Math 108 B

Fall 2025, University of California, Santa Barbara

Instructor Name: Jitendra Rathore

Homework - 5

Due Date : December 4, 2025

Notation: $\mathcal{L}(V)$ denote the set of all linear operators on V.

1. Show that if T and U are positive semi-definite operators such that $T^2 = U^2$, then $T = U$.

2. Give an example of a linear operator T on a finite dimensional vector space such that T is not nilpotent, but zero is the only eigenvalue of T.

3. Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

4. Suppose $T \in \mathcal{L}(V)$, m is a positive integer, $v \in V$, and $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that the vectors

$$v, Tv, T^2v, \ldots, T^{m-1}v$$

are linearly independent.

Theorem (Test for Diagonalizability): Let T be a linear operator on a finite-dimensional vector space V. Then T is diagonalizable if and only if the minimal polynomial of T is of the form

$$p(t) = (t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_k),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of T.

You can use the above theorem as a fact to answer the following questions.

5. Let A be $M_{n\times n}(\mathbb{C})$ such that $A^3 = A$. Show that A is diagonalizable.

6. Let A be any matrix in $M_{2\times 2}(\mathbb{C})$ such that

$$A^2 - 3A + 2I = 0.$$

Show that A is similar to

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

7. Suppose $T \in \mathcal{L}(V)$ is nilpotent and diagonalizable. Show that $T = 0$.