Lecture Orthogonal Complements

Overview of Lecture

- 📚 In this lecture, we introduce the concept of the **orthogonal complement** of a subset in an inner product space. Given a subset S of V, the orthogonal complement $S^\perp$ consists of all vectors in V that are orthogonal to every vector in S.
- 🧩 We will explore basic properties of orthogonal complements, including how they form subspaces and simple examples in $\mathbb{R}^3$.
- 🎯 A major result is the **Orthogonal Decomposition Theorem (Theorem 6.6)**, which states that any vector in V can be uniquely written as the sum of a vector in W and a vector in $W^\perp$, where W is a subspace of V. The component in W is called the **orthogonal projection** of the vector onto W.
- 🔥 Understanding orthogonal projections will be important for later applications, such as solving **least squares problems** in Section 6.3.

Definition.

Let S be a nonempty subset of an inner product space V.

The **orthogonal complement** of S, denoted $S^\perp$, is the set of all vectors in V that are orthogonal to every vector in S. That is,

$$S^\perp = \{x \in V : \langle x, y\rangle = 0 \text{ for all } y \in S\}.$$

Exercise 1. Show $S^\perp$ is a subspace of V for any subset S of V.

ANS: Let $x, z \in S^\perp$, $c \in F$. we want to show $cx + z \in S^\perp$

Note $\langle cx+z, y\rangle = c\langle x, y\rangle + \langle z, y\rangle = c \cdot 0 + 0 = 0$, $\forall y \in S$.

Also $0 \in S^\perp$. Thus $S^\perp$ is a subspace of V.

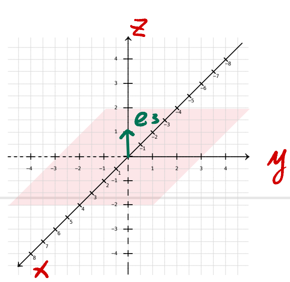
Example 2. Verify that $\{0\}^\perp = V$ and $V^\perp = \{0\}$ for any inner product space V.

ANS: By def, $\{0\}^\perp = \{x \in V \mid \langle x, y\rangle = 0 \text{ for all } y \in \{0\}\}$.

It's straightforward to see $V \subseteq \{0\}^\perp$ Thus $\{0\}^\perp = V$.

We leave $V^\perp = \{0\}$ as exercise.

Example 3. If $V = \mathbb{R}^3$ and $S = \{e_3\}$, then $S^\perp$ equals the xy-plane.



$S^\perp = \{x \in \mathbb{R}^3 \mid \langle x, e_3\rangle = 0\}$

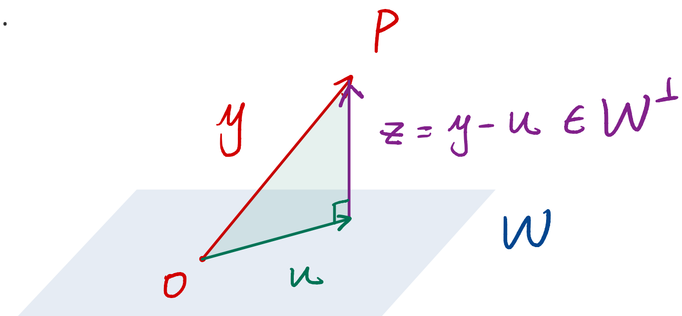
$\langle e_1, e_3\rangle = \langle e_2, e_3\rangle = 0$.

Discussion: Distance from a Point to a Subspace

Suppose we are given a point P in $\mathbb{R}^3$ and a plane W through the origin.

We want to find the **distance from P to the plane W**.

Let y be the vector from the origin 0 to the point P. The goal is to find the vector $u \in W$ that is **closest** to y. This means we want to minimize the distance $\|y - u\|$.



From the geometry, the difference $z = y - u$ is the vector from the point on the plane to the point P, and it is **perpendicular to the plane W**.

Therefore, $z \in W^\perp$, and we obtain the decomposition:

$$y = u + z, \quad \text{with } u \in W, \ z \in W^\perp.$$

This gives both a **geometric** and **algebraic** motivation for studying orthogonal complements. Finding the vector in W closest to y is the same as projecting y orthogonally onto W.

(See the above figure a visual representation of this idea.)

- To find the point in a subspace W closest to a vector y, we decompose y into a sum of vectors from W and $W^\perp$. The following theorem gives a general method for doing this when W is finite-dimensional.

Theorem 6.6. (Orthogonal Decomposition)

Let W be a finite-dimensional subspace of an inner product space V, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^\perp$ such that $y = u + z$.

Furthermore, if $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for W, then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i.$$

Proof: Let $\{v_1, \cdots, v_k\}$ be an orthonormal basis for W.

and let $u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$. as give above.

Consider $z = y - u$.

Then $u \in W$ since u is a linear combination of basis element for W.

7. Let β be a basis for a subspace W of an inner product space V, and let $z \in$ V. Prove that $z \in W^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

Next, we show $z \in W^{\perp}$. By HW#5 Question 1, we show z is orthogonal to each $v_j \in \{v_1, \cdots, v_k\}$

Then

$$\begin{aligned}
\langle z, v_j \rangle &= \left\langle y - \sum_{i=1}^{k} \langle y, v_i \rangle v_i ,\ v_j \right\rangle \\
&= \langle y, v_j \rangle - \sum_{i=1}^{k} \langle \langle y, v_i \rangle v_i,\ v_j \rangle \\
&= \langle y, v_j \rangle - \sum_{i=1}^{k} \langle y, v_i \rangle \underline{\langle v_i, v_j \rangle} \qquad \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \\
&= \langle y, v_j \rangle - \langle y, v_j \rangle \\
&= 0, \quad \forall\ v_j \in \{v_1, \cdots, v_k\}.
\end{aligned}$$

Thus $z \perp v_j$, $j = 1, \cdots, k$

Finally, we show the uniqueness of the decomposition.

$u - u' = z' - z \in W^{\perp}$

If $y = u + z = u' + z'$, where, $u' \in W$, $z' \in W^{\perp}$.

Then $u - u' \in W \cap W^{\perp}$, and $z - z' \in W \cap W^{\perp}$

and $W \cap W^{\perp} = \{0\}$.

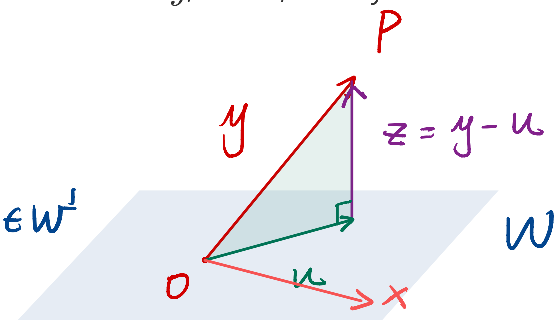
Thus $u = u'$ and $z = z'$ and the decomposition of $y = u + z$ is unique.

Corollary.

In the notation of **Theorem 6.6**, the vector u is the unique vector in W that is "closest" to y; that is, for any $x \in W$,

$$\|y - x\| \geq \|\overset{z}{\overset{\parallel}{y - u}}\|,$$

and this inequality is an equality if and only if $x = u$.



Proof: Let $y = u + z$ as in Thm 6.6. where $u \in W$, $z \in W^\perp$

Let $x \in W$. then

$\|y-x\|^2 = \|u+z-x\|^2 = \|u-x+z\|^2 = \|u-x\|^2 + \|z\|^2 + 2\mathrm{Re}\langle \underset{W}{u-x}, \underset{W^\perp}{z} \rangle$ (by proof of Thm 6.2 (d))

$= \|u-x\|^2 + \|z\|^2 \geq \|z\|^2 = \|y-u\|^2. \Rightarrow \|y-x\| \geq \|y-u\|$

If $\|y-x\| = \|y-u\|$ then the above inequality is an equality.

So we have $\|u-x\|^2 + \|z\|^2 = \|z\|^2. \Rightarrow \|u-x\|^2 = 0$

$\Rightarrow u = x$. If $x = u$, then $\|y-x\| = \|y-u\|$.

Remark.

The vector u in the corollary is called the **orthogonal projection** of y onto W. We will see the importance of orthogonal projections in the application to least squares in Section 6.3.

Theorem 6.7

Suppose that $S = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then:

(a) S can be extended to an orthonormal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

(b) If $W = \mathrm{span}(S)$, then $S_1 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is an orthonormal basis for $W^\perp$.

(c) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^\perp)$.

Proof: (a) From linear algebra 1, we know S can be extended to an ordered basis.

$S' = \{v_1, v_2, \cdots, v_k, w_{k+1}, \cdots, w_n\}$.

Then we can apply GS process to S' to get $\bar{S}$

By HW#4 Q4. the vectors $v_1, \cdots, v_k$ remains the same as in S. Moreover $\bar{S}$ spans V. Normalize the vectors in $\bar{S}$, we obtain an orthonormal basis $\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V.

(b) As $S_1 = \{v_{k+1}, \cdots, v_n\}$ is an orthogonal set, by Corollary 2 in Lecture 4, we know S_1 is a linearly independent set.

Then we show $\operatorname{Span}(S_1) = W^\perp$

By construction of S_1, we know $\forall x \in \operatorname{Span}(S_1)$. $x \in W^\perp$. Thus $\operatorname{Span}(S_1) \subseteq W^\perp$. Then we show $W^\perp \subseteq \operatorname{span}(S_1)$. Let $x \in W^\perp$, then as $x \in V$, we have $x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$. Since $x \in W^\perp$, we know $\langle x, v_i \rangle = 0$, if $1 \le i \le k$. Thus $x = \sum_{i=k+1}^{n} \langle x, v_i \rangle v_i \in \operatorname{Span}(S_1)$

Therefore S_1 is a basis of $W^\perp$

Recall

Definition. Basis

A subset β of a vector space V is called a **basis** if it satisfies two properties:

- β is **linearly independent**.
- β **spans** V (i.e., $\operatorname{span}(\beta) = V$).

If β is a basis for V, we also say that the vectors in β **form a basis** for V.

(c) If W is a subspace of V. of dimension k, then by (a) and (b),

$$\dim(V) = n = k + (n-k) = \dim(W) + \dim(W^\perp).$$

Example 4.

Let $W = \operatorname{span}(\{e_1, e_2\})$ in $\mathbb{F}^3$.

Then $x = (a, b, c) \in W^\perp$ if and only if

- $0 = \langle x, e_1 \rangle = a$ and
- $0 = \langle x, e_2 \rangle = b$.

Thus, $x = (0, 0, c)$, and therefore $W^\perp = \operatorname{span}(\{e_3\})$.

One can also deduce the same result by noting that $e_3 \in W^\perp$ and, from part (c), that $\dim(W^\perp) = 3 - 2 = 1$.