CS70 Modular Arithmetic

Kelvin Lee

kelvinlee@berkeley.edu

September 21, 2020

Overview

Basic Definitions

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In the modular space, the **multiplicative inverse** of x mod m is y if $xy \equiv 1 \pmod{m}$.

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Theorem (Existence of multiplicative inverse)

$gcd\,(x, m) = 1 \implies x$ *has a multiplicative inverse modulo m and it is* **unique**.

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$$\begin{aligned} \text{gcd }(16, 10) &= \text{gcd }(10, 6) \\ &= \text{gcd }(6, 4) \\ &= \text{gcd }(4, 2) \\ &= \text{gcd }(2, 0) \\ &= 2. \end{aligned}$$

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- This uses back substitutions repetitively so that the final expression is in terms of x and y.

Problem Time!