Nilpotent Operators and Generalized Eigenvectors

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Throughout this note, we denote the base field by $\mathbb{F}$, which is $\mathbb{R}$ or $\mathbb{C}$. All the vector spaces are considered over $\mathbb{F}$.

1 Nilpotent Operators

Definition 1.1. A linear operator $T : V \to V$ on a vector space V is called *nilpotent* if $T^k = 0$ for some positive integer k.

A matrix N is said to be nilpotent matrix if $N^k = 0$ for some positive integer k.

Exercise 1.2. Verify that $N^2 = 0$ for $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

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Example 1.3. The matrix $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is nilpotent and $N^3 = 0$.

Proposition 1.4 (Eigenvalues of a Nilpotent Operator)**.** *Let $N : V \to V$ be a nilpotent linear operator on a finite-dimensional vector space V. Then the only eigenvalue of N is 0.*

Proof. Since N is nilpotent, there exists a positive integer k such that $N^k = 0$. Let λ be an eigenvalue of N with corresponding nonzero eigenvector v, so that

$$Nv = \lambda v.$$

Applying N^{k-1} to both sides gives

$$N^{k-1}(Nv) = N^k v = \lambda^k v.$$

But $N^k = 0$, so $0 = \lambda^k v$. Since $v \neq 0$, we must have $\lambda^k = 0$, and therefore $\lambda = 0$. Hence, 0 is the only eigenvalue of N. $\square$

Proposition 1.5 (Existence of Stabilization)**.** *Let $N : V \to V$ be a linear operator on a vector space V. If for some integer $m \geq 1$,*

$$\ker(N^m) = \ker(N^{m+1}),$$

then

$$\ker(N^m) = \ker(N^{m+1}) = \ker(N^{m+2}) = \cdots .$$

Proof. Suppose $\ker(N^m) = \ker(N^{m+1})$. Let $v \in \ker(N^{m+2})$, i.e., $N^{m+2}v = 0$. Then $N^{m+1}(Nv) = 0 \implies Nv \in \ker(N^{m+1}) = \ker(N^m)$. Hence, $N^{m+1}(v) = N^m(Nv) = 0$, so $v \in \ker(N^{m+1})$. This shows $\ker(N^{m+2}) \subseteq \ker(N^{m+1})$, and the reverse inclusion is obvious. By induction, the equality holds for all higher powers. This completes the proof. □

Proposition 1.6. *Let $N : V \to V$ be a linear operator on an n-dimensional vector space V. Then there exists $m \leq n$ such that*

$$\ker(N^m) = \ker(N^{m+1}) = \ker(N^{m+2}) = ...$$

Proof. Consider the increasing sequence of integers:

$$\dim(\ker(N)) \leq \dim(\ker(N^2)) \leq \dots.$$

Note that all integers in this sequence are less than n. As the sequence is non-decreasing and bounded above by n, it must stabilize at some $m \leq n$, i.e., $\dim(\ker(N^m)) = \dim(\ker(N^{m+1}))$. Hence, $\ker(N^m) = \ker(N^{m+1})$. We are now done by the previous proposition. □

Theorem 1.7. *Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V of dimension n. Then*

$$V = \ker(T^n) \oplus \text{Range}(T^n),$$

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Proof. Let m be an integer such that $\ker(T^m) = \ker(T^{m+1})$. This integer exist by the Proposition 1.6 and $m \leq n$. We first claim that $\ker(T^m) \cap \text{Range}(T^m) = 0$. Consider $v \in \ker(T^m) \cap \text{Range}(T^m)$. Then there exists $u \in V$ such that $v = T^m(u)$. Since $v \in \ker(T^m)$, we have $T^m(v) = T^m(T^m(u)) = T^{2m}(u) = 0$. But by Proposition 1.6, $\ker(T^{2m}) = \ker(T^m)$, so $u \in \ker(T^m)$. Hence, $v = T^m(u) = 0$. Thus, $\ker(T^m) \cap \text{Range}(T^m) = \{0\}$.

We are now left to show that $V = \ker(T^m) + \text{Range}(T^m)$. By the Rank-Nullity theorem, we have $n = dim(\ker(T^m)) + dim(\text{Range}(T^m))$. Since $\ker(T^m) + \text{Range}(T^m) \subseteq V$, and hence $V = \ker(T^m) + \text{Range}(T^m)$. This completes the proof. □

Proposition 1.8 (Nilpotent Operator of Finite Dimension)**.** *Let $N : V \to V$ be a nilpotent linear operator on a finite-dimensional vector space V of dimension n. Then*

$$N^n = 0.$$

Proof. Since N is nilpotent, there exists a positive integer k such that $N^k = 0$. Consider the increasing sequence of null spaces

$$\ker(N) \subseteq \ker(N^2) \subseteq \cdots \subseteq \ker(N^m) \subseteq \cdots.$$

Because V is finite-dimensional, this sequence must stabilize; that is, there exists $m \leq n$ such that

$$\ker(N^m) = \ker(N^{m+1}) = \cdots = V.$$

Now since $N^k = 0$ and hence $\ker(N^k) = V$. Hence, we can conlude that

$$N^n = 0.$$

This completes the proof. □

2 Generalized Eigenvectors

Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. A nonzero vector $v \in V$ is an **eigenvector** of T corresponding to eigenvalue $\lambda \in \mathbb{F}$ if

$$T(v) = \lambda v.$$

Sometimes a linear operator T is not diagonalizable, i.e., it does not have enough eigenvectors to form a basis. In such cases, we need vectors v satisfying

$$(T - \lambda I)^k v = 0 \quad \text{for some } k > 1,$$

called generalized eigenvectors.

Definition 2.1 (Generalized Eigenvector)**.** Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let $\lambda \in \mathbb{F}$ be a scalar. A nonzero vector $v \in V$ is called a **generalized eigenvector** of T corresponding to the eigenvalue λ if there exists a positive integer k such that

$$(T - \lambda I)^k v = 0.$$

Remark 2.2. Every ordinary eigenvector is a generalized eigenvector for $k = 1$. Although k is allowed to be arbitrary integer but we will see later that $k = \dim(V)$.

Definition 2.3. The set consists of all generalized eigenvectors corresponding to λ and the zero vector forms a subspace of V, called the **generalized eigenspace** of T associated with λ:

$$G_\lambda = \ker\big((T - \lambda I)^m\big),$$

where m is large enough that the null space stabilizes, i.e., $\ker((T - \lambda I)^m) = \ker((T - \lambda I)^{m+1})$.

As every eigenvector is also a generalized eigenvector, so we have $E_\lambda \subseteq G_\lambda$, where E_λ is the eigenspace corresponding to the eigenvalue λ.

Proposition 2.4 (Generalized Eigenspace as a Null Space)**.** *Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over a field $\mathbb{F}$, and let $\lambda \in \mathbb{F}$. Then the generalized eigenspace of T corresponding to λ is given by*

$$G_\lambda = \ker\big((T - \lambda I)^n\big),$$

where $n = \dim V$. In particular, G_λ is a subspace of V.

Proof. For each positive integer k, define

$$V_k = \ker\big((T - \lambda I)^k\big).$$

Then we have the increasing sequence of subspaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots.$$

Since V is finite-dimensional, this sequence must stabilize by the stabilization proposition; that is, there exists an integer $m \leq n$ such that

$$V_m = V_{m+1} = V_{m+2} = \cdots.$$

By definition, the generalized eigenspace G_λ is the union of all vectors v such that $(T - \lambda I)^k v = 0$ for some k. Hence,

$$G_\lambda = \bigcup_{k \geq 1} V_k = V_m.$$

Since $V_m = V_{m+1} = \cdots$, we can take $m = n = \dim V$. Therefore,

$$G_\lambda = \ker\big((T - \lambda I)^n\big).$$

This completes the proof. $\square$