

T-Invariant Subspaces and Restriction Operators

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1 Definition: T-Invariant Subspaces

Let T be a linear operator on a vector space V . A subspace W of V is called a **T-invariant subspace** of V if $T(W) \subseteq W$. This means that if $v \in W$, then $T(v)$ must also be in W .

1.1 Examples

For a linear operator T on a vector space V , the following subspaces are always T -invariant:

- The zero subspace: $\{0\}$.
- The entire vector space: V .
- The range of T : $R(T)$.
- The null space of T : $N(T)$.
- The eigenspace E_λ for any eigenvalue λ of T .

Example: Let T be the linear operator on \mathbb{R}^3 defined by

$$T(a, b, c) = (a + b, b + c, 0).$$

The plane $\{(x, y, 0) : x, y \in \mathbb{R}\}$ is T -invariant. The line $\{(x, 0, 0) : x \in \mathbb{R}\}$ is also T -invariant.

2 Definition: T-Cyclic Subspace

Let T be a linear operator on a vector space V , and let x be a nonzero vector in V . The subspace

$$W = \text{span}\{x, T(x), T^2(x), \dots\}$$

is called the **T-cyclic subspace** generated by x .

- W is always T -invariant.
- W is the smallest T -invariant subspace of V containing x .

Example (Differentiation): Let T be the operator on polynomials defined by $T(f(x)) = f'(x)$. Then for x^2 :

$$T(x^2) = 2x, \quad T^2(x^2) = 2, \quad T^3(x^2) = 0.$$

So the T -cyclic subspace generated by x^2 is $\text{span}\{x^2, x, 1\}$.

3 Restriction of an Operator and Characteristic Polynomials

3.1 Restriction Operator

If W is a T -invariant subspace of V , then the restriction $T_W : W \rightarrow W$ is defined by

$$T_W(w) = T(w) \quad \text{for all } w \in W.$$

It is easy to observe that T_W is linear transformation.

3.2 Characteristic Polynomial and Invariant Subspaces

Theorem. Let T be a linear operator on a finite-dimensional vector space V , and let W be a T -invariant subspace of V . Then the characteristic polynomial of the restriction T_W divides the characteristic polynomial of T .

Hint: This follows from the determinant formula for block upper triangular matrices:

$$M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \Rightarrow \det(M) = \det(A) \det(C).$$

Remark: We need this result in the later lectures when we will study normal and self-adjoint operators.

3.3 Cyclic Subspace and Characteristic Polynomial

Let W be the T -cyclic subspace generated by v , and let $\dim(W) = k$. Then

$$\beta = \{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$$

is a basis for W .