Corollary If V has a finite basis, then all bases of V have the same cardinality.

Proof Suppose that S and T are bases of V and suppose that T is finite with $|T| = n$.

Then T spans V and each finite subset S_0 of S is linearly independent.
By Steinitz Replacement Theorem, $|S_0| \leq n$ for each such S_0.

Hence S is finite with $|S| =: m \leq n$.

Similarly $n \leq m$, so $n = m$. $\square$

Def. Let V be an F-vector space. If V has a finite basis, the dimension of V, denoted by $\dim(V)$, is the number of elements in any basis for V.
(This is well-defined by the previous corollary.)

Prop. Let $\dim(V) = n$.

(a) Every linearly independent subset of V contains at most n elements.

(b) Every linearly independent subset of V with n elements is a basis.

(c) Every spanning subset of V has at least n elements.

(d) Every spanning subset of V with n elements is a basis.

Proof. Let S be a basis for V (so $|S| = n$).

(a) Follows from Steinitz Replacement Theorem, since S spans V.

(b) By (a), a linearly independent subset of V with n elements is maximal with respect to being linearly independent, hence a basis.

(c) Follows from Steinitz Replacement Theorem, since S is linearly independent.

(d) By (c), a spanning subset of V with n elements is minimal with respect to spanning V, hence a basis. $\square$

Prop. Let V be a finite-dimensional vector space and let X be a linearly independent subset of V. Then X can be extended to a basis of V (i.e., there exists a basis S of V such that $X \subseteq S$).

Proof. Let Y be a basis of V. Apply the Steinitz Replacement Theorem to X and Y. □

Prop. If U is a proper subspace of a f.d.v.s. V, then $\dim(U) < \dim(V)$.

Proof. A basis X of U is linearly independent but does not span V, so X has fewer than $\dim V$ elements. □

Prop. (Sum - Intersection formula) Suppose that U_1, U_2 are finite-dimensional subspaces of a vector space V. Then $U_1 + U_2$ is finite-dimensional and

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Proof. Suppose that $\dim(U_1 \cap U_2) = k$. Let $Y = \{u_1, \dots, u_k\}$ be a basis of $U_1 \cap U_2$.

$$\text{Extend } Y \text{ to } \begin{cases} \text{a basis } X_1 = \{u_1, \dots, u_k, v_{k+1}, \dots, v_m\} \text{ of } U_1 \\ \text{a basis } X_2 = \{u_1, \dots, u_k, w_{k+1}, \dots, w_n\} \text{ of } U_2 \end{cases}$$

where $m = \dim(U_1)$ and $n = \dim(U_2)$.

<u>Claim</u> $X = \{u_1, \dots, u_k, v_{k+1}, \dots, v_m, w_{k+1}, \dots, w_n\}$ is a basis of $U_1 + U_2$.

Clearly, $\operatorname{span} X = U_1 + U_2$.

Suppose that

$$\sum_{i=1}^{k} a_i u_i + \sum_{i=k+1}^{m} b_i v_i + \sum_{i=k+1}^{n} c_i w_i = 0 \qquad (\dagger)$$

for some $a_i, b_i, c_i \in F$.

Then

$$\underbrace{\sum_{i=k+1}^{n} c_i w_i}_{\in U_2} = \underbrace{-\left(\sum_{i=1}^{k} a_i u_i + \sum_{i=k+1}^{m} b_i v_i \right)}_{\in U_1}$$

So $\sum_{i=k+1}^{n} c_i w_i \in U_1 \cap U_2$. Therefore, $\exists\, d_1, \dots, d_k \in F$ such that

$$\sum_{i=k+1}^{n} c_i w_i = \sum_{i=1}^{k} d_i u_i,$$

so

$$\sum_{i=1}^{k} d_i u_i + \sum_{i=k+1}^{n} (-c_i) w_i = 0$$

Since $X_2 = \{u_1, \dots, u_k, w_{k+1}, \dots, w_n\}$ is linearly independent, it follows that

$$d_1 = \dots = d_k = c_{k+1} = \dots = c_n = 0.$$

Now $(\dagger)$ implies that

$$a_1 = \dots = a_k = b_{k+1} = \dots = b_m = 0,$$

since $X_1 = \{u_1, \dots, u_k, v_{k+1}, \dots, v_n\}$ is linearly independent.

Therefore X is linearly independent. Thus X is a basis of $U_1 + U_2$.

Hence

$$\dim(U_1 + U_2) = |X| = m + n - k = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2). \qquad \square$$