Prop. Let U, V, W be f.d.v.s. over F. Let $m = \dim(U)$, $n = \dim(V)$, $p = \dim(W)$, and let α, β, γ be ordered bases for U, V, W respectively.

(i) If $\sigma \in \mathcal{L}(V, U)$ has matrix $A = {}_\alpha[\sigma]_\beta$ and $\tau \in \mathcal{L}(W, V)$ has matrix $B = {}_\beta[\tau]_\gamma$, then $\sigma\tau \in \mathcal{L}(W, U)$ has matrix $C = {}_\alpha[\sigma\tau]_\gamma$ given by the matrix product AB.

(ii) If $\sigma \in \mathcal{L}(V, U)$ has matrix $A = {}_\alpha[\sigma]_\beta$ and $y \in V$ has β-coordinate vector

$$[y]_\beta = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix},$$

then $\sigma(y) \in U$ has α-coordinate vector given by

$$[\sigma(y)]_\alpha = A\,[y]_\beta.$$

Proof. (i) Follows from previous discussion.

(ii) To find $[\sigma(y)]_\alpha$, we write $\sigma(y)$ as a linear combination of the elements in the basis $\alpha = (e_1, \dots, e_m)$. Say $\beta = (f_1, \dots, f_n)$ and $A = (a_{ij})$. Then

$$\begin{aligned} \sigma(y) &= \sigma\left(\sum_{j=1}^{n} \lambda_j f_j\right) \\ &= \sum_{j=1}^{n} \lambda_j \sigma(f_j) \\ &= \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} a_{ij} e_i\right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \lambda_j e_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \lambda_j\right) e_i \end{aligned}$$

So
$$[\sigma(y)]_\alpha = \begin{bmatrix} \Sigma a_{1j}\lambda_j \\ \vdots \\ \Sigma a_{mj}\lambda_j \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = A[y]_\beta .$$
□

Properties of matrix multiplication

Prop.

(i) Matrix multiplication is associative; i.e., if $A \in M_{m\times n}(F)$, $B \in M_{n\times p}(F)$, $C \in M_{p\times q}(F)$, then
$$A(BC) = (AB)C.$$

(ii) Matrix multiplication is distributive; i.e., if $A, B \in M_{m\times n}(F)$, $C, D \in M_{n\times p}(F)$, then
$$A(C+D) = AC + AD$$
$$(A+B)C = AC + BC.$$

(iii) For each positive integer r, define the $r\times r$ identity matrix as the matrix $I_r \in M_r(F)$ with all diagonal entries equal to 1 and all other entries equal to zero (i.e., $I_r = (\delta_{ij})$). Then, for all $A \in M_{m\times n}(F)$,
$$I_m A = A = A I_n.$$

Proof 1 (By calculation) (i) Say $A = (a_{ij})$, $B = (b_{jk})$, $C = (c_{kl})$.

Then

$$
\begin{aligned}
(A(BC))_{i,l} &= \sum_{j=1}^{n} a_{ij} (BC)_{j,l} \\
&= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} c_{kl} \right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ij} b_{jk} c_{kl} \\
&= \sum_{k=1}^{p} \sum_{j=1}^{n} a_{ij} b_{jk} c_{kl} \\
&= \sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} \\
&= \sum_{k=1}^{p} (AB)_{i,k} \, c_{kl} \\
&= ((AB)C)_{i,l}
\end{aligned}
$$

(ii), (iii) Exercise.

$\square$

Proof 2 (Using structure of linear maps)

(i) Consider the diagram

$$F^q \xrightarrow{\tau} F^p \xrightarrow{\sigma} F^n \xrightarrow{\rho} F^m$$

where ρ, σ, τ have matrices A, B, C for the standard bases. Then

$$
\begin{aligned}
A(BC) &= {}_{\mathcal{E}_m}[\rho]_{\mathcal{E}_n} \left({}_{\mathcal{E}_n}[\sigma]_{\mathcal{E}_p} \, {}_{\mathcal{E}_p}[\tau]_{\mathcal{E}_q} \right) \\
&= {}_{\mathcal{E}_m}[\rho]_{\mathcal{E}_n} \cdot {}_{\mathcal{E}_n}[\sigma\tau]_{\mathcal{E}_q} \\
&= {}_{\mathcal{E}_m}[\rho(\sigma\tau)]_{\mathcal{E}_q} \\
&= {}_{\mathcal{E}_m}[(\rho\sigma)\tau]_{\mathcal{E}_q} \\
&= {}_{\mathcal{E}_m}[\rho\sigma]_{\mathcal{E}_p} \cdot {}_{\mathcal{E}_p}[\tau]_{\mathcal{E}_q} \\
&= \left({}_{\mathcal{E}_m}[\rho]_{\mathcal{E}_n} \, {}_{\mathcal{E}_n}[\sigma]_{\mathcal{E}_p} \right) {}_{\mathcal{E}_p}[\tau]_{\mathcal{E}_q} \\
&= (AB)C.
\end{aligned}
$$

$\mathcal{E}_r$ = standard ordered basis of F^r

(ii) Let $\sigma_1 : F^n \longrightarrow F^m$, $\sigma_2 : F^n \longrightarrow F^m$, $\tau_1 : F^p \longrightarrow F^n$, $\tau_2 : F^p \longrightarrow F^n$ be the linear maps with matrices A, B, C, D respectively in the standard bases. Then

$$(A+B)C = ([\sigma_1]+[\sigma_2])[\tau_1]$$

$$= [\sigma_1+\sigma_2]\cdot[\tau_1]$$

$$= [(\sigma_1+\sigma_2)\tau_1]$$

$$= [\sigma_1\tau_1 + \sigma_2\tau_1]$$

$$= [\sigma_1\tau_1] + [\sigma_2\tau_1]$$

$$= [\sigma_1][\tau_1] + [\sigma_2][\tau_1]$$

$$= AC + BC.$$

Similarly, $A(C+D) = AC + AD$.

(iii) I_n represents the identity map in standard basis $\mathcal{E}_n = (e_1, \dots, e_n)$

$$\mathbb{1}_{F^n} : F^n \longrightarrow F^n$$

$$e_j \longmapsto \sum_{i=1}^{n} \delta_{ij} e_i = e_j.$$

Let $\sigma : F^n \longrightarrow F^m$ be the linear map with matrix A in the standard bases. Then

$$A I_n = [\sigma][\mathbb{1}_{F^n}]$$

$$= [\sigma \mathbb{1}_{F^n}]$$

$$= [\sigma]$$

$$= A.$$

Similarly $I_m A = A$.

Examples

Def. Let $n \in \mathbb{Z}_{\geq 0}$ and let F be a field. Let $P_n(F)$ denote the set of all polynomials in the formal variable X of degree at most n; i.e.,

$$P_n(F) = \{ a_0 + a_1 X + \cdots + a_n X^n \mid a_0, a_1, \ldots, a_n \in F \}.$$

Then $P_n(F)$ is an F-v.s. isomorphic to F^{n+1}.

Example Consider the differentiation map

$$D : P_2(F) \longrightarrow P_1(F)$$
$$a_0 + a_1 X + a_2 X^2 \longmapsto a_1 + 2a_2 X$$

Let $\alpha = (1, X)$, $\beta = (1, X, X^2)$ be the standard ordered bases for $P_1(F)$ and $P_2(F)$, respectively. Then

$$D(1) = 0 \cdot 1 + 0 \cdot X$$
$$D(X) = 1 \cdot 1 + 0 \cdot X$$
$$D(X^2) = 0 \cdot 1 + 2 \cdot X$$

So $${}_{\alpha}[D]_{\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Change of basis

Given a linear map $T: U \to V$ between f.d.v.s. over F with $m = \dim V$, $n = \dim U$, and given ordered bases α, β of U, V respectively, we can construct a matrix

$$A = {}_{\beta}[T]_{\alpha} \in M_{m\times n}(F)$$

such that the following diagram commutes.

$$\begin{tikzcd}
x \arrow[d, mapsto] & U \arrow[r, "T"] \arrow[d, "\wr"] & V \arrow[d, "\wr"] & y \arrow[d, mapsto] \\
{[x]_\alpha} & F^{n\times 1} \arrow[r, "T_A"] & F^{m\times 1} & {[y]_\beta}
\end{tikzcd}$$

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \longmapsto A\cdot\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

Q1 How does changing α and/or β affect the matrix A?

Q2 What can we deduce about matrices from this diagram?

Def. Let V be a fdvs over F with $\dim(V) = m$. Let β, β' be ordered bases for V. We call

$${}_{\beta'}[\mathbb{1}_V]_{\beta} \in M_m(F)$$

the change of basis matrix from β to β'. (Here $\mathbb{1}_V : V \to V$ is the identity map.)