📘 Homework #1

Due: October 15, 2025 **Electronic textbook:** Click here

📌 Since there are inconsistencies in exercise numbering between Edition 4 and Edition 5, the homework problems are listed explicitly below to avoid confusion.

Question 1. Label each of the following statements as **true** or **false**.

(1) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.

(2) There exists a square matrix with no eigenvectors.

(3) Eigenvalues must be nonzero scalars.

(4) Any two eigenvectors are linearly independent.

(5) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.

Question 2. For the following linear operator T on a vector space V, compute:

- the **determinant** of T, and
- the **characteristic polynomial** of T.

(1) $V = \mathbb{R}^3$, and

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - 3b + 2c \\ -2a + b + c \\ 4a - c \end{pmatrix}$$

Question 3.

For each of the following linear operators T on a vector space V and ordered basis β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

(1) $V = \mathbb{R}^3$

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}, \quad \beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

(2) $V = P_2(\mathbb{R})$

$$T(a + bx + cx^2) = (-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2$$

$$\beta = \{x - x^2, \ -1 + x^2, \ -1 - x + x^2\}$$

Question 4.

For each linear operator T on V, find the eigenvalues of T and an ordered basis β such that $[T]_\beta$ is diagonal.

(1) $V = \mathbb{R}^2$,

$$T(a, b) = (-2a + 3b, \ -10a + 9b)$$

(2) $V = P_2(\mathbb{R})$,

$$T(f(x)) = xf'(x) + f(2)x + f(3)$$

Question 5.

Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

Question 6.

A **scalar matrix** is a matrix of the form λI for some scalar λ. That is, a diagonal matrix with all diagonal entries equal.

(a) Prove that if A is similar to a scalar matrix λI, then $A = \lambda I$.

(b) Show that a diagonalizable matrix with only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Question 7.

Prove that for any square matrix A, the matrices A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

Question 8.

(a) Let T be a linear operator on V, and let x be an eigenvector of T with eigenvalue λ. Prove that for any positive integer m, x is an eigenvector of T^m with eigenvalue λ^m.

(b) State and prove the analogous result for matrices.

Question 9.

Let T be the linear operator on $M_{n\times n}(\mathbb{R})$ defined by

$$T(A) = A^t.$$

(a) Show that ± 1 are the only eigenvalues of T.

(b) Describe the eigenvectors corresponding to each eigenvalue of T.

(c) Find a basis β for $M_{2\times 2}(\mathbb{R})$ such that $[T]_\beta$ is diagonal.

(d) Find a basis β for $M_{n\times n}(\mathbb{R})$ such that $[T]_\beta$ is diagonal when $n > 2$.

Question 10.

Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$, and deduce that A is invertible if and only if $a_0 \neq 0$.