Minimal polynomial and Jordan Forms

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Throughout this note, we denote the base field by $\mathbb{F}$, which is $\mathbb{R}$ or $\mathbb{C}$. All the vector spaces are considered over $\mathbb{F}$.

We also saw that any matrix over over complex number is similar to a block diagonal matrix, where each block is upper triangular matrices with eigenvalues on diagonal. Using the results on generalized eigen-space, we have also proven the Cayley-Hamilton Theorem, which says that the characteristic polynomial of T becomes zero when evaluated on T. We now study the minimal polynomial which is the smallest degree monic polynomial which annihilates T. Using the result on generalized eigenspace and minimal polynomial we study Jordan forms.

1 Minimal Polynomial

Definition 1.1. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. A monic polynomial $m_T(t)$ of least degree such that

$$m_T(T) = 0$$

is called the **minimal polynomial** of T. Equivalently, $m_T(t)$ is the unique monic polynomial of smallest degree such that $m_T(t) = 0$. Recall a polynomial is said to be monic if coefficient of higher degree term is 1.

Example 1.2. Consider the polynomials

$$p_1(t) = t^3 - 2t + 1, \qquad p_2(t) = 3t^2 + t - 5.$$

- $p_1(t)$ is **monic** as the coefficient of t^3 is 1.
- $p_2(t)$ is **not monic** because the leading coefficient of t^2 is $3 \neq 1$.

Remark 1.3. (**Uniqueness of monic polynomial**) Note that there exist a unique monic polynomial $m(t)$ of least degree such that $m(T) = 0$. If $m'(t)$ is any other polynomial which also has this property, then using division algorithm, we have $m(t) = m'(t)r(t) + s(t)$, where $s(t)$ is a polynomial of degree less than of $m'(t)$. Since $m(T) = m'(T) = 0$, which implies that $s(T) = 0$, which is a contradiction. This shows that $m'(t)$ divides $m(t)$, and similar argument shows that $m'(t)$ divides $m(t)$ and since both are monic, and hence $m(t) = m'(t)$.

Example 1.4. Consider the 3×3 zero matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is

$$p_A(t) = \det(A - tI) = \det \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix} = -t^3.$$

But the minimal polynomial is

$$m_A(t) = t,$$

since

$$m_A(A) = A = 0 \quad \text{and} \quad m_A(t) = t$$

is the unique monic polynomial of smallest degree satisfying $m_A(A) = 0$.

The following is very nice and useful fact that the minimal polynomial divides the characterstic polynomial.

Proposition 1.5. *Let $T : V \to V$ be a linear operator on finite dimensional complex vector space. Then the minimal polynomial $m(t)$ of divides the characteristic polynomial $p(t)$.*

Proof. By division algorithm, we get $p(t) = m(t)q(t) + s(t)$, where $s(t)$ is a polynomial of degree less than that of $m(t)$. Since $p(T) = 0$ by Cayley-Hamilton theorem and $m(T) = 0$ by definition of $m(t)$. Hence, we get that $s(t) = 0$, which is contradiction as $m(t)$ is the least degree polynomial which annihilates T. Hence $s(t) = 0$. This shows that $m(t)$ divides $p(t)$. This completes the proof. $\square$

We now show that the minimal polynomial and characteristic polynomial has same zeros.

Proposition 1.6. *Then the characteristic polynomial $p_T(t)$ and the minimal polynomial $m_T(t)$ have exactly the same zeros.*

Proof. Since $m(t)$ divides $p(t)$ and hence every root of $m(t)$ is also a root of $p(t)$. Conversely, let λ be an eigenvalue of T and choose a nonzero eigenvector v with $Tv = \lambda v$. Since $m(T) = 0$, we have

$$0 = m_T(T)v = m_T(\lambda)v.$$

As $v \neq 0$, this implies that $m(\lambda) = 0.$. Hence λ is also a root of $m_T(t)$. This completes the proof.

Corollary 1.7. *Let $T : V \to V$ be a linear operator on a finite–dimensional complex vector space V. The eigenvalues of T are precisely the roots of the minimal polynomial.*

Proof. This proof easily follows from the previous proposition. $\square$

Remark 1.8. If $p(t)$ does not have any repeated roots, then $p(t) = m(t)$.

Example 1.9.

$$T : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \qquad T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3,\ 6z_2 + 2z_3,\ 7z_3).$$

With respect to the standard basis, the matrix of T is

$$A = [T] = \begin{pmatrix} 6 & 3 & 4 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{pmatrix}.$$

Since A is upper triangular, and its characteristic polynomial is $p_T(t) = (t-6)^2(t-7)$. For the minimal polynomial, first note that minimal polynomial must have the same zeros as the characteristic polynomial and also $m(t)$ divides $p(t)$, and hence $m(t)$ must be the following two forms:

$$m_T(t) = (t-6)^2(t-7), \text{ or } m_T(t) = (t-6)(t-7)$$

Now we can check easily that $(A-6I)(A-7I) \neq 0$, and hence the minimal polynomial is $m_T(t) = (t-6)^2(t-7)$.

Example 1.10. If $T : V \to V$ be a linear operator on a finite dimensional complex vector space V. If $p(t) = (t-\lambda_1)...(t-\lambda_n)$ where all λ_i are distinct. Then $m(t) = p(t)$. But it may possible that operator have repeated eigenvalues and yet the $m(t) = p(t)$. See the next example.

Example 1.11. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

The characteristic polynomial is $p_A(t) = (t-4)^2$.
Hence the minimal polynomial $m(t)$ is either $(t-4)$ or $(t-4)^2$. Since $A - 4I \neq 0$, hence the minimal polynomial must be

$$m_A(t) = (t-4)^2.$$

Hence in this example the minimal polynomial and the characteristic polynomial coincide.

2 Jordan Form

In the last lecture, we prove that any matrix over complex is similar to block diagonal matrix with each block is upper triangular matrix. We recall the statement for reader's convenience.

Theorem 2.1. *Let $T : V \to V$ be a linear operator on a finite–dimensional complex vector space V with dinstinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ having multiplicities $d_1, d_2, \ldots, d_m$ respectively. Then there exists a basis of V with respect to which T has a* block diagonal *matrix of the form*

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix},$$

where each block A_j is a $d_j \times d_j$ upper triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & * & \cdots & * \\ 0 & \lambda_j & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_j \end{pmatrix},$$

.

We now show that each such upper triangular matrix block A_j is infact has only possible non-zero entries in diagonal and line just above diagonal, known as Jordan block. We begin with a definition.

Definition 2.2. A basis of V is called a **Jordan basis** for a linear operator $T : V \to V$ if there exists a basis of V with respect to which T has a *block diagonal* matrix of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix},$$

where each block A_j is a $d_j \times d_j$ upper triangular matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ 0 & 0 & \lambda_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix}.$$

We call such blocks A_j are **Jordan blocks**. Thus, the matrix of T in a **Jordan basis** is a block diagonal matrix whose blocks are Jordan blocks.

Theorem 2.3. ***(Jordan Canonical Form for Operator)*** *Suppose V is finite dimensional complex vector space and T be a linear operator on V. Then T has a Jordan basis.*

Proof. This needs a proof and we are referring the reader to see the Theorem 8.60 in the book *Linear Algebra Done Right* by Sheldon Axler. □

Corollary 2.4 (**Jordan Canonical Form for Matrices**)**.** *Any complex matrix $A \in M_{n\times n}(\mathbb{C})$ is similar to a block diagonal matrix*

$$A \sim \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix},$$

where each block A_j is a Jordan block.

Remark 2.5. Note that here all eigenvalues λ_i are not necessarily distinct. In other words, two Jordan block's may correspond to same eigenvalue.