CS70 Counting

Kelvin Lee

kelvinlee@berkeley.edu

October 12, 2020

Overview

1. Rules of Counting
2. Stars and Bars
3. Binomial Theorem
4. Combinatorial Proofs
5. Principle of Inclusion-Exclusion
6. Problems
7. Summary/Tips

First Rule of Counting

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$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

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Zeroth Rule of Counting:

If a set A has a bijection relationship with a set B, then $|A| = |B|$.

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- Useful for **with replacement** but **order doesn't matter** type of problems.

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- Proofs by stories: same story from multiple perspectives.
- Proving an identity by counting the same thing in two different ways.
- Useful identity:
$$\binom{n}{k} = \binom{n}{n-k}.$$
- Choosing k objects to include is equivalent to choosing $n-k$ objects to exclude.

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RHS: Total number of subsets of a set of size n.

LHS: The number of ways to choose a subset of size i is $\binom{n}{i}$. To find the total number of subsets, we simply add all the cases when $i = 0, 1, 2, \ldots, n$.

$\square$

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Let $A_1, \ldots, A_n$ be arbitrary subsets of the same finite set A. Then,

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\} : |S| = k} \left| \cap_{i \in S} A_i \right| .$$

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Principle of Inclusion-Exclusion(Simplified):

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

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SP19 MT2 6.3

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Solution:

- There are $\binom{n}{3}$ ways to pick positions for $1, 2, 3$. For the positions picked, we place the three numbers in a way such that the conditions are met, i.e, we place them in the order of $3, 1, 2$.

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- Now for the remaining numbers, there are $(n - 3)!$ to arrange them.

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- Now for the remaining numbers, there are $(n-3)!$ to arrange them.
- Finally, by the **first rule of counting**, we have $\boxed{\frac{n!}{6}}$ permutations.

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SP18 MT2 5

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- Each edge requires 2 vertices, so $\binom{6}{2} = 15$ ways to choose an edge.

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- If every vertex has degree 1, then we can only have 3 edges.
- Each edge requires 2 vertices, so $\binom{6}{2} = 15$ ways to choose an edge.
- After choosing the first edge, we have 4 vertices remaining, so there are $\binom{4}{2} = 6$ ways to choose the second edge and similarly $\binom{2}{2} = 1$ way to choose the final edge.

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- If every vertex has degree 1, then we can only have 3 edges.
- Each edge requires 2 vertices, so $\binom{6}{2} = 15$ ways to choose an edge.
- After choosing the first edge, we have 4 vertices remaining, so there are $\binom{4}{2} = 6$ ways to choose the second edge and similarly $\binom{2}{2} = 1$ way to choose the final edge.
- However, since order doesn't matter, by the **second rule of counting**, we divide by $3! = 6$. So our final answer is $\boxed{15}$.

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- We are choosing two sets of 3 vertices. There are $\binom{6}{3} \ \binom{\cdot 3}{3} = 20$ ways.

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- We are choosing two sets of 3 vertices. There are $\binom{6}{3}\binom{3}{3} = 20$ ways.
- But order doesn't matter here again. So we divide by 2!. Thus, the answer is $\boxed{10}$.

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- However, it doesn't matter where we start, so divide by 6.
- The direction in which we travel along the cycle also doesn't matter, so divide by 2.
- Thus, our answer is $\frac{6!}{2\cdot 6} = \boxed{60}$.

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- So $n = m$ and $k = z$ in this case.

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- This is a stars and bars problem where we have $z-1$ bars and m stars.
- So $n=m$ and $k=z$ in this case.
- Thus, the answer is $\binom{n+k-1}{k-1} = \boxed{\binom{m+z-1}{z-1}}$.

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- **RHS:**

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Solution:

- **LHS:** the number of ternary strings of length n.
- **RHS:** There are $\binom{n}{i}$ positions of the 2's, and there are 2^{n-i} possible patterns of 0 and 1's in the remaining positions. The sum gives you all the ternary strings.

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Tips:

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- Identify which categories does the problem fall under.
- Double check answers by using two different counting approaches.
- Check for overcounting.
- Relax and have fun!