1 Model

The model takes the form

$$\boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}) + \boldsymbol{v}_k \tag{1}$$

$$\boldsymbol{y}_k = \boldsymbol{H}\boldsymbol{x}_{k-1} + \boldsymbol{w}_k \tag{2}$$

where $\boldsymbol{v}_k$ and $\boldsymbol{w}_k$ are white, uncorrelated, zero mean Gaussian noise vectors with covariances of $\boldsymbol{Q}$ and $\boldsymbol{R}$, respectively.

1.1 Pendulum example

Let $\boldsymbol{x}_k = [a_k \quad b_k]^T$ where a_k is the pendulum angle and b_k is the angular velocity,

$$\boldsymbol{f}(\boldsymbol{x}_k) = \begin{bmatrix} a_k + b_k\Delta \\ -b_k - g\sin(a_k)\Delta \end{bmatrix} \tag{3}$$

The linearisation used for the EKF is

$$\boldsymbol{F}(\boldsymbol{x}_k) = \begin{bmatrix} 1 & \Delta \\ -g\cos(a_k) & 1 \end{bmatrix} \tag{4}$$

The observation matrix is $H = [1 \quad 0]$, so that only the first state element is measured.

2 PCRB

The PCRB is computed from the Fisher information matrix computed on the joint density $p(\boldsymbol{y}, \boldsymbol{x})$,

$$\boldsymbol{J} = \mathbb{E}\left[\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}^T \log p(\boldsymbol{y}, \boldsymbol{x})\right] \tag{5}$$

where the expectation is taken over $\boldsymbol{y}$ and $\boldsymbol{x}$. A recursive formula was developed in **[?]** and is based on the transition density,

$$p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = \mathcal{N}(\boldsymbol{f}(\boldsymbol{x}_{k-1})), \boldsymbol{Q}) \tag{6}$$

$$= (2\pi)^{-d/2}|\boldsymbol{Q}|^{-1/2}\exp\left(-\frac{1}{2}(\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1}))^T\boldsymbol{Q}^{-1}(\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1}))\right) \tag{7}$$

The PCRB is calculated using the logarithm of the transition density,

$$\log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = C - \frac{1}{2}(\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1}))^T\boldsymbol{Q}^{-1}(\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1})) \tag{8}$$

The derivatives are

$$\nabla^T_{\boldsymbol{x}_k} \log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = -(\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1}))^T \boldsymbol{Q}^{-1} \tag{9}$$

$$\nabla_{\boldsymbol{x}_k} \nabla^T_{\boldsymbol{x}_k} \log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = -\boldsymbol{Q}^{-1} \tag{10}$$

$$\nabla_{\boldsymbol{x}_{k-1}} \nabla^T_{\boldsymbol{x}_k} \log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = \boldsymbol{F}(\boldsymbol{x}_{k-1}) \boldsymbol{Q}^{-1} \tag{11}$$

$$\nabla^T_{\boldsymbol{x}_{k-1}} \log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = (\boldsymbol{x}_k - \boldsymbol{f}(\boldsymbol{x}_{k-1}))^T \boldsymbol{Q}^{-1} \boldsymbol{F}(\boldsymbol{x}_{k-1}) \tag{12}$$

$$\nabla_{\boldsymbol{x}_{k-1}} \nabla^T_{\boldsymbol{x}_{k-1}} \log p(\boldsymbol{x}_k|\boldsymbol{x}_{k-1}) = -\boldsymbol{F}(\boldsymbol{x}_{k-1})^T \boldsymbol{Q}^{-1} \boldsymbol{F}(\boldsymbol{x}_{k-1}) \tag{13}$$

Now the terms in the PCRB are

$$\boldsymbol{U}_k = \mathbb{E}\left[\boldsymbol{F}(\boldsymbol{x}_{k-1})^T \boldsymbol{Q}^{-1} \boldsymbol{F}(\boldsymbol{x}_{k-1})\right] \tag{14}$$

$$\boldsymbol{V}_k = \mathbb{E}\left[-\boldsymbol{F}(\boldsymbol{x}_{k-1})^T \boldsymbol{Q}^{-1}\right] \tag{15}$$

$$\boldsymbol{W}_k = \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H} + \boldsymbol{Q}^{-1} \tag{16}$$

2.1 Monte Carlo approximation

The expectations in (14) and (15) can be approximated by numerical integration using many trajectory realisations. That is,

$$\boldsymbol{U}_k \approx \frac{1}{M} \sum_{i=1}^{M} \boldsymbol{F}(\boldsymbol{x}^{(i)}_{k-1})^T \boldsymbol{Q}^{-1} \boldsymbol{F}(\boldsymbol{x}^{(i)}_{k-1}) \tag{17}$$

$$\boldsymbol{V}_k \approx \frac{1}{M} \sum_{i=1}^{M} \boldsymbol{F}(\boldsymbol{x}^{(i)}_{k-1})^T \boldsymbol{Q}^{-1} \tag{18}$$

2.2 Pendulum simulations

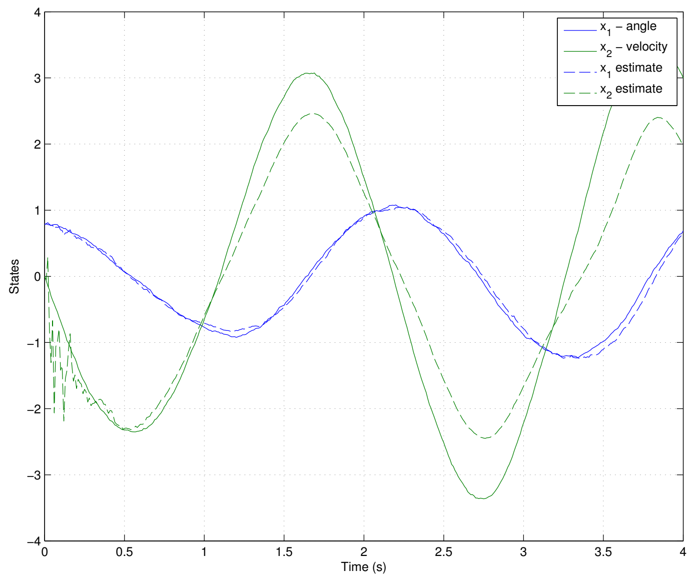

Figure 1: Trajectory of a pendulum and EKF estimates

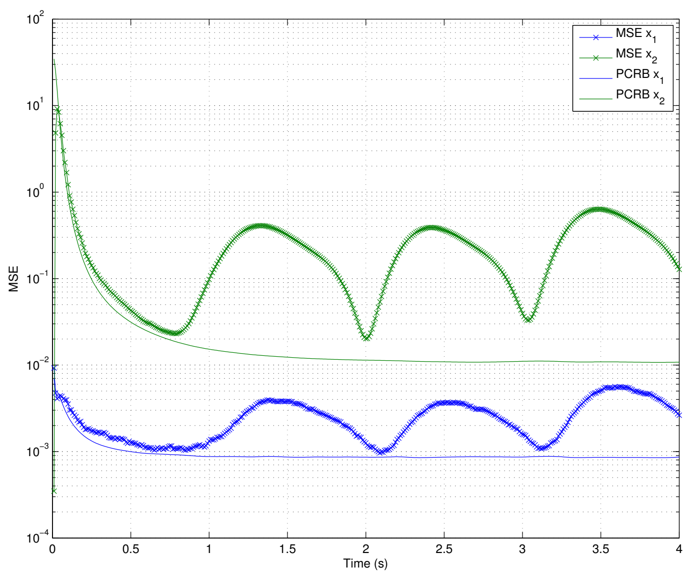

Figure 2: MSE of EKF estimates (averaged over 1000 realisations) compared to the PCRB. The suboptimal EKF is dependent on the accuracy of the linearisation at different angles, an optimal filter would achieve the CRB at all times