Theorem Let V be an F-v.s. and let U be a subspace of V.

(i) There is a unique F-v.s. structure on V/U such that the quotient map $\pi: V \to V/U$ is linear. Wrt this structure, the kernel of π is U.

(ii) (Universal property) Given any linear map $T: V \to W$ (where W is any F-v.s.) such that $U \leq \ker T$, there exists a unique linear map $\sigma: V/U \to W$ such that $T = \sigma \circ \pi$.

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "T"'] & V/U \arrow[d, dashed, "\sigma"] \\ & W \end{tikzcd}$$

Proof. (i) We define addition and scalar multiplication on V/U as follows:

- given $s, t \in V/U$, choose $x, y \in V$ such that $s = [x]$, $t = [y]$ and define $s + t = [x+y]$
- given $s \in V/U$ and $\lambda \in F$, choose $x \in V$ such that $s = [x]$ and define $\lambda s = [\lambda x]$.

We need to check these operations are well-defined (i.e., they do not depend on the choice of representatives).

Let $s \in V/U$, $t \in V/U$, and suppose $x, x', y, y' \in V$ satisfy

$$s = [x] = [x'], \quad t = [y] = [y'].$$

Then $x - x' \in U$, $y - y' \in U$, so

$$(x+y) - (x'+y') = (x - x') + (y - y') \in U.$$

Therefore $[x+y] = [x'+y']$. This shows that addition on V/U is well-defined.

Similarly, we can check that scalar multiplication is well-defined.

Now one can easily verify that V/U is an F-vector space under these operations and that $\pi: V \to V/U$ is linear. For instance, to check that addition on V/U is associative, note that, for all $s=[x]$, $t=[y]$, $w=[z] \in V/U$,

$$\begin{aligned}(s+t)+w &= ([x]+[y])+[z] \\ &= [x+y]+[z] \\ &= [(x+y)+z] \\ &= [x+(y+z)] \qquad \text{(because $+$ is associative on V)} \\ &= [x]+[y+z] \\ &= [x]+([y]+[z]) \\ &= s+(t+w).\end{aligned}$$

We leave the remaining verifications as an exercise.

Now we need to prove uniqueness, i.e., we need to prove that the F-vector space structure on V/U that we have just defined is the unique F-vector space structure on V/U for which $\pi: V \to V/U$ is linear.

Indeed, let $(V/U, \tilde{+}, \tilde{\cdot})$ be an F-vector space structure on V/U for which $\pi: V \to V/U$ is linear. Let $s=[x]$, $t=[y] \in V/U$. Then

$$s \tilde{+} t = [x] \tilde{+} [y] = \pi(x) \tilde{+} \pi(y) = \pi(x+y) = [x+y] = s+t.$$

So $\tilde{+}$ agrees with the previously defined addition $+$ on V/U. Similarly the scalar multiplication $\tilde{\cdot}$ agrees with the previously defined scalar multiplication on V/U.

It remains to prove that $\ker(\pi) = U$. To see this, note that for any $x \in V$,

$$
\begin{aligned}
x \in \ker(\pi) &\Leftrightarrow \pi(x) = 0_{V/U} \\
&\Leftrightarrow [x] = [0] \\
&\Leftrightarrow x - 0 \in U \\
&\Leftrightarrow x \in U.
\end{aligned}
$$

(ii) Suppose $T : V \to W$ is a linear map such that $\ker(T) \supseteq U$.

We define $\sigma : V/U \to W$ by setting $\sigma([x]) = T(x)$ for all $[x] \in V/U$.

We need to check that σ is well-defined. Suppose $[x] = [x'] \in V/U$. Then $x - x' \in U \subseteq \ker(T)$, so

$$T(x) = T(x' + (x - x')) = T(x') + T(x - x') = T(x') + 0 = T(x').$$

This shows that σ is well-defined.

Now we check that σ is linear. Let $\lambda, \mu \in F$, $[x], [y] \in V/U$. Then

$$
\begin{aligned}
\sigma(\lambda[x] + \mu[y]) &= \sigma([\lambda x + \mu y]) \\
&= T(\lambda x + \mu y) \\
&= \lambda T(x) + \mu T(y) \\
&= \lambda \sigma([x]) + \mu \sigma([y]).
\end{aligned}
$$

Now we prove that $\sigma \circ \pi = T$. Let $x \in V$. Then

$$(\sigma \circ \pi)(x) = \sigma(\pi(x)) = \sigma([x]) = T(x).$$

To show uniqueness, suppose $\sigma': V/U \to W$ is a linear map satisfying $\sigma' \circ \pi = T$. Then $\sigma' \circ \pi = \sigma \circ \pi$. Since π is surjective, this implies that $\sigma' = \sigma$. $\square$

<u>Corollary</u>. Let V be a f.d.v.s. over F and let U be a subspace of V. Then V/U is a f.d.v.s. and $\dim(V/U) = \dim V - \dim U$.

<u>Proof</u>. Since $\pi: V \longrightarrow V/U$ is surjective, $\mathrm{im}(\pi) = V/U$. By Rank-Nullity theorem,

$$\dim(V/U) = \mathrm{rank}(\pi) = \dim(V) - \mathrm{nullity}(\pi) = \dim(V) - \dim(U). \qquad \square$$

<u>Prop</u>. Let V be an F-v.s., let U be a subspace of V and let $\pi: V \longrightarrow V/U$ denote the canonical projection. The function

$$\left\{ \begin{array}{l} \text{subspaces of } V \\ \text{containing } U \end{array} \right\} \longrightarrow \{ \text{subspaces of } V/U \}$$
$$W \longmapsto \pi(W)$$

is well-defined and bijective with inverse given by $\Delta \mapsto \pi^{-1}(\Delta)$. Moreover, this bijection preserves inclusions, i.e., if W_1, W_2 are subspaces of V containing U, then $W_1 \subseteq W_2$ if and only if $\pi(W_1) \subseteq \pi(W_2)$.