Feasibility of Machine Learning

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1 Introduction

In this note, we are going to address the feasibility of learning from probability side of perspective. We will show how limited known data set $\mathbb{D}$ reveal enough information about the unknown target function f. Before the discussion of the feasibility of learning we need to introduce the very most important **Hoeffding Inequality** first.

2 Hoeffding Inequality

For any sample size N

$$P\left(|\overline{X} - E(\overline{X})| > \epsilon\right) \leq 2e^{-2\epsilon^2 N} \qquad , for\, any\, \epsilon > 0 \tag{1}$$

Here is what each notation means

- N : Sample size
- $P()$: Probability of an event
- $\overline{X}$: $\dfrac{X_1 + \cdots + X_N}{N}$ $\qquad$ X_i is i.i.d random variable
- $E(\overline{X})$: Expectation of $\overline{X}$
- ϵ : Any positive value that we chose

The inequality above says that, as long as N gets large enough $\overline{X}$ will approximate to $E(\overline{X})$. That is we can infer $E(\overline{X})$ by $\overline{X}$. We are going to use this inequality to explain why machine learning is feasible.

3 Applying Hoeffding Inequality to Machine Learning Feasibility Problem

We first define In-sample error and out-of-sample error which are corresponding to $\overline{X}$ and $E(\overline{X})$ respectively.

In-sample error

$$E_{in}(h) = \frac{1}{N}\sum_{i=1}^{N}\left[isTrue\left(h(x_i) \neq f(x_i)\right)\right] \tag{2}$$

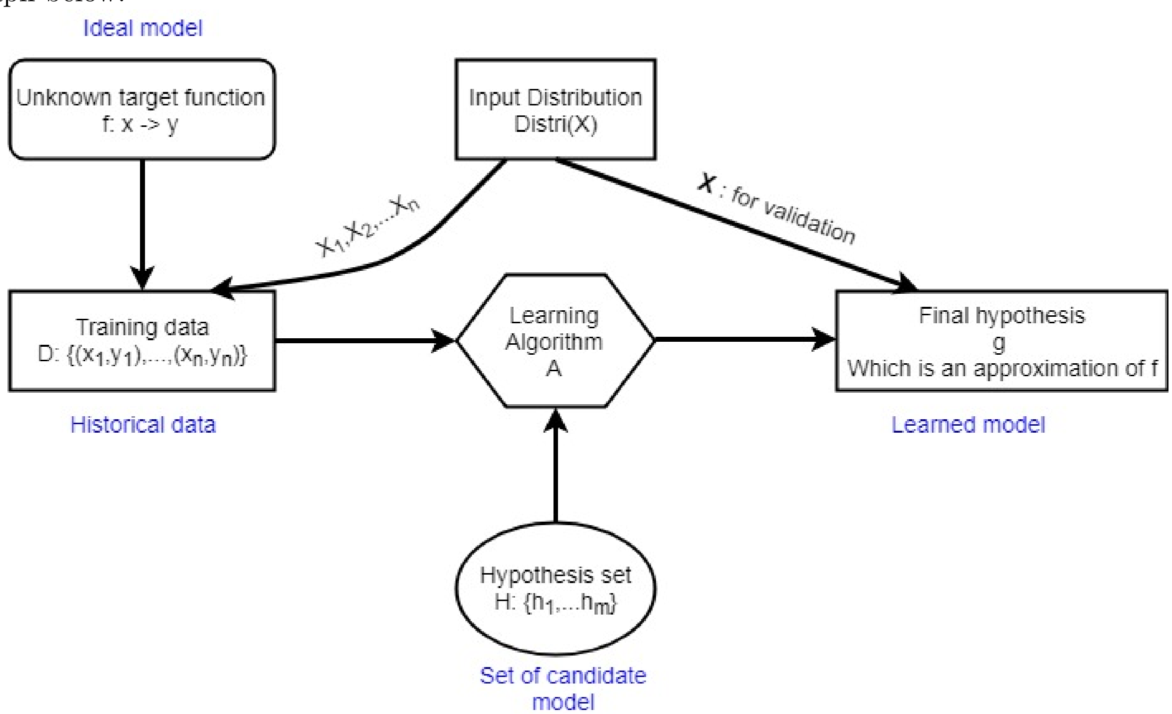
Out-of-sample error

$$E_{out}(h) = P[h(x) \neq f(x)] \tag{3}$$

We plug them into Hoeffding Inequality and get

$$P\left(|E_{in}(h) - E_{out}(h)| > \epsilon\right) \leq 2e^{-2\epsilon^2 N} \qquad , for\, any\, \epsilon > 0 \tag{4}$$

As of the original Hoeffding Inequality, we can say that $E_{in}(h) \rightarrow E_{out}(h)$ as N increase. But, before we can state this conclusion, we must ensure that the sampling data(training data) $\mathbb{D}$ fulfill some requirements. For the reason that the random variables$(X_1, X2, \cdots, X_N)$ that we used to calculate $\overline{X}$ are all identical independent random variables(i.i.d). So, the training data of our machine learning model need to follow this requirement as well. To generate a training data set with i.i.d random variables, we need an input distribution **Distri(X)** which can produce i.i.d $(X_1, X2, \cdots, X_N)$ for us. The function of this input distribution is illustrated in the structure graph below.



Now we can say that $E_{in}(h) \approx E_{out}(h)$ as N gets large enough. But it doesn't implies that $h \approx f$. What we need to do now is to find out a g from the hypothesis set $\mathbb{H}$, so that $E_{in}(g)$ is as small as possible. Then, we have $E_{in}(g) \approx 0 \rightarrow E_{out}(g) \approx 0$. By the definition of $E_{out}(g)$, we know that $g \approx f$ if $E_{out}(g) \approx 0$.

Here, the feasibility of machine learning is thus split into two questions.

1. How to make $E_{in}(g)$ close enough to $E_{out}(g)$.

2. How to make $E_{in}(g)$ as small as possible.

But, There are still some problems that we need to clarify about g. The probability upper bound of g is larger than h for the reason that we manually chose it from the hypothesis set. This manual selection make g easier to become a poor estimator. We will prove this in the following section.

4 Problems Caused by the Size of Hypothesis Set

We start by introducing to basic probability rules.

$$\textbf{Rule 1.} \qquad If A \Rightarrow B, then P(A) \leq P(B)$$
$$\textbf{Rule 2.} \qquad P(A_1|A_2|\cdots|A_m) \leq P(A_1) + P(A_2) + \cdots + P(A_m)$$

Now, we are going to illustrate the problem caused by manual selection of g. Suppose there are m hypothesis in $\mathbb{H}$. Then we have

$$\begin{array}{ll} P & (|E_{in}(h_1) - E_{out}(h_1)| > \epsilon) \leq 2e^{-2\epsilon^2 N} \qquad , for\, any\, \epsilon > 0 \\ P & (|E_{in}(h_2) - E_{out}(h_2)| > \epsilon) \leq 2e^{-2\epsilon^2 N} \qquad , for\, any\, \epsilon > 0 \\ \cdots & \\ P & (|E_{in}(h_m) - E_{out}(h_m)| > \epsilon) \leq 2e^{-2\epsilon^2 N} \qquad , for\, any\, \epsilon > 0 \end{array}$$

Because g is one of h_i we have

$$\begin{array}{rll} |E_{in}(g) - E_{out}(g)| > \epsilon \Rightarrow & \text{"} & |E_{in}(h_1) - E_{out}(h_1)| > \epsilon \\ & or & |E_{in}(h_2) - E_{out}(h_2)| > \epsilon \\ & \cdots & \\ & or & |E_{in}(h_m) - E_{out}(h_m)| > \epsilon\text{"} \end{array}$$

By applying Rule 1 and Rule 2, we easily get

$$\begin{aligned} P[|E_{in}(g) - E_{out}(g)| > \epsilon] &\leq \sum_{i=1}^{m} P[|E_{in}(h_i) - E_{out}(h_i)| > \epsilon] && (5) \\ &\leq 2me^{-2\epsilon^2 N} && (6) \end{aligned}$$

As is clearly shown on equation (6), The upper bound of g increase as the size hypothesis set $\mathbb{H}$ increase. Here comes the dilemma of machine learning. If we want $E_{in}(g)$ to be smaller, we need to increase the size of $\mathbb{H}$. But this will make $E_{in}(g)$ deviate from $E_{out}(g)$. If we want $E_{in}(g)$ to be closer to $E_{out}(g)$, we need to decrease the size of $\mathbb{H}$. But this will make $E_{in}(g)$ gets larger and lead to the deviation of g from f. We will discuss how to balance the two sides of a coin in later notes.

5 References

Almost all of the materials of this note are from Professor Hsuan-Tien Lin , NTU. If you wan to know more information about Machine Learning Foundation, please refer to Professor Lin's homesite.