**Rk.** Let $U, V$ be f.d.v.s. over $F$ with $\dim(U) = n = \dim(V)$.

Let $\alpha = (e_1, \dots, e_n)$, $\beta = (f_1, \dots, f_n)$ be ordered bases of $U, V$ respectively.

Let $T : U \to V$ be a linear map. Let $P = {}_\beta[T]_\alpha$. Then

$$T \text{ is injective} \Leftrightarrow \operatorname{rank}(T) = n \Leftrightarrow T \text{ is surjective},$$

so $\quad T$ is an isomorphism $\Leftrightarrow \operatorname{rank}(T) = n \quad (= \operatorname{rank} P)$.

In this case

$$P\,{}_\alpha[T^{-1}]_\beta = {}_\beta[\mathbb{1}_V]_\beta = I_n = {}_\alpha[\mathbb{1}_U]_\alpha = {}_\alpha[T^{-1}]_\beta\, P,$$

so $\quad {}_\alpha[T^{-1}]_\beta = P^{-1}$.

**Theorem** (Normal form) Let $A \in M_{m\times n}(F)$ and let $r = \operatorname{rank}(A)$. Then there exist invertible matrices $P \in M_{m\times m}(F)$ and $Q \in M_{n\times n}(F)$ such that

$$PAQ = \underbrace{\begin{bmatrix} \underbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}}_{r} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ \ddots \\ 1 \end{matrix}}\right\} r & \\ & \\ & \end{bmatrix}}_{n} \left.\vphantom{\begin{bmatrix} 1 \\ 1 \\ \ddots \\ 1 \\ \\ \\ \\ \end{bmatrix}}\right\} m$$

has $(i,i)$-entry $1$ for $i = 1, 2, \dots, r$ and all other entries zero.

*Proof.* Let $U = F^n$, $V = F^m$. Let $\alpha$, $\beta$ be the standard ordered bases for $U$, $V$ respectively. Let $I_A : U \to V$ be the (unique) linear map with ${}_{\beta}[I_A]_{\alpha} = A$. Let $r = \operatorname{rank}(I_A)$, $s = \operatorname{nullity}(I_A)$. By Rank-Nullity Theorem, $r + s = n$. Let $\{g_{r+1}, \dots, g_n\}$ be a basis for $\ker(I_A)$. Extend it to form an ordered basis for $U$: $\alpha' = (g_1, \dots, g_r, g_{r+1}, \dots, g_n)$.

For $i = 1, \dots, r$, let $h_i = I_A(g_i)$. As seen in the proof of the Rank-Nullity theorem, $(h_1, \dots, h_r)$ is an ordered basis for $\operatorname{im}(I_A)$. Extend it to an ordered basis for $V$: $\beta' = (h_1, \dots, h_r, h_{r+1}, \dots, h_m)$.

Then

$$
{}_{\beta'}[I_A]_{\alpha'} = \left[\begin{array}{cc} \underbrace{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}}_{r} \Bigg\}\, r & \\ & \end{array}\right] \quad \left( = \Big[\, [I_A(g_1)]_{\beta'} \ \cdots \ [I_A(g_n)]_{\beta'} \,\Big] \right)
$$

has $(i,i)$-entry $1$ and all other entries zero.

Let $P = {}_{\beta'}[\mathbb{1}_V]_{\beta}$ (change of basis matrix from $\beta$ to $\beta'$)

$Q = {}_{\alpha}[\mathbb{1}_U]_{\alpha'}$ (change of basis matrix from $\alpha'$ to $\alpha$)

Then $PAQ = {}_{\beta'}[\mathbb{1}_V]_{\beta}\, {}_{\beta}[I_A]_{\alpha}\, {}_{\alpha}[\mathbb{1}_U]_{\alpha'} = {}_{\beta'}[I_A]_{\alpha'}$. $\square$

__Aside__ Given $A \in M_{m \times n}(F)$, there exists an invertible matrix $R \in M_m(F)$ such that $RA$ is in __reduced row echelon form__ (rref). The rref is uniquely determined by $A$.

__Products__

Let $U_1, U_2, \dots, U_k$ be $F$-vector spaces. Let $V = U_1 \times U_2 \times \cdots \times U_k$ (the Cartesian product of $U_1, U_2, \dots, U_k$).

Given $(u_1, \dots, u_k), (v_1, \dots, v_k) \in V$ and $\lambda \in F$, we define

- $(u_1, \dots, u_k) + (v_1, \dots, v_k) = (u_1 + v_1, \dots, u_k + v_k)$
- $\lambda(u_1, \dots, u_k) = (\lambda u_1, \dots, \lambda u_k)$

Under these operations, $V$ is an $F$-vector space, called the __direct product__ of $U_1, U_2, \dots, U_k$.

There are canonical projection maps

$$\begin{aligned} \pi_i : V &\longrightarrow U_i \\ (u_1, \dots, u_k) &\longmapsto u_i \end{aligned}$$

and canonical inclusion maps

$$\begin{aligned} \iota_i : U_i &\longrightarrow V \\ x &\longmapsto (0_{U_1}, \dots, 0_{U_{i-1}}, x, 0_{U_{i+1}}, \dots, 0_{U_k}). \end{aligned}$$

**Prop.** With notations as above,

$$V = \bar{U}_1 \oplus \cdots \oplus \bar{U}_k, \quad \text{where} \quad \bar{U}_i := \iota_i(U_i).$$

**Proof.** Given $x = (u_1, \ldots, u_k) \in V$, we can write

$$x = \iota_1(u_1) + \cdots + \iota_k(u_k),$$

so $V = \bar{U}_1 + \cdots + \bar{U}_k$. Let $i \in \{1, \ldots, k\}$. Since elements of $\sum_{j \neq i} \bar{U}_j$ have $i$-th component equal to $0_{U_i}$ while elements of $\bar{U}_i$ have $j$-th component equal to $0_{U_j}$ for all $j \neq i$, we have

$$\bar{U}_i \cap \Big(\sum_{j \neq i} \bar{U}_j\Big) = \{0_V\} \qquad (0_V = (0_{U_1}, \ldots, 0_{U_k}))$$

So $V = \bar{U}_1 \oplus \cdots \oplus \bar{U}_k$ by the direct sum criterion. $\square$

**Rk** $V = U_1 \times \cdots \times U_k$ is also called the <u>external direct sum</u> of $U_1, \ldots, U_k$. However, the notion of direct product and external direct sum differ for an infinite set of vector spaces.

# Quotients

Let $V$ be an $\mathbb{F}$-v.s. and let $U$ be a subspace. We define a relation $\sim$ on $V$ by declaring that for all $x, y \in V$,

$$x \sim y \quad \text{if and only if} \quad x - y \in U.$$

**Lemma.** This is an equivalence relation on $V$.

*Proof.*

- Reflexivity. Let $x \in V$. Since $x - x = 0 \in U$, we have $x \sim x$.
- Symmetry. Let $x, y \in V$ and suppose $x \sim y$. Then $x - y \in U$. Since $U$ is a subspace, $y - x = -1(x-y) \in U$. Therefore $y \sim x$.
- Transitivity. Let $x, y, z \in V$ and suppose $x \sim y$ and $y \sim z$. Then $x - y \in U$, $y - z \in U$. Since $U$ is a subspace, $x - z = (x-y) + (y-z) \in U$. So $x \sim z$. $\square$

Given $x \in V$, the equivalence class of $x$ with respect to $\sim$ is the set

$$[x]_U = x + U = \{ x + y \mid y \in U \}.$$

We may also denote it by $[x]$ or $\bar{x}$ if $U$ is clear from context.

We denote by $V/U$ the quotient set of $V$ by $\sim$, i.e., the set of equivalence classes wrt $\sim$:
$$V/U = \{ [x] \mid x \in V \}$$

Note that, for any $x, x' \in V$, $[x] = [x']$ if and only if $x - x' \in U$.

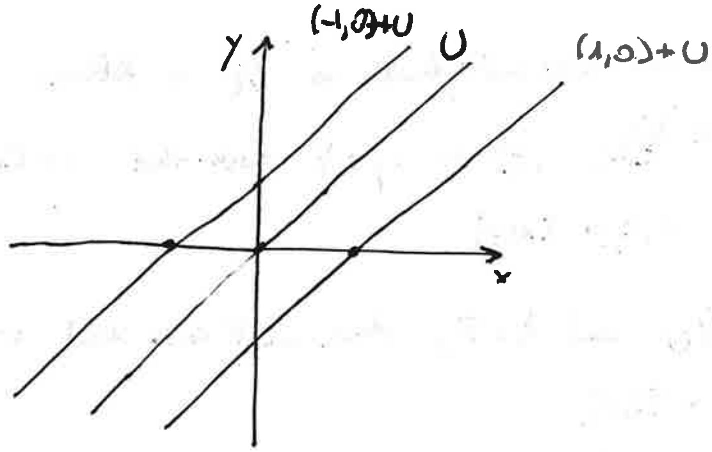
The map

$$\pi : V \longrightarrow V/U$$
$$x \longmapsto [x]$$

is called the <u>canonical quotient map</u>.

<u>Example</u>. Let $V = \mathbb{R}^2$, $U = \{(a,b) \in V \mid a = b\}$. Then

- $(1,0) + U = \{(\alpha+1, \alpha) \mid \alpha \in \mathbb{R}\} = (0,-1) + U$
- $(0,0) + U = \{(\alpha,\alpha) \mid \alpha \in \mathbb{R}\} = U$
- $(-1,0) + U = \{(\alpha-1,\alpha) \mid \alpha \in \mathbb{R}\}$.




$V/U$ is the set of lines in $\mathbb{R}^2$ parallel to $U$

Since each of these lines intersects the x-axis at a unique point, $V/U$ itself "looks like" a line, i.e., a 1-dimensional v.s. over $\mathbb{R}$.