Linear maps

Def. Let U, V be F-vector spaces. A map

$$T: U \to V$$

is a <u>linear map</u> or an <u>F-vector space homomorphism</u> if it respects addition and scalar multiplication, i.e., if

(i) $T(x+y) = T(x) + T(y) \quad \forall x, y \in U$, and

(ii) $T(\lambda x) = \lambda T(x) \quad \forall \lambda \in F, \forall x \in U.$

Rk T is linear if and only if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \qquad \forall \lambda, \mu \in F; \ \forall x, y \in U.$$

(Exercise.)

Examples

1) For any F-v.s. V, the identity map

$$\mathbb{1}_V : V \longrightarrow V$$
$$x \longmapsto x$$

and the zero map

$$\mathbb{0}_V : V \longrightarrow V$$
$$x \longmapsto 0$$

are linear.

2) For any $n \geq 1$, the map

$$F^n \longrightarrow F$$
$$(x_1, \dots, x_n) \longmapsto x_n$$

is linear.

3) If V is the space of functions $[0,1] \longrightarrow \mathbb{R}$, then

$$ev_{1/2} : V \longrightarrow \mathbb{R}$$
$$f \longmapsto f(1/2)$$

is a linear map.

4) For any F-v.s. V and for any $v_1, \dots, v_n \in V$,

$$F^n \longrightarrow V$$
$$(\lambda_1, \dots, \lambda_n) \longmapsto \lambda_1 v_1 + \dots + \lambda_n v_n$$

is a linear map.

Prop. Let U, V be F-vector spaces. Let $T: U \to V$ be a linear map.

(i) If $\lambda_1, \dots, \lambda_n \in F$, $u_1, \dots, u_n \in U$, then

$$T(\lambda_1 u_1 + \dots + \lambda_n u_n) = \lambda_1 T(u_1) + \dots + \lambda_n T(u_n)$$

(ii) $T(0_U) = 0_V$.

Proof.

(i) Use the definition of linear map and induction on n.

(ii) We have

$$T(0_U) = T(0_U + 0_U) = T(0_U) + T(0_U).$$

Therefore $T(0_U) = 0_V$.

Def. Let $T: U \to V$ be a linear map.

(i) The kernel (or null space) of T is

$$\ker T := T^{-1}(\{0\}) = \{x \in U \mid T(x) = 0\}$$

(ii) The image (or range) of T is

$$\operatorname{im} T := T(U) = \{T(x) \mid x \in U\}.$$

Prop. Let $T: V_1 \to V_2$ be an F-linear map.

(i) If U_1 is a subspace of V_1, then $T(U_1)$ is a subspace of V_2.

(ii) If U_2 is a subspace of V_2, then $T^{-1}(U_2)$ is a subspace of V_1.

Proof.

(i) Let U_1 be a subspace of V_1. By definition, $T(U_1) = \{T(x) \mid x \in U_1\}$.

Since U_1 is a subspace of V_1, $0_{V_1} \in U_1$ and therefore

$0_{V_2} = T(0_{V_1}) \in T(U_1)$.

Let $\lambda, \mu \in F$ and let $v, w \in T(U_1)$. Then $v = T(x)$, $w = T(y)$ for some $x, y \in U_1$. Then

$$\lambda v + \mu w = \lambda T(x) + \mu T(y) = T(\lambda x + \mu y) \in T(U_1)$$

By the subspace criterion, we conclude that $T(U_1)$ is a subspace of V_2.

(ii) Let U_2 be a subspace of V_2. By definition, $T^{-1}(U_2) = \{x \in V_1 \mid T(x) \in U_2\}$.

Since $T(0_{V_1}) = 0_{V_2} \in U_2$, we have that $0_{V_1} \in T^{-1}(U_2)$.

Let $\lambda, \mu \in F$ and let $x, y \in T^{-1}(U_2)$. Then $T(x), T(y) \in U_2$.

Therefore

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \in U_2,$$

so $\lambda x + \mu y \in T^{-1}(U_2)$.

By the subspace criterion, we conclude that $T^{-1}(U_2)$ is a subspace of V_1. $\square$

Corollary. Let $T: V_1 \to V_2$ be a linear map.

(i) $\operatorname{im} T$ is a subspace of V_2.

(ii) $\ker T$ is a subspace of V_1.

Recall A function $f: X \to Y$ is

(i) injective if $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$,

(ii) surjective if $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$,

(iii) bijective if f is both injective and surjective.

Prop. Let $T: U \to V$ be a linear map. Then T is injective if and only if $\ker T = \{0\}$.

Proof.

($\Rightarrow$) Suppose that T is injective. Let $x \in \ker T$. Then $T(x) = 0 = T(0)$. By injectivity $x = 0$. Therefore $\ker T = \{0\}$.

($\Leftarrow$) Suppose that $\ker T = \{0\}$. Let $x, y \in U$ and suppose that $T(x) = T(y)$. Then

$$T(x-y) = T(x) - T(y) = 0.$$

Therefore $x - y \in \ker T = \{0\}$, so $x - y = 0$ and $x = y$. $\square$

__Lemma__ Let $T: U \to V$ be an F-linear map. Let $S \subseteq U$ be a spanning set of U. Then $T(S)$ is a spanning set of $\operatorname{im} T$.

__Proof.__ Let $y \in \operatorname{im} T$. Then $x = T(x)$ for some $x \in U$. Since $\operatorname{span} S = U$, $\exists \lambda_1, \dots, \lambda_n \in F$ and $u_1, \dots, u_n \in S$ s.t. $x = \lambda_1 u_1 + \dots + \lambda_n u_n$. Then

$$y = T(x) = \lambda_1 T(u_1) + \dots + \lambda_n T(u_n) \in \operatorname{span}(T(S)).$$ $\square$

__Theorem__ (Rank-Nullity theorem). Let U be a f.d. v.s. over F. Let $T: U \to V$ be a linear map. Then $\ker T$ and $\operatorname{im} T$ are finite-dimensional and

$$\dim(\ker T) + \dim(\operatorname{im} T) = \dim U.$$

__Proof.__ Since $\ker(T)$ is a subspace of U and U is finite-dimensional, it follows that $\ker T$ is finite-dimensional of dimension at most $\dim U$.

Let $k = \dim(\ker T)$, $m = \dim U$.

Let $X = \{x_1, \dots, x_k\}$ be a basis of $\ker T$ and extend it to a basis $S = \{x_1, \dots, x_k, y_{k+1}, \dots, y_m\}$ of U.

__Claim__: $\{T(y_{k+1}), \dots, T(y_m)\}$ is a basis for $\operatorname{im}(T)$ (of cardinality $m-k$)

By the previous lemma, $T(S)$ is a spanning set for $\operatorname{im}(T)$.

Since $T(x_1) = \dots = T(x_k) = 0$, we have

$$T(S) = \{0, T(y_{k+1}), \dots, T(y_m)\}.$$

Therefore $\{T(y_{k+1}), \dots, T(y_m)\}$ spans $\operatorname{im} T$.

Now suppose that

$$\mu_{k+1} T(y_{k+1}) + \dots + \mu_m T(y_m) = 0 \qquad \text{for some } \mu_{k+1}, \dots, \mu_m \in F.$$

Then

$$T(\mu_{k+1} y_{k+1} + \dots + \mu_m y_m) = 0,$$

so

$$\mu_{k+1} y_{k+1} + \dots + \mu_m y_m \in \ker T.$$

Then

$$\mu_{k+1} y_{k+1} + \dots + \mu_m y_m = \lambda_1 x_1 + \dots + \lambda_k x_k \qquad \text{for some } \lambda_1, \dots, \lambda_k \in F.$$

Therefore

$$\sum_{i=1}^{k} \lambda_i x_i + \sum_{i=k+1}^{m} (-\mu_i) y_i = 0.$$

Since S is linearly independent, we must have $\lambda_1 = \dots = \lambda_k = \mu_{k+1} = \dots = \mu_m = 0$.

Therefore $\{T(y_{k+1}), \dots, T(y_m)\}$ is linearly independent (and contains $m-k$ distinct elements).

Hence $\{T(y_{k+1}), \dots, T(y_m)\}$ is a basis of $\operatorname{im}(T)$ (of cardinality $m-k$).

Finally, we have

$$\dim(\ker T) + \dim(\operatorname{im} T) = k + (m-k) = m = \dim U. \qquad \square$$

Rk $\dim(\ker T)$ is called the <u>nullity</u> of T

$\dim(\operatorname{im} T)$ is called the <u>rank</u> of T

Corollary Let U, V be f.d.v.s. and let $T: U \to V$ be a linear map.

(i) If $\dim U > \dim V$, then T is not injective.

(ii) If $\dim U < \dim V$, then T is not surjective.

Proof.

(i) By Rank-Nullity Theorem

$$\begin{aligned} \dim(\ker(T)) &= \dim U - \dim(\operatorname{im}(T)) \\ &\geq \dim U - \dim V \\ &> 0 \end{aligned}$$

So $\ker T \neq \{0\}$ and therefore T is not injective.

(ii) By Rank Nullity Theorem

$$\begin{aligned} \dim \operatorname{im}(T) &= \dim U - \dim(\ker T) \\ &\leq \dim U \\ &< \dim V \end{aligned}$$

So $\operatorname{im}(T)$ is a proper subspace of V. $\square$