Lecture #11: Counting and Probability 2
Summary

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Summary

This lecture is based on Epp's book chapter 9.
Hence, the section numbering is according to the book.

9.5 Counting Subsets of a Set: Combinations

- *r*-combination, *r*-permutation, permutations of a set with repeat elements, partitions of a set into *r* subsets

9.6 *r*-Combinations with Repetition Allowed

- Multiset
- Formula to use depends on whether (1) order matters, (2) repetition is allowed

9.7 Pascal's Formula and the Binomial Theorem

9.8 Probability Axioms and Expected Value

- Probability axioms, complement of an event, general union of two events, expected value

9.9 Conditional Probability, Bayes' Formula, and Independent Events

Reference: Epp's Chapter 9 Counting and Probability

Summary

Definition: *r*-combination

Let n and r be non-negative integers with $r \leq n$.

An ***r*-combination** of a set of n elements is a subset of r of the n elements.

$\binom{n}{r}$, read "*n* choose *r*", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Other symbols used are $C(n, r)$, ${}_nC_r$, $C_{n,r}$, or ${}^nC_r$.

Theorem 9.5.1 Formula for $\binom{n}{r}$

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where n and r are non-negative integers with $r \leq n$.

Theorem 9.5.2 Permutations with sets of indistinguishable objects

Suppose a collection consists of *n* objects of which

n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
:
n_k are of type *k* and are indistinguishable from each other

and suppose that $n_1 + n_2 + \ldots + n_k = n$. Then the number of distinguishable permutations of the *n* objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{\boldsymbol{n}!}{\boldsymbol{n_1}!\boldsymbol{n_2}!\boldsymbol{n_3}!\cdots\boldsymbol{n_k}!}$$

Definition: Multiset

An ***r*-combination with repetition allowed**, or **multiset of size *r***, chosen from a set *X* of *n* elements is an unordered selection of elements taken from *X* with repetition allowed.

If $X = \{x_1, x_2, \cdots, x_n\}$, we write an *r*-combination with repetition allowed as $[x_{i_1}, x_{i_2}, \cdots, x_{i_r}]$ where each x_{i_j} is in *X* and some of the x_{i_j} may equal each other.

Theorem 9.6.1 Number of *r*-combinations with Repetition Allowed

The number of *r*-combination with repetition allowed (multisets of size *r*) that can be selected from a set of *n* elements is:

$$\binom{r+n-1}{r}$$

This equals the number of ways *r* objects can be selected from *n* categories of objects with repetitions allowed.

Summary

9.6 r-Combinations with Repetition Allowed

Which formula to use?

	Order Matters	Order Does Not Matter
Repetition Is Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Is Not Allowed	$P(n, k)$	$\binom{n}{k}$

Theorem 9.7.1 Pascal’s Formula

Let n and r be positive integers, $r \le n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Theorem 9.7.2 Binomial Theorem

Given any real numbers a and b and any non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

Theorem 6.3.1 Number of elements in a Power Set

If a set X has n $(n \ge 0)$ elements, then $\wp(X)$ has 2^n elements.

Probability Axioms

Let S be a sample space. A probability function P from the set of all events in S to the set of real numbers satisfies the following axioms: For all events A and B in S,

1. $0 \le P(A) \le 1$
2. $P(\emptyset) = 0$ and $P(S) = 1$
3. If A and B are disjoint ($A \cap B = \emptyset$), then $P(A \cup B) = P(A) + P(B)$

Probability of the Complement of an Event

If A is any event in a sample space S, then

$$P(\bar{A}) = 1 - P(A)$$

Probability of a General Union of Two Events

If A and B are any events in a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Summary

Definition: Expected Value

Suppose the possible outcomes of an experiment, or random process, are real numbers $a_1, a_2, a_3, \cdots, a_n$ which occur with probabilities $p_1, p_2, p_3, \cdots, p_n$. The **expected value** of the process is

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + a_3 p_3 + \cdots + a_n p_n$$

Linearity of Expectation

The expected value of the sum of random variables is equal to the sum of their individual expected values, regardless of whether they are independent. For random variables X and Y,

$$E[X+Y] = E[X] + E[Y]$$

For random variables $X_1, X_2, \cdots, X_n$ and constants $c_1, c_2, \cdots, c_n$,

$$E\left[\sum_{i=1}^{n} c_i \cdot X_i\right] = \sum_{i=1}^{n} (c_i \cdot E[X_i])$$

Summary

Definition: Conditional Probability

Let A and B be events in a sample space S. If $P(A) \neq 0$, then the **conditional probability of *B* given *A***, denoted $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad 9.9.1$$

$$P(A \cap B) = P(B|A) \cdot P(A) \quad 9.9.2$$

$$P(A) = \frac{P(A \cap B)}{P(B|A)} \quad 9.9.3$$

Theorem 9.9.1 Bayes' Theorem

Suppose that a sample space S is a union of mutually disjoint events $B_1, B_2, B_3, \ldots, B_n$.

Suppose A is an event in S, and suppose A and all the B_i have non-zero probabilities.

If k is an integer with $1 \leq k \leq n$, then

$$P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + \cdots + P(A|B_n) \cdot P(B_n)}$$

Definition: Independent Events

If A and B are events in a sample space S, then A and B are **independent**, if and only if,

$$P(A \cap B) = P(A) \cdot P(B)$$

Definition: Pairwise Independent and Mutually Independent

Let A, B and C be events in a sample space S. A , B and C are **pairwise independent**, if and only if, they satisfy conditions 1 – 3 below. They are **mutually independent** if, and only if, they satisfy all four conditions below.

1. $P(A \cap B) = P(A) \cdot P(B)$
2. $P(A \cap C) = P(A) \cdot P(C)$
3. $P(B \cap C) = P(B) \cdot P(C)$
4. $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

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