

Def. Given F -vector spaces U and V , let $\mathcal{L}(U, V)$ denote the set of all linear maps from U to V .

Prop. $\mathcal{L}(U, V)$ is an F -vector space. More precisely:

(a) given $\tau, \sigma \in \mathcal{L}(U, V)$, the sum

$$\begin{aligned}\tau + \sigma: U &\longrightarrow V \\ x &\longmapsto \tau(x) + \sigma(x)\end{aligned}$$

is in $\mathcal{L}(U, V)$.

(b) given $\tau \in \mathcal{L}(U, V)$ and $\lambda \in F$, the product

$$\begin{aligned}\lambda\tau: U &\longrightarrow V \\ x &\longmapsto \lambda \cdot \tau(x)\end{aligned}$$

is in $\mathcal{L}(U, V)$.

(c) $\mathcal{L}(U, V)$ is an F -vector space w.r.t. these operations.

Proof. Let $x, y \in U$ and let $\alpha, \beta \in F$. Then we have:

$$\begin{aligned}(a) \quad (\tau + \sigma)(\alpha x + \beta y) &= \tau(\alpha x + \beta y) + \sigma(\alpha x + \beta y) \\ &= \alpha \tau(x) + \beta \tau(y) + \alpha \sigma(x) + \beta \sigma(y) \\ &= \alpha (\tau(x) + \sigma(x)) + \beta (\tau(y) + \sigma(y)) \\ &= \alpha ((\tau + \sigma)(x)) + \beta ((\tau + \sigma)(y)).\end{aligned}$$

hence $\tau + \sigma$ is a linear map.

$$\begin{aligned}
 (b) \quad (\lambda T)(\alpha x + \beta y) &= \lambda (T(\alpha x + \beta y)) \\
 &= \lambda (\alpha T(x) + \beta T(y)) \\
 &= \alpha \cdot \lambda T(x) + \beta \cdot \lambda T(y) \\
 &= \alpha \cdot (\lambda T)(x) + \beta \cdot (\lambda T)(y)
 \end{aligned}$$

hence λT is a linear map.

(c) Addition in $\mathcal{L}(U, V)$ is clearly commutative and the zero map $\mathbb{O}: U \rightarrow V, u \mapsto \mathbb{O}_v$ clearly satisfies $\mathbb{O} + T = T = T + \mathbb{O}$ $\forall T \in \mathcal{L}(U, V)$.

Let $T, \sigma, \rho \in \mathcal{L}(U, V)$ and $x \in U$. Then

$$\begin{aligned}
 [(T + \sigma) + \rho](x) &= (T + \sigma)(x) + \rho(x) \\
 &= (T(x) + \sigma(x)) + \rho(x) \\
 &= T(x) + (\sigma(x) + \rho(x)) \\
 &= T(x) + (\sigma + \rho)(x) \\
 &= [T + (\sigma + \rho)](x)
 \end{aligned}$$

Since this equality holds for all $x \in U$, it shows that

$$(T + \sigma) + \rho = T + (\sigma + \rho).$$

Since this is true for all $T, \sigma, \rho \in \mathcal{L}(U, V)$, this establishes the associativity of "+" in $\mathcal{L}(U, V)$.

Now $[T + (-1)T](x) = T(x) + [(-1)T](x)$

$$= T(x) + (-1)T(x)$$

$$= 0$$

$$= \dots = [(-1)T + T](x)$$

hence $T + (-1)T = 0 = (-1)T + T$, so inverses exist in $\mathcal{L}(U, V)$.

Thus, $\mathcal{L}(U, V)$ is an abelian group under $+$.

Let $T, \sigma \in \mathcal{L}(U, V)$ and let $\lambda \in F$. Then, for all $x \in U$,

$$\begin{aligned} [\lambda(T + \sigma)](x) &= \lambda[(T + \sigma)(x)] \\ &= \lambda[T(x) + \sigma(x)] \\ &= \lambda \cdot T(x) + \lambda \cdot \sigma(x) \\ &= (\lambda T)(x) + (\lambda \sigma)(x) \\ &= [\lambda T + \lambda \sigma](x), \end{aligned}$$

so $\lambda(T + \sigma) = \lambda T + \lambda \sigma$.

Similarly, one can check (exercise)

- $(\lambda + \mu)T = \lambda T + \mu T \quad \forall \lambda, \mu \in F, \forall T \in \mathcal{L}(U, V)$
- $(\lambda \mu)T = \lambda(\mu T) \quad \forall \lambda, \mu \in F, \forall T \in \mathcal{L}(U, V)$
- $1_F T = T \quad \forall T \in \mathcal{L}(U, V)$

□

Prop. Let U, V, W, Z be F -vector spaces.

(a) If $T \in \mathcal{L}(U, V)$ and $\sigma \in \mathcal{L}(V, W)$, then the composition

$\sigma T = \sigma \circ T : U \rightarrow W$ (defined as $(\sigma T)(x) = \sigma(T(x)) \quad \forall x \in U$)
is in $\mathcal{L}(U, W)$.

(b) If $T_1, T_2 \in \mathcal{L}(U, V)$ and $\sigma \in \mathcal{L}(V, W)$, then

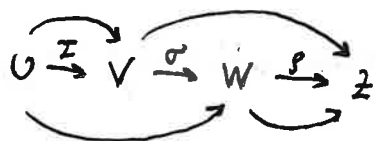
$$\sigma(T_1 + T_2) = \sigma T_1 + \sigma T_2.$$

(c) If $T \in \mathcal{L}(U, V)$, and $\sigma_1, \sigma_2 \in \mathcal{L}(V, W)$, then

$$(\sigma_1 + \sigma_2)T = \sigma_1 T + \sigma_2 T.$$

(d) If $T \in \mathcal{L}(U, V)$, $\sigma \in \mathcal{L}(V, W)$ and $\rho \in \mathcal{L}(W, Z)$, then

$$\rho(\sigma T) = (\rho \sigma)T.$$



Proof.

(a) Let $x, y \in U$ and $\alpha, \beta \in F$. Then

$$\begin{aligned} (\sigma T)(\alpha x + \beta y) &= \sigma(T(\alpha x + \beta y)) \\ &= \sigma(\alpha T(x) + \beta T(y)) \\ &= \alpha \sigma(T(x)) + \beta \sigma(T(y)) \\ &= \alpha(\sigma T)(x) + \beta(\sigma T)(y), \end{aligned}$$

so $\sigma T \in \mathcal{L}(U, W)$.

(b) Let $x \in U$. Then

$$\begin{aligned} [\sigma(z_1 + z_2)](x) &= \sigma((z_1 + z_2)(x)) \\ &= \sigma(z_1(x) + z_2(x)) \\ &= \sigma(z_1(x)) + \sigma(z_2(x)) \\ &= (\sigma z_1)(x) + (\sigma z_2)(x) \\ &= (\sigma z_1 + \sigma z_2)(x). \end{aligned}$$

Hence, $\sigma(z_1 + z_2) = \sigma z_1 + \sigma z_2$.

(c) Similar to (b).

(d) Follows by associativity of functions. \square

Def. A ring is an abelian group R (written additively) together with a binary operation $\cdot: R \rightarrow R$ satisfying the following:

(i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall x, y, z \in R$

(ii) $\left. \begin{aligned} x \cdot (y + z) &= x \cdot y + x \cdot z \\ (y + z) \cdot x &= y \cdot x + z \cdot x \end{aligned} \right\} \quad \forall x, y, z \in R$

(iii) there exists an element $1_R \in R$ such that

$$1_R \cdot x = x = x \cdot 1_R \quad \forall x \in R.$$

Examples

- 1) \mathbb{Z} is a ring under standard addition and multiplication.
- 2) For any positive integer n , $\mathbb{Z}/n\mathbb{Z}$ is a ring under addition and multiplication modulo n .
- 3) Any field is a ring.
- 4) The set of $n \times n$ matrices with coefficients in any ring is again a ring under matrix addition and multiplication.
- 5) By the previous proposition, if U is any F -vector space then $\mathcal{L}(U, U)$ is a ring under addition and composition of functions. Note that the multiplicative identity is the identity map $I_U \in \mathcal{L}(U, U)$.
We call $\mathcal{L}(U, U)$ the ring of endomorphisms of U , and we denote it by $\text{End}(U)$. An element $T \in \mathcal{L}(U, U)$ is called an endomorphism of U .