CS70 Continuous Probability

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Overview

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Remark: $f(x)$ is not probability of anything and does not have to be bounded by 1! One example is $U[0, \frac{1}{2}]$ where $f(x) = 2 > 1$. $f(x)$ actually represents the density, or **probability per unit length** vicinity of x!

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$$f(x) = \frac{dF(x)}{dx}.$$

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We interpret $f(x, y)$ as the **probability per unit area** in the vicinity of (x, y).

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$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \left(\int_{-\infty}^{\infty} xf(x)dx\right)^2.$$

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&= \frac{(b-a)^2}{12}.
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Problem Time!