Lecture   The Adjoint of a Linear Operator

Introduction to the Adjoint Operator

- In Section 6.1, we defined the conjugate transpose A^* of a matrix A.
- Now, for a linear operator T on an inner product space V, we define a related linear operator called the **adjoint** of T.
- The matrix representation of the adjoint with respect to any orthonormal basis β for V is $[T]_\beta^*$.
- We will soon see a clear analogy between complex conjugation and the adjoint of a linear operator.

Preliminary Result

- Let V be an inner product space, and let $y \in V$.
- Define $g : V \to F$ by

$$g(x) = \langle x, y \rangle.$$

- It is clear that g is linear. (since $\langle , \rangle$ is linear in the first component)

i.e. $g(cu+v) = \langle cu+v, y \rangle = c\langle u, y \rangle + \langle v, y \rangle = cg(u) + g(v)$.

- More interestingly, if V is finite-dimensional, **every** linear transformation from V into F is of this form:

Theorem 6.8

Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Then there exists a unique vector $y \in V$ such that

$$g(x) = \langle x, y \rangle \quad \text{for all } x \in V.$$

Proof: Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V.

and consider $y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i$

Then define $h : V \to F$ by $h(x) = \langle x, y \rangle$. $\forall x \in V$

So h is a linear transformation

Also, for each basis element v_j, $1 \le j \le n$ $\quad \delta_{ij} = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$

$$h(v_j) = \left\langle v_j, \sum_{i=1}^{n} \overline{g(v_i)}\, v_i \right\rangle = \sum_{i=1}^{n} \overline{\overline{g(v_i)}} \langle v_j, v_i \rangle$$

$$= \sum_{i=1}^{n} g(v_i)\, \delta_{ij} = g(v_j)$$

Thus g and h are the same on the basis element in β.

Then $\forall x \in V$, $x = \sum_{i=1}^{n} a_i v_i$, since h and g are linear.

$$\underline{h(x)} = h\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i h(v_i) = \sum_{i=1}^{n} a_i g(v_i) = g\left(\sum_{i=1}^{n} a_i v_i\right) = \underline{g(x)}$$

So $h = g$. Thus

$$g(x) = h(x) = \langle x, y \rangle, \text{ where } y = \sum_{i=1}^{n} \overline{g(v_i)}\, v_i,$$

For uniqueness, assume $\exists y'$ such that.

$$g(x) = \langle x, y' \rangle = \langle x, y \rangle.$$

Then Th 6.1 (e) $\quad y' = y$.

Example 1

- Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(a_1, a_2) = 2a_1 + a_2.$$

- Clearly, g is a linear transformation.
- Let $\beta = \{e_1, e_2\}$ be the standard basis, and define

$$y = g(e_1)e_1 + g(e_2)e_2 = 2e_1 + e_2 = (2, 1),$$

as in the proof of Theorem 6.8.

- Then

$$g(a_1, a_2) = \langle (a_1, a_2), (2, 1) \rangle = 2a_1 + a_2.$$

Theorem 6.9

Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^* : V \to V$ such that

$$\langle T(x), y \rangle = \langle x, T^*(y) \rangle \quad \text{for all } x, y \in V. \qquad \circledast$$

Furthermore, T^* is linear.

Idea of the proof:

- Given y, consider a function $g: V \to F$ by

$$g(x) = \langle T(x), y \rangle \quad \text{(show } g \text{ is linear)}.$$

- Then by Thm 6.8, $\exists ! \, y'$ s.t. $g(x) = \langle x, y' \rangle$
- Then define $T^* : V \to V$ by $T(y) = y'$. Then $\circledast$ holds by construction
- Show linearity and uniqueness of T^*

Proof: Let $y \in V$. Consider $g: V \to F$ by

$$g(x) = \langle T(x), y \rangle$$

Then g is linear. In fact, let $x_1, x_2 \in V$. $c \in F$.

then $g(cx_1+x_2) = \langle T(cx_1+x_2), y\rangle = \langle cT(x_1)+T(x_2), y\rangle$

$= c\langle T(x_1), y\rangle + \langle T(x_2), y\rangle$

$= c\,g(x_1) + g(x_2).$

Then by Thm 6.8, there exists a unique $y' \left(= \sum_{i=1}^{n} \overline{g(v_i)}\, v_i = \sum_{i=1}^{n} \overline{\langle T(v_i), y\rangle} v_i\right)$

s.t. $\overset{\langle T(x), y\rangle}{\overset{\shortparallel}{g(x)}} = \langle x, y'\rangle$

Define $T^*: V \longrightarrow V$ by

$$T^*(y) = y' \left(= \sum_{i=1}^{n} \overline{g(v_i)}\, v_i = \sum_{i=1}^{n} \overline{\langle T(v_i), y\rangle} v_i\right)$$

Then (⊛ is satisfied by T^*:)

$g(x) = \underline{\langle T(x), y\rangle} = \langle x, y'\rangle = \underline{\langle x, T^*(y)\rangle}$

We show T^* is linear: let $y_1, y_2 \in V$, $c \in F$

Then $\forall x \in V$.

$\langle x, T^*(cy_1+y_2)\rangle = \langle T(x), cy_1+y_2\rangle$

$= \bar{c}\langle T(x), y_1\rangle + \langle T(x), y_2\rangle$

$= \bar{c}\langle x, T^*(y_1)\rangle + \langle x, T^*(y_2)\rangle$

$= \langle x, cT^*(y_1) + T^*(y_2)\rangle,$

Then by Thm 6.1 (e), $T^*(cy_1+y_2) = cT^*(y_1) + T^*(y_2)$.

To show T^* is unique, if $G: V \to V$ is linear s.t.

$\underset{\underset{\langle x, T^*(y)\rangle}{\shortparallel}}{\langle T(x), y\rangle} = \langle x, G(y)\rangle, \ \forall x.$

Then again by Thm 6.1 (e), $T^*(y) = G(y)$, $\forall y$. So $G = T^*$

$\square$

Definition: Adjoint Operator

The linear operator T^* described in Theorem 6.9 is called the **adjoint** of the operator T.

- The symbol T^* is read as "**T star**."

Remark

- The adjoint T^* is the unique operator on V satisfying

$$\langle T(x), y\rangle = \langle x, T^*(y)\rangle$$

  for all $x, y \in V$.
- Note that we also have

$$\langle x, T(y)\rangle = \overline{\langle T(y), x\rangle} = \overline{\langle y, T^*(x)\rangle} = \langle T^*(x), y\rangle.$$

- Therefore,

$$\boxed{\langle x, T(y)\rangle = \langle T^*(x), y\rangle}$$

  for all $x, y \in V$.
- Symbolically, we may think of **adding a * to T** when shifting its position inside the inner product symbol.

The next theorem is a useful result for computing adjoints.

Theorem 6.10

Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then

$$[T^*]_\beta = [T]_\beta^*.$$

Proof: Let $A = [T]_\beta$, $B = [T^*]_\beta$. we want to show $B = A^*$.

Let $\beta = \{v_1, v_2, \cdots, v_n\}$. Then by Corollary in Lecture 4.

$$B_{ij} = \langle T^*(v_j), v_i\rangle = \overline{\langle v_i, T^*(v_j)\rangle} = \overline{\langle T(v_i), v_j\rangle}$$

$$= \overline{A_{ji}} = (A^*)_{ij}$$

Thus $B = A^*$.

Corollary

Let A be an $n \times n$ matrix. Then

$$L_{A^*} = (L_A)^*.$$

Proof: Let β be the standard ordered basis for $\mathbb{F}^n$

Then $[L_A]_\beta = A$.

So $[(L_A)^*]_\beta = [L_A]_\beta^* = A^* = [L_{A^*}]_\beta$

Thus $L_{A^*} = (L_A)^*$.

Exercise.

Let T be the linear operator on $\mathbb{C}^2$ defined by

$$T(a_1, a_2) = (2ia_1 + 3a_2,\ a_1 - a_2).$$

If β is the standard ordered basis for $\mathbb{C}^2$: $\beta = \{(1,0), (0,1)\}$.

1. Find the matrix $[T]_\beta$.
2. Compute $[T^*]_\beta$.
3. Determine an explicit formula for $T^*(a_1, a_2)$.

1. $T(1,0) = (2i, 1) = 2i(1,0) + 1\cdot(0,1)$

$T(0,1) = (3,-1) = 3(1,0) - 1(0,1)$

Thus $[T]_\beta = \begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix}$

2. $[T^*]_\beta = [T]_\beta^* = \begin{bmatrix} -2i & 1 \\ 3 & -1 \end{bmatrix}$

3. From 2, we know $T^*(a_1, a_2) = (-2ia_1 + a_2,\ 3a_1 - a_2)$.

The following theorem suggests an analogy between the conjugates of complex numbers and the adjoints of linear operators.

Theorem 6.11

Let V be an inner product space, and let T and U be linear operators on V whose adjoints exist. Then:

(a) $T+U$ has an adjoint, and

$$(T+U)^* = T^* + U^*.$$

(b) cT has an adjoint, and

$$(cT)^* = \overline{c}T^* \quad \text{for any } c \in F.$$

(c) TU has an adjoint, and

$$(TU)^* = U^*T^*.$$

(d) T^* has an adjoint, and

$$(T^*)^* = T.$$

(e) The identity operator I has an adjoint, and

$$I^* = I.$$

Proof: (a) $\langle (T+U)x, y\rangle = \langle T(x)+U(x), y\rangle$

$= \langle x, T^*(y)\rangle + \langle x, U^*(y)\rangle$

$= \langle x, T^*(y) + U^*(y)\rangle$

$= \langle x, (T^*+U^*)y\rangle$

Thus $(T+U)^*$ exists and $(T+U)^* = T^*+U^*$

(b). $\langle cT(x), y\rangle = \langle T(x), \bar{c}y\rangle = \langle x, \bar{c}T^*(y)\rangle$

Thus $(cT)^*$ exists and $(cT)^* = \bar{c}T^*$.

(c) $\langle TU(x), y\rangle = \langle T(U(x)), y\rangle = \langle U(x), T^*y\rangle$

$= \langle x, U^*T^*y\rangle$

Thus $(TU)^*$ exists and $(TU)^* = U^*T^*$

(d) $\langle T^*(x), y \rangle = \langle x, T(y) \rangle$

Thus $(T^*)^*$ exists and $(T^*)^* = T$.

(e) $\langle I(x), y \rangle = \langle x, y \rangle = \langle x, I(y) \rangle$

Thus I^* exists and $I^* = I$.

Corollary

Let A and B be $n \times n$ matrices. Then:

(a) $(A+B)^* = A^* + B^*$.

(b) $(cA)^* = \overline{c}A^* \quad \text{for all } c \in F$.

(c) $(AB)^* = B^*A^*$.

(d) $(A^*)^* = A$.

(e) $I^* = I$.

Proof: Exercise. See Pg 357 for a proof of c).