NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF STATISTICS AND DATA SCIENCE

ST2334 PROBABILITY AND STATISTICS

MID-SEMESTER TEST SAMPLE PAPER 2

(SEMESTER I, AY 2023/2024)

TIME ALLOWED: 60 MINUTES

Suggested Solutions

INSTRUCTIONS TO STUDENTS

1. Please write your student number only. **Do not write your name.**
2. This assessment contains 15 questions and comprises 7 printed pages.
3. The total marks is 25; marks are equal distributed for all questions.
4. Please answer ALL questions.
5. Calculators may be used.
6. This is an **OPEN BOOK** assessment. Only **HARD COPIES** of materials are allowed.

1. **Multiple Choice: Choose one answer only**

   $A \cup (B \cap C) = ?$

   (a) $(A \cup B) \cap (A \cup C)$

   (b) $(A \cup B) \cap C$

   (c) $A \cup B' \cup C'$

   (d) $(A \cap B) \cup (A \cap C)$

   Solution
   (a)

2. **Fill in the blank**

   How many ways are there to choose an arbitrary number of students (including the possibility of choosing 0 student) from 6 students?

   (Provide your answer in numerical form.)

   Solution
   For each student, there are 2 possibilities: "chosen" or "not chosen". So the total number of possibilities is $2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$.

3. **Fill in the blank**

   Suppose

   $$P(A') = 1/2, \quad P(B) = 3/8, \quad \text{and}, \quad P(B'|A) = 3/4.$$

   Find $P(B \cap A)$.

   (Provide your answer in decimal form and round it to three decimal places if necessary)

   Solution
   0.125.

   $$\begin{aligned} P(B' \cap A) &= P(B'|A)P(A) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}, \\ P(B \cap A) &= P(A) - P(B' \cap A) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8} = 0.125. \end{aligned}$$

4. **Fill in the blank**

   A group of 8 friends A,B,C,D,E,F,G,H go to a restaurant. Due to safe-distancing measures, the group needs to split up into two groups of 4. How many ways are there to split the group such that A and B are together but away from C?

   (Provide your answer in numerical form.)

   Solution
   Except A, B, C, there are 5 people left. The group with A and B only has two more slots, the groups are set if and only if we select two more people out of 5 to fill in the slots of the group with A, B, and the rest 3 are with C. So the number of ways is $\binom{5}{2} = 10$.

5. **MULTIPLE RESPONSE: CHOOSE ALL ANSWERS THAT APPLY**

Let A and B be two events. Which of the following statements is/are true?

(a) If $A \neq B$, then $P(A) \neq P(B)$.

(b) If A and B are independent, then we must have $P(A \cup B) = 1 - \{1 - P(A)\}\{1 - P(B)\}$.

(c) If $P(A) = 1 - P(B')$, then $P(A) = P(B)$.

(d) $(A \cap B') \cup (A' \cap B) = \emptyset$, then $A = B$.

SOLUTION
Answer: (b), (c), (d).

6. **MULTIPLE CHOICE: CHOOSE ONE ANSWER ONLY**

Consider the following statements about Peter whom you have not met before.

(A): He is not married.

(B): He is not married and smokes.

(C): He is married.

(D): He is married and does not smoke.

You are to assign probabilities to these statements. Which answer below is consistent with the laws of probability?

(a) $P(A) = 0.45$, $P(B) = 0.5$, $P(C) = 0.55$, $P(D) = 0.4$

(b) $P(A) = 0.45$, $P(B) = 0.1$, $P(C) = 0.6$, $P(D) = 0.3$

(c) $P(A) = 0.45$, $P(B) = 0.2$, $P(C) = 0.55$, $P(D) = 0.5$

(d) $P(A) = 0.45$, $P(B) = 0.4$, $P(C) = 0.55$, $P(D) = 0.6$

SOLUTION
Answer: (c).

7. **TRUE/FALSE**

Let A and B be mutually exclusive events. If $P(A) = 0.1$, $P(B) = 0.01$, then A and B are not independent.

- TRUE
- FALSE

SOLUTION
TRUE; If otherwise, $P(A \cap B) = P(A)P(B) > 0$, which contradicts that A and B are mutually exclusive.

8. **TRUE/FALSE**

Cumulative distribution function can not take on values greater than 1 or smaller than 0.

- TRUE

- FALSE

SOLUTION
TRUE

9. **FILL IN THE BLANK**

Suppose that random variable X has the cumulative distribution function given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{9}, & 0 \le x \le 3 \\ 1, & x > 3 \end{cases}$$

Compute $P(X = 1.5)$.

(Provide your answer in decimal form and round it to two decimal places if necessary.)

SOLUTION
From the c.d.f., it is a continuous random variable. Therefore $P(X = 1.5) = 0$.

10. **FILL IN THE BLANK**

Let X be a random variable, whose cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.2, & 0 \le x < 2 \\ 0.6, & 2 \le x < 3 \\ 0.7, & 3 \le x < 5 \\ 1 & x \ge 5 \end{cases}.$$

Compute $E(X)$.

(Provide your answer in decimal form and round it to two decimal places if necessary.)

SOLUTION
X is a discrete RV, whose p.m.f. is given by $f(x) = 0.2, 0.4, 0.1, 0.3$, for $x = 0, 2, 3, 5$.

$$E(X) = 0 \times 0.2 + 2 \times 0.4 + 3 \times 0.1 + 5 \times 0.3 = 2.6.$$

11. **FILL IN THE BLANK**

Let X have probability mass function given by the following table.

x	0	2	5	6
$f(x)$	0.3	0.5	0.1	0.1

Compute $E(X)$.

(Provide your answer in decimal form and round it to two decimal places if necessary.)

SOLUTION
$E(X) = 0(0.3) + 2(0.5) + 5(0.1) + 6(0.1) = 2.1.$

12. **MULTIPLE CHOICE: CHOOSE ONE ANSWER ONLY**

Let X be a random variable. Which of the following statement is **INCORRECT**?

(a) If $P(X=1)=0.1$ and $E(X)$ exists, then we must have $E(X^2)>(E(X))^2$.

(b) If $V(X)>0$, then for any x, $P(X=x)<1$.

(c) By the definition of the random variable, the range of X is a subset of $\mathbb{R}$; therefore, it is impossible that $P(X=x)=0$ for any $x \in \mathbb{R}$.

(d) There are cases under which $E(X)$ does not exist.

SOLUTION

Answer: (c)

- (a) is correct because $V(X)=E(X^2)-[E(X)]^2$ and $P(X=1)=0.1$ implies $V(X)>0$ (since otherwise $P(X=E(X))=1$).
- (b) is correct with similar reason to (a).
- For any continuous RV, we must have $P(X=x)=0$ for any $x \in \mathbb{R}$.
- For example, X is a random variable with p.d.f.

$$f(x)=\begin{cases} \frac{1}{2x^2}, & \text{for } |x| \geq 1, \\ 0, \text{elsewhere} \end{cases}$$

Then

$$E(X)=\int_{-\infty}^{1} x \cdot \frac{1}{2x^2} dx + \int_{1}^{\infty} x \cdot \frac{1}{2x^2} dx = -\infty + \infty,$$

which implies $E(X)$ does not exist.

13. **FILL IN THE BLANK**

A service station has both self-service and full-service islands. On each island, there is a single regular unleaded pump with two hoses. Let X denote the number of hoses being used on the self-service island at a particular time, and let Y denote the number of hoses on the full-service island in use at that time. The joint probability mass function of X and Y is given in the table below.

x	y		
	0	1	2
0	0.10	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.30

Compute $E(X|Y=1)$.

(Provide your answer in decimal form and round it to two decimal places if necessary.)

SOLUTION

1.26.

Focusing on the column of $y = 1$, the sum of these numbers leads to $P(Y = 1) = 0.04 + 0.20 + 0.14 = 0.38$. So, we obtain the conditional distribution of $X|Y = 1$:

$$\begin{aligned} P(X = 0|Y = 1) &= 0.04/0.38; \\ P(X = 1|Y = 1) &= 0.20/0.38; \\ P(X = 2|Y = 2) &= 0.14/0.08. \end{aligned}$$

Therefore, we obtain

$$E(X|Y = 1) = 0(0.04/0.38) + 1(0.20/0.38) + 2(0.14/0.38) = 0.48/0.38 = 1.26.$$

14. **TRUE/FALSE**

Let $f(x,y)$ be the joint probability function of a discrete random vector (X,Y). If $f_X(1) = 0$, then $f(1,y) = 0$ for any y being a real number.

- TRUE
- FALSE

SOLUTION
TRUE
$f_X(1) = 0$ implies $\sum_{y \in R_Y} f(1,y) = 0$, but $f(1,y) \geq 0$; therefore $f(1,y) = 0$ for all $y \in R_Y$, and thus $f(1,y) = 0$ for all real numbers y.

15. The joint probability function of (X,Y) is given by

$$f(x,y) = \begin{cases} \frac{1}{8}(x+y) & 0 \leq x \leq 2; 0 \leq y \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

Compute $P(Y \geq 1|X \geq 1)$.

(Provide your answer in decimal form and round it to three decimal places if necessary.)

SOLUTION
0.6.

We need to compute

$$P(Y \geq 1|X \geq 1) = \frac{P(Y \geq 1; X \geq 1)}{P(X \geq 1)}.$$

We shall evaluate the numerator and denominator separately.

The marginal density of X is given by

$$f_X(x) = \int_0^2 \frac{1}{8}(x+y)dx = \frac{1}{8}(2x+2) = \frac{1}{4}(x+1).$$

Therefore

$$P(X \geq 1) = \int_1^2 \frac{1}{4}(x+1)dx = 0.625.$$

For the numerator:

$$P(X \geq 1; Y \geq 1) = \int_1^2 \int_1^2 \frac{1}{8}(x+y)dxdy = 0.375.$$

As a consequence

$$P(Y \geq 1 | X \geq 1) = \frac{0.375}{0.625} = 0.6.$$

END OF PAPER