

5. Comments on Q15 (graded by Ben)

(a) Some let $x = 2k + 1$ and $xy = 2l$, which can give $xy = (2k + 1)y = 2l$ and then $y = 2(l - ky)$. Since l, k, y are all integers, y is even number. This is accepted.

Some divides into 2 cases when proving by contraposition: x is even or xy is odd. This is wrong division by cases, should discuss x's even or odd, and derive xy's parity when needed.

Some discuss xy's parity with respect to y's without claiming contradiction when y is odd. Half mark is deducted since that doesn't form a formal proof by showing conclusion clearly.

It's wrong to use the same variable to represent x's and y's parity and equate them.

(b) The question basically asks to prove that any integer that is divisible by 2 and 9 is divisible by 6.

Taking any element a from the LHS set, it is wrong to write $2 \mid a \wedge 9 \mid a \Rightarrow 18 \mid a$ <u>without justification</u>. Why? Though this is true for this case, it is not true that in general, $p \mid a \wedge q \mid a \Rightarrow pq \mid a$ (counterexample: $6 \mid 18 \wedge 9 \mid 18 \not\Rightarrow 54 \mid 18$).

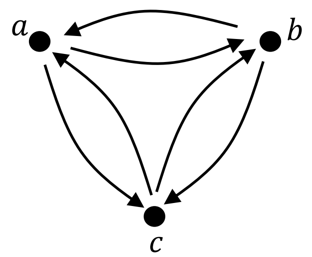
It is also wrong to use a single variable k to conclude $a = 2k = 9k$.

Some directly write $2 \mid a \wedge 9 \mid a$ which is not right. They should be inferred by considering a is an element of the LHS set and use definition of set intersection.

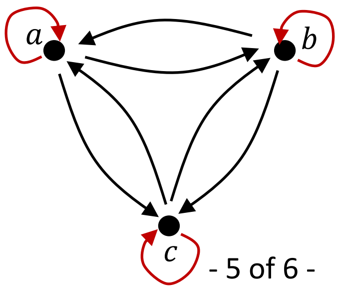
6. Comments on Q16 (graded by Aaron)

(a) Many students are able to prove that R is not transitive by giving a counterexample.

(b) I hope students drew some diagram to get an idea. Diagrams are often useful. For example, let's take a set $A = \{a, b, c\}$ instead of $\mathbb{Z}$. Then you may draw the directed graph for the relation R on A:



Note that every element has an arrow going to every other element. As shown in part (a), R is not transitive. How about S, which is a reflexive closure of R? We obtain S by adding a loop around every element:



Consider the definition of transitivity: $\forall x, y, z \in A\ (x\ S\ y \wedge y\ S\ z \Rightarrow x\ S\ z)$.
We need only to consider the cases $x = z$ and $x \neq z$. For the former, we know S is reflexive (since it is the reflexive closure of R) and hence $x\ S\ z$ (which is $x\ S\ x$) is true. For the latter, we know $x\ R\ z$ from the definition of R and since $R \subseteq S$, $x\ R\ z \Rightarrow x\ S\ z$.

7. Comments on Q17 (graded by Aaron)

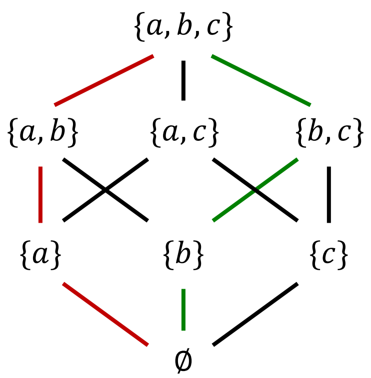
(a) I thought part (a) is really easy as all you need to do is to draw the Hasse diagram and then identify the minimal and maximal elements. We have this in tutorial. Many students left out the element 7 in the minimal elements and maximal elements.

(b) The set $\mathbb{N}$ is an infinite set so now you can't draw the complete Hasse diagram. But using the definition of divisibility, 1 is the (only) minimal element (and hence it is the smallest element) as 1 divides every natural number. Most students got this right. Some students added prime numbers as the minimal elements, which is incorrect.

Only a few students got the maximal element 0. Most students put "none" as their answer.

This is an interesting case where the minimal is 1 and the maximal is 0.

(c) We like to give new definitions to test whether students are able to understand definitions. Here, we have the set $\wp(\{a, b, c\})$ which is $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. The Hasse diagram for the partial order $\subseteq$ on this set is as follows (and this is similar to Lecture 6 slide 79!):



A maximal chain is hence a path you can trace from $\emptyset$ to $\{a, b, c\}$. There are 6 such paths, two of which are highlighted in colour. You need to write out only two of the six maximal chains.

Prepared by Aaron Tan with inputs from Hon Wai and Ben Wu
14 March 2022