Generalized Eigenspaces

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Throughout this note, we denote the base field by $\mathbb{F}$, which is $\mathbb{R}$ or $\mathbb{C}$. All the vector spaces are considered over $\mathbb{F}$.

We have learn so far that it is possible for a vector space V to be not written as direct sum of eigenspaces. Infact, this happens when there are eigenvalues whose algebraic multiplicity is not equal to the dimension of eigenspace corresoponding to that eigenvalue. We will see in this lecture, if we work with generalized eigenspace, we can always write V as direct sum of generalized eigen subspaces. We first quickly recall few definitions.

1 Generalized eigenspace

Definition 1.1 (Generalized Eigenvector)**.** Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V, and let $\lambda \in \mathbb{F}$ be a scalar. A nonzero vector $v \in V$ is called a **generalized eigenvector** of T corresponding to the eigenvalue λ if there exists a positive integer k such that

$$(T - \lambda I)^k v = 0.$$

Definition 1.2. The set consists of all generalized eigenvectors corresponding to λ and the zero vector forms a subspace of V, called the **generalized eigenspace** of T associated with λ:

$$G_\lambda = \ker\big((T - \lambda I)^m\big),$$

where m is large enough that the null space stabilizes, i.e., $\ker((T - \lambda I)^m) = \ker((T - \lambda I)^{m+1})$.

Note that every eigenvector is also a generalized eigenvector, so we have $E_\lambda \subseteq G_\lambda$, where E_λ is the eigenspace corresponding to the eigenvalue λ.

Proposition 1.3 (Generalized Eigenspace as a Null Space)**.** *Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V over a field $\mathbb{F}$, and let $\lambda \in \mathbb{F}$. Then the generalized eigenspace of T corresponding to λ is given by*

$$G_\lambda = \ker\big((T - \lambda I)^n\big),$$

where $n = \dim V$. In particular, G_λ is a subspace of V.

Proof. For each positive integer k, define

$$V_k = \ker\big((T - \lambda I)^k\big).$$

Then we have the increasing sequence of subspaces

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq \cdots .$$

Since V is finite-dimensional, this sequence must stabilize by the stabilization proposition; that is, there exists an integer $m \leq n$ such that

$$V_m = V_{m+1} = V_{m+2} = \cdots .$$

By definition, the generalized eigenspace G_λ is the union of all vectors v such that $(T - \lambda I)^k v = 0$ for some k. Hence,

$$G_\lambda = \bigcup_{k \geq 1} V_k = V_m.$$

Since $V_m = V_{m+1} = \cdots$, we can take $m = n = \dim V$. Therefore,

$$G_\lambda = \ker\big((T - \lambda I)^n\big).$$

This completes the proof. □

The next result shows that generalized eigenvector are linearly independent.

Proposition 1.4. *Let $T : V \to V$ be a linear operator on a finite–dimensional vector space. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of T, and for each i let $v_i \neq 0$ be a generalized eigenvector corresponding to λ_i, i.e. $(T - \lambda_i I)^{m_i} v_i = 0$ for some $m_i \in \mathbb{N}$. Then the vectors $v_1, \ldots, v_k$ are linearly independent.*

Proof. Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \tag{1}$$

for some scalars $a_1, \ldots, a_k$. We will show that all $a_i = 0$. Define the polynomial

$$p(x) = \prod_{j=2}^{k} (x - \lambda_j)^{m_j}.$$

Since $p(T)$ is linear operator and hence on applying on equation (1), we obtain

$$a_1\, p(T) v_1 + \sum_{j=2}^{k} a_j\, p(T) v_j = 0.$$

Note that $p(T)(v_j) = 0$ for $j \geq 2$ because

$$(T - \lambda_j I)^{m_j} v_j = 0.$$

Hence, we have $a_1\, p(T) v_1 = 0$. Since $p(T)(v_1) = \prod_{j=2}^{k} (\lambda_1 - \lambda_j)^{m_j}$ (Easy check !). Since $\lambda_1 \neq \lambda_j$ for all $j \geq 2$. Hence, $a_1\, p(T) v_1 = 0$ implies $a_1 = 0$. Similary we can prove $a_j = 0$ for $j \geq 2$. This completes the proof.

□

Proposition 1.5. *Let $N : V \to V$ be a nilpotent linear operator on an n–dimensional vector space V. Then there exists a basis of V such that the matrix of N has the form*

$$\begin{pmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ 0 & 0 & 0 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

that is, an upper triangular matrix with all diagonal entries equal to 0.

Proof. Since N is nilpotent and V is finite dimensional of dimension n, so $N^n = 0$. In fact the sequence of kernels

$$\ker N \subseteq \ker N^2 \subseteq \cdots \subseteq \ker N^n$$

stabilizes, and in fact $\ker N^n = V$.

We construct a basis of V by keep extending a basis from $\mathrm{Ker}(N)$ via this increasing sequence of subspaces. We begin by choosing a basis for $\ker N$ say β_1. Since $\ker N \subseteq \ker N^2$, we may extend this basis of $\ker N$ to a basis of $\ker N^2$. Continuing inductively, having obtained a basis for $\ker N^j$, we extend it to a basis for $\ker N^{j+1}$. At the n–th stage, because $\ker N^n = V$, this process yields a basis

$$\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$$

of V.

By construction, for each j, the subspace $\ker N^j$ is spanned by the first d_j vectors of this basis. In particular, if $v_i \in \ker N^j$, then $Nv_i \in \ker N^{j-1}$, which is spanned by first $j-1$ vector. Hence, for each basis vector v_i, the vector Nv_i is a linear combination of $v_1, v_2, \ldots, v_{i-1}$. So the column corresponding to Nv_i has only possible non-zero entries in first $j-1$ entries. This shows that, relative to the basis $\mathcal{B}$, the matrix representation of N is upper triangular: the i–th column of the matrix (representing Nv_i) has nonzero entries only in the first $i-1$ rows. Since the diagonal entries of an upper triangular matrix are precisely the eigenvalues of the operator which are zero as N is nilpotent. Therefore, all diagonal entries must be zero. This completes the proof. □

We next show that each generalized eigenspaces are T-invariant.

Proposition 1.6. *Let $T : V \to V$ be a linear operator on a complex finite dimensional vector space V, and let λ be an eigenvalue of T. Then the generalized eigenspace*

$$G_\lambda := \bigcup_{m \geq 1} \ker\big(T - \lambda I\big)^m$$

is a T–invariant subspace of V.

Proof. Let $v \in G_\lambda$. By definition there exists some $m \geq 1$ with

$$\big(T - \lambda I\big)^m v = 0.$$

Since T and $T - \lambda I$ commute (indeed $T(T - \lambda I) = (T - \lambda I)T$), they also commute with every polynomial in $T - \lambda I$. In particular

$$\big(T - \lambda I\big)^m T = T\big(T - \lambda I\big)^m.$$

Apply $\big(T-\lambda I\big)^m$ to Tv:

$$\big(T-\lambda I\big)^m(Tv) = T\big(T-\lambda I\big)^m v = T\cdot 0 = 0.$$

Thus $Tv \in \ker\big(T-\lambda I\big)^m \subseteq G_\lambda$. Since v was an arbitrary element of G_λ, we conclude $T(G_\lambda) \subseteq G_\lambda$, so G_λ is T–invariant. □

Hence this proposition make sure, we can define the restriction operator

$$T_{G_\lambda} : G_\lambda \to G_\lambda, \qquad T_{G_\lambda}(v) = Tv.$$

Proposition 1.7. *Let $T : V \to V$ be a linear operator on a n-dimensional complex vector space, and let λ be an eigenvalue of T. Then the operator $(T_{G_\lambda} - \lambda I)$ is nilpotent on G_λ.*

Proof. First note that G_λ is $(T_{G_\lambda} - \lambda I)$-invariant subspace. As if $v \in G_\lambda$, then $(T_{G_\lambda} - \lambda I)(v) = T_{G_\lambda}(v) - v \in G_\lambda$, since G_λ is T_λ-invaraint subspace. We now show that the operator $(T_{G_\lambda} - \lambda I)$ is nilpotent on G_λ. Consider the operator

$$N := T_{G_\lambda} - \lambda I.$$

For any $v \in G_\lambda$ we have

$$N^n v = (T_{G_\lambda} - \lambda I)^n v = (T-\lambda I)^n v = 0.$$

Thus $N^n = 0$ on all of G_λ, so N is nilpotent. Therefore $(T_{G_\lambda} - \lambda I)$ is a nilpotent operator on G_λ. □

We have summarized above results in the form of the following main theorem.

Theorem 1.8. *Let $T : V \to V$ be a linear operator on a finite–dimensional complex vector space V. For each eigenvalue λ of T, let*

$$G_\lambda = \ker(T-\lambda I)^n, \qquad n = \dim V,$$

denote the generalized eigenspace corresponding to λ. Then:

1. *The generalized eigenspace G_λ is a T–invariant subspace of V.*
2. *For each eigenvalue λ, the restriction operator*

   $$T_{G_\lambda} : G_\lambda \to G_\lambda, \qquad T_{G_\lambda}(v) = Tv,$$

   satisfies that $(T_{G_\lambda} - \lambda I)$ is nilpotent on G_λ.
3. *The vector space V is the direct sum of the generalized eigenspaces:*

   $$V = \bigoplus_{\lambda \in \sigma(T)} G_\lambda,$$

   where $\sigma(T)$ denotes the set of eigenvalues of T.

Proof. The part (1) and (2) follows from previous two propositions. So we only need to show the part (3). We prove the statement by induction on $n = \dim V$.

Base case: $n = 1$. If $\dim V = 1$, then T has exactly one eigenvalue λ, and clearly $V = G_\lambda$. The statement holds.

Induction step. Assume the proposition holds for all vector spaces of dimension $< n$, and let $\dim V = n > 1$. Since T is a linear operator on a non-zero complex vector space dimensional space, it has at least one eigenvalue; call it λ_1.

Consider the operator $A := T - \lambda_1 I$. Since $\dim V = n$, the sequences

$$\ker A \subseteq \ker A^2 \subseteq \cdots \subseteq \ker A^n, \qquad \operatorname{Ran}(A^n) \subseteq \operatorname{Ran}(A^{n-1}) \subseteq \cdots$$

stabilize by dimension arguments. In particular,

$$G_{\lambda_1} = \ker A^n.$$

A standard fact (proved using stabilization of kernels and images) is the decomposition

$$V = \ker A^n \oplus \operatorname{Ran}(A^n) = G_{\lambda_1} \oplus \operatorname{Ran}(T - \lambda_1 I)^n.$$

Let

$$W := \operatorname{Ran}(T - \lambda_1 I)^n.$$

Then W is T-invariant, and $\dim W < n$ because A^n is not the zero operator on V unless $T = \lambda_1 I$ (in which case the conclusion is trivial). Moreover, the restriction

$$T|_W : W \to W$$

has the same eigenvalues as T except possibly λ_1, and its generalized eigenspaces within W coincide with those of T for all $\lambda \neq \lambda_1$.

By the induction hypothesis applied to $T|_W$, we obtain

$$W \;=\; \bigoplus_{\lambda \neq \lambda_1} G_\lambda,$$

where the direct sum is taken over all eigenvalues $\lambda \neq \lambda_1$ of T.

Finally, combining with the earlier decomposition, we get

$$V = G_{\lambda_1} \oplus W = G_{\lambda_1} \oplus \bigoplus_{\lambda \neq \lambda_1} G_\lambda = \bigoplus_{\lambda \in \sigma(T)} G_\lambda.$$

This completes the induction and the proof. $\square$