Positive and Unitary Operators

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1 Recall: Normal and Self-adjoint Operators

Lets recall what we have done in the last class. Let V be a finite-dimensional inner product space over $\mathbb{C}$. For a linear operator $T : V \to V$, then T^* is the unique operator satisfying

$$\langle Tv, w\rangle = \langle v, T^*w\rangle, \quad \forall v, w \in V.$$

Definition 1. An operator T is called **normal** if $TT^* = T^*T$.

Remark. Normal operators includes self-adjoint. Normality ensures that T and T^* "commute," which is a strong symmetric condition.

Definition 2. T is **self-adjoint** if $T = T^*$.

Such operators generalize real symmetric matrices. When $F = \mathbb{R}$, they are precisely the symmetric matrices. In the last lecture, we have also proved Spectral theorems for self-adjoint operators and normal operators. We again restate theorems as follows:

Theorem 1. *(Spectral Theorem for Normal Operators) Let T be a linear operator on a finite dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting of eigenvectors.*

Remark. In particular, normal operators on finite dimensional complex inner product space are diagonalizable.

Remark. We only used $\mathbb{C}$ to ensure that characteristic polynomial splits. We can replace the hypothesis "Complex" with "real + characteristic polynomial splits"

Theorem 2. *(Spectral Theorem for Self-adjoint Operators) Let T be a linear operator on a finite dimensional real inner product space V. Then T is self-adjoint iff there exist an orthonormal basis for V consisting of eigenvectors of T. .*

2 Positive and Positive Definite Operators

We now refine the notion of positivity for operators.

Definition 3. Let T be a self-adjoint operator on an inner product space V.

- T is called **positive semidefinite** if

$$\langle Tv, v\rangle \geq 0 \quad \text{for all } v \in V.$$

- T is called **positive definite** if

$$\langle Tv, v\rangle > 0 \quad \text{for all } v \neq 0.$$

Proposition 1. *If T is positive semidefinite (resp. definite), then all eigenvalues of T are real and nonnegative (resp. positive).*

Proof. Since T is self-adjoint, the spectral theorem ensures the existence of an orthonormal basis of eigenvectors of T, and that all eigenvalues are real.

Let λ be an eigenvalue with eigenvector $v \neq 0$. Then

$$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \lambda\langle v, v\rangle.$$

By positivity, $\langle Tv, v\rangle \geq 0$, while $\langle v, v\rangle > 0$. Thus $\lambda \geq 0$. □

Remark. A positive definite operator has strictly positive eigenvalues.

Theorem 3 (Existence of Positive Square Root)**.** *If T is positive semidefinite, then there exists a positive semidefinite operator S such that*

$$S^2 = T.$$

*We call S, a **positive square root** of T and write $S = \sqrt{T}$.*

Proof. By the spectral theorem, $T = UDU^{-1}$ where U is matrix consisting of orthonormal basis and hence $UU^* = U^*U = I$. So $U^{-1} = U^*$ and $T = UDU^*$. Here D is a diagonal with entries are eigenvalues and hence are non-negative. Let $\sqrt{D}$ be the diagonal matrix obtained by taking square roots of the diagonal entries of D. Then define

$$S = U\sqrt{D}U^*.$$

Clearly $S^2 = (U\sqrt{D}U^{-1})^2 = U\sqrt{D}^2U^{-1} = UDU^{-1} = T$, and S is also positive semidefinite. □

3 Unitary Operators

Definition 4. An operator U is **unitary** if $U^*U = UU^* = I$. If the base field is $\mathbb{R}$, then we say U is an **orthogonal** operator.

Note that such U preserves inner products and therefore lengths as shown in the following proposition.

Proposition 2. *A unitary operator U preserves lengths: $\|Uv\| = \|v\|$ for all $v \in V$.*

Proof. By definition, U is unitary, so $U^*U = I$. Then for any $v \in V$,

$$\|Uv\|^2 = \langle Uv, Uv\rangle = \langle v, U^*Uv\rangle = \langle v, v\rangle = \|v\|^2.$$

Taking square roots gives $\|Uv\| = \|v\|$. Thus, unitary operators preserve lengths. □

Proposition 3. *The eigenvalues of a unitary operator lie on the unit circle.*

Proof. Let U be a unitary operator on V, so $U^*U = I$. Suppose λ is an eigenvalue of U with corresponding eigenvector $v \neq 0$. Then

$$Uv = \lambda v.$$

Taking norms of both sides and using that U preserves norms, we get

$$\|v\| = \|Uv\| = \|\lambda v\| = |\lambda| \cdot \|v\|.$$

Since $\|v\| \neq 0$, we may divide both sides by $\|v\|$ to obtain

$$|\lambda| = 1.$$

Thus, every eigenvalue of U has length 1, meaning that all eigenvalues of U lie on the unit circle in the complex plane. □

Proposition 4. *Let U be a linear operator on finite dimensional inner product space. Then the following statements are equivalent:*

1. *U preserves inner products: $\langle Uv, Uw\rangle = \langle v, w\rangle$ for all $v, w \in V$.*
2. *$U^*U = UU^* = I$.*
3. *U maps any orthonormal basis of V to another orthonormal basis.*

Proof. **(1 ⇒ 2):** For any v, w, $\langle v, U^*Uw\rangle = \langle Uv, Uw\rangle = \langle v, w\rangle = \langle v, Iw\rangle$. Since this holds for all v, we have $U^*U(w) = w$ for all w. Hence $U^*U = I$ and similarly, we can show that $UU^* = I$.

(2 ⇒ 1): For any v, w, $\langle Uv, Uw\rangle = \langle v, U^*Uw\rangle = \langle v, w\rangle$.

(1 ⇔ 3): Let $\{e_1, \dots, e_n\}$ be an orthonormal basis. If U preserves inner product, then $\langle Ue_i, Ue_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$, so $\{Ue_1, \dots, Ue_n\}$ is orthonormal basis. Conversely, assume U maps orthonormal basis $\{e_1, \dots, e_n\}$ to another orthonormal set $\{f_1, \dots, f_n\}$, i.e., $Ue_i = f_i$ and $\langle f_i, f_j\rangle = \delta_{ij}$. Let $v = \sum_i \alpha_i e_i$ and $w = \sum_j \beta_j e_j$ be arbitrary vectors.

$$Uv = \sum_i \alpha_i Ue_i = \sum_i \alpha_i f_i, \quad Uw = \sum_j \beta_j Ue_j = \sum_j \beta_j f_j.$$

By linearity of the inner product,

$$\langle Uv, Uw\rangle = \left\langle \sum_i \alpha_i f_i, \sum_j \beta_j f_j \right\rangle = \sum_{i,j} \alpha_i \overline{\beta_j} \langle f_i, f_j\rangle = \sum_i \alpha_i \overline{\beta_i} = \langle v, w\rangle.$$

Hence, U preserves inner products.

□

Unitary Form of Spectral Theorem

If T is normal, then there exists a unitary U such that $T = UDU^*$, where D is diagonal.

Theorem 4 (Spectral Theorem for Unitary Operators)**.** *Let T be a linear operator on a finite-dimensional real inner product space V. Then V has an orthonormal basis consisting of eigenvectors of T, and the corresponding eigenvalues have absolute value 1, if and only if T is self-adjoint (Hermitian) and orthogonal.*

Proof. (⇒) Suppose V has an orthonormal basis of eigenvectors of T with eigenvalues λ_i such that $|\lambda_i| = 1$. Let $\{v_1, \dots, v_n\}$ be this orthonormal basis. Then T is diagonalizable by a unitary (or orthogonal) matrix U with $T = UDU^*$, where D is diagonal with entries λ_i satisfying $|\lambda_i| = 1$. Then $T^* = UD^*U^* = UD^{-1}U^* = T^{-1}$, so T is unitary. Moreover, if T is real and $T = T^T$, then T is self-adjoint (orthogonal).

(⇐) Suppose T is self-adjoint and unitary. Then T is normal, so by the spectral theorem for normal (or self-adjoint) operators, T is diagonalizable by an orthonormal basis. Let λ be any eigenvalue with eigenvector v. Since T is unitary, $\|Tv\| = \|v\|$, hence $|\lambda| = 1$. Therefore, there exists an orthonormal basis of eigenvectors with eigenvalues on the unit circle. □

4 Singular Values and SVD

Proposition 5. *For any linear operator T on an inner product space V, the operator T^*T is positive.*

Proof. Let $x \in V$ be arbitrary. Using the property of T^*, we have

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2 \geq 0.$$

This shows that T^*T is positive.

□

Corollary 1. *For any linear operator $T : V \to V$, $\sqrt{T^*T}$ always exist.*

Definition 5. The **singular values** of T are the non-negative square roots of the eigenvalues of T^*T.

Theorem 5 (Singular Value Decomposition)**.** *Let A be any $n \times n$ matrix with real or complex entries. Then there exist*

- *a unitary (orthogonal, in the real case) matrix $U \in \mathbb{C}^{n\times n}$,*
- *a unitary (orthogonal) matrix $V \in \mathbb{C}^{n\times n}$, and*
- *a diagonal matrix*
$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \dots, \sigma_r, 0, \dots, 0),$$
where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$,

such that
$$A = U\Sigma V^*.$$

The numbers $\sigma_1, \dots, \sigma_r$ are called the singular values *of A.*