Midterm exam solutions

MATH 108A, SPRING 2025

NAME: ____________________________ PERM NUMBER: _______________

- *The time for this exam is **75 minutes**.*
- *The exam has **4 problems**. Each of them is worth 10 points.*
- *No notes, calculators or electronic devices are allowed during the exam.*
- *You can use the blank pages at the end as scratch paper.*

Problem 1. [10 points] Let V be an F-vector space. Let v_1, v_2 and v_3 be three distinct elements of V such that $\{v_1, v_2, v_3\}$ is linearly independent. Let $w \in V$. Prove that if the set

$$\{v_1 + w, v_2 + w, v_3 + w\}$$

is linearly dependent, then $w \in \operatorname{span}\{v_1, v_2, v_3\}$.

Solution. Suppose that the set $\{v_1 + w, v_2 + w, v_3 + w\}$ is linearly dependent. Then there exist scalars $\lambda_1, \lambda_2, \lambda_3 \in F$ not all zero such that

$$\lambda_1(v_1 + w) + \lambda_2(v_2 + w) + \lambda_3(v_3 + w) = 0.$$

Then

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = -(\lambda_1 + \lambda_2 + \lambda_3)w.$$

Let $\gamma = \lambda_1 + \lambda_2 + \lambda_3$. If $\gamma = 0$, the above equation gives a non-trivial linear dependence relation between v_1, v_2 and v_3, contradicting the assumption that $\{v_1, v_2, v_3\}$ is linearly independent. Therefore $\gamma \neq 0$ and we obtain

$$w = -\gamma^{-1}(\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) \in \operatorname{span}\{v_1, v_2, v_3\}.$$

Problem 2.

(a) [5 points] Let V be a 7-dimensional F-vector space. Prove that there does not exist a linear map $\tau : V \to V$ such that $\ker(\tau) = \mathrm{im}(\tau)$.

(b) [5 points] Let V be a finite-dimensional F-vector space. Let U_1, U_2, U_3 be subspaces of V such that $\dim(U_1) + \dim(U_2) + \dim(U_3) > 2\dim(V)$. Prove that $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Solution.

(a) Suppose there exists such a linear map $\tau : V \to V$. Then, by the Rank-Nullity Theorem,

$$7 = \dim(V) = \dim(\ker(\tau)) + \dim(\mathrm{im}(\tau)) = 2\dim(\ker(\tau)),$$

which is a contradiction since 7 is odd.

(b) By the Sum-Intersection formula,

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \geq \dim(U_1) + \dim(U_2) - \dim(V).$$

Applying the Sum-Intersection formula again, we obtain

$$\begin{aligned}
\dim(U_1 \cap U_2 \cap U_3) &= \dim((U_1 \cap U_2) \cap U_3) \\
&= \dim(U_1 \cap U_2) + \dim(U_3) - \dim((U_1 \cap U_2) + U_3) \\
&\geq \dim(U_1 \cap U_2) + \dim(U_3) - \dim(V) \\
&\geq \dim(U_1) + \dim(U_2) + \dim(U_3) - 2\dim(V) > 0.
\end{aligned}$$

Therefore $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Problem 3. [10 points] Let U, V and W be F-vector spaces. Suppose that U and V are finite-dimensional. Let $\tau : U \to V$ and $\sigma : V \to W$ be linear maps. Prove that

$$\operatorname{rank}(\sigma\tau) \leq \min\{\operatorname{rank}(\sigma), \operatorname{rank}(\tau)\}.$$

Solution. First we note that $\operatorname{im}(\sigma\tau) = \sigma(\operatorname{im}(\tau))$. Indeed,

$$\operatorname{im}(\sigma\tau) = \{(\sigma\tau)(x) \mid x \in U\} = \{\sigma(\tau(x)) \mid x \in U\} = \{\sigma(y) \mid y \in \operatorname{im}(\tau)\}.$$

Consider the linear map $\sigma_1 = \sigma_{|\operatorname{im}(\tau)} : \operatorname{im}(\tau) \to W$. By the previous observation, we have that

$$\operatorname{im}(\sigma_1) = \sigma(\operatorname{im}(\tau)) = \operatorname{im}(\sigma\tau).$$

Applying the Rank-Nullity theorem to σ_1, we deduce

$$\dim(\operatorname{im}(\sigma_1)) = \dim(\operatorname{im}(\tau)) - \dim(\ker(\sigma_1)) \leq \dim(\operatorname{im}(\tau)) = \operatorname{rank}(\tau).$$

Combined with the previous equation, this yields

$$\operatorname{rank}(\sigma\tau) = \dim(\operatorname{im}(\sigma\tau)) = \dim(\operatorname{im}(\sigma_1)) \leq \operatorname{rank}(\tau).$$

Also, we have that $\operatorname{im}(\sigma\tau) = \sigma(\operatorname{im}(\tau)) \subseteq \sigma(V) = \operatorname{im}(\sigma)$, so

$$\operatorname{rank}(\sigma\tau) = \dim(\operatorname{im}(\sigma\tau)) \leq \dim(\operatorname{im}(\sigma)) = \operatorname{rank}(\sigma).$$

Hence, we conclude that

$$\operatorname{rank}(\sigma\tau) \leq \min\{\operatorname{rank}(\sigma), \operatorname{rank}(\tau)\}.$$

Problem 4. We denote by $\mathcal{P}_2(\mathbb{R})$ the $\mathbb{R}$-vector space of all polynomials in the variable X with coefficients in $\mathbb{R}$ of degree at most 2. That is,

$$\mathcal{P}_2(\mathbb{R}) = \{a_0 + a_1 X + a_2 X^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}.$$

Consider the ordered bases $\alpha = (1, X, X^2)$ and $\alpha' = (1, (X-1), (X-1)^2)$ for $\mathcal{P}_2(\mathbb{R})$.

(a) [5 points] Determine the change-of-basis matrices ${}_{\alpha}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha'}$ and ${}_{\alpha'}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha}$.

(b) [5 points] Consider the linear map

$$\begin{aligned} \varphi : \mathcal{P}_2(\mathbb{R}) &\longrightarrow \mathbb{R}^4 \\ p(X) &\longmapsto (p(0), p(1), p(2), p(3)). \end{aligned}$$

Let $\varepsilon = (e_1, e_2, e_3, e_4)$ be the standard ordered basis for $\mathbb{R}^4$. Determine the matrix ${}_{\varepsilon}[\varphi]_{\alpha}$.

Solution.

(a) To find ${}_{\alpha}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha'}$, we note that

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2, \\ X - 1 &= (-1) \cdot 1 + 1 \cdot X + 0 \cdot X^2, \\ (X-1)^2 &= 1 \cdot 1 + (-2) \cdot X + 1 \cdot X^2. \end{aligned}$$

Thus

$${}_{\alpha}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha'} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, to find ${}_{\alpha'}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha}$, we note that

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot (X-1) + 0 \cdot (X-1)^2, \\ X &= 1 \cdot 1 + 1 \cdot (X-1) + 0 \cdot X^2, \\ X^2 = ((X-1)+1)^2 &= 1 \cdot 1 + 2 \cdot (X-1) + 1 \cdot (X-1)^2. \end{aligned}$$

Thus

$${}_{\alpha'}[\mathbb{1}_{\mathcal{P}_2(\mathbb{R})}]_{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) We have

$$\begin{aligned}
\varphi(1) &= (1,1,1,1) = 1e_1 + 1e_2 + 1e_3 + 1e_4 \\
\varphi(X) &= (0,1,2,3) = 0e_1 + 1e_2 + 2e_3 + 3e_4 \\
\varphi(X^2) &= (0,1,4,9) = 0e_1 + 1e_2 + 4e_3 + 9e_4.
\end{aligned}$$

Therefore

$$ {}_{\varepsilon}[\varphi]_{\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}. $$