Homework 4

Problem 1. Let U, V and W be F-vector spaces. Let $\tau : U \to V$ and $\sigma : V \to W$ be linear maps.

(a) Prove that $\ker(\tau) \subseteq \ker(\sigma\tau)$.

(b) Prove that $\operatorname{im}(\sigma\tau) \subseteq \operatorname{im}(\sigma)$.

Solution.

(a) Let $u \in \ker(\tau)$. Then u is an element in U such that $\tau(u) = 0$. Therefore $(\sigma\tau)(u) = \sigma(\tau(u)) = \sigma(0) = 0$, so $u \in \ker(\sigma\tau)$.

(b) Let $w \in \operatorname{im}(\sigma\tau)$. Then $w = (\sigma\tau)(x)$ for some $x \in U$. Thus $w = \sigma(\tau(x))$, i.e., w is the image by σ of the element $\tau(x) \in V$. So $w \in \operatorname{im}(\sigma)$.

Problem 2. Let U and V be finite-dimensional F-vector spaces and let W be any F-vector space. Let $\tau : U \to V$ and $\sigma : V \to W$ be linear maps. Prove that

$$\operatorname{nullity}(\sigma\tau) \leq \operatorname{nullity}(\sigma) + \operatorname{nullity}(\tau).$$

Hint: Show that the restriction of τ to $\ker(\sigma\tau)$ gives a linear map $\tau_1 : \ker(\sigma\tau) \to \ker(\sigma)$ and apply the Rank-Nullity theorem to τ_1.

Solution. Let $x \in \ker(\sigma\tau)$. Then $\sigma(\tau(x)) = (\sigma\tau)(x) = 0$. Therefore $\tau(x) \in \ker(\sigma)$. Since $x \in \ker(\sigma\tau)$ was arbitrary, we see that the restriction $\tau_{|\ker(\sigma\tau)} : \ker(\sigma\tau) \to V$ gives a linear map

$$\begin{aligned} \tau_1 : \ker(\sigma\tau) &\to \ker(\sigma) \\ x &\mapsto \tau(x). \end{aligned}$$

By Problem 1, we know that $\ker(\tau) \subseteq \ker(\sigma\tau)$. Therefore $\ker(\tau_1) = \ker(\tau)$. Using the Rank-Nullity theorem for the linear map τ_1, we obtain

$$\dim(\ker(\sigma\tau)) = \operatorname{nullity}(\tau_1) + \operatorname{rank}(\tau_1) = \operatorname{nullity}(\tau) + \operatorname{rank}(\tau_1).$$

Since the codomain of τ_1 is $\ker(\sigma)$, we have that

$$\operatorname{rank}(\tau_1) = \dim(\operatorname{im}(\tau_1)) \leq \dim(\ker(\sigma)) = \operatorname{nullity}(\sigma).$$

Altogether, we obtain

$$\operatorname{nullity}(\sigma\tau) = \dim(\ker(\sigma\tau)) = \operatorname{nullity}(\tau) + \operatorname{rank}(\tau_1) \leq \operatorname{nullity}(\tau) + \operatorname{nullity}(\sigma).$$

Problem 3. Let U and V be F-vector spaces and let W be a subspace of V. Let $\mathcal{L}(U, V)$ denote the F-vector space of all linear maps from U to V. Prove that the set

$$\mathcal{M} = \{\tau \in \mathcal{L}(U, V) \mid \operatorname{im}(\tau) \subseteq W\}$$

is a subspace of $\mathcal{L}(U, V)$.

Solution. First note that the zero map

$$\begin{aligned} \mathbb{0}_{U,V} : U &\to V \\ u &\mapsto 0_V \end{aligned}$$

is linear and $\operatorname{im}(\mathbb{0}_{U,V}) = \{0_V\} \subseteq W$. So $\mathbb{0}_{U,V} \in \mathcal{M}$ and therefore $\mathcal{M}$ is non-empty. Now let $\tau_1, \tau_2 \in \mathcal{M}$ and $\lambda_1, \lambda_2 \in F$. Fix any $u \in U$. By definition of $\mathcal{M}$, we have that $\tau_1(u), \tau_2(u) \in W$. Thus, since W is a subspace of V, we have that $\lambda_1\tau_1(u) + \lambda_2\tau_2(u) \in W$. Therefore

$$(\lambda_1\tau_1 + \lambda_2\tau_2)(u) = \lambda_1\tau_1(u) + \lambda_2\tau_2(u) \in W.$$

Since u was arbitrary, this shows that $\operatorname{im}(\lambda_1\tau_1 + \lambda_2\tau_2) \subseteq W$. Therefore, $\lambda_1\tau_1 + \lambda_2\tau_2 \in \mathcal{M}$. By the subspace criterion, we conclude that $\mathcal{M}$ is a subspace of $\mathcal{L}(U, V)$.

Problem 4. Let $\alpha = (v_1, v_2, v_3, v_4)$ be an ordered basis of an $\mathbb{R}$-vector space V.

(a) Prove that $\beta = (v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$ is also an ordered basis of V.

(b) If $x \in V$ has α-coordinate vector $[x]_\alpha = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, what is the β-coordinate vector $[x]_\beta$?

Solution.

(a) Since $\dim(V) = 4$, any spanning subset of V with exactly 4 elements is a basis for V. Thus, it suffices to show that β spans V. Let $v \in V$ be an arbitrary element. Since α is an ordered basis, there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in F$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4.$$

Let

$$\mu_1 := \lambda_1 \tag{1}$$
$$\mu_2 := \lambda_2 - \lambda_1 \tag{2}$$
$$\mu_3 := \lambda_3 - \lambda_2 + \lambda_1 \tag{3}$$
$$\mu_4 := \lambda_4 - \lambda_3 + \lambda_2 - \lambda_1 \tag{4}$$

Then $\mu_1 + \mu_2 = \lambda_2$, $\mu_2 + \mu_3 = \lambda_3$ and $\mu_3 + \mu_4 = \lambda_4$. Therefore

$$\begin{aligned} \mu_1(v_1 + v_2) + \mu_2(v_2 + v_3) + \mu_3(v_3 + v_4) + \mu_4 v_4 &= \mu_1 v_1 + (\mu_1 + \mu_2)v_2 + (\mu_2 + \mu_3)v_3 + (\mu_3 + \mu_4)v_4 \\ &= \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 \\ &= v. \end{aligned}$$

This shows that v lies in the span of β. Since v was arbitrary, we conclude that β spans V. As mentioned before, this makes β an ordered basis for V.

(b) Suppose that $x \in V$ has α-coordinate vector $[x]_\alpha = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$. This means that $x = 1v_1 + 2v_2 + 1v_3 + 1v_4$.

Using equations (1)-(4), we deduce that

$$\begin{aligned} x &= 1(v_1 + v_2) + (2 - 1)(v_2 + v_3) + (1 - 2 + 1)(v_3 + v_4) + (1 - 1 + 2 - 1)v_4 \\ &= 1(v_1 + v_2) + 1(v_2 + v_3) + 0(v_3 + v_4) + 1v_4. \end{aligned}$$

Thus $[x]_\beta = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.