CS70 Geometric and Poisson Distributions

Kelvin Lee

kelvinlee@berkeley.edu

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Overview

Geometric Distribution

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- **Variance:**

$$\mathrm{Var}(X) = \lambda.$$

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Problem Time!