

# Practice final solutions

MATH 108A, SPRING 2025

NAME: \_\_\_\_\_

PERM NUMBER: \_\_\_\_\_

- *The time for this exam is **3 hours**.*
- *The exam has **6 problems**. Each of them is worth 10 points.*
- *No notes, books, calculators or electronic devices are allowed during the exam.*
- *You can use the blank pages at the end as scratch paper.*
- *Unless otherwise stated, in each problem  $F$  stands for an arbitrary field.*



**Problem 1.** [10 points] Let  $V$  and  $W$  be finite-dimensional  $F$ -vector spaces. Let  $U$  be a subspace of  $V$  such that

$$\dim(W) \geq \dim(V) - \dim(U).$$

Prove that there exists  $\tau \in \mathcal{L}(V, W)$  such that  $\ker(\tau) = U$ .

*Solution.*

Let  $r = \dim(U)$ ,  $n = \dim(V)$ ,  $m = \dim(W)$ . By assumption we have  $m \geq n - r$ . Let  $(x_1, \dots, x_r)$  be an ordered basis for  $U$  and extend it to an ordered basis  $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$  for  $V$ . Let  $(w_1, \dots, w_m)$  be an ordered basis for  $W$ . Let  $\tau \in \mathcal{L}(V, W)$  be the linear map defined by

$$\begin{aligned}\tau(x_i) &= 0 \quad \text{for } i = 1, \dots, r \\ \tau(y_j) &= w_j \quad \text{for } j = 1, \dots, n - r.\end{aligned}$$

Note that this makes sense since  $m \geq n - r$ .

We claim that  $U = \ker \tau$ . Indeed, by definition we have that  $x_1, \dots, x_r \in \ker \tau$  and therefore  $U = \text{span}\{x_1, \dots, x_r\} \subseteq \ker \tau$ . To prove the opposite inclusion, let  $v \in \ker \tau$ . Since  $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$  is an ordered basis for  $V$ , we can write

$$v = \sum_{i=1}^r \lambda_i x_i + \sum_{j=1}^{n-r} \mu_j y_j$$

for some  $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_{n-r} \in F$ . Then

$$0 = \tau(v) = \tau\left(\sum_{i=1}^r \lambda_i x_i + \sum_{j=1}^{n-r} \mu_j y_j\right) = \sum_{i=1}^r \lambda_i \tau(x_i) + \sum_{j=1}^{n-r} \mu_j \tau(y_j) = \sum_{j=1}^{n-r} \mu_j w_j.$$

Since  $(w_1, \dots, w_{n-r})$  is linearly independent, it follows that  $\mu_1 = \dots = \mu_{n-r} = 0$  and therefore

$$v = \sum_{i=1}^r \lambda_i x_i \in U.$$



**Problem 2.** [10 points] Let  $U$ ,  $V$  and  $W$  be  $F$ -vector spaces. Let  $\tau \in \mathcal{L}(U, V)$  and let  $\sigma \in \mathcal{L}(V, W)$ . Prove that the following statements are equivalent:

- (i)  $\ker(\sigma \circ \tau) = \ker(\tau)$ ;
- (ii)  $\ker(\sigma) \cap \text{im}(\tau) = \{0\}$ .

*Solution.*

((i)  $\Rightarrow$  (ii)) Suppose that  $\ker(\sigma \circ \tau) = \ker(\tau)$ . Let  $y \in \ker(\sigma) \cap \text{im}(\tau)$ . Since  $y \in \text{im}(\tau)$ , we have that  $y = \tau(x)$  for some  $x \in U$ . Now, since  $y \in \ker(\sigma)$ , we have that

$$0 = \sigma(y) = \sigma(\tau(x)) = (\sigma \circ \tau)(x),$$

so  $x \in \ker(\sigma \circ \tau)$ . Since  $\ker(\sigma \circ \tau) = \ker(\tau)$ , we also have that  $x \in \ker(\tau)$ , so  $y = \tau(x) = 0$ .

((ii)  $\Rightarrow$  (i)) Suppose that  $\ker(\sigma) \cap \text{im}(\tau) = \{0\}$ . Recall that the inclusion  $\ker(\tau) \subseteq \ker(\sigma \circ \tau)$  is true in general: if  $x \in \ker(\tau)$ , then  $\tau(x) = 0$  and therefore  $(\sigma \circ \tau)(x) = \sigma(\tau(x)) = \sigma(0) = 0$ , so  $x \in \ker(\sigma \circ \tau)$ . Now we prove the opposite inclusion. Let  $x \in \ker(\sigma \circ \tau)$ . Then  $(\sigma \circ \tau)(x) = 0$ , so  $\sigma(\tau(x)) = 0$ . Then  $\tau(x) \in \ker(\sigma) \cap \text{im}(\tau)$ , so  $\tau(x) = 0$  and therefore  $x \in \ker(\tau)$ .



**Problem 3.** [10 points] Let  $V$  be an  $F$ -vector space. Let  $U$  and  $W$  be subspaces of  $V$  and suppose that  $V = U + W$ . Prove that the map

$$\begin{aligned}\tau : V/(U \cap W) &\longrightarrow V/U \times V/W \\ x + U \cap W &\longmapsto (x + U, x + W)\end{aligned}$$

is well-defined and that it is an isomorphism of  $F$ -vector spaces.

*Solution.*

Consider the map

$$\begin{aligned}\varphi : V &\longrightarrow V/U \times V/W \\ x &\longmapsto (x + U, x + W).\end{aligned}$$

First we show that  $\varphi$  is linear. Let  $x, y \in V$  and let  $\lambda, \mu \in F$ . Then

$$\begin{aligned}\varphi(\lambda x + \mu y) &= (\lambda x + \mu y + U, \lambda x + \mu y + W) \\ &= (\lambda x + U, \lambda x + W) + (\mu y + U, \mu y + W) \\ &= \lambda(x + U, x + W) + \mu(y + U, y + W) \\ &= \lambda\varphi(x) + \mu\varphi(y).\end{aligned}$$

Now we show that  $\ker \varphi = U \cap W$ . Indeed,

$$\begin{aligned}\ker \varphi &= \{x \in V \mid (x + U, x + W) = (0 + U, 0 + W)\} \\ &= \{x \in V \mid x \in U \text{ and } x \in W\} \\ &= U \cap W.\end{aligned}$$

Now we prove that  $\varphi$  is surjective. Let  $(y + U, z + W) \in V/U \times V/W$ . We need to find  $x \in V$  such that  $x + U = y + U$  and  $x + W = z + W$ . Equivalently, we need to find  $x \in V$  such that  $x = y + u$  for some  $u \in U$  and  $x = z + w$  for some  $w \in W$ . Since  $V = U + W$ , there exist  $u' \in U$  and  $w' \in W$  such that  $u' + w' = y - z$ . Take  $u = -u'$  and  $w = w'$ . Clearly  $u \in U$  and  $w \in W$ . Also, we have that  $-u + w = y - z$ , so  $y + u = z + w$ . Define  $x := y + u = z + w$ . Then  $x$  satisfies  $x + U = y + U$  and  $x + W = z + W$ , so  $\varphi(x) = (y + U, z + W)$ .

Finally, by the First Isomorphism Theorem, we conclude that the map

$$\begin{aligned}\tau : V/(U \cap W) &= V/\ker(\varphi) \longrightarrow \text{im}(\varphi) = V/U \times V/W \\ x + U \cap W &= x + \ker(\varphi) \longmapsto \varphi(x) = (x + U, x + W)\end{aligned}$$

is a well-defined isomorphism of  $F$ -vector spaces.



**Problem 4.** Let  $\alpha = (e_1, e_2, e_3)$  be the standard ordered basis for  $\mathbb{R}^3$  and let  $\alpha^* = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$  be the ordered basis for  $(\mathbb{R}^3)^*$  dual to  $\alpha$ . Let  $\beta = (e_1, e_1 + e_2, e_1 + e_2 + e_3)$ , which is another ordered basis for  $\mathbb{R}^3$ , and let  $\beta^* = (\psi_1, \psi_2, \psi_3)$  be the ordered basis for  $(\mathbb{R}^3)^*$  dual to  $\beta$ .

- (a) [5 points] Write each element of  $\beta^*$  as a linear combination of the elements of  $\alpha^*$ .
- (b) [5 points] Consider the subspace  $U = \text{span}\{e_1 + e_2 + e_3\}$  of  $\mathbb{R}^3$ . Find a basis for  $U^\perp$ , expressing each element of this basis as a linear combination of the elements of  $\alpha^*$ .

*Solution.*

- (a) Let  $\theta \in (\mathbb{R}^3)^*$ . Since  $\alpha^*$  is an ordered basis for  $(\mathbb{R}^3)^*$ , there exist unique scalars  $\lambda_1, \lambda_2, \lambda_3 \in F$  such that

$$\theta = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \lambda_3 \varepsilon_3.$$

Evaluating at  $e_1, e_2, e_3$  we find

$$\lambda_1 = \theta(e_1), \quad \lambda_2 = \theta(e_2), \quad \lambda_3 = \theta(e_3),$$

so  $\theta = \theta(e_1)\varepsilon_1 + \theta(e_2)\varepsilon_2 + \theta(e_3)\varepsilon_3$ . We apply this observation to express the elements  $\psi_1, \psi_2, \psi_3 \in (\mathbb{R}^3)^*$  as linear combinations of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$ .

Let  $v_1 = e_1, v_2 = e_1 + e_2, v_3 = e_1 + e_2 + e_3$ . Since the ordered basis  $(\psi_1, \psi_2, \psi_3)$  is dual to the ordered basis  $(v_1, v_2, v_3)$ , we know that

$$\psi_i(v_j) = \delta_{ij} \quad \text{for } i = 1, 2, 3, j = 1, 2, 3.$$

Then we have

$$\begin{aligned} \psi_1(e_1) &= \psi_1(v_1) = 1, \\ \psi_1(e_2) &= \psi_1(v_2 - v_1) = \psi_1(v_2) - \psi_1(v_1) = 0 - 1 = -1, \\ \psi_1(e_3) &= \psi_1(v_3 - v_2) = \psi_1(v_3) - \psi_1(v_2) = 0 - 0 = 0, \end{aligned}$$

so  $\psi_1 = \varepsilon_1 - \varepsilon_2$ .

Similarly

$$\begin{aligned} \psi_2(e_1) &= \psi_2(v_1) = 0, \\ \psi_2(e_2) &= \psi_2(v_2 - v_1) = \psi_2(v_2) - \psi_2(v_1) = 1 - 0 = 1, \\ \psi_2(e_3) &= \psi_2(v_3 - v_2) = \psi_2(v_3) - \psi_2(v_2) = 0 - 1 = -1, \end{aligned}$$

so  $\psi_2 = \varepsilon_2 - \varepsilon_3$ .

Finally

$$\begin{aligned}\psi_3(e_1) &= \psi_3(v_1) = 0, \\ \psi_3(e_2) &= \psi_3(v_2 - v_1) = \psi_3(v_2) - \psi_3(v_1) = 0 - 0 = 0, \\ \psi_3(e_3) &= \psi_3(v_3 - v_2) = \psi_3(v_3) - \psi_3(v_2) = 1 - 0 = 1,\end{aligned}$$

so  $\psi_3 = \varepsilon_3$ .

- (b) We keep the notations introduced in (a). In particular, we have that  $U = \text{span}\{v_3\}$ . Note that  $\dim(U^\perp) = \dim(\mathbb{R}^3) - \dim(U) = 3 - 1 = 2$ . Note also that  $\psi_1(v_3) = \psi_2(v_3) = 0$ , so  $\psi_1, \psi_2 \in U^\perp$ . Therefore  $\{\psi_1, \psi_2\}$  is a linearly independent subset of  $U^\perp$ . Since  $\dim(U^\perp) = 2$ ,

$$\{\psi_1, \psi_2\} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

is a basis for  $U^\perp$ .

**Problem 5.** [10 points] Let  $V$  be a finite-dimensional  $F$ -vector space. Let  $\sigma \in \text{End}(V)$ . Let  $U$  be a  $\sigma$ -invariant subspace of  $V$ . Let  $\sigma|_U \in \text{End}(U)$  and  $\bar{\sigma} \in \text{End}(V/U)$  be the endomorphisms of  $U$  and  $V/U$ , respectively, induced by  $\sigma$ . Let  $\lambda \in F$ . Prove that  $\lambda$  is an eigenvalue of  $\sigma$  if and only if  $\lambda$  is an eigenvalue of  $\sigma|_U$  or an eigenvalue of  $\bar{\sigma}$ .

*Solution.* Let  $\tau = \sigma - \lambda \mathbb{1}_V$ . Since  $U$  is  $\sigma$ -invariant, it is also  $\tau$ -invariant, so we can consider the endomorphisms  $\tau|_U \in \text{End}(U)$  and  $\bar{\tau} \in \text{End}(V/U)$  induced by  $\tau$ . Note that  $\tau|_U = \sigma|_U - \lambda \mathbb{1}_U$  and  $\bar{\tau} = \bar{\sigma} - \lambda \mathbb{1}_{V/U}$ . Thus

$$\begin{aligned}\lambda \text{ is an eigenvalue of } \sigma &\iff \tau \text{ is not injective}, \\ \lambda \text{ is an eigenvalue of } \sigma|_U &\iff \tau|_U \text{ is not injective}, \\ \lambda \text{ is an eigenvalue of } \bar{\sigma} &\iff \bar{\tau} \text{ is not injective}.\end{aligned}$$

Hence, it suffices to prove that

$$\tau \text{ is not injective} \iff \tau|_U \text{ is not injective or } \bar{\tau} \text{ is not injective}$$

or equivalently that

$$\tau \text{ is injective} \iff \tau|_U \text{ is injective and } \bar{\tau} \text{ is injective}$$

We prove the last biimplication.

( $\Rightarrow$ ) Suppose that  $\tau$  is injective. Then clearly  $\tau|_U$  is injective. Also, since  $V$  is finite-dimensional and  $\tau$  is injective, we have that  $\tau$  is surjective, and it follows easily that  $\bar{\tau}$  is surjective. Since  $V/U$  is finite-dimensional, this implies that  $\bar{\tau}$  is injective.

( $\Leftarrow$ ) Suppose that  $\tau|_U$  and  $\bar{\tau}$  are both injective. Let  $x \in \ker \tau$ . Then  $\tau(x) = 0$ , so  $\bar{\tau}(x + U) = \tau(x) + U = 0 + U$ . Thus  $x + U \in \ker \bar{\tau}$ . Since  $\bar{\tau}$  is injective, it follows that  $x + U = 0 + U$ , so  $x \in U$ . Now we have that  $\tau|_U(x) = \tau(x) = 0$ . Since  $\tau|_U$  is injective, we deduce that  $x = 0$ . Hence,  $\ker \tau = \{0\}$  and therefore  $\tau$  is injective.



**Problem 6.** [10 points] Let  $V$  be a 3-dimensional  $\mathbb{R}$ -vector space. Let  $(v_1, v_2, v_3)$  be an ordered basis for  $V$ . Let  $\tau \in \text{End}(V)$  be defined by

$$\tau(v_1) = 2v_1, \quad \tau(v_2) = v_1 + 2v_2, \quad \tau(v_3) = 2v_1 + v_2 + 3v_3.$$

Determine whether  $\tau$  is diagonalizable. If  $\tau$  is diagonalizable, find an ordered basis for  $V$  with respect to which the matrix of  $\tau$  is diagonal. If  $\tau$  is not diagonalizable, prove so.

*Solution.*

Let  $\alpha = (v_1, v_2, v_3)$  and let  $A = {}_{\alpha}[\tau]_{\alpha}$ . Note that

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since  $A$  is upper-triangular, the eigenvalues of  $\tau$  are the entries on the diagonal, which are 2 and 3. We need to find the corresponding eigenspaces. Recall that to find  $E(\lambda, \tau)$  we need to solve the linear system  $(A - \lambda I_3)\mathbf{x} = \mathbf{0}$ .

To find  $E(2, \tau)$ , we need to solve

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$\left. \begin{array}{l} y + 2z = 0 \\ z = 0 \\ z = 0 \end{array} \right\},$$

which yields  $y = z = 0$ . Thus

$$\text{Null}(A - 2I_3) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

and therefore  $E(2, \tau) = \text{span}\{v_1\}$ . Since  $\{v_1\}$  is clearly a basis for  $E(2, \tau)$ , in particular we have  $\dim(E(2, \tau)) = 1$ .

To find  $E(3, \tau)$ , we need to solve

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, equivalently,

$$\begin{array}{rcl} -x & + & y & + & 2z = 0 \\ & -y & + & z & = 0 \\ & & 0 & = & 0 \end{array} \left. \right\},$$

which yields  $y = z$  and  $x = 3z$ . Thus

$$\text{Null}(A - 3I_3) = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and therefore  $E(3, \tau) = \text{span}\{3v_1 + v_2 + v_3\}$ . Since  $\{3v_1 + v_2 + v_3\}$  is clearly a basis for  $E(2, \tau)$ , in particular we have  $\dim(E(3, \tau)) = 1$ .

Since  $\dim(E(2, \tau)) + \dim(E(3, \tau)) < \dim(V)$ , we conclude that  $\tau$  is not diagonalizable.