Solutions to Exam 2018-2019 Semester 2

1. (a) Let $f(x) = 1384(1-2x)e^{3x}$, $-\infty < x < \infty$. Find the absolute maximum value of f. Give your answer correct to the nearest integer.

   (b) Let m and n denote two positive even integers with $m < n$. It is known that the area of the region between the graphs of $y = 2\cos x$ and $y = \sin 2x$ from $x = m\pi$ to $x = (n+1)\pi$ is equal to 8554. Find the exact value of $n - m$.

   Answer. (a) 1521, (b) 2138.

   Solution. (a) First $f'(x) = 1384(-2e^{3x} + 3(1-2x)e^{3x}) = 1384e^{3x}(1-6x)$. We have $f'(x) > 0$ if $x < \frac{1}{6}$; and $f'(x) < 0$ if $x > \frac{1}{6}$. By the first derivative test, f has the absolute maximum at $x = \frac{1}{6}$. The absolute maximum value is $f(\frac{1}{6}) = 1384 \times \frac{2}{3} \times e^{\frac{1}{2}} = 1521.22 \approx 1521$.

   (b) Let's find the intersection points of the two curves. We have $\sin 2x = 2\cos x \Leftrightarrow 2\sin x\cos x = 2\cos x \Leftrightarrow \cos x(\sin x - 1) = 0 \Leftrightarrow x = k\pi + \frac{\pi}{2}$, where k is an integer. The function $|\sin 2x - 2\cos x|$ is clearly periodic of period 2π. Therefore, we first consider the interval $[m\pi, m\pi + 2\pi]$. Here m is an even integer. We have

   $\int_{m\pi}^{m\pi+2\pi} |\sin 2x - 2\cos x|\, dx = \int_0^{2\pi} |\sin 2x - 2\cos x|\, dx$
   $= \int_0^{\frac{\pi}{2}} 2\cos x - \sin 2x\, dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin 2x - 2\cos x\, dx + \int_{\frac{\pi}{2}}^{2\pi} 2\cos x - \sin 2x\, dx$
   $= [2\sin x + \frac{1}{2}\cos 2x]_0^{\frac{\pi}{2}} + [-\frac{1}{2}\cos 2x - 2\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} + [2\sin x + \frac{1}{2}\cos 2x]_{\frac{3\pi}{2}}^{2\pi}$
   $= (2 - \frac{1}{2} - 0 - \frac{1}{2}) + (\frac{1}{2} + 2 - \frac{1}{2} + 2) + (0 + \frac{1}{2} + 2 + \frac{1}{2}) = 1 + 4 + 3 = 8.$

   Also consider the last interval $[n\pi, (n+1)\pi]$. Here n is an even integer. We have $\int_{n\pi}^{(n+1)\pi} |\sin 2x - 2\cos x|\, dx = \int_0^{\pi} |\sin 2x - 2\cos x|\, dx = \int_0^{\frac{\pi}{2}} 2\cos x - \sin 2x\, dx + \int_{\frac{\pi}{2}}^{\pi} \sin 2x - 2\cos x\, dx = [2\sin x + \frac{1}{2}\cos 2x]_0^{\frac{\pi}{2}} + [-\frac{1}{2}\cos 2x - 2\sin x]_{\frac{\pi}{2}}^{\pi} = (2 - \frac{1}{2} - 0 - \frac{1}{2}) + (-\frac{1}{2} - 0 - \frac{1}{2} + 2) = 2.$

   Consequently,
   $\int_{m\pi}^{(n+1)\pi} |\sin 2x - 2\cos x|\, dx = \int_{m\pi}^{n\pi} |\sin 2x - 2\cos x|\, dx + \int_{n\pi}^{(n+1)\pi} |\sin 2x - 2\cos x|\, dx$
   $= \frac{n-m}{2} \times 8 + 2 = 4(n-m) + 2 = 8554$. Thus $n - m = 2138$.

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2. (a) Let $P(x)$ denote the degree two Taylor polynomial of the function $\ln(2 + \tan x)$ at $x = 0$. Find the value of $P(\frac{9}{10})$. Give your answer correct to two decimal places.

   (b) Find the directional derivative of the function $f(x,y,z) = 4xyz - 2x^2 + y^2 + z^2 + 321$ at the point $(1,1,2)$ in the direction of the vector which joins $(2,3,1)$ to $(1,2,3)$. Give your answer correct to two decimal places.

Answer. (a) 1.04, (b) −2.67.

Solution. (a) Let $f(x) = \ln(2+\tan x)$. Then $f'(x) = \frac{\sec^2 x}{2+\tan x}$, and

$$f''(x) = \frac{2\sec x \sec x \tan x(2+\tan x) - \sec^2 x \sec^2 x}{(2+\tan x)^2}.$$

Thus $f(0) = \ln 2, f'(0) = \frac{1}{2}, f''(0) = -\frac{1}{4}$. The degree two Taylor polynomial of f at $x = 0$ is $P(x) = \ln 2 + \frac{1}{2}x - \frac{1}{8}x^2$. Therefore, $P(\frac{9}{10}) = \ln 2 + \frac{1}{2}\frac{9}{10} - \frac{1}{8}(\frac{9}{10})^2 = 1.04$.

(b) The vector which joins $(2,3,1)$ to $(1,2,3)$ is $\langle 1,2,3\rangle - \langle 2,3,1\rangle = \langle -1,-2,2\rangle$ with length $\sqrt{(-1)^2+(-2)^2+2^2} = 3$. The unit vector along this direction is $\langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$.

We have $\nabla f = \langle 4yz-4x, 4xz+2y, 4xy+2z\rangle$. Thus $\nabla f(1,1,2) = \langle 4,10,8\rangle$.

Therefore the required directional derivative is $\langle 4,10,8\rangle \cdot \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle = -\frac{8}{3} = -2.67$.

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3. (a) It is known that the function $f(x,y) = 3xy - x^2 - y^3 - 5$ has exactly one local maximum point at (a,b). If $a+b = \frac{m}{n}$ where m and n are two positive integers without any common factors, find the exact value of $m+n$.

   (b) The region R lies above the paraboloid $z = 4-x^2-y^2$ and below the paraboloid $z = 8-3x^2-3y^2$. Find the volume of R. Give your answer correct to two decimal places.

   Answer. (a) 29, (b) 12.57.

   Solution. (a) First $f_x = 3y-2x, f_y = 3x-3y^2$. We need to solve the system $3y-2x = 0, 3x-3y^2 = 0$. Substituting $x = \frac{3}{2}y$ into the second equation, we have $\frac{9}{2}y - 3y^2 = 0$. That is $\frac{3}{2}y(3-2y) = 0$ so that $y = 0, \frac{3}{2}$. Using $x = \frac{3}{2}y$, if $y = 0$, then $x = 0$; if $y = \frac{3}{2}$, then $x = \frac{9}{4}$. Thus there are two critical points $(0,0)$, $(\frac{9}{4}, \frac{3}{2})$.

   We have $f_{xx} = -2, f_{yy} = -6y, f_{xy} = 3$, and $D(x,y) = (-2)(-6y) - 3^2 = 12y-9$.

   At $(\frac{9}{4}, \frac{3}{2})$, we have $f_{xx}(\frac{9}{4}, \frac{3}{2}) = -2 < 0, D(\frac{9}{4}, \frac{3}{2}) = 9$. By the second derivative test, f has a local maximum at $(\frac{9}{4}, \frac{3}{2})$. By the statement of the question, we know f has exactly one local maximum point. Thus $(\frac{9}{4}, \frac{3}{2})$ is the point (a,b). We may this reject the the critical point $(0,0)$. Therefore, $a+b = \frac{9}{4} + \frac{3}{2} = \frac{15}{4}$ and $m+n = 15+4 = 29$.

   (b) Equating the two equations of the paraboloids, we have $4-x^2-y^2 = 8-3x^2-3y^2 \Leftrightarrow x^2+y^2 = 2$, which is a circle with radius $\sqrt{2}$. That is the two paraboloids intersect in a circle $x^2+y^2 = 2, z = 2$. Let D on the xy-plane be the circular disk $D = \{(x,y) \mid x^2+y^2 \le 2\}$.

   Thus $V = \iint_D (8-3x^2-3y^2)-(4-x^2-y^2)\,dxdy = \iint_D 4-2x^2-2y^2\,dxdy$. Using polar coordinates, we have $V = \int_0^{2\pi}\int_0^{\sqrt{2}} (4-2r^2)r\,drd\theta = 2\pi[2r^2 - \frac{1}{2}r^4]_{r=0}^{r=\sqrt{2}} = 2\pi(4-2) = 4\pi = 12.57$.

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4. (a) Evaluate $\int_{-2}^{0}\int_{0}^{x^2} e^{y-\frac{1}{3}y^{\frac{3}{2}}}\,dydx$. Give your answer correct to two decimal places.

(b) At time $t = 0$ a tank contains 20 pounds of salt dissolved in 120 gallons of water. Assume that water containing 0.5 pound of salt per gallon is entering the tank at a rate of 4 gallons per minute and the well stirred solution is leaving the tank at the same rate. Find the amount of salt in the tank at time $t = 16$ minutes. Give your answer in pounds correct to two decimal places.

Answer. (a) 5.59, (b) 36.53.

Solution. (a) The region of integration is

$$R = \{(x,y) \mid y \leq x^2,\ -2 \leq x \leq 0\} = \{(x,y) \mid -2 \leq x \leq -\sqrt{y},\ 0 \leq y \leq 4\}.$$

Thus $\int_{-2}^{0}\int_{0}^{x^2} e^{y-\frac{1}{3}y^{\frac{3}{2}}}\,dydx = \int_{0}^{4}\int_{-2}^{-\sqrt{y}} e^{y-\frac{1}{3}y^{\frac{3}{2}}}\,dxdy = \int_{0}^{4}\left[xe^{y-\frac{1}{3}y^{\frac{3}{2}}}\right]_{x=-2}^{x=-\sqrt{y}} dy$
$= \int_{0}^{4}(2-y^{\frac{1}{2}})e^{y-\frac{1}{3}y^{\frac{3}{2}}}\,dy = \int_{0}^{4} 2e^{y-\frac{1}{3}y^{\frac{3}{2}}}\,d\left(y-\frac{1}{3}y^{\frac{3}{2}}\right) = 2\left[e^{y-\frac{1}{3}y^{\frac{3}{2}}}\right]_{0}^{4} = 2(e^{\frac{4}{3}}-1)$
$= 5.59.$

(b) First note that the volume of the solution remains constant which is 120 gallons. Let Q be the amount of salt in pound at time t. The concentration of salt in the solution is $Q/120$ pound per gallon. Suppose at time $t+dt$, the amount of salt is $Q+dQ$. Then

$$dQ = \text{salt input} - \text{salt output} = 4 \times 0.5 \times dt - 4 \times \frac{Q}{120} \times dt.$$

Thus

$$\frac{dQ}{dt} = 2 - \frac{Q}{30}, \quad \text{or equivalently} \quad \frac{dQ}{dt} + \frac{Q}{30} = 2.$$

The general solution to this first order linear DE is $Q = 60 + Ce^{-\frac{t}{30}}$. Since $Q(0) = 20$, we have $20 = 60 + C$ so that $C = -40$. Consequently, $Q = 60 - 40e^{-\frac{t}{30}}$. Therefore, $Q(16) = 60 - 40e^{-\frac{16}{30}} = 36.53$ pounds.

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5. (a) Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{y^3}{x^2}, \text{ with } x > 0, y > 0 \text{ and } y(1) = \sqrt{\frac{5}{7}}.$$

Find the value of $y(\frac{3}{2})$. Give your answer correct to two decimal places.

(b) The growth of the sandhill crane population follows a logistic model with a birth rate per capita of 10% per year. Initially at time $t = 0$ there were 1521 sandhill cranes. It is known that at time $t = 10$ years there were 2019 sandhill cranes. How many sandhill cranes will there be after a very long time? Give your answer correct to the nearest integer.

Answer. (a) 0.43, (b) 2494.

Solution. (a) This is a Bernoulli equation with $n = 3$. Let $u = y^{-2}$. The equation becomes

$$u' - \frac{4}{x}u = -\frac{2}{x^2}.$$

This is a first order linear differential equation. An integrating factor is $e^{\int -\frac{4}{x}\,dx} = e^{-4\ln x} = x^{-4}$. Multiplying through the above equation by x^{-4}, we have

$$(x^{-4}u)' = -\frac{2}{x^6}.$$

Integrating, $x^{-4}u = \frac{2}{5x^5} + C$. That is $u = Cx^4 + \frac{2}{5x}$. Therefore, $y = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{Cx^4 + \frac{2}{5x}}}$. Since $y(1) = \sqrt{\frac{5}{7}}$, we have $\sqrt{\frac{5}{7}} = \frac{1}{\sqrt{C + \frac{2}{5}}}$ so that $C = 1$. Consequently, $y = \frac{1}{\sqrt{x^4 + \frac{2}{5x}}}$. Therefore, $y(\frac{3}{2}) = \frac{1}{\sqrt{(\frac{3}{2})^4 + \frac{2}{5(\frac{3}{2})}}} = \frac{1}{\sqrt{(\frac{3}{2})^4 + \frac{4}{15}}} = 0.43$.

(b) Let $N(t)$ be the number of sandhill crane at time t in years. The logistic model gives $\frac{1}{N} = \frac{s}{B} + Ce^{-Bt}$, where s, C are constants and $B = 10\% = 0.1$ is the birth rate per capita. Thus $\frac{1}{N} = 10s + Ce^{-t/10}$, $t \geq 0$.

$N(0) = 1521 \Rightarrow \frac{1}{1521} = 10s + C$, $N(10) = 2019 \Rightarrow \frac{1}{2019} = 10s + Ce^{-1}$. Subtracting the two equations, we obtain

$\frac{1}{1521} - \frac{1}{2019} = C(1 - e^{-1}) \Rightarrow C = \frac{498}{1521 \times 2019 \times (1 - e^{-1})} = \frac{166}{1023633(1 - e^{-1})} = 0.000256545$.

Therefore, $10s = \frac{1}{1521} - C$. As $t \to \infty$, $N \to N_\infty = \frac{1}{10s} = \frac{1}{\frac{1}{1521} - C} = \frac{1}{\frac{1}{1521} - \frac{166}{1023633(1 - e^{-1})}}$ $= 2494.28 = 2494$ to the nearest integer.

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