To show uniqueness, suppose $\sigma': V/U \to W$ is a linear map satisfying $\sigma' \circ \pi = T$. Then $\sigma' \circ \pi = \sigma \circ \pi$. Since π is surjective, this implies that $\sigma' = \sigma$. $\square$

<u>Corollary</u>. Let V be a f.d.v.s. over F and let U be a subspace of V. Then V/U is a f.d.v.s. and $\dim(V/U) = \dim V - \dim U$.

<u>Proof</u>. Since $\pi: V \longrightarrow V/U$ is surjective, $\operatorname{im}(\pi) = V/U$. By Rank-Nullity theorem,

$$\dim(V/U) = \operatorname{rank}(\pi) = \dim(V) - \operatorname{nullity}(\pi) = \dim(V) - \dim(U).$$ $\square$

<u>Prop</u>. Let V be an F-v.s., let U be a subspace of V and let $\pi: V \to V/U$ denote the canonical projection. The function

$$\left\{ \begin{matrix} \text{subspaces of } V \\ \text{containing } U \end{matrix} \right\} \longrightarrow \{ \text{subspaces of } V/U \}$$
$$W \longmapsto \pi(W)$$

is well-defined and bijective with inverse given by $\Delta \mapsto \pi^{-1}(\Delta)$. Moreover, this bijection preserves inclusions, i.e., if W_1, W_2 are subspaces of V containing U, then $W_1 \subseteq W_2$ if and only if $\pi(W_1) \subseteq \pi(W_2)$.

Proof. To check that the function

$$\left\{\begin{matrix}\text{subspaces of } V\\ \text{containing } U\end{matrix}\right\} \longrightarrow \left\{\text{subspaces of } V/U\right\}$$

$$W \longmapsto \pi(W)$$

is well-defined, we need to check that if W is a subspace of V containing U, then $\pi(W)$ is a subspace of V/U. This holds because π is a linear map and we have seen that the image of a subspace by a linear map is again a subspace.

Now we check that the function

$$\left\{\text{subspaces of } V/U\right\} \longrightarrow \left\{\begin{matrix}\text{subspaces of } V\\ \text{containing } U\end{matrix}\right\}$$

$$\Delta \longmapsto \pi^{-1}(\Delta)$$

is well-defined. Let Δ be a subspace of V/U. Then $\pi^{-1}(\Delta)$ is a subspace of V, since the preimage of a subspace by a linear map is again a subspace. Moreover,

$$U = \pi^{-1}(\{\bar{0}\}) \subseteq \pi^{-1}(\Delta),$$

so $\pi^{-1}(\Delta)$ is a subspace of V containing U.

Now we prove that the two functions above are inverses of each other. For that we need to check:

(i) if W is a subspace of V containing U, then $\pi^{-1}(\pi(W)) = W$;

(ii) if Δ is a subspace of V/U, then $\pi(\pi^{-1}(\Delta)) = \Delta$.

To prove (i), let W be a subspace of V containing U. The inclusion $W \subseteq \pi^{-1}(\pi(W))$ is clear. Now let $x \in \pi^{-1}(\pi(W))$. Then $\pi(x) \in \pi(W)$. Therefore $\pi(x) = \pi(u)$ for some $u \in W$. Thus $\pi(x-u) = \bar{0}$, so $x - u \in \ker \pi = U \subseteq W$. Therefore $x = (x-u) + u \in W$.

Condition (ii) just follows from the surjectivity of π.

Finally, we prove that this bijection preserves inclusions. Let W_1, W_2 be subspaces of V containing U. If $W_1 \subseteq W_2$, then clearly $\pi(W_1) \subseteq \pi(W_2)$. Conversely, if $\pi(W_1) \subseteq \pi(W_2)$, then $W_1 = \pi^{-1}(\pi(W_1)) \subseteq \pi^{-1}(\pi(W_2)) = W_2$. $\square$

Theorem (First isomorphism theorem) Let $T: V \to W$ be a linear map. Then the map

$$\bar{T}: V/_{\ker T} \longrightarrow \operatorname{im} T$$
$$[x] \longmapsto T(x)$$

is a well-defined isomorphism.

Proof. $\bar{T}$ is a well-defined linear map by the proof of the universal property of $V/_{\ker T}$ and the fact that it clearly takes values in $\operatorname{im} T$. Surjectivity is obvious. To prove injectivity, let $[x] \in V/_{\ker T}$. Then

$$\bar{T}([x]) = 0 \Rightarrow T(x) = 0 \Rightarrow x \in \ker T \Rightarrow [x] = [0]. \qquad \square$$

Theorem (Second Isomorphism theorem) Let U, W be subspaces of an F-vector space V. Then

$$U/_{U \cap W} \cong (U+W)/_W.$$

Proof Consider the map

$$T: U \longrightarrow (U+W)/_W$$
$$x \longmapsto [x]_W$$

Let $\lambda, \mu \in F$, $x, y \in U$. Then

$$T(\lambda x + \mu y) = [\lambda x + \mu y]_W = \lambda [x]_W + \mu [y]_W = \lambda T(x) + \mu T(y).$$

So T is linear.

Let $s \in (U+W)/W$. Then $s = [u+w]_W$ for some $u \in U$, $w \in W$.

But $[u+w]_W = [u]_W = \mathcal{I}(u)$. So $\mathcal{I}$ is surjective.

Now let $x \in U$. Then

$$x \in \ker \mathcal{I} \iff [x]_W = [0]_W \iff x \in W \iff x \in U \cap W.$$

because we are already assuming $x \in U$.

Therefore $\ker \mathcal{I} = U \cap W$.

Thus, by the 1st Isom. Thm., the map

$$\begin{aligned} \overline{\mathcal{I}} : U/_{U\cap W} &\longrightarrow (U+W)/_W \qquad (= \operatorname{im} \mathcal{I}) \\ [x]_{U\cap W} &\longmapsto [x]_W \end{aligned}$$

is an isomorphism. $\square$

<u>Remark</u> If U, W are finite-dimensional, from the 2nd Isom. Thm. we deduce

$$\begin{aligned} \dim(U) - \dim(U\cap W) &= \dim\left(U/_{U\cap W}\right) \\ &= \dim\left((U+W)/_W\right) \\ &= \dim(U+W) - \dim(W), \end{aligned}$$

2nd Isom. Thm.

so we recover the Sum-Intersection formula.

__Theorem__ (Third Isomorphism theorem) Let U, W be subspaces of a vector space V. Suppose that $U \subseteq W$. Then

$$(V/U)\big/(W/U) \simeq V/W.$$

__Proof.__ Consider the diagram

$$\begin{array}{ccc} x & \longmapsto & x+U = [x]_U \\ V & \xrightarrow{\pi_U} & V/U \end{array}$$

$$\begin{array}{ccc} x & & \\ & \searrow^{\pi_W} & V/W \\ & \searrow & x+W = [x]_W \end{array}$$

Since $\ker(\pi_W) = W \supseteq U$, the universal property for the quotient V/U implies that there exists a unique linear map $\sigma: V/U \to V/W$ such that $\sigma \circ \pi_U = \pi_W$.

$$\begin{array}{ccccc} V & \xrightarrow{\pi_U} & V/U & & [x]_U \\ & \searrow^{\pi_W} & \downarrow \sigma & & \downarrow \\ & & V/W & & [x]_W \end{array}$$

The map $\sigma: V/U \to V/W$ is given by $\sigma(x+U) = x+W$ for all $x \in V$.

Now σ is surjective since $\sigma \circ \pi_U = \pi_W$ is surjective. Moreover

$$\begin{aligned} \ker(\sigma) &= \{x+U \in V/U \mid \sigma(x+U) = 0+W\} \\ &= \{x+U \in V/U \mid x+W = W\} \\ &= \{x+U \in V/U \mid x \in W\} \\ &= W/U. \end{aligned}$$

By the 1st Isom. Thm.

$$\left(V/U\right)/\left(W/U\right) = \left(V/U\right)/\ker \sigma$$

$$\cong \operatorname{im} \sigma$$

$$= V/W.$$

$\square$

Dual spaces

Def. Let V be an F-vector space. The dual space of V, denoted V^* or $V^\vee$, is $\mathcal{L}(V, F) = \{\theta : V \to F \mid \theta \text{ is linear}\}$.

As seen earlier, this is an F-vector space. The elements of V^* are called linear functionals on V. Its zero element is the linear map $V \to F$ given by $v \mapsto 0_F$ for all $v \in V$.

Remark If V is finite-dimensional, then

$$\dim V^* = \dim\left(\mathcal{L}(V,F)\right) = (\dim V)\cdot(\dim F) = \dim V.$$

Lemma Let V, U be f.d.v.s. over F. Let $W \subsetneq V$ be a proper subspace of V. Let $x \in V \setminus W = \{v \in V \mid v \notin W\}$. and $u \in U$. Then there exists $\alpha \in \mathcal{L}(V, U)$ such that

- $W \leq \ker(\alpha)$,
- $\alpha(x) = u$.

Proof. Let $r = \dim W$ and let $\{e_1, \dots, e_r\}$ be a basis for W. Since

$x \notin W = \text{span}\{e_1, \dots, e_r\}$,

the set $\{e_1, \dots, e_r, x\}$ is linearly independent. Set $e_{r+1} := x$. Extend $\{e_1, \dots, e_{r+1}\}$ to a basis $\{e_1, \dots, e_{r+1}, \dots, e_n\}$ for V (where $n = \dim V$).

Let $\alpha : V \to U$ be the unique linear map such that

$$\alpha(e_i) = \begin{cases} 0 & \text{if } i \neq r+1 \\ u & \text{if } i = r+1. \end{cases}$$

That is, $\alpha(\lambda_1 e_1 + \dots + \lambda_n e_n) = \lambda_{r+1} u$ for all $\lambda_1, \dots, \lambda_n \in F$. Clearly $\alpha(x) = u$. Also, since $\alpha(e_i) = 0$ for $i = 1, \dots, r$, $\alpha(W) = \alpha(\text{span}\{e_1, \dots, e_r\}) = \{0\}$, so $W \leq \ker \alpha$. □

Corollary. Let V be a f.d.v.s. over F. Then for all $x \in V \setminus \{0\}$ there exists $\theta \in V^*$ such that $\theta(x) = 1_F$.

Proof Apply the lemma with $W = \{0\}$, $U = F$, $u = 1_F$. □