Def. A matrix $A = (a_{ij}) \in M_n(F)$ is diagonal if $a_{ij} = 0$ for $i \neq j$.

$$A = \begin{bmatrix} \lambda_1 & & \bigcirc \\ & \lambda_2 & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{bmatrix}$$

Def. Let $\sigma \in End(V)$ and let $\lambda \in F$. The <u>eigenspace</u> of σ corresponding to λ is defined to be

$$E(\lambda, \sigma) := \ker(\sigma - \lambda \mathbb{1}_V) = \left\{ \begin{matrix} \text{eigenvectors of } \sigma \\ \text{with eigenvalue } \lambda \end{matrix} \right\} \cup \{0\}.$$

Lemma Let $\sigma \in End(V)$. Let $\lambda_1, \dots, \lambda_m \in F$ be distinct eigenvalues of σ.

Then $E(\lambda_1, \sigma) + \dots + E(\lambda_m, \sigma) = E(\lambda_1, \sigma) \oplus \dots \oplus E(\lambda_m, \sigma)$.

In particular, $\dim\left(E(\lambda_1, \sigma) + \dots + E(\lambda_m, \sigma)\right) = \dim(E(\lambda_1, \sigma)) + \dots + \dim(E(\lambda_m, \sigma))$.

Proof Let $v_i \in E(\lambda_i, \sigma)$ for $i = 1, \dots, m$ be elements such that $v_1 + \dots + v_m = 0$. Note that each v_i is either zero or an eigenvector of eigenvalue λ_i. Since eigenvectors with distinct eigenvalues are l.i., $v_1 = \dots = v_m = 0$. $\square$

Def. Let V be a f.d.v.s. over F. Let $\sigma \in \mathrm{End}(V)$. We say that σ is <u>diagonalizable</u> if there exists an ordered basis β for V such that ${}_\beta[\sigma]_\beta$ is a diagonal matrix.

Def A matrix $A \in M_n(F)$ is diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.

Rk Let α be an ordered basis for V and let $A = {}_\alpha[\sigma]_\alpha$. Then

σ is diagonalizable $\Leftrightarrow$ there exists an ordered basis β for V s.t. ${}_\beta[\sigma]_\beta$ is diagonal

$\Leftrightarrow$ there exists an ordered basis β for V s.t. ${}_\beta[1_V]_\alpha\, {}_\alpha[\sigma]_\alpha\, {}_\alpha[1_V]_\beta$ is diagonal

$\Leftrightarrow$ there exists an invertible matrix P s.t. $P^{-1}AP$ is diagonal

$\Leftrightarrow$ A is diagonalizable

Example Let $F = \mathbb{R}$, $V = \mathbb{R}^2$ and let $\beta = (e_1, e_2)$ be the standard ordered basis for V. Let $\sigma \in \mathrm{End}(V)$ be such that

$$\sigma(e_1) = e_1 \quad \text{and} \quad \sigma(e_2) = e_1 + 2e_2.$$

Then ${}_\beta[\sigma]_\beta = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Let $A = {}_\beta[\sigma]_\beta$.

Since A is upper-triangular, the eigenvalues of σ are $\lambda_1 = 1$ and $\lambda_2 = 2$.

The corresponding eigenspaces are $\ker(\sigma - \lambda_1 1_V)$ and $\ker(\sigma - \lambda_2 1_V)$.

We compute

$$\mathrm{Null}\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\mathrm{Null}\left(\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix} - 2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \mathrm{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

So $v_1 = e_1$ and $v_2 = e_1 + e_2$ are eigenvectors for λ_1 and λ_2, respectively.

Let $\gamma = (v_1, v_2)$. Then

$${}_\gamma[\sigma]_\gamma = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Note also that the change-of-basis matrix from γ to β is

$$P := {}_\beta[\mathbb{1}_V]_\gamma = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

so $$P^{-1} A P = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Theorem Let $\sigma \in End(V)$ and let $\lambda_1, \ldots, \lambda_m \in F$ be the distinct eigenvalues of σ. Then the following are equivalent:

(i) σ is diagonalizable

(ii) V has an ordered basis of eigenvectors.

(iii) there exist one-dimensional σ-invariant subspaces $U_1, \ldots, U_n \leq V$ such that

$$V = U_1 \oplus \cdots \oplus U_n$$

(iv) $V = E(\lambda_1, \sigma) \oplus \cdots \oplus E(\lambda_m, \sigma)$

(v) $\dim V = \dim(E(\lambda_1, \sigma)) + \cdots + \dim(E(\lambda_m, \sigma))$.

Proof.

((i) $\Leftrightarrow$ (ii)) We have ${}_{\beta}[\sigma]_{\beta} = \begin{bmatrix} \mu_1 & & O \\ & \mu_2 & \\ & & \ddots \\ O & & & \mu_n \end{bmatrix}$ for some ordered basis $\beta = (v_1, \ldots, v_n)$

if and only if $\sigma(v_i) = \mu_i v_i \quad \forall i = 1, \ldots, n$ (i.e., β consists of eigenvectors)

((ii) $\Rightarrow$ (iii)) If $\beta = (v_1, \ldots, v_n)$ is an ordered basis of eigenvectors, we can take $U_i = \operatorname{span}\{v_i\}$.

((ii) $\Leftarrow$ (iii)) Suppose $V = U_1 \oplus \cdots \oplus U_n$ for some one-dimensional σ-invariant subspaces $U_1, \ldots, U_n \leq V$. For $i = 1, \ldots, n$, choose a non-zero $v_i \in U_i$. Then $v_1, \ldots, v_n$ are eigenvectors and $\beta = (v_1, \ldots, v_n)$ is an ordered basis.

((ii) $\Rightarrow$ (iv)) Let $\beta = (v_1, \dots, v_n)$ be an ordered basis of eigenvectors for V. Relabeling the vectors $v_1, \dots, v_n$, we may assume wlog that

$$\beta = (\beta_1, \dots, \beta_m)$$

where β_i consists of eigenvectors with eigenvalue λ_i. Then

$$V = \operatorname{span}\{\beta_1\} + \dots + \operatorname{span}\{\beta_m\} \subseteq E(\lambda_1, \sigma) + \dots + E(\lambda_m, \sigma)$$

so

$$V = E(\lambda_1, \sigma) + \dots + E(\lambda_m, \sigma) \underset{\substack{\uparrow \\ \text{seen before}}}{=} E(\lambda_1, \sigma) \oplus \dots \oplus E(\lambda_m, \sigma)$$

((iv) $\Rightarrow$ (v)) Take dimensions.

((v) $\Rightarrow$ (ii)) For $i = 1, \dots, m$, let β_i be an ordered basis for $E(\lambda_i, \sigma)$. Then $\beta = (\beta_1, \dots, \beta_m)$ is a linearly independent list of $n = \dim(V)$ vectors, so β is an ordered basis for V consisting of eigenvectors.

<u>Corollary</u> If σ has n distinct eigenvalues, then σ is diagonalizable.

<u>Example</u> $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not diagonalizable since $\operatorname{nullity}(B - I_3) = 1 < 3$.

Problem Let V be a f.d.v.s. over F. Let $\sigma \in End(V)$. Let $U \leq V$ be a σ-invariant subspace. Let $\sigma|_U \in End(U)$ and $\bar{\sigma} \in End(V/U)$ be the endomorphisms of U and V/U induced by σ. Prove that if σ is diagonalizable, then both $\sigma|_U$ and $\bar{\sigma}$ are diagonalizable. Is the converse true?

Solution Suppose that σ is diagonalizable. Let $\lambda_1, \ldots, \lambda_m \in F$ be the distinct eigenvalues of σ. Since σ is diagonalizable, we have that

$$V = E(\lambda_1, \sigma) \oplus \cdots \oplus E(\lambda_m, \sigma). \qquad (*)$$

Note that $E(\lambda_i, \sigma|_U) = E(\lambda_i, \sigma) \cap U$ for $i = 1, \ldots, m$.

Let $x \in U$. By $(*)$, there exist unique $v_i \in E(\lambda_i, \sigma)$, $i = 1, \ldots, m$, such that $x = v_1 + \cdots + v_m$. By HW 8, Problem 1, since U is σ-invariant we have $v_1, \ldots, v_m \in U$.
This shows that

$$U = E(\lambda_1, \sigma|_U) \oplus \cdots \oplus E(\lambda_m, \sigma|_U),$$

so $\sigma|_U$ is diagonalizable.

Now let $y + U \in V/U$. By $(*)$, there exist unique $w_i \in E(\lambda_i, \sigma)$, $i = 1, \ldots, m$ such that $y = w_1 + \cdots + w_m$. Note that

$$\bar{\sigma}(w_i + U) = \sigma(w_i) + U = \lambda_i w_i + U = \lambda_i (w_i + U),$$

so $w_i + U \in E(\lambda_i, \bar{\sigma})$. Thus

$$y + U = (w_1 + U) + \cdots + (w_m + U) \in E(\lambda_1, \bar{\sigma}) + \cdots + E(\lambda_m, \bar{\sigma}).$$

This shows that

$$V/U = E(\lambda_1, \bar{\sigma}) + \cdots + E(\lambda_m, \bar{\sigma})$$

so $\bar{\sigma}$ is diagonalizable.

The converse is false. As a counterexample, let $\alpha = (e_1, e_2)$ be the standard ordered basis for $\mathbb{R}^2$ and consider the endomorphism $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\sigma(e_1) = e_1, \qquad \sigma(e_2) = e_1 + e_2.$$

Note that $\quad {}_\alpha[\sigma]_\alpha = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$

Let $U = \operatorname{span}\{e_1\}$. Then U is σ-invariant and clearly $\sigma|_U$ and $\bar{\sigma}$ are both diagonalizable. However, you can easily show that σ is not diagonalizable.