Prop. Let V, β, β' be as above. Then

(i) ${}_{\beta'}[\mathbb{1}_V]_\beta [y]_\beta = [y]_{\beta'}$ for all $y \in V$.

(ii) ${}_{\beta'}[\mathbb{1}_V]_\beta \, {}_{\beta}[\mathbb{1}_V]_{\beta'} = I_m = {}_{\beta}[\mathbb{1}_V]_{\beta'} \, {}_{\beta'}[\mathbb{1}_V]_{\beta}$.

(iii) If $T : U \to V$ is a linear map where U is a f.d.v.s. over F with $\dim(U) = n$ and α, α' are ordered bases for U, then

$$ {}_{\beta'}[T]_{\alpha'} = {}_{\beta'}[\mathbb{1}_V]_\beta \, {}_\beta[T]_\alpha \, {}_\alpha[\mathbb{1}_U]_{\alpha'}. $$

Proof.

(i) ${}_{\beta'}[\mathbb{1}_V]_\beta [y]_\beta = [\mathbb{1}_V(y)]_{\beta'} = [y]_{\beta'}$.

(ii) ${}_{\beta'}[\mathbb{1}_V]_\beta \, {}_\beta[\mathbb{1}_V]_{\beta'} = {}_{\beta'}[\mathbb{1}_V \mathbb{1}_V]_{\beta'} = {}_{\beta'}[\mathbb{1}_V]_{\beta'} = I_m \quad \left(\underset{\beta'}{V} \xrightarrow{\mathbb{1}_V} \underset{\beta}{V} \xrightarrow{\mathbb{1}_V} \underset{\beta'}{V} \right)$

Similarly ${}_\beta[\mathbb{1}_V]_{\beta'} \, {}_{\beta'}[\mathbb{1}_V]_\beta = I_m$.

(iii) ${}_{\beta'}[\mathbb{1}_V]_\beta \, {}_\beta[T]_\alpha \, {}_\alpha[\mathbb{1}_U]_{\alpha'} = {}_{\beta'}[\mathbb{1}_V T \mathbb{1}_U]_{\alpha'} = {}_{\beta'}[T]_{\alpha'}$

$$\left(\underset{\alpha'}{U} \xrightarrow{\mathbb{1}_U} \underset{\alpha}{U} \xrightarrow{T} \underset{\beta}{V} \xrightarrow{\mathbb{1}_V} \underset{\beta'}{V} \right)$$

$\square$

<u>Example</u> Consider the <u>differentiation</u> map

$$D: P_2(F) \longrightarrow P_1(F)$$
$$a_0 + a_1 X + a_2 X^2 \mapsto a_1 + 2a_2 X.$$

Let $\alpha = (1, X)$, $\alpha' = (1, 1+X)$ (ordered bases for $P_1(F)$)

$\beta = (1, X, X^2)$ (ordered basis for $P_2(F)$)

We saw that

$${}_{\alpha}[D]_{\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now consider the change of basis matrix from α to α':

$$1 = 1 \cdot 1 + 0 \cdot (1+X)$$
$$X = (-1) \cdot 1 + 1 \cdot (1+X)$$

So ${}_{\alpha'}[\mathbb{1}_{P_1(F)}]_{\alpha} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Then
$$\begin{aligned} {}_{\alpha'}[D]_{\beta} &= {}_{\alpha'}[\mathbb{1}_{P_1(F)}]_{\alpha} \, {}_{\alpha}[D]_{\beta} \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Matrix equivalence

Def. Let $A \in M_{m\times n}(F)$. The column space of A, denoted by $\mathrm{Col}(A)$, is the subspace of $M_{m\times 1}(F) = F^{m\times 1}$ spanned by the columns of A; i.e., if

$$A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix},$$

then $\mathrm{Col}(A) = \mathrm{span}\{\vec{v}_1, \dots, \vec{v}_n\}$. The row space of A, denoted by $\mathrm{Row}(A)$, is the subspace of $M_{1\times n}(F) = F^{1\times n}$ spanned by the rows of A. The null space of A is defined as

$$\mathrm{Null}(A) := \{\vec{x} \in F^{n\times 1} \mid A\vec{x} = \vec{0}\}.$$

The column rank, row rank and nullity of A are the dimensions of $\mathrm{Col}(A)$, $\mathrm{Row}(A)$ and $\mathrm{Null}(A)$, respectively.

Prop. Let $A \in M_{m\times n}(F)$ and let $T_A : F^n \to F^m$ be the associated linear map (i.e., ${}_{\mathcal{E}_m}[T_A]_{\mathcal{E}_n} = A$).

(i) The canonical isomorphism

$$[\cdot]_{\mathcal{E}_m} : F^m \xrightarrow{\sim} F^{m\times 1}, \quad (\lambda_1, \dots, \lambda_m) \mapsto \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix}$$

("take coordinates in the standard basis")

restricts to an isomorphism

$$F^m \supseteq \mathrm{im}(T_A) \xrightarrow{\sim} \mathrm{Col}(A) \subseteq F^{m\times 1}.$$

(ii) The canonical isomorphism $[\cdot]_{\mathcal{E}_n} : \mathbb{F}^n \longrightarrow \mathbb{F}^{n\times 1}$ restricts to an isomorphism

$$\mathbb{F}^n \supseteq \ker(T_A) \xrightarrow{\sim} \text{Null}(A) \leq \mathbb{F}^{n\times 1}.$$

In particular,

- $\text{nullity}(A) = \text{nullity}(T_A)$
- $\text{column rank}(A) = \text{rank}(T_A)$
- $\text{column rank}(A) + \text{nullity}(A) = n.$

Proof.

(i) Let $\mathcal{E}_n = (\vec{e}_1, \dots, \vec{e}_n)$ be the standard ordered basis for $\mathbb{F}^n$ and let $\mathcal{E}_m = (\vec{f}_1, \dots, \vec{f}_m)$ be the standard ordered basis for $\mathbb{F}^m$. Then

$$\text{im}(T_A) = \text{span}\left\{ T_A(\vec{e}_1), \dots, T_A(\vec{e}_n) \right\}$$

$$\xrightarrow[\sim]{[\cdot]} \text{span}\left\{ [T_A(\vec{e}_1)]_{\mathcal{E}_m}, \dots, [T_A(\vec{e}_n)]_{\mathcal{E}_m} \right\}$$

$$= \text{span}\left\{ {}_{\mathcal{E}_m}[T_A]_{\mathcal{E}_n} [\vec{e}_1]_{\mathcal{E}_n}, \dots, {}_{\mathcal{E}_m}[T_A]_{\mathcal{E}_n} [\vec{e}_n]_{\mathcal{E}_n} \right\}$$

$$= \text{span}\left\{ A[\vec{e}_1]_{\mathcal{E}_n}, \dots, A[\vec{e}_n]_{\mathcal{E}_n} \right\}$$

Note that $[\vec{e}_i]_{\mathcal{E}_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ i-th entry. Let $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$.

Then $\text{span}\left\{ A[\vec{e}_1]_{\mathcal{E}_n}, \dots, A[\vec{e}_n]_{\mathcal{E}_n} \right\} = \text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = \text{Col}(A).$

(ii) Exercise.

The remaining claims are immediate. $\square$

<u>Theorem</u> (column rank = row rank). Let $A \in M_{m\times n}(F)$. Then the column rank of A is equal to the row rank of A. (From now on we call it simply the <u>rank</u> of A).

<u>Proof.</u> Let r = row rank of A, s = column rank of A.

Let $\vec{v}_1, \dots, \vec{v}_s \in F^{m\times 1}$ form a basis of $Col(A)$. Let $\vec{a}^j$ denote the j-th column of A. Then, for each $j = 1, \dots, n$,

$$\vec{a}^j = \sum_{k=1}^{s} c_{kj}\vec{v}_k = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_s \end{bmatrix}\begin{bmatrix} c_{1j} \\ \vdots \\ c_{sj} \end{bmatrix} \quad \text{for some } c_{1j}, \dots, c_{sj} \in F.$$

So

$$A = \begin{bmatrix} \vec{a}^1 & \cdots & \vec{a}^n \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_s \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{s1} & \cdots & c_{sn} \end{bmatrix}}_{C}$$

Then the rows of A are linear combinations of the rows of C, so $Row(A) \leq Row(C)$. Since the rows of C form a spanning set for $Row(C)$, we deduce $\dim(Row(C)) \leq s$. Therefore

$$r = \dim(Row(A)) \leq \dim(Row(C)) \leq s.$$

Similarly one can show that $s \leq r$, so we conclude that $r = s$. $\square$

Def. Let $P \in M_{m\times n}(F)$. A matrix $Q \in M_{n\times m}(F)$ is said to be a <u>right inverse</u> of P if $P\cdot Q = I_m$. Similarly, a matrix $R \in M_{n\times m}(F)$ is said to be a <u>left inverse</u> of P if $RP = I_n$.

<u>Prop.</u> Let $P \in M_{m\times n}(F)$.

(i) There exists a right inverse of P if and only if $\operatorname{rank}(P) = m$ (if and only if T_P is surjective).

(ii) There exists a left inverse of P if and only if $\operatorname{rank}(P) = n$ (if and only if T_P is injective).

<u>Proof</u>

(i) ($\Rightarrow$) Suppose $Q \in M_{n\times m}(F)$ is a right inverse of P. Then $PQ = I_m$. Then the columns of I_m are linear combinations of the columns of P, so

$$F^{m\times 1} = Col(I_m) \subseteq Col(P) \subseteq F^{m\times 1}$$

and therefore $Col(P) = F^{m\times 1}$. Thus $\operatorname{rank}(P) = \dim(F^{m\times 1}) = m$.

($\Leftarrow$) Suppose $\operatorname{rank}(P) = m$. Then $\dim(Col(P)) = m$. Since $Col(P) \subseteq F^{m\times 1}$ and $\dim(F^{m\times 1}) = m$, it follows that $Col(P) = F^{m\times 1}$. Let $P = [\vec{v}_1 \cdots \vec{v}_n]$, $I_m = [\vec{e}_1 \cdots \vec{e}_m]$.

Since $\operatorname{span}\{\vec{v}_1, \dots, \vec{v}_n\} = F^{m\times 1}$, there exist $c_{jk} \in F$, $1 \le j \le n$, $1 \le k \le m$, s.t. $\vec{e}_k = \sum_{j=1}^{n} c_{jk}\vec{v}_j$ for each $k = 1, \dots, m$.

Let $Q = (q_{jk}) \in M_{n\times m}(F)$. Then

$$PQ = \begin{bmatrix} | & & | \\ \sum_{j=1}^{n} q_{j1}\vec{v}_j & \cdots & \sum_{j=1}^{n} q_{jm}\vec{v}_j \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \vec{e}_1 & \cdots & \vec{e}_m \\ | & & | \end{bmatrix} = I_m.$$

(ii) Exercise. □

Prop. Let $P \in M_n(F)$. The following are equivalent:

(i) there exists a unique right inverse Q of P;

(ii) there exists a unique left inverse R of P;

(iii) $\operatorname{rank}(P) = n$.

Moreover, when these conditions hold, then $Q = R$. We say in this case that P is <u>invertible</u> and call Q the <u>inverse</u> of P, denoted P^{-1}.

Proof. By the previous proposition,

$$Q \text{ exists} \iff \operatorname{rank}(P) = n \iff R \text{ exists}.$$

It remains to show that, when $\operatorname{rank}(P) = n$, Q and R are unique and $Q = R$. For that, note that

$$Q = I_n Q = (RP)Q = R(PQ) = R I_n = R.$$

Now, if Q' is any right inverse of P, then the same argument applied to Q' and R shows that $Q' = R$. So $Q' = R = Q$.

Similarly, if R' is any left inverse of P, $R' = Q = R$. □