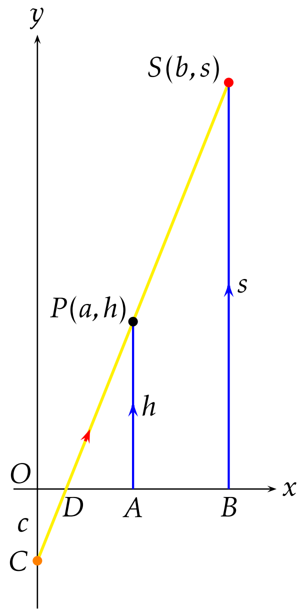
Solutions to Exam 2019-2020 Semester 2

1. In this question all length measurements are in meters and time measurements are in seconds. Let a, b, c denote three positive constants with $b > a$. A light source is placed at the point $(0, -c)$. At time $t = 0$ a particle starts at the point $(a, 0)$ moving upwards along the line $x = a$ in such a way that at time t its height h from the starting point $(a, 0)$ is directly proportional to t^2. It is observed that $h = 28$ at time $t = 3$. If the line $x = b$ represents a screen and the speed of the shadow of the projection of the particle onto the screen is equal to 188 meter per second when $h = 68$, find the value of the ratio $\frac{b}{a}$. Give your answer correct to two decimal places.

   Answer. 6.46.

   Solution. In the figure below, $A = (a, 0)$, $B = (b, 0)$, P is the particle and S is its shadow on the screen.



   The equation of the line PC is $\frac{y+c}{x} = \frac{h+c}{a}$. Thus $y = \frac{1}{a}(h+c)x - c$. When $x = b$, we have $y = s$. Therefore, $s = \frac{1}{a}(h+c)b - c$. Equivalently, $s = \frac{b}{a}h + \frac{c}{a}(b-a)$.

   As $h = kt^2$ and $h(3) = 28$, we have $28 = 9k$ so that $k = \frac{28}{9}$. Thus $h = \frac{28}{9}t^2$.

   When $h = 68$, we have $68 = \frac{28}{9}t^2$ so that $t = \sqrt{\frac{9\times 68}{28}}$.

   As $s = \frac{b}{a}\frac{28}{9}t^2 + \frac{c}{a}(b-a)$, we have $\frac{ds}{dt} = \frac{b}{a}\frac{2\times 28}{9}t$.

   At $t = \sqrt{\frac{9\times 68}{28}}$, we are given $\frac{ds}{dt} = 188$. Therefore, $188 = \frac{b}{a}\frac{2\times 28}{9}\sqrt{\frac{9\times 68}{28}}$.

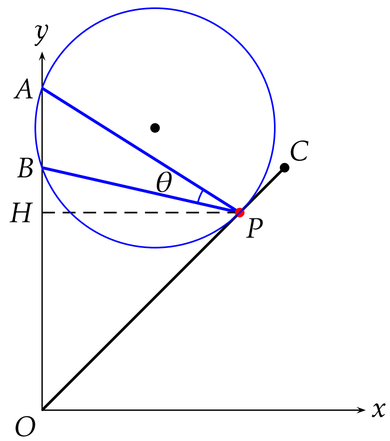
   From this, $\frac{b}{a} = 188 \times \frac{9}{2\times 28} \times \sqrt{\frac{28}{9\times 68}} = \frac{141}{2\sqrt{119}} = 6.46$. ∎

2. Let O denote the origin $(0,0)$, A denote the point $(0,2020)$, B denote the point $(0,1521)$ and C denote the point $(1521,1521)$. Let P denote a point on the line segment OC such that P is between O and C, $P \neq O$ and $P \neq C$. Let θ denote the angle APB measured in DEGREES. Note that $0 < \theta < 90°$. Find the maximum value of θ. Give your answer correct to two decimal places.

   (Important: Leave out the degree symbol ° when you enter your answer as the computer cannot recognize it. For e.g. if your answer is 1.23°, then you just enter 1.23 for your answer.)

   Answer. 19.40.

   Solution. Let $P = (t,t)$. Let $b = 1521$, $a = 2020$. Let H be the foot of the perpendicular from P onto OA. Then $\theta = \angle APB = \tan^{-1}\frac{t}{b-t} - \tan^{-1}\frac{t}{a-t}$, $0 < t < b < a$.



$\frac{d\theta}{dt} = \frac{\frac{1}{b-t}-\frac{t}{(b-t)^2}}{1+(\frac{t}{b-t})^2} - \frac{\frac{1}{a-t}-\frac{t}{(a-t)^2}}{1+(\frac{t}{a-t})^2} = \frac{(a-b)(ab-2t^2)}{((a-t)^2+t^2)((b-t)^2+t^2)}$.

Thus $\frac{d\theta}{dt} = 0 \Leftrightarrow ab - 2t^2 = 2(\sqrt{\frac{ab}{2}} + t)(\sqrt{\frac{ab}{2}} - t) = 0 \Leftrightarrow t = \sqrt{\frac{ab}{2}}$.

Also $0 < t < \sqrt{\frac{ab}{2}} \Rightarrow \frac{d\theta}{dt} > 0$ and $\sqrt{\frac{ab}{2}} < t < b \Rightarrow \frac{d\theta}{dt} < 0$. By the first derivative test, θ has an absolute maximum at $t = \sqrt{\frac{ab}{2}} = \sqrt{\frac{2020\times 1521}{2}} = 39\sqrt{1010} = 1239.439$.

The maximum value of θ is $\tan^{-1}\frac{1239.439}{1521-1239.439} - \tan^{-1}\frac{1239.439}{2020-1239.439} = 0.338645$ radian = 19.40 degree.

Remark. The position of P for the maximum value of θ can be shown to be the point of tangency of the circle through A, B and tangent to the line OC. Thus $(\sqrt{2}t)^2 = OP^2 = OA \times OB = 2020 \times 1521$. ■

3. Let $f(x) = 2(\sin(\frac{\pi}{4} - 864x))(\sin(\frac{\pi}{4} + 288x))$. Find the value of the definite integral

$$\int_0^{10\pi} |f(x)|\,dx,$$

where $|z|$ denotes the absolute value of z. Give your answer correct to two decimal places.

Answer. 25.98.

Solution. First we simplify f as follow.

$$\begin{aligned} f(x) &= 2\sin(\tfrac{\pi}{4} - 864x)\sin(\tfrac{\pi}{4} + 288x) \\ &= \cos(1152x) - \cos(\tfrac{\pi}{2} - 576x) \\ &= \cos(1152x) - \sin(576x) \\ &= 1 - 2\sin^2(576x) - \sin(576x) \\ &= \tfrac{9}{8} - 2(\sin(576x) + \tfrac{1}{4})^2. \end{aligned}$$

Thus f has a period of $\frac{2\pi}{576}$. Let's first restrict f on $[0, \frac{2\pi}{576}]$.

Now $f(x) = 0 \Leftrightarrow \frac{9}{8} - 2(\sin(576x) + \frac{1}{4})^2 = 0 \Leftrightarrow \sin(576x) = \frac{1}{2}, -1 \Leftrightarrow 576x = \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{3\pi}{2} + 2k\pi \;\; k = 0, 1, 2 \ldots.$

Thus the solutions for $f(x) = 0$ in $[0, \frac{2\pi}{576}]$ are $x = \frac{\pi}{6\times 576}, \frac{5\pi}{6\times 576}, \frac{3\pi}{2\times 576}$.

Also $f'(x) = -4(\sin(576x) + \frac{1}{4})\cos(567x)$ so that $f'(\frac{3\pi}{2\times 576}) = 0$. This means f has a double root at $x = \frac{3\pi}{2\times 576}$.

x	$0 \le x < \frac{\pi}{6\times 576}$	$\frac{\pi}{6\times 576}$	$\frac{\pi}{6\times 576} < x < \frac{5\pi}{6\times 576}$	$\frac{5\pi}{6\times 576}$
$f(x)$	+	0	−	0

x	$\frac{5\pi}{6\times 576} < x < \frac{3\pi}{2\times 576}$	$\frac{3\pi}{2\times 576}$	$\frac{3\pi}{2\times 576} < x < \frac{2\pi}{576}$
$f(x)$	+	0	+

Therefore, $\int_0^{\frac{2\pi}{576}} |f(x)|\,dx = \int_0^{\frac{\pi}{6\times 576}} f(x)\,dx - \int_{\frac{\pi}{6\times 576}}^{\frac{5\pi}{6\times 576}} f(x)\,dx + \int_{\frac{5\pi}{6\times 576}}^{\frac{2\pi}{576}} f(x)\,dx$.

As $\int \cos(1152x) - \sin(576x)\,dx = \frac{1}{1152}\sin(1152x) + \frac{1}{576}\cos(576x) + C$, we have

$$\begin{aligned} \int_0^{\frac{2\pi}{576}} |f(x)|\,dx &= \Big[\tfrac{1}{1152}\sin(1152x) + \tfrac{1}{576}\cos(576x)\Big]_0^{\frac{\pi}{6\times 576}} \\ &\quad - \Big[\tfrac{1}{1152}\sin(1152x) + \tfrac{1}{576}\cos(576x)\Big]_{\frac{\pi}{6\times 576}}^{\frac{5\pi}{6\times 576}} \\ &\quad + \Big[\tfrac{1}{1152}\sin(1152x) + \tfrac{1}{576}\cos(576x)\Big]_{\frac{5\pi}{6\times 576}}^{\frac{2\pi}{576}} \\ &= \tfrac{1}{1152}(\tfrac{\sqrt{3}}{2} - 0) + \tfrac{1}{576}(\tfrac{\sqrt{3}}{2} - 1) \\ &\quad - \tfrac{1}{1152}(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}) - \tfrac{1}{576}(-\tfrac{\sqrt{3}}{2} - \tfrac{\sqrt{3}}{2}) \\ &\quad + \tfrac{1}{1152}(0 + \tfrac{\sqrt{3}}{2}) + \tfrac{1}{576}(1 + \tfrac{\sqrt{3}}{2}) \\ &= \tfrac{\sqrt{3}}{192}. \end{aligned}$$

As $10\pi = 2880\times\frac{2\pi}{576}$, we have $\int_0^{10\pi}|f(x)|\,dx = 2880\int_0^{\frac{2\pi}{576}}|f(x)|\,dx = \frac{2880\sqrt{3}}{192} = 15\sqrt{3} = 25.98$. ■

4. Let
$$f(x) = \frac{x+1}{x^2+2x+7}.$$
Find the value of $f^{(9)}(-1)$, (i.e. the 9-th derivative of f at $x=-1$). Give your answer correct to two decimal places.

Answer. 46.67.

Solution. Expanding f into Taylor series about $x=-1$, we have

$$\begin{aligned} f(x) &= \tfrac{x+1}{x^2+2x+7} = \tfrac{x+1}{6+(x+1)^2} = \tfrac{x+1}{6}\tfrac{1}{1+(\frac{x+1}{\sqrt{6}})^2} \\ &= \tfrac{x+1}{6}\left(1-(\tfrac{x+1}{\sqrt{6}})^2+(\tfrac{x+1}{\sqrt{6}})^4-(\tfrac{x+1}{\sqrt{6}})^6+(\tfrac{x+1}{\sqrt{6}})^8-\cdots\right), \text{ for } |x+1|<\sqrt{6}. \end{aligned}$$

Thus $f^{(9)}(-1) = \frac{9!}{6(\sqrt{6})^8} = \frac{9!}{6^5} = \frac{140}{3} = 46.67$. ■

5. It is known that the power series
$$\sum_{n=0}^{\infty} c_n x^n$$
has a positive radius of convergence larger than $\frac{1}{20}$. It is also known that $c_0 = 1, c_1 = 1$ and the equation
$$c_{n+2} = 18c_{n+1} - 28c_n,$$
holds for all non-negative integers $n = 0,1,2,3,\ldots$. If
$$\sum_{n=1}^{\infty}\frac{c_n}{(68)^n} = \frac{a}{b},$$
where a and b are two positive integers with no common factors, find the exact value of $a+b$.

Answer. 867.

Solution. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, $x\in(-\frac{1}{20},\frac{1}{20})$. We wish to find $f(\frac{1}{68}) - c_0$.

For $n\ge 0$, we have $c_{n+2} = 18c_{n+1} - 28c_n$.

Thus $\sum_{n=0}^{\infty} c_{n+2}x^{n+2} = \sum_{n=0}^{\infty} 18c_{n+1}x^{n+2} - \sum_{n=0}^{\infty} 28c_n x^{n+2}$.

That is $-c_0 - c_1 x + \sum_{n=0}^{\infty} c_n x^n = -18c_0 x + 18x\sum_{n=0}^{\infty} c_n x^n - 28x^2\sum_{n=0}^{\infty} c_n x^n$.

Thus $(28x^2 - 18x + 1)f(x) = (-18c_0 + c_1)x + c_0$. As $c_0 = c_1 = 1$, we have $f(x) = \frac{-17x+1}{28x^2-18x+1}$. Hence, $f(x) - c_0 = \frac{-17x+1}{28x^2-18x+1} - 1$. Therefore $f(\frac{1}{68}) - c_0 = \frac{10}{857}$. Thus $a + b = 867$.

6. Let A, B and C denote the three points $(101, 0, 0), (0, 202, 0)$ and $(0, 0, 303)$ respectively. Let S denote the plane that passes through the three points A, B and C. Let M denote the mid-point of the line segment AB and let N denote the mid-point of the line segment BC. A line L_1 is drawn on the plane S such that L_1 passes through M and L_1 is perpendicular to AB. Another line L_2 is drawn on the plane S such that L_2 passes through N and L_2 is perpendicular to BC. If (a, b, c) denote the point of intersection of L_1 and L_2, find the value of $a + b + c$. Give your answer correct to two decimal places.

   Answer. 234.98

   Solution. First we have $M = (\frac{101}{2}, 101, 0), N = (0, 101, \frac{303}{2})$. The equation of the plane S is $\frac{x}{101} + \frac{y}{202} + \frac{z}{303} = 1$. Thus a normal vector to S is $\mathbf{n} = \langle \frac{1}{101}, \frac{1}{202}, \frac{1}{303} \rangle$. The vector along BA is $\mathbf{a} = \langle 101, -202, 0 \rangle$. The vector along BC is $\mathbf{b} = \langle 0, -202, 303 \rangle$.

   Therefore, a vector along L_1 is $\mathbf{a} \times \mathbf{n} = \langle 101, -202, 0 \rangle \times \langle \frac{1}{101}, \frac{1}{202}, \frac{1}{303} \rangle = \langle -\frac{2}{3}, -\frac{1}{3}, \frac{5}{2} \rangle$.

   A vector along L_2 is $\mathbf{b} \times \mathbf{n} = \langle 0, -202, 303 \rangle \times \langle \frac{1}{101}, \frac{1}{202}, \frac{1}{303} \rangle = \langle -\frac{13}{6}, 3, 2 \rangle$.

   A parametric equation for L_1 is $x = \frac{101}{2} - \frac{2}{3}t, y = 101 - \frac{1}{3}t, z = \frac{5}{2}t$.

   A parametric equation for L_2 is $x = -\frac{13}{6}s, y = 101 + 3s, z = \frac{303}{2} + 2s$.

   Equating the two equations, we have

   $\frac{101}{2} - \frac{2}{3}t = -\frac{13}{6}s,$

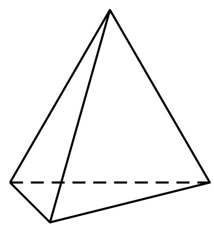
   $101 - \frac{1}{3}t = 101 + 3s,$

   $\frac{5}{2}t = \frac{303}{2} + 2s.$

   Then $s = -\frac{303}{49}$, $t = \frac{2727}{49}$.

   The intersection point of L_1 and L_2 is $(\frac{13}{6} \times \frac{303}{49}, 101 - 3 \times \frac{303}{49}, \frac{303}{2} - 2 \times \frac{303}{49}) = (\frac{1313}{98}, \frac{4040}{49}, \frac{13635}{98})$. Sum of the three coordinates $= \frac{11514}{49} = 234.98$. ■

7. Let n denote a positive constant. Let S denote a tetrahedron (i.e. a triangular pyramid: for your reference a picture of an example of such a solid is shown below) with its four vertices at the points $(-6, -6, -(\frac{2020}{1521})^n), (9, 6, 3), (-6, 0, -9)$, and $(6, 0, -6)$. If the volume of S is equal to 297, find the value of n. Give your answer correct to two decimal places.



Answer. 13.17.

Solution. Let $A = (-6, -6, -(\frac{2020}{1521})^n)$, $B = (9, 6, 3)$, $C = (-6, 0, -9)$, $O = (6, 0, -6)$.

Let $\mathbf{a} = \boldsymbol{OA} = \langle -12, -6, -(\frac{2020}{1521})^n + 6\rangle$, $\mathbf{b} = \boldsymbol{OB} = \langle 3, 6, 9\rangle$, $\mathbf{c} = \boldsymbol{OC} = \langle -12, 0, -3\rangle$.

The area of $\triangle OBC = \frac{1}{2}\|\mathbf{b}\times\mathbf{c}\|$.

The height from A to the base $\triangle OBC = \left|\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|}\right|$.

The volume of the tetrahedron $ABCD = \frac{1}{3}\times$ height $\times$ area $\triangle OBC$.

Therefore the volume of the tetrahedron $ABCD = \frac{1}{3}\left|\mathbf{a}\cdot\frac{\mathbf{b}\times\mathbf{c}}{|\mathbf{b}\times\mathbf{c}|}\right|\frac{1}{2}\|\mathbf{b}\times\mathbf{c}\| = \frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

We have $\mathbf{b}\times\mathbf{c} = \langle 3, 6, 9\rangle\times\langle -12, 0, -3\rangle = \langle -18, -99, 72\rangle$.

Thus $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{1}{6}\left|\langle -12, -6, -(\frac{2020}{1521})^n + 6\rangle\cdot\langle -18, -99, 72\rangle\right|$

$= \frac{1}{6}|216 + 594 - 72(\frac{2020}{1521})^n + 432| = \frac{1}{6}|1242 - 72(\frac{2020}{1521})^n|$

$= |207 - 12(\frac{2020}{1521})^n|$.

Therefore, $|207 - 12(\frac{2020}{1521})^n| = 297 \Leftrightarrow 207 - 12(\frac{2020}{1521})^n = \pm 297$. Thus $(\frac{2020}{1521})^n = 42 \Leftrightarrow n = \frac{\ln 42}{\ln 2020 - \ln 1521} = 13.17$. That is $A = (-6, -6, -42)$. ∎

8. Let S denote a plane. It is known that S passes through the point $(10, 15, 5)$ and that the vector joining $(10, 15, 5)$ to $(18, 8, 11)$ is perpendicular to S. Let $f(x, y, z)$ denote a differentiable function of three variables defined in the following way: at the point (x, y, z) we draw a line L passing through this point and perpendicular to the plane S, if (a, b, c) denotes the point of intersection of L and S, then we define $f(x, y, z) = a + b + c$. Find the directional derivative of f at the point $(1001, 1521, 2020)$ in the direction of the vector joining $(1001, 1521, 2020)$ to $(1000, 1522, 2021)$. Give your answer correct to two decimal places.

   Answer. 0.82.

   Solution. A normal vector to the plane S is $\langle 18, 8, 11\rangle - \langle 10, 15, 5\rangle = \langle 8, -7, 6\rangle$. Thus an equation of S is $\langle x - 10, y - 15, z - 5\rangle\cdot\langle 8, -7, 6\rangle = 0$. That is $8x - 7y + 6z - 5 = 0$.

   Let (x_0, y_0, z_0) be a point in $\mathbb{R}^3$. A parametric equation of the line ℓ through (x_0, y_0, z_0) and perpendicular to S is given by $x = x_0 + 8t, y = y_0 - 7t, z = z_0 + 6t$.

   To find its intersection with S, we substitute this parametric equation into the equation of S. Thus $8(x_0 + 8t) - 7(y_0 - 7t) + 6(z_0 + 6t) - 5 = 0$ so that $t = -\frac{1}{149}(8x_0 - 7y_0 + 6z_0 - 5)$.

   Thus the intersection point between ℓ and S is

   $(x_0 - \frac{8}{149}(8x_0 - 7y_0 + 6z_0 - 5), y_0 + \frac{7}{149}(8x_0 - 7y_0 + 6z_0 - 5), z_0 - \frac{6}{149}(8x_0 - 7y_0 + 6z_0 - 5))$.

   The sum of the three coordinates is

$x_0 - \frac{8}{149}(8x_0 - 7y_0 + 6z_0 - 5) + y_0 + \frac{7}{149}(8x_0 - 7y_0 + 6z_0 - 5) + z_0 - \frac{6}{149}(8x_0 - 7y_0 + 6z_0 - 5)$
$= \frac{1}{149}(93x_0 + 198y_0 + 107z_0 + 35)$.

Therefore, $f(x,y,z) = \frac{1}{149}(93x + 198y + 107z + 35)$. Then $\nabla f = \frac{1}{149}\langle 93, 198, 107\rangle$.

The unit vector in the direction of the vector joining the point $(1001, 1521, 2020)$ to the point $(1000, 1522, 2021)$ is $\mathbf{u} = \frac{1}{\sqrt{3}}\langle -1, 1, 1\rangle$.

At $(1001, 1521, 2020)$, $D_{\mathbf{u}} f = \frac{1}{149}\langle 93, 198, 107\rangle \cdot \frac{1}{\sqrt{3}}\langle -1, 1, 1\rangle = \frac{212}{149\sqrt{3}} = 0.82$. ■

9. Let $f(x,y,z)$ denote a differentiable function of three variables. It is known that $f(1,2,3) = 11$, $f(1.1, 1.7, 3.1) = 16$, $f(1.2, 2.2, 3.3) = 18$ and $f(0.9, 2.1, 2.9) = 20$. Using directional derivatives, estimate the maximum rate of change of f at the point $(1,2,3)$. Give your answer correct to two decimal places.

   Answer. 872.87.

   Solution. Let's derive the following. Suppose the point P in $\mathbb{R}^3$ changes to the point P' so that $\triangle\mathbf{v} = \boldsymbol{P}' - \boldsymbol{P}$. (Here $\boldsymbol{P}$ denotes the position vector of the point P.) Thus $|\boldsymbol{PP}'| = \|\triangle\mathbf{v}\|$ and the unit vector along $\boldsymbol{PP}'$ is $\frac{\triangle\mathbf{v}}{|\triangle\mathbf{v}|}$. Let $\triangle f = f(P') - f(P)$. Recall that $D_{\frac{\triangle\mathbf{v}}{|\triangle\mathbf{v}|}} f(P) = \nabla f(P) \cdot \frac{\triangle\mathbf{v}}{|\triangle\mathbf{v}|}$. Therefore, $\triangle f \approx D_{\frac{\triangle\mathbf{v}}{|\triangle\mathbf{v}|}} f(P)\|\triangle\mathbf{v}\| = \nabla f(P) \cdot \triangle\mathbf{v}$. Summarizing, at the point P, we have

$$\boxed{\nabla f \cdot \triangle\mathbf{v} \approx \triangle f}$$

   Let $\nabla f(1,2,3) = \langle a, b, c\rangle$.

   Let $\triangle\mathbf{v}_1 = \langle 1.1, 1.7, 3.1\rangle - \langle 1,2,3\rangle = \langle 0.1, -0.3, 0.1\rangle$.

   Let $\triangle\mathbf{v}_2 = \langle 1.2, 2.2, 3.3\rangle - \langle 1,2,3\rangle = \langle 0.2, 0.2, 0.3\rangle$.

   Let $\triangle\mathbf{v}_3 = \langle 0.9, 2.1, 2.9\rangle - \langle 1,2,3\rangle = \langle -0.1, 0.1, -0.1\rangle$.

   By the above equation, we have $0.1a - 0.3b + 0.1c \approx 16 - 11 = 5$, $0.2a + 0.2b + 0.3c = 18 - 11 = 7$, $-0.1a + 0.1b - 0.1c \approx 20 - 11 = 9$.

   Solving the system of equations:

$$\begin{cases} 0.1a - 0.3b + 0.1c & = & 5 \\ 0.2a + 0.2b + 0.3c & = & 7 \\ -0.1a + 0.1b - 0.1c & = & 9 \end{cases}$$

   We obtain the approximate values $a = -690, b = -70, c = 530$. Thus $\|\nabla f(P)\| \approx \sqrt{(-690)^2 + (-70)^2 + 530^2} = 10\sqrt{7619} = 872.87$.

   Remark. An example of such a function is the linear function $f(x,y,z) = -690x - 70y + 530z - 749$. ■

10. Let a denote a positive constant. Let S denote a pentagon on the plane $18x+38y-az+1521=0$. It is known that the projection of S on the xy-plane is another pentagon with vertices at $(0,0,0),(80,0,0),(100,60,0),(50,80,0)$, and $(0,30,0)$. If the area of S is equal to 8888, find the value of a. Give your answer correct to two decimal places.

    Answer. 34.63.

    Solution. The area of the projection of S onto the xy-plane can be calculated to be 5650. A unit normal vector to S is $\mathbf{n} = \frac{1}{\sqrt{18^2+38^2+(-a)^2}}\langle 18,38,-a\rangle = \frac{1}{\sqrt{1768+a^2}}\langle 18,38,-a\rangle$.

    Thus the angle θ between $\mathbf{n}$ and $\langle 0,0,1\rangle$ is given by $\cos\theta = \frac{1}{\sqrt{1768+a^2}}\langle 18,38,-a\rangle\cdot\langle 0,0,1\rangle$. That is $\cos\theta = \frac{-a}{\sqrt{1768+a^2}}$. Therefore, $\left|\frac{-a}{\sqrt{1768+a^2}}\right| = \frac{5650}{8888}$. Since $a>0$, we have $\frac{a}{\sqrt{1768+a^2}} = \frac{5650}{8888}$. We obtain $a = 5650\sqrt{442/11768511} = 34.63$. ∎

11. Let a and b denote two positive constants such that $a>b$ and $a+b=88$. Let R denote the finite region in the first quadrant of the xy-plane bounded by the x-axis, the y-axis, the circle $x^2+y^2=a^2$, and the line $x=b$. It is known that the surface area of the portion of the cylinder $x^2+z^2=a^2$ above R is equal to 1521. Find the exact value of a^2+b^2.

    Answer. 4702.

    Solution. The portion of the cylinder above the xy-plane has the equation $z=\sqrt{a^2-x^2}$ over the region R. We have $z_x = \frac{-x}{\sqrt{a^2-x^2}}, z_y=0$. The surface area of the portion of the cylinder over R is

    $$\iint_R \sqrt{1+z_x^2+z_y^2}\,dxdy = \iint_R \frac{a}{\sqrt{a^2-x^2}}\,dxdy = \int_0^b\int_0^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2}}\,dydx$$
    $$= \int_0^b \frac{a\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\,dx = \int_0^b a\,dx = ab.$$

    Therefore $ab=1521$ and $a+b=88$. Thus $a^2+b^2=(a+b)^2-2ab=88^2-2\times 1521=4702$. ∎

12. Let a denote a positive constant. Let R denote the finite triangular plane region on the xy-plane with vertices at $(0,0),(1010,a)$ and $(2020,3a)$. Let D denote the solid region under the hyperbolic paraboloid $z=xy$ and over the plane region R. If the volume of D is equal to 88, Find the value of a. Give your answer correct to two decimal places.

    Answer. 4.56.

    Solution. Let $O=(0,0), A=(1010,a)$ and $B=(2020,3a)$. The equation of the line OA is $y=\frac{ax}{1010}$, the equation of the line OB is $y=\frac{3ax}{2020}$ and the equation of the line AB is $\frac{y-a}{x-1010} = \frac{3a-a}{2020-1010} \Leftrightarrow y = \frac{ax}{505}-a$. Thus the volume of D is

$$\iint_R xy\,dA = \int_0^{1010}\int_{\frac{ax}{1010}}^{\frac{3ax}{2020}} xy\,dydx + \int_{1010}^{2020}\int_{\frac{ax}{505}-a}^{\frac{3ax}{2020}} xy\,dydx$$

$$= \int_0^{1010}\left[\frac{xy^2}{2}\right]_{\frac{ax}{1010}}^{\frac{3ax}{2020}} dx + \int_{1010}^{2020}\left[\frac{xy^2}{2}\right]_{\frac{ax}{505}-a}^{\frac{3ax}{2020}}$$

$$= \int_0^{1010}\frac{a^2x^3}{2}[(\tfrac{3}{2020})^2-(\tfrac{1}{1010})^2]\,dx + \int_{1010}^{2020}\frac{a^2x}{2}[(\tfrac{3x}{2020})^2-(\tfrac{x}{505}-1)^2]\,dx$$

$$= \int_0^{1010}\frac{5a^2x^3}{2\times 2020^2}\,dx + \int_{1010}^{2020}\frac{a^2x}{2}[\tfrac{9x^2}{2020^2}-\tfrac{x^2}{505^2}+\tfrac{2x}{505}-1]\,dx$$

$$= \int_0^{1010}\frac{5a^2x^3}{2\times 2020^2}\,dx + \int_{1010}^{2020}\frac{a^2x}{2}[-\tfrac{7x^2}{2020^2}+\tfrac{2x}{505}-1]\,dx$$

$$= \frac{5a^2}{2\times 2020^2}\int_0^{1010} x^3\,dx$$

$$-\frac{7a^2}{2\times 2020^2}\int_{1010}^{2020} x^3\,dx + \frac{a^2}{505}\int_{1010}^{2020} x^2\,dx - \frac{a^2}{2}\int_{1010}^{2020} x\,dx$$

$$= \frac{5a^2\times 1010^4}{8\times 2020^2} - \frac{7a^2(2020^4-1010^4)}{8\times 2020^2} + \frac{a^2(2020^3-1010^3)}{3\times 505} - \frac{a^2(2020^2-1010^2)}{4}$$

$$= \frac{1275125a^2}{8} + \frac{15556525a^2}{24} = \frac{4845475a^2}{6}.$$

Thus $\frac{4845475a^2}{6} = 8^8$. From this we get $a = \frac{4096\sqrt{6}}{505\sqrt{19}} = 4.56$. ■

13. Let a and n denote two positive constants with $n < \frac{3}{2}$. A perfectly spherical rain drop with volume a at time $t = 0$ second falls through very dry air and it evaporates in such a way that it always keeps its perfectly spherical shape and that the rate of reduction of its volume is directly proportional to the n-th power of its surface area. It is observed that the volume of the rain drop is equal to $\frac{1}{25}a$ at time $t = 30$ seconds and that the raindrop completely disappears at time $t = 80$ seconds. Find the value of n. Give your answer correct to two decimal places.

Answer. 1.28.

Solution. The volume V and the surface area S of a sphere of radius r are given by $V = \frac{4\pi}{3}r^3$ and $S = 4\pi r^2$ respectively. Thus $S = 4\pi(\frac{3V}{4\pi})^{\frac{2}{3}}$. We are given $\frac{dV}{dt} = -kS^n$, where k is a positive constant.

Thus $\frac{dV}{dt} = -k(4\pi(\frac{3V}{4\pi})^{\frac{2}{3}})^n = -k(4\pi)^{\frac{n}{3}}(3V)^{\frac{2n}{3}} = -k(36\pi)^{\frac{n}{3}}V^{\frac{2n}{3}}$.

That is $\int V^{-\frac{2n}{3}}\,dV = \int -k(36\pi)^{\frac{n}{3}}\,dt$. Here $n < \frac{3}{2}$ is needed to ensure $V^{-\frac{2n}{3}} \neq V^{-1}$.

Therefore, $\frac{V^{1-\frac{2n}{3}}}{1-\frac{2n}{3}} = -k(36\pi)^{\frac{n}{3}}t + C$. $V(0) = a \Rightarrow C = \frac{a^{1-\frac{2n}{3}}}{1-\frac{2n}{3}} \Rightarrow \frac{V^{1-\frac{2n}{3}}}{1-\frac{2n}{3}} = -k(36\pi)^{\frac{n}{3}}t + \frac{a^{1-\frac{2n}{3}}}{1-\frac{2n}{3}}$.

Rearranging, $a^{1-\frac{2n}{3}} - V^{1-\frac{2n}{3}} = k(1-\frac{2n}{3})(36\pi)^{\frac{n}{3}}t$. (13.1)

$V(30) = \frac{a}{25} \Rightarrow a^{1-\frac{2n}{3}} - (\frac{a}{25})^{1-\frac{2n}{3}} = k(1-\frac{2n}{3})(36\pi)^{\frac{n}{3}}30$. (13.2)

$V(80) = 0 \Rightarrow a^{1-\frac{2n}{3}} = k(1-\frac{2n}{3})(36\pi)^{\frac{n}{3}}80$. (13.3)

Dividing (13.2) by (13.3), we obtain $1 - \frac{1}{25^{1-\frac{2n}{3}}} = \frac{3}{8} \Leftrightarrow \frac{1}{25^{1-\frac{2n}{3}}} = \frac{5}{8} \Leftrightarrow 25^{\frac{2n}{3}} = \frac{125}{8} \Leftrightarrow n = \frac{3}{2}\left(\frac{\ln 125 - \ln 8}{\ln 25}\right) = 1.28$.

Remark. By (13.3), $k(1-\frac{2n}{3})(36\pi)^{\frac{n}{3}} = \frac{a^{1-\frac{2n}{3}}}{80}$. Thus we have $a^{1-\frac{2n}{3}} - V^{1-\frac{2n}{3}} = \frac{a^{1-\frac{2n}{3}}}{80}t \Leftrightarrow V^{1-\frac{2n}{3}} = a^{1-\frac{2n}{3}}(1-\frac{t}{80}) \Leftrightarrow V = a(1-\frac{t}{80})^{\frac{3}{3-2n}}$. For $n = \frac{3}{2}\left(\frac{\ln 125 - \ln 8}{\ln 25}\right) = 1.28$, we obtain $V = a(1-\frac{t}{80})^{6.84862}$. ■

14. Let a denote a positive constant. Let $y(x)$ denote the solution to the differential equation

$$\frac{dy}{dx} = \frac{x^2 + axy + y^2}{xy}$$

with $x > 0, y > 0, y(1) = \frac{1}{a}$ and $y(2) = \frac{2020}{a}$. Find the value of a. Give your answer correct to two decimal places.

Answer. 38.04.

Solution. Let $y = vx$. Then $y' = v'x + v$. Thus the given DE can be expressed as $v'x + v = \frac{x^2+ax^2v+x^2v^2}{x^2v} = \frac{1+av+v^2}{v} = v^{-1} + a + v$. That is $v'x = v^{-1} + a \Leftrightarrow \frac{v'}{v^{-1}+a} = \frac{1}{x} \Leftrightarrow \frac{vv'}{1+av} = \frac{1}{x} \Leftrightarrow \frac{vdv}{1+av} = \frac{dx}{x}$. Integrating, $\int \frac{vdv}{1+av} = \int \frac{dx}{x} \Leftrightarrow \int \frac{1}{a}\left(1 - \frac{1}{1+av}\right)dv = \int \frac{dx}{x} \Leftrightarrow \frac{1}{a}(v - \frac{1}{a}\ln|1+av|) + C = \ln|x| \Leftrightarrow \frac{v}{a} + C = \ln|x| + \frac{1}{a^2}\ln|1+av| \Leftrightarrow \frac{v}{a} + C = \ln|x||1+av|^{\frac{1}{a^2}}$.

Since $a > 0, x > 0, y > 0$, we have $v = y/x > 0$ so that $\frac{v}{a} + C = \ln(x(1+av)^{\frac{1}{a^2}}) \Leftrightarrow e^{\frac{v}{a}+C} = x(1+av)^{\frac{1}{a^2}} \Leftrightarrow e^{av+a^2C} = x^{a^2}(1+av) \Leftrightarrow Ae^{av} = x^{a^2}(1+av)$, where $A = e^{a^2C}$ is a constant. Therefore, the general solution is

$$Ae^{\frac{ay}{x}} = x^{a^2}(1 + \frac{ay}{x}).$$

As $y(1) = \frac{1}{a}$, we have $Ae = 2$ so that $A = \frac{2}{e}$. Thus the solution is

$$2e^{\frac{ay}{x}-1} = x^{a^2}(1 + \frac{ay}{x}).$$

As $y(2) = \frac{2020}{a}$, we have $2e^{1010-1} = 2^{a^2}(1+1010)$. That is $2e^{1009} = 2^{a^2}(1011) \Leftrightarrow 2^{a^2} = \frac{2e^{1009}}{1011} \Leftrightarrow a^2 = \frac{\ln(\frac{2e^{1009}}{1011})}{\ln 2} = \frac{1009+\ln 2-\ln 1011}{\ln 2}$. Therefore, $a = \sqrt{\frac{1009+\ln 2-\ln 1011}{\ln 2}} = 38.04$. ■