Solutions Homework 5

Problem 1. Let U, V and W be F-vector spaces. Let $\phi_1 : U \to U \times V$ and $\phi_2 : V \to U \times V$ be the natural inclusion maps. Prove that the map

$$\begin{aligned} \Psi : \mathcal{L}(U \times V, W) &\longrightarrow \mathcal{L}(U, W) \times \mathcal{L}(V, W) \\ \tau &\longmapsto (\tau\phi_1, \tau\phi_2) \end{aligned}$$

is an isomorphism of F-vector spaces.

Solution.

We need to prove that Ψ is linear, injective and surjective.

Linearity. Let $\tau, \sigma \in \mathcal{L}(U \times V, W)$ and $\lambda \in F$. Then

$$\begin{aligned} \Psi(\sigma + \mu) &= \big((\sigma + \tau) \circ \phi_1, (\sigma + \tau) \circ \phi_2\big) && \text{(definition of } \Psi\text{)} \\ &= \big(\sigma\phi_1 + \tau\phi_1, \sigma\phi_2 + \tau\phi_2\big) && \text{(distributive law of composition)} \\ &= (\sigma\phi_1, \sigma\phi_2) + (\tau\phi_1, \tau\phi_2) && \text{(definition of + in } \mathcal{L}(U, W) \times \mathcal{L}(V, W)\text{)} \\ &= \Psi(\sigma) + \Psi(\tau) && \text{(definition of } \Psi\text{)} \end{aligned}$$

and

$$\begin{aligned} \Psi(\lambda\sigma) &= \big((\lambda\sigma)\phi_1, (\lambda\sigma)\phi_2\big) && \text{(definition of } \Psi\text{)} \\ &= \big(\lambda(\sigma\phi_1), \lambda(\sigma\phi_2)\big) && \text{(compatibility of composition with scalar multiplication)} \\ &= \lambda(\sigma\phi_1, \sigma\phi_2) && \text{(definition of scalar multiplication in } \mathcal{L}(U, W) \times \mathcal{L}(V, W)\text{)} \\ &= \lambda\Psi(\sigma) && \text{(definition of } \Psi\text{).} \end{aligned}$$

Injectivity. Let $\tau \in \mathcal{L}(U \times V, W)$ and suppose $\Psi(\tau) = (\tau\phi_1, \tau\phi_2) = (0, 0)$. Then $\tau\phi_1(u) = 0$ and $\tau\phi_2(v) = 0$ for all $u \in U$, $v \in V$. Therefore, for all $(u, v) \in U \times V$,

$$\begin{aligned} \tau(u, v) &= \tau(u, 0) + \tau(0, v) \\ &= \tau\phi_1(u) + \tau\phi_2(v) \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

so $\tau = 0$. Therefore $\ker \Psi = \{0\}$, so Ψ is injective.

Surjectivity. Let $(\gamma_1, \gamma_2) \in \mathcal{L}(U, W) \times \mathcal{L}(V, W)$. Define a map

$$\begin{aligned} \tau : U \times V &\longrightarrow W \\ (u, v) &\longmapsto \gamma_1(u) + \gamma_2(v). \end{aligned}$$

We claim that τ is linear (i.e., τ is an element of $\mathcal{L}(U \times V, W)$). Let $(u, v), (u', v') \in U \times V$ and $\lambda, \lambda' \in F$. Then

$$\begin{aligned} \tau\big(\lambda(u, v) + \lambda'(u', v')\big) &= \tau(\lambda u + \lambda' u', \lambda v + \lambda' v') && \text{(definition of } \cdot \text{ and + in } U \times V\text{)} \\ &= \gamma_1(\lambda u + \lambda' u') + \gamma_2(\lambda v + \lambda' v') && \text{(definition of } \tau\text{)} \\ &= \lambda\gamma_1(u) + \lambda'\gamma_2(u) + \lambda\gamma_2(v) + \lambda'\gamma_2(v') && \text{(linearity of } \gamma_1,\ \gamma_2\text{)} \\ &= \lambda\big(\gamma_1(u) + \gamma_2(v)\big) + \lambda'\big(\gamma_1(u') + \gamma_2(v')\big) && \text{(vector space structure of } W\text{)} \\ &= \lambda\tau(u, v) + \lambda'\tau(u', v') && \text{(definition of } \tau\text{)} \end{aligned}$$

so τ is linear. By construction, $\tau\phi_1(u) = \tau(u, 0) = \gamma_1(u) + \gamma_2(0) = \gamma_1(u)$ for all $u \in U$, which implies that $\tau\phi_1 = \gamma_1$. Similarly, we can show that $\tau\phi_2 = \gamma_2$. Therefore $\Psi(\tau) = (\gamma_1, \gamma_2)$. This shows that Ψ is surjective.

Problem 2. Let $U = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y + 5z = 0\}$, which we regard as a subspace of the $\mathbb{R}$-vector space $\mathbb{R}^3$. Let $\mathbf{x}_0 \in \mathbb{R}^3$. Prove that there exists some $d \in \mathbb{R}$ such that

$$\mathbf{x}_0 + U = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y + 5z = d\}.$$

Solution.

Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ and let $d = 2x_0 + 3y_0 + 5z_0$. We will prove that

$$\mathbf{x}_0 + U = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y + 5z = d\}.$$

($\subseteq$) Let $(x_1, y_1, z_1) \in \mathbf{x}_0 + U$. Then

$$(x_1, y_1, z_1) = (x_0, y_0, z_0) + (x_2, y_2, z_2) = (x_0 + x_2, y_0 + y_2, z_0 + z_2)$$

for some $(x_2, y_2, z_2) \in U$. Since $(x_2, y_2, z_2) \in U$, we have that $2x_2 + 3y_2 + 5z_2 = 0$. Therefore

$$2x_1 + 3y_1 + 5z_1 = 2(x_0 + x_2) + 3(y_0 + y_2) + 5(z_0 + z_2) = (2x_0 + 3y_0 + 5z_0) + (2x_2 + 3y_2 + 5z_2) = d + 0 = d,$$

so

$$(x_1, y_1, z_1) \in \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y + 5z = d\}.$$

($\supseteq$) Let $(x_1, y_1, z_1) \in \{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y + 5z = d\}$. Then $2x_1 + 3y_1 + 5z_1 = d$. Define

$$(x_2, y_2, z_2) := (x_1, y_1, z_1) - (x_0, y_0, z_0) = (x_1 - x_0, y_1 - y_0, z_1 - z_0).$$

Then we have

$$2x_2 + 3y_2 + 5z_2 = 2(x_1 - x_0) + 3(y_1 - y_0) + 5(z_1 - z_0) = (2x_1 + 3y_1 + 5z_1) - (2x_0 + 3y_0 + 5z_0) = d - d = 0,$$

so $(x_2, y_2, z_2) \in U$. Hence,

$$(x_1, y_1, z_1) = (x_0, y_0, z_0) + (x_2, y_2, z_2) \in (x_0, y_0, z_0) + U.$$

Problem 3. Let V be an F-vector space and let U and W be subspaces of V. Let $x, y \in V$. Prove that if $x + U \subseteq y + W$, then $U \subseteq W$.

Solution.

Suppose that $x + U \subseteq y + W$. Since $x = x + 0 \in x + U$, we have $x \in y + W$. Therefore $x = y + w_0$ for some $w_0 \in W$, so $y - x = w_0 \in W$. Now let $u \in U$. Then $x + u \in x + U$, so $x + u \in y + W$. Therefore $x + u = y + w$ for some $w \in W$. Hence

$$u = y - x + w = w_0 + w \in W.$$

Since u was an arbitrary element of U, this shows that $U \subseteq W$.

Problem 4. Let V be an F-vector space and let U be a subspace of V. Given an element $x \in V$, we denote by $[x]$ the equivalence class $x + U$ in V/U. Let $u_1, \ldots, u_n, v_1, \ldots, v_m \in V$. Suppose that $(u_1, \ldots, u_n)$ is an ordered basis for U and $([v_1], \ldots, [v_m])$ is an ordered basis for V/U. Prove that $(u_1, \ldots, u_n, v_1, \ldots, v_m)$ is an ordered basis for V.

Solution.

We will prove that $(u_1, \ldots, u_n, v_1, \ldots, v_m)$ is linearly independent and spans V. (Alternatively, you can prove one of these two conditions and justify that $\dim(V) = n + m$.)

First we show linear independence. Suppose that

$$\lambda_1 u_1 + \cdots + \lambda_n u_n + \mu_1 v_1 + \cdots + \mu_m v_m = 0 \tag{$\dagger$}$$

for some $\lambda_1, \dots, \lambda_n, \mu_1 \dots, \mu_m \in F$. Then

$$[\lambda_1 u_1 + \cdots + \lambda_n u_n + \mu_1 v_1 + \cdots + \mu_m v_m] = [0].$$

Since $\lambda_1 u_1 + \cdots + \lambda_n u_n \in U$, it follows that

$$[\mu_1 v_1 + \cdots + \mu_m v_m] = [0],$$

which can be rewritten as

$$\mu_1 [v_1] + \cdots + \mu_m [v_m] = [0].$$

Since $([v_1], \dots, [v_m])$ is an ordered basis for V/U, and in particular linearly independent, we deduce that $\mu_1 = \cdots = \mu_m = 0$. Now from (†) we deduce that

$$\lambda_1 u_1 + \cdots + \lambda_n u_n = 0.$$

Since $(u_1, \dots, u_n)$ is an ordered basis for the subspace U, and in particular linearly independent, we deduce that $\lambda_1 = \cdots = \lambda_n = 0$. Therefore $(u_1, \dots, u_n, v_1, \dots, v_m)$ is linearly independent.

Now we prove that $(u_1, \dots, u_n, v_1, \dots, v_m)$ spans V. Let $y \in V$. Since $([v_1], \dots, [v_m])$ spans V/U, there exist $\mu_1, \dots, \mu_m \in F$ such that

$$[y] = \mu_1 [v_1] + \cdots + \mu_m [v_m].$$

Then $[y - (\mu_1 v_1 + \cdots + \mu_m v_m)] = [0]$, so $y - (\mu_1 v_1 + \cdots + \mu_m v_m) \in U$. Now, since $(u_1, \dots, u_n)$ spans U, there exist $\lambda_1, \dots, \lambda_n \in F$ such that

$$y - (\mu_1 v_1 + \cdots + \mu_m v_m) = \lambda u_1 + \cdots + \lambda_n u_n$$

and therefore

$$y = \lambda_1 u_1 + \cdots + \lambda_n u_n + \mu_1 v_1 + \cdots + \cdots \mu_m v_m.$$