Solutions Homework 2

Problem 1. Prove or give a counterexample: if V is an F-vector space and $x, y, z, w \in V$ are any four elements such that $\{x, y, z, w\}$ spans V, then

$$\{x - y, y - z, z - w, w\}$$

also spans V.

Solution. We show that the statement is true. Let $v \in V$. Since $\{x, y, z, w\}$ spans V, there exist $a, b, c, d \in F$ such that $v = ax + by + cz + dw$. Now define $\alpha, \beta, \gamma, \delta \in F$ via

$$\begin{aligned} \alpha &:= a \\ \beta &:= a + b \\ \gamma &:= a + b + c \\ \delta &= a + b + c + d \end{aligned}$$

Then $\beta - \alpha = b$, $\gamma - \beta = c$ and $\delta - \gamma = d$. Therefore

$$\begin{aligned} \alpha(x - y) + \beta(y - z) + \gamma(z - w) + \delta w &= \alpha x + (\beta - \alpha)y + (\gamma - \beta)z + (\delta - \gamma)z \\ &= ax + by + cz + dw \\ &= v. \end{aligned}$$

This shows that v lies in the span of $\{x - y, y - z, z - w, w\}$.

Problem 2. Let V be an F-vector space and let S and T be two subsets of V. Prove that

$$\mathrm{Span}(S \cup T) = \mathrm{Span}(S) + \mathrm{Span}(T).$$

(To simplify the problem, you can assume if you want that S and T are finite.)

Solution. We show that each side is a subset of the other.

$\boxed{\subseteq}$ Let $v \in \mathrm{Span}(S \cup T)$. Then there exist $n \in \mathbb{Z}_{\geq 0}$, $\lambda_1, \ldots, \lambda_n \in F$ and $v_1, v_2, \ldots, v_n \in S \cup T$ such that

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Each v_i lies in S or T (or both). Let $I \subseteq \{1, 2, \ldots, n\}$ be the subset of all indices i such that $v_i \in S$. Let $J \subseteq \{1, 2, \ldots, n\}$ denote the complement of I in $\{1, 2, \ldots, n\}$. Then $v_j \in T$ for any $j \in J$. Let

$$x := \sum_{i \in I} \lambda_i v_i \qquad y := \sum_{j \in J} \lambda_j v_j$$

where x is taken to be zero if $I = \varnothing$ (and similarly for y). By definition, $x \in \mathrm{Span}(S)$ and $y \in \mathrm{Span}(T)$. Now

$$\begin{aligned} v &= \lambda_1 v_1 + \ldots + \lambda_n v_n \\ &= \sum_{i \in I} \lambda_i v_i + \sum_{j \in J} \lambda_j v_j \\ &= x + y \in \mathrm{Span}(S) + \mathrm{Span}(T). \end{aligned}$$

$\boxed{\supseteq}$ Let $v \in \mathrm{Span}(S) + \mathrm{Span}(T)$. Then

$$v = x + y$$

for some $x \in \mathrm{Span}(S)$ and $y \in \mathrm{Span}(T)$. Then clearly $x, y \in \mathrm{Span}(S \cup T)$. Since $\mathrm{Span}(S \cup T)$ is a subspace of V, it follows that

$$v = x + y \in \mathrm{Span}(S \cup T).$$

Problem 3. Let V be an F-vector space. Let S be a non-empty and linearly independent subset of V and let $w \in V$. Prove that if the set

$$\{v + w \mid v \in S\}$$

is linearly dependent, then $w \in \mathrm{Span}(S)$. (To simplify the problem, you can assume if you want that S is finite.)

Solution. Let $T := \{v + w \mid v \in S\}$. Suppose that T is linearly dependent. Then there exist $n \in \mathbb{Z}_{\geq 1}$, distinct elements $x_1, \ldots, x_n \in T$ and scalars $\lambda_1, \ldots, \lambda_n \in F$ not all of which are zero such that

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0 \tag{‡}$$

By definition of T, there exist $v_1, \ldots, v_n \in S$ such that $x_i = v_i + w$ for $i = 1, \ldots, n$. Since the elements $x_1, \ldots, x_n$ are distinct, the elements $v_1, \ldots, v_n$ are also distinct. We can rewrite (‡) as

$$(\lambda_1 + \ldots + \lambda_n)w = -\big(\lambda_1 v_1 + \ldots + \lambda_n v_n\big) \tag{$*$}$$

Let $\gamma := \lambda_1 + \ldots + \lambda_n$. If $\gamma = 0$, then $(*)$ provides a non-trivial linear dependence relation between distinct elements of S, contradicting the linear independence of S. Thus $\gamma \neq 0$, and we can multiply both sides of $(*)$ by γ^{-1} to obtain an expression for w as a linear combination of $v_1, \ldots, v_n$, i.e., $w \in \mathrm{Span}(S)$.

Problem 4. Suppose that U and W are subspaces of an F-vector space V such that $V = U \oplus W$. Prove that if S is a basis for U and T is a basis for W, then $S \cup T$ is a basis for V. (To simplify the problem, you can assume if you want that S and T are finite.)

Solution. Let S and T be bases for U and W, respectively. Note that, since $V = U \oplus W$, we know that $V = U + W$ and $U \cap W = \{0\}$.

We first show that $S \cup T$ spans V. Indeed, by the previous problem, we have that

$$\mathrm{Span}(S \cup T) = \mathrm{Span}(S) + \mathrm{Span}(T) = U + W = V.$$

Now we show that $S \cup T$ is linearly independent. Suppose that for some $n \in \mathbb{Z}_{\geq 1}$ there exist $a_1, \ldots, a_n \in F$ and distinct $v_1, \ldots, v_n \in S \cup T$ such that

$$0 = a_1 v_1 + \ldots + a_n v_n.$$

We can assume, without loss of generality, that $v_1, \ldots, v_m \in S$ and $v_{m+1}, \ldots, v_n \in T$ for some $0 \leq m \leq n$. Rearranging the previous equation, we get that

$$a_1 v_1 + \ldots + a_m v_m = -(a_{m+1} v_{m+1} + \ldots + a_n v_n) \in U \cap W = \{0\}.$$

Therefore

$$a_1 v_1 + \ldots + a_m v_m = 0, \text{ and } a_{m+1} v_{m+1} + \ldots + a_n v_n = 0.$$

Since both S and T are linearly independent sets, we conclude that

$$a_1 = \ldots = a_m = 0,$$

and that

$$a_{m+1} = \ldots = a_n = 0.$$

Therefore $S \cup T$ is linearly independent.

Problem 5. Let V be an F-vector space and let $U_1, \ldots, U_m$ be finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim(U_1) + \cdots + \dim(U_m).$$

Solution. Let $S_1, \ldots, S_m$ be bases for $U_1, \ldots, U_m$, respectively. Since the subspaces $U_1, \ldots, U_m$ are finite-dimensional, the sets $S_1, \ldots, S_m$ are finite with $|S_i| = \dim(U_i)$ for each $i = 1, \ldots, m$. By problem 3 and an easy induction argument, we have that

$$\operatorname{Span}(S_1 \cup \cdots \cup S_m) = \operatorname{Span}(S_1) + \cdots + \operatorname{Span}(S_m) = U_1 + \cdots + U_m.$$

Therefore $S_1 \cup \cdots \cup S_m$ is a (finite) spanning set for $U_1 + \cdots + U_m$. Thus $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq |S_1 \cup \cdots \cup S_m| \leq |S_1| + \cdots + |S_m| = \dim(U_1) + \cdots + \dim(U_m).$$