Math 104
Real Analysis

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Contents

LECTURE 1

THE REAL NUMBER SYSTEMS

1 Natural Numbers $\mathbb{N}$

Definition 1 (Peano Axioms (Peano Postulates))**.** The properties of the set of natural numbers, denoted $\mathbb{N}$, are as follows:

(i) 1 belongs to $\mathbb{N}$.

(ii) If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$.

(iii) 1 is not the successor of any element in $\mathbb{N}$.

(iv) If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.

(v) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal to $\mathbb{N}$.

Remark. The last axiom is the basis of mathematical induction. Let $P_1, P_2, P_3, \ldots$ be a list of propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, \ldots$ are true provided

- P_1 is true. (Basis for induction)
- $P_n \implies P_{n+1}$. (Induction step)

2 Rational Numbers $\mathbb{Q}$

Definition 2 (Rational Numbers)**.** The set of **rational numbers**, denoted $\mathbb{Q}$, is defined by

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid n, m \in \mathbb{Z}, n \neq 0 \right\},$$

which supports addition, multiplication, subtraction, and division.

Remark. $\mathbb{Q}$ is a very nice algebraic system. However, there is no rational solution to equations like $x^2 = 2$.

Definition 3 (Algebraic Number)**.** A number is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $c_0, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Remark. Rational numbers are always algebraic numbers.

Theorem 4 (Rational Zeros Theorem)**.** Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equations

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1, c_n, c_0 \neq 0$. Let $r = \frac{c}{d}$ where $\gcd(c,d) = 1$. Then $c \mid c_0$ and $d \mid c_n$. In simpler terms, the only rational candidates for solutions to the equation have the form $\frac{c}{d}$ where c is a factor of c_0 and d is a factor of c_n.

Proof. Plug in $r = \frac{c}{d}$ to the equation, we get

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0.$$

Then we multiply by d^n on both sides and get

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Solving for $c_0 d^n$, we obtain

$$c_0 d^n = -c \left(c_n c^n + c_{n-1}^{n-2} + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}\right).$$

Then it follows that $c \mid c_0 d^n$. Since $\gcd(c,d) = 1$, c can only divide c_0.
Now let's instead solve for $c_n c^n$, then we have

$$c_n c^n = -d \left(c_{n-1} c^{n-1} + c_{n-2} c^{n-2} d + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}\right).$$

Thus $d \mid c_n c^n$, which implies $d \mid c_n$ because $\gcd(c,d) = 1$. □

Corollary 5. Consider

$$x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. Since the Rational Zeros Theorem states that d must divide c_n, which is 1 in this case, r is an integer and it divides c_0. □

Example 6. $\sqrt{2}$ is not a rational number.

Proof. Using Corollary 5, if $r = \sqrt{2}$ is rational, then $\sqrt{2}$ must be an integer, which is a contradiction. □

3 Real Numbers $\mathbb{R}$

4 Reading (Rudin's)

4.1 Ordered Sets

Definition 7 (Order)**.** Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$, then one and only one of the statements
$$s < y, \quad x = y, \quad , y < x$$
is true.
- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 8 (Ordered Set)**.** An **ordered set** is a set S in which an order is defined.

For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

4.2 Fields

Definition 9 (Field)**.** A **field** is a set F with two operations: *addition* and *multiplication*, which satisfy the following **field axioms**:

(A) **Axioms for addition**

- (A1) If $x, y \in F$, then $x + y \in F$.
- (A2) (Commutativity) $\forall x, y \in F$, $x + y = y + x$.
- (A3) (Associativity) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$.
- (A4) (Identity) $\forall x \in F$, $0 + x = x$.
- (A5) (Inverse) $\forall x \in F$, there exists a corresponding $-x \in F$ such that
$$x + (-x) = 0.$$

(M) **Axioms for multiplication**

- (M1) If $x, y \in F$, then $xy \in F$.
- (M2) (Commutativity) $\forall x, y \in F$, $xy = yx$.
- (M3) (Associativity) $\forall x, y, z \in F$, $(xy)z = x(yz)$.
- (M4) (Identity) $\forall x \in F$, $1x = x$.
- (M5) (Inverse) $\forall x \in F$, there exists a corresponding $\frac{1}{x} \in F$ such that
$$x\left(\frac{1}{x}\right) = 1.$$

(D) **The distributive law**

$$\forall x, y, z \in F, x(y + z) = xy + xz.$$

Definition 10 (Ordered Field)**.** An **ordered field** is a field F which is also an *ordered set*, such that

(i) if $y < z$ and $x, y, z \in F$, $x + y < x + z$,

(i) if $x, y > 0$ and $x, y \in F$, $xy > 0$.

LECTURE 2

BASIC TOPOLOGY

Definition 11.

Theorem 12.