Name: ______________________________ **Perm Number:** ______________________________

MATH 108B **Fall 2025**

Midterm **Time Limit: 75 Minutes**

- **DO NOT open the exam booklet until you are told to begin. You should write your name and perm number at the top and read the instructions.**
- Show and organize all of your work in a neat and coherent way in the space provided. If you wish for something to not be graded, please strike it out or erase neatly. We will only grade work that is written on the exam paper.
- You may only use the notes you have written down on a 3” by 5” note card. Do not use any book, lecture notes, and/or any other study guides you have created. Do not use a calculator, cell phone, computer, or any other smart device.
- If you have any questions, please raise your hand, and I will come over to assist you.
- When you have completed your test, hand it to the front, and enjoy the rest of your day!

Problem	Points	Score
1	10	
2	8	
3	10	
4	10	
5	6	
6	6	
Total:	50	

1. (10 points) Determine whether the following statements are true or false. Write only **True** or **False**. Do not provide justification for your answer.

   (i) Let A be a 4×4 matrix over $\mathbb{R}$ such that A has an eigenvalue 2. Then one of the eigenvalues of A^3 is 8.

   (ii) Let V be an inner product space over $\mathbb{C}$. Let x and y be two non-zero orthogonal vectors, then x and y are linearly independent vectors.

   (iii) Let V be an inner product space over $\mathbb{C}$. Suppose there exist a vector x such that $\langle x, y\rangle = \langle x, z\rangle$ for all $y, z \in V$. Then $x = 0$.

   (iv) The sum of any two eigenvectors of an operator T is always an eigenvector of T.

   (v) Let $A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \in M_{2\times 2}(\mathbb{C})$. Then $A^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

2. (8 points) (a) [4 points] Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(b) [4 points] Let $V = M_{2\times 2}(\mathbb{R})$. Is the matrix

$$\begin{pmatrix} 0 & 2025 \\ 2025 & 0 \end{pmatrix}$$

diagonalizable? Justify your answer.

3. (10 points) Let $V = P_1(\mathbb{R})$ be the two dimensional vector space of all polynomial of degree at most 1. Let $T : V \longrightarrow V$ be a linear operator given by $T(ax + b) = (-6a + 2b)x + (-6a + b)$. Find eigenvalues of T and an ordered basis β such that $[T]_\beta$ is diagonal.

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4. (10 points) (i) [4 points] Provide reason why the following function $\langle , \rangle : V \times V \longrightarrow \mathbb{R}$ defined as

$$\langle A , B \rangle = tr(A + B),$$

is not an inner product space on $V = M_{2\times 2}(\mathbb{R})$, where $tr(A)$ denotes the trace of the matrix A.

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(ii) [6 points] Let V be an inner product space, and suppose that x and y are two orthogonal vectors in V. Prove that

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

5. (6 points) Let $V = \mathbb{R}^4$. Let

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

be a set of linearly indepedent vectors. Use the Gram-Schmidt process to compute the orthogonal vectors v_1, v_2 and v_3 such that $\mathrm{Span}(S) = \mathrm{Span}\left(\{v_1, v_2, v_3\}\right)$.

6. (6 points) Let V be an inner product space, and let T and U be linear operator on V. Show that

$$(T + U)^* = T^* + U^*.$$

Extra Space

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