Solutions to Midterm

MA1521 Calculus for Computing

Time allowed: 1 hour 15 mins.

The test is open book. You may use online graphic calculator.

Answer all 7 questions. Each question carries 10 marks.

Justify your answers and show your steps clearly.

1. Let a and b be integers. It is known that

$$\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+\frac{a}{3}+\frac{b}{x^2}\right)=0.$$

Determine the value of $a+b$.

Answer. 2.

Solution. First $\lim_{x\to 0}\left(\frac{\sin 2x}{x^3}+\frac{a}{3}+\frac{b}{x^2}\right)=0$ is equivalent to $\lim_{x\to 0}\left(\frac{\sin 2x+bx}{x^3}\right)=-\frac{a}{3}$. Applying L'Hôpital's Rule, we have $\lim_{x\to 0}\left(\frac{\sin 2x+bx}{x^3}\right)=\lim_{x\to 0}\left(\frac{2\cos 2x+b}{3x^2}\right)$. The latter limit exists only if $\lim_{x\to 0}(2\cos 2x+b)=2+b=0$. That is $b=-2$. When $b=-2$, we apply L'Hôpital's Rule twice to get $\lim_{x\to 0}\left(\frac{2\cos 2x-2}{3x^2}\right)=\lim_{x\to 0}\left(\frac{-4\sin 2x}{6x}\right)=\lim_{x\to 0}\left(\frac{-8\cos 2x}{6}\right)=-\frac{4}{3}$. Therefore, $-\frac{a}{3}=-\frac{4}{3}$ so that $a=4$ and $a+b=2$.

2. Let $f(x)=\sqrt{3x+\sqrt{x}}$ for $x>0$. An equation of the tangent line to the graph of $f(x)$ at $x=1$ is of the form $ax-by+9=0$, where a and b are integers. Find the value of $a+b$.

Answer. 15.

Solution. $f'(x)=\frac{1}{2\sqrt{3x+\sqrt{x}}}\left(3+\frac{1}{2\sqrt{x}}\right)$. Thus $f'(1)=\frac{7}{8}$. Also $f(1)=2$. Hence, an equation of the tangent line to the graph of f at $(1,2)$ is given by $y-2=\frac{7}{8}(x-1)$, or equivalently, $7x-8y+9=0$. Thus $a=7$, $b=8$ and $a+b=15$.

3. A robot X moves from left to right along the positive x-axis whose speed at time t is given by $5(1-\frac{1}{t+1})$ meters per min. Another robot Y moves upward along the positive y-axis whose speed at time t is given by $12(1-\frac{1}{t+1})$ meters per min. At time $t=0$ min, they both start moving from rest at the origin O. The distance between the two robots at time $t=12$ min is increasing at the rate of R meters per min. Determine the value of R.

Answer. 12.

Solution. The distance measured in meters of X from O at time t min is $x(t) = \int 5 - \frac{5}{t+1}\,dt = 5t - 5\ln(t+1) + C$. When $t = 0$, the distance from O is zero. Thus $C = 0$. Therefore, $x(t) = 5t - 5\ln(t+1)$.

Similarly, the distance measured in meters of Y from O at time t min is $y(t) = 12t - 12\ln(t+1)$.

Therefore, the distance between them at time t min is given by

$$L = \sqrt{(5t - 5\ln(t+1))^2 + (12t - 12\ln(t+1))^2}.$$

Simplifying, $L = 13(t - \ln(t+1))$, for $t \geq 0$. Thus $\frac{dL}{dt} = 13(1 - \frac{1}{t+1})$. When $t = 12$, $R = \frac{dL}{dt}\Big|_{t=12} = 12$.

4. A farmer wishes to employ tomato pickers to harvest 42500 tomatoes. Each picker can harvest 625 tomatoes per hour and is paid \$6 per hour. In addition, the farmer must pay a supervisor \$10 per hour and pay the union \$10 for each picker employed. How many pickers should the farmer employ to minimize the cost of harvesting the tomatoes? Your answer should be a positive integer.

   Answer. 8.

   Solution. Let x be the number of pickers. Each picker picks $\frac{42500}{x}$ tomatoes. Each picker spends $\frac{42500}{625x}$ hours in picking the tomatoes. The farmer needs to pay each picker $\frac{42500\times 6}{625x}$ dollars. Thus the cost of hiring x pickers is $\frac{42500\times 6}{625} = 408$ dollars.

   The cost of hiring the supervisor is $\frac{42500\times 10}{625x} = \frac{680}{x}$ dollars. The farmer also needs to pay $10x$ dollars to the union.

   Therefore, the total cost in dollars is $C(x) = 408 + \frac{680}{x} + 10x$, $x > 0$.

   Then $C'(x) = -\frac{680}{x^2} + 10$. Thus $C'(x) = 0 \Leftrightarrow x = \sqrt{68} = 8.25$.

   Also For $0 < x < \sqrt{68}$, $C'(x) < 0$ and for $x > \sqrt{68}$, $C'(x) > 0$. Thus by the first derivative test, C has an absolute minimum at $x = \sqrt{68}$. As $\sqrt{68}$ is not a whole number, we look for the value of C at nearby integers $x = 8$ and 9. We have $C(8) = 573$ dollars and $C(9) = \frac{5162}{9} = 573.56$ dollars. Comparing these 2 values, we find that the minimum cost is attained at $x = 8$, and the job is done in 8.5 hours.

5. The curve $y^4 = 36(y^2 - x^2)$ has a shape like a figure 8. Find the area of the region enclosed by the two loops of the curve.

   Answer. 48.

   Solution. The curve is symmetric about the x-axis and also about the y-axis since for any point (x, y) on the curve, the points $(x, -y), (-x, y), (-x, -y)$ are also on the curve. So we may simply consider the curve in the first quadrant in which $x, y \geq 0$.

Solving x in terms of y, we obtain $x = y\sqrt{1-\frac{y^2}{36}}$. Thus $0 \le y \le 6$. Therefore, the area of the region bounded by the curve and the y-axis in the first quadrant is given by $\int_0^6 y\sqrt{1-\frac{y^2}{36}}\,dy = \left[-12(1-\frac{y^2}{36})^{\frac{3}{2}}\right]_0^6 = 12$. Hence the required area is $4\times 12 = 48$.

6. Let $f(x) = \dfrac{1}{10}\displaystyle\int_{\frac{\pi}{2}}^{x} \sqrt{2+\sin t+\sin^2 t}\,dt$.

   Show that f^{-1} exists by proving that f is increasing on $\mathbb{R}$. Find also the value of $(f^{-1})'(0)$.

   Answer. 5.

   Solution. By the fundamental theorem of calculus, $f'(x) = \frac{1}{10}\sqrt{2+\sin x+\sin^2 x} \ge \frac{1}{10}\sqrt{2+\sin x} \ge \frac{1}{10}\sqrt{2-1} > 0$. Therefore, f is increasing on $\mathbb{R}$. Thus f is injective and f^{-1} exists.

   Note that $f^{-1}(0) = x \Leftrightarrow f(x) = 0 \Leftrightarrow \frac{1}{10}\int_{\frac{\pi}{2}}^{x}\sqrt{2+\sin t+\sin^2 t}\,dt = 0$.

   Since $\frac{1}{10}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{2+\sin t+\sin^2 t}\,dt = 0$ and f is injective, we have $x = \frac{\pi}{2}$.

   Therefore, $f^{-1}(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(\frac{\pi}{2})} = \frac{1}{\frac{1}{10}\sqrt{2+\sin\frac{\pi}{2}+\sin^2\frac{\pi}{2}}} = 5$.

7. It is known that the improper integral $\displaystyle\int_0^1 \frac{1}{x^2} - \frac{1}{(x+1)[\ln(x+1)]^2}\,dx = \frac{p-\ln 8}{\ln 4}$.

   Determine the value of p. Justify your answer.

   Answer. 2.

   Solution.

$$
\begin{aligned}
&\int_0^1 \frac{1}{x^2} - \frac{1}{(x+1)[\ln(x+1)]^2}\,dx \\
&= \lim_{b\to 0^+}\int_b^1 \frac{1}{x^2} - \frac{1}{(x+1)[\ln(x+1)]^2}\,dx \\
&= \lim_{b\to 0^+}\left[-\frac{1}{x} + \frac{1}{\ln(x+1)}\right]_b^1 \\
&= -1 + \frac{1}{\ln 2} + \lim_{b\to 0^+}\left(\frac{1}{b} - \frac{1}{\ln(b+1)}\right).
\end{aligned}
$$

By L'Hôpital's rule, $\displaystyle\lim_{b\to 0^+}\left(\frac{1}{b} - \frac{1}{\ln(b+1)}\right) = \lim_{b\to 0^+}\frac{\ln(b+1)-b}{b\ln(b+1)} = \lim_{b\to 0^+}\frac{\frac{d}{db}(\ln(b+1)-b)}{\frac{d}{db}(b\ln(b+1))} =$ $\displaystyle\lim_{b\to 0^+}\frac{\frac{1}{b+1}-1}{\ln(b+1)+\frac{b}{b+1}} = \lim_{b\to 0^+}\frac{-b}{(b+1)\ln(b+1)+b} = \lim_{b\to 0^+}\frac{\frac{d}{db}(-b)}{\frac{d}{db}((b+1)\ln(b+1)+b)}$

$$= \lim_{b \to 0^+} \frac{-1}{\ln(b+1)+1+1} = -\tfrac{1}{2}.$$

Therefore, the value of the improper integral is $-1+\frac{1}{\ln 2}-\frac{1}{2} = -\frac{3}{2}+\frac{1}{\ln 2} = \frac{2-3\ln 2}{2\ln 2} = \frac{2-\ln 8}{\ln 4}$. Consequently $p = 2$.