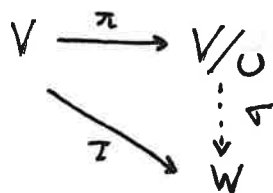


Theorem Let V be an F.v.s. and let U be a subspace of V .

(i) There is a unique F.v.s. structure on V/U such that the quotient map $\pi: V \rightarrow V/U$ is linear. Wrt this structure, the kernel of π is U .

(ii) (Universal property) Given any linear map $Z: V \rightarrow W$ (where W is any F.v.s.) such that $U \subseteq \ker Z$, there exists a unique linear map $\sigma: V/U \rightarrow W$ such that $Z = \sigma \circ \pi$.



Proof. (i) We define addition and scalar multiplication on V/U as follows:

- given $s, t \in V/U$, choose $x, y \in V$ such that $s = [x]$, $t = [y]$ and define $s+t = [x+y]$

- given $s \in V/U$ and $\lambda \in F$, choose $x \in V$ such that $s = [x]$ and define $\lambda s = [\lambda x]$.

We need to check these operations are well-defined (i.e., they do not depend on the choice of representatives).

Let $s \in V/U$, $t \in V/U$, and suppose $x, x', y, y' \in V$ satisfy

$$s = [x] = [x'], \quad t = [y] = [y'].$$

Then $x-x' \in U$, $y-y' \in U$, so

$$(x+y) - (x'+y') = (x-x') + (y-y') \in U.$$

Therefore $[x+y] = [x'+y']$. This shows that addition on V/U is well-defined.

Similarly, we can check that scalar multiplication is well-defined.

Now one can easily verify that V/U is an F -vector space under these operations and that $\pi: V \rightarrow V/U$ is linear. For instance, to check that addition on V/U is associative, note that, for all $s = [x]$, $t = [y]$, $w = [z] \in V/U$,

$$\begin{aligned}(s+t)+w &= ([x]+[y])+[z] \\&= [x+y]+[z] \\&= [(x+y)+z] \\&= [x+(y+z)] && (\text{because } + \text{ is associative on } V) \\&= [x]+[y+z] \\&= [x]+([y]+[z]) \\&= s+(t+w).\end{aligned}$$

We leave the remaining verifications as an exercise.

Now we need to prove uniqueness, i.e., we need to prove that the F -vector space structure on V/U that we have just defined is the unique F -vector space structure on V/U for which $\pi: V \rightarrow V/U$ is linear.

Indeed, let $(V/U, \tilde{+}, \tilde{\cdot})$ be an F -vector space structure on V/U for which $\pi: V \rightarrow V/U$ is linear. Let $s = [x]$, $t = [y] \in V/U$. Then

$$s \tilde{+} t = [x] \tilde{+} [y] = \pi(x) \tilde{+} \pi(y) = \pi(x+y) = [x+y] = s+t.$$

So $\tilde{+}$ agrees with the previously defined addition $+$ on V/U . Similarly the scalar multiplication $\tilde{\cdot}$ agrees with the previously defined scalar multiplication on V/U .

It remains to prove that $\ker(\pi) = U$. To see this, note that for any $x \in V$,

$$x \in \ker(\pi) \Leftrightarrow \pi(x) = 0_{V/U}$$

$$\Leftrightarrow [x] = [0]$$

$$\Leftrightarrow x - 0 \in U$$

$$\Leftrightarrow x \in U.$$

(ii) Suppose $Z: V \rightarrow W$ is a linear map such that $\ker(Z) \supseteq U$.

We define $\sigma: V/U \rightarrow W$ by setting $\sigma([x]) = Z(x)$ for all $[x] \in V/U$.

We need to check that σ is well-defined. Suppose $[x] = [x'] \in V/U$.

Then $x - x' \in U \subseteq \ker(Z)$, so

$$Z(x) = Z(x' + (x - x')) = Z(x') + Z(x - x') = Z(x') + 0 = Z(x').$$

This shows that σ is well-defined.

Now we check that σ is linear. Let $\lambda, \mu \in F$, $[x], [y] \in V/U$. Then

$$\begin{aligned}\sigma(\lambda[x] + \mu[y]) &= \sigma([\lambda x + \mu y]) \\ &= Z(\lambda x + \mu y) \\ &= \lambda Z(x) + \mu Z(y) \\ &= \lambda \sigma([x]) + \mu \sigma([y]).\end{aligned}$$

Now we prove that $\sigma \circ \pi = Z$. Let $x \in V$. Then

$$(\sigma \circ \pi)(x) = \sigma(\pi(x)) = \sigma([x]) = Z(x).$$

To show uniqueness, suppose $\sigma': V/U \rightarrow W$ is a linear map satisfying $\sigma' \circ \pi = \tau$.

Then $\sigma' \circ \pi = \sigma \circ \pi$. Since π is surjective, this implies that $\sigma' = \sigma$. \square

Corollary. Let V be a f.d.v.s. over F and let U be a subspace of V . Then V/U is a f.d.v.s. and $\dim(V/U) = \dim V - \dim U$.

Proof. Since $\pi: V \rightarrow V/U$ is surjective, $\text{im}(\pi) = V/U$. By Rank-Nullity theorem, $\dim(V/U) = \text{rank}(\pi) = \dim(V) - \text{nullity}(\pi) = \dim(V) - \dim(U)$. \square

Prop. Let V be an F.v.s., let U be a subspace of V and let $\pi: V \rightarrow V/U$ denote the canonical projection. The function

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{subspaces of } V \\ \text{containing } U \end{array} \right\} & \longrightarrow & \left\{ \text{subspaces of } V/U \right\} \\ W & \longmapsto & \pi(W) \end{array}$$

is well-defined and bijective with inverse given by $\Delta \mapsto \pi^{-1}(\Delta)$. Moreover, this bijection preserves inclusions, i.e., if W_1, W_2 are subspaces of V containing U , then $W_1 \subseteq W_2$ if and only if $\pi(W_1) \subseteq \pi(W_2)$.