Solutions Homework 7

Problem 1. Let V be an F-vector space. Let U and W be subspaces of V. Prove that

$$(U+W)^{\perp} = U^{\perp} \cap W^{\perp}.$$

Solution.
($\subseteq$) Let $\theta \in (U+W)^{\perp}$. Then $\theta(x) = 0$ for all $x \in U+W$. Since $U \subseteq U+W$ and $W \subseteq U+W$, it follows that $\theta(u) = 0$ for all $u \in U$ and $\theta(w) = 0$ for all $w \in W$. Therefore $\theta \in U^{\perp}$ and $\theta \in W^{\perp}$, i.e., $\theta \in U^{\perp} \cap W^{\perp}$.
($\supseteq$) Let $\theta \in U^{\perp} \cap W^{\perp}$. Then $\theta(u) = 0$ for all $u \in U$ and $\theta(w) = 0$ for all $w \in W$. Now let $x \in U+W$. We can write $x = y+z$ for some $y \in U$ and $z \in W$. Thus $\theta(x) = \theta(y+z) = \theta(y) + \theta(z) = 0+0 = 0$. Since this holds for all $x \in U+W$, we conclude that $\theta \in (U+W)^{\perp}$.

Problem 2. Let V be a finite-dimensional F-vector space. Let U and W be subspaces of V. Prove that

$$(U \cap W)^{\perp} = U^{\perp} + W^{\perp}.$$

Solution.
First we show that $U^{\perp} + W^{\perp} \subseteq (U \cap W)^{\perp}$. Let $\theta \in U^{\perp} + W^{\perp}$. Then $\theta = \phi + \psi$ for some $\phi \in U^{\perp}$ and $\psi \in W^{\perp}$. Let $x \in U \cap W$. Then $x \in U$ and $x \in W$, so $\phi(x) = 0$ and $\psi(x) = 0$ and therefore $\theta(x) = \phi(x) + \psi(x) = 0 + 0 = 0$. Since this holds for all $x \in U \cap W$, we deduce that $\theta \in (U \cap W)^{\perp}$.

Now using, in this order, the Sum-Intersection formula for $U^{\perp}$ and $W^{\perp}$, the previous problem, the relation between the dimension of a subspace and the dimension of its annihilator, some rearrangement of terms, the Sum-Intersection formula for U and W, and again the relation between the dimension of a subspace and the dimension of its annihilator, we obtain

$$\begin{aligned}
\dim(U^{\perp} + W^{\perp}) &= \dim(U^{\perp}) + \dim(W^{\perp}) - \dim(U^{\perp} \cap W^{\perp}) \\
&= \dim(U^{\perp}) + \dim(W^{\perp}) - \dim((U+W)^{\perp}) \\
&= (\dim(V) - \dim(U)) + (\dim(V) - \dim(W)) - (\dim(V) - \dim(U+W)) \\
&= \dim(V) - (\dim(U) + \dim(W) - \dim(U+W)) \\
&= \dim(V) - \dim(U \cap W) \\
&= \dim((U \cap W)^{\perp})).
\end{aligned}$$

Hence, we have shown that $U^{\perp} + W^{\perp} \subseteq (U \cap W)^{\perp}$ and $\dim(U^{\perp} + W^{\perp}) = \dim((U \cap W)^{\perp})$, so we conclude that $U^{\perp} + W^{\perp} = (U \cap W)^{\perp}$.

Problem 3. Let $\sigma : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by

$$\sigma(x,y,z) = (4x + 5y + 6z, 7x + 8y + 9z).$$

Let (φ_1, φ_2) be the dual ordered basis of the standard ordered basis of $\mathbb{R}^2$ and let (ψ_1, ψ_2, ψ_3) be the dual ordered basis of the standard ordered basis of $\mathbb{R}^3$.

(a) Write down explicitly the maps $\sigma^*(\varphi_1), \sigma^*(\varphi_2) \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$. (More precisely, write the maps $\sigma^*(\varphi_1)$ and $\sigma^*(\varphi_2)$ in the form $\sigma^*(\varphi_1)(x,y,z) = a_1x + b_1y + c_1z$ and $\sigma^*(\varphi_2)(x,y,z) = a_2x + b_2y + c_2z$ with $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$.)

(b) Write $\sigma^*(\varphi_1)$ and $\sigma^*(\varphi_2)$ as linear combinations of the basis (ψ_1, ψ_2, ψ_3).

Solution.

(a) The linear functionals $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ are defined by $\varphi_1(u,v) = u$ and $\varphi_2(u,v) = v$ for all $(u,v) \in \mathbb{R}^2$. Therefore, for all $(x,y,z) \in \mathbb{R}^2$,

$$\begin{aligned}\sigma^*(\varphi_1)(x,y,z) &= (\varphi_1 \circ \sigma)(x,y,z) = \varphi_1(\sigma(x,y,z)) = 4x + 5y + 6z,\\ \sigma^*(\varphi_2)(x,y,z) &= (\varphi_1 \circ \sigma)(x,y,z) = \varphi_1(\sigma(x,y,z)) = 7x + 8y + 9z.\end{aligned}$$

(b) The linear functionals $\psi_1 : \mathbb{R}^3 \to \mathbb{R}$, $\psi_2 : \mathbb{R}^3 \to \mathbb{R}$ and $\psi_3 : \mathbb{R}^3 \to \mathbb{R}$ are defined by $\psi_1(x,y,z) = x$, $\psi_2(x,y,z) = y$ and $\psi_3(x,y,z) = z$ for all $(x,y,z) \in \mathbb{R}^2$. Therefore, for all (x,y,z),

$$\begin{aligned}\sigma^*(\varphi_1)(x,y,z) &= 4x + 5y + 6z = 4\psi_1(x,y,z) + 5\psi_2(x,y,z) + 6\psi_3(x,y,z) = (4\psi_1 + 5\psi_2 + 6\psi_3)(x,y,z)\\ \sigma^*(\varphi_2)(x,y,z) &= 7x + 8y + 9z = 7\psi_1(x,y,z) + 8\psi_2(x,y,z) + 9\psi_3(x,y,z) = (7\psi_1 + 8\psi_2 + 9\psi_3)(x,y,z),\end{aligned}$$

so we conclude that

$$\begin{aligned}\sigma^*(\varphi_1) &= 4\psi_1 + 5\psi_2 + 6\psi_3,\\ \sigma^*(\varphi_2) &= 7\psi_1 + 8\psi_2 + 9\psi_3.\end{aligned}$$

Problem 4. Let V be an F-vector space. The *bidual space of* V, denoted by V^{**}, is the dual space of V^*, i.e., $V^{**} = (V^*)^*$. For each $v \in V$, we define

$$\begin{aligned}\Psi(v) : V^* &\longrightarrow F\\ \theta &\longmapsto \theta(v).\end{aligned}$$

(a) Prove that $\Psi(v) \in V^{**}$ for all $v \in V$.

(b) Prove that the map

$$\begin{aligned}\Psi : V &\longrightarrow V^{**}\\ v &\longmapsto \Psi(v)\end{aligned}$$

is a linear map.

(c) Prove that if V is finite-dimensional, then $\Psi : V \to V^{**}$ is an isomorphism.

Solution.

(a) Let $v \in V$. We need to show that $\Psi(v)$ is linear. Let $\lambda, \mu \in F$ and let $\theta, \phi \in V^*$. Then

$$\Psi(v)(\lambda\theta + \mu\phi) = (\lambda\theta + \mu\phi)(v) = \lambda\theta(v) + \mu\phi(v) = \lambda\Psi(v)(\theta) + \mu\Psi(v)(\phi).$$

(b) Let $\lambda, \mu \in F$ and let $u, v \in V$. Let $\theta \in V^*$. Then

$$\Psi(\lambda u + \mu v)(\theta) = \theta(\lambda u + \mu v) = \lambda\theta(u) + \mu\theta(v) = \lambda\Psi(u)(\theta) + \mu\Psi(v)(\theta) = (\lambda\Psi(u) + \mu\Psi(v))(\theta).$$

Since this holds for all $\theta \in V^*$, we deduce that $\Psi(\lambda u + \mu v) = \lambda\Psi(u) + \mu\Psi(v)$.

(c) Suppose that V is finite-dimensional. Then $\dim(V^{**}) = \dim(V^*) = \dim(V)$. Also, Ψ is linear by part (b). Therefore, to prove that Ψ is an isomorphism it suffices to show that it is injective.

Let $v \in \ker\Psi$. Then $\Psi(v) = 0$, which means that $\Psi(v)(\theta) = 0$ for all $\theta \in V^*$. By the definition of $\Psi(v)$, this means that $\theta(v) = 0$ for all $\theta \in V^*$. As seen in class, this implies that $v = 0$.