__Corollary.__ Let $A \in F^{m\times n}$, $B \in F^{n \times p}$. Then

$$(AB)^T = B^T A^T.$$

__Proof.__ Let $U = F^p$, $V = F^n$, $W = F^m$. Let α, β, γ be the standard ordered bases of U, V, W respectively. Let $\alpha^*, \beta^*, \gamma^*$ be the ordered bases for U^*, V^*, W^* dual to α, β, γ respectively. Let $T \in \mathcal{L}(U, V)$, $\sigma \in \mathcal{L}(V, W)$ be the linear maps with ${}_{\beta}[T]_{\alpha} = B$, ${}_{\gamma}[\sigma]_{\beta} = A$.

We know that ${}_{\gamma}[\sigma \circ T]_{\alpha} = {}_{\gamma}[\sigma]_{\beta}\,{}_{\beta}[T]_{\alpha} = AB.$

By the previous proposition,

$${}_{\beta^*}[\sigma^*]_{\gamma^*} = A^T, \quad {}_{\alpha^*}[T^*]_{\beta^*} = B^T, \quad {}_{\alpha^*}[(\sigma \circ T)^*]_{\gamma^*} = (AB)^T.$$

Since $(\sigma \circ T)^* = T^* \circ \sigma^*$, we have

$$(AB)^T = {}_{\alpha^*}[(\sigma \circ T)^*]_{\gamma^*} = {}_{\alpha^*}[T^* \circ \sigma^*]_{\gamma^*} = {}_{\alpha^*}[T^*]_{\beta^*}\,{}_{\beta^*}[\sigma^*]_{\gamma^*} = B^T A^T.$$

$\square$

Prop Let U, V be f.d.v.s. over F. Let $n = \dim U$, $m = \dim V$. Let α, β be ordered bases for U, V respectively. Let α^*, β^* be the ordered bases for U^*, V^* dual to α, β respectively.

(i) Let $\theta \in V^*$. Then $[\theta]_{\beta^*} = \left({}_{1_F}[\theta]_\beta \right)^T$.

(ii) Let $\theta \in V^*$ and let $\tau \in \mathcal{L}(U, V)$. Let $A = {}_\beta[\tau]_\alpha$ and $v = {}_{1_F}[\theta]_\beta \in F^{1 \times m}$. Then $[\tau^*(\theta)]_{\alpha^*} = (v \cdot A)^T$.

Proof. (i) Say $\beta = (f_1, \dots, f_m)$, $\beta^* = (\phi_1, \dots, \phi_m)$.

Suppose $\theta = \lambda_1 \phi_1 + \dots + \lambda_m \phi_m$, where $\lambda_1, \dots, \lambda_m \in F$.

This means that

$$[\theta]_{\beta^*} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}.$$

Also, for $i = 1, \dots, m$,

$$\theta(f_i) = \langle \lambda_1 \phi_1 + \dots + \lambda_m \phi_m, f_i \rangle = \lambda_i,$$

so $${}_{1_F}[\theta]_\beta = [\lambda_1 \ \cdots \ \lambda_n].$$

Hence $$[\theta]_{\beta^*} = \left({}_{1_F}[\theta]_\beta \right)^T.$$

(ii) $$[\sigma^*(\theta)]_{\alpha^*} = [\theta\circ\sigma]_{\alpha^*}$$

$$= \left({}_{1_F}[\theta\circ\sigma]_{\alpha}\right)^T \qquad \text{(by part (a))}$$

$$= \left({}_{1_F}[\theta]_{\beta}\cdot {}_{\beta}[\sigma]_{\alpha}\right)^T$$

$$= (v\cdot A)^T. \qquad \square$$

Rk It can be helpful to think of elements in the dual space as "row vectors" with dual maps acting on them by right matrix multiplication (ie, we forget about the transposes above).

Lemma Let U, V, W be f.d.v.s. over F. Let $\sigma \in \mathcal{L}(U,V)$ and let $\varphi \in \mathcal{L}(U,W)$. Then, there exists $\tau \in \mathcal{L}(V,W)$ such that $\tau\sigma = \varphi$ if and only if $\ker(\sigma) \leq \ker(\varphi)$.

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\sigma"'] & W \\ V \arrow[ur, dashed, "\tau"'] & \end{tikzcd}$$

Rk In particular, taking $W = F$, we see that given $\eta \in U^*$, there exists $\theta \in V^*$ such that $\sigma^*(\theta) = \theta\circ\sigma = \eta$ if and only if $\ker(\sigma) \leq \ker(\eta)$, i.e.,

$$\eta \in \operatorname{im}(\sigma^*) \iff \ker(\sigma) \leq \ker(\eta).$$

Prop. Let $\sigma \in \mathcal{L}(U,V)$. Then $\operatorname{im}(\sigma^*) = \ker(\sigma)^\perp$.

Proof. Let $\eta \in U^*$. Then

$$\begin{aligned}
\eta \in \operatorname{im}(\sigma^*) &\iff \ker(\sigma) \subseteq \ker(\eta) \\
&\iff \eta(x) = 0 \quad \forall x \in \ker(\sigma) \\
&\iff \langle \eta, x \rangle = 0 \quad \forall x \in \ker(\sigma) \\
&\iff \eta \in \ker(\sigma)^\perp.
\end{aligned}$$

Therefore $\operatorname{im}(\sigma^*) = \ker(\sigma)^\perp$. $\square$

Rk. Previously, we also proved that $\ker(\sigma^*) = \operatorname{im}(\sigma)^\perp$.

Prop. Let $\sigma \in \mathcal{L}(U,V)$. Then:

(i) σ is injective if and only if σ^* is surjective.

(ii) σ is surjective if and only if σ^* is injective.

Proof.

$$\begin{aligned}
\text{(i)}\quad \sigma \text{ is injective} &\iff \ker(\sigma) = \{0\} \\
&\iff \ker(\sigma)^\perp = U^* \\
&\iff \operatorname{im}(\sigma^*) = U^* \\
&\iff \sigma^* \text{ is surjective.}
\end{aligned}$$

(ii) σ is surjective $\Leftrightarrow$ $\mathrm{im}(\sigma) = V$

$\Leftrightarrow$ $\mathrm{im}(\sigma)^{\perp} = \{0\}$

$\Leftrightarrow$ $\ker(\sigma^*) = \{0\}$

$\Leftrightarrow$ σ^* is injective.

Example Let U, V be $\mathbb{R}$-vector spaces. Suppose $\dim(U) = 3$, $\dim(V) = 2$. Let $\alpha = (e_1, e_2, e_3)$, $\beta = (f_1, f_2)$ be ordered bases for U, V, respectively. Let $\alpha^* = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\beta^* = (\phi_1, \phi_2)$ be the ordered bases for U^*, V^* dual to α, β, respectively. Let $\sigma \in \mathcal{L}(U, V)$ be the linear map with ${}_{\beta}[\sigma]_{\alpha} = A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & -4 \end{bmatrix}$.

- $\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \end{bmatrix} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$,

  so $\mathrm{im}(\sigma) = \mathrm{span}\{f_1 + 2f_2\}$.

- $\mathrm{Null}(A) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

  $= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; x + 2y - 2z = 0,\; 2x + 4y - 4z = 0 \right\}$

  $= \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

so $\ker(\sigma) = \mathrm{span}\{-2e_1 + e_2,\ 2e_1 + e_3\}$.

- $\mathrm{Row}(A) = \mathrm{span}\{[1\ 2\ -2],\ [2\ 4\ -4]\}$
$= \mathrm{span}\{[1\ 2\ -2]\}$,

$\mathrm{Col}(A^T) = \mathrm{span}\left\{\begin{bmatrix}1\\2\\-2\end{bmatrix}\right\}$

so $\mathrm{im}(\sigma^*) = \mathrm{span}\{\varepsilon_1 + 2\varepsilon_2 - 2\varepsilon_3\}$

Identify $\mathbb{R}^{1\times 3}$ with $(\mathbb{R}^{3\times 1})^*$ by identifying each element $[\alpha\ \beta\ \gamma] \in \mathbb{R}^{1\times 3}$ with the linear map

$$\mathbb{R}^{3\times 1} \longrightarrow \mathbb{R}$$
$$\begin{bmatrix}x\\y\\z\end{bmatrix} \longmapsto [\alpha\ \beta\ \gamma]\begin{bmatrix}x\\y\\z\end{bmatrix} = \alpha x + \beta y + \gamma z.$$

Similarly, identify $\mathbb{R}^{1\times 2}$ with $(\mathbb{R}^{2\times 1})^*$. Then:

- $\mathrm{Null}(A)^\perp = \left\{[\alpha\ \beta\ \gamma] \;\middle|\; [\alpha\ \beta\ \gamma]\begin{bmatrix}x\\y\\z\end{bmatrix} = 0 \text{ for all } \begin{bmatrix}x\\y\\z\end{bmatrix} \in \mathrm{Null}(A)\right\}$

$$= \left\{[\alpha\ \beta\ \gamma] \;\middle|\; [\alpha\ \beta\ \gamma]\begin{bmatrix}-2\\1\\0\end{bmatrix} = [\alpha\ \beta\ \gamma]\begin{bmatrix}2\\0\\1\end{bmatrix} = 0\right\}$$

$$= \{[\alpha\ \beta\ \gamma] \mid -2\alpha + \beta = 0,\ 2\alpha + \gamma = 0\}$$

$$= \mathrm{span}\{[1\ 2\ -2]\}.$$

Then $\ker(\sigma)^{\perp} = \text{span}\{\varepsilon_1 + 2\varepsilon_2 - 2\varepsilon_3\}$.

Note that $\text{Row}(A) = \text{Null}(A)^{\perp}$, so $\text{im}(\sigma^*) = \ker(\sigma)^{\perp}$.

- $\text{Null}^T(A) := \{[u\ v] \mid [u\ v]A = [0\ 0\ 0]\}$ | $\text{Null}(A^T) = \text{span}\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}$

$= \text{span}\{[2\ -1]\}$

so $\ker(\sigma^*) = \text{span}\{2\phi_1 - \phi_2\}$.

- $\text{Col}(A)^{\perp} = \left\{[u\ v] \mid [u\ v]\begin{bmatrix}x\\y\end{bmatrix} \text{ for all } \begin{bmatrix}x\\y\end{bmatrix} \in \text{Col}(A)\right\}$

$= \left\{[u\ v] \mid [u\ v]\begin{bmatrix}1\\2\end{bmatrix} = 0\right\}$

$= \text{span}\{[2\ -1]\}$

So $\text{im}(\sigma)^{\perp} = \text{span}\{2\phi_1 - \phi_2\}$.

Note that $\text{Null}^T(A) = \text{Col}(A)^{\perp}$, so $\ker(\sigma^*) = \text{im}(\sigma)^{\perp}$.