<u>Prop.</u> Let V be an F-vector space. Let $S = \{v_1, \ldots, v_n\}$ be a finite subset of V. Then S is a basis of V if and only if every element of V can be written uniquely in the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_1, \ldots, \lambda_n \in F$.

<u>Proof</u>

($\Rightarrow$) Suppose S is a basis of V. Let $x \in V$. Since $\operatorname{span} S = V$,

$$x = \lambda_1 v_1 + \cdots + \lambda_n v_n \quad \text{for some } \lambda_1, \ldots, \lambda_n \in F.$$

To show uniqueness, suppose that

$$x = \lambda_1 v_1 + \cdots + \lambda_n v_n = \lambda_1' v_1 + \cdots + \lambda_n' v_n \quad \text{with } \lambda_1, \ldots, \lambda_n, \lambda_1', \ldots, \lambda_n' \in F.$$

Then

$$(\lambda_1 - \lambda_1') v_1 + \cdots + (\lambda_n - \lambda_n') v_n = 0.$$

Since S is linearly independent, it follows that $\lambda_i = \lambda_i'$ for $i = 1, \ldots, n$.

($\Leftarrow$) Suppose that every element of V can be written uniquely in the form $\lambda_1 v_1 + \cdots + \lambda_n v_n$ with $\lambda_1, \ldots, \lambda_n \in F$. Then clearly $\operatorname{span} S = V$. Now suppose $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$ for some $\lambda_1, \ldots, \lambda_n \in F$. Then

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0 = 0 \cdot v_1 + \cdots + 0 \cdot v_n.$$

By uniqueness, $\lambda_1 = \cdots = \lambda_n = 0$, so S is linearly independent. $\square$

Rk. If $S = \{v_1, \dots, v_n\}$ is a basis for V and we set

$$U_i = \{\lambda v_i \mid \lambda \in F\} = \operatorname{span}\{v_i\},$$

then

$$V = U_1 \oplus \cdots \oplus U_n.$$

Lemma Let V be an F-vector space, let S be a finite subset of V and let $y \in V$. If $S \cup \{y\}$ spans V and $y \in \operatorname{span} S$, then $\operatorname{span} S = V$.

Proof. Say $S = \{v_1, \dots, v_n\}$ and $y = \mu_1 v_1 + \dots + \mu_n v_n$ for some $\mu_i \in F$. Let $x \in V$. Since $S \cup \{y\}$ spans V, $\exists \lambda_1, \dots, \lambda_n, \lambda \in F$ such that

$$x = \lambda_1 v_1 + \dots + \lambda_n v_n + \lambda y.$$

Then

$$\begin{aligned} x &= \lambda_1 v_1 + \dots + \lambda_n v_n + \lambda(\mu_1 v_1 + \dots + \mu_n v_n) \\ &= (\lambda_1 + \lambda\mu_1) v_1 + \dots + (\lambda_n + \lambda\mu_n) v_n \in \operatorname{span} S. \end{aligned}$$

Hence, $\operatorname{span} S = V$. $\square$

Prop. Let V be an F-vector space. Suppose that V is spanned by some finite set S. Then S has a subset which is a basis for V.

Proof. Choose a subset $S' = \{v_1, \dots, v_n\}$ of S (where $v_1, \dots, v_n$ are distinct elements) with the least number of elements among the subsets of S spanning V.

If S' is not linearly independent, then $\exists \lambda_1, \dots, \lambda_n \in F$ not all zero such that

$$\sum_{i=1}^{n} \lambda_i v_i = 0.$$

Suppose $\lambda_r \neq 0$. Then

$$v_r = -\sum_{i \neq r} \lambda_r^{-1} \lambda_i v_i \in \operatorname{span}(S' \setminus \{v_r\}).$$

By the previous lemma, $S' \setminus \{v_r\}$ spans V, which contradicts the minimality of S'.

Hence, S' is linearly independent and therefore a basis for V. $\square$

Corollary A finite set S is a basis for V if and only if it is minimal with respect to the property of spanning V (i.e., S spans V and no proper subset of S spans V).

Proof. Exercise.

Prop. A finite set $S = \{v_1, \dots, v_n\}$ is a basis for V if and only if it is maximal with respect to being linearly independent (i.e., S is linearly independent but no $S' \subseteq V$ with $S \subsetneq S'$ is linearly independent).

Proof

($\Leftarrow$) Suppose that S is maximal w.r.t. being linearly independent.

It suffices to show that S spans V.

Clearly $S \subseteq \operatorname{span} S$. Let $x \in V \setminus S$. Then $S \subsetneq S \cup \{x\}$, so $S \cup \{x\}$ is not linearly independent. So $\exists \lambda_1, \dots, \lambda_n, \lambda \in F$ not all zero such that

$$\sum_{i=1}^{n} \lambda_i v_i + \lambda x = 0.$$

If $\lambda = 0$, then since S is linearly independent it would follow that $\lambda_1 = \dots = \lambda_n = 0$, contradicting the assumption that $\lambda_1, \dots, \lambda_n, \lambda$ are not all zero.

So $\lambda \neq 0$ and therefore

$$x = -\sum_{i=1}^{n} \lambda^{-1} \lambda_i v_i \in \operatorname{span}(S).$$

Hence $\operatorname{span} S = V$.

($\Rightarrow$) Suppose that S is a basis. Let $x \in V \setminus S$. Since $\operatorname{span} S = V$, we have that $x \in \operatorname{span} S$, so $S \cup \{x\}$ is not linearly independent. $\square$

Theorem (Steinitz Replacement Theorem). Let V be an F-vector space.

Let $X = \{x_1, \dots, x_r\}$ be a linearly independent subset of V (with $x_1, \dots, x_r$ distinct) and let Y be a spanning subset of V with $|Y| = s$. Then $r \leq s$ and $\exists y_{r+1}, \dots, y_s \in Y$ such that $\{x_1, \dots, x_r, y_{r+1}, \dots, y_s\}$ spans V.

Proof. We proceed by induction on r.

If $r = 0$ (so $X = \emptyset$) the statement is clearly true. ($0 \leq s$ and we can take $y_1, \dots, y_s$ to be the s distinct elements of Y).

Let $r > 0$ and suppose that the result is known for $r-1$. Since X is linearly independent, so is the subset $\{x_1, \dots, x_{r-1}\}$. By induction hypothesis,

(a) $r-1 \leq s$

(b) $\exists y_r, \dots, y_s \in Y$ such that $\operatorname{span}\{x_1, \dots, x_{r-1}, y_r, \dots, y_s\} = V$.

So we may write

$$x_r = \sum_{i=1}^{r-1} \lambda_i x_i + \sum_{j=r}^{s} \mu_j y_j .$$

(*Note*: The first sum is zero if $r=1$; the second sum is zero if $r-1 = s$.)

If the second sum is zero, then $x_r \in \operatorname{span}\{x_1, \dots, x_{r-1}\}$, which contradicts the assumption that X is linearly independent.

Hence

(a) $r-1 < s$, so $r \le s$, and

(b) $\exists k \in \{r, \dots, s\}$ such that $\mu_k \neq 0$. Renumbering the y_j's if necessary, we may assume that $\mu_r \neq 0$.

Thus

$$y_r = -\mu_r^{-1}\left(\sum_{i=1}^{r-1} \lambda_i x_i + \sum_{j=r+1}^{s} \mu_j y_j - x_r\right),$$

i.e., y_r is a linear combination of $S := \{x_1, \dots, x_r, y_{r+1}, \dots, y_s\}$.

Since $\{x_1, \dots, x_{r-1}, y_r, \dots, y_s\}$ spans V, so does $\{x_1, \dots, x_{r-1}, x_r, y_r, \dots, y_s\} = S \cup \{y_r\}$.

Now, since $y_r \in \operatorname{span} S$, it follows that $\operatorname{span} S = V$.

$\square$