Lecture Inner Products and Norms

📌 **Progress Update**

- ✅ We've completed Sections **5.1** and **5.2** from Chapter 5.
- 🔁 We will return to **Section 5.4** (*Invariant Subspaces and the Cayley-Hamilton Theorem*) before diving into Chapter 7 on Canonical Forms, as the material will be more relevant at that point.

📚 **Next Topic: Chapter 6 – Inner Product Spaces**

Over the next few weeks, we will explore:

- 📏 Inner product spaces
- 📐 Orthogonal projections
- 📉 Least-squares approximations
- 🧮 Selected topics in applied linear algebra

In **Chapter 6,** all vector spaces are over $\mathbb{R}$ or $\mathbb{C}$ unless otherwise stated.

- Many applications of mathematics involve **measurement**—understanding magnitude or relative size.
- We define **distance and length** in vector spaces using **inner product spaces**, which generalize the dot product from multivariable calculus.
- This allows us to extend geometric ideas like **angle**, **length**, and **perpendicularity** from $\mathbb{R}^2$ and $\mathbb{R}^3$ to more general real and complex vector spaces.
- A **Euclidean vector space** is a finite-dimensional inner product space over the real numbers—keep $\mathbb{R}^n$ in mind as the basic example throughout this chapter.
- Similarly, $\mathbb{C}^n$ with the standard Hermitian inner product serves as the basic example for inner product spaces over $\mathbb{C}$.

Definition. Inner Product

Let V be a vector space over F (either $\mathbb{R}$ or $\mathbb{C}$). An **inner product** on V is a function that assigns to each ordered pair of vectors $x, y \in V$ a scalar $\langle x, y\rangle \in F$, satisfying the following properties for all $x, y, z \in V$ and all scalars $c \in F$:

$\langle\ ,\ \rangle : V \times V \longrightarrow F$

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$.

(b) $\langle cx, y\rangle = c\langle x, y\rangle$.

(c) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ (where the bar denotes complex conjugation).

(d) If $x \neq 0$, then $\langle x, x\rangle$ is a **positive real number**.

Remarks.

- Condition **(c)** simplifies to $\langle x, y\rangle = \langle y, x\rangle$ when $F = \mathbb{R}$.
- Conditions **(a)** and **(b)** state that the inner product is **linear in the first component**.
- It is easy to show that if $a_1, a_2, \ldots, a_n \in F$ and $y, v_1, v_2, \ldots, v_n \in V$, then

$$\left\langle \sum_{i=1}^{n} a_i v_i, y \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, y\rangle.$$

Example 1. Standard Inner Product on F^n

Recall we assume $F = \mathbb{R}$ or $\mathbb{C}$ in this chapter.

For $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in F^n, define

$$\langle x, y\rangle = \sum_{i=1}^{n} a_i \bar{b}_i$$

We check $<,>$ defined above satisfies the properties in the def:

Let $x = (a_1, a_2, \cdots, a_n)$, $y = (b_1, b_2, \cdots, b_n)$, $z = (c_1, c_2, \cdots, c_n)$.

(a) Then $\langle x+z, y\rangle = \langle a_1+c_1, a_2+c_2, \cdots, a_n+c_n, b_1, b_2, \cdots, b_n\rangle$

$$= \sum_{i=1}^{n} (a_i + c_i)\bar{b}_i = \sum_{i=1}^{n} a_i \bar{b}_i + \sum_{i=1}^{n} c_i \bar{b}_i$$

$$= \langle x, y\rangle + \langle z, y\rangle$$

(b) $\langle cx, y\rangle = \sum_{i=1}^{n} c a_i \bar{b}_i = c \sum_{i=1}^{n} a_i \bar{b}_i = c\langle x, y\rangle$

(c) $\langle x, y\rangle = \sum_{i=1}^{n} a_i \overline{b_i} = \sum_{i=1}^{n} \overline{\overline{a_i}}\, \overline{b_i} = \overline{\sum_{i=1}^{n} b_i \overline{a_i}} = \overline{\langle y, x\rangle}$

(d) We assume $F = \mathbb{C}$, and

$$a_j = s_j + i t_j, \text{ for some } s_j, t_j \in \mathbb{R}, \quad \forall j = 1, \cdots, n.$$

Then $a_j \overline{a_j} = (s_j + i t_j)(s_j - i t_j) = s_j^2 + t_j^2, \ \forall j$

If $x \neq \vec{0}$, then not all s_i, t_j are 0's. Thus

$$\langle x, x\rangle = \sum_{i=1}^{n} a_i \overline{a_i} = \sum_{i=1}^{n} (s_i^2 + t_j^2) > 0$$

and it is a real number.

In general, if $a = s + it \in \mathbb{C}$, then

$$a \overline{a} = (s+it)(s-it) = s^2 + t^2 = |a|^2$$

where $|a|$ is the length/norm of $a \in \mathbb{C}$ in the usual sense.

Note: We might have two distinct inner product spaces with the same underlying vector space, as Example 2 shows.

Example 2. Scaling an Inner Product

If $\langle x, y \rangle$ is an inner product on a vector space V and $r > 0$, then we can define a new inner product by

$$\langle x, y \rangle' = r\langle x, y \rangle.$$

However, if $r \leq 0$, condition **(d)** would fail, so the result would not be a valid inner product.

The verification of $\langle x, y \rangle'$ is an inner product is straightforward.

E.g., for property (a):

$$\langle x+z, y \rangle' = r\langle x+z, y \rangle = r\langle x, y \rangle + r\langle z, y \rangle = \langle x, y \rangle' + \langle z, y \rangle'$$

Example 3. Inner Product on Function Space $C([0,1])$

Let $V = C([0,1])$, the vector space of real-valued continuous functions on $[0,1]$. For $f, g \in V$, define

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt.$$

- Since the integral is linear in f, properties **(a)** and **(b)** follow immediately.
- Property **(c)** is also clear, as $\langle f, g \rangle = \langle g, f \rangle$.
- If $f \neq 0$, then by continuity, f^2 is bounded away from zero on some subinterval of $[0,1]$, so

$$\langle f, f \rangle = \int_0^1 [f(t)]^2\,dt > 0.$$

Thus $A^* = \overline{A^t} = \bar{A}^t$

Definition. Conjugate Transpose / Adjoint

Let $A \in \mathrm{M}_{m\times n}(F)$. We define the conjugate transpose or adjoint of A to be the $n \times m$ matrix A^* such that $(A^*)_{ij} = \bar{A}_{ji}$ for all i, j.

Eg: If $A = \begin{bmatrix} 2 & 3+i \\ 6+2i & 5+3i \end{bmatrix}$, then

$$A^* = \begin{bmatrix} 2 & 6-2i \\ 3-i & 5-3i \end{bmatrix}$$

Exercise 4. Frobenius Inner Product

Let $V = M_{n\times n}(F)$, and define $\langle A, B\rangle = \mathrm{tr}(B^* A)$ for $A, B \in V$. (Recall that the trace of a matrix A is given by $\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}$.)

Verify that this defines an inner product on V.

Let $A, B, C \in V$, $c \in F$. Then

(a) $\langle A+B, C\rangle = tr(C^*(A+B)) = tr(C^*A + C^*B)$

$= tr(C^*A) + tr(C^*B)$

$= \langle A, C\rangle + \langle B, C\rangle$

(b) $\langle cA, B\rangle = tr(B^*(cA)) = c\, tr(B^*A) = c\langle A, B\rangle$

(c) $\langle A, B\rangle = tr(B^*A) = tr(\overline{B^t} A) = tr\left((\overline{B^t} A)^t\right) = tr(\overline{A^t} B^{tt})$

$= \overline{tr(A^*B)} = \overline{\langle B, A\rangle}$

(d) $\langle A, A\rangle = tr(A^*A) = \sum_{i=1}^{n} (A^*A)_{ii}$

$= \sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik} A_{ki}$

Recall

$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} \overline{A}_{ki} A_{ki}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2$$

If $A \neq O_{n\times n}$, then $A_{ki} \neq 0$ for some $k, i \in \{1, \cdots, n\}$.

Thus $\langle A, A \rangle = \sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ki}|^2 > 0$.

Recall some basic properties of the trace :

Let $A, B, C \in M_{n\times n}(F)$ and $c \in F$, then

(1) Linearity :

$$tr(cA+B) = c\,tr(A) + tr(B)$$

(2) Invariant under transposition :

$$tr(A) = tr(A^t)$$

(3) Trace of matrix product :

$$tr(AB) = tr(BA)$$

(4) Cyclic Property

$$tr(ABC) = tr(BCA) = tr(CAB)$$

Theorem 6.1.

Let V be an inner product space. Then for all $x, y, z \in V$ and $c \in F$, the following hold:

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \overline{c}\langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$

(d) $\langle x, x \rangle = 0$ if and only if $x = 0$

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$

Remark.

From **(a)** and **(b)**, we see that the inner product is **conjugate-linear in the second component**.

A function $f : V \to W$ is called **conjugate-linear** (or **antilinear**) if for all $x, y \in V$ and $c \in F$, it satisfies:

- $f(x+y) = f(x) + f(y)$
- $f(cx) = \overline{c}\, f(x)$

This contrasts with linearity, where the scalar factor remains unchanged.

Proof:

(a) $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle}$

$= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$

$= \langle x, y \rangle + \langle x, z \rangle$

(b) $\langle x, cy \rangle = \overline{\langle cy, x \rangle} = \overline{c\langle y, x \rangle} = \overline{c}\, \overline{\langle y, x \rangle} = \overline{c}\langle x, y \rangle$

(c) $\langle x, 0 \rangle = \langle x, x-x \rangle = \langle x, x \rangle - \langle x, x \rangle = 0$

Similarly $\langle 0, x \rangle = 0$

(d) "$\Rightarrow$" The proof follows by taking the contrapositive of the property (d) in the def. of inner product.

"$\Leftarrow$" If $x = 0$, then $\langle 0, 0 \rangle = 0$ by part (c) above.

(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$,

then

$$0 = \langle x, y \rangle - \langle x, z \rangle = \langle x, y - z \rangle$$

In particular, choose $x = y - z$, then

$$0 = \langle y - z, y - z \rangle$$

By part (d), we know $y - z = 0$

So $y = z$.

Recall that if $x = (a, b, c) \in \mathbb{R}^3$, then the length of x is given by

$$\|x\| = \sqrt{a^2 + b^2 + c^2} = \sqrt{\langle x, x \rangle}.$$

We now generalize this definition of length to any inner product space.

Definition. (Norm / Length)

Let V be an inner product space. For $x \in V$, the **norm** or **length** of x is defined by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Example 5. Euclidean Norm on F^n

Let $V = F^n$. If $x = (a_1, a_2 \ldots, a_n)$, then

$$\|x\| = \|(a_1, a_2 \ldots, a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$

is the Euclidean definition of length. Note that if $n = 1$, we have $\|a\| = |a|$.

We now show that the familiar properties of length in $\mathbb{R}^n$—such as scaling, positivity, and the triangle inequality—hold more generally in any inner product space. These are summarized below.

Theorem 6.2.

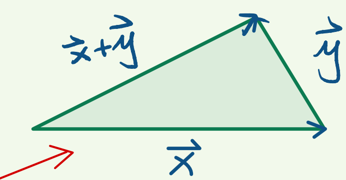
Let V be an inner product space over F. Then for all $x, y \in V$ and $c \in F$, the following properties hold:

(a) $\|cx\| = |c| \cdot \|x\|$

(b) $\|x\| = 0$ if and only if $x = 0$. In any case, $\|x\| \geq 0$

(c) (**Cauchy–Schwarz Inequality**) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

(d) (**Triangle Inequality**) $\|x + y\| \leq \|x\| + \|y\|$



Remark: Note if $V = \mathbb{R}^2$ or $\mathbb{R}^3$, then (c) is easy to show if we recall $\langle x, y \rangle = |x| \cdot |y| \cos\theta$

Proof:

(a) $\|cx\| = \sqrt{\langle cx, cx \rangle} = \sqrt{c\bar{c}\langle x, x \rangle} = \sqrt{|c|^2 \langle x, x \rangle} = |c| \cdot \|x\|$

(b) $\Rightarrow$ If $\|x\| = \sqrt{\langle x, x \rangle} = 0$, then $\langle x, x \rangle = 0$

Thus $x = 0$ by Thm 6.1 (d).

$\Leftarrow$ If $x = 0$, then by Thm 6.1 (d), $\|x\| = \sqrt{\langle x, x \rangle} = 0$

As $\langle x, x\rangle$ is a nonnegative real number.

$$\|x\| = \sqrt{\langle x, x\rangle} \geq 0$$

(c) The result is straightforward if $y=0$. If $y \neq 0$, then $\forall c \in F$, we have

$$\begin{aligned}
0 &\leq \langle x - cy, x - cy\rangle \\
&= \langle x, x-cy\rangle - c\langle y, x-cy\rangle \\
&= \langle x,x\rangle - \bar{c}\langle x,y\rangle - c\langle y,x\rangle + c\bar{c}\langle y,y\rangle
\end{aligned}$$

Particularly, we choose

$$c = \frac{\langle x,y\rangle}{\langle y,y\rangle} \in F$$

Then

$$\bar{c}\langle x,y\rangle = \frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle x,y\rangle = \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$c\langle y,x\rangle = \frac{\langle x,y\rangle}{\langle y,y\rangle}\overline{\langle x,y\rangle} = \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$c\bar{c}\langle y,y\rangle = \frac{\langle x,y\rangle}{\langle y,y\rangle}\cdot\frac{\overline{\langle x,y\rangle}}{\langle y,y\rangle}\langle y,y\rangle = \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

Recall if $c \in \mathbb{C}$ then $c\bar{c} = |c|^2$

Apply this to $c = \langle x,y\rangle$

Thus

$$0 \leq \langle x,x\rangle - \bar{c}\langle x,y\rangle \cancel{- c\langle y,x\rangle + c\bar{c}\langle y,y\rangle} = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$\Rightarrow \|x\|^2 \cdot \|y\|^2 - |\langle x,y\rangle|^2 \geq 0$$

$$\Rightarrow |\langle x,y\rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x,y\rangle| < \|x\| \cdot \|y\|$$

(Note every term is nonnegative so we take the square roots both sides)

(d) We compute

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + \overset{\overline{\langle x,y\rangle}}{\langle y,x\rangle} + \langle x,y\rangle + \langle y,y\rangle$$

real part of $\langle x,y\rangle \in \mathbb{C}$

$$= \|x\|^2 + 2\mathrm{Re}(\langle x,y\rangle) + \|y\|^2$$

Note if $c = s + it \in \mathbb{C}$

Then

$$c + \bar{c} = s + it + s - it = 2s = 2\mathrm{Re}(c)$$

$$\leq \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \quad \text{(by part c Cauchy-Schwarz)}$$

$$= (\|x\| + \|y\|)^2$$

Aagain by taking square roots on both sides, we have

$$\|x+y\| \leq \|x\| + \|y\|$$

Example 6. Coordinate Forms of Cauchy–Schwarz and Triangle Inequalities in F^n

For F^n, we may apply parts **(c)** and **(d)** of Theorem 6.2 to the standard inner product to obtain the following well-known inequalities:

Cauchy–Schwarz Inequality in F^n

$$\left|\sum_{i=1}^{n} a_i\,\overline{b_i}\right| \le \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2}$$

Triangle Inequality in F^n

$$\left(\sum_{i=1}^{n} |a_i + b_i|^2\right)^{1/2} \le \left(\sum_{i=1}^{n} |a_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |b_i|^2\right)^{1/2}$$

Recall from previous courses such as **Math 6A** that two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are said to be **perpendicular** (or **orthogonal**) if their dot product is zero. We now generalize this notion to any inner product space:

Definitions. Orthogonal, Orthonormal, Unit Vector

Let V be an inner product space.

- Two vectors $x, y \in V$ are **orthogonal** (or **perpendicular**) if

$$\langle x, y\rangle = 0.$$

- A subset $S \subseteq V$ is **orthogonal** if every pair of distinct vectors in S is orthogonal.
- A vector $x \in V$ is a **unit vector** if $\|x\| = 1$.
- A subset $S \subseteq V$ is **orthonormal** if it is orthogonal and every vector in S is a unit vector.

Remarks. By the above definition, if $S = \{v_1, v_2, \ldots\}$, then S is **orthonormal** if and only if

$$\langle v_i, v_j\rangle = \delta_{ij},$$

where δ_{ij} is the **Kronecker delta**, defined by $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

Also note:

- Multiplying vectors by nonzero scalars does **not** affect orthogonality.
- If x is any nonzero vector, then $\dfrac{x}{\|x\|}$ is a **unit vector**. The process of multiplying a nonzero vector by the reciprocal of its length is called **normalizing**.

Example 7. Normalizing an Orthogonal Set

In F^3, the set

$$\{(1,1,0),\ (1,-1,1),\ (-1,1,2)\}$$

is an **orthogonal** set of nonzero vectors, but it is **not orthonormal**.

However, if we **normalize** each vector, we obtain the orthonormal set

$$\left\{\frac{1}{\sqrt{2}}(1,1,0),\ \frac{1}{\sqrt{3}}(1,-1,1),\ \frac{1}{\sqrt{6}}(-1,1,2)\right\}.$$

The following exercise is adapted from the examples in Section 6.1 of the textbook. Students who have taken **Math 6B** or analysis courses such as **Math 117** may recognize its connection to **Fourier series**.

Exercise. Orthonormal System in a Function Space

Let H be the space of continuous complex-valued functions on the interval $[0, 2\pi]$, with inner product defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,\overline{g(t)}\,dt.$$

(a) Verify that this defines a valid inner product on H.

(b) For each integer n, define

$$f_n(t) = e^{int}, \quad 0 \le t \le 2\pi,$$

and let

$$S = \{f_n : n \in \mathbb{Z}\}.$$

Show that:

- **(i)** If $m \neq n$, then $\langle f_m, f_n\rangle = 0$
- **(ii)** For all n, we have $\langle f_n, f_n\rangle = 1$

(c) Conclude that S is an **orthonormal set** in H.

(*Hint:* Use the identity $\overline{e^{int}} = e^{-int}$ and evaluate the integral of $e^{i(m-n)t}$ over $[0, 2\pi]$.)

Note: You can find the solutions to this in the book Section 6.1.