Solutions Homework 3

Problem 1. Let U be the subspace of $\mathbb{R}^5$ defined as

$$U = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \,|\, x_1 = 3x_2,\ x_3 = 7x_4\right\}.$$

(a) Find a basis for U.

(b) Extend the basis in part (a) to a basis of $\mathbb{R}^5$.

(c) Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution. Let $\vec{0} = (0, 0, 0, 0, 0)$ denote the zero vector in $\mathbb{R}^5$.

(a) Let $v_1, v_2, v_3 \in \mathbb{R}^5$ be defined as

$$v_1 = (3, 1, 0, 0, 0), \quad v_2 = (0, 0, 7, 1, 0), \quad v_3 = (0, 0, 0, 0, 1).$$

We claim that $S := \{v_1, v_2, v_3\}$ is a basis for U.

Suppose that for some $\lambda_1, \lambda_2, \lambda_3 \in F$, we have

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \vec{0}.$$

Then $(3\lambda_1, \lambda_1, 7\lambda_2, \lambda_2, \lambda_3) = (0, 0, 0, 0, 0)$ and looking at the second, fourth and fifth components, we deduce that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus S is linearly independent.

Now let $v \in U$ be an arbitrary element. Then $v = (x_1, x_2, x_3, x_4, x_5)$ for some $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$ with $x_1 = 3x_2$ and $x_3 = 7x_4$. Writing $a = x_2$, $b = x_4$, $c = x_5$, we see that

$$\begin{aligned} v &= (3a, a, 7b, b, c) \\ &= av_1 + bv_2 + cv_3 \in \operatorname{Span}(S). \end{aligned}$$

Thus S is a basis for U.

(b) Let $v_4, v_5 \in \mathbb{R}^5$ be defined as

$$v_4 = (1, 0, 0, 0, 0), \quad v_5 = (0, 0, 1, 0, 0).$$

We claim that $T := \{v_1, v_2, v_3, v_4, v_5\}$ is a basis for $\mathbb{R}^5$. Since $\dim(\mathbb{R}^5) = 5$, it suffices to show that T is linearly independent. To this end, suppose $\lambda_1, \ldots, \lambda_5 \in \mathbb{R}$ are such that

$$\lambda_1 v_1 + \ldots + \lambda_5 v_5 = \vec{0}.$$

Then

$$(3\lambda_1 + \lambda_4, \lambda_1, 7\lambda_2 + \lambda_5, \lambda_2, \lambda_3) = (0, 0, 0, 0, 0).$$

Looking at the second, fourth and fifth components, we deduce that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. From this, the first and third components tell us that $\lambda_4 = \lambda_5 = 0$. Thus T is linearly independent and hence a basis for $\mathbb{R}^5$.

(c) Let $W := \operatorname{Span}\{v_4, v_5\}$. Then

$$\begin{aligned} U + W &= \operatorname{Span}(S) + \operatorname{Span}\{v_4, v_5\} \\ &= \operatorname{Span}(T) = \mathbb{R}^5. \end{aligned}$$

Now let $v \in U \cap W$. Since $v \in U$, we have that $v = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, and, since $v \in W$, we have that $v = \lambda_4 v_4 + \lambda_5 v_5$ for some $\lambda_4, \lambda_5 \in \mathbb{R}$. Thus

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \lambda_4 v_4 + \lambda_5 v_5$$

and therefore

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + (-\lambda_4) v_4 + (-\lambda_5) v_5 = \vec{0}.$$

Since T is a basis for $\mathbb{R}^5$ (and in particular linearly independent) it follows that

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.$$

Hence $v = \vec{0}$. We conclude that $U \cap W = \{0\}$.

Since $U + W = \mathbb{R}^5$ and $U \cap W = \{0\}$, we have that $\mathbb{R}^5 = U \oplus W$.

Problem 2. Use the sum-intersection formula to establish the following.

(a) Suppose that U and W are both four-dimensional subspaces of a six-dimensional vector space V. Prove that there exist two elements in $U \cap W$ such that neither is a scalar multiple of the other.

(b) Suppose that V is a ten-dimensional vector space and U_1, U_2, U_3 are any three subspaces of V such that $\dim(U_1) = \dim(U_2) = \dim(U_3) = 7$. Prove that $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Solution.

(a) By the sum-intersection formula, we have

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W),$$

so

$$\dim(U \cap W) = \dim(U) + \dim(W) - \dim(U + W) \geq \dim(U) + \dim(W) - \dim(V) = 2.$$

Therefore, we can find a linearly independent subset $S = \{x_1, x_2\}$ of $U \cap W$ of cardinality 2. Since S is linearly independent, neither of the vectors x_1, x_2 is a scalar multiple of the other. Indeed, if for instance $x_1 = \lambda x_2$ for some $\lambda \in F$, then $x_1 + (-\lambda) x_2$ yields a non-trivial linear dependence relation for S, contradicting the fact that S is linearly independent. So x_1 is not a scalar multiple of x_2, and similarly x_2 is not a scalar multiple of x_1.

(b) By the sum-intersection formula,

$$\dim(U_1 \cap U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \geq \dim(U_1) + \dim(U_2) - \dim(V) = 4$$

and

$$\begin{aligned}\dim(U_1 \cap U_2 \cap U_3) = \dim((U_1 \cap U_2) \cap U_3) &= \dim(U_1 \cap U_2) + \dim(U_3) - \dim((U_1 \cap U_2) + U_3) \\ &\geq \dim(U_1 \cap U_2) + \dim(U_3) - \dim(V) \geq 4 + 7 - 10 = 1.\end{aligned}$$

Therefore $U_1 \cap U_2 \cap U_3 \neq \{0\}$.

Problem 3. Let U and V be F-vector spaces. Suppose that U is finite-dimensional of dimension $n \geq 1$. Let $S = \{u_1, \ldots, u_n\}$ be a basis for U. Given any map $f : S \to V$ (of sets), show that there is a unique linear map

$$\tau_f : U \to V$$

such that $\tau_f(u_i) = f(u_i)$ for all $i = 1, \ldots, n$.

Solution. Since S is a basis for U, every vector $x \in U$ can be written uniquely in the form $x = \lambda_1 u_1 + \cdots + \lambda_n u_n$ with $\lambda_1, \ldots, \lambda_n \in F$. Thus, we can define a map $\tau_f : U \to V$ by

$$\tau_f(\lambda_1 u_1 + \cdots + \lambda_n u_n) = \lambda_1 f(u_1) + \cdots \lambda_n f(u_n).$$

To check that τ_f is linear, let $x, y \in U$ and let $\alpha, \beta \in F$. Write $x = \lambda_1 u_1 + \cdots + \lambda_n u_n$, $y = \mu_1 u_1 + \cdots + \mu_n u_n$, where $\lambda_1 \ldots, \lambda_n, \mu_1, \ldots, \mu_n \in F$. Then

$$\begin{aligned}
\tau_f(\alpha x + \beta y) &= \tau_f(\alpha(\lambda_1 u_1 + \cdots + \lambda_n u_n) + \beta(\mu_1 u_1 + \cdots \mu_n u_n)) \\
&= \tau_f((\alpha\lambda_1 + \beta\mu_1)u_1 + \cdots + (\alpha\lambda_n + \beta\mu_n)u_n) \\
&= (\alpha\lambda_1 + \beta\mu_1)f(u_1) + \cdots + (\alpha\lambda_n + \beta\mu_n)f(u_n) \\
&= \alpha(\lambda_1 f(u_1) + \cdots + \lambda_n f(u_n)) + \beta(\mu_1 f(u_1) + \cdots + \mu_n f(u_n)) \\
&= \alpha\tau_f(x) + \beta\tau_f(y).
\end{aligned}$$

Finally, to check that τ_f is unique, suppose that $\tau_f' : U \to V$ is any linear map such that $\tau_f'(u_i) = f(u_i)$ for $i = 1, \ldots, n$. Let $x = \lambda_1 u_1 + \cdots + \lambda_n u_n \in U$. Then, by linearity,

$$\tau_f'(x) = \lambda_1 \tau_f'(u_1) + \cdots + \lambda_n \tau_f'(u_n) = \lambda_1 f(u_1) + \cdots + \lambda_n f(u_n) = \tau_f(x).$$

Since this holds for all $x \in U$, we conclude that $\tau_f' = \tau_f$, so τ_f is unique.