NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING

MID-TERM TEST
AY2020/21 Semester 2

CS1231/CS1231S — DISCRETE STRUCTURES

6 March 2021 Time Allowed: 1 hour 30 minutes

INSTRUCTIONS

1. This assessment paper contains **SIXTEEN (16)** questions (excluding question 0) in **THREE (3)** parts and comprises **SEVEN (7)** printed pages.
2. Answer **ALL** questions.
3. This is an **OPEN BOOK** assessment.
4. The maximum mark of this assessment is 50.
5. You are to submit a **single pdf file** (size $\leqslant$ 20MB) to your submission folder on LumiNUS.
6. Your submitted file should be named after your **Student Number** (eg: A1234567X.pdf) and your Student Number should also be written at the top of the first page of your submitted file.
7. Limit your answers to **TWO pages** if possible, or at most THREE pages.
8. Do not write your name in your submitted file.

——— END OF INSTRUCTIONS ———

1	2	3	4	5	6	7
C	**A**	**B**	**C**	**D**	**C**	**B**

8	9	10	11	12
A,C,D	**C,D**	**A,B,D**	**C,D,E**	**C,E**

0. **Check that you have done the following:**
   (a) Submission folder consists of a **single pdf file** and no other files. [1 mark]
   (b) File named correctly with **Student Number** (eg: A1234567X.pdf). [1 mark]
   (c) Student Number written **on top of the first page** of submitted file. [1 mark]

Part A: Multiple Choice Questions (Total: 14 marks)

Each multiple choice question (MCQ) is worth <u>two marks</u> and has exactly **one** correct answer. You are advised to write your answers on a **single line** to conserve space. For example:

1 A 2 B 3 C 4 D ...

Please write in **CAPITAL LETTERS**.

1. Given this statement:

   "If Aiken can do it, then Dueet can do it."

   Which of the following is logically equivalent to the above statement?

   A. "Aiken can do it" is a necessary condition for "Dueet can do it."

   B. "If Dueet can do it, then Aiken can do it."

   C. "Aiken can do it only if Dueet can do it."

   D. "Dueet can do it only if Aiken can do it."

   E. None of (A), (B), (C), (D) is logically equivalent to the given statement.

   Answer: C

   $p \rightarrow q$ means "p is a sufficient condition for q", or "p only if q".

2. The **reciprocal**, or **multiplicative inverse**, of a real number x is a real number y such that $xy = 1$.

   Knowing that every non-zero real number has a reciprocal, which of the following statements is TRUE?

   A. $\forall x \in \mathbb{R}\,\big((x = 0) \vee \exists y \in \mathbb{R}\,(xy = 1)\big)$.

   B. $\forall x \in \mathbb{R}\,\big((x \neq 0) \wedge \exists y \in \mathbb{R}\,(xy = 1)\big)$.

   C. $\forall x \in \mathbb{R}\,\big((x = 0) \wedge \exists y \in \mathbb{R}\,(xy \neq 1)\big)$.

   D. $\forall x \in \mathbb{R}\,\big((x \neq 0) \vee \exists y \in \mathbb{R}\,(xy = 1)\big)$.

   E. None of (A), (B), (C), (D) is true.

   Answer: A

   Counterexample for (B): $x = 0$; for (C): $x = 1$; for (D): $x = 0$.

3. Which of the following is/are true?

(i) $\overline{(\overline{A} \cup B) \cap (\overline{B} \cup C)} \cup (\overline{A} \cup C) = \mathbb{Z}$ for all sets $A, B, C \subseteq \mathbb{Z}$.

(ii) $\overline{A \setminus (B \cup C)} \subseteq \overline{A} \cap (B \cup C)$ for all sets $A, B, C \subseteq \mathbb{Z}$.

A. (i) and (ii) are both true.

B. (i) is true but (ii) is false.

C. (i) is false but (ii) is true.

D. (i) and (ii) are both false.

Answer: B

$\overline{(\overline{A} \cup B) \cap (\overline{B} \cup C)} \cup (\overline{A} \cup C)$

$= \{x \in \mathbb{Z} : \sim\big((x \notin A \vee x \in B) \wedge (x \notin B \vee x \in C)\big) \vee (x \notin A \vee x \in C)\}$

$= \{x \in \mathbb{Z} : (x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in C) \rightarrow (x \in A \rightarrow x \in C)\}$

$= \{x \in \mathbb{Z} : \mathbf{true}\}$

$= \mathbb{Z}.$

If $A = B = C = \{1\}$, then $\overline{A \setminus (B \cup C)} = \mathbb{Z} \nsubseteq \emptyset = \overline{A} \cap (B \cup C)$.

4. Which of the following is/are true?

(i) There are **distinct** partitions $\mathcal{C}_1, \mathcal{C}_2$ of $\mathbb{Z}$ such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$.

(ii) There are **distinct** partitions $\mathcal{C}_1, \mathcal{C}_2$ of $\mathbb{Z}$ such that $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.

A. (i) and (ii) are both true.

B. (i) is true but (ii) is false.

C. (i) is false but (ii) is true.

D. (i) and (ii) are both false.

Answer: C

For (i), we show that if $\mathcal{C}_1, \mathcal{C}_2$ are partitions of $\mathbb{Z}$ such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\mathcal{C}_2 \subseteq \mathcal{C}_1$.

Let $S_2 \in \mathcal{C}_2$. Being a component of a partition, this S_2 must be nonempty. Take $x \in S_2$. This must be contained in a component of the partition $\mathcal{C}_1$, say S_1. Then $S_1 \in \mathcal{C}_1 \subseteq \mathcal{C}_2$. Thus $S_1 \in \mathcal{C}_2$. Now x is an element of S_1 and S_2, both of which are components of the partition $\mathcal{C}_2$. So we must have $S_2 = S_1 \in \mathcal{C}_1$.

For (ii), let $\mathcal{C}_1 = \{\mathbb{Z}\}$ and $\mathcal{C}_2 = \big\{\{2k : k \in \mathbb{Z}\}, \{2k+1 : k \in \mathbb{Z}\}\big\}$.

5. Define square root and exponentiation on $\mathbb{Z}_3$ as follows.

- For all $[x] \in \mathbb{Z}_3$, define $\sqrt{[x]}$ to be the unique $[y] \in \mathbb{Z}_3$ such that $[y] \cdot [y] = [x]$.
- For all $[x], [y] \in \mathbb{Z}_3$ with $x, y > 0$, define $[x]^{[y]} = [x^y]$.

Are square root and exponentiation well defined here?

A. Both square root and exponentiation are well defined here.

B. Square root is well defined here, but exponentiation is not.

C. Exponentiation is well defined here, but square root is not.

D. Neither square root nor exponentiation is well defined here.

Answer: D

Square root is not well defined because $[1] \cdot [1] = [1] = [4] = [2] \cdot [2]$, but $[1] \neq [2]$.

Exponentiation is not well defined because $[2] = [5]$ but $[2^2] = [4] = [1] \neq [2] = [32] = [2^5]$.

6. Which of the following is/are true?

(i) For every set U of subsets of $\mathbb{Z}$, the subset relation $\subseteq$ on U is a total order.

(ii) For all $S \subseteq \mathbb{Z}^+$, the usual order $\leqslant$ on S is a linearization of the divisibility relation $|$ on S.

A. (i) and (ii) are both true.

B. (i) is true but (ii) is false.

C. (i) is false but (ii) is true.

D. (i) and (ii) are both false.

Answer: C

A counterexample for (i) is $U = \{\{1\}, \{2\}\}$.
For (ii), note that for all $a, b \in \mathbb{Z}^+$, if $a \mid b$, then $a \leqslant b$.

7. Which of the following is/are true?

(i) Whenever $\preccurlyeq$ is a partial order on a set A, there are no $n \in \mathbb{Z}^+$ and no $c_0, c_1, \dots, c_n \in A$ such that $c_0 \prec c_1 \prec \cdots \prec c_n = c_0$.

(ii) Whenever $\preccurlyeq$ is a partial order on a set A, there are no $n \in \mathbb{Z}^+$ and no $c_0, c_1, \dots, c_n \in A$ such that $c_0 \not\succcurlyeq c_1 \not\succcurlyeq \cdots \not\succcurlyeq c_n = c_0$.

A. (i) and (ii) are both true.

B. (i) is true but (ii) is false.

C. (i) is false but (ii) is true.

D. (i) and (ii) are both false.

Answer: B

(i) is proved in line 4 of the proof of Proposition 7.4.6. For (ii), consider Example 7.3.5.

Part B: Multiple Response Questions [Total: 15 marks]

Each multiple response question (MRQ) is worth three marks and may have one answer or multiple answers. Write out **all** correct answers. For example, if you think that A, B, C are the correct answers, write A, B, C. Only if you get all the answers correct will you be awarded three marks. **No partial credit will be given for partially correct answers.**

You are advised to write your answers on a **single line** to conserve space. For example:

8 A,B   9 B,D   10 C   11 A,B,C,D   ...

Please write in **CAPITAL LETTERS**.

8. The exclusive-or operation, denoted by $\oplus$, is defined as follows:

p	q	$p \oplus q$
true	true	false
true	false	true
false	true	true
false	false	false

Given that p, q and r are statement variables, which of the following is/are true?

A. $p \oplus p \equiv q \oplus q$

B. $(p \oplus p) \oplus p \equiv (q \oplus q) \oplus q$

C. $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r)$

D. $(p \oplus {\sim}p) \oplus p \equiv (p \oplus p) \oplus {\sim}p$

Answer: A, C, D

A. $p \oplus p \equiv q \oplus q \equiv false$

B. $(p \oplus p) \oplus p \equiv p \not\equiv q \equiv (q \oplus q) \oplus q$

C. $(p \oplus q) \oplus r \equiv p \oplus (q \oplus r) \equiv ({\sim}p \wedge {\sim}q \wedge r) \vee ({\sim}p \wedge q \wedge {\sim}r) \vee (p \wedge {\sim}q \wedge {\sim}r) \vee (p \wedge q \wedge r)$ [$\oplus$ is associative]

D. $(p \oplus {\sim}p) \oplus p \equiv (p \oplus p) \oplus {\sim}p \equiv {\sim}p$

9. Let $A = \{-2, -1, 0, 1, 2\}, B = \{0, 1, 2\}$ and $C = \{-4, -3, -2\}$.

Let $|x|$ denote the absolute value of x, i.e.

$$|x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Which of the following is/are TRUE?

A. $\forall x, y \in A, \forall z \in B \; (|x - y| \leq z^2)$.

B. $\forall x \in A, \exists y \in B, \forall z \in C \; (|x - y| \geq |z|)$.

C. $\forall x, y \in C, \exists z \in B \; (|x - y| \leq z)$.

D. $\exists z \in B, \forall x, y \in C \; (|x - y| \leq z)$.

Answer: C, D

A. Counterexample: $x = -2, y = 2, z = 0$. Then $|-2-2| = 4 \nleq 0 = 0^2$.

B. Counterexample: If $x = 0$ and $z = -4$, then there is no y that makes $(|x-y| \geq |z|)$, or $|y| \geq |-4|$.

C. For any $x, y \in C$, we may pick $z = 2$ to satisfy $|x-y| \leq z$.

D. Let $z = 2$, then $|x-y| \leq 2\ \forall x, y \in C$.

10. Let the domain of discourse be this set $S = \{1,2,4,8,16,32,64\}$ and define $P(x,y)$ and $Q(x,y)$ as follows:

$$P(x,y): xy = x$$

$$Q(x,y): x|y$$

where $x|y$ means "x divides y"; in other words, $y = kx$ for some $k \in \mathbb{Z}$.

Which of the following is/are TRUE?

A. $\forall x\ \forall y\ P(x,y) \equiv \forall x\ \forall y\ Q(x,y)$

B. $\forall x\ \exists y\ P(x,y) \equiv \forall x\ \exists y\ Q(x,y)$

C. $\exists x\ \forall y\ P(x,y) \equiv \exists x\ \forall y\ Q(x,y)$

D. $\exists x\ \exists y\ P(x,y) \equiv \exists x\ \exists y\ Q(x,y)$

Answer: A, B, D

A. Both $\forall x\ \forall y\ P(x,y)$ and $\forall x\ \forall y\ Q(x,y)$ are false. Counterexample: $x = 4, y = 2$.

B. Both $\forall x\ \exists y\ P(x,y)$ and $\forall x\ \exists y\ Q(x,y)$ are true. For $\forall x\ \exists y\ P(x,y)$,the required y is 1. For $\forall x\ \exists y\ Q(x,y)$, the required y is x.

C. $\exists x\ \forall y\ P(x,y)$ is false; counterexample: $y = 2$. $\exists x\ \forall y\ Q(x,y)$ is true; the required x is 1.

D. Both $\exists x\ \exists y\ P(x,y)$ and $\exists x\ \exists y\ Q(x,y)$ are true. Example, $x = y = 1$.

11. Consider the congruence-mod-12 relation on $\mathbb{Z}$, i.e., the equivalence relation $\sim$ on $\mathbb{Z}$ satisfying, for all $x, y \in \mathbb{Z}$,

$$x \sim y \quad \Leftrightarrow \quad x \equiv y \pmod{12}.$$

Which of the following is/are equal to $[6] + [9]$?

A. $[-15]$.

B. $[1]$.

C. $[3]$.

D. $[15]$.

E. $[27]$.

Answer: C, D, E

$[6] + [9] = [15] = [3] = [27]$. This is equal to neither $[-15]$ nor $[1]$.

12. Let $A = \{1,2,3,4,5\}$. Consider the partial order

$$R = \{(x,x) : x \in A\} \cup \{(1,4),(1,5),(2,1),(2,4),(2,5),(3,1),(3,4),(3,5)\}$$

on A. Which of the following is/are true with respect to this partial order?

A. 1 is a minimal element.

B. 1 is a smallest element.

C. 2 is a minimal element.

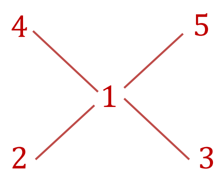
D. 2 is a smallest element.

E. 3 is a minimal element.

F. 3 is a smallest element.

Answer: C, E

A Hasse diagram:



Part C: There are 4 questions in this part [Total: 18 marks]

13. Given the following argument, where p, q, r and s are statement variables, determine whether the argument is valid or invalid. Explain your answer with working. (Answer with no explanation will not earn any mark.) [3 marks]

$$(p \vee q) \rightarrow r$$
$$(q \wedge r) \rightarrow (p \vee s)$$
$$(p \vee \sim r \vee s) \rightarrow q$$
$$\therefore (q \vee s) \rightarrow p$$

Answer: **It is not valid.** Counterexample: $p = false, q = r = s = true$.

Explanation:

$$(p \vee q) \rightarrow r \equiv (false \vee true) \rightarrow true \equiv true \rightarrow true \equiv true$$

$$(q \wedge r) \rightarrow (p \vee s) \equiv (true \wedge true) \rightarrow (false \vee true) \equiv true \rightarrow true \equiv true$$

$$(p \vee \sim r \vee s) \rightarrow q \equiv ((false \vee false) \vee true) \rightarrow true \equiv (false \vee true) \rightarrow true \equiv true \rightarrow true \equiv true$$

$$(q \vee s) \rightarrow p \equiv (true \vee true) \rightarrow false \equiv true \rightarrow false \equiv false$$

14. An integer is either even or odd, but not both. A **perfect square** is an integer that is a square of some integer (eg: 1, 4, 9, 16, 25). An **odd perfect square** is a perfect square that is odd (eg; 1, 9, 25).

    You are given the following three theorems T1, T2 and T3 which you may quote in your answer without proof. You have proved T1 in tutorial 1 question 10.

    $\forall n \in \mathbb{Z}, n^2$ is odd if and only if n is odd. (T1)

    $\forall n \in \mathbb{Z}, n^2$ is even if and only if n is even. (T2)

    The sum of two odd integers is even. (T3)

    Prove the following claim, justifying your steps wherever appropriate:

    The sum of two odd perfect squares is never a perfect square. [4 marks]

Answer: Proof by contradiction.

1. Suppose to the contrary, that there are 2 odd perfect squares x and y such that $x + y = z$ is a perfect square.
2. Let $x = a^2, y = b^2$, and $z = c^2$, for some $a, b, c \in \mathbb{Z}$.
3. Since x and y are odd, so are a and b (by T1).
4. Hence, $a = 2k + 1$ and $b = 2m + 1$ for some $k, m \in \mathbb{Z}$ (by definition of odd numbers).
5. Moreover, since x and y are odd, z must be even (by T3).
6. Since $z = c^2$ is even, therefore c is also even (by T2).
7. Hence, $c = 2n$ for some $n \in \mathbb{Z}$ (by definition of even numbers).
8. Substituting (4) and (7) into $x + y = z$, we have

   $(2k + 1)^2 + (2m + 1)^2 = (2n)^2$

   $4k^2 + 4k + 1 + 4m^2 + 4m + 1 = 4n^2$

   $4(k^2 + k + m^2 + m) + 2 = 4n^2$

Alternatives for step 9:

9. Dividing both sides by 2, we have: $2(k^2 + k + m^2 + m) + 1 = 2n^2$.
   Since $(k^2 + k + m^2 + m) \in \mathbb{Z}$ (by closure of integers under $\times$ and $+$), LHS is odd (by definition of odd numbers) and RHS is even (by definition of even numbers), hence contradiction.

9. Dividing both sides by 4, we have: $(k^2 + k + m^2 + m) + \frac{1}{2} = n^2$.
   Since $(k^2 + k + m^2 + m) \in \mathbb{Z}$ (by closure of integers under $\times$ and $+$), LHS $\notin \mathbb{Z}$ and RHS $\in \mathbb{Z}$, hence contradiction.

10. So, the suppositionthat $x + y = z$ is a perfect square is false.
11. Therefore, the sum of two odd perfect squares is never a perfect square.

15. Consider the equivalence relation $\sim$ on $\mathcal{P}(\{1,2,3\})$ defined by setting
$$A \sim B \quad \Leftrightarrow \quad |A| = |B|$$
for all $A, B \in \mathcal{P}(\{1,2,3\})$. Write down in roster notation **all** the equivalence classes. No working is required. [3 marks]

Answer: $\{\{\}\}, \{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{2,3\},\{1,3\}\}, \{\{1,2,3\}\}$

16. Let R be the relation on $\mathbb{Q}$ satisfying, for all $x, y \in \mathbb{Q}$,
$$x\ R\ y \quad \Leftrightarrow \quad xy \in \mathbb{Z}.$$

(a) Is R reflexive?

(b) Is R symmetric?

(c) Is R antisymmetric?

(d) Is R transitive?

For each of the questions above, if you answer yes, then prove your claim; if you answer no, then give a counterexample (and no further explanation is needed). [8 marks]

Answer:

(a) No. Since $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \notin \mathbb{Z}$, we know $\left(\frac{1}{2} \not{R}\ \frac{1}{2}\right)$.

(b) Yes, as shown below.

1. Let $x, y \in \mathbb{Z}$ such that $x\ R\ y$.
2. Then $xy \in \mathbb{Z}$ by the definition of R.
3. So $yx = xy \in \mathbb{Z}$ too.
4. This tells us $y\ R\ x$ by the definition of R.

(c) No. Since $2 \cdot \frac{1}{2} = 1 \in \mathbb{Z}$ and $\frac{1}{2} \cdot 2 = 1 \in \mathbb{Z}$, we know $2\ R\ \frac{1}{2}$ and $\frac{1}{2}\ R\ 2$, but $2 \neq \frac{1}{2}$.

(d) No. Since $\frac{1}{2} \cdot 2 = 1 \in \mathbb{Z}$ and $2 \cdot \frac{1}{2} = 1 \in \mathbb{Z}$, we know $\frac{1}{2}\ R\ 2$ and $2\ R\ \frac{1}{2}$. However, since $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \notin \mathbb{Z}$, we have $\left(\frac{1}{2} \not{R}\ \frac{1}{2}\right)$.

=== END OF PAPER ===