CS1231S TUTORIAL #6

Functions

Q7.

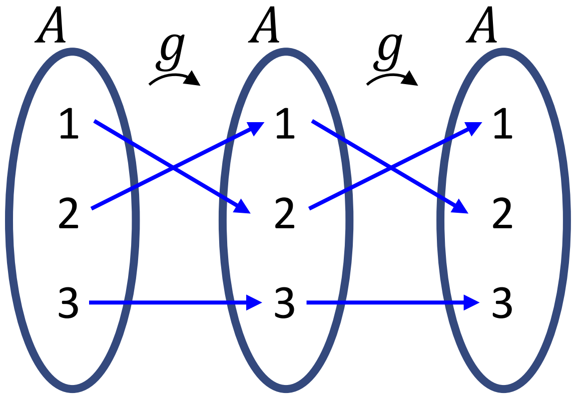
Let $A = \{1, 2, 3\}$. The *order* of a bijection $f : A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that

$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$

Define functions $g, h : A \to A$ by setting, for all $x \in A$,

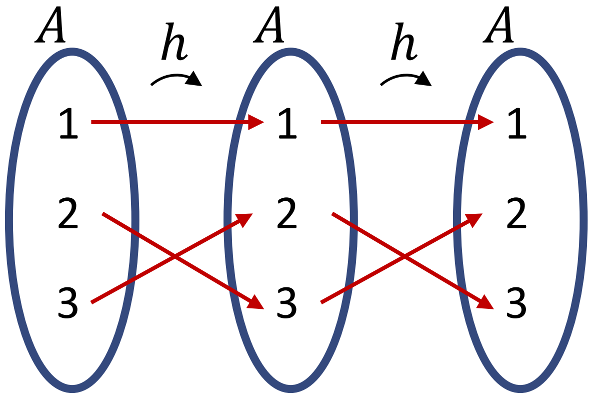
$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$

Find the orders of g, h, $g \circ h$, and $h \circ g$.



$g \circ g(1) = 1$
$g \circ g(2) = 2$
$g \circ g(3) = 3$

∴ order of g = 2



$h \circ h(1) = 1$
$h \circ h(2) = 2$
$h \circ h(3) = 3$

∴ order of h = 2

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Is $(g \circ h) \circ (g \circ h) = \mathrm{id}_A$?

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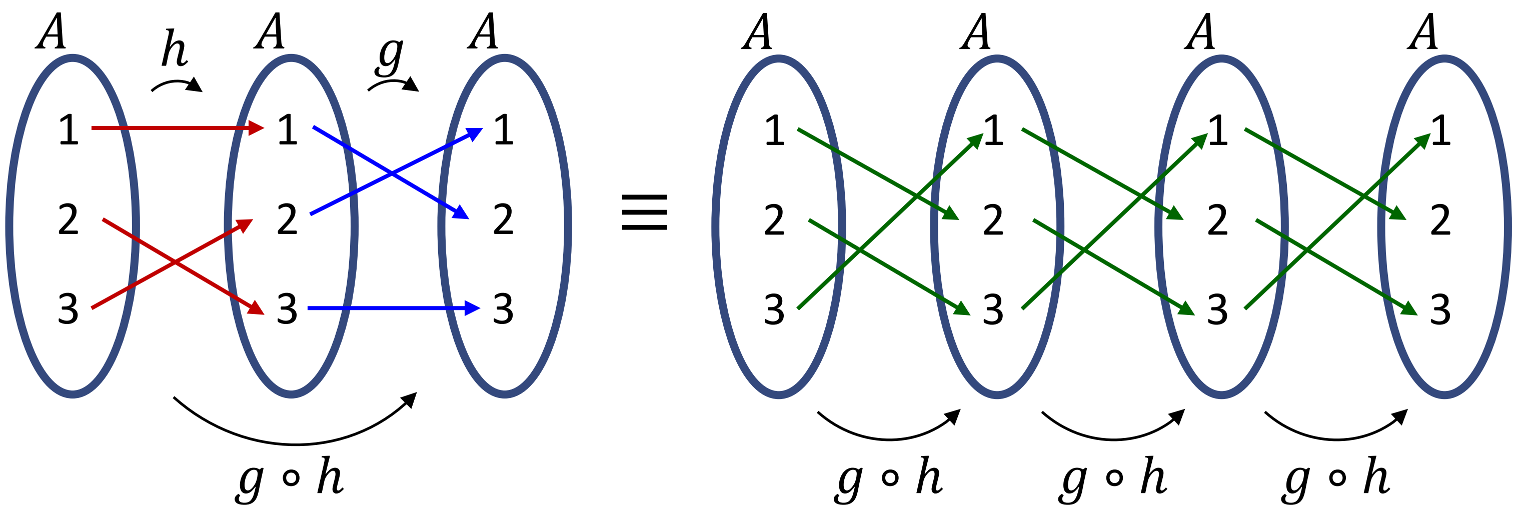

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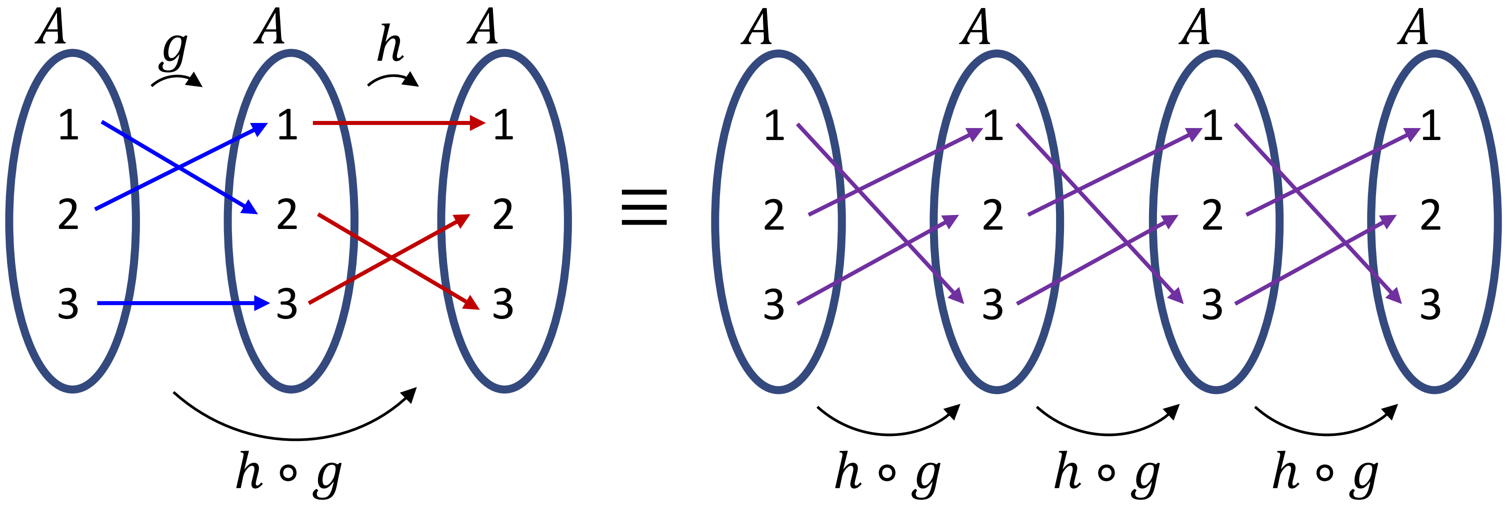
Is $(h \circ g) \circ (h \circ g) = \mathrm{id}_A$?

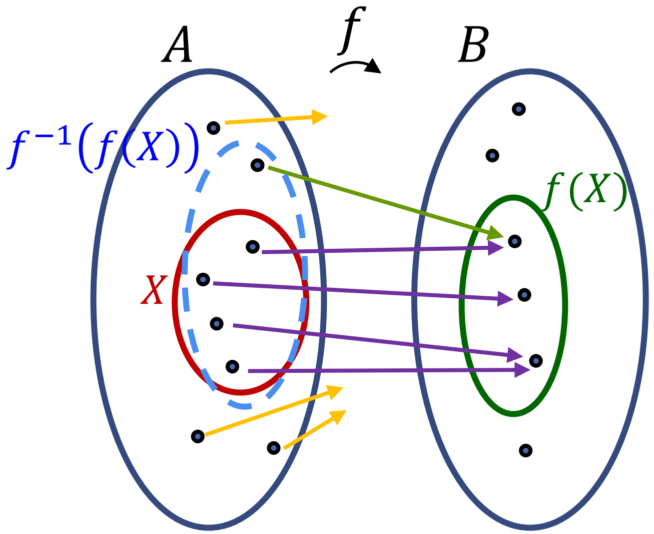
Find the orders of g, h, $g \circ h$, and $h \circ g$.

Is $(h \circ g) \circ (h \circ g) \circ (h \circ g) = \mathrm{id}_A$?



Q8. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(a) Compare the sets X and $f^{-1}(f(X))$.
Is one always a subset of the other?



Is $X \subseteq f^{-1}(f(X))$ always true? Yes

1. Let $x \in X$.
2. Then $f(x) \in f(X)$ by the definition of $f(X)$.
3. So $x \in f^{-1}(f(X))$ by the definition of $f^{-1}(f(X))$.

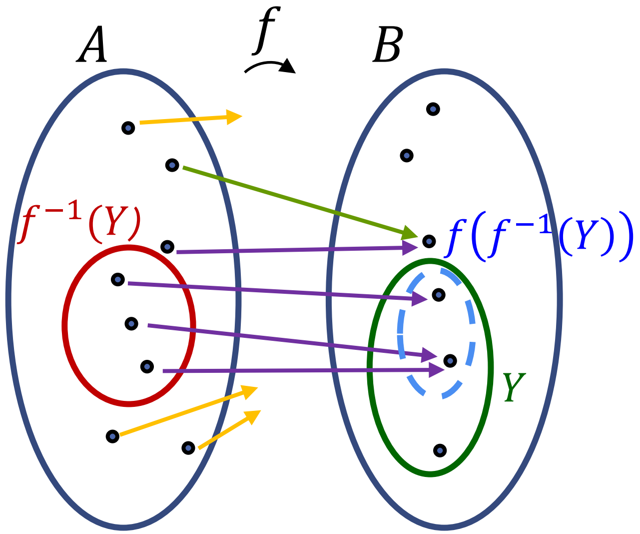
Is $f^{-1}(f(X)) \subseteq X$ always true? No

1. Consider $f: \{-1, 1\} \to \{0\}$ where $f(-1) = 0 = f(1)$, and $X = \{1\}$.
2. Note $f(X) = \{f(1)\} = \{0\}$.
3. Since $f(-1) = 0$, we know $-1 \in f^{-1}(\{0\}) = f^{-1}(f(X))$.
4. As $-1 \notin \{1\} = X$, we deduce that $f^{-1}(f(X)) \not\subseteq X$.

Setwise image and preimage. Let $f: A \to B$.
(1) If $X \subseteq A$, then let $f(X) = \{f(x): x \in X\}$.
(2) If $Y \subseteq B$, then let $f^{-1}(Y) = \{x \in A: f(x) \in Y\}$

Q8. Let $f: A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.

(b) Compare the sets Y and $f(f^{-1}(Y))$.
Is one always a subset of the other?



Is $f(f^{-1}(Y)) \subseteq Y$ always true? Yes

1. Take any $y \in f(f^{-1}(Y))$.
2. There is some $x \in f^{-1}(Y)$ s.t. $y = f(x)$ by the def^n of $f(f^{-1}(Y))$.
3. As $x \in f^{-1}(Y)$, we get $y' \in Y$ which makes $y' = f(x)$.
4. Since f is a function, $y = f(x) = y' \in Y$.

Is $Y \subseteq f(f^{-1}(Y))$ always true? No

1. Consider $f: \{0\} \to \{-1, 1\}$ where $f(0) = 1$, and $Y = \{-1\}$.
2. Note that no $x \in \{0\}$ makes $f(X) = -1$.
3. So $f^{-1}(Y) = \emptyset$ by the definition of $f^{-1}(Y)$.
4. This entails $f(f^{-1}(Y)) = \emptyset \not\supseteq \{-1\} = Y$ by the definition of $f(f^{-1}(Y))$.

Setwise image and preimage. Let $f: A \to B$.
(1) If $X \subseteq A$, then let $f(X) = \{f(x): x \in X\}$.
(2) If $Y \subseteq B$, then let $f^{-1}(Y) = \{x \in A: f(x) \in Y\}$