📘 Homework # 4

Due: November 11 at 11:59 PM **Electronic textbook:** Click here

📌 Since there are inconsistencies in exercise numbering between Edition 4 and Edition 5, the homework problems are listed explicitly below to avoid confusion.

Question 1. Let β be a basis for a subspace W of an inner product space V, and let $z \in V$. Prove that $z \in \mathrm{W}^{\perp}$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

Let F_1 and F_2 be fields. Recall a function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$.

Question 2. Let $V = C([-1,1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Question 3. For each of the following inner product spaces V (over F) and linear transformations $\mathrm{g}: V \rightarrow F$, find a vector y such that $\mathrm{g}(x) = \langle x, y\rangle$ for all $x \in V$.

(a) $V = \mathbb{R}^3,\ \mathrm{g}(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$

(b) $V = \mathbb{C}^2,\ \mathrm{g}(z_1, z_2) = z_1 - 2z_2$

(c) $V = P_2(R)$ with $\langle f(x), h(x)\rangle = \int_0^1 f(t)h(t)dt,\ \mathrm{g}(f) = f(0) + f'(1)$

Question 4. For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

(a) $V = \mathbb{R}^2, T(a,b) = (2a+b, a-3b), x = (3,5)$.

(b) $V = \mathbb{C}^2, T(z_1, z_2) = (2z_1 + iz_2, (1-i)z_1), x = (3-i, 1+2i)$.

Question 5. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Question 6. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.

(a) $N(T^*T) = N(T)$. Deduce that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.

(b) $\operatorname{rank}(T) = \operatorname{rank}(T^*)$. Deduce from (a) that $\operatorname{rank}(TT^*) = \operatorname{rank}(T)$.

(c) For any $n \times n$ matrix A, $\operatorname{rank}(A^* A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$.

Question 7. Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

Question 8. Suppose that A is an $m \times n$ matrix in which no two columns are identical. Prove that $A^* A$ is a diagonal matrix if and only if every pair of columns of A is orthogonal.
