Lecture #10: Counting and Probability I Summary

Aaron Tan

AY2021/22 Semester 2

Summary

This lecture is based on Epp's book chapter 9.
Hence, the section numbering is according to the book.

9.1 Introduction

- Random process, sample space, event and probability

9.2 Possibility Trees and the Multiplication Rule

- Possibility trees
- The multiplication/product rule
- Permutations

9.3 Counting Elements of Disjoint Sets

- The addition/sum rule
- The difference rule
- The inclusion/exclusion rule

9.4 The Pigeonhole Principle (PHP)

- Pigeonhole principle, general pigeonhole principle

Reference: Epp's Chapter 9 Counting and Probability

Summary

9.1 Introduction

Definitions

A **sample space** is the set of all possible outcomes of a random process or experiment.
An **event** is a subset of a sample space.

Notation

For a finite set A, $|A|$ denotes the number of elements in A.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability** of E, denoted $P(E)$, is

$$P(E) = \frac{\text{The number of outcomes in } E}{\text{The total number of outcomes in } S} = \frac{|E|}{|S|}$$

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \le n$, then there are $n - m + 1$ integers from m to n inclusive.

Summary

Theorem 9.2.1 The Multiplication/Product Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,

the second step can be performed in n_2 ways

(regardless of how the first step was performed),

:

the k^{th} step can be performed in n_k ways

(regardless of how the preceding steps were performed),

Then the entire operation can be performed in

$$n_1 \times n_2 \times n_3 \times \ldots \times n_k \text{ ways.}$$

Summary

Theorem 9.2.2 Permutations

The number of permutations of a set with n ($n \geq 1$) elements is $\boldsymbol{n!}$

Definition

An ***r*-permutation** of a set of n elements is an ordered selection of r elements taken from the set.
The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

Theorem 9.2.3 *r*-permutations from a set of *n* elements

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1) \qquad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \text{second version}$$

Summary

9.3 Counting Elements of Disjoint Sets

Theorem 9.3.1 The Addition/Sum Rule

Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then $|A| = |A_1| + |A_2| + \ldots + |A_k|$.

Theorem 9.3.2 The Difference Rule

If A is a finite set and $B \subseteq A$, then $|A \setminus B| = |A| - |B|$.

Formula for the Probability of the Complement of an Event

If S is a finite sample space and A is an event in S, then $P(\bar{A}) = 1 - P(A)$.

Theorem 9.3.3 The Inclusion/Exclusion Rule for 2 or 3 Sets

If A, B, and C are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Summary

Pigeonhole Principle (PHP)

A function from one finite set to a smaller finite set cannot be one-to-one:
There must be at least 2 elements in the domain that have the same image in the co-domain.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then
there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Generalized Pigeonhole Principle (Contrapositive Form)

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k,
if for each $y \in Y$, $f^{-1}(\{y\})$ has at most k elements,
then X has at most km elements; in other words, $n \leq km$.

END OF FILE