CS3236 Semester 2 2022/23:
Midterm (Total 50 Marks)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You are given 1 hour and 30 minutes for this assessment. You are allowed one sheet of A4 paper, printed or written on both sides. Calculators are not permitted.

1. [**Entropy and Mutual Information**]

   (a) **(18 Marks)** Consider the pair (Z, X) described as follows for some $p \in (0,1)$:

   - $Z \sim \text{Bernoulli}(p)$ (i.e., $P_Z(1) = p$ and $P_Z(0) = 1 - p$)
   - Given $Z = 0$, the conditional distribution of X is uniform on $\{1,2,3,4\}$
   - Given $Z = 1$, the conditional distribution of X is uniform on $\{5,6,7,8,9,10,11,12\}$

   Recall the definition $H_2(q) = q \log_2 \frac{1}{q} + (1-q) \log_2 \frac{1}{1-q}$, which you may use in your answers if you wish.

   (i) Compute $H(Z)$, $H(X|Z=0)$, $H(X|Z=1)$, $H(X|Z)$, $H(Z|X)$, $H(X)$, $H(X,Z)$ and $I(X;Z)$, expressing your answers in terms of p when needed.

   (ii) Show that there exists at least one choice of p such that $H(X|Z=1) > H(X)$, and another choice of p such that $H(X|Z=1) < H(X)$.

   (iii) Do there exist values of $p \in [0,1]$ such that $H(X|Z=0) > H(X)$? Explain briefly.

   (b) **(6 Marks)** Suppose that for some triplet of random variables (X, Y, Z), we know that $I(Y;Z) - I(X;Z) = 1$. Can we conclude that $I(Y;Z|X) - I(X;Z|Y)$ is less than 1, equal to 1, or greater than 1, or is it the case that two or more of these are possible based on the information given? Explain your answer.

(There is more space for your answer on the following page)

(Extra space for answering Q1)

(If you still need more space, please use Pages 5 and 6)

2. **[Source Coding Algorithms]**

In both parts (a) and (b) below, you should assume that $P_X(x) > 0$ for all x.

(a) **(12 Marks)** This question concerns Shannon-Fano coding. We use letters (a, b, c, ...) to denote the symbols in the source.

Notes: *• The following are 4 separate questions, and none of the information from one should be assumed in any of the other 3. • When the question mentions "potentially taking values", this means that the corresponding quantity may (or may not) take different values depending on the alphabet size, the unspecified P_X values, etc., and your task is to identify all values that may occur when those vary.*

Let $\ell(a), \ell(b), \dots$ denote the codeword lengths of the Shannon-Fano code corresponding to source P_X, and answer the following:

(i) If we (only) know that $P_X(a) = 8P_X(b)$ and $\ell(a) = 2$, then what integer values could the length $\ell(b)$ potentially take?

(ii) If we (only) know that $P_X(a) = 6P_X(b)$ and $\ell(a) = 2$, then what integer values could $\ell(b)$ potentially take?

(iii) If there are 10 symbols (i.e., a–j) and symbol a is the **least** likely among them, what values could the length $\ell(a)$ potentially take?

(iv) If there are 10 symbols (i.e., a–j) and symbol a is the **most** likely among them, what values could the length $\ell(a)$ potentially take?

(Extra space on Pages 5 and 6 if needed)

(b) **(14 Marks)** This question concerns Huffman coding. Recall that the Huffman algorithm repeatedly merges two nodes to create a new node whose value sums those of the two being merged. Let $P_X(\cdot)$ be the source distribution (with an unspecified alphabet size). *(Note: The following are 4 separate questions, and none of the information from one should be assumed in any of the other 3)*

(i) If $P_X(a) = \frac{1}{2}$, explain why the Huffman algorithm can only merge symbol a's node with another node (possibly formed from earlier merging) whose value is also $\frac{1}{2}$.

(ii) If we (only) know that $P_X(a) = 0.2$, then there are many possible values of the node that a is eventually merged with (depending on the alphabet size and the other P_X values). Find the entire set of such possible values, explaining your answer.

(iii) Can the Huffman code lengths potentially satisfy $\sum_x 2^{-\ell(x)} > 1$? Explain.

(iv) Can the Huffman code lengths potentially satisfy $\sum_x 2^{-\ell(x)} < 1$? Explain.

(Extra space on Pages 5 and 6 if needed)

(Extra space for any question you need it for – **please clearly indicate the question number & letter here, and on the relevant earlier page(s) write "See Page 5"***)*

(Extra space for any question you need it for – **please clearly indicate the question number & letter here, and on the relevant earlier page(s) write "See Page 6"** *)*