Invariant subspaces

Def. Let V be an F-v.s. Let $\sigma \in \mathrm{End}(V)$ and let U be a subspace of V. We say that U is <u>invariant under σ</u> if $\sigma(U) \subseteq U$, i.e., if $\sigma(x) \in U$ for all $x \in U$.

Rk. We do not necessarily have $\sigma(x) = x$ for all $x \in U$, nor do we have $\sigma(U) = U$.

Def. Let $\sigma \in \mathrm{End}(V)$ and let $U \subseteq V$ be a σ-invariant subspace.

(i) We call

$$\begin{aligned} \sigma_{|U} : U &\longrightarrow U \\ x &\longmapsto \sigma(x) \end{aligned}$$

the endomorphism of U induced by σ.

(ii) Let $\pi_U : V \to V/U$ denote the quotient map. Since $\sigma(U) \subseteq U$, we have $\pi_U(\sigma(U)) \subseteq \pi(U) = \{0_{V/U}\}$. Therefore, $U \subseteq \ker(\pi_U \circ \sigma)$. So by the universal property of the quotient V/U, there exist a unique linear map $\bar{\sigma} : V/U \to V/U$ such that $\bar{\sigma} \circ \pi_U = \pi_U \circ \sigma$.

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[dddd, bend right] & \sigma(x) \arrow[dddd, bend left] \\
V \arrow[r, "\sigma"] \arrow[d, "\pi_U"] & V \arrow[d, "\pi_U"] \\
V/U \arrow[r, "\bar{\sigma}"] & V/U \\
\\
x+U \arrow[r, mapsto] & \sigma(x)+U
\end{tikzcd}$$

We call $\bar{\sigma}$ the endomorphism of V/U induced by σ.

Example For any $\sigma \in \mathrm{End}(V)$, the subspaces $\{0\}$, V, $\ker(\sigma)$ and $\mathrm{im}(\sigma)$ of V are σ-invariant.

Example Let $\sigma \in \mathrm{End}(V)$ and let $U \le V$ be a σ-invariant subspace.
Let $(v_1, \dots, v_r)$ be an ordered basis for U and extend it to an ordered basis $\beta = (v_1, \dots, v_n)$ for V (here $r = \dim U$ and $n = \dim V$). Then

$$
\begin{array}{cc}
 & \begin{array}{cc} \overbrace{\hphantom{AAA}}^{r} & \overbrace{\hphantom{AAA}}^{n-r} \end{array} \\
{}_{\beta}[\sigma]_{\beta} = \begin{array}{c} r\{ \\ n-r\{ \end{array} & \left[\begin{array}{c|c} A & B \\ \hline O & D \end{array}\right]
\end{array}
$$

Indeed, for $j \in \{1, \dots, r\}$,

$$\sigma(v_j) = \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{r} a_{ij} v_i + \sum_{i=r+1}^{n} a_{ij} v_i$$

for some $a_{ij} \in F$. Since $v_j \in U$ and U is σ-invariant,

$$\sigma(v_j) \in U = \mathrm{span}\{v_1, \dots, v_r\},$$

i.e., $a_{ij} = 0$ for $r+1 \le i \le n$.

More precisely, say $\alpha = (v_1, \dots, v_r)$ and $\gamma = ([v_{r+1}], \dots, [v_n])$, which are ordered bases for U and V/U, respectively. Then above one has

$$A = {}_{\alpha}[\sigma|_U]_{\alpha} \quad \text{and} \quad D = {}_{\gamma}[\bar{\sigma}]_{\gamma}$$

where $\bar{\sigma} : V/U \to V/U$ is the induced quotient operator. (exercise)

Example Let V be a f.d.v.s. over F, let $\sigma \in End(V)$, and suppose that

$$V = U_1 \oplus \cdots \oplus U_k$$

for some σ-invariant subspaces $U_1, \dots, U_k$ of V. Let $m_i = \dim U_i$ and let β_i be an ordered basis for U_i. Then $\beta := (\beta_1, \dots, \beta_k)$ (the concatenation of $\beta_1, \dots, \beta_k$) is an ordered basis for V and

$${}_{\beta}[\sigma]_{\beta} = \begin{bmatrix} \overbrace{\boxed{A_1}}^{m_1}\left.\vphantom{\boxed{A_1}}\right\}{\scriptstyle m_1} & & & 0 \\ & \overbrace{\boxed{A_2}}^{m_2}\left.\vphantom{\boxed{A_2}}\right\}{\scriptstyle m_2} & & \\ & & \ddots & \\ 0 & & & \overbrace{\boxed{A_k}}^{m_k}\left.\vphantom{\boxed{A_k}}\right\}{\scriptstyle m_k} \end{bmatrix}$$

is a block diagonal matrix with blocks $A_i = {}_{\beta_i}[\sigma|_{U_i}]_{\beta_i}$, which are of size $m_i \times m_i$.

If each $m_i = 1$, then ${}_{\beta}[\sigma]_{\beta}$ is a diagonal matrix.

Eigenvectors, eigenvalues, eigenspaces

Def. Let $\sigma \in End(V)$. An element $v \in V$ is an eigenvector if $v \neq 0$ and $\sigma(v) = \lambda v$ for some $\lambda \in F$, called the eigenvalue of σ attached to v.

A scalar $\mu \in F$ is an eigenvalue of σ if μ is the eigenvalue attached to some eigenvector of σ.

Rk. A non-zero element $v \in V$ is an eigenvector of σ if and only if $\mathrm{span}\{v\}$ is a σ-invariant subspace of V.

Lemma Let $\sigma \in End(V)$ and let $\lambda \in F$. Then λ is an eigenvalue of σ if and only if $\sigma - \lambda 1\!\!1_V$ is not injective ($\Leftrightarrow \sigma - \lambda 1\!\!1_V$ is not surjective in case V is finite-dimensional).

Proof. ($\Rightarrow$) Suppose λ is an eigenvalue and let $v \in V \setminus \{0\}$ be an eigenvector with eigenvalue λ. Then $\sigma(v) = \lambda v$, so

$$(\sigma - \lambda 1\!\!1_V)(v) = \sigma(v) - \lambda 1\!\!1_V(v) = \sigma(v) - \lambda v = 0,$$

so $0 \neq v \in \ker(\sigma - \lambda 1\!\!1_V)$. Thus $\sigma - \lambda 1\!\!1_V$ is not injective.

($\Leftarrow$) Suppose $\sigma - \lambda 1\!\!1_V$ is not injective. Then there exists a non-zero element $v \in \ker(\sigma - \lambda 1\!\!1_V)$. This element satisfies

$$0 = (\sigma - \lambda 1\!\!1_V)(v) = \sigma(v) - \lambda v,$$

so $\sigma(v) = \lambda v$. Thus λ is an eigenvalue. $\square$

Prop Let $\sigma \in \mathrm{End}(V)$. If $v_1, \dots, v_m \in V$ are eigenvectors with distinct eigenvalues $\lambda_1, \dots, \lambda_m$, then $(v_1, \dots, v_m)$ is linearly independent (i.e., $v_1, \dots, v_m$ are distinct and $\{v_1, \dots, v_m\}$ is linearly independent).

Proof. The elements $v_1, \dots, v_m$ are clearly distinct since they have distinct eigenvalues. Suppose, for contradiction, that $\{v_1, \dots, v_m\}$ is linearly dependent. Since $v_1 \neq 0$ by definition of eigenvector, there exists $r \geq 2$ such that $v_r \in \mathrm{span}\{v_1, \dots, v_{r-1}\}$. Let $k \geq 2$ be the smallest such integer. Then $v_k \in \mathrm{span}\{v_1, \dots, v_{k-1}\}$ and $\{v_1, \dots, v_{k-1}\}$ is linearly independent.

Since $v_k \in \mathrm{span}\{v_1, \dots, v_{k-1}\}$,

$$v_k = \sum_{i=1}^{k-1} \mu_i v_i \qquad \text{for some } \mu_1, \dots, \mu_{k-1} \in F.$$

Applying $\sigma - \lambda_k \mathbb{1}_V$, we deduce

$$\begin{aligned}
0 = \sigma(v_k) - \lambda_k v_k &= \sigma\left(\sum_{i=1}^{k-1} \mu_i v_i\right) - \lambda_k \left(\sum_{i=1}^{k-1} \mu_i v_i\right) \\
&= \sum_{i=1}^{k-1} \mu_i \sigma(v_i) - \sum_{i=1}^{k-1} \mu_i \lambda_k v_i \\
&= \sum_{i=1}^{k-1} \mu_i \lambda_i v_i - \sum_{i=1}^{k-1} \mu_i \lambda_k v_i \\
&= \sum_{i=1}^{k-1} \mu_i (\lambda_i - \lambda_k) v_i .
\end{aligned}$$

Since $\{v_1, \dots, v_{k-1}\}$ is linearly independent and $\lambda_i \neq \lambda_k$ for all $i \in \{1, \dots, k-1\}$, it follows that $\mu_1 = \dots = \mu_{k-1} = 0$. But then

$$v_k = \sum_{i=1}^{k-1} \mu_i v_i = 0,$$

which is a contradiction. $\square$

<u>Corollary</u>. If V is finite-dimensional, then any endomorphism $\sigma \in End(V)$ has at most $\dim(V)$ distinct eigenvalues.

<u>Proof</u>. This follows immediately from the previous proposition and the fact that a linearly independent set has at most $\dim(V)$ elements. $\square$

Let V be an F-v.s. and let $\sigma \in End(V)$. For any integer $m \geq 1$, we define $\sigma^m = \underbrace{\sigma \circ \cdots \circ \sigma}_{m \text{ times}}$. We also define $\sigma^0 = \mathbb{1}_V$.

If σ is an isomorphism, we can also define σ^m for integers $m \leq -1$:

$$\sigma^m = (\sigma^{-1})^{-m} = \underbrace{\sigma^{-1} \circ \cdots \circ \sigma^{-1}}_{-m \text{ times}}.$$

With these definitions

$$\sigma^m \cdot \sigma^n = \sigma^{m+n} \quad \text{for all } m, n \in \mathbb{Z}_{\geq 0} \quad (\text{and for all } m, n \in \mathbb{Z} \text{ when } \sigma \text{ is an isomorphism})$$

If $P(X) = \sum_{i=0}^{n} a_i X^i \in P_n(F)$ is a polynomial, we define

$$P(\sigma) = \sum_{i=0}^{n} a_i \sigma^i \in End(V).$$

<u>Rk</u> Let $P_1(X)$, $P_2(X)$ be polynomials on X with coefficients in F.

If $P(X) = P_1(X) \cdot P_2(X)$ (product of polynomials)

then $P(\sigma) = P_1(\sigma) \cdot P_2(\sigma)$ (product / composition of endomorphisms).

If $Q(X) = P_1(X) + P_2(X)$ (addition of polynomials)

then $Q(\sigma) = P_1(\sigma) + P_2(\sigma)$ (addition of endomorphisms)

<u>Example</u> $X^2 + 1 = (X - i)(X + i) \Rightarrow \sigma^2 + 1 = (\sigma - i)(\sigma + i)$