CS1231S Assignment #1

AY2021/22 Semester 2

Deadline: Monday, 14 February 2022, 1:00pm

ANSWERS

IMPORTANT: Please read the instructions below

This is a graded assignment worth 10% of your final grade. There are **seven questions** and total score is 20 marks. Please work on it by yourself, not in a group or in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own), or sending your answers to others, or other forms of academic dishonesty will be penalised with a straight zero for the assignment, and possibly an F grade for the module.

You are to submit your assignment to **LumiNUS Files**. A submission folder has been created for you at Files > Assignment #1 > Your tutorial group > Your personal folder.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink if it is handwritten, or font size smaller than 11 if it is typed) or marks may be deducted.

You are to submit **a SINGLE pdf file**, where each page is A4 size. Do not submit multiple files or files in other formats. If you submit multiple files, we will grade only the last submission.

You may test out your submission folder before the deadline, but make sure you remove any test files you have submitted earlier.

Late submission will NOT be accepted. We will set the closing time of the submission folders to slightly later than 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last minute; the system may get sluggish due to overload and you will miss the deadline.

Note the following as well:

- Name your pdf file with your **Student Number**. Your student number begins with 'A' (eg: A0234567X). (Do not mix up your student number with your NUSNET-id which begins with 'e'.) A penalty of one mark will be given if your file is not named according to this instruction.
- At the top of the first page of your submission, write your **Name** and **Tutorial Group**. **Both** must be present and correct, or a penalty of one mark will be given.
- To keep the submitted file short, you may submit your answers without including the questions.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

* for Android: https://fossbytes.com/best-android-scanner-apps/

* for iphone:
https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on the **LumiNUS > Assignments** forum so that everybody can read the answers to your queries.

Note: Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.

You may assume that every integer is even or odd, but not both. (Lecture 1 slide 27)

Question 1. (2 marks)

Given the following argument:

> "Aiken is a pianist or a writer. If he is a pianist, then he has a keen sense of hearing. Aiken does not have a keen sense of hearing. Therefore, he is a writer."

Using the following variables, write the above argument using logical statements and show that the above argument is valid. Use rules of inference instead of the critical rows method. Quote the rule of inference used in every step of your proof.

p: Aiken is a pianist.
w: Aiken is a writer.
h: Aiken has a keen sense of hearing.

Answer:

The argument is:

$$p \vee w$$
$$p \rightarrow h$$
$$\sim h$$
$$\therefore w$$

Proof:

1. $(p \rightarrow h) \wedge \sim h \rightarrow \sim p$ (by modus tollens)
2. $\sim p \wedge (p \vee w) \rightarrow w$ (by elimination)
3. Therefore, the argument is valid.

Comment for students:

Best to answer in 2 parts. Part (1) write the argument, part (2) prove the argument is valid.

Question 2. (3 marks)

You have proved that the argument in question 1 is valid by using rules of inference. Now, convert the argument into a conditional statement (refer to Tutorial #1 Additional Notes) and use the laws of logical equivalences in Theorem 2.1.1 and the implication law to prove that the argument in question 1 is valid. Remember to quote the law used at every step.

Answer:

The conditional statement is $(p \vee w) \wedge (p \to h) \wedge {\sim}h \to w$ (the order of the three premises depends on what students wrote in their answer for question 1; the order does not matter anyway). We are to show that this conditional statement is a tautology.

$(p \vee w) \wedge (p \to h) \wedge {\sim}h \to w$

1.	$\equiv (p \vee w) \wedge ({\sim}p \vee h) \wedge {\sim}h \to w$	by the implication law
2.	$\equiv (p \vee w) \wedge \big(({\sim}p \vee h) \wedge {\sim}h\big) \to w$	by the associative law
3.	$\equiv (p \vee w) \wedge \big({\sim}h \wedge ({\sim}p \vee h)\big) \to w$	by the commutative law
4.	$\equiv (p \vee w) \wedge \big(({\sim}h \wedge {\sim}p) \vee ({\sim}h \wedge h)\big) \to w$	by the distributive law
5.	$\equiv (p \vee w) \wedge \big(({\sim}h \wedge {\sim}p) \vee (h \wedge {\sim}h)\big) \to w$	by the commutative law
6.	$\equiv (p \vee w) \wedge \big(({\sim}h \wedge {\sim}p) \vee \mathbf{false}\big) \to w$	by the negation law
7.	$\equiv (p \vee w) \wedge ({\sim}h \wedge {\sim}p) \to w$	by the identity law
8.	$\equiv {\sim}\big((p \vee w) \wedge ({\sim}h \wedge {\sim}p)\big) \vee w$	by the implication law
9.	$\equiv \big({\sim}(p \vee w) \vee {\sim}({\sim}h \wedge {\sim}p)\big) \vee w$	by De Morgan's law
10.	$\equiv \Big(({\sim}p \wedge {\sim}w) \vee \big({\sim}({\sim}h) \vee {\sim}({\sim}p)\big)\Big) \vee w$	by De Morgan's law (x2)
11.	$\equiv \big(({\sim}p \wedge {\sim}w) \vee (h \vee p)\big) \vee w$	by the double negative law (x2)
12.	$\equiv w \vee \big(({\sim}p \wedge {\sim}w) \vee (h \vee p)\big)$	by the commutative law
13.	$\equiv \big(w \vee ({\sim}p \wedge {\sim}w)\big) \vee (h \vee p)$	by the associative law
14.	$\equiv \big((w \vee {\sim}p) \wedge (w \vee {\sim}w)\big) \vee (h \vee p)$	by the distributive law
15.	$\equiv \big((w \vee {\sim}p) \wedge \mathbf{true}\big) \vee (h \vee p)$	by the negation law
16.	$\equiv (w \vee {\sim}p) \vee (h \vee p)$	by the identity law
17.	$\equiv (w \vee {\sim}p) \vee (p \vee h)$	by the commutative law
18.	$\equiv w \vee ({\sim}p \vee p) \vee h$	by the associative law
19.	$\equiv w \vee (p \vee {\sim}p) \vee h$	by the commutative law
20.	$\equiv w \vee \mathbf{true} \vee h$	by the negation law
21.	$\equiv w \vee (\mathbf{true} \vee h)$	by the associative law
22.	$\equiv w \vee (h \vee \mathbf{true})$	by the commutative law
23.	$\equiv (w \vee h) \vee \mathbf{true}$	by the associative law
24.	$\equiv \mathbf{true}$	by the universal bound law

Since $(p \vee w) \wedge (p \to h) \wedge {\sim}h \to w$ is a tautology, the argument is valid.

Question 3. (3 marks)

Prove the following statement:

"Given any irrational number x and any real number y,
at least one of $x + y$ or $x - y$ is irrational."

Answer:

Proof (by contradiction)

1. Suppose not, that is, neither $x + y$ nor $x - y$ is irrational. In other words, both $x + y$ and $x - y$ are rational.
2. So $x + y = \frac{p}{q}$ and $x - y = \frac{r}{s}$ for some $p, q, r, s \in \mathbb{Z}$ and $q \neq 0$ and $s \neq 0$ (by the definition of rational numbers).
3. Then $(x + y) + (x - y) = 2x = \frac{p}{q} + \frac{r}{s} = \frac{ps+rq}{qs}$ (by basic algebra).
4. So $x = \frac{ps+rq}{2qs} \in \mathbb{Q}$ as $ps + rq, 2qs \in \mathbb{Z}$ (by closure of integers under $\times$ and $+$) and $2qs \neq 0$ (as $q \neq 0$ and $s \neq 0$).
5. Hence x is a rational number (by the definition of rational numbers), contradicting x is irrational.
6. Therefore, given any irrational number x and any real number y, at least one of $x + y$ or $x - y$ is irrational.

Comment for students:

Some students used the definition of rational number in writing "a rational number $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$" but omitted the part on "$q \neq 0$". This is wrong.

Question 4. (4 marks)

Let $A = \{-1,0,1\}$, $B = \{1,4\}$ and $C = \{-2,-1,0,1,2\}$. Indicate whether each of the following statements is true or false with proof or explanation.

(a) $\forall x \in A\ \forall y \in C\ (xy \in B \to x = y)$;

Answer: **True.**
The only 2 cases when $xy \in B$ are $x = y = 1$ and $x = y = -1$.

(b) $\exists x \in A\ \forall y \in B\ \exists z \in C\ \big((y \neq 0) \land (y > xz)\big)$;

Answer: **True.**
The required x is -1 or 0 or 1 (only one is needed).
Pick $x = -1$, then $\forall y \in B = \{1,4\}$, $\exists z = 0$ (or 1 or 2) $\in C$ s.t. $(y \neq 0) \land (y > xz)$.
Or, pick $x = 0$, then $\forall y \in B = \{1,4\}$, $\exists z = 0$ (or 1 or 2) $\in C$ s.t. $(y \neq 0) \land (y > xz)$.
Or, pick $x = 1$, then $\forall y \in B = \{1,4\}$, $\exists z = -1$ (or -2) $\in C$ s.t. $(y \neq 0) \land (y > xz)$.

(c) $\exists x \in A\ \forall y \in B\ \exists z \in C\ (z > xy)$;

Answer: **True.**
The required x is -1 or 0 (only one is needed).
Pick $x = -1$, $\forall y \in B$, $\exists z = 0$ (or 1 or 2) such that $z > -y$.
Or, pick $x = 0$, $\forall y \in B$, $\exists z = 1$ (or 2) such that $z > 0$.

Comment for students:
For parts (b) and (c), some students began with "When $y = 1$, let $x = \cdots$; when $y = 4$, let $x = \cdots$;" This is wrong. This is for statement "$\forall y\ \exists x$". What we have here is "$\exists x\ \forall y$", so we should fix a particular value for x, then prove the statement for all values of y.

(d) $\forall x \in C\ \big((\forall y \in B\ xy \in B) \to (x^2 = x)\big)$.

Answer: **True.**

Case $x = -2, -1$ or 2: $(\forall y \in B\ xy \in B)$ is false (counterexample: $y = 4$), therefore
$\big((\forall y \in B\ xy \in B) \to (x^2 = x)\big)$ is vacuously true.

Case $x = 0$: $(\forall y \in B\ xy \in B)$ is false (as $0 \notin B$), therefore
$\big((\forall y \in B\ xy \in B) \to (x^2 = x)\big)$ is vacuously true.

Case $x = 1$: $(\forall y \in B\ xy \in B)$ is true and $(x^2 = x)$ is also true, therefore
$\big((\forall y \in B\ xy \in B) \to (x^2 = x)\big)$ is true.

In all cases, $\big((\forall y \in B\ xy \in B) \to (x^2 = x)\big)$ is true

Question 5. (2 marks)

Let S be the set of students and M the set of modules. For all $s \in S$ and $x, y \in M$, define the following predicates:

- $Prereq(x, y)$: "module x is the prerequisite of module y";
- $Taken(s, x)$: "student s has taken module x";
- $Taking(s, x)$: "student s is currently taking module x".

A database stores information about prerequisite among the modules, for example: $Prereq(CS1101S, CS2100), Prereq(CS2100, CS2106)$, etc.

Write out two rules (universal conditional statements) so that when given a module that a particular student is currently taking, you are able to infer what modules that student has already taken. For example, if you are given $Taking(Dueet, CS2106)$, then you are able to infer that Dueet has taken CS1101S and CS2100.

Answer:

- $\forall s \in S\ \forall x, y \in M\ \big(Prereq(x, y) \wedge Taking(s, y) \rightarrow Taken(s, x)\big)$

  (If module x is the prerequisite of y and s is taking y, then s has taken module x.)
- $\forall s \in S\ \forall x, y \in M\ \big(Prereq(x, y) \wedge Taken(s, y) \rightarrow Taken(s, x)\big)$.

  (If module x is the prerequisite of y and s has taken y, then s has taken module x.)

Question 6. (2 marks)

For each $k \in \mathbb{Z}^+$, let $A_k = \{n \in \mathbb{Z}_{\geqslant 2} : k = mn \text{ for some } m \in \mathbb{Z}_{\geqslant 2}\}$.
Write down each of the following sets in roster notation:

(a) $\bigcup_{k=3}^{10} A_k$;

(b) $\bigcap_{k=3}^{10} A_k$.

Answers:

Working: $A_3 = \emptyset, A_4 = \{2\}, A_5 = \emptyset, A_6 = \{2,3\}, A_7 = \emptyset, A_8 = \{2,4\}, A_9 = \{3\}, A_{10} = \{2,5\}$.

(a) $\bigcup_{k=3}^{10} A_k = \{2,3,4,5\}$.

(b) $\bigcap_{k=3}^{10} A_k = \emptyset$.

Question 7. (4 marks)

Prove the following statements, where $\wp(X)$ denotes the power set of X. Do not use diagrams in your proofs.

(a) For every set A, if $A \subseteq \emptyset$, then $A = \emptyset$. [1 mark]

(b) For all sets A and B, $\wp(A \cap B) = \wp(A) \cap \wp(B)$. [3 marks]

Answers:

(a) **Direct proof:**

1. Suppose $A \subseteq \emptyset$.
2. Since $\emptyset \subseteq A$ is always true, we have $(A \subseteq \emptyset) \wedge (\emptyset \subseteq A) \Rightarrow A = \emptyset$ (by set equality).

Proof by contradiction:

1. Suppose not, that is, $A \subseteq \emptyset$ and $A \neq \emptyset$.
2. Then there is an element $x \in A$.
3. Then $x \in \emptyset$ since $A \subseteq \emptyset$ (by the definition of $\subseteq$).
4. This contradicts the fact that $\emptyset$ has no element.
5. Therefore, if $A \subseteq \emptyset$, then $A = \emptyset$.

(b)

1. $X \in \wp(A \cap B) \Leftrightarrow X \subseteq A \cap B$ (by the definition of power set)
2. $\Leftrightarrow \forall x \in X\ (x \in A \cap B)$ (by the definition of $\subseteq$)
3. $\Leftrightarrow \forall x \in X\ (x \in A \wedge x \in B)$ (by the definition of $\cap$)
4. $\Leftrightarrow \big(\forall x \in X\ (x \in A)\big) \wedge \big(\forall x \in X\ (x \in B)\big)$ (by the definition of $\wedge$)
5. $\Leftrightarrow (X \subseteq A) \wedge (X \subseteq B)$ (by the definition of $\subseteq$)
6. $\Leftrightarrow \big(X \in \wp(A)\big) \wedge \big(X \in \wp(B)\big)$ (by the definition of power set)
7. $\Leftrightarrow X \in \wp(A) \cap \wp(B)$ (by the definition of $\cap$)
8. Therefore for all sets A and B, $\wp(A \cap B) = \wp(A) \cap \wp(B)$.

Comment for students:

Some students used the following without proof:

$$X \subseteq A \cap B \Leftrightarrow X \subseteq A \wedge X \subseteq B$$

The "only if" direction of this can be quoted with Theorem 6.2.1 Inclusion of Intersection; however the "if" direction has to be separately proved.

Some students used the following definition as a substitute for the above. The two are not the same.

$$X \in A \cap B \Leftrightarrow X \in A \wedge X \in B$$

Alternative proof:

1. Let S be a set containing all sets.
2. $\wp(A \cap B) = \{X \in S : X \subseteq A \cap B\}$ (by the definition of power set)
3. $= \{X \in S : \forall y\ (y \in X \Rightarrow y \in A \cap B)\}$ (by the definition of ⊆)
4. $= \{X \in S : \forall y\ (y \in X \Rightarrow y \in A \wedge y \in A)\}$ (by the definition of ∩)
5. $= \{X \in S : \forall y\ \big(y \notin X \vee (y \in A \wedge y \in A)\big)\}$ (by the implication law)
6. $= \{X \in S : \forall y\ (y \notin X \vee y \in A) \wedge (y \notin X \vee y \in B)\}$ (by the distributive law)
7. $= \{X \in S : \forall y\ (y \in X \Rightarrow y \in A) \wedge (y \in X \Rightarrow y \in B)\}$ (by the implication law)
8. $= \{X \in S : X \subseteq A \wedge X \subseteq B\}$ (by the definition of ⊆)
9. $= \{X \in S : X \in \wp(A) \wedge X \in \wp(B)\}$ (by the definition of power set)
10. $= \{X \in S : X \in \wp(A) \cap \wp(B)\}$ (by the definition of ∩)
11. $= \wp(A) \cap \wp(B)$
12. Therefore for all sets A and B, $\wp(A \cap B) = \wp(A) \cap \wp(B)$.

=== End of paper ===