CS3236 Semester 2 2019/20:
Midterm (Total 50 Marks)

Name:______________________________

Matriculation Number:_______________

Score:______________________________

You are given 1 hour and 30 minutes for this quiz. You are allowed one sheet of A4 paper, printed or written on both sides. Calculators are not needed.

If you run out of space for some questions, you may continue on the final pages. (Write "See final pages" in the original space, and clearly label questions on the final pages.)

1. [**Entropy and Mutual Information**]

   (a) **(10 Marks)** Suppose that $X \sim \mathrm{Bernoulli}(p)$ (i.e., $P_X(1) = p$ and $P_X(0) = 1 - p$), and that the conditional distribution of Y given X is as follows: $P_{Y|X}(0|0) = 1$, $P_{Y|X}(1|0) = 0$, $P_{Y|X}(0|1) = \delta$, and $P_{Y|X}(1|1) = 1 - \delta$.
   Compute $H(Y)$, $H(Y|X)$, and $I(X;Y)$, expressing your answers in terms of p and δ, and using the definition $H_2(q) = q \log_2 \frac{1}{q} + (1-q) \log_2 \frac{1}{1-q}$ in your answers if you wish.

(b) **(6 Marks)** Let X and Y be discrete random variables with alphabets $\mathcal{X}$ and $\mathcal{Y}$, and suppose that $P_X(x) > 0$ for all $x \in \mathcal{X}$. Prove that if $H(Y|X) = 0$, then it must be the case that Y is a deterministic function of X (i.e., for all $x \in \mathcal{X}$, some value of $y \in \mathcal{Y}$ has $P_{Y|X}(y|x) = 1$ and the rest have $P_{Y|X}(y|x) = 0$).

(c) **(6 Marks)** Give an example of a collection of four random variables (X_1, X_2, Y_1, Y_2) such that $I(X_1, X_2; Y_1, Y_2) > I(X_1; Y_1) + I(X_2; Y_2)$. Explain your answer.

2. [**Source Coding Algorithms**]

   (a) **(8 Marks)** Apply the Huffman algorithm to the source with symbols (a, b, c, d, e, f, g, h) and corresponding probabilities $(0.05, 0.26, 0.2, 0.02, 0.04, 0.09, 0.29, 0.05)$. Show the full Huffman tree, and write down the codeword associated with each symbol.

(b) **(10 Marks)** Let X be a discrete random variable on the alphabet $\mathcal{X} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ with probability mass function P_X such that $\min_{x \in \mathcal{X}} P_X(x) > 0$. For a symbol code $C(\cdot)$, let $\ell_{\min}(C) = \min_{x \in \mathcal{X}} \ell(x)$ be the length of its shortest codeword.

(i) Explain why the Huffman code can give $\ell_{\min}(C) = 1$ for some P_X.

(ii) Explain why the Huffman code cannot give $\ell_{\min}(C) > 3$ for any P_X.

(iii) Could the Shannon-Fano code give $\ell_{\min}(C) > 3$ for some P_X? Explain.

(c) **(10 Marks)** Now consider the same setup as part (b), but instead consider $\ell_{\max}(C) = \max_{x \in \mathcal{X}} \ell(x)$, the length of the longest codeword.

(i) There exists an integer A such that the Huffman code can give $\ell_{\max}(C) = A$ for some P_X, but can never give $\ell_{\max}(C) < A$ for any P_X. Find a suitable choice of A and explain why these claims are true.

(ii) There exists an integer B such that the Huffman code can give $\ell_{\max}(C) = B$ for some P_X, but can never give $\ell_{\max}(C) > B$ for any P_X. Find a suitable choice of B and explain why these claims are true.

(iii) Could the Shannon-Fano code give $\ell_{\max}(C) > 100$ for some P_X? Explain.

[Use this page for any extra working if you run out of space. You must clearly write “See final pages” for any question continued here, and here you must clearly indicate each exact question and part (e.g., 2(c)).]