Fundamental Theorem of Algebra

Every non-constant polynomial on the variable X with coefficients in $\mathbb{C}$ has at least one complex root.

Equivalently: Every polynomial of degree n on the variable X with coefficients in $\mathbb{C}$ has precisely n complex roots counted with multiplicity; i.e., if $P(X)$ is a polynomial of degree n with coefficients in $\mathbb{C}$, then $\exists a, \lambda_1, \dots, \lambda_n \in \mathbb{C}$ s.t. $P(X) = a(X-\lambda_1)\cdots(X-\lambda_n)$.

Rk Fields satisfying this property are called <u>algebraically closed fields</u>.

Prop. Let F be an algebraically closed field and let $V \neq \{0\}$ be a f.d.v.s. over F. Then every $\sigma \in \mathrm{End}(V)$ has at least one eigenvalue.

Proof. Choose $u \in V \setminus \{0\}$ and consider the list $(u, \sigma(u), \dots, \sigma^n(u))$, where $n = \dim V$. Then this list is linearly dependent, so $\exists a_0, a_1, \dots, a_n \in F$ not all zero such that $a_0 X + a_1 \sigma(u) + \dots + a_n \sigma^n(u) = 0$.

Let m be the largest index s.t. $a_m \neq 0$. Consider the polynomial $P(X) = a_0 + a_1 X + \dots + a_m X^m$. Since F is alg. closed, $P(X) = a_m (X-\lambda_1)\cdots(X-\lambda_m)$ for some $\lambda_1, \dots, \lambda_m \in F$. Thus $P(\sigma) = a_m(\sigma - \lambda_1 \mathbb{1}_V)\cdots(\sigma - \lambda_m \mathbb{1}_V)$. Since $u \neq 0$ and

$$P(\sigma)u = a_0 u + a_1 \sigma(u) + \dots + a_n \sigma^n(u) = 0,$$

the endomorphism $P(\sigma)$ has non-trivial kernel, so it is not injective.

Since $P(\sigma) = (a_m 1\!\!1_V) \circ (\sigma - \lambda_1 1\!\!1_V) \circ \dots \circ (\sigma - \lambda_m 1\!\!1_V)$ and $a_m 1\!\!1_V$ is injective, we deduce that at least one of the endomorphisms $\sigma - \lambda_1 1\!\!1_V, \dots, \sigma - \lambda_m 1\!\!1_V$ is not injective. Say $\sigma - \lambda_k 1\!\!1_V$ is not injective. Then λ_k is an eigenvalue of σ. $\square$

__Def.__ A square matrix $A = (a_{ij}) \in M_n(F)$ is upper-triangular if $a_{ij} = 0$ for $i > j$.

$$A = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

__Rk__ Let V be a f.d. v.s. over F. Let $n = \dim V$ and let $\beta = (v_1, \dots, v_n)$ be an ordered basis for V. Let $\sigma \in End(V)$. Then ${}_\beta[\sigma]_\beta$ is upper-triangular if and only if $\sigma(v_j) \in \operatorname{span}\{v_1, \dots, v_j\}$ for all $j = 1, \dots, n$.

__Prop.__ Let F be an alg. closed field. Let $V \neq \{0\}$ be a f.d. v.s. over F. Let $\sigma \in End(V)$. Then there exists an ordered basis β for V such that ${}_\beta[\sigma]_\beta$ is upper-triangular.

__Proof.__ We proceed by induction on $n := \dim V$.

For $n = 1$ the result is trivial (all 1×1 matrices are upper-triangular)

Now let V be an F-v.s. with $\dim V = n > 1$ and suppose the result is true for vector spaces of smaller dimension. Let $\sigma \in End(V)$.

By the previous proposition, σ has an eigenvalue $\lambda_1 \in F$. Let $v_1 \in V$ be an eigenvector with eigenvalue λ_1 and let $U = \operatorname{span}\{v_1\}$.

Since $v_1 \neq 0$ (by definition of eigenvector), we have $\dim(U) = 1$, so $\dim(V/U) = \dim V - \dim U = n-1$. Also, since v_1 is an eigenvector, the subspace U is σ-invariant. Hence σ induces an endomorphism $\bar{\sigma}: V/U \to V/U$.

By the induction hypothesis, there exists an ordered basis $\gamma = (v_2+U, \dots, v_n+U)$ for V/U such that ${}_\gamma[\bar{\sigma}]_\gamma$ is upper-triangular. Let $\beta = (v_1, v_2, \dots, v_n)$, which is an ordered basis for V. Then

$$ {}_\beta[\sigma]_\beta = \left[\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & {}_\gamma[\bar{\sigma}]_\gamma & \\ 0 & & & \end{array}\right] $$

is upper-triangular. $\square$

Prop. Let $V \neq \{0\}$ be a f.d.v.s. over F. Let $\sigma \in \text{End}(V)$ and suppose that β is an ordered basis for V such that ${}_{\beta}[\sigma]_{\beta}$ is upper-triangular. Then σ is invertible (i.e., an isomorphism) if and only if all diagonal entries of ${}_{\beta}[\sigma]_{\beta}$ are non-zero.

Proof. We proceed by induction on $n := \dim V$.

For $n = 1$, clearly

$$\sigma \text{ is invertible} \iff {}_{\beta}[\sigma]_{\beta} = [\lambda_1] \text{ with } \lambda_1 \neq 0.$$

Let V be an F-v.s. with $\dim V = n > 1$ and suppose the result is true for F-vector spaces of smaller dimension. Let $\sigma \in \text{End}(V)$ and let $\beta = (v_1, \dots, v_n)$ be an ordered basis for V s.t. ${}_{\beta}[\sigma]_{\beta}$ is upper-triangular, say

$${}_{\beta}[\sigma]_{\beta} = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Let $U = \text{span}\{v_1\}$. Note that v_1 is an eigenvector of eigenvalue λ_1. In particular U is σ-invariant. Let $\alpha = (v_1)$, $\gamma = (v_2 + U, \dots, v_n + U)$, which are ordered bases for U and V/U, respectively. Then

$${}_{\alpha}[\sigma|_U]_{\alpha} = [\lambda_1] \quad \text{and} \quad {}_{\gamma}[\bar{\sigma}]_{\gamma} = \begin{bmatrix} \lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

where $\sigma|_U : U \to U$ and $\bar{\sigma} : V/U \to V/U$ are the endomorphisms of U and V/U induced by σ.

($\Rightarrow$) Suppose σ is invertible. Then σ is injective, and therefore so is $\sigma|_U$. Since $\sigma|_U : U \to U$ is an endomorphism of a f.d.v.s., it follows that $\sigma|_U$ is invertible. By the $n=1$ case, we deduce $\lambda_1 \neq 0$.

Also, since σ is invertible, it is in particular surjective, and it follows easily that $\bar{\sigma} : V/_U \to V/_U$ is surjective. Since $V/_U$ is f.d., again it follows that $\bar{\sigma}$ is invertible. By the induction hypothesis, we deduce $\lambda_j \neq 0$ for $j = 2, \dots, n$.

Altogether, $\lambda_j \neq 0$ for $j = 1, \dots, n$.

($\Leftarrow$) Suppose $\lambda_j \neq 0$ for $j = 1, \dots, n$. By the induction hypothesis, it follows that $\sigma|_U : U \to U$ and $\bar{\sigma} : V/_U \to V/_U$ are invertible. We need to show $\sigma : V \to V$ is invertible.

Since V is f.d., it suffices to show injectivity.

Let $x \in \ker \sigma$. Then $\sigma(x) = 0$, so

$$\bar{\sigma}(x+U) = \sigma(x) + U = 0 + U,$$

i.e., $x + U \in \ker \bar{\sigma}$. Since $\bar{\sigma}$ is injective, we deduce that $x + U = 0 + U$ and therefore $x \in U$. Thus we have

$$\sigma|_U(x) = \sigma(x) = 0.$$

Since $\sigma|_U$ is injective, we deduce $x = 0$. Thus $\ker \sigma = \{0\}$ and therefore σ is injective. $\square$

<u>Corollary</u> Let V be a f.d.v.s over F. Let $\sigma \in \mathrm{End}(V)$ and suppose that β is an ordered basis for V such that ${}_{\beta}[\sigma]_{\beta}$ is upper-triangular. Then the eigenvalues of σ are the diagonal entries of ${}_{\beta}[\sigma]_{\beta}$.

<u>Proof</u>. Let $n = \dim(V)$ and say

$$ {}_{\beta}[\sigma]_{\beta} = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{bmatrix}. $$

Let $\lambda \in F$. Then

$$ {}_{\beta}[\sigma - \lambda 1\!\!1_V]_{\beta} = \begin{bmatrix} \lambda_1 - \lambda & & * \\ & \lambda_2 - \lambda & \\ 0 & \ddots & \lambda_n - \lambda \end{bmatrix}. $$

Then

$$ \begin{aligned} \lambda \text{ is an eigenvalue of } \sigma &\iff \sigma - \lambda 1\!\!1_V \text{ is not invertible} \\ &\iff \lambda_i - \lambda = 0 \text{ for some } i = 1, \ldots, n \\ &\iff \lambda = \lambda_i \text{ for some } i = 1, \ldots, n. \end{aligned} $$

$\square$