📘 Lecture Review

🔍 What Everybody Should Know Before Math 108B

This is a concise review of the fundamental concepts you should feel comfortable with before diving into **Lecture 1. Eigenvalues and Eigenvectors**.

Presumably, your Math 108A course covered the following topics:

1. **Abstract vector spaces** and **subspaces**
2. **Span** and **linear independence**
3. **Basis** and **dimension**
4. **Linear maps** (also known as linear transformations, linear operators)
5. **Determinant**
6. **Eigenvalues** and **eigenvectors** *(We'll review this last one in detail for Lecture 1)*

📌 In this note, we will briefly review Topics 1–5 to clarify definitions, fix notations, and ensure consistency moving forward.

You've probably learned and proved many results in your Math 108A classes, but our main focus here is on the **definitions**, as they form the foundation from which most results are derived.

📝 Naturally, there are other subtle concepts that we assume you are already familiar with. When necessary, we'll include brief reviews of those topics within individual lectures.

1. Vector spaces and subspaces

Definition. Vector Space

A **vector space** (or **linear space**) V over a field F consists of a set on which two operations (called **addition** and **scalar multiplication**) are defined so that for each pair of elements $x, y \in V$, there is a unique element $x + y \in V$, and for each $a \in F$, the element $ax \in V$ satisfies the following conditions:

(VS 1) For all $x, y \in V$,

$$x + y = y + x \quad \text{(commutativity of addition)}$$

(VS 2) For all $x, y, z \in V$,

$$(x + y) + z = x + (y + z) \quad \text{(associativity of addition)}$$

(VS 3) There exists an element $0 \in V$ such that

$$x + 0 = x \quad \text{for each } x \in V$$

(VS 4) For each $x \in V$, there exists an element $y \in V$ such that

$$x + y = 0$$

(VS 5) For each $x \in V$,

$$1x = x$$

(VS 6) For all $a, b \in F$ and $x \in V$,

$$(ab)x = a(bx)$$

(VS 7) For all $a \in F$ and $x, y \in V$,

$$a(x + y) = ax + ay$$

(VS 8) For all $a, b \in F$ and $x \in V$,

$$(a + b)x = ax + bx$$

The elements $x + y$ and ax are called the **sum** of x and y, and the **product** of a and x, respectively.

Examples of Vector Spaces

1. F^n: the set of all n-tuples with entries from a field F.

   If $u=(a_1,a_2,\ldots,a_n)\in \mathrm{F}^n, v=(b_1,b_2\ldots,b_n)\in \mathrm{F}^n$, $c\in F$
   - Addtion: $u+v=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n)$
   - Scalar-multiplication: $cu=(ca_1,ca_2,\ldots,ca_n)$
2. $\mathrm{M}_{m\times n}(F)$: the set of all $m\times n$ matrices with entries from a field F, with the usual operations of matrix addition and scalar multiplication.
3. $\mathrm{P}_n(F)$: the polynomials in $\mathrm{P}(F)$ of degree at most n, with the usual addition and scalar multiplication.

Definition. Subspace

A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Theorem 1.3. Let V be a vector space and W a subset of V . Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.
(a) $0\in\ W$.
(b) $x+y\in\ W$ whenever $x\in\ W$ and $y\in\ W$.
(c) $cx\in W$ whenever $c\in F$ and $x\in W$.

Exericise 1. Prove that the set of $m\times n$ upper triangular matrices is a subspace of $M_{m\times n}(F)$.

Exercise 2. Prove that a subset W of a vector space V is a subspace of V if and only if $0\in W$ and $ax+y\in W$ whenever $a\in F$ and $x,y\in W$.

2. Span and Linear Independence

Definition. Span of a set

Let S be a nonempty subset of a vector space V . The **span** of S, denoted $\operatorname{span}(S)$, is the set consisting of all linear combinations of the vectors in S. For convenience, we define $\operatorname{span}(\varnothing) = \{0\}$.

Example.

In $\mathbb{R}^3$, the span of $(1,0,0),(0,1,0)$ is the set of all vectors of the form $(a,b,0)$, which lie in the xy-plane. This span is a **subspace** of $\mathbb{R}^3$, and more generally, the span of any set of vectors is always a subspace.

Definition: Span or Generating Set

A subset S of a vector space V **generates** (or **spans**) V if $\operatorname{span}(S) = V$. In this case, we also say that the vectors in S **generate** or **span** V.

Exercise. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Definition. Linearly Dependent and Independent Sets

A subset S of a vector space V is called **linearly dependent** if there exist finitely many distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n = 0.$$

In this case, we also say that the vectors in S are linearly dependent.

A subset of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors in such a set are linearly independent.

Theorem 1.7. Let S be a linearly independent subset of a vector space V , and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \mathrm{span}(S)$.

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3. Basis and Dimension

A **basis** is a fundamental building block of a vector space.

Definition. Basis

A subset β of a vector space V is called a **basis** if it satisfies two properties:

- β is **linearly independent**.
- β **spans** V (i.e., $\mathrm{span}(\beta) = V$).

If β is a basis for V, we also say that the vectors in β **form a basis** for V.

Examples.

1. In F^n, define the vectors

$$e_1 = (1, 0, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \quad e_n = (0, 0, \ldots, 0, 1).$$

   The set $\{e_1, e_2, \ldots, e_n\}$ is easily verified to be a **basis** for F^n and is called the **standard basis** of F^n.
2. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the i-th row and j-th column. Then the set

$$\{E^{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$$

   forms a **basis** for the vector space $M_{m\times n}(F)$.
3. In $P_n(F)$, the set

$$\{1, x, x^2, \ldots, x^n\}$$

   forms a **basis** for the vector space $P_n(F)$. This is called the **standard basis** of $P_n(F)$.

Theorem 1.8. Let V be a vector space and $u_1, u_2, \ldots, u_n$ be distinct vectors in V. Then $\beta = \{u_1, u_2, \ldots, u_n\}$ is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Corollary 1. Let V be a vector space having a finite basis. Then all bases for V are finite, and every basis for V contains the same number of vectors.

Definition. Dimension

A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique integer n such that every basis for V contains exactly n elements is called the **dimension** of V and is denoted by $\dim(V)$. A vector space that is not finite-dimensional is called **infinite-dimensional**.

Examples

1. The vector space F^n has dimension n.
2. The vector space $\mathrm{M}_{m\times n}(F)$ has dimension mn.
3. The vector space $\mathrm{P}_n(F)$ has dimension $n+1$.

Corollary 2. Let V be a vector space with dimension n.
(a) Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.

(b) Any linearly independent subset of V that contains exactly n vectors is a basis for V.

(c) Every linearly independent subset of V can be extended to a basis for V, that is, if L is a linearly independent subset of V, then there is a basis β of V such that $L \subseteq \beta$.

Exercise. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

4. Linear Maps

Definition. Linear Transformation / Linear Map / Linear Operator

Let V and W be vector spaces over the same field F. A function $T : V \to W$ is called a **linear transformation** (also known as a **linear map** or **linear operator**) from V to W if, for all $x, y \in V$ and $c \in F$, the following properties hold:

(a) $T(x + y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

Definition. Null Space / Kernel and Range / Image

Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

- The **null space** (or **kernel**) of T, denoted $N(T)$, is the set of all vectors $x \in V$ such that $T(x) = 0$; that is,

$$N(T) = \{x \in V : T(x) = 0\}.$$

- The **range** (or **image**) of T, denoted $R(T)$, is the subset of W consisting of all images of vectors in V under T; that is,

$$R(T) = \{T(x) : x \in V\}.$$

Definition. Rank and Nullity

Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

If both the null space $N(T)$ and the range $R(T)$ are finite-dimensional, we define:

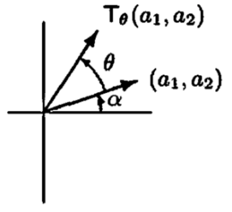
- the **nullity** of T, denoted $\operatorname{nullity}(T)$, as $\dim N(T)$
- the **rank** of T, denoted $\operatorname{rank}(T)$, as $\dim R(T)$

Examples

1. **Rotation**

   For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0,0)$, and $T_\theta(0,0) = (0,0)$.

   Then T_θ is a linear transformation called the **rotation by θ**.



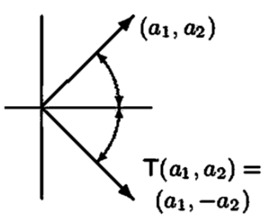
(a) Rotation

$$\mathrm{T}_\theta (a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta)$$

2. **Reflection**

   Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$.

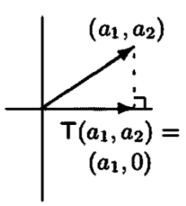
   This transformation is called the **reflection about the x-axis**.



(b) Reflection

3. **Projection**

   Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$.
   This transformation is called the **projection onto the x-axis**.



(c) Projection

4. Define $T : \mathrm{M}_{m\times n}(F) \to \mathrm{M}_{n\times m}(F)$ by $T(A) = A^t$, the transpose of A.

   Then T is a linear transformation.

5. Let V be the space of all real-valued functions on $\mathbb{R}$ with derivatives of all orders. Define $T : V \to V$ by $T(f) = f'$. Then T is a **linear transformation**, since differentiation is linear:

$$T(ag + h) = (ag + h)' = ag' + h' = aT(g) + T(h).$$

6. Let $V = C(\mathbb{R})$, the space of continuous real-valued functions. For fixed $a < b$, define $T : V \to \mathbb{R}$ by

$$T(f) = \int_a^b f(t)\, dt.$$

Then T is **linear**, as integration preserves linear combinations:

$$T(af + g) = \int_a^b (af + g)(t)\, dt = aT(f) + T(g).$$

Theorem 2.1. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

Then $N(T)$ is a subspace of V, and $R(T)$ is a subspace of W.

Theorem 2.2. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then

$$R(T) = \mathrm{span}(T(\beta)) = \mathrm{span}\left(\{T(v_1), T(v_2), \ldots, T(v_n)\}\right).$$

Theorem 2.3 (Dimension Theorem).

Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. If V is finite-dimensional, then

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V).$$

Theorem 2.4. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation.

Then T is **one-to-one** if and only if $N(T) = \{0\}$.

Theorem 2.5. Let V and W be finite-dimensional vector spaces with $\dim(V) = \dim(W)$, and let $T : V \to W$ be a linear transformation.

Then the following statements are equivalent:

(a) T is one-to-one

(b) T is onto

(c) $\operatorname{rank}(T) = \dim(V)$

Theorem 2.6.

Let V and W be vector spaces over a field F, and suppose $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

Given any $w_1, w_2, \ldots, w_n$ in W, there exists a **unique** linear transformation $T : V \to W$ such that

$$T(v_i) = w_i \quad \text{for } i = 1, 2, \ldots, n.$$

Definition. Ordered Basis

Let V be a finite-dimensional vector space.

An **ordered basis** for V is a basis for V endowed with a specific order; that is, an ordered basis is a finite sequence of linearly independent vectors in V that generates V.

Definition. Coordinate Vector Relative to an Ordered Basis

Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space V.

For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i.$$

The **coordinate vector of x relative to β**, denoted $[x]_\beta$, is defined by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Example. Coordinate Vector
Let $V = P_2(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ (standard ordered basis).

Let $f(x) = 4 + 6x - 7x^2$. Then

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}.$$

Definition. Matrix Representation of a Linear Transformation

Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_m\}$, respectively.

Let $T : V \to W$ be linear. For each $j = 1, 2, \dots, n$, there exist scalars a_{ij} such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$

The $m \times n$ matrix $A = [T]_\beta^\gamma$ with entries $A_{ij} = a_{ij}$ is called the **matrix representation of T in the ordered bases β and γ**.

Example. Matrix Representation of a Linear Map
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(a_1, a_2) = (a_1 + 3a_2,\ 0,\ 2a_1 - 4a_2).$$

Using standard bases β and γ for $\mathbb{R}^2$ and $\mathbb{R}^3$, we compute:

- $T(1, 0) = (1, 0, 2)$
- $T(0, 1) = (3, 0, -4)$

So the matrix representation is

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}.$$

If we change the basis $\gamma' = \{e_3, e_2, e_1\}$, the new matrix is

$$[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

Example. Derivative as a Linear Transformation
Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$.
Using the standard ordered bases, we compute:

- $T(1) = 0, \qquad T(x) = 1, \qquad T(x^2) = 2x, \qquad T(x^3) = 3x^2.$

So the matrix representation is

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Definition. Addition and Scalar Multiplication of Transformations

Let $T, U : V \to W$ be arbitrary functions, where V and W are vector spaces over a field F, and let $a \in F$.

We define:

- $(T + U)(x) = T(x) + U(x)$ for all $x \in V$,
- $(aT)(x) = a \cdot T(x)$ for all $x \in V$.

Theorem 2.7.

Let V and W be vector spaces over a field F, and let $T, U : V \to W$ be linear. Then:

(a) For all $a \in F$, the map $aT + U$ is linear.

(b) The set of all linear transformations from V to W forms a vector space over F.

Theorem 2.9.
Let V, W, and Z be vector spaces over the same field F, and let $T : V \to W$ and $U : W \to Z$ be linear.
Then the composition $UT : V \to Z$ is linear.

Theorem 2.10.
Let V be a vector space. Let $T, U_1, U_2 \in \mathcal{L}(V)$. Then:

(a) $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$
(b) $T(U_1U_2) = (TU_1)U_2$
(c) $TI = IT = T$, where I is the identity map.
(d) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a

Definition. Matrix Multiplication
Let A be an $m \times n$ matrix and B an $n \times p$ matrix.
The **product** AB is the $m \times p$ matrix defined by:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}, \quad \text{for } 1 \le i \le m,\ 1 \le j \le p.$$

Example. Matrix Multiplication

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \end{pmatrix}$$

Note: Matrix multiplication is **not commutative**; $AB \neq BA$ in general.

Theorem 2.11.
Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then:

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Example 2. Composition of Integral and Derivative
Let $U : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be $U(f) = f'$, and
$T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be $T(f)(x) = \int_0^x f(t)\,dt$.
Then $UT = I$ on $P_2(\mathbb{R})$.

Using standard ordered bases, we verify:

$$[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = I_\beta.$$

Theorem 2.12.
Let A be an $m \times n$ matrix, B and C be $n \times p$ matrices, and D, E be $q \times m$ matrices. Then:

(a) $A(B + C) = AB + AC$, and $(D + E)A = DA + EA$
(b) $a(AB) = (aA)B = A(aB)$ for all scalars a
(c) $I_m A = A = AI_n$

Corollary.
Let A be an $m \times n$ matrix, and B_i, C_i be $n \times p$ and $q \times m$ matrices respectively. Then:

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i A B_i, \quad \left(\sum_{i=1}^{k} a_i C_i\right) A = \sum_{i=1}^{k} a_i C_i A.$$

Theorem 2.13.
Let A be an $m \times n$ matrix and B an $n \times p$ matrix.
Let u_j and v_j denote the jth columns of AB and B, respectively. Then:

(a) $u_j = Av_j$
(b) $v_j = Be_j$, where e_j is the jth standard basis vector of $\mathbb{F}^p$

Theorem 2.14.
Let V and W be finite-dimensional vector spaces with ordered bases β and γ, and let $T : V \to W$ be linear. Then for any $u \in V$,

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta.$$

Example. Application of Matrix Representation
Let $T(f) = f'(x)$ on $P_3(\mathbb{R}) \to P_2(\mathbb{R})$ with standard ordered bases.
Let $p(x) = 2 - 4x + x^2 + 3x^3$, so $T(p(x)) = -4 + 2x + 9x^2$. Then:

- $[T(p(x))]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$
- $[T]_\beta^\gamma [p(x)]_\beta = A[p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$

Theorem 2.15.
Let A be an $m \times n$ matrix over F. Then $L_A : F^n \to F^m$ is linear. Furthermore:

(a) $[L_A]_\beta^\gamma = A$
(b) $L_A = L_B$ if and only if $A = B$
(c) $L_{A+B} = L_A + L_B$, and $L_{aA} = aL_A$
(d) If $T : F^n \to F^m$ is linear, then there exists a unique matrix C such that $T = L_C$. In fact, $C = [\mathrm{T}]_\beta^\gamma$.
(e) $L_{AE} = L_A L_E$ for any E
(f) If $m = n$, then $L_{I_n} = I_{F^n}$

Definition. Invertible Linear Transformation

Let V and W be vector spaces, and let $T : V \to W$ be linear. A function $U : W \to V$ is called an **inverse** of T if

$$TU = I_W \quad \text{and} \quad UT = I_V.$$

If such a function U exists, we say that T is **invertible**, and the inverse of T is unique and is denoted by T^{-1}.

Facts for Invertible Functions T and U

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$

Theorem 2.17.
Let V and W be vector spaces, and let $T : V \to W$ be a linear and invertible transformation.
Then the inverse map $T^{-1} : W \to V$ is also linear.

Corollary.
Let $T : V \to W$ be an invertible linear transformation.

Then V is finite-dimensional if and only if W is finite-dimensional.

In this case, $\dim(V) = \dim(W)$.

Definition. Invertible Matrix

Let A be an $n \times n$ matrix.
We say that A is **invertible** if there exists an $n \times n$ matrix B such that

$$AB = BA = I.$$

The matrix B is called the **inverse** of A, and is denoted by A^{-1}.

Theorem 2.19.
Let V and W be finite-dimensional vector spaces over the same field.
Then V is isomorphic to W if and only if

$$\dim(V) = \dim(W).$$

Definition. Isomorphism

Let V and W be vector spaces.

We say that V is **isomorphic** to W if there exists an invertible linear transformation $T : V \to W$.

Such a transformation is called an **isomorphism**.

Example. Define $T : F^2 \to P_1(F)$ by

$$T(a_1, a_2) = a_1 + a_2 x.$$

It is easy to verify that T is an isomorphism.

Therefore, $F^2 \cong P_1(F)$.

Theorem 2.20.
Let V and W be finite-dimensional vector spaces over a field F, with $\dim(V) = n$ and $\dim(W) = m$.

Let β and γ be ordered bases for V and W, respectively.

Then the map

$$\Phi_\beta^\gamma : \mathcal{L}(V, W) \to M_{m\times n}(F), \quad \Phi_\beta^\gamma(T) = [T]_\beta^\gamma,$$

is an isomorphism.

5. Determinant

Definition. Minor and Cofactor

Let A be an $n \times n$ matrix.

- The **minor** of entry a_{ij} is denoted by A_{ij} and is the determinant of the matrix obtained by deleting the ith row and jth column from A.
- The **cofactor** of a_{ij} is defined as $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Definition. Determinant Expansion

Let A be an $n \times n$ matrix with $n \geq 2$. Then:

- By expanding along the ith row:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

- By expanding along the jth column:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Properties. Determinants

Let A be an $n \times n$ matrix:

- If A has a zero row or column, then $\det(A) = 0$.
- If we get matrix B by interchanging two rows of A then $\det(B) = -\det(A)$.
- If we get matrix B by multipying one row of A by $k \neq 0$ then $\det(B) = k\det(A)$.
- Adding a multiple of one row to another: $\det(B) = \det(A)$.
- Scalar multiple: $\det(kA) = k^n \det(A)$.
- Transpose: $\det(A^T) = \det(A)$.
- Product: $\det(AB) = \det(A)\det(B)$.
- Inverse (if A invertible):

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$