CS1231S: Discrete Structures
Tutorial #9: Counting and Probability I
(Week 11: 28 March – 1 April 2022)

I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

   (a) What is the probability that the first ball is blue and the second is white?
   (b) What is the probability that both balls drawn are white?
   (c) What is the probability that the second ball drawn is blue?

D2. Calculate
   (a) the probability that a randomly chosen positive three-digit integer is a multiple of 6.
   (b) the probability that a randomly chosen positive four-digit integer is a multiple of 7.

D3. In Tutorial 7 D3, you are asked to write down all possible functions $\{1,2,3\} \to \{4,5\}$. How many possible functions $f: A \to B$ are there if $|A| = n$ and $|B| = k$?

D4. Assuming that all years have 365 days and all birthdays occur with equal probability. What is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least 50%?

   Write out the equation to solve for n and write a program to compute n.
   (This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

II. Tutorial Questions

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?
   (We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

2. A pack of cards consists of 52 cards with 4 suits: spades (♠), hearts (♥), diamonds (♦) and clubs (♣). Each suit has 13 cards: 2, 3, 4, 5, 6, 7, 8, 9, Ten, Jack, Queen, King and Ace.

   You draw a sequence of 5 cards from a pack of cards with replacement.

   (a) How many sequences will have at least one Queen?

   (b) How many sequences will have at least one Queen or one King?

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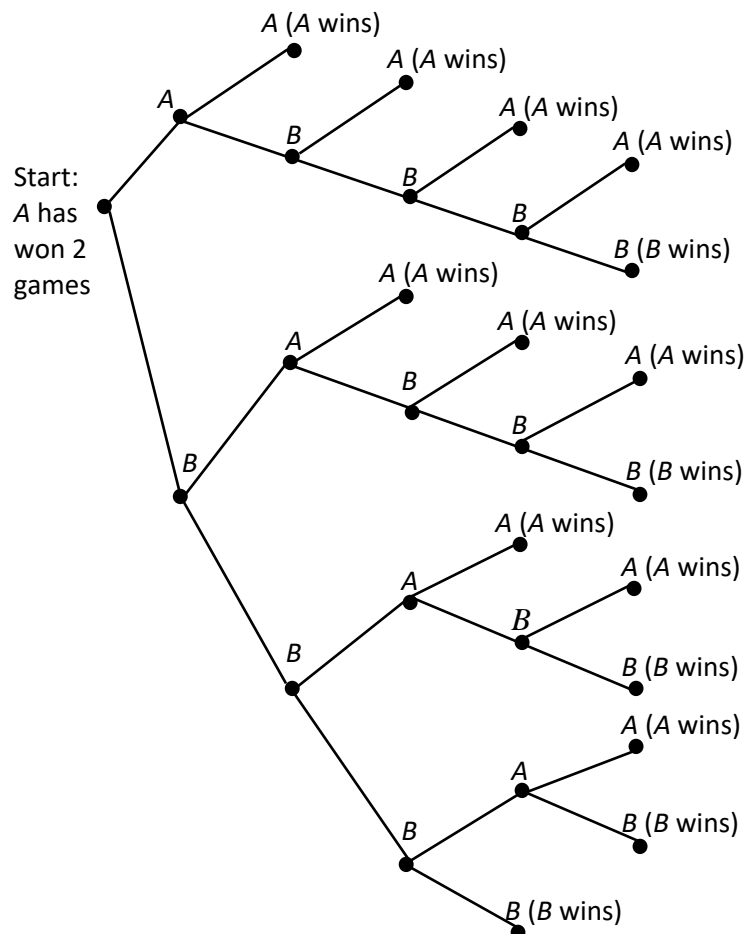
   (We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

   Answer:
   15 ways.

   The following are the 15 ways, excluding the first two games which are won by team A:

   (1) *AA*, (2) *ABA*, (3) *ABBA*, (4) *ABBBA*, (5) *ABBBB*, (6) *BAA*, (7) *BABA*, (8) *BABBA*, (9) *BABBB*, (10) *BBAA*, (11) *BBABA*, (12) *BBABB*, (13) *BBBAA*, (15) *BBBAB*, (15) *BBBB*.

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   You draw a sequence of 5 cards from a pack of cards with replacement.

   (a) How many sequences will have at least one Queen?

   (b) How many sequences will have at least one Queen or one King?

   Answers:

   (a) Number of 5-card sequences = 52^5.
   Number of 5-card sequences with no Queen = 48^5.
   Therefore, number of 5-card sequences with at least one Queen = $52^5 - 48^5 =$ **125, 400, 064**.

   (b) Number of 5-card sequences with no Queen and no King = 44^5.
   Therefore, number of 5-card sequences with at least one Queen or one King = $52^5 - 44^5 =$ **215, 287, 808**.

3. There are 789 CS students in SoC. Among them, 672 are taking CS1231S, 629 are taking CS1101S, 153 are taking MA1101R, 608 are taking CS1231S and CS1101S, 87 are taking CS1231S and MA1101R, 53 are taking CS1101S and MA1101R, and 46 are taking all three modules.

   How many CS students are not taking any of these three modules?

   Answer:

   1. Let A, B and C be the sets of CS students taking CS1231S, CS1101S and MA1101R respectively.

   2. By the inclusion/exclusion rule (theorem 9.3.3),

   $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
   $$= 672 + 629 + 153 - 608 - 87 - 53 + 46 = 752.$$

   3. $|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{(A \cup B \cup C)}| = |U| - |A \cup B \cup C| = 789 - 752 = 37.$

   4. There are **37** CS students who are not taking any of these three modules.

4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

An integer k is in the correct position if it is at the kth position in the permutation. For example, the permutation 3, 2, 4, 1, 5 has integers 2 and 5 in their correct positions, and the permutation 12, 1, 3, 9, 10, 8, 7, 6, 2, 4, 11, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.
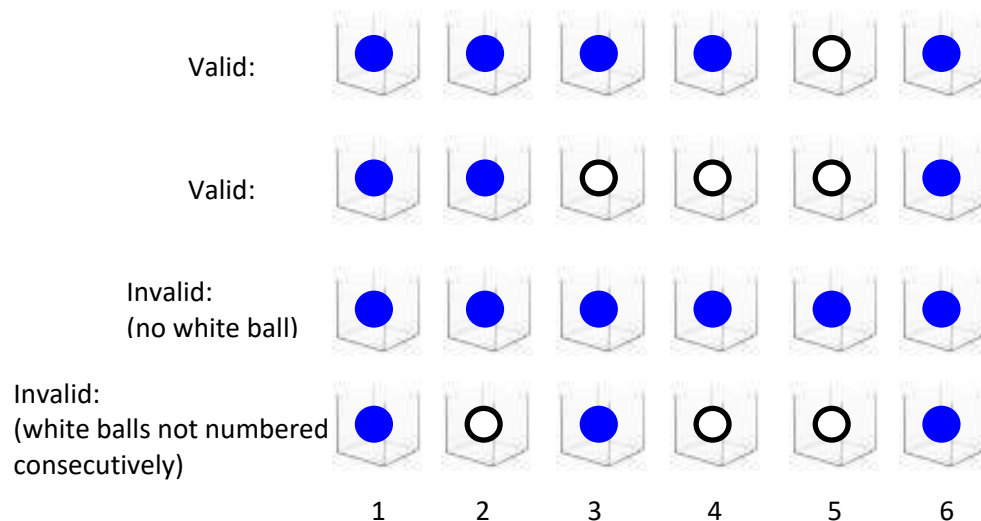
Answer:

1. Let $|P_k|$ be the number of permutations with integer k in its correct position.
2. $|P_1| = |P_2| = |P_3| = (n - 1)!$
3. $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| == (n - 2)!$
4. $|P_1 \cap P_2 \cap P_3| = (n - 3)!$
5. By the inclusion/exclusion rule (theorem 9.3.3),
$$|P_1 \cup P_2 \cup P_3| = 3(n - 1)! - 3(n - 2)! + (n - 3)! = (3n^2 - 12n + 13)(n - 3)!$$

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

(For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

Some examples for $n = 6$ are shown below for your reference.

Answer:

1. For k ($1 \leq k \leq n$) consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways.

2. Therefore, total number of ways is $\sum_{k=1}^{n}(n - k + 1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

6. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

What if we seat them around a circular table? Examine the figure below.
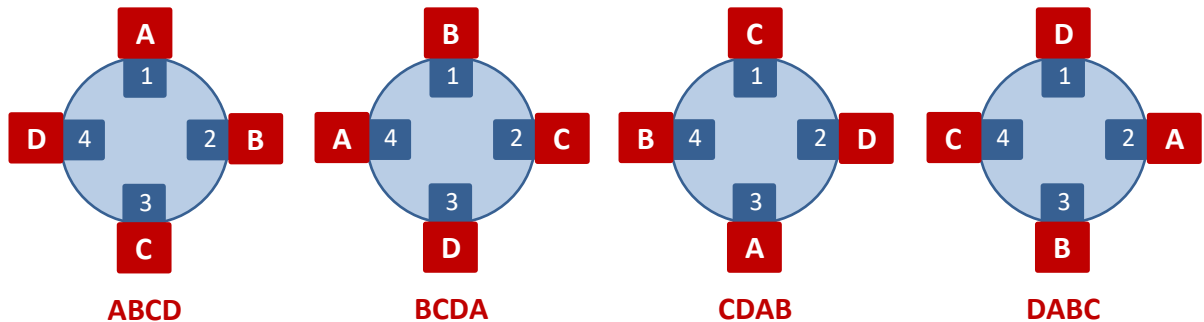
ABCD   **BCDA**   **CDAB**   **DABC**

The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

Answer the following questions:

a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

c. In how many ways can $n-1$ people sit around a circular table with n chairs?

Answers:

a. 8 boys can be seated in a circle in 7! ways. There are 8 spaces between the boys, which can be occupied by 4 girls in P(8,4) ways. Hence, total number of ways = 7! × P(8,4) = 5040 × 1680 = **8467200**.

b. There are 5! = 120 ways for 6 people to sit around a circular table. There are 2 × 4! = 48 ways for Eric and Freddy to sit together. Therefore, the answer is 120 – 48 = **72** ways.

c. Treat the empty chair as just another person, therefore there are $(\boldsymbol{n-1})!$ ways to seat $n-1$ people around a table with n chairs.

7. [Past year's exam question.]
   Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

   Answer:

   1. Divide the unit square into 25 equal smaller squares of side 1/5 each.

   2. Then at least one of these small squares would contain at least three points. (Otherwise, if every square contains two points or less, the total number of points is no more than $2 \times 25 = 50$, which contradicts our assumptions that there are 51 points – Generalised PHP.)

   3. Now, the circle circumvented around the small square with the three points inside also contains these three points and has radius

   $$\sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \sqrt{\frac{2}{100}} = \sqrt{\frac{1}{50}} < \sqrt{\frac{1}{49}} = \frac{1}{7}$$

8. This is the famous chess master problem to illustrate the use of the Pigeonhole Principle. Try it out yourself before googling for the answer.

   A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire herself, she decides not to play more than 12 games during any one week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

   Answer:

   1. There are 77 days in 7 weeks. Define P_i for $1 \leq i \leq 77$ as the total number of games the chess master has played from day 1 up to and including day i.

   2. Hence $1 \leq P_1 < P_2 < \cdots < P_{77} \leq 132$ as the chess master plays at least one game per day and at most $11 \times 12 = 132$ games altogether. Note that all the P_i's are distinct.

   3. Define $Q_i = P_i + 21$. Note that all the Q_i's are distinct too. Note also that the largest possible value for Q_i is $132 + 21 = 153$.

   4. There are 154 numbers in the P_i's and Q_i's, but each P_i or Q_i can take a value in the range from 1 to 153 inclusive. So, there are 154 pigeons and 153 pigeonholes.

   5. By PHP, two of the numbers must be equal. Hence, $P_j = Q_i = P_i + 21$ for some i, j.

   6. Therefore, the chess master has played exactly 21 games in the consecutive block from day $i + 1$ to day j.

9. [CS1231 Past Year's Exam Question]
   You wish to select five persons from seven men and six women to form a committee that includes at least three men.

   (a) In how many ways can you form the committee?

   (b) If you randomly choose five persons to form the committee, what is the probability that you will get a committee with at least three men? Give your answer correct to 4 significant figures.

   Answers:

   (a) Let E be the event that the committee includes at least three men, that is, it has (three men and two women), or (four men and one woman) or (five men).

   Therefore, $|E| = \binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5} = 35 \times 15 + 35 \times 6 + 21 = \mathbf{756}$.

   (b) Let S be the sample space. Then $|S| = \binom{13}{5} = 1287$.
   Therefore, $P(E) = \frac{|E|}{|S|} = \frac{756}{1287} = \mathbf{58.74\%}$ (or $\mathbf{0.5874}$).

10. [AY2020/21 Semester 2 Exam Question]
    How many integer solutions for x_1, x_2, x_3 and x_4 does the following equation have, given that $x_i \geq 2^i + i$, for $1 \leq i \leq 4$?

    $$x_1 + x_2 + x_3 + x_4 = 56.$$

    Answer:

    $x_1 \geq 2^1 + 1 = 3$; $x_2 \geq 2^2 + 2 = 6$; $x_3 \geq 2^3 + 3 = 11$; $x_4 \geq 2^4 + 4 = 20$.

    $3 + 6 + 11 + 20 = 40$.

    Let $y_i = x_i - (2^i + i)$, then $y_1 + y_2 + y_3 + y_4 = 16$, where $y_i \geq 0$.

    This becomes a multiset problem with $n = 4, r = 16$.

    $$\binom{r + n - 1}{r} = \binom{16 + 4 - 1}{16} = \binom{19}{16} = \mathbf{969}.$$