Def. Let U and V be $\mathbb{F}$-vector spaces. An <u>isomorphism</u> from U to V is a bijective linear map from U to V. We say that U and V are <u>isomorphic</u>, written $U \simeq V$, if there exists an isomorphism from U to V.

Prop. Let U, V, W be $\mathbb{F}$-vector spaces. Then:

(i) $U \simeq U$ (reflexive)

(ii) If $U \simeq V$, then $V \simeq U$ (symmetric)

(iii) If $U \simeq V$ and $V \simeq W$, then $U \simeq W$ (transitive)

Proof. (i) $\mathbb{1}_U : U \to U$ is an isomorphism.

(ii) Let $I : U \to V$ be an isomorphism. In particular, I is bijective, so there exists an inverse map $I^{-1} : V \to U$ which is again bijective. We need to show that I^{-1} is linear.

Let $x, y \in V$ and let $\lambda, \mu \in \mathbb{F}$. Note that, since I is linear,

$$I(\lambda I^{-1}(x) + \mu I^{-1}(y)) = \lambda I(I^{-1}(x)) + \mu I(I^{-1}(y))$$
$$= \lambda x + \mu y.$$

Applying I^{-1} to both sides, we get

$$\lambda I^{-1}(x) + \mu I^{-1}(y) = I^{-1}(\lambda x + \mu y).$$

Therefore I^{-1} is linear.

(iii) Let $\tau : U \to V$ and $\sigma : V \to W$ be isomorphisms. We have seen that ~~the~~ composition of two linear maps is again a linear map, so $\sigma\tau : U \to W$ is linear. It is also bijective, since the composition of two bijective maps is a bijective map. Thus $\sigma\tau$ is an isomorphism. $\square$

__Prop.__ Let U be a finite-dimensional F-vector space and let $n = \dim U$. Then $U \cong F^n$.

__Proof__ Let $S = \{v_1, \dots, v_n\}$ be a basis for U. Then, the "coordinate map"

$$\begin{aligned} \tau : F^n &\longrightarrow U \\ (\lambda_1, \dots, \lambda_n) &\longmapsto \lambda_1 v_1 + \dots + \lambda_n v_n \end{aligned}$$

is

- linear (exercise)
- surjective since S spans U
- injective since S is linearly independent.

So τ is an isomorphism. $\square$

Matrices and linear maps

Def. Let F be a field. Let $m, n \in \mathbb{Z}_{>0}$. An <u>$m \times n$ matrix</u> over F is a function

$$A : \{1, \dots, m\} \times \{1, \dots, n\} \longrightarrow F$$
$$(i, j) \longmapsto a_{ij}.$$

We write A pictorially as $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ or simply as $A = (a_{ij})$.

The set of all $m \times n$ matrices is denoted by $F^{m \times n}$ or $M_{m \times n}(F)$. If $m = n$, we can write $M_n(F)$.

<u>Remark</u>. Let $A = (a_{ij}), B = (b_{ij}) \in M_{m \times n}(F)$ and let $\lambda \in F$. We define:

- $A + B := (a_{ij} + b_{ij}) \in M_{m \times n}(F)$.
- $\lambda A := (\lambda a_{ij}) \in M_{m \times n}(F)$.

Under these operations, $M_{m \times n}(F)$ is an F-vector space. Its zero vector is the $m \times n$ matrix with all entries equal to zero, which we denote by $O_{m \times n}$ or $\mathbb{O}$ if the size is clear.

<u>Remark</u> We will distinguish between F^n, $F^{n \times 1}$ and $F^{1 \times n}$. Of course, these spaces are naturally isomorphic.

Def. Let V be a finite-dimensional F-vector space. Let $n = \dim(V)$. An ordered basis for V is an ordered n-tuple $(v_1, \dots, v_n)$ of elements in V such that $\{v_1, \dots, v_n\}$ is a basis for V.

Given an ordered basis $\beta = (v_1, \dots, v_n)$ for V, we have seen that the map

$$\begin{aligned} F^n &\longrightarrow V \\ (\lambda_1, \dots, \lambda_n) &\longmapsto \lambda_1 v_1 + \dots + \lambda_n v_n \end{aligned}$$

is an isomorphism. If $v = \lambda_1 v_1 + \dots + \lambda_n v_n$, we call $\lambda_1, \dots, \lambda_n$ the coordinates of v wrt β and we write

$$[v]_\beta = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}.$$

The $n \times 1$ matrix $[v]_\beta$ is called the β-coordinate vector of v.

Now let U, V be f.d.v.s over F. Let $n = \dim U$, $m = \dim V$. Let $\beta = (e_1, \dots, e_n)$, $\gamma = (f_1, \dots, f_m)$ be ordered bases of U, V respectively. Let $T \in \mathcal{L}(U, V)$. Then, for each $j = 1, \dots, n$,

$$T(e_j) = \sum_{i=1}^{m} a_{ij} f_i \qquad (\dagger)$$

for some $a_{ij} \in F$. So $(\dagger)$ associates with T a matrix

$${}_\gamma[T]_\beta := (a_{ij}) \in M_{m \times n}(F).$$

Conversely, given a matrix $A \in M_{m \times n}(F)$, the equations $(\dagger)$ define a unique linear map $T_A \in \mathcal{L}(U, V)$; i.e., there exists a unique linear map $T_A \in \mathcal{L}(U, V)$ such that $T_A(e_j) = \sum_{i=1}^{m} a_{ij} f_i$ for each $j = 1, \dots, n$ (exercise: prove this).

Prop. With notations as above, the map

$$\begin{aligned} \mathcal{L}(U, V) &\longrightarrow M_{m \times n}(F) \\ T &\longmapsto {}_\gamma[T]_\beta \end{aligned}$$

is an isomorphism of F-vector spaces.

Proof. Let $T, \sigma \in \mathcal{L}(U, V)$ and let $\lambda, \mu \in F$. Write ${}_\gamma[T]_\beta = (a_{ij})$, ${}_\gamma[\sigma]_\beta = (b_{ij})$.

For each $j = 1, \dots, n$,

$$\begin{aligned} (\lambda T + \mu\sigma)(e_j) &= \lambda T(e_j) + \mu\sigma(e_j) \\ &= \lambda \left(\sum_{i=1}^{m} a_{ij} f_i \right) + \mu \left(\sum_{i=1}^{m} b_{ij} f_i \right) \\ &= \sum_{i=1}^{m} (\lambda a_{ij} + \mu b_{ij}) f_i. \end{aligned}$$

Therefore

$$ {}_\gamma[\lambda T + \mu \sigma]_\beta = (\lambda a_{ij} + \mu b_{ij}) = \lambda (a_{ij}) + \mu (b_{ij}) = \lambda\, {}_\gamma[T]_\beta + \mu\, {}_\gamma[\sigma]_\beta . $$

This shows linearity. Bijectivity follows from the previous discussion. □

Prop. (Basis for $M_{m\times n}(\mathbb{F})$). For $1 \le i \le m$, $1 \le j \le n$, let $E_{ij} \in M_{m\times n}(\mathbb{F})$ denote the matrix with (i,j)-entry equal to 1 and all other entries equal to zero. Then $\{E_{ij} \mid 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $M_{m\times n}(\mathbb{F})$. In particular, if U, V are f.d.v.s. over $\mathbb{F}$ with $n = \dim U$ and $m = \dim V$, then

$$ \dim \mathcal{L}(U,V) = mn. $$

Proof. If $\lambda_{ij} \in F$ for $1 \le i \le m$ and $1 \le i \le n$, then

$$ \sum_{\substack{1 \le i \le m \\ 1 \le j \le n}} \lambda_{ij} E_{ij} = (\lambda_{ij}) $$

(the matrix with (i,j)-entry equal to λ_{ij} for each $1 \le i \le m$, $1 \le j \le n$).

Hence any matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{F})$ can be written uniquely in the form $\sum_{i,j} \lambda_{ij} E_{ij}$ with $\lambda_{ij} \in \mathbb{F}$, namely by taking $\lambda_{ij} = a_{ij}$. □

Let U, V, W be F-vector spaces. Let $\alpha = (e_1, \dots, e_m)$, $\beta = (f_1, \dots, f_n)$, $\gamma = (g_1, \dots, g_p)$ be ordered bases of U, V, W respectively.

Let $\sigma \in \mathcal{L}(V, U)$, $\tau \in \mathcal{L}(W, V)$. Then $\sigma\tau \in \mathcal{L}(W, U)$. We wish to determine the matrix of $\sigma\tau$ in terms of the matrices of σ and τ.

Write
$$ {}_\alpha[\sigma]_\beta = (a_{ij}) =: A \in M_{m\times n}(F) $$
$$ {}_\beta[\tau]_\gamma = (b_{jk}) =: B \in M_{n\times p}(F) $$

This means that
$$ \sigma(f_j) = \sum_{i=1}^{m} a_{ij} e_i \qquad \text{for } j = 1, \dots, n $$
$$ \tau(g_k) = \sum_{j=1}^{n} b_{jk} f_j \qquad \text{for } k = 1, \dots, p. $$

To find the matrix
$$ C := {}_\alpha[\sigma\tau]_\gamma \in M_{m\times p}(F), $$
we evaluate $\sigma\tau$ on each element of the ordered basis γ of W.

For each $k = 1, \dots, p$, we have
$$ \begin{aligned} (\sigma\tau)(g_k) &= \sigma(\tau(g_k)) \\ &= \sigma\left(\sum_{j=1}^{n} b_{jk} f_j\right) \\ &= \sum_{j=1}^{n} b_{jk}\, \sigma(f_j) \qquad \text{(by linearity)} \\ &= \sum_{j=1}^{n} b_{jk} \left(\sum_{i=1}^{m} a_{ij} e_i\right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} b_{jk} e_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} b_{jk}\right) e_i \end{aligned} $$

Define $\quad c_{ik} := \sum_{j=1}^{n} a_{ij} b_{jk}$. Then

$$(\sigma\tau)(g_k) = \sum_{i=1}^{m} c_{ik} e_i \qquad \text{for } k=1,\dots,p,$$

so $\quad C = {}_{\alpha}[\sigma\tau]_{\gamma} = (c_{ik}) \in M_{m\times p}(F).$

Pictorially, the (i,k)-entry of ${}_{\alpha}[\sigma\tau]_{\gamma}$ is obtained by taking the "product" of the i-th row of $A = {}_{\alpha}[\sigma]_{\beta}$ and the k-th column of $B = {}_{\beta}[\tau]_{\gamma}$.

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}] \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}.$$

Def. (Matrix products) If $A = (a_{ij}) \in M_{m\times n}(F)$ and $B = (b_{jk}) \in M_{n\times p}(F)$, the product AB is the matrix $C = (c_{ik}) \in M_{m\times p}(F)$ defined by

$$c_{ik} := \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \text{for } 1 \le i \le m,\ 1 \le k \le p.$$

Rk Products of matrices only exist when the sizes are compatible.

Rk Let $\vec{a}^1, \dots, \vec{a}^n \in F^{m\times 1}$ denote the columns of A and let $\vec{b}_1, \dots, \vec{b}_n \in F^{1\times p}$ denote the rows of B, i.e., $A = \left[\vec{a}^1 \cdots \vec{a}^n\right]$, $B = \begin{bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_n \end{bmatrix}$.

Then the k-th column of C is $b_{1k}\vec{a}^1 + \dots + b_{nk}\vec{a}^n$

and the i-th row of C is $a_{i1}\vec{b}_1 + \dots + a_{in}\vec{b}_n$.