📘 Homework # 2

Due: October 28 at 11:59 PM **Electronic textbook:** [Click here]

📌 Since there are inconsistencies in exercise numbering between Edition 4 and Edition 5, the homework problems are listed explicitly below to avoid confusion.

Question 1. Label the following statements as **true** or **false**.

(1) An inner product is a scalar-valued function on the set of ordered pairs of vectors.

(2) An inner product space must be over the field of real or complex numbers.

(3) An inner product is linear in both components.

(4) There is exactly one inner product on the vector space $\mathbb{R}^n$.

(5) Only square matrices have a conjugate-transpose.

(6) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.

Question 2. Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in $\mathbb{C}^3$. Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

Question 3. In $\mathbb{C}^2$, show that $\langle x, y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

Question 4. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b), (c,d)\rangle = ac - bd$ on $\mathbb{R}^2$.

(b) $\langle A, B\rangle = \mathrm{tr}(A+B)$ on $\mathrm{M}_{2\times 2}(R)$.

(c) $\langle f(x), g(x)\rangle = \displaystyle\int_0^1 f'(t)g(t)dt$ on $\mathrm{P}(R)$, where $'$ denotes differentiation.

Question 5. Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

Question 6. Prove the parallelogram law on an inner product space V; that is, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in \mathrm{V}$$

What does this equation state about parallelograms in $\mathbb{R}^2$?

Question 7. Suppose that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner products on a vector space V. Prove that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$ is another inner product on V .

Question 8. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*)$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A ?

(b) Let A be an $n \times n$ matrix. Prove that the representation in **(a)** is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

Question 9. Let $V = \mathrm{F}^n$, and let $A \in \mathrm{M}_{n\times n}(F)$.

(a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in V$.

(b) Suppose that for some $B \in \mathrm{M}_{n\times n}(F)$, we have $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathrm{T}(x) = Ax$ and $\mathrm{U}(x) = A^*x$. Show that $[\mathrm{U}]_\beta = [\mathrm{T}]_\beta^*$ for any orthonormal basis β for V.

Question 10. Let V be a vector space over $\mathbb{C}$, and suppose that $[\cdot,\cdot]$ is a real inner product on V, where V is regarded as a vector space over $\mathbb{R}$, such that $[x, ix] = 0$ for all $x \in V$. Let $\langle \cdot,\cdot \rangle$ be the complex-valued function defined by

$$\langle x, y\rangle = [x, y] + i[x, iy] \quad \text{for } x, y \in V.$$

Prove that $\langle \cdot,\cdot \rangle$ is a complex inner product on V.