Direct sums

Rk If $U_1, \dots, U_n$ are subspaces of an F-vector space V, then we define

$$U_1 + \dots + U_n := \{x_1 + \dots + x_n \mid x_i \in U_i \text{ for } 1 \leq i \leq n\}.$$

We showed that it is a subspace of V for $n = 2$, and it can be proved for all $n \geq 1$ by induction.

Def. Let V be an F-vector space and let $U_1, \dots, U_n$ be subspaces of V. We say that V is the (internal) direct sum of $U_1, \dots, U_n$, written

$$V = U_1 \oplus \dots \oplus U_n,$$

if each element of V can be written uniquely in the form $u_1 + \dots + u_n$ where $u_i \in U_i$ for all $1 \leq i \leq n$.

Prop. The following are equivalent

(i) $V = U_1 \oplus \dots \oplus U_n$

(ii) $V = U_1 + \dots + U_n$ and $U_i \cap \left(\sum_{j \neq i} U_j\right) = \{0\}$ for all $i = 1, \dots, n$.

Proof.

(i) $\Rightarrow$ (ii) Suppose (i) holds. Then clearly $V = U_1 + \dots + U_n$.

Fix i and let $v \in U_i \cap \left(\sum_{j \neq i} U_j\right)$. Then

$$v = \sum_{j \neq i} u_j \qquad \text{for some } u_j \in U_j \text{ for } j \in \{1, \dots, n\} \setminus \{i\}.$$

Then $$0 = \overset{U_1}{\overset{\cup}{0}} + \dots + \overset{U_{i-1}}{\overset{\cup}{0}} + \overset{U_i}{\overset{\cup}{0}} + \overset{U_{i+1}}{\overset{\cup}{0}} + \dots + \overset{U_n}{\overset{\cup}{0}}$$

$$= u_1 + \dots + u_{i-1} + (-v) + u_{i+1} + \dots + u_n.$$

Since $V = U_1 \oplus \dots \oplus U_n$, this implies $-v = 0$ and therefore $v = 0$.

<u>(ii) ⇒ (i)</u> Let $v \in V$ and suppose that

$$v = u_1 + \dots + u_n = u_1' + \dots + u_n'$$

where $u_j, u_j' \in U_j$ for $j = 1, \dots, n$.

Then for each $i \in \{1, \dots, n\}$,

$$\underbrace{u_i - u_i'}_{\in\, U_i} = \underbrace{\sum_{j \neq i} (u_j' - u_j)}_{\in\, \sum_{j \neq i} U_j} \in U_i \cap \Big(\sum_{j \neq i} U_j\Big).$$

Therefore $u_i - u_i' = 0$ and so $u_i = u_i'$. This proves uniqueness (existence follows trivially from $U_1 + \dots + U_n = V$). $\square$

<u>Example</u> Let $F = \mathbb{R}$, $V = \mathbb{R}^2$. Let

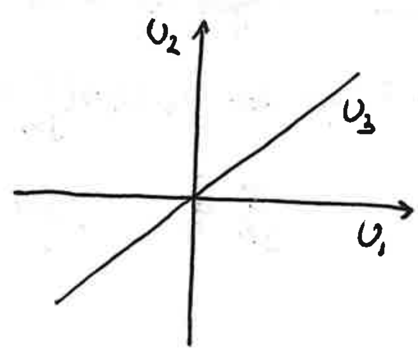
$$U_1 = \{(x,0) \mid x \in \mathbb{R}\}, \quad U_2 = \{(0,y) \mid y \in \mathbb{R}\}, \quad U_3 = \{(z,z) \mid z \in \mathbb{R}\}.$$

Then $V = U_1 \oplus U_2 = U_2 \oplus U_3 = U_1 \oplus U_3$.

Also $V = U_1 + U_2 + U_3$, but

$V \neq U_1 \oplus U_2 \oplus U_3$

(e.g. $(0,0) = (1,0) + (0,1) + (-1,-1)$
$= (0,0) + (0,0) + (0,0)$)



Linear independence

Def. Let S be a subset of an F-vector space V. We say that S is *linearly dependent* if $\exists \lambda_1, \dots, \lambda_n \in F$ not all zero and distinct $v_1, \dots, v_n \in S$ s.t. $\sum_{i=1}^{n} \lambda_i v_i = 0$.

Otherwise, we say that S is *linearly independent*.

So S is linearly independant if and only if for all distinct $v_1, \dots, v_n \in S$

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \text{ for some } \lambda_i \in F \Rightarrow \lambda_i = 0, \ i = 1, \dots, n$$

Examples

1) Let $v \in V$ be a non-zero vector. Then, for all $\lambda \in F$,

$$\lambda v = 0 \Rightarrow \lambda = 0.$$

Therefore $\{v\}$ is linearly independent.

2) Any subset of V containing 0_V is linearly dependent, since, for instance,
$$1 \cdot 0_V = 0_V.$$

3) Let $V = F^n$. For $1 \leq i \leq n$, let $\vec{e}_i \in F^n$ be the n-tuple with i-th entry 1 and all other entries 0, i.e.,

$$\vec{e}_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th entry}}}{1}, 0, \dots, 0)$$

Let $\lambda_1, \dots, \lambda_n \in F$. Then

$$\sum_{i=1}^{n} \lambda_i \vec{e}_i = (\lambda_1, \dots, \lambda_n)$$

so $\displaystyle\sum_{i=1}^{n} \lambda_i \vec{e}_i = \vec{0} \Rightarrow \lambda_1 = \dots = \lambda_n = 0.$

Hence $\{\vec{e}_1, \dots, \vec{e}_n\}$ is linearly independent.

<u>Convention</u> The empty set is linearly independent.

<u>Prop.</u> Let V be an F-vector space and let $S = \{v_1, \dots, v_n\}$ be a finite subset of V with $|S| = n \geq 2$. TFAE:

(i) S is linearly dependent

(ii) $v_1 = 0$ <u>or</u> for some $r \in \{2, \dots, n\}$ the element v_r is a linear combination of $v_1, \dots, v_{r-1}$.

(iii) For some $r \in \{1, \dots, n\}$, v_r is a linear combination of $S \setminus \{v_r\}$.

<u>Proof</u>

<u>(i) $\Rightarrow$ (ii)</u> Suppose S is linearly dependent. Then

$\lambda_1 v_1 + \dots + \lambda_n v_n = 0$ for some $\lambda_1, \dots, \lambda_n \in F$ not all zero.

Let r be the largest integer s.t. $\lambda_r \neq 0$. If $r = 1$, then $\lambda_1 v_1 = 0$, so $v_1 = 0$.

If $r > 1$, then $v_r = -\displaystyle\sum_{i=1}^{r-1} \lambda_r^{-1} \lambda_i v_i$.

(ii) $\Rightarrow$ (iii) Easy.

(iii) $\Rightarrow$ (i) Let $r \in \{1, \dots, n\}$ and suppose that v_r is a linear combination of $S \setminus \{v_r\}$.

Then $v_r = \sum_{i \neq r} \lambda_i v_i$ for some $\lambda_i \in F$, $i \neq r$.

Taking $\lambda_r = -1$, we have
$$\sum_{i=1}^{n} \lambda_i v_i = 0 \quad \text{and} \quad \lambda_r \neq 0.$$

$\square$

Recall If $S = \{v_1, \dots, v_n\}$ is a finite subset of a vector space V, the subspace of V spanned by S is the set of all linear combinations of elements of S, denoted by $\operatorname{span} S$.

More generally, if S is an arbitrary subset of an F-vector space V, we denote by $\operatorname{span} S$ the set of all linear combinations of finitely many elements of S; i.e., an element $x \in V$ is in $\operatorname{span} S$ if and only if $\exists \lambda_1, \dots, \lambda_n \in F$ and $v_1, \dots, v_n \in S$ such that $x = \lambda_1 v_1 + \dots + \lambda_n v_n$.

As in the finite case, $\operatorname{span} S$ is a subspace of V (exercise).

Def. Let V be an F-vector space. A subset S of V is a <u>basis</u> of V if

(i) S is linearly independent, and

(ii) $\operatorname{span} S = V$.

We say that V is <u>finite-dimensional</u> if it has a finite basis; otherwise, we say that V is <u>infinite-dimensional</u>.

<u>Examples</u>

1) $\mathbb{C}$ as an $\mathbb{R}$-vector space has $\{1, i\}$ as a basis. Indeed, every $z \in \mathbb{C}$ can be written as $z = a + bi$ for some $a, b \in \mathbb{R}$, so $\operatorname{span}\{1, i\} = \mathbb{C}$, and for all $c, d \in \mathbb{R}$

$$c + di = 0 \Rightarrow c = d = 0,$$

so $\{1, i\}$ is linearly independent.

2) For any field F, F^n has $\{\vec{e}_1, \dots, \vec{e}_n\}$ as a basis. We call it the <u>standard basis</u> of F^n.

3) By convention, the zero vector space $\{0\}$ has the empty set as a basis.