## Annihilators

**Def.** Let $S$ be a subset of an $F$-vector space $V$. The annihilator of $S$ is the set

$$S^\perp := \{ \theta \in V^* \mid \langle \theta, x \rangle = 0 \quad \forall x \in S \}.$$

It is easy to see that $S^\perp$ is a subspace of $V^*$. Indeed, $0_{V^*} \in S^\perp$ (since $\langle 0_{V^*}, x \rangle = 0$ for all $x \in V$) and for any $\theta, \phi \in S^\perp$, scalars $\lambda, \mu \in F$ and $x \in S$, we have

$$\langle \lambda\theta + \mu\phi, x \rangle = \lambda \langle \theta, x \rangle + \mu \langle \phi, x \rangle = \lambda \cdot 0 + \mu \cdot 0 = 0$$

**Prop** Let $V$ be a f.d.v.s. over $F$. Let $W \leq V$ be a subspace. Then

$$\dim(W) + \dim(W^\perp) = \dim V.$$

**Proof.** Let $r = \dim(W)$ and $n = \dim(V)$. Let $(e_1, \ldots, e_r)$ be an ordered basis for $W$. Extend it to an ordered basis $(e_1, \ldots, e_n)$ for $V$. Let $(\varepsilon_1, \ldots, \varepsilon_n)$ be the dual basis associated with $(e_1, \ldots, e_n)$. For each $i = r+1, \ldots, n$,

$$\langle \varepsilon_i, e_j \rangle = 0 \quad \text{for all } 1 \leq j \leq r,$$

so $\langle \varepsilon_i, x \rangle = 0$ for all $x \in W$. Thus $\varepsilon_{r+1}, \ldots, \varepsilon_n \in W^\perp$.

We will prove that $(\varepsilon_{r+1}, \ldots, \varepsilon_n)$ is a basis for $W^\perp$. Since $(\varepsilon_1, \ldots, \varepsilon_n)$ is a basis for $V^*$, in particular it is linearly independent and therefore $(\varepsilon_{r+1}, \ldots, \varepsilon_n)$ is also linearly independent. We need to show $\operatorname{span}\{\varepsilon_{r+1}, \ldots, \varepsilon_n\} = W^\perp$.

Let $\phi \in W^\perp$. Since $(\varepsilon_1, \ldots, \varepsilon_n)$ is an ordered basis for $V^*$, $\phi = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$ for some $\lambda_1, \ldots, \lambda_n \in F$. Also, since $e_1, \ldots, e_r \in W$ and $\phi \in W^\perp$, we know that $\langle \phi, e_j \rangle = 0$ for $1 \leq j \leq r$. Therefore, for $1 \leq j \leq r$,

$$0 = \langle \phi, e_j \rangle = \langle \sum_{i=1}^{n} \lambda_i \varepsilon_i, e_j \rangle = \sum_{i=1}^{n} \lambda_i \langle \varepsilon_i, e_j \rangle = \lambda_j.$$

Hence $\phi = \lambda_{r+1} \varepsilon_{r+1} + \cdots + \lambda_n \varepsilon_n \in \operatorname{span}\{\varepsilon_{r+1}, \ldots, \varepsilon_n\}$.

Since we have seen that $(\varepsilon_{r+1}, \ldots, \varepsilon_n)$ is an ordered basis for $W^\perp$, we have $\dim(W^\perp) = n - r$. Therefore

$$\dim(W) + \dim(W^\perp) = r + (n-r) = n = \dim(V).$$

$\square$

Corollary. Let $V$ be a f.d.v.s. over $F$. Let $S \subseteq V$ be a subset of $V$. Then $S^{\perp} = \operatorname{span}(S)^{\perp}$. In particular, $\dim(S^{\perp}) = \dim(V) - \dim(\operatorname{span}(S))$.

Proof. Exercise.

## Dual maps

Def. Let $U, V$ be f.d.v.s. over $F$ and let $\sigma \in \mathcal{L}(U,V)$. The dual of $\sigma$ is the linear map $\sigma^* \in \mathcal{L}(V^*, U^*)$ defined by $\sigma^*(\theta) = \theta \circ \sigma$ for all $\theta \in V^*$.

$$\begin{tikzcd} V \arrow[r, "\theta"] & F \\ U \arrow[u, "\sigma"] \arrow[ur, dashed, "\theta\circ\sigma"'] & \end{tikzcd}$$

Exercise. Prove that $\sigma^*$ is indeed a linear map from $V^*$ to $U^*$.

Prop. The dualizing map

$$\begin{aligned} \Psi : \mathcal{L}(U,V) &\longrightarrow \mathcal{L}(V^*, U^*) \\ \sigma &\longmapsto \sigma^* \end{aligned}$$

is an isomorphism of $F$-vector spaces.

Proof. First we show that $\Psi$ is linear. Let $\lambda, \mu \in F$ and let $\sigma, \tau \in \mathcal{L}(U,V)$. We want to check that the elements

$$\Psi(\lambda\sigma + \mu\tau) = (\lambda\sigma + \mu\tau)^* \quad \text{and} \quad \lambda\Psi(\sigma) + \mu\Psi(\tau) = \lambda\sigma^* + \mu\tau^*$$

of $\mathcal{L}(V^*, U^*)$ are equal.

Let $\theta \in V^*$. Then

$$\Psi(\lambda\sigma + \mu\tau)(\theta) = \theta \circ (\lambda\sigma + \mu\tau) \qquad \text{(by def. of dual map)}$$

$$= \lambda(\theta \circ \sigma) + \mu(\theta \circ \tau) \qquad \text{(by linearity of } \theta\text{)}$$

$$= \lambda\, \Psi(\sigma)(\theta) + \mu\, \Psi(\tau)(\theta) \qquad \text{(by def. of dual map)}$$

$$= \left(\lambda\, \Psi(\sigma) + \mu\, \Psi(\tau)\right)(\theta) \qquad \text{(by def. of } F\text{-vector space structure on } \mathcal{L}(V^*, U^*)\text{)}$$

Since this holds for all $\theta \in V^*$, we conclude that

$$\Psi(\lambda\sigma + \mu\tau) = \lambda\, \Psi(\sigma) + \mu\, \Psi(\tau)$$

Next we show that $\Psi$ is injective. Let $\sigma \in \ker \Psi$. Then $\sigma^* = \Psi(\sigma) = 0$.

Now

$$\sigma^* = 0_{\mathcal{L}(V^*, U^*)} \iff \sigma^*(\theta) = 0 \quad \forall \theta \in V^*$$

$$\iff \theta \circ \sigma = 0 \quad \forall \theta \in V^*$$

$$\iff (\theta \circ \sigma)(u) = 0 \quad \forall \theta \in V^*, \ \forall u \in U.$$

$$\iff \theta(\sigma(u)) = 0 \quad \forall \theta \in V^*, \ \forall u \in U.$$

Suppose $\sigma(x) \neq 0$ for some $x \in U$. Then, as we have seen, there exists $\varphi \in V^*$ such that $\varphi(\sigma(x)) \neq 0$, which contradicts the fact that $\theta(\sigma(u)) = 0 \ \forall \theta \in V^*, \forall u \in U$.

So $\sigma(u) = 0$ for all $u \in U$, i.e., $\sigma = 0$. Thus $\ker \Psi = \{0\}$ and $\Psi$ is injective.

Surjectivity follows by noting that

$$\begin{aligned}\dim \mathcal{L}(U, V) &= \dim(U) \cdot \dim(V) \\ &= \dim(U^*) \cdot \dim(V^*) \\ &= \dim \mathcal{L}(V^*, U^*).\end{aligned}$$

$\square$

<u>Prop</u>. Let $U, V, W$ be f.d. v.s. over $F$. Let $\sigma \in \mathcal{L}(U, V)$, $\tau \in \mathcal{L}(V, W)$.

Then $(\tau \circ \sigma)^* = \sigma^* \circ \tau^*$.

$$\begin{tikzcd} U \arrow[r, "\sigma"] \arrow[rr, bend right, "\tau \circ \sigma"'] & V \arrow[r, "\tau"] & W \end{tikzcd}$$

$$\begin{tikzcd} U^* & V^* \arrow[l, "\sigma^*"'] & W^* \arrow[l, "\tau^*"'] \arrow[ll, bend left, "\sigma^* \circ \tau^*"] \end{tikzcd}$$

<u>Proof</u>. Let $\theta \in W^*$. Then

$$\begin{aligned}(\sigma^* \circ \tau^*)(\theta) &= \sigma^*(\tau^*(\theta)) \\ &= \sigma^*(\theta \circ \tau) \\ &= (\theta \circ \tau) \circ \sigma \\ &= \theta \circ (\tau \circ \sigma) \\ &= (\tau \circ \sigma)^*(\theta).\end{aligned}$$

Since this holds for all $\theta \in W^*$, we conclude $\sigma^* \circ \tau^* = (\tau \circ \sigma)^*$. $\square$

**Prop.** Let $U, V$ be f.d.v.s. over $F$. Let $\sigma \in \mathcal{L}(U,V)$. Then

(a) $\ker(\sigma^*) = \operatorname{im}(\sigma)^\perp$

(b) $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma^*)$

**Proof.**

(a) Let $\theta \in V^*$. Then

$$\begin{aligned}
\theta \in \ker(\sigma^*) &\iff \sigma^*(\theta) = 0 \\
&\iff \theta \circ \sigma = 0 \\
&\iff (\theta \circ \sigma)(u) = 0 \quad \forall u \in U \\
&\iff \theta(\sigma(u)) = 0 \quad \forall u \in U \\
&\iff \theta(v) = 0 \quad \forall v \in \operatorname{im}(\sigma) \\
&\iff \theta \in \operatorname{im}(\sigma)^\perp.
\end{aligned}$$

Thus $\ker(\sigma^*) = \operatorname{im}(\sigma)^\perp$.

(b) By part (a),

$$\begin{aligned}
\operatorname{nullity}(\sigma^*) &= \dim(\operatorname{im}(\sigma)^\perp) \\
&= \dim V - \dim(\operatorname{im}(\sigma)) \\
&= \dim V - \operatorname{rank}(\sigma).
\end{aligned}$$

By the Rank-Nullity theorem applied to $\sigma^*$,

$$\begin{aligned}
\operatorname{nullity}(\sigma^*) &= \dim V^* - \operatorname{rank}(\sigma^*) \\
&= \dim V - \operatorname{rank}(\sigma^*).
\end{aligned}$$

Combining both equalities,

$$\dim V - \operatorname{rank}(\sigma) = \operatorname{nullity}(\sigma^*) = \dim V - \operatorname{rank}(\sigma^*),$$

so $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma^*)$. $\square$

## Matrix representation of dual maps

**Def.** Let $A \in M_{m\times n}(F)$. The transpose of $A$ is the matrix $A^T \in M_{n\times m}(F)$ with $(j,i)$-th entry equal to the $(i,j)$-th entry of $A$ for $1 \le i \le m$, $1 \le j \le n$.

**Prop.** Let $U, V$ be f.d.v.s. over $F$. Let $\alpha = (e_1, \dots, e_n)$, $\beta = (f_1, \dots, f_m)$ be ordered bases for $U, V$ respectively. (where $n = \dim U$, $m = \dim V$). Let $\alpha^*, \beta^*$ be the ordered bases for $U^*, V^*$ dual to $\alpha, \beta$ respectively. Then for any $\sigma \in \mathcal{L}(U,V)$

$$_{\alpha^*}[\sigma^*]_{\beta^*} = \left( _{\beta}[\sigma]_{\alpha} \right)^T.$$

Proof. Say $\alpha^* = (\varepsilon_1, \dots, \varepsilon_n)$, $\beta^* = (\phi_1, \dots, \phi_m)$.

Let $A = (a_{ij}) = {}_{\beta}[\sigma]_{\alpha} \in F^{m \times n}$

$B = (b_{ji}) = {}_{\alpha^*}[\sigma^*]_{\beta^*} \in F^{n \times m}$.

This means that

$$\sigma(e_j) = \sum_{k=1}^{m} a_{kj} f_k \quad \text{for all } j \in \{1, \dots, n\},$$

$$\sigma^*(\phi_i) = \sum_{l=1}^{m} b_{li} \varepsilon_l \quad \text{for all } i \in \{1, \dots, m\}.$$

Then, for $i = 1, \dots, m$, $j = 1, \dots, n$,

$$\langle \phi_i, \sigma(e_j) \rangle = \langle \phi_i, \sum_{k=1}^{m} a_{kj} f_k \rangle = \sum_{k=1}^{m} a_{kj} \underbrace{\langle \phi_i, f_k \rangle}_{\delta_{ik}} = a_{ij}$$

$$\langle \sigma^*(\phi_i), e_j \rangle = \langle \sum_{l=1}^{n} b_{li} \varepsilon_l, e_j \rangle = \sum_{l=1}^{n} b_{li} \underbrace{\langle \varepsilon_l, e_j \rangle}_{\delta_{lj}} = b_{ji}.$$

Also, by definition,

$$\langle \phi_i, \sigma(e_j) \rangle = \phi_i(\sigma(e_j)) = (\phi_i \circ \sigma)(e_j)$$

$$= \sigma^*(\phi_i)(e_j) = \langle \sigma^*(\phi_i), e_j \rangle.$$

Altogether,

$$a_{ij} = \langle \phi_i, \sigma(e_j) \rangle = \langle \sigma^*(\phi_i), e_j \rangle = b_{ji} \quad \text{for all } i = 1, \dots, m, \; j = 1, \dots, n,$$

so $B = A^T$. $\square$