Practice midterm solutions MATH 108A, SPRING 2025

NAME: ______________________ PERM NUMBER: ______________

- *The time for this exam is **75 minutes**.*
- *The exam has **4 problems**. Each of them is worth 10 points.*
- *No notes, calculators or electronic devices are allowed during the exam.*
- *You can use the blank pages at the end as scratch paper.*
- *If you need extra space for any of the problems, you can use the blank pages at the end after directing the reader.*

Problem 1. [10 points] Let V be an F-vector space. Let v_1 and v_2 be distinct elements of V such that $\{v_1, v_2\}$ is linearly independent. Let w be an element of V such that

$$w \in \operatorname{span}\{v_1 + w, v_2 + w\}.$$

Prove that $w \in \operatorname{span}\{v_1, v_2\}$.

Solution. Since $w \in \operatorname{span}\{v_1 + w, v_2 + w\}$, there exist scalars $\alpha, \beta \in F$ such that

$$w = \alpha(v_1 + w) + \beta(v_2 + w). \tag{$\dagger$}$$

Then

$$(1 - \alpha - \beta)w = \alpha v_1 + \beta v_2.$$

If $1 - \alpha - \beta = 0$, then ($\dagger$) implies that $\alpha v_1 + \beta v_2 = 0$. Since v_1, v_2 are linearly independent, this implies that $\alpha = \beta = 0$. But then $0 = 1 - \alpha - \beta = 1 - 0 - 0 = 1$, which is a contradiction. Thus $1 - \alpha - \beta \neq 0$.

Let $\gamma = (1 - \alpha - \beta)^{-1}$. Multiplying both sides of ($\dagger$) by γ, we obtain

$$w = \gamma\alpha v_1 + \gamma\beta v_2 \in \operatorname{span}\{v_1, v_2\}.$$

Problem 2.

(a) [5 points] Let V be a 5-dimensional F-vector space. Prove that there does not exist a linear map $\tau : V \to V$ such that $\ker(\tau) = \operatorname{im}(\tau)$.

(b) [5 points] Let V be a finite-dimensional F-vector space. Let U_1, U_2, U_3 be subspaces of V such that $\dim(U_1) + \dim(U_2) + \dim(U_3) > 2\dim(V)$. Prove that $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Solution.

(a) Suppose there exists such a linear map $\tau : V \to V$. Then, by the Rank-Nullity Theorem,

$$5 = \dim(V) = \dim(\ker(\tau)) + \dim(\operatorname{im}(\tau)) = 2\dim(\ker(\tau)),$$

which is a contradiction since 5 is odd.

(b) By the Sum-Intersection formula,

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \geq \dim(U_1) + \dim(U_2) - \dim(V).$$

Applying the Sum-Intersection formula again, we obtain

$$\begin{aligned}
\dim(U_1 \cap U_2 \cap U_3) &= \dim((U_1 \cap U_2) \cap U_3) \\
&= \dim(U_1 \cap U_2) + \dim(U_3) - \dim((U_1 \cap U_2) + U_3) \\
&\geq \dim(U_1 \cap U_2) + \dim(U_3) - \dim(V) \\
&\geq \dim(U_1) + \dim(U_2) + \dim(U_3) - 2\dim(V) > 0.
\end{aligned}$$

Therefore $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Problem 3. Let U be an F-vector space. Let $\tau \in \mathrm{End}(U)$ be an endomorphism of U such that $\tau^2 = \tau$ (where $\tau^2 = \tau \circ \tau$). Let $\mathbb{1}_U \in \mathrm{End}(U)$ denote the identity endomorphism of U.

(a) [5 points] Prove that $\mathrm{im}(\tau) = \ker(\mathbb{1}_U - \tau)$.

(b) [5 points] Prove that

$$U = \ker(\tau) \oplus \mathrm{im}(\tau) = \ker(\tau) \oplus \ker(\mathbb{1}_U - \tau).$$

Solution.

(a) First we show that $\mathrm{im}(\tau) \subseteq \ker(\mathbb{1}_U - \tau)$. Let $y \in \mathrm{im}(\tau)$. Then $y = \tau(x)$ for some $x \in U$. Therefore

$$(\mathbb{1}_U - \tau)(y) = y - \tau(y) = \tau(x) - \tau(\tau(x)) = \tau(x) - \tau^2(x) = \tau(x) - \tau(x) = 0,$$

so $y \in \ker(\mathbb{1}_U - \tau)$.

Now we show that $\ker(\mathbb{1}_U - \tau) \subseteq \mathrm{im}(\tau)$. Let $y \in \ker(\mathbb{1}_U - \tau)$. Then $(\mathbb{1}_U - \tau)(y) = 0$, so $y - \tau(y) = 0$ and therefore $y = \tau(y) \in \mathrm{im}(\tau)$.

Thus, we have proved that $\mathrm{im}(\tau) = \ker(\mathbb{1}_U - \tau)$.

(b) By part (a), we only need to prove that $U = \ker(\tau) \oplus \mathrm{im}(\tau)$. Equivalently, we need to prove that $U = \ker(\tau) + \mathrm{im}(\tau)$ and $\ker(\tau) \cap \mathrm{im}(\tau) = \{0\}$.

Let $x \in U$. Note that

$$\tau(x - \tau(x)) = \tau(x) - \tau(\tau(x)) = \tau(x) - \tau^2(x) = \tau(x) - \tau(x) = 0,$$

so $x - \tau(x) \in \ker(\tau)$. Therefore

$$x = (x - \tau(x)) + \tau(x) \in \ker(\tau) + \mathrm{im}(\tau).$$

Thus, we deduce that $U = \ker(\tau) + \mathrm{im}(\tau)$.

Now let $y \in \ker(\tau) \cap \mathrm{im}(\tau)$. Since $y \in \mathrm{im}(\tau)$, we have that $y = \tau(x)$ for some $x \in U$. Since also $y \in \ker(\tau)$, we have that

$$0 = \tau(y) = \tau(\tau(x)) = \tau^2(x) = \tau(x) = y.$$

Thus, we deduce that $\ker(\tau) \cap \mathrm{im}(\tau) = \{0\}$.

Problem 4. Let V denote the $\mathbb{R}$-vector space consisting of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, i.e.,

$$V = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable}\}.$$

Let W be the subspace of V spanned by $\{\cos(x), \sin(x)\}$.

(a) [5 points] Prove that $\{\cos(x), \sin(x)\}$ is a linearly independent set (and therefore a basis for the subspace W).

(b) [5 points] Consider the ordered basis $\alpha = (\cos(x), \sin(x))$ for W. Consider the differentiation map

$$\begin{aligned} D : V &\longrightarrow V \\ f &\longmapsto f'. \end{aligned}$$

Prove that the linear map D restricts to a linear map $D_W : W \to W$ and find ${}_{\alpha}[D_W]_{\alpha}$. (You do not need to justify that D is linear.)

Solution.

(a) Let $\mathbf{O} : \mathbb{R} \to \mathbb{R}$ denote the zero function. Suppose that $\lambda \cos(x) + \mu \sin(x) = \mathbf{O}$ for some $\lambda, \mu \in \mathbb{R}$. Evaluating at $x = 0$, we deduce that $\lambda = 0$, whereas evaluating at $x = \pi/2$ we deduce that $\mu = 0$. Therefore $\{\cos(x), \sin(x)\}$ is linearly independent.

(b) We need to show that $D(W) \subseteq W$. Since $\{\cos(x), \sin(x)\}$ spans W, it suffices to show that $D(\cos(x)) \in W$ and $D(\sin(x)) \in W$. This follows from the following:

$$\begin{aligned} D(\cos(x)) &= -\sin(x) = 0 \cdot \cos(x) + (-1)\sin(x) \\ D(\sin(x)) &= \cos(x) = 1 \cdot \cos(x) + 0 \cdot \sin(x). \end{aligned}$$

Moreover, this shows that

$${}_{\alpha}[D_W]_{\alpha} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$