Notation For $x \in V$, $\theta \in V^*$, we often write $\langle \theta, x \rangle$ for $\theta(x)$.

Prop. Let V be a f.d. v.s. over F.

(a) For $\theta \in V^*$ fixed, the map $x \mapsto \langle \theta, x \rangle$ from V to F is linear.

(b) For $x \in V$ fixed, the map $\theta \mapsto \langle \theta, x \rangle$ from V^* to F is linear.

(c) If $\theta \in V^*$ satisfies $\langle \theta, x \rangle = 0$ for all $x \in V$, then $\theta = 0$ (i.e., θ equals the zero map $V \to F$).

(d) If $x \in V$ satisfies $\langle \theta, x \rangle = 0$ for all $\theta \in V^*$, then $x = 0$.

Proof

(a) Saying that $x \mapsto \langle \theta, x \rangle$ is linear is the same as saying that θ is linear.

(b) Let $\lambda, \mu \in F$, $\theta, \phi \in V^*$. Then

$$\begin{aligned}\langle \lambda\theta + \mu\phi, x \rangle &= (\lambda\theta + \mu\phi)(x) \\ &= \lambda\theta(x) + \mu\phi(x) \\ &= \lambda\langle \theta, x \rangle + \mu \langle \phi, x \rangle.\end{aligned}$$

(c) Clear.

(d) This follows from the previous corollary. $\square$

Rk Properties (a) and (b) are referred to as saying that

$$\langle \cdot, \cdot \rangle : V^* \times V \longrightarrow F$$
$$(\theta, x) \longmapsto \langle \theta, x \rangle = \theta(x)$$

is a bilinear map.

Properties (c) and (d) are referred to as saying that this bilinear map is non-degenerate.

Prop. (Dual basis) Let V be a f.d.v.s. over F. Let $\beta = (e_1, \dots, e_n)$ be an ordered basis for V (where $n = \dim V$). Then there exists a unique ordered basis $\beta^* = (\varepsilon_1, \dots, \varepsilon_n)$ for V^* such that

$$\langle \varepsilon_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The ordered basis β^* is called the dual basis of β.

Proof. Since $(e_1, \dots, e_n)$ is an ordered basis for V, there exist unique linear functionals

$$\varepsilon_i \in V^* = \mathcal{L}(V, F) \quad \text{such that} \quad \varepsilon_i(e_j) = \langle \varepsilon_i, e_j \rangle = \delta_{ij} \quad \text{for all}$$

$1 \leq j \leq n$, $1 \leq i \leq n$. We need to prove that $\beta^* = (\varepsilon_1, \dots, \varepsilon_n)$ is an ordered basis for V^*. Since we know that $\dim V^* = \dim V = n$, it suffices to show that the elements $\varepsilon_1, \dots, \varepsilon_n$ are linearly independent.

Suppose $\lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n = 0$ for some $\lambda_1, \dots, \lambda_n \in F$. Then, for $j = 1, \dots, n$,

$$0 = \langle \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n, e_j \rangle = \lambda_1 \langle \varepsilon_1, e_j \rangle + \cdots + \lambda_n \langle \varepsilon_n, e_j \rangle = \lambda_j .$$

So $\lambda_1 = \cdots = \lambda_n = 0$. $\square$

Exercise Keep the notations in the proposition. Let $\theta \in V^*$. Show that

$$[\theta]_{\beta'} = \begin{bmatrix} \theta(e_1) \\ \vdots \\ \theta(e_n) \end{bmatrix}.$$