Solutions Homework 1

Problem 1. Let V be an F-vector space. Prove that $-(-v) = v$ for all $v \in V$.

Solution. Let $v \in V$. By definition, the additive inverse $-v$ of v satisfies

$$(-v) + v = v + (-v) = 0.$$

But this identity also implies that v is the additive inverse of $-v$, i.e., that $v = -(-v)$.

Problem 2. Let V be the $\mathbb{R}$-vector space of real-valued functions on the open interval $(-4, 4)$. Let

$$U = \{f \in V \mid f \text{ is differentiable and } f'(-1) = 3f(2)\}.$$

Prove that U is a subspace of V.

Solution. First note that the zero function 0_V (defined by $0_V(t) = 0$ for all $t \in (-4, 4)$) is in U, since it is differentiable and $0_V'(-1) = 0 = 3 \cdot 0_V(2)$. In particular, the set U is non-empty.

Now let $g, h \in U$ and let $\lambda, \mu \in \mathbb{R}$. Since $g, h \in U$, we know that the functions g and h are differentiable and satisfy $g'(-1) = 3g(2)$ and $h'(-1) = 3h(2)$. From calculus, we know that the function $\lambda g + \mu h$ is also differentiable and $(\lambda g + \mu h)'(t) = \lambda g'(t) + \mu h'(t)$ for all $t \in (-4, 4)$. Therefore

$$(\lambda g + \mu h)'(-1) = \lambda g'(-1) + \mu h'(-1) = 3\lambda g(2) + 3\mu h(2) = 3(\lambda g + \mu h)(2).$$

Thus $\lambda g + \mu h \in U$. By the subspace criterion, we conclude that U is a subspace of V.

Problem 3. Let V be an F-vector space and let U_1, U_2 and W be subspaces of V. Prove that if W contains both U_1 and U_2, then W contains $U_1 + U_2$.

Solution. Suppose that W contains both U_1 and U_2. Let $x \in U_1 + U_2$. Then $x = x_1 + x_2$ for some $x_1 \in U_1$, $x_2 \in U_2$. Since $U_1 \subseteq W$ and $U_2 \subseteq W$, we know that $x_1 \in W$ and $x_2 \in W$. Thus, since W is a subspace, it follows that $x = x_1 + x_2 \in W$. Hence, $U_1 + U_2 \subseteq W$.

Problem 4. Prove or give a counterexample: if U_1, U_2, W are subspaces of an F-vector space V such that

$$U_1 + W = U_2 + W,$$

then $U_1 = U_2$.

Solution. The statement is false. As a counterexample, let F be any field, let $V = F^2$, and define

$$U_1 = \{(x, 0) \in F^2 \mid x \in F\}, \quad U_2 = \{(x, x) \in F^2 \mid x \in F\}, \quad W = \{(0, x) \in F^2 \mid x \in F\}.$$

We claim that $U_1 + W = U_2 + W = V$. Clearly $U_1 + W \subseteq V$ and $U_2 + W \subseteq V$. Let $(x, y) \in V$, where $x, y \in F$. Then we can write $(x, y) = (x, 0) + (0, y)$. Since $(x, 0) \in U_1$ and $(0, y) \in W$, this shows that $(x, y) \in U_1 + W$. Also, we can write $(x, y) = (x, x) + (0, y - x)$. Since $(x, x) \in U_2$ and $(0, y - x) \in W$, this shows that $(x, y) \in U_2 + W$. Thus $U_1 + W = U_2 + W = V$. However, $U_1 \neq U_2$, since, for instance, $(1, 1) \in U_2$ but $(1, 1) \notin U_1$.

Problem 5. Give an example of a non-empty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication but U is not a subspace of $\mathbb{R}^2$.

Solution. Let $U_1 = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$, $U_2 = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ and let $U = U_1 \cup U_2$. The sets U_1 and U_2 are clearly non-empty, so $U_1 \cup U_2$ is also non-empty. Moreover, the sets U_1 and U_2 are subspaces of $\mathbb{R}^2$. Let $\lambda \in \mathbb{R}$ and let $v \in U$. Then $v \in U_1$ or $v \in U_2$. If $v \in U_1$, then since U_1 is a subspace we have that $\lambda v \in U_1$. Similarly, if $v \in U_2$, then $\lambda v \in U_2$. Thus, in both cases, $\lambda v \in U$. This shows that U is closed under scalar multiplication. However, U is not a subspace. Indeed, $(1, 0) \in U_1$ and $(0, 1) \in U_2$, so $(1, 0) \in U$ and $(0, 1) \in U$. However $(1, 1) \notin U_1$ and $(1, 1) \notin U_2$, so $(1, 0) + (0, 1) = (1, 1) \notin U$, which shows that U is not a subspace.