

Solutions Homework 8

Problem 1. Let V be an F -vector space. Let $\sigma \in \text{End}(V)$. Let U be a σ -invariant subspace of V . Let $v_1, \dots, v_m \in V$ be eigenvectors of σ with distinct eigenvalues $\lambda_1, \dots, \lambda_m \in F$. Prove that if $v_1 + \dots + v_m \in U$, then $v_1, \dots, v_m \in U$.

Solution. We proceed by induction on m .

If $m = 1$, then the result is clear.

Let $m > 1$ and suppose that the result is true for $m - 1$. Let $v_1, \dots, v_m \in V$ be eigenvectors of σ with distinct eigenvalues $\lambda_1, \dots, \lambda_m \in F$ and suppose that $v_1 + \dots + v_m \in U$. Since U is invariant under σ , it is also invariant under $\sigma - \lambda_m \mathbb{1}_V$. Therefore

$$(\sigma - \lambda_m \mathbb{1}_V)(v_1 + \dots + v_m) \in U.$$

We have that

$$\begin{aligned} (\sigma - \lambda_m \mathbb{1}_V)(v_1 + \dots + v_m) &= (\sigma - \lambda_m \mathbb{1}_V)(v_1) + \dots + (\sigma - \lambda_m \mathbb{1}_V)(v_m) \\ &= (\lambda_1 - \lambda_m)v_1 + \dots + (\lambda_{m-1} - \lambda_m)v_{m-1} \end{aligned}$$

Note that $\lambda_i - \lambda_m \neq 0$ for all $i \in \{1, \dots, m-1\}$, so the vectors $(\lambda_1 - \lambda_m)v_1, \dots, (\lambda_{m-1} - \lambda_m)v_{m-1}$ are eigenvectors of σ with eigenvalues $\lambda_1, \dots, \lambda_{m-1}$, respectively (and in particular with distinct eigenvalues). By the induction hypothesis, it follows that

$$(\lambda_1 - \lambda_m)v_1, \dots, (\lambda_{m-1} - \lambda_m)v_{m-1} \in U.$$

Since $\lambda_i - \lambda_m \neq 0$ for all $i \in \{1, \dots, m-1\}$, it follows that $v_1, \dots, v_{m-1} \in U$. Finally, since $v_1 + \dots + v_m \in U$, we also have

$$v_m = (v_1 + \dots + v_m) + (-v_1) + \dots + (-v_{m-1}) \in U.$$

Problem 2. Let V be a finite-dimensional F -vector space. Let $\sigma \in \text{End}(V)$, let $\sigma^* \in \text{End}(V^*)$ be the corresponding dual map, and let $\lambda \in F$. Prove that λ is an eigenvalue of σ if and only if λ is an eigenvalue of σ^* .

Solution. Note that $(\sigma - \lambda \mathbb{1}_V)^* = \sigma^* - \lambda \mathbb{1}_{V^*}^* = \sigma^* - \lambda \mathbb{1}_{V^*}$. Also note that, since V is finite-dimensional, we know that $\sigma - \lambda \mathbb{1}_V$ is injective if and only if it is surjective. Therefore

$$\begin{aligned} \lambda \text{ is an eigenvalue of } \sigma &\iff \sigma - \lambda \mathbb{1}_V \text{ is injective} \\ &\iff \sigma - \lambda \mathbb{1}_V \text{ is surjective} \\ &\iff \sigma^* - \lambda \mathbb{1}_{V^*} \text{ is injective} \\ &\iff \lambda \text{ is an eigenvalue of } \sigma^*. \end{aligned}$$

Problem 3. Let V be an \mathbb{R} -vector space. Let $\sigma \in \text{End}(V)$. Prove that 9 is an eigenvalue of σ^2 if and only if 3 or -3 is an eigenvalue of σ .

Solution.

(\Rightarrow) Suppose that 9 is an eigenvalue of σ^2 . Then $\sigma^2 - 9\mathbb{1}_V$ is not injective. Since $\sigma^2 - 9\mathbb{1}_V = (\sigma - 3\mathbb{1}_V)(\sigma + 3\mathbb{1}_V)$ we deduce that one of the endomorphisms $\sigma - 3\mathbb{1}_V$ and $\sigma + 3\mathbb{1}_V$ is not injective (otherwise $\sigma^2 - 9\mathbb{1}_V$ would be injective, since the composition of injective functions is an injective function). Therefore 3 or -3 is an eigenvalue of σ .

(\Leftarrow) Suppose that 3 or -3 is an eigenvalue of σ . Then there exists a non-zero element $v \in V$ such that $\sigma(v) = \lambda v$, with $\lambda = 3$ or $\lambda = -3$. Then $\sigma^2(v) = \sigma(\sigma(v)) = \sigma(\lambda v) = \lambda \sigma(v) = \lambda^2 v = 9v$, so 9 is an eigenvalue of σ^2 .

Problem 4. Let V be a finite-dimensional F -vector space. Let $\sigma \in \text{End}(V)$. Let U be a σ -invariant subspace of V . Let $\bar{\sigma} : V/U \rightarrow V/U$ be the endomorphism of V/U induced by σ . Prove that every eigenvalue of $\bar{\sigma}$ is an eigenvalue of σ .

Solution. Let λ be an eigenvalue of $\bar{\sigma}$. Let $\tau = \sigma - \lambda \mathbb{1}_V$. Since U is invariant under σ , it is also invariant under τ and the endomorphism of V/U induced by τ is $\bar{\tau} = \bar{\sigma} - \lambda \mathbb{1}_{V/U}$.

Since λ is an eigenvalue of $\bar{\sigma}$, we know that $\bar{\tau}$ is not injective. Since V/U is finite-dimensional, this is equivalent to saying that $\bar{\tau}$ is not surjective.

We claim that τ is also not surjective. Indeed, suppose that τ is surjective. Let $y + U \in V/U$. Then there exists $x \in V$ such that $\tau(x) = y$ and therefore there exists $x + U \in V/U$ such that $\bar{\tau}(x + U) = \tau(x) + U = y + U$. Thus $\bar{\tau}$ is surjective, which is a contradiction.

Thus we have that τ is not surjective. Since V is finite-dimensional, τ is also not injective, so λ is an eigenvalue of σ .