CS1231S TUTORIAL #5

Relations & Partial orders

Learning objectives of this tutorial

Partial orders

- Determining whether a relation is antisymmetric
- Determining whether a relation is asymmetric
- Reasoning about partial orders
- Drawing Hasse diagrams of partial orders
- Knowing about comparability and compatibility of elements in a partial order
- Distinguishing minimal/maximal elements from smallest/largest elements
- Understanding linearizations

Q1. Lexicographic order of strings.

Let $A = \{a, b\}$. Define a relation R on A by: $x\ R\ y$ iff $x = y$ or x comes before y for all $x, y \in A$. S to be the set of all strings over alphabet A. Define the partial order $\preccurlyeq$ on S to be the lexicographic order specified in Theorem 8.5.1.

For any two strings in S, $a_1 a_2 \cdots a_m$ and $b_1 b_2 \cdots b_n$, where $m, n \in \mathbb{Z}^+$,

1. If $m \leq n$ and $a_i = b_i$ for all $i = 1,2, \cdots, m$, then $a_1 a_2 \cdots a_m \preccurlyeq b_1 b_2 \cdots b_n$.
2. If for some integer k with $1 \leq k \leq m$ and $1 \leq k \leq n$, $a_i = b_i$ for all $i = 1,2, \cdots, k-1$, and $a_k \neq b_k$, but $a_k\ R\ b_k$, then $a_1 a_2 \cdots a_m \preccurlyeq b_1 b_2 \cdots b_n$.
3. If ε is the null string and s is any string in S, then $\varepsilon \preccurlyeq s$.

(a) $aab \preccurlyeq aaba$ (a) True by (1).

(b) $bbab \preccurlyeq bba$ (b) False.

(c) $\varepsilon \preccurlyeq aba$ (c) True by (3).

(d) $ababb \preccurlyeq abb$ (d) True by (2).

(e) $bbab \preccurlyeq bbaa$ (e) False.

(f) $ababa \preccurlyeq ababaa$ (f) True by (1).

(g) $bbaba \preccurlyeq bbabb$ (g) True by (2).

Q2. Let R be a binary relation on a non-empty set A. Let $x, y \in A$.
Define a relation S on A by: $x\ S\ y \Leftrightarrow x = y \lor x\ R\ y$ for all $x, y \in A$.
Show that (a) S is reflexive; (b) $R \subseteq S$; and
(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.
What is this relation S called?

(a)

1. Let $x \in A$.
2. $x = x$, so $x\ S\ x$ by the definition of S.
3. Therefore, S is reflexive.

Let R be a relation on a set A.
R is **reflexive** iff $\forall x \in A\ (xRx)$.

Q2.

Let R be a binary relation on a non-empty set A. Let $x, y \in A$.
Define a relation S on A by: $x\ S\ y \Leftrightarrow x = y \lor x\ R\ y$ for all $x, y \in A$.
Show that (a) S is reflexive; (b) $R \subseteq S$; and
(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.
What is this relation S called?

(b)

1. Suppose $(x, y) \in R$, that is, $x\ R\ y$.
2. So $x\ S\ y$ by the definition of S.
3. So $(x, y) \in S$.
4. Therefore, $R \subseteq S$ by the definition of $\subseteq$.

Aim: To arrive at $(x, y) \in S$.

Q2.

Let R be a binary relation on a non-empty set A. Let $x, y \in A$.
Define a relation S on A by: $x\ S\ y \Leftrightarrow x = y \vee x\ R\ y$ for all $x, y \in A$.
Show that (a) S is reflexive; (b) $R \subseteq S$; and
(c) if S' is another reflexive relation on A and $R \subseteq S'$, then $S \subseteq S'$.
What is this relation S called?

(c)

1. Suppose $(x, y) \in S$.
2. Then $x\ S\ y$, which means $x = y \vee x\ R\ y$ by the defn of S.
3. Case 1: $x = y$
   - 3.1. Then $x\ S'\ y$ since S' is reflexive.
   - 3.2. So $(x, y) \in S'$.
4. Case 2: $x\ R\ y$
   - 4.1. Then $(x, y) \in R \subseteq S'$.
   - 4.2. Then $(x, y) \in S'$.

Aim:

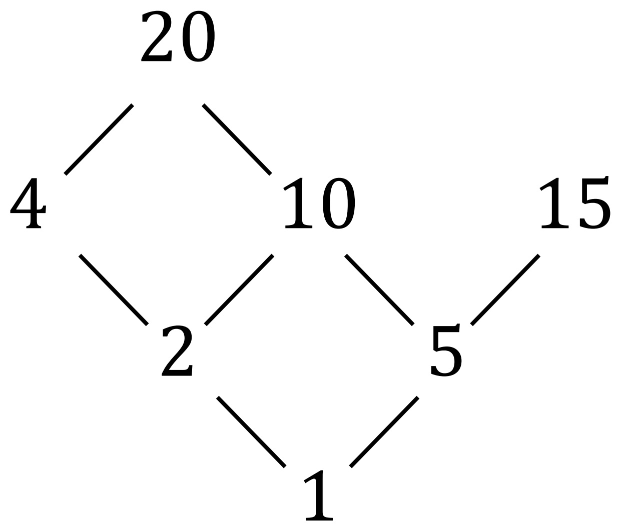
5. In all cases, $(x, y) \in S'$.
6. Therefore, $S \subseteq S'$.

S is called the **reflexive closure** of R.

It is the smallest relation on A that is reflexive and contains R as a subset.

Q3. Consider the “divides” relation on the following set. Draw a Hasse diagram, and find all minimal, maximal, smallest and largest elements.

(a) $A = \{1,2,4,5,10,15,20\}$.



Minimal: 1

Maximal: $15, 20$

Smallest: 1

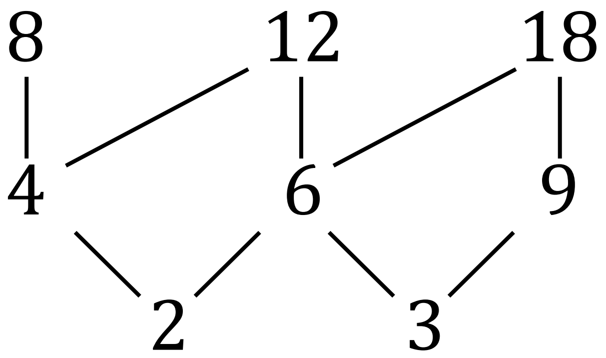
Largest: None

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$
2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Q3. Consider the “divides” relation on the following set. Draw a Hasse diagram, and find all minimal, maximal, smallest and largest elements.

(b) $A = \{2,3,4,6,8,9,12,18\}$.



Minimal: 2,3

Maximal: 8,12, 18

Smallest: None

Largest: None

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$
2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Q4. Let A be a set and $\wp(A)$ the power set of A. Prove that the binary relation $\subseteq$ on $\wp(A)$ is a partial order.

1. (Reflexivity) Take any $S \in \wp(A)$,
   1.1. $S \subseteq S$ by the definition of subset.
   1.2. Hence $\subseteq$ is reflexive.
2. (Antisymmetry) Take any $S, T \in \wp(A)$,
   2.1. Suppose $S \subseteq T$ and $T \subseteq S$.
   2.2. Then $S = T$ by the definition of set equality.
   2.3. Hence $\subseteq$ is antisymmetric.
3. (Transitivity)
   3.1. $\subseteq$ is transitive by Theorem 6.2.1.
4. Therefore $\subseteq$ on $\wp(A)$ is a partial order.

Theorem 6.2.1.
For all sets A, B, C,
$A \subseteq B \wedge B \subseteq C \Rightarrow A \subseteq C$.

Let R be a relation on a set A.
1. R is **reflexive** iff $\forall x \in A\ (xRx)$.
2. R is **antisymmetric** iff $\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$.
3. R is **transitive** iff $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

Q5. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B \left((a,b)\ R\ (c,d) \Leftrightarrow (a \leq c) \wedge (b \leq d) \right).$$

(a) Prove that R is a partial order.

Let R be a relation on a set A.

1. R is **reflexive** iff $\forall x \in A\ (xRx)$.
2. R is **antisymmetric** iff $\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$.

1. (Reflexivity) Take any $(a,b) \in B \times B$,
   1.1. $a \leq a$ and $b \leq b$.
   Aim: 1.2. So $(a,b)\ R\ (a,b)$ by the definition of R.
   1.3. Hence R is reflexive.

2. (Antisymmetry) Take any $(a,b),(c,d) \in B \times B$,
   2.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (a,b)$.
   2.2. Then $\underline{a \leq c}, \underline{b \leq d}, \underline{c \leq a}$ and $\underline{d \leq b}$ by the definition of R.
   2.3. Then $\underline{a = c}$ and $\underline{b = d}$ by the antisymmetry of $\leq$.
   Aim: 2.4. So $(a,b) = (c,d)$ by equality of ordered pairs.
   2.5. Hence R is antisymmetric.

Q5. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B\ \big((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \wedge (b \le d)\big).$$

(a) Prove that R is a partial order.

3. (Transitivity) Take any $(a,b),(c,d),(e,f) \in B \times B$,
   - 3.1. Suppose $(a,b)\ R\ (c,d)$ and $(c,d)\ R\ (e,f)$.
   - 3.2. Then $\underline{a \le c}$, $\underline{b \le d}$, $\underline{c \le e}$ and $\underline{d \le f}$ by the definition of R.
   - 3.3. Then $\underline{a \le e}$ and $\underline{b \le f}$ by the transitivity of $\le$.
   - Aim: 3.4. So $(a,b)\ R\ (e,f)$ by the definition of R .
   - 3.5. Hence R is transitive.
4. Therefore R on $B \times B$ is a partial order.

Let R be a relation on a set A.
1. R is **reflexive** iff $\forall x \in A\ (xRx)$.
2. R is **antisymmetric** iff $\forall x,y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$.
3. R is **transitive** iff $\forall x,y,z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

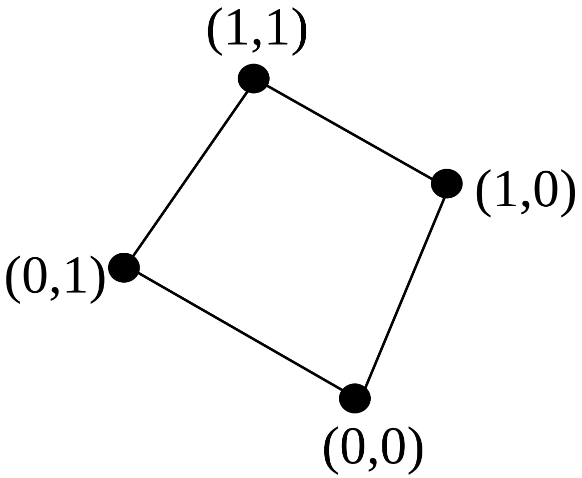
Q5. Let $B = \{0,1\}$ and define the binary relation R on $B \times B$ as follows:

$$\forall (a,b),(c,d) \in B \times B \left((a,b)\ R\ (c,d) \Leftrightarrow (a \le c) \land (b \le d) \right).$$

(b) Draw the Hasse diagram for R.

(c) Find the maximal, largest, minimal and smallest elements.

(d) Is $(B \times B, R)$ well-ordered?



Maximal:	$(1,1)$
Largest:	$(1,1)$
Minimal:	$(0,0)$
Smallest:	$(0,0)$

No well-ordered. Reason: It is not even a total order, as $(0,1)$ and $(1,0)$ are not comparable.

Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c \in A$.

1. c is a **maximal element** of A iff $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x.)$
2. c is a **minimal element** of A iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
3. c is the **largest element** of A iff $\forall x \in A\ (x \preccurlyeq c)$.
4. c is the **smallest element** of A iff $\forall x \in A\ (c \preccurlyeq x)$.

Q6. Let R be a binary relation on a set A.

- R is antisymmetric iff $\forall x, y \in A\ (x\ R\ y \land y\ R\ x \Rightarrow x = y)$.
- R is asymmetric iff $\forall x, y \in A\ (x\ R\ y \Rightarrow y \not R\ x)$.

Find a binary relation on A that is …

(a) both asymmetric and antisymmetric.

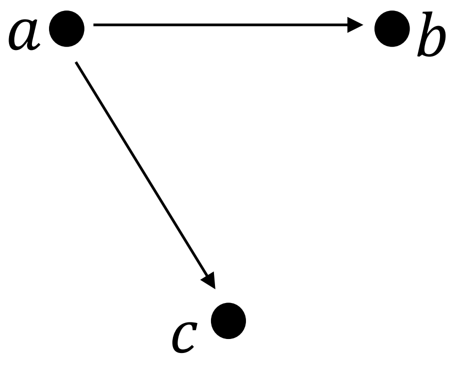
(b) not asymmetric but antisymmetric.

(c) asymmetric but not antisymmetric.

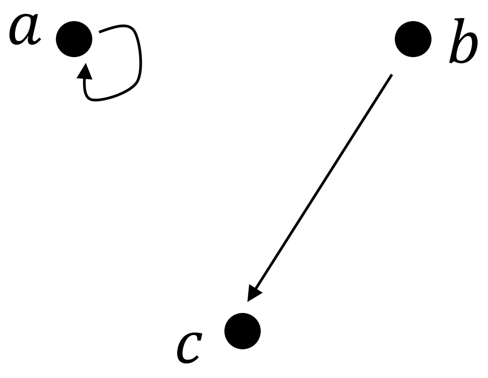
(d) neither asymmetric nor antisymmetric.

Let $A = \{a, b, c\}$.

(a) $R = \{(a, b), (a, c)\}$.



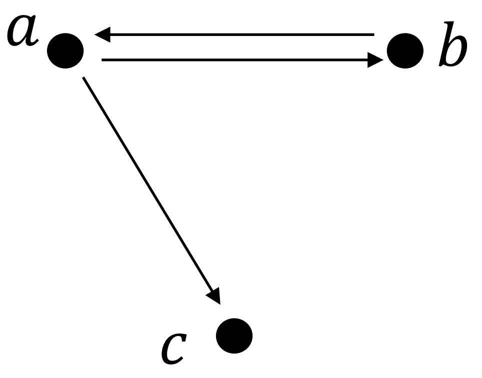
(b) $R = \{(a, a), (b, c)\}$.



(c)

?

(d) $R = \{(a, b), (b, a), (a, c)\}$.



Q6. Let R be a binary relation on a set A.

- R is antisymmetric iff $\forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$.
- R is asymmetric iff $\forall x, y \in A\ (x\ R\ y \Rightarrow y\ \not R\ x)$.

Find a binary relation on A that is …

(c) asymmetric but not antisymmetric.

To prove:
Every asymmetric relation is antisymmetric.

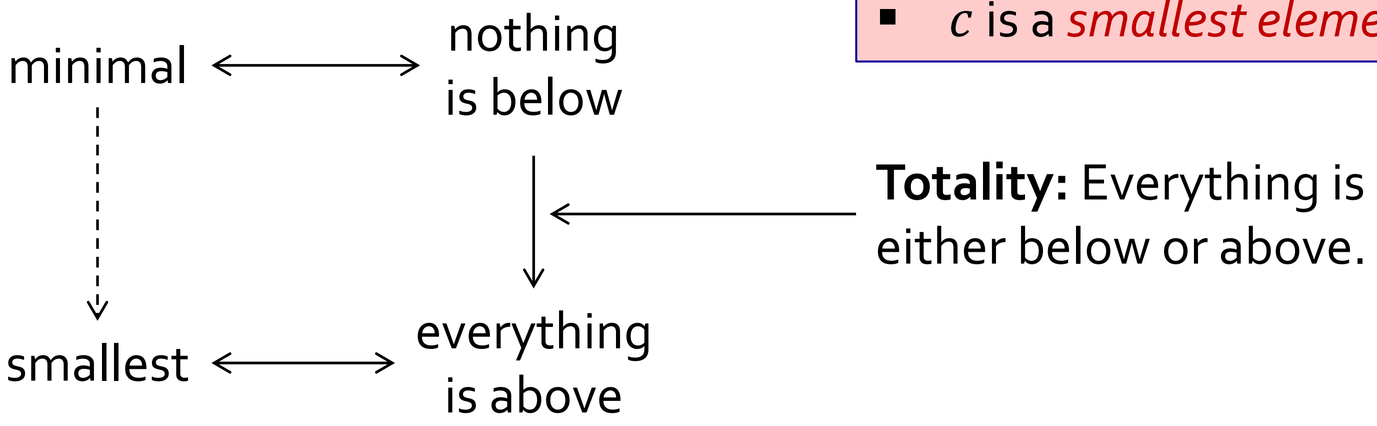
1. Take any binary relation R on a set A.
2. Suppose R is asymmetric.,
   2.1. Then $\forall x, y \in A\ (x\ R\ y \Rightarrow y\ \not R\ x)$ by the definition of asymmetry.
   2.2. $\equiv \forall x, y \in A\ (x\ \not R\ y \vee y\ \not R\ x)$ by the implication law.
   2.3. $\Rightarrow \forall x, y \in A\ \big((x\ \not R\ y \vee y\ \not R\ x) \vee x = y\big)$ by generalization.
   2.4. $\equiv \forall x, y \in A\ ({\sim}(x\ R\ y \wedge y\ R\ x) \vee x = y)$ by the De Morgan's law.
   Aim: 2.5. $\equiv \forall x, y \in A\ (x\ R\ y \wedge y\ R\ x \Rightarrow x = y)$ by the implication law.
3. Line 2.5 is the definition of antisymmetry, hence R is antisymmetric.

Q7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

A relation R on a set A is a *total order* iff
R is a partial order and $\forall x, y \in A\ (x\ R\ y \lor y\ R\ x)$.

Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

- c is a *minimal element* iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
- c is a *smallest element* iff $\forall x \in A\ (c \preccurlyeq x)$.



Q7. Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

A relation R on a set A is a *total order* iff
R is a partial order and $\forall x, y \in A\ (x\ R\ y \lor y\ R\ x)$.

Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

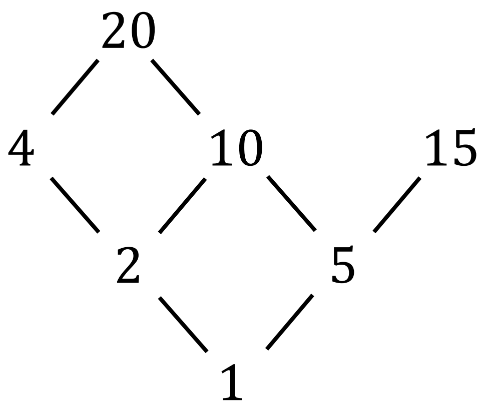
- c is a *minimal element* iff $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.
- c is a *smallest element* iff $\forall x \in A\ (c \preccurlyeq x)$.

1. Let $c \in A$ that is minimal with respect to $\preccurlyeq$
2. Pick any $x \in A$.
3. As $\preccurlyeq$ is a total order, either $x \preccurlyeq c$ or $c \preccurlyeq x$.
4. Case 1: $x \preccurlyeq c$.
   4.1. Then $x = c$ by the minimality of c.
   4.2. So $c \preccurlyeq x$ by the reflexibility of $\preccurlyeq$.
5. Case 2: $c \preccurlyeq x$.
   5.1. Then $c \preccurlyeq x$.

Aim:

6. So $c \preccurlyeq x$ in all cases, i.e., c is the smallest element.

Q8. Consider the “divides” relation on $A = \{1, 2, 4, 5, 10, 15, 20\}$.
List out the pairs of distinct elements in A that are
(a) comparable; (b) compatible.



Let $\preccurlyeq$ be a partial order on a set A, and $a, b \in A$.

- a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- a, b are *compatible* if there is $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(a) Comparable:

$\{1,2\}, \{1,5\}, \{1,4\}, \{1,10\}, \{1,15\}, \{1,20\}$,
$\{2,4\}, \{2,10\}, \{2,20\}, \{5,10\}, \{5,15\}, \{5,20\}, \{4,20\}, \{10,20\}$.

(b) Compatible:

$\{1,2\}, \{1,5\}, \{1,4\}, \{1,10\}, \{1,15\}, \{1,20\}$,
$\{2,4\}, \{2,5\}, \{2,10\}, \{2,20\}, \{5,4\}, \{5,10\}, \{5,15\}, \{5,20\}$,
$\{4,10\}, \{4,20\}, \{10,20\}$.

Q9. State whether the following is true or false and justify your answer.

Let $\preccurlyeq$ be a partial order on a set A, and $a, b \in A$.

- a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- a, b are *compatible* if there is $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(a) In all partially ordered set, any two comparable elements are compatible.

Yes

The 2 cases are symmetrical, so may just use WLOG to prove one of the cases.

1. Let $a, b \in A$ such that a, b are comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ by the definition of comparability.
3. Case 1: $a \preccurlyeq b$.
   - 3.1. Let $c = b$.
   - 3.2. Then $a \preccurlyeq b = c$ by assumption and $b \preccurlyeq b = c$ by reflexivity.
   - 3.3. So a, b are compatible by the definition of compatibility.
4. Case 2: $b \preccurlyeq a$.
   - 4.1. Let $c = a$.
   - 4.2. Then $b \preccurlyeq a = c$ by assumption and $a \preccurlyeq a = c$ by reflexivity.
   - 4.3. So a, b are compatible by the definition of compatibility.
5. So a, b are compatible in any case.

Q9. State whether the following is true or false and justify your answer.

Let $\preccurlyeq$ be a partial order on a set A, and $a, b \in A$.
- a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- a, b are *compatible* if there is $c \in A$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(b) In all partially ordered set, any two compatible elements are comparable.

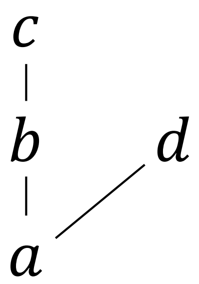
No

1. Consider the "divides" relation | on $\mathbb{Z}^+$, which is a partial order.
2. Then 2 and 3 are compatible as $2 \mid 6$ and $3 \mid 6$.
3. But 2 and 3 are not comparable as $2 \nmid 3$ and $3 \nmid 2$.

Q10. Let $A = \{a, b, c, d\}$. Consider the following partial order on A:

$$R = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (c,c), (d,d)\}.$$

Hasse diagram of R



Hasse diagrams of all the linearizations of R

Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfises the following condition for all $x, y \in A$:
If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.

A relation R on a set A a *total order* iff
R is a partial order and $\forall x, y \in A\ (x\ R\ y \lor y\ R\ x)$.

A *linearization* of a partial order $\preccurlyeq$ on a set A is a total order $\preccurlyeq^*$ on A such that

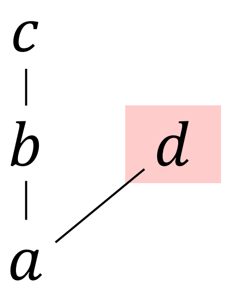
$$\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y).$$

Kahn's Algorithm. Pick out a minimal element and place it at the bottom of the total order. Repeat until nothing is left.

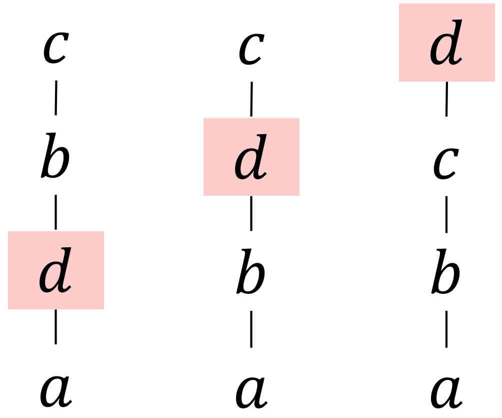
Q10. Let $A = \{a, b, c, d\}$. Consider the following partial order on A:

$$R = \{(a,a), (a,b), (a,c), (a,d), (b,b), (b,c), (c,c), (d,d)\}.$$

Hasse diagram of R



Hasse diagrams of all the linearizations of R



Let $\preccurlyeq$ be a partial order on a set A. A *Hasse diagram* of $\preccurlyeq$ satisfises the following condition for all $x, y \in A$:
If $x \prec y$ and no $z \in A$ is such that $x \prec z \prec y$, then x is placed below y and there is a line joining x to y, else no line joins x to y.

A relation R on a set A a *total order* iff
R is a partial order and $\forall x, y \in A\ (x\ R\ y \vee y\ R\ x)$.

A *linearization* of a partial order $\preccurlyeq$ on a set A is a total order $\preccurlyeq^*$ on A such that
$$\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y).$$

Kahn's Algorithm. Pick out a minimal element and place it at the bottom of the total order. Repeat until nothing is left.

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