CS1231S Assignment #2

AY2021/22 Semester 2

Deadline: Monday, 4 April 2022, 1:00pm

ANSWERS

IMPORTANT: Please read the instructions below

This is a graded assignment worth 10% of your final grade. There are **four questions** and total score is 20 marks. Please work on it by yourself, not in a group or in collaboration with anybody. Anyone found committing plagiarism (submitting other's work as your own), or sending your answers to others, or other forms of academic dishonesty will be penalised with a straight zero for the assignment, and possibly an F grade for the module.

You are to submit your assignment to **LumiNUS Files**. A submission folder has been created for you at Files > Assignment #2 > Your tutorial group > Your personal folder.

Your answers may be typed or handwritten. Make sure that it is legible (for example, don't use very light pencil or ink if it is handwritten, or font size smaller than 11 if it is typed) or marks may be deducted.

You are to submit **a SINGLE pdf file**, where each page is A4 size. Do not submit multiple files or files in other formats. If you submit multiple files, we will grade only the last submission.

You may test out your submission folder before the deadline, but make sure you remove any test files you have submitted earlier. You must also remove your assignment 1 submission as we have recycled the submission folders.

Late submission will NOT be accepted. We will set the closing time of the submission folders to slightly later than 1pm to give you a few minutes of grace, but in your mind, you should treat **1pm** as the deadline. If you think you might be too busy on the day of the deadline, please submit earlier. Also, avoid submitting in the last minute; the system may get sluggish due to overload and you will miss the deadline.

Note the following as well:

- Name your pdf file with your **Student Number**. Your student number begins with 'A' (eg: A0234567X). (Do not mix up your student number with your NUSNET-id which begins with 'e'.) A penalty of one mark will be given if your file is not named according to this instruction.
- At the top of the first page of your submission, write your **Name** and **Tutorial Group**. **Both** must be present and correct, or a penalty of one mark will be given.
- To keep the submitted file short, you may submit your answers without including the questions.
- As this is an assignment given well ahead of time, we expect you to work on it early. You should submit **polished work**, not answers that are untidy or appear to have been done in a hurry, for example, with scribbling and cancellation all over the places. Marks may be deducted for untidy work.

To combine all pages into a single pdf document for submission, you may find the following scanning apps helpful if you intend to scan your handwritten answers:

* for Android: https://fossbytes.com/best-android-scanner-apps/

* for iphone:
https://www.switchingtomac.com/tutorials/ios-tutorials/the-best-ios-scanner-apps-to-scan-documents-images/

If you need any clarification about this assignment, please do NOT email us or post on telegram, but post on the **LumiNUS > Assignments** forum so that everybody can read the answers to your queries.

Note: Do not use any methods that have not been covered in class. When using a theorem or result that has appeared in class (lectures or tutorials), please quote the theorem number/name, the lecture and slide number, or the tutorial number and question number in that tutorial, failing which marks may be deducted. Remember to use numbering and give justification for important steps in your proof, or marks may be deducted.

Question 1. (5 marks)

Given the following statement:

> "For any integer n, and any positive integer d, there exist integers q and r such that $n = dq + r$, where $0 \leq r < d$."

(a) Use Mathematical Induction to prove the above statement for $n \in \mathbb{N}$.

(b) Using the result of (a), prove the above statement for $n \in \mathbb{Z}$.

(a) [3 marks]

Answer:

1. Let $P(n) \equiv (\forall d \in \mathbb{Z}^+ \; \exists q, r \in \mathbb{Z}$ such that $n = dq + r$, where $0 \leq r < d)$, for any $n \in \mathbb{N}$.
2. Basis step: $n = 0$
   2.1. For any $d \in \mathbb{Z}^+$, let $q = r = 0$.
   2.2. Then $n = 0 = dq + r = d(0) + 0$.
   2.3. Hence $P(0)$ is true.
3. Inductive step: Assume $P(k)$ is true for any $k \in \mathbb{N}$, i.e. $\forall d \in \mathbb{Z}^+ \; \exists q, r \in \mathbb{Z}$ such that $k = dq + r$, where $0 \leq r < d$.
   3.1. Then $k + 1 = (dq + r) + 1$ (by the inductive hypothesis).
   3.2. Case 1: $r < d - 1$
      3.2.1. Let $r' = r + 1$.
      3.2.2. Hence $k + 1 = (dq + r) + 1 = dq + (r + 1) = dq + r'$.
      3.2.3. Note that $r' = r + 1 < d$. ← this is important, to ensure that $k + 1$ can be expressed in the desired form
      3.2.4. Hence $P(k + 1)$ is true.
   3.3. Case 2: $r = d - 1$
      3.3.1. Then $k + 1 = (dq + d - 1) + 1 = dq + d = d(q + 1) + 0$.
      3.3.2. Note that $q + 1$ is an integer since q is an integer (closure of integers under +).
      3.3.3. Hence $P(k + 1)$ is true.
   3.4. In both cases, $P(k + 1)$ is true.
4. Therefore, $\forall n \in \mathbb{N} \; P(n)$ is true by MI.

Alternative answer (by strong MI)

1. Let $P(n) \equiv (\forall d \in \mathbb{Z}^+\ \exists q, r \in \mathbb{Z}$ such that $n = dq + r$, where $0 \leq r < d)$, for any $n \in \mathbb{N}$.

2. Basis step: $n = 0$
   2.1. For any $d \in \mathbb{Z}^+$, let $q = r = 0$.
   2.2. Then $n = 0 = dq + r = d(0) + 0$.
   2.3. Hence $P(0)$ is true.

3. Inductive hypothesis: Assume $P(n)$ is true for all $n \leq k$ for some positive integer k.

4. Inductive step: We want to show $P(k + 1)$ is true.
   4.1. Case 1: $d > k + 1$
      4.1.1. Let $q = 0, r = k + 1$.
      4.1.2. Therefore, $k + 1 = qd + r$, where $0 \leq r = k + 1 < d$.
      4.1.3. Hence $P(k + 1)$ is true.
   4.2. Case 2: $d \leq k + 1$
      4.2.1. Hence $0 \leq k + 1 - d$.
      4.2.2. Since $d > 0$, we have $k + 1 - d < k + 1$.
      4.2.3. Therefore, $0 \leq k + 1 - d \leq k$ (by 4.2.1, 4.2.2 and since $k + 1 - d \in \mathbb{Z}$).
      4.2.4. By the induction hypothesis, $k + 1 - d = ds + t$ for some integers s, t where $0 \leq t < d$.
      4.2.5. Therefore, $k + 1 = d(s + 1) + t$.
      4.2.6. Hence $P(k + 1)$ is true.
   4.3. In both cases, $P(k + 1)$ is true.

5. Therefore, $\forall n \in \mathbb{N}\ P(n)$ is true by strong MI.

(b) [2 marks]
Answer:

(It suffices to show $P(n)$ is true for all negative integers n.)

1. Suppose $m \in \mathbb{Z}^-$, then $m = -n$ for some $n \in \mathbb{Z}^+$ such that $n = dq + r$, where $q, r \in \mathbb{Z}$ and $0 \leq r < d$ (from part (a)).
2. Now, $m = -n = -(dq + r) = d(-q) - r$ by basic algebra.
3. Case 1: $r = 0$
   4.1. Then $m = d(-q) + 0$ where $-q \in \mathbb{Z}$ (by closure of integers under $\times$) and $0 \leq 0 < d$.
4. Case 2: $0 < r < d$
   4.1. Then $m = d(-q) - r = d(-q - 1) + (d - r)$ where $-q - 1 \in \mathbb{Z}$ (by closure of integers under $-$) and $0 < (d - r) < d$. ← again, this is important, that $(d - r)$ is in the desired range of values
5. Therefore in all cases $P(n)$ is true for $n \in \mathbb{Z}^-$.
6. With part (a), $P(n)$ is true for $n \in \mathbb{Z}$.

The purpose for part (b) is to use part (a) result, so there is no need to do MI all over again for the negative integers.

Question 2. (5 marks)

Let $A = \{a, b\}$ and S be the set of all strings over A. Define the function $f : S \to S$ as follows:

$\forall s \in S$, $f(s)$ is a new string that replaces the leftmost occurrence of a in s with b.
If a does not occur in s, then $f(s) = s$.

Examples: $f(\varepsilon) = \varepsilon$; $f(a) = b$; $f(bbb) = bbb$; $f(bbaab) = bbbab$.

Another function $g : S \to \mathbb{N}$ is defined as follows:

$\forall s \in S$, $g(s) =$ (the number of a's in s) $-$ (the number of b's in s), or 0 if the difference is negative.

Examples: $g(\varepsilon) = 0$; $g(abaa) = 2$; $g(bb) = 0$; $g(babaa) = 1$; $g(abba) = 0$.

(a) Is f injective? Prove or disprove. (Mark will not be awarded with a mere "yes" or "no" answer without justification.)

Answer: No. $f(a) = b = f(b)$ but $a \neq b$. [1 mark]

(b) What is $(f \circ f)(abaa)$?

Answer: $(f \circ f)(abaa) = f\big(f(abaa)\big) = f(bbaa) = \boldsymbol{bbba}$. [1 mark]

(c) Is $f \circ f$ injective? Prove or disprove. (Mark will not be awarded with a mere "yes" or "no" answer without justification.)

Answer: No. $(f \circ f)(aab) = f\big(f(aab)\big) = f(bab) = bbb = (f \circ f)(bbb)$ but $aab \neq bbb$. [1 mark]

(d) Is g surjective? Prove or disprove. (Mark will not be awarded with a mere "yes" or "no" answer without justification.)

Answer: Yes.
Proof (by construction) (other constructions possible)

1. Take any $n \in \mathbb{N}$.
2. Let $s = aa \cdots a$ (exactly n a's). (Note: n is possibly 0. If so, $s = \varepsilon$.)
3. Hence, $s \in S$ since its members are all a's.
4. Now, $g(s) = n - 0 = n$ by the definition of g.
5. Therefore, g is surjective.

[2 marks]

The trick is to construct a suitable $s \in S$ that works. Other constructions are possible, eg. a string containing $(n + k)$ a's followed by k b's.

Question 3. (4 marks)

Define $f: \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\} \to \mathbb{N}$ by setting $f(S)$ to be the smallest element of S whenever $S \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$. (Note that $\mathcal{P}(A)$ denotes the power set of A.)

Prove or disprove the following:

(a) The function f has an inverse.

(b) $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{N}$.

Answers:

(a) **False.**

$f(\{0\}) = 0 = f(\{0,1\})$, but $\{0\} \neq \{0,1\}$. So f is not injective, and hence not bijective. It follows that f does not have an inverse (by Theorem 7.2.3) [2 marks]

Theorem 7.2.3: If $f: X \to Y$ is a bijection, then $f^{-1}: Y \to X$ is also a bijection. In other words, $f: X \to Y$ is bijective iff f has an inverse.

(b) **True.**

To prove $f^{-1}(\{0\})$ is uncountable.

1. Note that $f^{-1}(\{0\}) = \{\{0\} \cup S : S \in \mathcal{P}(\mathbb{Z}^+)\}$.
2. If S_1 and S_2 are different elements of $\mathcal{P}(\mathbb{Z}^+)$, then $\{0\} \cup S_1 \neq \{0\} \cup S_2$.
3. So $|f^{-1}(\{0\})| = |\mathcal{P}(\mathbb{Z}^+)|$.
4. As $\mathcal{P}(\mathbb{Z}^+)$ is uncountable (by tutorial 8 question 9), hence $f^{-1}(\{0\})$ is also uncountable.

Tutorial 8 Question 9: Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.

[2 marks]

Alternative proof (proof by contradiction)

1. Suppose $f^{-1}(\{x\})$ is countable for all $x \in \mathbb{N}$.
2. Let $n \in \mathbb{N}$, then $f^{-1}(\{n\})$ is countable (by line 1).
3. Claim: $f^{-1}(\{n\})$ is infinite.
   - 3.1. Consider the set
   $$T = \{\{n\}, \{n, n+1\}, \{n, n+1, n+2\}, \{n, n+1, n+2, n+3\}, \ldots\}$$
   $$= \{\{(n+k): 0 \leq k \leq l\} : l \in \mathbb{N}\}$$
   - 3.2. Claim: T is countably infinite.
     - 3.2.1. We will show this claim by finding a bijection $g: T \to \mathbb{N}$.
     - 3.2.2. Let $g: T \to \mathbb{N}$ be defined by $g(\{n, \ldots, n+l\}) = l$.
     - 3.2.3. Let $g^*: \mathbb{N} \to T$ be defined by $g^*(l) = \{(n+k): 0 \leq k \leq l\}$.
     - 3.2.4. Then for any $x \in T, y \in \mathbb{N}$, we have $y = g(x) \Leftrightarrow x = g^*(y)$.
     - 3.2.5. $\therefore$ g^* is the inverse of g.
     - 3.2.6. $\therefore$ g is a bijection from T to $\mathbb{N}$.

3.2.7. $\therefore |T| = |\mathbb{N}|$ and hence T is countably infinite.

3.3. Claim: $T \subseteq f^{-1}(\{n\})$

3.3.1. Let Y be an arbitrary element of T.

3.3.2. $f(Y) = n$ since the smallest element of Y is n by our construction of T.

3.3.3. $\therefore Y \in f^{-1}(\{n\})$ by the definition of setwise preimage.

3.3.4. Since $Y \in T \Rightarrow Y \in f^{-1}(\{n\})$, therefore $T \subseteq f^{-1}(\{n\})$.

3.4. As T is infinite (line 3.2) and $T \subseteq f^{-1}(\{n\})$ (line 3.3), hence $f^{-1}(\{n\})$ is infinite.

4. Since $f^{-1}(\{n\})$ is countably infinite (by 2 and 3), we can get a sequence of sets $A_1, A_2, A_3, \ldots$ in which every element of $f^{-1}(\{n\})$ appears exactly once (Lecture 9, Proposition 9.1).
5. Consider the set $B = \{n\} \cup \{(n+k) \mid (n+k) \notin A_k,\ k \in \mathbb{Z}^+\}$.

5.1. For any A_k, $B \neq A_k$

5.1.1. If $(n+k) \in A_k$, then $(n+k) \notin B$.

5.1.2. If $(n+k) \notin A_k$, then $(n+k) \in B$.

5.2. However, $B \in f^{-1}(\{n\})$ by our construction of B, since the smallest element of B would be n.

5.3. This contradicts line 4.

6. Therefore $f^{-1}(\{x\})$ is uncountable for some $x \in \mathbb{N}$.

Alternative answer:

Lemma:

$$\bigcup_{i\in\mathbb{N}} f^{-1}(\{i\}) = \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$$

Proof of lemma:

1. We know that $f: \mathcal{P}(\mathbb{N})\backslash\{\emptyset\} \to \mathbb{N}$.
2. Consider the setwise preimage of $\mathbb{N}$ under f.
   2.1. Since $\mathbb{N}$ is the codomain of f, then $f^{-1}(\mathbb{N}) = \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$.
3. Since

$$\mathbb{N} = \bigcup_{i\in\mathbb{N}} \{i\}$$

   we have

$$\bigcup_{i\in\mathbb{N}} f^{-1}(\{i\}) = f^{-1}\left(\bigcup_{i\in\mathbb{N}} \{i\}\right) = f^{-1}(\mathbb{N}) = \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$$

We will show that $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{N}$ by contradiction.

1. Assume for contradiction that $f^{-1}(\{x\})$ is countable for all $x \in \mathbb{N}$.
2. Therefore,

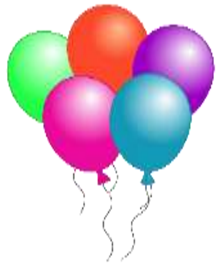
$$\bigcup_{i\in\mathbb{N}} f^{-1}(\{i\})$$

   Is also countable (countable union of countable sets are countable, by Lecture #9 slide 30).
3. By the above lemma, $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$ is countable.
4. By tutorial 8 question 6, there is a bijection $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\} \cup \{\emptyset\} \to \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$, and hence $\mathcal{P}(\mathbb{N})$ (which is $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\} \cup \{\emptyset\}$) is countable.
5. This contradicts the fact that $\mathcal{P}(\mathbb{N})$ is uncountable (by tutorial 8 question 9).
6. Therefore, $f^{-1}(\{n\})$ is uncountable for some $n \in \mathbb{N}$.

Tutorial 8 Question 6: Let B be a (not necessarily countable) infinite set and C be a finite set. Define a bijection $B \cup C \to B$.

Tutorial 8 Question 9: Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.

Question 4. (6 marks)



The Aiken Dueet balloon shop sells balloons with five colours with at least 20 balloons of each colour.

(a) How many different combinations of 20 balloons can you choose?

(b) If Aiken Dueet has only 5 red balloons but at least 20 of each of the other colours, how many combinations of 20 balloons can you choose?

(c) If Aiken Dueet has only 5 red balloons and 5 green balloons but at least 20 of each of the other colours, how many combinations of 20 balloons can you choose?

Show your workings.

Answers:

The number of multisets of size r that can be selected from a set of n elements is $\binom{r+n-1}{r}$.

(a) Here, $r = 20, n = 5$, so $\binom{24}{20} = \mathbf{10626}$. [1 mark]

(b) Let $R_{\geq 6}$ denote the set of selections containing at least 6 red balloons, $R_{\leq 5}$ the set of selections containing at most 5 red balloons, and T the set of selections with no restriction.

$T = R_{\leq 5} \cup R_{\geq 6}$ and $R_{\leq 5} \cap R_{\geq 6} = \emptyset$.

$|R_{\geq 6}| = \binom{14+5-1}{14} = \binom{18}{14} = 3060.$

Therefore $|R_{\leq 5}| = |\mathrm{T}| - |R_{\geq 6}| = 10626 - 3060 = \mathbf{7566}$. [2 marks]

(c) Let $G_{\geq 6}$ denote the set of selections containing at least 6 green balloons, and $G_{\leq 5}$ the set of selections containing at most 5 green balloons.

$|G_{\geq 6}| = \binom{14+5-1}{14} = \binom{18}{14} = 3060.$

$|R_{\geq 6} \cap G_{\geq 6}| = \binom{8+5-1}{8} = \binom{12}{8} = 495.$

$|R_{\geq 6} \cup G_{\geq 6}| = |R_{\geq 6}| + |G_{\geq 6}| - |R_{\geq 6} \cap G_{\geq 6}| = 3060 + 3060 - 495 = 5625.$

$|R_{\leq 5} \cap G_{\leq 5}| = |T| - |\overline{R_{\leq 5} \cap G_{\leq 5}}| = |T| - |R_{\geq 6} \cup G_{\geq 6}| = 10626 - 5626 = \mathbf{5001}$. [3 marks]

=== End of paper ===