CS70 Modular Arithmetic

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September 21, 2020

Overview

1. Basic Definitions
2. Multiplicative Inverse
3. Euclid's Algorithm
4. Extended Euclid's algorithm

Basic Definitions

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Theorem (Existence of multiplicative inverse)

$gcd\,(x, m) = 1 \implies x$ *has a multiplicative inverse modulo m and it is* **unique**.

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Compute gcd(16,10):

$$\begin{aligned} \text{gcd }(16, 10) &= \text{gcd }(10, 6) \\ &= \text{gcd }(6, 4) \\ &= \text{gcd }(4, 2) \\ &= \text{gcd }(2, 0) \\ &= 2. \end{aligned}$$

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- This uses back substitutions repetitively back substitute values so that the final expression is in terms of x and y.

Problem Time!