Normal and Self-adjoint operators

We have seen that for an operator to be diagonalizable, it is necessary and sufficient condition for the vector space V to possess a basis of eigenvectors. Suppose V be an inner product space. We will study certain types of operators which admits "a nice" set of basis vector.

Lemma 1: Let T be a linear operator on a finite dimensional inner product space V. If T has an eigenvector, then so does T^*.

Proof:- Suppose that v is an eigenvector of T with corresponding eigenvalue λ.

Then for any $x \in V$,

$$\begin{aligned}
\langle v, (T^* - \bar{\lambda} I)(x) \rangle &= \langle v, (T - \lambda I)^*(x) \rangle \\
&= \langle (T - \lambda I)(v), x \rangle \\
&= \langle Tv - \lambda v, x \rangle \qquad [\because Tv = \lambda v] \\
&= \langle 0, x \rangle \\
&= 0
\end{aligned}$$

Hence, v is orthogonal to set $\{(T^* - \bar{\lambda} I)(x) \mid x \in V\} = W$

Since, $v \perp W$ so, $W \neq V$

$\Rightarrow$ $T^* - \bar{\lambda} I$ is not onto, and since V is finite dimensio

$\therefore$ $T^* - \bar{\lambda} I$ is not one-one.

$\therefore$ $\exists\ y_{\neq 0} \in V$ s.t. $(T^* - \bar{\lambda} I)(y) = 0$

$\therefore$ $T^* y = \bar{\lambda} y$.

$\therefore$ T^* has an eigenvector. ▨

Let $T: V \to V$ be a linear operator on a vector space V. Recall a subspace W of V is said to be T-invariant subspace if $T(W) \subseteq W$. We denote $T_W : W \to W$ be the restriction of T to W.

Theorem 2 (Schur): Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

Proof: We prove this by induction on n = dimension of V.

If $n = 1$, then it is clear.

Assume $n > 1$. Suppose that the result is true for linear operators on $(n-1)$ dimensional inner product spaces whose characteristic polynomial splits.

Since characteristic polynomial splits so T has an eigenvalue and hence by previous lemma T^* has an eigenvector say z. We can assume z is unit vector (by changing z to $\frac{z}{\|z\|}$).

i.e. $$T^*z = \lambda z$$

Set $W = \text{Span}(\{z\})$. We show that $W^\perp$ is T-invariant.

To show this, let $y \in W^\perp$ and $x = cz \in W$ for any $c \in F$.

Then

$$\begin{aligned}
\langle Ty, x \rangle &= \langle y, T^*x \rangle \\
&= \langle y, T^*(cz) \rangle \\
&= \langle y, cT^*z \rangle \\
&= \langle y, c\lambda z \rangle \\
&= \overline{c\lambda} \langle y, z \rangle \qquad (\because y \in W^\perp) \\
&= 0
\end{aligned}$$

$$\therefore \quad T(W^\perp) \subseteq W^\perp.$$

So one can consider $T_{W^\perp} : W^\perp \longrightarrow W^\perp$.

Since characteristic polynomial of $T_{W^\perp}$ divides characteristic polynomial of $T : V \to V$. Hence the characteristic polynomial of $T_{W^\perp}$ splits. Since $\dim(W^\perp) = n-1$ (as $\dim W = 1$).

So, by induction hypothesis, there exist an orthonormal basis γ of $W^\perp$ s.t. $[T_{W^\perp}]_\gamma$ is upper triangular.

Define $\beta = \gamma \cup \{z\}$.

$$\text{Then} \quad [T]_\beta = \begin{pmatrix} \underbrace{[T]_\gamma}_{n-1 \times n-1} & * \\ 0 & \underbrace{*}_{1\times 1} \end{pmatrix}$$

Note that β is an orthonormal basis and $[T]_\beta$ is upper triangular matrix. — $\blacksquare$

<u>Remark:</u> If $F = \mathbb{C}$, then characteristic polynomial always splits.

<u>Ques:</u> Let T be a linear operator on finite dimensional inner product space V. Does there exist an orthonormal basis of eigenvectors of T?

~~if~~ Suppose such a basis β exist; then

$[T]_\beta$ is a diagonal matrix, and

hence $[T^*]_\beta = [T]_\beta^*$ is also a diagonal matrix.

Since diagonal matrix commutes so,

$$[T^*]_\beta [T]_\beta^* = [T]_\beta^* [T^*]_\beta .$$

$\boxed{\text{So we concluded that if } V \text{ possess an orthonormal basis of eigenvectors of } T, \text{ then } TT^* = T^*T.}$

Definition:- Let V be an inner product space, and let T be a linear operator on V. We say that T is **normal** if $TT^* = T^*T$. An $n \times n$ real or complex matrix A is **normal** if $AA^* = A^*A$.

Proposition: T is normal iff $[T]_\beta$ is normal, where β is an orthonormal basis.

Proof: Let T be normal and β be orthonormal basis then

$$[T^*]_\beta = [T]_\beta^* \qquad -(1)$$

Now if $TT^* = T^*T$

then $[T]_\beta [T^*]_\beta = [T^*]_\beta [T]_\beta$

Using eqⁿ (1), we get

$$[T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta$$

$\therefore$ $[T]_\beta$ is normal.

Conversly, if $[T]_\beta$ is normal

i.e. $[T]_\beta [T]_\beta^* = [T]_\beta^* [T]_\beta$

$\therefore$ $[T]_\beta [T^*]_\beta = [T^*]_\beta [T]_\beta$.

$\therefore$ $[TT^*]_\beta = [T^*T]_\beta$

Since β is basis so, $TT^* = T^*T$.

<u>Example:1:</u> Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be rotation by θ, where $0 < \theta < \pi$.
Let β be the standard basis. Then

$$A = [T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

It is clear that $AA^* = A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So, T is normal operator.

Remark: The above operator T does not possess eigenvectors. Even normal operator may not possess eigenvectors.

Example 2: Suppose that A is real symmetric (i.e. $A^T = A$) or A is real skew symmetric (i.e. $A^T = -A$). Then A is normal.

if $A^T = A$, then $AA^* = AA^T = AA = A^2$ and $A^*A = A^TA = A^2$.

Also, if $A^T = -A$, then $AA^* = AA^T = -A^2$ and $A^*A = A^TA = -A^2$.

Theorem:- Let V be an inner product space, and let T be normal operator on V. Then followings holds:

(a) $\|T(x)\| = \|T^*(x)\| \quad \forall\ x \in V$.

(b) $T - cI$ is normal for every $c \in F$.

(c) If x is an eigenvector of T, then x is also an eigenvector of T^*. In fact if $T(x) = \lambda x$, then $T^*(x) = \bar{\lambda} x$.

(d) If λ_1 and λ_2 are distinct eigenvalues of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal.

Proof:- (a) For any $x \in V$, we have

$$\|T(x)\|^2 = \langle T(x), T(x) \rangle$$

$$= \langle T^*T(x), x \rangle \quad \text{(definition of adjoint)}$$

$$= \langle TT^*(x), x \rangle \quad (\because T \text{ is normal})$$

$$= \langle T^*(x), T^*(x) \rangle$$

$$= \|T^*(x)\|^2$$

$$\therefore \quad \|T(x)\| = \|T^*(x)\| \quad \forall\, x \in V.$$

(b) Let $c \in F$.

Then $(T - cI)(T - cI)^* = (T - cI)(T^* - \bar{c}I)$

$$= TT^* - T(\bar{c}I) - cIT^* - c\bar{c}I$$

$$= T^*T - (\bar{c}I)T - T^*cI - c\bar{c}I$$

$$= (T^* - \bar{c}I)(T - cI).$$

$$= (T - cI)^*(T - cI)$$

$\therefore$ $(T - cI)$ is normal.

(c) Suppose $T(x) = \lambda x$ for some $x \in V$.

Then $(T-\lambda I)$ is normal

So, by (a), we have

$$\|(T-\lambda I)(x)\| = \|(T-\lambda I)^*(x)\|$$

$$\Rightarrow \quad \|Tx - \lambda x\| = \|(T^* - \bar{\lambda} I)(x)\|$$

$$\Rightarrow \quad 0 = \|T^* x - \bar{\lambda} x\|$$

$$\Rightarrow \quad T^* x = \bar{\lambda} x$$

$\therefore$ x is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

(d) Let $Tx_1 = \lambda_1 x_1$, $\lambda_1 \neq \lambda_2$ $x_1, x_2 \in V$.

and $Tx_2 = \lambda_2 x_2$

Then

$$\begin{aligned}
\lambda_1 \langle x_1, x_2 \rangle &= \langle \lambda_1 x_1, x_2 \rangle \\
&= \langle Tx_1, x_2 \rangle \\
&= \langle x_1, T^* x_2 \rangle \\
&= \langle x_1, \bar{\lambda}_2 x_2 \rangle \\
&= \lambda_2 \langle x_1, x_2 \rangle .
\end{aligned}$$

Since $\lambda_1 \neq \lambda_2$ so, $\langle x_1, x_2 \rangle = 0$ ▨

<u>Theorem</u>: Let T be a linear operator on a finite dimensional complex inner product space V. Then T is normal iff there exists an orthonormal basis for V consisting

of eigenvectors of T.

Remark: In particular, normal operators on finite dimensional complex inner product space are diagonalizable.

Proof: As over $\mathbb{C}$, the characteristic polynomial of T splits. Then by Schur's theorm, there exist an orthonormal basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V such that

$$[T]_\beta \text{ is upper triangular.}$$

We can always assume that v_1 is an eigenvector (extend the basis by begin with v_1).

Assume that $v_1, v_2, \ldots, v_{k-1}$ are eigenvectors of T

We claim that v_k is also an eigenvector of T.

Then this claim implies that all basis vectors are eigenvectors and we will be done.

Let $1 \leq j < k$, and λ_j denote the eigenvalue of T corresponding v_j. i.e. $T v_j = \lambda_j v_j$

$$\Rightarrow \quad T^* v_j = \bar{\lambda}_j v_j.$$

Since $A = [T]_\beta$ is upper triangular matrix.

$$T(v_k) = A_{1k} v_1 + A_{2k} v_2 + \cdots + A_{jk} v_j + \cdots + A_{kk} v_k$$

Also, since β is orthonormal basis so,

$$\begin{aligned} A_{jk} &= \langle T(v_k), v_j \rangle. \\ &= \langle v_k, T^* v_j \rangle \\ &= \langle v_k, \bar{\lambda}_j v_j \rangle \\ &= \lambda_j \langle v_k, v_j \rangle \\ &= 0 \qquad [\because j < k] \end{aligned}$$

$\therefore$ $A_{jk} = 0$ for $j < k$.

$\therefore$ $T(v_k) = A_{kk} v_k$

$\therefore$ v_k is an eigenvector.

Converse, if $\exists$ $\beta = \{v_1, \ldots, v_n\}$, an orthonormal basis of eigenvectors, then $[T]_\beta$ is diagonal matrix

Hence $[T]_\beta$ is normal matrix and so is T. $\boxtimes$

Key-observation: 1). We only used $\mathbb{C}$ to ensure that characteristic polynomial splits. So one can replace the hypothesis "Complex" with "real + characteristic polynomial splits".

2). $T: \mathbb{R}^2 \to \mathbb{R}^2$ given as $[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where β is standard basis, does not possess eigenvectors.

Hence, $\mathbb{R}^2$ does not possess orthonormal basis of eigenvectors, even T is normal.

So, normality is not sufficient to guarantee the existence of an orthonormal basis of eigenvectors.

The problem here is that it does not even have eigenvectors. because characteristic polynomial does not split.

3). Normality over $\mathbb{C}$ is necessary and sufficient condition for existence of orthonormal basis consisting of eigenvectors.

For real, we need a stronger notion called "self-adjoint."

Definition: (Self-Adjoint or Hermitian) Let T be a linear operator on an inner product space V. We say T is self-adjoint (or Hermitian) if $T = T^*$.

An $n \times n$ matrix real or complex matrix A is Self-adjoint (or Hermitian) if $A = A^*$.

It is clear that if β is an orthonormal basis, then T is Self-adjoint iff $[T]_\beta$ is self-adjoint.

For real matrices, it means that $A^T = A$ ie A is symmetric matrix.

The next result says that self-adjoint operator on finite dimension on real inner product space has eigenvalues.

Theorem:- Let T be a self-adjoint operator on a finite dimensional inner product space V. Then

(a) Every eigenvalue of T is real.

(b) Suppose that V is real inner product space. Then the characteristic polynomial of T splits.

<u>Proof</u>:- (a) Suppose $T(x) = \lambda x$, $x \neq 0$,

Then $T^*(x) = \bar{\lambda} x$

Since $T = T^*$

So $\lambda x = \bar{\lambda} x$

$(\lambda - \bar{\lambda}) x = 0$

$\therefore \lambda = \bar{\lambda}$

$\therefore \lambda$ is real.

(b). Let $n = \dim(V)$, β be an orthonormal basis for V and $A = [T]_\beta$. Then A is self-adjoint.

Consider

$$T_A : \mathbb{C}^n \longrightarrow \mathbb{C}^n.$$

$$T_A(x) = Ax$$

Since $[T_A]_\gamma = A$, where γ is standard basis.

As A is self-adjoint so, T_A is self-adjoint.

Hence eigenvalues of T_A are real.

Since characteristic polynomial of T_A splits (as over $\mathbb{C}$)

so, it factors into $(t-\lambda_1) \ldots (t-\lambda_k)$ $\lambda_i \in \mathbb{R}$

But T_A & A has same characteristic polynomial

Therefore, the characteristic polynomial of T splits.

$- \boxtimes$

Theorem: Let T be a linear operator on a finite dimensional real inner product space V. Then T is self-adjoint iff there exist an orthonormal basis β for V consisting of eigenvectors of T.

Proof: Suppose T is self-adjoint. By the previous lemma, the characterstic polynomial of T splits. Then by Schur's theorem, there exist an orthonormal basis β for V s.t. $A = [T]_\beta$ is upper triangular.

But $A^* = [T]_\beta^* = [T^*]_\beta = [T]_\beta = A$

So, A and A^* are both upper triangular

$\therefore$ A is a diagonal matrix.

$\therefore$ β must consists of eigenvectors of T.

Conversly, if β is an orthonormal basis consisting of eigenvectors, then $A = [T]_\beta$ is diagonal.

So, $A = A^*$

Since, $A^* = [T]_\beta^* = [T^*]_\beta$ is also diagonal

$$\therefore [T^*]_\beta = [T]_\beta^* = A^* = A = [T]_\beta$$

$\therefore$ T is self-adjoint.

Note that real symmetric matrices are self-adjoint, and self-adjoint are normal.

Warning: Complex symmetric matrices may not be normal!

Take $A = \begin{pmatrix} i & i \\ i & 1 \end{pmatrix}$ and $A^* = \begin{pmatrix} -i & -i \\ -i & 1 \end{pmatrix}$

$A^T = A$ So A is complex symmetric but A is not normal as $AA^* \neq A^*A$.

So, complex symmetric matrices need not be normal.