Lecture 5: Set Theory

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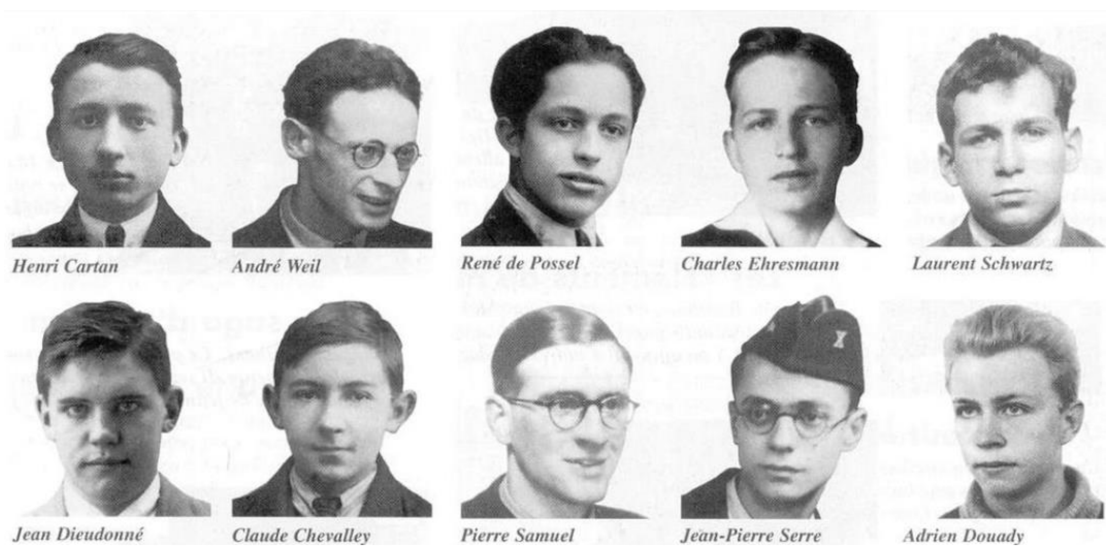
"For whereas in the past it was thought that every branch of mathematics depended on its own particular intuition which provided its concepts and prime truths, nowadays it is known to be possible, logically speaking, to derive practically the whole of known mathematics from a single source – the Theory of Sets."

~Nicolas Bourbaki

Elements of Mathematics: Theory of Sets



Why is Set Theory important for Computer Science?

- It is a useful tool for formalizing and reasoning about computation and the objects of computation.
- It is indivisible from logic where Computer Science has its roots.
- Applications
  - Data structures
  - Database theory – databases as relations over sets
  - Formal language theory
  - Machine learning
  - etc.

5.1 Definitions

- Set-roster notation; set-builder notation; replacement notation.
- Membership; cardinality.
- Subsets; proper subsets; empty set; singleton.
- Ordered pairs; Cartesian products.
- Set equality; Venn diagrams; operations on sets (union, intersection, difference, complement).
- Partitions of sets; power sets.

5.2 Properties of Sets

- Some subset relations
- Procedural versions of set definitions
- Set identities

Reference: Epp's Chapter 6 Set Theory

5.1 Definitions

5.1.1. Definitions

- A set is a unordered collection of objects.
- The objects are called members or elements of the set.

$S = \{ 3, a, \sqrt{5} \}$

Days = { Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday }

Set-Roster Notation

A set may be specified by writing all of its elements between braces.

Examples: {1, 2, 3}, {1, 2, 3, …, 100}, {1, 2, 3, …}.

(The symbol ⋯ is called an **ellipsis** and is read "and so forth".)

- Order and duplicates do not matter.

$$\{ 9, 8, 7 \} = \{ 7, 9, 8 \} = \{ 7, 8, 7, 9, 9, 7, 7 \}$$

We distinguish between **set** and **multiset**. Eg: {a, b, c} and {a, b, b, c} are two different multisets. Multisets are not in the scope of this lecture.

Definition: Membership of a Set (Notation: $\in$)

If S is a set, the notation $x \in S$ means that x is an element of S.
($x \notin S$ means x is not an element of S.)

- Eg: $b \in \{a, b, c\}$; $456 \in \{1, 2, 3, \ldots, 999\}$; $f \notin \{a, b, c\}$.

Definition: Cardinality of a Set (Notation: $|S|$)

The cardinality of a set S, denoted as $|S|$, is the size of the set, that is, the number of elements in S.

- Eg: $|\{a, b, c\}| = 3$.

⚠ The notation $|\ |$ can also mean absolute value, so the meaning depends on context. Sometimes, cardinality of a set S is denoted by $n(S)$ or $\#S$. We will use the notation $|S|$ in CS1231S.

a. Is $\{a\} \in \{a, \{a, b\}, c\}$?

No. The 3 members of $\{a, \{a, b\}, c\}$ are a, $\{a, b\}$ and c.

b. What is the cardinality of the set $\{1, \{1,2\}, \{3, \{4,5\}, 6\}, \{2,3\}\}$?

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c. For each integer n, let $U_n = \{n, -n\}$.
Find $|U_1|$, $|U_{-5}|$ and $|U_0|$.

$|U_1| = |\{1, -1\}| = 2$;
$|U_{-5}| = |\{-5, 5\}| = 2$;
$|U_0| = |\{0, -0\}| = |\{0\}| = 1$.

Table 5.1. Common Sets

Symbol	Meaning
$\mathbb{N}$	The set of all natural numbers {0, 1, 2, 3,…}*
$\mathbb{Z}$	The set of all integers
$\mathbb{Q}$	The set of all rational numbers
$\mathbb{R}$	The set of all real numbers
$\mathbb{C}$	The set of all complex numbers
$\mathbb{Z}^+$	The set of all positive integers
$\mathbb{Z}^-$	The set of all negative integers
$\mathbb{Z}_{\geq 0}$	The set of all non-negative integer

$\mathbb{Q}^+, \mathbb{Q}^-, \mathbb{Q}_{\geq m}$, $\mathbb{R}^+, \mathbb{R}^-, \mathbb{R}_{\geq m}$, etc. are defined similarly.

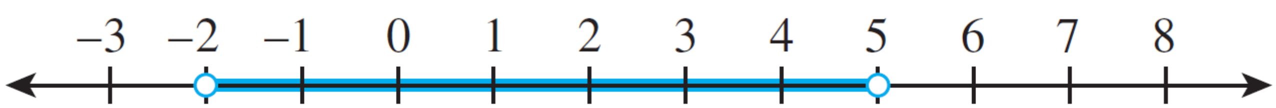
*: In this module we define the set $\mathbb{N}$ to include zero.

Zero is neither positive nor negative.

Describe each of the following sets.

a. $\{x \in \mathbb{R} : -2 < x < 5\}$

The open interval of real numbers (strictly) between -2 and 5, pictured as follows:



b. $\{x \in \mathbb{Z} : -2 < x < 5\}$

The set of all integers (strictly) between -2 and 5, that is, { -1, 0, 1, 2, 3, 4}.

c. $\{x \in \mathbb{Z}^+ : -2 < x < 5\}$

The set { 1, 2, 3, 4}.

Another way to specify a set is to describe its elements. This is called the set-builder notation.

Set-Builder Notation

Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted

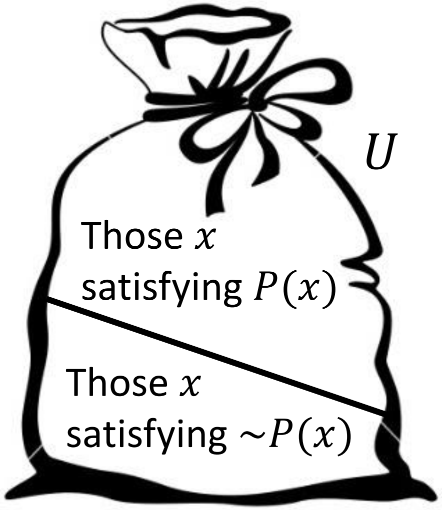
$$\{x \in U : P(x)\} \quad \text{or} \quad \{x \in U \mid P(x)\}$$

which is read as "the set of all x in U such that $P(x)$ (is true)".

- Example: $\{x \in \mathbb{Z}_{\geq 0} : Even(x)\}$, where $Even(x)$ means "x is even", is the set of non-negative even integers.

To check whether an object z is an element of the set $S = \{x \in U : P(x)\}$,

- If $z \in U$ and $P(z)$ is true, then $z \in S$.
- If $z \notin U$, then $z \notin S$.
- If $\sim P(z)$, then $z \notin S$.



Yet another way to specify a set uses the replacement notation.

Replacement Notation

Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted

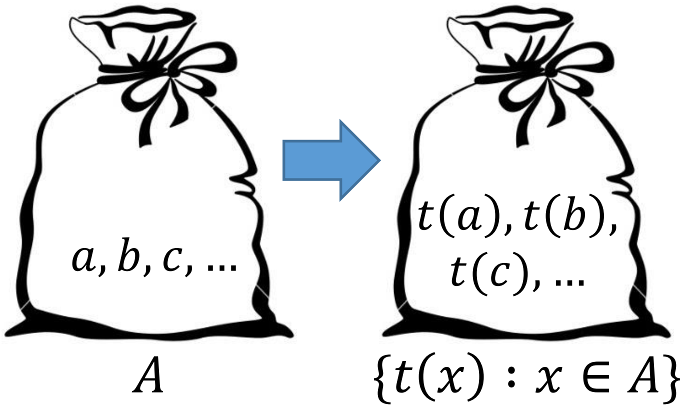
$$\{t(x) : x \in A\} \quad \text{or} \quad \{t(x) \mid x \in A\}$$

which is read as "the set of all $t(x)$ where $x \in A$".

- Eg: The elements of $S = \{x + 1 : \ x \in \mathbb{Z}_{\geq 0}\}$ are precisely those $x + 1$ where $x \in \mathbb{Z}_{\geq 0}$, i.e., the positive integers. So, $1 = 0 + 1 \in S$ but $0 \notin S$.

To check whether an object z is an element of $S = \{t(x) : x \in A\}$:

- If there is an $x \in A$ such that $t(x) = z$, then $z \in S$, else $z \notin S$.



5.1.2. Subsets, Proper Subsets, Empty Set and Singleton

Definition: Subset

Let A and B be sets. A is a **subset** of B, written $A \subseteq B$, iff every element of A is also an element of B.

Symbolically:

$$A \subseteq B \text{ iff } \forall x\ (x \in A \Rightarrow x \in B)$$

Another way of saying "A is a subset of B" is "A is contained in B".

If $A \subseteq B$, we may also write $B \supseteq A$ which reads as "B contains A" or "B includes A".

Avoid the symbol $\subset$ as it means different things to different people .

Definition: Proper Subset

Let A and B be sets. A is a **proper subset** of B, denoted $A \subsetneq B$, iff $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is proper or strict.

- It follows from the definition of subset that for a set *A* not to be a subset of a set *B* means that there is at least one element of *A* that is not an element of *B*.
- Symbolically: $A \not\subseteq B \Leftrightarrow \exists x\ (x \in A \land x \notin B)$.

$\subseteq$	$\subsetneq$	$\not\subseteq$
Subset	Proper subset	Not subset

- A set with no element, {}, is an empty set, denoted as ∅.

Theorem 6.2.4

An empty set is a subset of every set, i.e. $\emptyset \subseteq A$ for all sets A.

- A set with exactly one element is called a singleton.

Distinction between ∈ and ⊆

Which of the following are true?

a. $2 \in \{1, 2, 3\}$ ✓

b. $\{2\} \in \{1, 2, 3\}$ ✗

c. $2 \subseteq \{1, 2, 3\}$ ✗

d. $\{2\} \subseteq \{1, 2, 3\}$ ✓

e. $\{2,3\} \subseteq \{1, 2, 3\}$ ✓

f. $\{2\} \subseteq \{\{1\}, \{2\}, \{3\}\}$ ✗

g. $\{2\} \in \{\{1\}, \{2\}, \{3\}\}$ ✓

5.1.3. Ordered Pairs

Definition: Ordered Pair

An **ordered pair** is an expression of the form (x, y).

Two ordered pairs (a, b) and (c, d) are equal iff $a = c$ and $b = d$.

Symbolically: $(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d)$.

a. Is $(1,2) = (2,1)$? No (although $\{1,2\} = \{2,1\}$)

b. Is $(3,0.5) = (\sqrt{9}, \frac{1}{2})$? (Assuming all values are positive.)

Yes

5.1.4. Cartesian Products

Definition: Cartesian Product

Given sets A and B, the **Cartesian product** of A and B, denoted $\boldsymbol{A \times B}$ and read "A cross B", is the set of all ordered pairs (a, b) where a is in A and b is in B.

Symbolically: $A \times B = \{(a, b) : a \in A \wedge b \in B\}$.

Let $A = \{1,2,3\}$ and $B = \{u, v\}$.

a. Find $A \times B$ $\quad A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. Find $B \times A$ $\quad B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. Find $B \times B$ $\quad B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. How many elements are there in $A \times B$, $B \times A$, and $B \times B$?

6, 6, and 4

Let $\mathbb{R}$ denote the set of all real numbers. Describe $\mathbb{R} \times \mathbb{R}$.

$\mathbb{R} \times \mathbb{R}$ is the set of all ordered pairs (x, y) where $x, y \in \mathbb{R}$.

Each ordered pair (x, y) can be said to correspond to a unique point in the plane where x and y indicate the horizontal and vertical positions of the point.

The term Cartesian plane is often used to refer to a plane with this coordinate system.

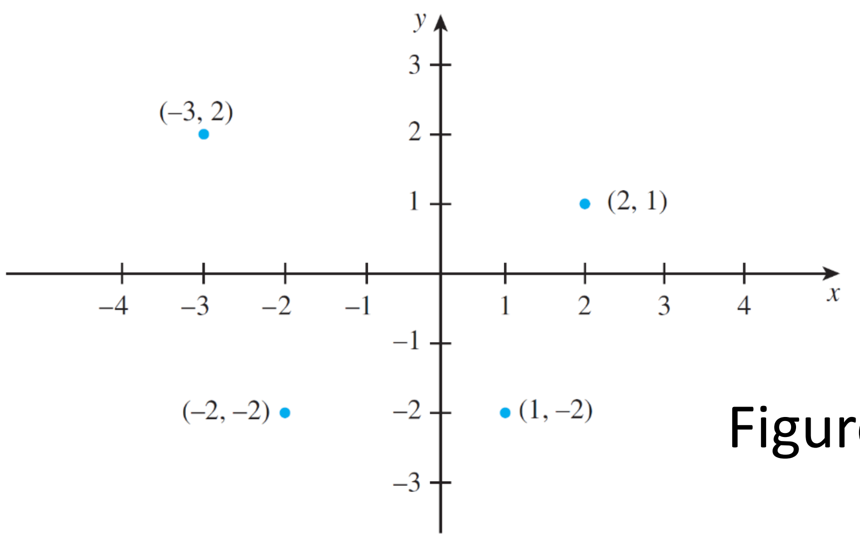


Figure 1.2.1 A Cartesian Plane

5.1.5. Definition: Set Equality

Sets *A* and *B* are equal if, and only if, they have exactly the same elements.

Definition: Set equality

Given sets A and B, A equals B, written $\boldsymbol{A = B}$ iff every element of A is in B and every element of B is in A.

Symbolically: $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$.

Basic method for proving that two sets are equal:

1. Let sets X and Y be given. To prove $X = Y$:
2. ($\subseteq$) Prove that $X \subseteq Y$.
3. ($\supseteq$) Prove that $Y \subseteq X$ (or $X \supseteq Y$).
4. From (2) and (3), conclude that $X = Y$.

a. Let $A = \{1, 2, 3\}, B = \{3, 1, 2\}$ and $C = \{1, 1, 2, 3, 3, 3\}$. Are they the same set?

Yes, they are the same set, each containing three elements: 1, 2 and 3.

b. Is $\{9\} = 9$?

No, $\{9\} \neq 9$ because $\{9\}$ is a set but 9 is not.

c. Is $\{9\} = \{\{9\}\}$?

No, $\{9\} \neq \{\{9\}\}$ because $\{9\}$ is a set with the element 9 whereas $\{\{9\}\}$ is a a set with the element $\{9\}$.

d. Is $\{9\} = \{9, \emptyset\}$?

No, $\{9\} \neq \{9, \emptyset\}$ because $\{9\}$ is a set with one element whereas $\{9, \emptyset\}$ is a set with two elements.
Alternatively, since $\{9, \emptyset\} \nsubseteq \{9\}$, hence $\{9\} \neq \{9, \emptyset\}$.

$$A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$$

From the definition of subset:

$$A \subseteq B \text{ iff } \forall x\, (x \in A \Rightarrow x \in B)$$

We have this alternative definition for set equality.

Definition: Set equality

Given sets A and B, A equals B, written $\boldsymbol{A = B}$ iff every element of A is in B and every element of B is in A.

Symbolically: $A = B \Leftrightarrow \forall x\, (x \in A \Leftrightarrow x \in B)$.

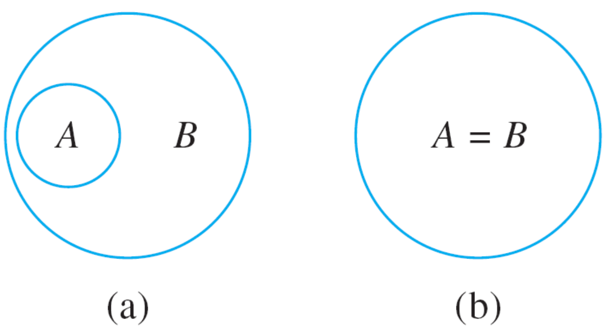
Example: Prove that $\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$.

Proof

1. $(\Rightarrow)$
   1.1. Take any $z \in \{x \in \mathbb{Z} : x^2 = 1\}$.
   1.2. Then $z \in \mathbb{Z}$ and $z^2 = 1$.
   1.3. So, $z^2 - 1 = (z-1)(z+1) = 0$. (by basic algebra)
   1.4. $\therefore z - 1 = 0$ or $z + 1 = 0$.
   1.5. $\therefore z = 1$ or $z = -1$.
   1.6. So, $z \in \{1, -1\}$.
2. $(\Leftarrow)$
   2.1. Take any $z \in \{1, -1\}$.
   2.2. Then $z = 1$ or $z = -1$.
   2.3. In either case, we have $z \in \mathbb{Z}$ and $z^2 = 1$.
   2.4. So, $z \in \{x \in \mathbb{Z} : x^2 = 1\}$.
3. Therefore, $\{x \in \mathbb{Z} : x^2 = 1\} = \{1, -1\}$. (from (1) and (2))

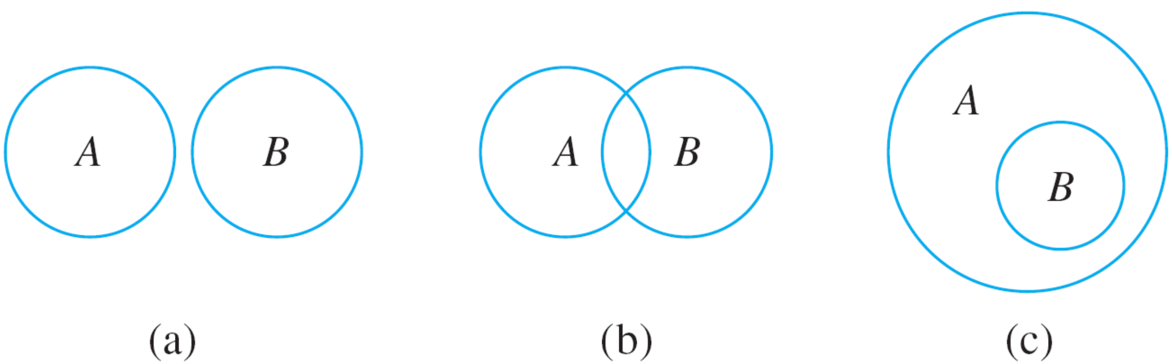
5.1.6. Venn Diagrams

If sets *A* and *B* are represented as regions in the plane, relationships between *A* and *B* can be represented by pictures, called **Venn diagrams,** introduced by the British mathematician John Venn in 1881.



$A \subseteq B$

Figure 6.1.1



$A \nsubseteq B$

Figure 6.1.2

5.1.7. Operations on Sets

Universal set:

Most mathematical discussions are carried on within some context. For example, in a certain situation all sets being considered might be sets of real numbers.

In such a situation, the set of real numbers would be called a **universal set** or a **universe of discourse** for the discussion.

Union, intersection, difference and complement.

Definitions

Let A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A} \cup \boldsymbol{B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A} \cap \boldsymbol{B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B} - \boldsymbol{A}$, or $\boldsymbol{B} \setminus \boldsymbol{A}$, is the set of all elements that are in B and not A.
4. The complement of A, denoted $\overline{\boldsymbol{A}}$, is the set of all elements in U that are not in A. (Note: Epp uses the notation A^c.)

Symbolically:

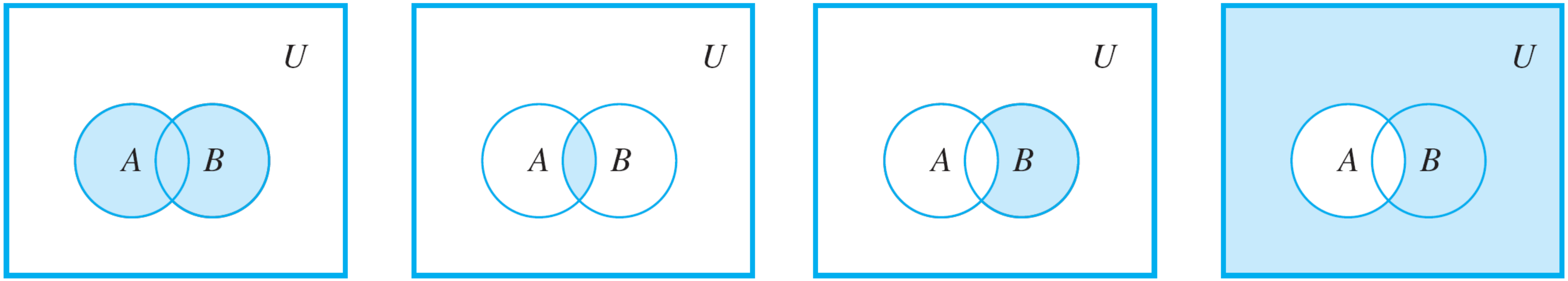
$$A \cup B = \{x \in U : x \in A \vee x \in B\},$$

$$A \cap B = \{x \in U : x \in A \wedge x \in B\},$$

$$B \setminus A = \{x \in U : x \in B \wedge x \notin A\},$$

$$\bar{A} = \{\mathrm{x} \in U \mid x \notin A\}.$$

Union, intersection, difference and complement.



Shaded region represents $A \cup B$.

Shaded region represents $A \cap B$.

Shaded region represents $B - A$.

Shaded region represents A^c.

Figure 6.1.4

Let the universal set be $U = \{a, b, c, d, e, f, g\}$ and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find

a. $A \cup B$ $\{a, c, d, e, f, g\}$

b. $A \cap B$ $\{e, g\}$

c. $B \setminus A$ $\{d, f\}$

d. $\bar{A}$ $\{b, d, f\}$

Note: In a context where U is the universal set (so that implicitly means $U \supseteq X$), the complement of X, denoted $\bar{X}$ or X^c, is defined by $\bar{X} = U \setminus X$.

Intervals of real numbers:

There is a convenient notation for subsets of real numbers that are intervals.

Notation

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$(a, \infty) = \{x \in \mathbb{R} : x > a\}, \qquad [a, \infty) = \{x \in \mathbb{R} : x \geq a\},$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\}, \qquad (-\infty, b] = \{x \in \mathbb{R} : x \leq b\}.$$

Note: Observe that the notation for the interval (a, b) is identical to the notation for the ordered pair (a, b). However, context makes it unlikely that the two will be confused.

The definitions of unions and intersections for more than two sets are very similar to the definitions for two sets.

- $\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n$
- $\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \cdots \cap A_n$

• Definition

Unions and Intersections of an Indexed Collection of Sets

Given sets $A_0,\ A_1,\ A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

5.1.8. Partitions of Sets

In many applications of set theory, sets are divided up into nonoverlapping (or *disjoint*) pieces. Such a division is called a *partition*.

Definition

Two sets are **disjoint** iff they have no elements in common.

Symbolically: A and B are disjoint iff $A \cap B = \emptyset$.

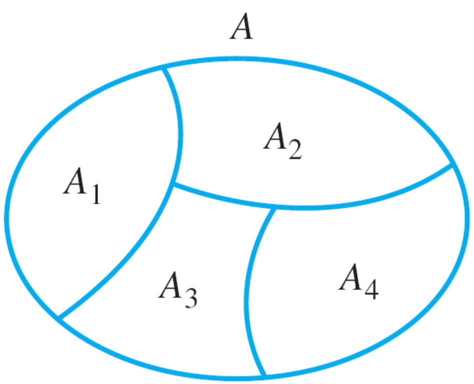
Definition

Sets $A_1, A_2, A_3, \cdots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) iff no two sets A_i and A_j with distinct subscripts have any elements in common, i.e. for all $i, j = 1,2,3, \cdots$

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

Suppose A, A_1, A_2, A_3, and A_4 are the sets of points represented by the regions shown in Figure 6.1.5.

Then A_1, A_2, A_3, and A_4 are subsets of A, and $A = A_1 \cup A_2 \cup A_3 \cup A_4$.



A Partition of a Set

Figure 6.1.5

Suppose further that boundaries are assigned to the regions representing A_1, A_2, A_3, and A_4 in such a way that these sets are mutually disjoint.

Then A is called a *union of mutually disjoint subsets*, and the collection of sets $\{A_1, A_2, A_3, A_4\}$ is said to be a *partition* of A.

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \leq r < d.$$

Examples:

- $n = 54, d = 4$
  $54 = 4 \cdot 13 + 2$; hence $q = 13$ and $r = 2$.
- $n = -27, d = 5$
  $-27 = 5 \cdot (-6) + 3$; hence $q = -6$ and $r = 3$.

a. Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

Yes. $A = A_1 \cup A_2 \cup A_3$ and the sets A_1, A_2, and A_3 are mutually disjoint.

b. Let $\mathbb{Z}$ be the set of all integers and let

$$T_0 = \{n \in \mathbb{Z} : n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbb{Z} : n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbb{Z} : n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbb{Z}$?

Yes. By the quotient-remainder theorem, every integer n can be written in exactly one of the three forms: $n = 3k$, or $n = 3k+1$, or $n = 3k + 2$ for some integer k. This implies that T_0, T_1 and T_2 are mutually disjoint and $\mathbb{Z} = T_0 \cup T_1 \cup T_2$.

5.1.9. Power Sets

There are various situations in which it is useful to consider the set of all subsets of a particular set.

The power set axiom guarantees that this is a set.

Definition

Given a set A, the **power set** of A, denoted $\wp(A)$, is the set of all subsets of A.

Let $A = \{x, y\}$. Find the power set of A, i.e. $\wp(A)$.

$\wp(A)$ is the set of all subsets of A. Therefore

$$\wp(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$$

The number of subsets of a set is the number of elements of its power set. Assume that set A is a finite set.

If A has n elements, then its power set, $\wp(A)$, has 2^n elements.

The proof uses mathematical induction and is based on the following observations. Suppose A is a set and $z \in A$.

- The subsets of A can be split into two groups: those that do not contain z and those that contain z.
- The subsets of A that do not contain z are the same as the subset of $A - \{z\}$.
- The subsets of A that do not contain z can be matched up one for one with the subsets of A that do contain z by matching each subset that does not contain z to the subset that contains z.
- Thus, there are as many subsets of A that contains z as there are subsets of A that do not contain z.

For instance, if $A = \{x, y, z\}$, the following table shows the correspondence between subsets of A that do not contain z and subsets of A that contain z.

Subsets of *A* that do not contain *z*		Subsets of *A* that contain *z*
$\emptyset$	$\leftrightarrow$	$\emptyset \cup \{z\} = \{z\}$
$\{x\}$	$\leftrightarrow$	$\{x\} \cup \{z\} = \{x, z\}$
$\{y\}$	$\leftrightarrow$	$\{y\} \cup \{z\} = \{y, z\}$
$\{x, y\}$	$\leftrightarrow$	$\{x, y\} \cup \{z\} = \{x, y, z\}$

Theorem 6.3.1

Suppose A is a finite set with n elements, then $\wp(A)$ has 2^n elements. In other words, $|\wp(A)| = 2^{|A|}$.

5.1.10. Ordered n-tuples and Cartesian Products (Revisit)

Definition

Let $n \in \mathbb{Z}^+$ and let $x_1, x_2, \cdots, x_n$ be (not necessarily distinct) elements. An **ordered n-tuple** is an expression of the form $(x_1, x_2, \cdots, x_n)$.
An **ordered pair** is an ordered 2-tuple; an **ordered triple** is an ordered 3-tuple.

Equality of two ordered n-tuples:

$$(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n) \Leftrightarrow x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n.$$

Definition

Given sets $A_1, A_2, \cdots, A_n$, the **Cartesian product** of $A_1, A_2, \cdots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \cdots, a_n \in A_n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}.$$

If A is a set, then $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ many } A\text{'s}}$.

Let $A_1 = \{x, y\}$, $A_2 = \{1,2,3\}$ and $A_3 = \{a, b\}$.

a. Find $A_1 \times A_2$.

$$A_1 \times A_2 = \{(x,1),(x,2),(x,3),(y,1),(y,2),(y,3)\}$$

b. Find $(A_1 \times A_2) \times A_3$.

$$(A_1 \times A_2) \times A_3 = \{(u, v) : u \in A_1 \times A_2 \text{ and } v \in A_3\}$$
$$= \{((x,1),a),((x,2),a),((x,3),a),((y,1),a),((y,2),a),((y,3),a),$$
$$((x,1),b),((x,2),b),((x,3),b),((y,1),b),((y,2),b),((y,3),b)\}$$

c. Find $A_1 \times A_2 \times A_3$.

$$A_1 \times A_2 \times A_3 = \{(u, v, w) : u \in A_1, v \in A_2 \text{ and } w \in A_3\}$$
$$= \{(x,1,a),(x,1,b),(x,2,a),(x,2,b),(x,3,a),(x,3,b),$$
$$(y,1,a),(y,1,b),(y,2,a),(y,2,b),(y,3,a),(y,3,b)\}$$

5.2 Properties of Sets

5.2.1 Properties of Sets

We begin by listing some set properties that involve subset relations.

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection*: For all sets A and B,

   (a) $A \cap B \subseteq A$ (b) $A \cap B \subseteq B$

2. *Inclusion in Union*: For all sets A and B,

   (a) $A \subseteq A \cup B$ (b) $B \subseteq A \cup B$

3. *Transitive Property of Subsets*: For all sets A, B and C,

   $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$.

Procedural versions of the definitions of the other set operations are derived similarly and are summarized below.

Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose a and b are elements of U.

1. $a \in X \cup Y \Leftrightarrow a \in X \vee a \in Y$
2. $a \in X \cap Y \Leftrightarrow a \in X \wedge a \in Y$
3. $a \in X - Y \Leftrightarrow a \in X \wedge a \notin Y$
4. $a \in \bar{X} \Leftrightarrow a \notin X$
5. $(a, b) \in X \times Y \Leftrightarrow a \in X \wedge b \in Y$

Note: In a context where U is the universal set (so that implicitly means $U \supseteq X$), the complement of X, denoted $\bar{X}$ or X^c, is defined by $\bar{X} = U \setminus X$.

5.2.2 Set Identities

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws*: For all sets A and B,
   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
2. *Associative Laws*: For all sets A, B and C,
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and (b) $(A \cap B) \cap C = A \cap (B \cap C)$.
3. *Distributive Laws*: For all sets A, B and C,
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. *Identity Laws*: For all sets A,
   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.
5. *Complement Laws*: For all sets A,
   (a) $A \cup \bar{A} = U$ and (b) $A \cap \bar{A} = \emptyset$.
6. *Double Complement Law*: For all sets A,
   $\bar{\bar{A}} = A$.

Theorem 6.2.2 Set Identities

7. *Idempotent Laws*: For all sets A,
   (a) $A \cup A = A$ and (b) $A \cap A = A$.
8. *Universal Bound Laws*: For all sets A,
   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
9. *De Morgan's Laws*: For all sets A and B,
   (a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ and (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
10. *Absorption Laws*: For all sets A and B,
   (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.
11. *Complements of U and ∅*:
   (a) $\bar{U} = \emptyset$ and (b) $\bar{\emptyset} = U$.
12. *Set Difference Law*: For all sets A and B,
   $A \setminus B = A \cap \bar{B}$.

Properties of Sets

Table 6.4.1 summarizes the main features of the logical equivalences from Theorem 2.1.1. and the set properties from Theorem 6.2.2. Notice how similar they are.

In fact, both are special cases of the same general structure, known as a *Boolean algebra*.

Logical Equivalences	Set Properties
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$

Table 6.4.1

Properties of Sets

a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) \equiv A$ b. $A \cap (A \cup B) \equiv A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Table 6.4.1 (continued)

5.2.3 Examples of Proofs involving Sets

Example #1: Prove the De Morgan's law below by working in the universal set U and using definition of set operations and laws for propositional logic.

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \text{ (De Morgan's law)}$$

Proof:

1. Let $z \in U$.
2. 2.1. $z \in \overline{A \cup B}$

2.2	$\Leftrightarrow$	$\sim(z \in A \cup B)$	by the definition of $\overline{}$
2.3	$\Leftrightarrow$	$\sim\big((z \in A) \vee (z \in B)\big)$	by the definition of $\cup$
2.4	$\Leftrightarrow$	$(z \notin A) \wedge (z \notin B)$	by De Morgan's Law for propositional logic
2.5	$\Leftrightarrow$	$(z \in \bar{A}) \wedge (z \in \bar{B})$	by the definition of $\overline{}$
2.6	$\Leftrightarrow$	$z \in \bar{A} \cap \bar{B}$	by the definition of $\cap$

5.2.3 Examples of Proofs involving Sets

Example #2: Using set identities (Theorem 6.2.2), prove the following: Under the universal set U, show that

$(A \cap B) \cup (A \setminus B) = A$ for all sets A, B.

Proof:

1. $(A \cap B) \cup (A \setminus B) = (A \cap B) \cup (A \cap \bar{B})$ by the Set Difference Law
2. $= A \cap (B \cup \bar{B})$ by the Distributive Law
3. $= A \cap U$ by the Complement Law
4. $= A$ by the Identity Law

END OF FILE