# Solutions Homework 6

**Problem 1.** Let $V$ be an $F$-vector space and let $U$ and $W$ be subspaces of $V$. Consider the map

$$\begin{aligned} \tau : V &\longrightarrow (V/U) \times (V/W) \\ x &\longmapsto (x+U, x+W). \end{aligned}$$

(a) Prove that $\tau$ is a linear map and $\ker \tau = U \cap W$.

(b) Prove that if $\tau$ is surjective, then $V = U + W$.

*Hint for (b):* To say that $\tau$ is surjective is to say that for every two elements $x, y \in V$, there exists an element $z \in V$ such that $z + U = x + U$ and $z + W = y + W$. Think of what this is saying when you take $y = 0$.

*Solution.*

(a) First we show that $\tau$ is linear. Let $v, v' \in V$ and let $\lambda, \lambda' \in F$. Then

$$\begin{aligned} \tau(\lambda v + \lambda' v') &= \big(\lambda v + \lambda' v' + U, \lambda v + \lambda' v' + W\big) \\ &= \big(\lambda v + U, \lambda v + W\big) + \big(\lambda' v' + U, \lambda' v' + W\big) \\ &= \lambda(v + U, v + W) + \lambda'(v' + U, v' + W) \\ &= \lambda\tau(v) + \lambda'\tau(v'). \end{aligned}$$

Now we observe that

$$\begin{aligned} \ker \tau &= \{x \in V \mid (x + U, x + W) = (0 + U, 0 + W)\} \\ & \{x \in V \mid x \in U \text{ and } x \in W\} \\ &= U \cap W. \end{aligned}$$

(b) Suppose that $\tau$ is surjective. Let $x \in V$ and consider the element $(x + U, 0 + W) \in (V/U) \times (V/W)$. Since $\tau$ is surjective, there exists $z \in V$ such that $\tau(z) = (x + U, 0 + W)$, or equivalently

$$(z + U, z + W) = (x + U, 0 + W).$$

Then $z + U = x + U$ and $z + W = 0 + W$. From the equality $z + U = x + U$ we deduce that $x - z \in U$, and from the equality $z + W = 0 + W$ we deduce that $z \in W$. Therefore

$$x = (x - z) + z \in U + W.$$

**Problem 2.** Let $V$ be an $F$-vector space. Let $U$ be a subspace of $V$ and suppose that $V/U$ is finite-dimensional.

(a) Prove that if $W$ is a finite-dimensional subspace of $V$ and $V = U + W$, then $\dim(W) \geq \dim(V/U)$.

(b) Prove that there exists a finite-dimensional subspace $W$ of $V$ such that $\dim(W) = \dim(V/U)$ and $V = U \oplus W$.

*Solution.*

(a) Suppose that $W$ is a finite-dimensional subspace of $V$ such that $V = U + W$. Let $n = \dim W$ and let $\{w_1, \ldots, w_n\}$ be a basis for $W$. We claim that $\{[w_1]_U, \ldots, [w_n]_U\}$ spans $V/U$, which in particular implies that $V/U$ is a finite-dimensional vector space of dimension at most $n$. Let $s \in V/W$. Then

$s = [x]_U$ for some $x \in V$. Since $V = U + W$, we have that $x = y + z$ for some $y \in U$ and $z \in W$. Thus $s = [y + z]_U = [z]_U$. Since $\{w_1, \ldots, w_n\}$ is a basis for $W$, there exist $\lambda_1, \ldots, \lambda_n \in F$ such that

$$z = \lambda_1 w_1 + \cdots + \lambda_n w_n.$$

Hence

$$s = [z]_U = \lambda_1 [w_1]_U + \cdots + \lambda_n [w_n]_U.$$

*Alternative proof.* Suppose that $W$ is a finite-dimensional subspace of $V$ such that $V = U + W$. Then, by the Second Isomorphism Theorem,

$$W/U \cap W \simeq (U + W)/U = V/U.$$

Therefore

$$\dim(V/U) = \dim(W/U \cap W) = \dim W - \dim(U \cap W) \leq \dim W.$$

(b) Let $n = \dim(V/U)$ and let $(s_1, \ldots, s_n)$ be an ordered basis for $V/U$. For each $i \in \{1, \ldots, n\}$, we can write $s = [x_i]_U$ for some $x_i \in V$. Let $W = \operatorname{span}\{x_1, \ldots, x_n\} \subseteq V$. We claim that $(x_1, \ldots, x_n)$ is an ordered basis for $W$. Clearly $(x_1, \ldots, x_n)$ spans $W$, so we just need to show linear independence. Suppose that

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0$$

for some $\lambda_1, \ldots, \lambda_n \in F$. Then

$$[\lambda_1 x_1 + \cdots + \lambda_n x_n]_U = [0]_U,$$

which we can rewrite as

$$\lambda_1 [x_1]_U + \cdots + \lambda_n [x_n]_U = [0]_U$$

or

$$\lambda_1 s_1 + \cdots + \lambda_n s_n = 0.$$

Since $(s_1, \ldots, s_n)$ is an ordered basis for $V/U$, and in particular linearly independent, it follows that $\lambda_1 = \cdots = \lambda_n = 0$. Thus $(x_1, \ldots, x_n)$ is an ordered basis for $W$ and therefore $\dim W = n = \dim(V/U)$.

Now we need to prove that $V = U \oplus W$. This is equivalent to showing that $V = U + W$ and $U \cap W = \{0\}$. To show that $V = U + W$, let $v \in V$. Since $(s_1, \ldots, s_n)$ spans $V/U$, there exist $\lambda_1, \ldots, \lambda_n \in F$ such that

$$[v]_U = \lambda_1 s_1 + \cdots + \lambda_n s_n = \lambda_1 [x_1]_U + \cdots + \lambda_n [x_n]_U = [\lambda_1 x_1 + \cdots + \lambda_n x_n]_U.$$

Define $w = \lambda_1 x_1 + \cdots + \lambda_n x_n$ and note that $w \in W$. Since $[v]_U = [w]_U$, we have that $v - w \in U$, so

$$v = (v - w) + w \in U + W.$$

Now we prove that $U \cap W = \{0\}$. Let $u \in U \cap W$. Since $u \in W$, there exist $\mu_1, \ldots, \mu_n \in F$ such that

$$u = \mu_1 x_1 + \cdots + \mu_n x_n.$$

On the other hand, since $u \in U$, we have that $[u]_U = [0]_U$. Therefore

$$[0]_U = [u]_U = [\mu_1 x_1 + \cdots + \mu_n x_n]_U = \mu_1 [x_1]_U + \cdots + \mu_n [x_n]_U = \mu_1 s_1 + \cdots + \mu_n s_n.$$

Since $(s_1, \ldots, s_n)$ is linearly independent, it follows that $\mu_1 = \ldots = \mu_n = 0$. Therefore

$$u = 0x_1 + \cdots + 0x_n = 0.$$

**Problem 3.** Let $V$ and $W$ be $F$-vectors spaces. Let $\tau : V \to W$ be a linear map. Suppose that $\operatorname{im}(\tau)$ is finite-dimensional and let $(w_1, \ldots, w_n)$ be an ordered basis for $\operatorname{im}(\tau)$. Hence, for each $v \in V$ there exist unique scalars $\varphi_1(v), \ldots, \varphi_n(v) \in F$ such that

$$\tau(v) = \varphi_1(v)w_1 + \cdots + \varphi_n(v)w_n,$$

thus defining functions $\varphi_1, \ldots, \varphi_n$ from $V$ to $F$. Prove that the functions $\varphi_1, \ldots, \varphi_n$ are linear functionals on $V$.

*Solution.* Let $\lambda, \mu \in F$ and let $x, y \in V$. Then, using the definition of the functions $\varphi_1, \ldots, \varphi_n$, we have

$$\tau(\lambda x + \mu y) = \varphi_1(\lambda x + \mu y)w_1 + \cdots + \varphi_n(\lambda x + \mu y)w_n.$$

On the other hand, using the linearity of $\tau$ and then again the definition of $\varphi_1, \ldots, \varphi_n$, we have

$$\begin{aligned}\tau(\lambda x + \mu y) &= \lambda\tau(x) + \mu\tau(y) \\ &= \lambda(\varphi_1(x)w_1 + \cdots + \varphi_n(x)w_n) + \mu(\varphi_1(y)w_1 + \cdots + \varphi_n(y)w_n) \\ &= (\lambda\varphi_1(x) + \mu\varphi_n(y))w_1 + \cdots + (\lambda\varphi_n(x) + \mu\varphi_n(y))w_n.\end{aligned}$$

Combining the previous two equations, we obtain

$$\varphi_1(\lambda x + \mu y)w_1 + \cdots + \varphi_n(\lambda x + \mu y)w_n = (\lambda\varphi_1(x) + \mu\varphi_1(x))w_1 + \cdots + (\lambda\varphi_n(x) + \mu\varphi_n(y))w_n.$$

Since $(w_1, \ldots, w_n)$ is an ordered basis for $\operatorname{im}(\tau)$, it follows that

$$\varphi_i(\lambda x + \mu y) = \lambda\varphi_i(x) + \mu\varphi_i(y)$$

for all $i \in \{1, \ldots, n\}$. Thus $\varphi_1, \ldots, \varphi_n$ are linear functionals on $V$.