

MATH 108 B

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Practice Set: *Self-adjoint, Normal, Positive definite, Unitary, Nilpotent Operators, Generalized Eigenvectors and Minimal Polynomial*

Notation: $\mathcal{L}(V)$ denote the set of all linear operators on V .

1 Normal, Self-adjoint, and Positive definite Operators

1. Let T and U be self-adjoint operators on an inner product space V . Prove that TU is self-adjoint if and only if $TU = UT$.
2. Let V be a complex inner product space, and let T be a linear operator on V . Define

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*).$$

- (a) Prove that T_1 and T_2 are self-adjoint and that

$$T = T_1 + iT_2.$$

- (b) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.
3. Assume that T be self-adjoint linear operator on a finite dimensional complex inner product space V with an adjoint T^* . Show that $\langle T(x), x \rangle$ is real for all $x \in V$.
4. If T and U are positive definite operators such that $TU = UT$, then TU is positive definite.
5. Show that if T and U are positive semi-definite operators such that $T^2 = U^2$, then $T = U$.
6. Read and understand the statements of Spectral Theorems for normal and self-adjoint operators.

2 Unitary Operators

Recall, A square matrix A is called *orthogonal* if $A^tA = AA^t = I$, and *unitary* if $A^*A = AA^* = I$. Here A^t denotes the transpose of the matrix A . We say two matrices A and B are unitarily equivalent (orthogonally equivalent) if and only if there exists a unitary (orthogonal) matrix P such that

$$A = P^*BP.$$

7. Let $A \in M_{n \times n}(\mathbb{C})$. Show that A is normal if and only if A is unitarily equivalent to a diagonal matrix.

8. Are the following two matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

are unitary equivalent?

9. Let A be a real $n \times n$ matrix. Then A is symmetric if and only if A is orthogonally equivalent to a real diagonal matrix.
10. Prove that a matrix that is both unitary and upper triangular must be a diagonal matrix.

3 Nipotent Operators, Generalized eigenvectors, Minimal Polynomial

11. Give an example of a linear operator T on a finite dimensional vector space such that T is not nilpotent, but zero is the only real eigenvalue of T .
12. Show that $I + N$ is invertible matrix if N is nilpotent matrix.

Theorem (Test for Diagonalizability): Let T be a linear operator on a finite-dimensional vector space V . Then T is diagonalizable if and only if the minimal polynomial of T is of the form

$$p(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_k),$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigenvalues of T .

You can use the above theorem as a fact to answer some of the questions.

13. Let A be $M_{n \times n}(\mathbb{C})$ such that $A^2 = A$. Show that A is diagonalizable.
14. Let A be any matrix in $M_{2 \times 2}(\mathbb{C})$ such that

$$A^2 - 3A + 2I = 0.$$

Assume that A is not a diagonal matrix. Show that A is similar to

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

15. Suppose $T \in \mathcal{L}(V)$ is nilpotent and diagonalizable. Show that $T = 0$.

16. Find the minimal polynomial of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

17. Let T be a nilpotent linear operator. Show that its minimal polynomial is $m_T(x) = x^k$ for some integer $k \geq 1$.