

Practice final solutions

MATH 108A, SPRING 2025

NAME: _____

PERM NUMBER: _____

- *The time for this exam is **3 hours**.*
- *The exam has **6 problems**. Each of them is worth 10 points.*
- *No notes, books, calculators or electronic devices are allowed during the exam.*
- *You can use the blank pages at the end as scratch paper.*
- *Unless otherwise stated, in each problem F stands for an arbitrary field.*

Problem 1. [10 points] Let V and W be finite-dimensional F -vector spaces. Let U be a subspace of V such that

$$\dim(W) \geq \dim(V) - \dim(U).$$

Prove that there exists $\tau \in \mathcal{L}(V, W)$ such that $\ker(\tau) = U$.

Solution.

Let $r = \dim(U)$, $n = \dim(V)$, $m = \dim(W)$. By assumption we have $m \geq n - r$. Let (x_1, \dots, x_r) be an ordered basis for U and extend it to an ordered basis $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ for V . Let (w_1, \dots, w_m) be an ordered basis for W . Let $\tau \in \mathcal{L}(V, W)$ be the linear map defined by

$$\begin{aligned}\tau(x_i) &= 0 \quad \text{for } i = 1, \dots, r \\ \tau(y_j) &= w_j \quad \text{for } j = 1, \dots, n - r.\end{aligned}$$

Note that this makes sense since $m \geq n - r$.

We claim that $U = \ker \tau$. Indeed, by definition we have that $x_1, \dots, x_r \in \ker \tau$ and therefore $U = \text{span}\{x_1, \dots, x_r\} \subseteq \ker \tau$. To prove the opposite inclusion, let $v \in \ker \tau$. Since $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ is an ordered basis for V , we can write

$$v = \sum_{i=1}^r \lambda_i x_i + \sum_{j=1}^{n-r} \mu_j y_j$$

for some $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_{n-r} \in F$. Then

$$0 = \tau(v) = \tau\left(\sum_{i=1}^r \lambda_i x_i + \sum_{j=1}^{n-r} \mu_j y_j\right) = \sum_{i=1}^r \lambda_i \tau(x_i) + \sum_{j=1}^{n-r} \mu_j \tau(y_j) = \sum_{j=1}^{n-r} \mu_j w_j.$$

Since (w_1, \dots, w_{n-r}) is linearly independent, it follows that $\mu_1 = \dots = \mu_{n-r} = 0$ and therefore

$$v = \sum_{i=1}^r \lambda_i x_i \in U.$$

Problem 2. [10 points] Let U , V and W be F -vector spaces. Let $\tau \in \mathcal{L}(U, V)$ and let $\sigma \in \mathcal{L}(V, W)$. Prove that the following statements are equivalent:

- (i) $\ker(\sigma \circ \tau) = \ker(\tau)$;
- (ii) $\ker(\sigma) \cap \operatorname{im}(\tau) = \{0\}$.

Solution.

((i) \Rightarrow (ii)) Suppose that $\ker(\sigma \circ \tau) = \ker(\tau)$. Let $y \in \ker(\sigma) \cap \operatorname{im}(\tau)$. Since $y \in \operatorname{im}(\tau)$, we have that $y = \tau(x)$ for some $x \in U$. Now, since $y \in \ker(\sigma)$, we have that

$$0 = \sigma(y) = \sigma(\tau(x)) = (\sigma \circ \tau)(x),$$

so $x \in \ker(\sigma \circ \tau)$. Since $\ker(\sigma \circ \tau) = \ker(\tau)$, we also have that $x \in \ker(\tau)$, so $y = \tau(x) = 0$.

((ii) \Rightarrow (i)) Suppose that $\ker(\sigma) \cap \operatorname{im}(\tau) = \{0\}$. Recall that the inclusion $\ker(\tau) \subseteq \ker(\sigma \circ \tau)$ is true in general: if $x \in \ker(\tau)$, then $\tau(x) = 0$ and therefore $(\sigma \circ \tau)(x) = \sigma(\tau(x)) = \sigma(0) = 0$, so $x \in \ker(\sigma \circ \tau)$. Now we prove the opposite inclusion. Let $x \in \ker(\sigma \circ \tau)$. Then $(\sigma \circ \tau)(x) = 0$, so $\sigma(\tau(x)) = 0$. Then $\tau(x) \in \ker(\sigma) \cap \operatorname{im}(\tau)$, so $\tau(x) = 0$ and therefore $x \in \ker(\tau)$.

Problem 3. [10 points] Let V be an F -vector space. Let U and W be subspaces of V and suppose that $V = U + W$. Prove that the map

$$\begin{aligned}\tau : V/(U \cap W) &\longrightarrow V/U \times V/W \\ x + U \cap W &\longmapsto (x + U, x + W)\end{aligned}$$

is well-defined and that it is an isomorphism of F -vector spaces.

Solution.

Consider the map

$$\begin{aligned}\varphi : V &\longrightarrow V/U \times V/W \\ x &\longmapsto (x + U, x + W).\end{aligned}$$

First we show that φ is linear. Let $x, y \in V$ and let $\lambda, \mu \in F$. Then

$$\begin{aligned}\varphi(\lambda x + \mu y) &= (\lambda x + \mu y + U, \lambda x + \mu y + W) \\ &= (\lambda x + U, \lambda x + W) + (\mu y + U, \mu y + W) \\ &= \lambda(x + U, x + W) + \mu(y + U, y + W) \\ &= \lambda\varphi(x) + \mu\varphi(y).\end{aligned}$$

Now we show that $\ker \varphi = U \cap W$. Indeed,

$$\begin{aligned}\ker \varphi &= \{x \in V \mid (x + U, x + W) = (0 + U, 0 + W)\} \\ &= \{x \in V \mid x \in U \text{ and } x \in W\} \\ &= U \cap W.\end{aligned}$$

Now we prove that φ is surjective. Let $(y + U, z + W) \in V/U \times V/W$. We need to find $x \in V$ such that $x + U = y + U$ and $x + W = z + W$. Equivalently, we need to find $x \in V$ such that $x = y + u$ for some $u \in U$ and $x = z + w$ for some $w \in W$. Since $V = U + W$, there exist $u' \in U$ and $w' \in W$ such that $u' + w' = y - z$. Take $u = -u'$ and $w = w'$. Clearly $u \in U$ and $w \in W$. Also, we have that $-u + w = y - z$, so $y + u = z + w$. Define $x := y + u = z + w$. Then x satisfies $x + U = y + U$ and $x + W = z + W$, so $\varphi(x) = (y + U, z + W)$.

Finally, by the First Isomorphism Theorem, we conclude that the map

$$\begin{aligned}\tau : V/(U \cap W) &= V/\ker(\varphi) \longrightarrow \text{im}(\varphi) = V/U \times V/W \\ x + U \cap W &= x + \ker(\varphi) \longmapsto \varphi(x) = (x + U, x + W)\end{aligned}$$

is a well-defined isomorphism of F -vector spaces.

Problem 4. Let $\alpha = (e_1, e_2, e_3)$ be the standard ordered basis for \mathbb{R}^3 and let $\alpha^* = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ be the ordered basis for $(\mathbb{R}^3)^*$ dual to α . Let $\beta = (e_1, e_1 + e_2, e_1 + e_2 + e_3)$, which is another ordered basis for \mathbb{R}^3 , and let $\beta^* = (\psi_1, \psi_2, \psi_3)$ be the ordered basis for $(\mathbb{R}^3)^*$ dual to β .

- (a) [5 points] Write each element of β^* as a linear combination of the elements of α^* .
- (b) [5 points] Consider the subspace $U = \text{span}\{e_1 + e_2 + e_3\}$ of \mathbb{R}^3 . Find a basis for U^\perp , expressing each element of this basis as a linear combination of the elements of α^* .

Solution.

- (a) Let $\theta \in (\mathbb{R}^3)^*$. Since α^* is an ordered basis for $(\mathbb{R}^3)^*$, there exist unique scalars $\lambda_1, \lambda_2, \lambda_3 \in F$ such that

$$\theta = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \lambda_3 \varepsilon_3.$$

Evaluating at e_1, e_2, e_3 we find

$$\lambda_1 = \theta(e_1), \quad \lambda_2 = \theta(e_2), \quad \lambda_3 = \theta(e_3),$$

so $\theta = \theta(e_1)\varepsilon_1 + \theta(e_2)\varepsilon_2 + \theta(e_3)\varepsilon_3$. We apply this observation to express the elements $\psi_1, \psi_2, \psi_3 \in (\mathbb{R}^3)^*$ as linear combinations of $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

Let $v_1 = e_1$, $v_2 = e_1 + e_2$, $v_3 = e_1 + e_2 + e_3$. Since the ordered basis (ψ_1, ψ_2, ψ_3) is dual to the ordered basis (v_1, v_2, v_3) , we know that

$$\psi_i(v_j) = \delta_{ij} \quad \text{for } i = 1, 2, 3, j = 1, 2, 3.$$

Then we have

$$\begin{aligned} \psi_1(e_1) &= \psi_1(v_1) = 1, \\ \psi_1(e_2) &= \psi_1(v_2 - v_1) = \psi_1(v_2) - \psi_1(v_1) = 0 - 1 = -1, \\ \psi_1(e_3) &= \psi_1(v_3 - v_2) = \psi_1(v_3) - \psi_1(v_2) = 0 - 0 = 0, \end{aligned}$$

so $\psi_1 = \varepsilon_1 - \varepsilon_2$.

Similarly

$$\begin{aligned} \psi_2(e_1) &= \psi_2(v_1) = 0, \\ \psi_2(e_2) &= \psi_2(v_2 - v_1) = \psi_2(v_2) - \psi_2(v_1) = 1 - 0 = 1, \\ \psi_2(e_3) &= \psi_2(v_3 - v_2) = \psi_2(v_3) - \psi_2(v_2) = 0 - 1 = -1, \end{aligned}$$

so $\psi_2 = \varepsilon_2 - \varepsilon_3$.

Finally

$$\psi_3(e_1) = \psi_3(v_1) = 0,$$

$$\psi_3(e_2) = \psi_3(v_2 - v_1) = \psi_3(v_2) - \psi_3(v_1) = 0 - 0 = 0,$$

$$\psi_3(e_3) = \psi_3(v_3 - v_2) = \psi_3(v_3) - \psi_3(v_2) = 1 - 0 = 1,$$

so $\psi_3 = \varepsilon_3$.

- (b) We keep the notations introduced in (a). In particular, we have that $U = \text{span}\{v_3\}$. Note that $\dim(U^\perp) = \dim(\mathbb{R}^3) - \dim(U) = 3 - 1 = 2$. Note also that $\psi_1(v_3) = \psi_2(v_3) = 0$, so $\psi_1, \psi_2 \in U^\perp$. Therefore $\{\psi_1, \psi_2\}$ is a linearly independent subset of U^\perp . Since $\dim(U^\perp) = 2$,

$$\{\psi_1, \psi_2\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

is a basis for U^\perp .

Problem 5. [10 points] Let V be a finite-dimensional F -vector space. Let $\sigma \in \text{End}(V)$. Let U be a σ -invariant subspace of V . Let $\sigma|_U \in \text{End}(U)$ and $\bar{\sigma} \in \text{End}(V/U)$ be the endomorphisms of U and V/U , respectively, induced by σ . Let $\lambda \in F$. Prove that λ is an eigenvalue of σ if and only if λ is an eigenvalue of $\sigma|_U$ or an eigenvalue of $\bar{\sigma}$.

Solution. Let $\tau = \sigma - \lambda \mathbb{1}_V$. Since U is σ -invariant, it is also τ -invariant, so we can consider the endomorphisms $\tau|_U \in \text{End}(U)$ and $\bar{\tau} \in \text{End}(V/U)$ induced by τ . Note that $\tau|_U = \sigma|_U - \lambda \mathbb{1}_U$ and $\bar{\tau} = \bar{\sigma} - \lambda \mathbb{1}_{V/U}$. Thus

$$\begin{aligned} \lambda \text{ is an eigenvalue of } \sigma &\iff \tau \text{ is not injective,} \\ \lambda \text{ is an eigenvalue of } \sigma|_U &\iff \tau|_U \text{ is not injective,} \\ \lambda \text{ is an eigenvalue of } \bar{\sigma} &\iff \bar{\tau} \text{ is not injective.} \end{aligned}$$

Hence, it suffices to prove that

$$\tau \text{ is not injective} \iff \tau|_U \text{ is not injective or } \bar{\tau} \text{ is not injective}$$

or equivalently that

$$\tau \text{ is injective} \iff \tau|_U \text{ is injective and } \bar{\tau} \text{ is injective}$$

We prove the last biimplication.

(\Rightarrow) Suppose that τ is injective. Then clearly $\tau|_U$ is injective. Also, since V is finite-dimensional and τ is injective, we have that τ is surjective, and it follows easily that $\bar{\tau}$ is surjective. Since V/U is finite-dimensional, this implies that $\bar{\tau}$ is injective.

(\Leftarrow) Suppose that $\tau|_U$ and $\bar{\tau}$ are both injective. Let $x \in \ker \tau$. Then $\tau(x) = 0$, so $\bar{\tau}(x + U) = \tau(x) + U = 0 + U$. Thus $x + U \in \ker \bar{\tau}$. Since $\bar{\tau}$ is injective, it follows that $x + U = 0 + U$, so $x \in U$. Now we have that $\tau|_U(x) = \tau(x) = 0$. Since $\tau|_U$ is injective, we deduce that $x = 0$. Hence, $\ker \tau = \{0\}$ and therefore τ is injective.

Problem 6. [10 points] Let V be a 3-dimensional \mathbb{R} -vector space. Let (v_1, v_2, v_3) be an ordered basis for V . Let $\tau \in \text{End}(V)$ be defined by

$$\tau(v_1) = 2v_1, \quad \tau(v_2) = v_1 + 2v_2, \quad \tau(v_3) = 2v_1 + v_2 + 3v_3.$$

Determine whether τ is diagonalizable. If τ is diagonalizable, find an ordered basis for V with respect to which the matrix of τ is diagonal. If τ is not diagonalizable, prove so.

Solution.

Let $\alpha = (v_1, v_2, v_3)$ and let $A = {}_\alpha[\tau]_\alpha$. Note that

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since A is upper-triangular, the eigenvalues of τ are the entries on the diagonal, which are 2 and 3. We need to find the corresponding eigenspaces. Recall that to find $E(\lambda, \tau)$ we need to solve the linear system $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$.

To find $E(2, \tau)$, we need to solve

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$\left. \begin{array}{lcl} y + 2z & = & 0 \\ z & = & 0 \\ z & = & 0 \end{array} \right\},$$

which yields $y = z = 0$. Thus

$$\text{Null}(A - 2I_3) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and therefore $E(2, \tau) = \text{span}\{v_1\}$. Since $\{v_1\}$ is clearly a basis for $E(2, \tau)$, in particular we have $\dim(E(2, \tau)) = 1$.

To find $E(3, \tau)$, we need to solve

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$\left. \begin{array}{rrcr} -x & + & y & + & 2z & = & 0 \\ & & -y & + & z & = & 0 \\ & & & & 0 & = & 0 \end{array} \right\},$$

which yields $y = z$ and $x = 3z$. Thus

$$\text{Null}(A - 3I_3) = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and therefore $E(3, \tau) = \text{span}\{3v_1 + v_2 + v_3\}$. Since $\{3v_1 + v_2 + v_3\}$ is clearly a basis for $E(3, \tau)$, in particular we have $\dim(E(3, \tau)) = 1$.

Since $\dim(E(2, \tau)) + \dim(E(3, \tau)) < \dim(V)$, we conclude that τ is not diagonalizable.