2. Subspaces

Def. A <u>subspace</u> of an F-vector space V is a subset U of V which is itself an F-vector space with respect to addition and scaler multiplication in V.

<u>Subspace criterion</u> A <u>non-empty</u> subset U of an F-vector space V is a subspace if and only if

$$\forall \lambda, \mu \in F, \quad \forall x, y \in U, \quad \lambda x + \mu y \in U.$$

<u>Proof</u>.

($\Rightarrow$) Suppose U is a subspace. Then

$$\left.\begin{array}{l} \lambda \in F,\ x \in U \Rightarrow \lambda x \in U \\ \mu \in F,\ y \in U \Rightarrow \mu y \in U \end{array}\right\} \Rightarrow \lambda x + \mu y \in U.$$

($\Leftarrow$) Suppose that

$$\forall \lambda, \mu \in F, \quad \forall x, y \in U, \quad \lambda x + \mu y \in U. \qquad (*)$$

Since $U \neq \emptyset$, $\exists x_0 \in U$. Taking $\lambda = \mu = 0$, $x = y = x_0$ in $(*)$, we see that

$$0_V = 0 \cdot x_0 + 0 \cdot x_0 \in U.$$

Taking $\lambda = \mu = 1$ in $(*)$, we see that

$$\forall x, y \in U, \quad x + y = 1 \cdot x + 1 \cdot y \in U,$$

so U is closed under addition.

Taking $\lambda = -1$, $\mu = 0$, $y = 0_V$ in $(*)$, we see that

$$\forall x \in U, \quad -x = (-1) x + 0 \cdot 0_V \in U,$$

so U is closed under taking additive inverses.

Finally, taking $\mu = 0$, $y = 0_V$ in $(*)$, we see that

$$\forall \lambda \in F, \quad \forall x \in U, \quad \lambda x = \lambda x + 0 \cdot 0_V \in U,$$

so U is closed under scalar multiplication.

$\square$

Examples

1) For any vector space V, the subsets $\{0_V\}$ and V are subspaces.

2) In F^n, the subset $\{(x_1, \dots, x_{n-1}, 0) \mid x_1, \dots, x_{n-1} \in F\}$ is a subspace.

3) If V is the space of functions $[0,1] \to \mathbb{R}$, the subset U of continuous functions $[0,1] \to \mathbb{R}$ is a subspace of V, since for $f, g : [0,1] \to \mathbb{R}$ continuous and $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g : [0,1] \to \mathbb{R}$ is continuous.

4) If V is a vector space over F and $v \in V$ is a fixed element, then $\{\lambda \cdot v \mid \lambda \in F\}$ is a subspace of V.

5) If $a_{ij} \in F$ for $1 \leq i \leq m$, $1 \leq j \leq n$, then the set of all $(x_1, \dots, x_n) \in F^n$ such that

$$\sum_{j=1}^{n} a_{ij} x_j = 0 \qquad \text{for all } 1 \leq i \leq m \qquad (*)$$

is a subspace of F^n (called the "solution space" for the system of equations $(*)$).

<u>Prop.</u> Let U_1, U_2 be subspaces of an F-vector space V.

(a) $U_1 \cap U_2$ is a subspace of V.

(b) More generally, the intersection of any family (finite or infinite) of subspaces is a subspace.

(c) $U_1 \cup U_2$ is not a subspace unless $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

(d) Write $U_1 + U_2 = \{x_1 + x_2 \mid x_1 \in U_1, x_2 \in U_2\}$. Then $U_1 + U_2$ is a subspace of V that contains U_1 and U_2.

<u>Proof</u>

(a) $U_1 \cap U_2 \neq \emptyset$ since $0 \in U_1$ and $0 \in U_2$.

Let $x, y \in U_1 \cap U_2$ and let $\lambda, \mu \in F$. Then since $x, y \in U_1$ and U_1 is a subspace we have $\lambda x + \mu y \in U_1$. Similarly $\lambda x + \mu y \in U_2$.

Therefore $\lambda x + \mu y \in U_1 \cap U_2$. Hence $U_1 \cap U_2$ is a subspace of V.

(b) Virtually identical to (a) (exercise)

(c) Suppose $\exists x \in U_1 \setminus U_2$ and $\exists y \in U_2 \setminus U_1$. Then

$$\begin{cases} x+y \in U_1 \Rightarrow y = (x+y) - x \in U_1 & \text{contradiction.} \\ x+y \in U_2 \Rightarrow x = (x+y) - y \in U_2 & \text{contradiction.} \end{cases}$$

Hence $x+y \notin U_1 \cup U_2$, and so $U_1 \cup U_2$ is not a subspace.

(d) Since U_2 is a subspace, $0 \in U_2$. Then

$$\forall x \in U_1, \quad x = x + 0 \in U_1 + U_2.$$

Hence $U_1 \subseteq U_1 + U_2$ (and so in particular $U_1 + U_2$ is non-empty).

Similarly $U_2 \subseteq U_1 + U_2$.

Now let $x, y \in U_1 + U_2$ and let $\lambda, \mu \in F$. Then we can write

$x = x_1 + x_2$, $y = y_1 + y_2$ for some $x_1, y_1 \in U_1$, $x_2, y_2 \in U_2$. Then

$$\begin{aligned} \lambda x + \mu y &= \lambda(x_1 + x_2) + \mu(y_1 + y_2) \\ &= \lambda x_1 + \lambda x_2 + \mu y_1 + \mu y_2 \\ &= \underbrace{(\lambda x_1 + \mu y_1)}_{\in U_1} + \underbrace{(\lambda x_2 + \mu y_2)}_{\in U_2} \in U_1 + U_2. \end{aligned}$$

$\square$

Def. Let V be an F-vector space. Let $v_1, \dots, v_n \in V$ and $x \in V$ be any elements. We say that x is a <u>linear combination</u> of $v_1, \dots, v_n$ if $\exists \lambda_1, \dots, \lambda_n \in F$ such that

$$x = \lambda_1 v_1 + \dots + \lambda_n v_n.$$

Prop. Let V be an F-vector space and let $v_1, \dots, v_n \in V$ be any elements. Then the set U of all linear combinations of $v_1, \dots, v_n$ is a subspace of V that contains $v_1, \dots, v_n$. We call U the <u>space spanned by $v_1, \dots, v_n$</u> and denote it by $\operatorname{span}\{v_1, \dots, v_n\}$.

Proof. First note that $v_i \in U$ for all i since

$$v_i = 0 \cdot v_1 + \dots + 0 \cdot v_{i-1} + 1 \cdot v_i + 0 \cdot v_{i+1} + \dots + 0 \cdot v_n.$$

In particular $U \neq \emptyset$.

Let $x = \sum_{i=1}^{n} \lambda_i v_i$, $y = \sum_{i=1}^{n} \mu_i v_i \in U$, where $\lambda_i, \mu_i \in F$, and let $\alpha, \beta \in F$. Then

$$\alpha x + \beta y = \alpha \left(\sum_{i=1}^{n} \lambda_i v_i \right) + \beta \left(\sum_{i=1}^{n} \mu_i v_i \right) = \sum_{i=1}^{n} \alpha \lambda_i v_i + \sum_{i=1}^{n} \beta \mu_i v_i = \sum_{i=1}^{n} (\alpha \lambda_i + \beta \mu_i) v_i$$

so $\alpha x + \beta y \in U$. $\square$