# Math 104
# Real Analysis

Instructor: Peng Zhou

Kelvin Lee

UC Berkeley

# Contents

LECTURE 1

# THE REAL NUMBER SYSTEMS

## 1 Natural Numbers $\mathbb{N}$

**Definition 1.1** (Peano Axioms (Peano Postulates)). The properties of the set of natural numbers, denoted $\mathbb{N}$, are as follows:

(i) 1 belongs to $\mathbb{N}$.

(ii) If $n$ belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$.

(iii) 1 is not the successor of any element in $\mathbb{N}$.

(iv) If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.

(v) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains $n$, must equal to $\mathbb{N}$.

**Remark.** The last axiom is the basis of mathematical induction. Let $P_1, P_2, P_3, \ldots$ be a list of propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, \ldots$ are true provided

- $P_1$ is true. (Basis for induction)
- $P_n \implies P_{n+1}$. (Induction step)

## 2 Rational Numbers $\mathbb{Q}$

**Definition 2.1** (Rational Numbers). The set of **rational numbers**, denoted $\mathbb{Q}$, is defined by

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid n, m \in \mathbb{Z}, n \neq 0 \right\},$$

which supports addition, multiplication, subtraction, and division.

**Remark.** $\mathbb{Q}$ is a very nice algebraic system. However, there is no rational solution to equations like $x^2 = 2$.

**Definition 2.2** (Algebraic Number)**.** A number is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $c_0, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

**Remark.** Rational numbers are always algebraic numbers.

**Theorem 2.3** (Rational Zeros Theorem)**.** Suppose $c_0, c_1, \ldots, c_n$ are integers and $r$ is a rational number satisfying the polynomial equations

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1, c_n, c_0 \neq 0$. Let $r = \frac{c}{d}$ where $\gcd(c, d) = 1$. Then $c \mid c_0$ and $d \mid c_n$. In simpler terms, the only rational candidates for solutions to the equation have the form $\frac{c}{d}$ where $c$ is a factor of $c_0$ and $d$ is a factor of $c_n$.

*Proof.* Plug in $r = \frac{c}{d}$ to the equation, we get

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0.$$

Then we multiply by $d^n$ on both sides and get

$$c_n c^n + c_{n-1}c^{n-1}d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Solving for $c_0 d^n$, we obtain

$$c_0 d^n = -c\left(c_n c^n + c_{n-1}^{n-2} + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}\right).$$

Then it follows that $c \mid c_0 d^n$. Since $\gcd(c, d) = 1$, $c$ can only divide $c_0$.
Now let's instead solve for $c_n c^n$, then we have

$$c_n c^n = -d\left(c_{n-1}c^{n-1} + c_{n-2}c^{n-2}d + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}\right).$$

Thus $d \mid c_n c^n$, which implies $d \mid c_n$ because $\gcd(c, d) = 1$. $\square$

**Corollary 2.4.** Consider

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides $c_0$.

*Proof.* Since the Rational Zeros Theorem states that $d$ must divide $c_n$, which is 1 in this case, $r$ is an integer and it divides $c_0$. $\square$

**Example 2.5.** $\sqrt{2}$ is not a rational number.

*Proof.* Using Corollary 5, if $r = \sqrt{2}$ is rational, then $\sqrt{2}$ must be an integer, which is a contradiction. $\square$

# 3 Real Numbers $\mathbb{R}$

## 3.1 The Completeness Axiom

**Definition 3.1** (Maximum/minimum)**.** Let $S$ be a nonempty subset of $\mathbb{R}$.

(i) If $S$ contains a largest element $s_0$(i.e., $s_0 \in S$, $s \leq s_0 \forall s \in S$), then $s_0$ is the **maximum** of $S$, denoted $s_0 = \max S$.

(i) If $S$ contains a smallest element, then it is called the **minimum** of $S$, denoted as $\min S$.

**Remark.**

- If $s_1, s_2$ are both maximum of $S$, then $s_1 \geq s_2, s_2 \geq s_1$, which implies that $s_1 = s_2$. Thus the maximum is **unique** if it exists.
- However, the maximum may not exist (e.g. $S = \mathbb{R}$).
- If $S \subset \mathbb{R}$ is a finite subset, then $\max S$ exists.

**Definition 3.2** (Upper/Lower bound)**.** Let $S$ be a nonempty subset of $\mathbb{R}$.

(i) If a real number $M$ satisfies $s \leq M$ for all $s \in S$, then $M$ is an **upper bound** of $S$ and $S$ is said to be *bounded above.*

(i) If a real number $m$ satisfies $\leq s$ for all $s \in S$, then $m$ is a **lower bound** of $S$ and $S$ is said to be *bounded below.*

(i) $S$ is said to be *bounded* if it is bounded above and bounded below. Thus $S$ is bounded if there exist real numbers $m$ and $M$ such that $S \subset [m, M]$.

**Definition 3.3** (Supremum/Infimum)**.** Let $S$ be a nonempty subset of $\mathbb{R}$.

- If $S$ is bounded above and $S$ has a least upper bound, then it is called the **supremum** of $S$, denoted by $\sup S$.
- If $S$ is bounded below and $S$ has a greatest lower bound, then it is called the **infimum** of $S$, denoted by $\inf S$.

**Remark.** If $S$ has a maximum, then $\max S = \sup S$. Similarly, if $S$ has a minimum, then $\min S = \inf S$. Also note that $\sup S$ and $\inf S$ need not belong to $S$.

**Example 3.4.** Suppose we have $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\max S$ does not exist and $\sup S = 1$.

*Proof.* Suppose for contradiction that it exists. Then it must be of the form $1 - \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. However,

$$1 - \frac{1}{n_0 + 1} > 1 - \frac{1}{n_0},$$

and $1 - \frac{1}{n_0+1} \in S$. Hence a contradiction. $\square$

**Theorem 3.5** (Completeness Axiom)**.** Every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

**Corollary 3.6.** Every nonempty subset $S \subset \mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

*Proof.* Consider the set $-S = \{-s \mid s \in S\}$. Since $S$ is bounded below there exists an $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$. This implies $-m \geq -s$ for all $s \in S$, so $-m \geq u$ for all $u \in -S$. Thus, $-S$ is bounded above by $-m$. The Completeness Axiom applies to $-S$, so $\sup -S$ exists.
Now we show that $\inf S = -\sup -S$. Let $s_0 = \sup -S$, we need to prove

$$-s_0 \leq s \qquad \text{for all } s \in S,$$

and if $t \leq s$ for all $s \in S$, then $t \leq -s_0$. The first inequality will show that $-s_0$ is a lower bound while the second inequality will show that $-s_0$ is the greatest lower bound, i.e., $-s_0 = \inf S$. The proofs of the two claims are left as an exercise. □

**Theorem 3.7** (Archimedean Property)**.** If $a, b > 0$, then $na > b$ for some positive integer $n$.

*Proof.* Suppose the property fails for some pair of $a, b > 0$. That is, for all $n \in \mathbb{N}$, we have $na \leq b$, meaning that $b$ is an upper bound for the set $S = \{na \mid n \in \mathbb{N}\}$. Using the Completeness Axiom, we can let $s_0 = \sup S$. Since $a > 0$, we have $s_0 - a < s_0$, so $s_0 - a$ cannot be an upper bound for $S$. It follows that $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$, which then implies that $s_0 < (n_0 + 1)a$. Since $(n_0 + 1)a$ is in $S$, $s_0$ is not an upper bound for $S$, which is a contradiction. □

**Theorem 3.8** (Denseness of $\mathbb{Q}$)**.** If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

*Proof.* We need to show that $a < \frac{m}{n} < b$ for some integers $m$ and $n$ where $n \neq 0$. Equivalently, we want

$$an < m < bn.$$

Since $b - a > 0$, the Archimedean property shows that there exists an $n \in \mathbb{N}$ such that

$$n(b - a) > 1 \implies bn - an > 1.$$

Now we need to show that there is an integer $m$ between $an$ and $bn$. □

# 4 $+\infty$ and $-\infty$

We adjoint $+\infty$ and $-\infty$ to $\mathbb{R}$ and extend our ordering to the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Explicitly, we have $-\infty \leq a \leq +\infty$ for all $a \in \mathbb{R} \cup \{-\infty, +\infty\}$.

**Remark.** $+\infty$ and $-\infty$ are not real numbers. Theorems that apply to real numbers would not work.

We define

$$\sup S = +\infty \qquad \text{if } S \text{ is not bounded above}$$

and

$$\inf S = -\infty \qquad \text{if } S \text{ is not bounded below.}$$

# 5 Reading (Rudin's)

## 5.1 Ordered Sets

**Definition 5.1** (Order)**.** Let $S$ be a set. An **order** on $S$ is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$, then one and only one of the statements

$$s < y, \quad x = y, \quad , y < x$$

  is true.

- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

**Definition 5.2** (Ordered Set)**.** An **ordered set** is a set $S$ in which an order is defined.

For example, $Q$ is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

## 5.2 Fields

**Definition 5.3** (Field)**.** A **field** is a set $F$ with two operations: *addition* and *multiplication*, which satisfy the following **field axioms**:

(A) **Axioms for addition**

(A1) If $x, y \in F$, then $x + y \in F$.
(A2) (Commutativity) $\forall x, y \in F$, $x + y = y + x$.
(A3) (Associativity) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$.
(A4) (Identity) $\forall x \in F$, $0 + x = x$.
(A5) (Inverse) $\forall x \in F$, there exists a corresponding $-x \in F$ such that

$$x + (-x) = 0.$$

(M) **Axioms for multiplication**

(M1) If $x, y \in F$, then $xy \in F$.
(M2) (Commutativity) $\forall x, y \in F$, $xy = yx$.
(M3) (Associativity) $\forall x, y, z \in F$, $(xy)z = x(yz)$.
(M4) (Identity) $\forall x \in F$, $1x = x$.
(M5) (Inverse) $\forall x \in F$, there exists a corresponding $\frac{1}{x} \in F$ such that

$$x\left(\frac{1}{x}\right) = 1.$$

(D) **The distributive law**

$$\forall x, y, z \in F, x(y + z) = xy + xz.$$

**Definition 5.4** (Ordered Field)**.** An **ordered field** is a field $F$ which is also an *ordered set*, such that

(i) if $y < z$ and $x, y, z \in F$, $x + y < x + z$,

(i) if $x, y > 0$ and $x, y \in F$, $xy > 0$.

# LECTURE 2

# SEQUENCES

## 6 Limits of Sequences

**Definition 6.1** (Sequence)**.** A **sequence** is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ where $m$ is usually 1 or 0.

One may wonder why do we care about sequence, and the answer is that sequences are useful for *approximation*.

**Definition 6.2.** A sequence $\{s_n\}$ of real numbers is said to **converge** to the real number $s$ if $\forall \epsilon > 0$, $\exists N > 0$ such that for all positive integers $n > N$, we have

$$|s_n - s| < \epsilon.$$

If $\{s_n\}$ converges to $s$, we write $\lim_{n\to\infty} s_n = s$, or simply $s_n \to s$, where $s$ is called the **limit** of the sequence. A sequence that does not converge to some real number is said to **diverge**.

## 7 Proofs of Limits

**Example 7.1.** Prove $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

*Scratch.* For any $\epsilon > 0$, we want

$$\left|\frac{1}{n^2} - 0\right| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff \frac{1}{\epsilon} < n^2 \iff \frac{1}{\sqrt{\epsilon}} < n.$$

Thus, we can just take $N = \frac{1}{\sqrt{\epsilon}}$. $\square$

*Proof.* Let $\epsilon > 0$ and $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $\left|\frac{1}{n^2} - 0\right| < \epsilon$. This proves our claim. $\square$

**Example 7.2.** Prove $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

*Scratch.* $\forall \epsilon > 0$, we need $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon$, which implies that

$$\left|\frac{21n + 7 - 21n + 12}{7(4n-4)}\right| < \epsilon \implies \left|\frac{19}{7(7n-4)}\right| < \epsilon.$$

Since $7n - 4 > 0$, we can remove the absolute value sign and have

$$\frac{19}{7\epsilon} < 7n - 4 \implies \frac{19}{49\epsilon} + \frac{4}{7} < n.$$

Thus, we have $N = \frac{19}{49\epsilon} + \frac{4}{7}$. □

*Proof.* Let $\epsilon > 0$ and let $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Then $n > N$ implies $n > \frac{19}{49\epsilon} + \frac{4}{7}$, hence $7n > \frac{19}{7\epsilon} + 4$, which gives us $\frac{19}{7(7n-4)} < \epsilon$, and thus $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon$. Then we are done. □

**Example 7.3.** Prove $\lim_{n\to\infty} 1 + \frac{1}{n}(-1)^n = 1$.

*Scratch.* $\forall \epsilon > 0$, we want $n$ large enough, such that

$$|a_n - 1| < \epsilon \iff \left|1 + \frac{1}{n}(-1)^n - 1\right| < \epsilon \iff \left|\frac{1}{n}(-1)^n\right| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

Just take $\alpha = \frac{1}{\epsilon}$, then $n > N \to |a_n - 1| < \epsilon$ □

# 8 Limit Theorems for Sequences

**Definition 8.1** (Bounded)**.** A sequence $\{s_n\}$ of real numbers is said to be **bounded** if the set $\{s_n \mid n \in \mathbb{N}\}$ is a bounded set, i.e., if there exists a constant $M$ such that $|s_n| \leq M$ for all $n$.

**Theorem 8.2.** Convergent sequences are bounded.

*Proof.* Let $\{s_n\}$ be a convergent sequence and let $s = \lim_{n\to\infty} s_n$. Let $\epsilon > 0$ be fixed. Then by convergence of the sequence, there exists an number $N \in \mathbb{N}$ such that

$$n > N \implies |s_n - s| < \epsilon.$$

By the triangle inequality we see that $n > N$ implies $|s_n| < |s| + \epsilon$. Define $M = \max\{|s| + \epsilon, |s_1|, \ldots, |s_N|\}$. Then $|s_n| \leq M$ for all $n \in \mathbb{N}$, so $\{s_n\}$ is a bounded sequence. □

**Theorem 8.3.** Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ such that $s_n \to s$ and $t_n \to t$. Let $k \in \mathbb{R}$ be a constant. Then

(i) $ks_n \to ks$.

(ii) $(s_n + t_n) \to s + t$.

(iii) $s_n t_n \to st$.

(iv) If $s_n \neq 0$ for all $n$, and if $s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$.

(v) If $s_n \neq 0$ and $s \neq 0$ for all $n$, then $\frac{t_n}{s_n} \to \frac{t}{s}$.

*Proof of (i).* Since the case where $k=0$ is trivial, we assume $k \neq 0$. Let $\epsilon > 0$ and we want to show that $|ks_n - ks| < \epsilon$ for large $n$. Since $\lim_{n\to\infty} = s$, there exists $N$ such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{|k|}.$$

Then

$$n > N \implies |ks_n - ks| < \epsilon.$$

□

*Proof of (ii).* Let $\epsilon > 0$. We need to show

$$|s_n + t_n - (s+t)| < \epsilon \quad \text{for large } n.$$

Using triangle inequality, we have $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t|$. Since $s_n \to s$, there exists $N_1$ such that

$$n > N_1 \implies |s_n - s| < \frac{\epsilon}{2}.$$

Similarly, there exists $N_2$ such that

$$n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then clearly

$$n > N \implies |s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

*Proof of (iii).* We use the identity

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s).$$

Given $\epsilon > 0$, there are integers $N_1, N_2$ such that

$$\begin{aligned} n > N_1 &\implies |s_n - s| < \sqrt{\epsilon} \\ n > N_2 &\implies |t_n - t| < \sqrt{\epsilon} \end{aligned}$$

If we take $N = \max\{N_1, N_2\}, n \geq N$ implies

$$|(s_n - s)(t_n - t)| < \epsilon$$

which implies that

$$\lim_{n\to\infty} (s_n - s)(t_n - t) = 0.$$

Applying (i) and (ii), we get

$$\lim_{n\to\infty} (s_n t_n - st) = 0.$$

□

*Proof of (iv).* Choosing $m$ such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \frac{1}{2}|s| \quad (n \geq m).$$

Given $\epsilon > 0$, there is an integer $N > m$ such that $n > N$ implies

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon.$$

Hence, for $n \geq N$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \epsilon.$$

□

*Proof of (v).* Using (iv), we have $\frac{1}{s_n} \to \frac{1}{s}$, and by (iii), we get

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} \frac{1}{s_n} \cdot t_n = \frac{1}{s} \cdot t = \frac{t}{s}.$$

□

**Theorem 8.4.**

(i) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(ii) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.

(iii) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(iv) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for $a > 0$.

*Proof of (i).* Let $\epsilon > 0$ and let $N = \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$. Then $n > N$ implies $n^p > \frac{1}{\epsilon}$ and thus $\epsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows $n > N$ implies $\left|\frac{1}{n^p} - 0\right| < \epsilon$. □

*Proof of (ii).* The case for $a = 0$ is trivial. Suppose that $a \neq 0$. Since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$ where $b > 0$. By the binomial theorem, we have $(1+b)^n \geq 1 + nb > nb$, then

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Consider $\epsilon > 0$ and let $N = \frac{1}{\epsilon b}$. Then $n > N$ implies $n > \frac{1}{\epsilon b}$ and thus $|a^n - 0| < \frac{1}{nb} < \epsilon$. □

*Proof of (iii).* Let $s_n = n^{\frac{1}{n}} - 1$. Then $s_n \geq 0$ and by the binomial theorem,

$$n = (1 + s_n)^n \geq \frac{n(n-1)}{2}s_n^2.$$

Hence,

$$0 \leq s_n \leq \sqrt{\frac{2}{n-1}} \implies s_n \to 0.$$

□

*Proof of (iv).* Suppose $a > 1$. Let $s_n = a^{\frac{1}{n}} - 1$. Then $s_n > 0$, and by the binomial theorem,

$$1 + ns_n \leq (1 + s_n)^n = a,$$

so that

$$0 < s_n \leq \frac{p-1}{n}.$$

Hence, $s_n \to 0$. The case for $a = 1$ is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals. □

### 8.1 Upper and lower limits

**Definition 8.5.** Let $\{s_n\}$ be a sequence of real numbers with the property that for every real $M$ there is an integer $N$ such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real $M$ there is an integer $N$ such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

## 9 Monotone Sequences and Cauchy Sequences

**Definition 9.1** (Monotone sequence)**.** A sequence $\{s_n\}$ of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all $n$, and $\{s_n\}$ is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all $n$. If $\{s_n\}$ is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a **monotone sequence** or a **monotonic sequence**.

**Theorem 9.2.** All bounded monotone sequences converge.

*Proof.* Let $\{s_n\}$ be a bounded increasing sequence, Let $= \{s \mid n \in \mathbb{N}\}$ and let $u = \sup S$, Since $S$ is bounded, $u$ represents a real number. We show $s_n \to u$. Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for $S$, there exists $N$ such that $s_N > u - \epsilon$.Since $\{s_n\}$ is increasing, $s_N \leq s_n$ for all $n \geq N$. Of course $s_n \leq u$ for all $n$, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. Hence $s_n \to u$. The proof for bounded decreasing sequences is left as an exercise. □

**Theorem 9.3.**

(i) If $\{s_n\}$ is an unbounded increasing sequence, then $s_n \to +\infty$.

(ii) If $\{s_n\}$ is an unbounded decreasing sequence, then $s_n \to -\infty$.

**Corollary 9.4.** If $\{s_n\}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.

*Proof.* Simply apply the previous two theorems. □

**Definition 9.5.** Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$$

**Theorem 9.6.** Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

(i) If $\lim s_n$ is defined (real, or $\pm\infty$), then

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and

$$\lim s_n = \liminf s_n = \limsup s_n.$$

**Definition 9.7** (Cauchy sequence)**.** A sequeunce $\{s_n\}$ of real numbers i called a **Cauchy sequeunce** if for each $\epsilon > 0$ there exists a number $N$ such that

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

**Lemma 9.8.** Convergent sequences are Cauchy sequences.

*Proof.* Suppose $\lim s_n = s$. Since the terms $s_n$ are close to $s$ for large $n$, they must also be close to each other; indeed

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|.$$

Let $\epsilon > 0$. Then there exists $N$ such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{2}.$$

Clearly we can also write

$$m > N \implies |s_m - s| < \frac{\epsilon}{2},$$

so

$$m, n > N \implies |s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{s_n\}$ is a Cauchy sequence. $\square$

**Lemma 9.9.** Cauchy sequences are bounded.

*Proof.* Let $\epsilon = 1$. By definition, we have $N$ in $\mathbb{N}$ such that

$$m, n > N \implies |s_n - s_m| < 1.$$

In particular, $|s_n - s_{N+1}| < 1$ for $n > N$, so $|s_n| < |s_{N+1}| + 1$ for $n > N$. If $M = \max\{|s_{N+1} + 1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

**Theorem 9.10.** A sequence is a convergent sequence if and only if it is a Cauchy sequence.

*Proof.* Earlier we have shown one direction in a lemma. We now only need to show that Cauchy sequences are convergent sequences. Consider a Cauchy sequence $\{s_n\}$ and it is bounded by previous lemma. We now need to show that

$$\liminf s_n = \limsup s_n.$$

Let $\epsilon > 0$. Since $\{s_n\}$ is a Cauchy sequence, there exists $N$ so that

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n \mid n > N\}$, so $v_N = \sup\{s_n \mid n > N\} \leq s_m + \epsilon$ for $m > N$. This, in turn, shows $v_N - \epsilon$ is a lower bound for $\{s_m \mid m > N\}$, so $v_N - \epsilon \leq \inf\{s_m \mid m > N\} = u_N$. Thus

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\limsup s_n \leq \liminf s_n$. Since $\limsup s_n \geq \liminf s_n$ always holds, we are done. $\square$

## 9.1 Subsequences

**Definition 9.11** (Subsequence)**.** Suppose $\{s_n\}_{n\in\mathbb{N}}$ is a sequence. A **subsequence** of this sequence is a sequence of the form $\{t_k\}_{k\in\mathbb{N}}$

**Theorem 9.12.** Every sequence $\{s_n\}$ has a monotonic subsequence.

*Proof.* We say that the $n$-th term is *dominant* if $s_m < s_n$ for all $m > n$. There are two cases:

*Case 1:* Suppose there are infinitely many dominant terms, and let $\{s_{nk}\}$ be any subsequence consisting solely of dominant terms. Then $s_{nk+1} < s_{n_k}$ for all $k$, so $\{s_{n_k}\}$ is a decreasing sequence.

*Case 2:* Suppose there are only finitely many dominant terms. Select $n_1$ so that $s_{n_1}$ is beyond all the dominant terms of the sequence. Then given $N \geq n_1$, there exists $m > N$ such that $s_m \geq s_N$. □

**Theorem 9.13** (Bolzano-Weierstrass Theorem)**.** Every bounded sequence has a convergent subsequence.

*Proof.* Using previous theorem, we have a monotonic subsequence. Since monotonic bounded sequence are convergent, we are done. □

*Alternative proof.* Suppose that $\{s_n\}$ is bounded. Then there exists $M > 0$ such that $|s_n| < M$ for all $n \in \mathbb{N}$. Let $A_1 = \{n \in \mathbb{N} \mid s_n \in [0, M]\}, B_1 = \{n \in \mathbb{N} \mid s_n \in [-M, 0]\}$. Since $A_1 \cup B_1 = \mathbb{N}$ is an infinite set, hence at least one of $A_1, B_1$ is infinite. WLOG assume that $A_1$ is infinite. We then cut $[0, M]$ into two halves, and repeat the same procedure, then at least one of $[0, M/2]$ and $[M/2, M]$ contains infinitely many points of the sequence. Then, we get a nested sequence of closed intervals,

$$I_1 \supset I_2 \supset \cdots, \qquad |I_{n+1}| = \frac{1}{2}|I_n|.$$

One can pick subsequence $\{s_{nk}\}$ such that for all $k$, $s_{nk}$ is in $I_k$, and $n_{k+1} > n_k$. Then this subsequence is Cauchy, hence is convergent. □

**Definition 9.14** (Subsequential limit)**.** A **subsequential limit** is any real number or symbol $\pm\infty$ that is the limit of some subsequence of $\{s_n\}$.

**Example 9.15.** Consider $\{s_n\}$ where $s_n = n^2(-1)^n$. The subsequence of even terms diverges to $+\infty$ where as that of odd terms diverges to $-\infty$. Hence, the set $\{-\infty, +\infty\}$ is the set of subsequential limits of $\{s_n\}$.

**Example 9.16.** Consider $\{r_n\}$, a list of all rational numbers. Every real number is a subsequential limit of $\{r_n\}$ as well as $\pm\infty$. Thus, $\mathbb{R} \cup \{-\infty, +\infty\}$ is the set of subsequential limits of $\{r_n\}$.

**Theorem 9.17.** Let $\{s_n\}$ be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$ and there exists a monotonic subsequence whose limit is $\liminf s_n$.

*Proof.* If $\{s_n\}$ is not bounded above, then a monotonic subsequence of $\{s_n\}$ has limit $\limsup s_n = +\infty$. Similarly, if $\{s_n\}$ is not bounded below, a monotonic subsuquence has limit $\liminf s_n$.
Consider the case that it is bounded above. Let $t = \limsup s_n$, and consider $\epsilon > 0$. There exists $N_0$ so that for $N \geq N_0$,

$$\sup\{s_n \mid n > N\} < t + \epsilon.$$

In particular, $s_n < t + \epsilon$ for all $n > N_0$. We now claim

$$\{n \in \mathbb{N} \mid |s_n - t| < \epsilon\} \text{ is infinite.}$$

Otherwise, there exists $N_1 > N_0$ □

**Theorem 9.18.** Let $\{s_n\}$ be any sequence in $\mathbb{R}$, and let $S$ denote the set of subsequential limits of $\{s_n\}$.

(i) $S$ is non-empty.

(ii) $\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.

(iii) $\lim s_n$ exists if and only if $S$ has exactly one element, namely $\lim s_n$.

*Proof.* (i) is an immediate consequence of the previous theorem.
To prove (ii), consider any limit $t$ of a subsequence $\{s_{nk}\}$ of $\{s_n\}$. By the □

## 10 lim sup's and lim inf's

Let $\{s_n\}$ be any sequence of real numbers, and let $S$ be the set of subsequential limits of $\{s_n\}$. Recall the following definition:

$$\limsup s_n = \lim_{N\to\infty} \sup s_n \mid n > N = \sup S$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf s_n \mid n > N = \inf S.$$

**Claim.**

$$\liminf s_n \leq \limsup s_n.$$

*Proof.* We know that

$$\sup{}_{n>N} s_n \geq \inf{}_{n>N} s_n.$$

Then take limit $N \to \infty$. □

**Claim.** If $\{s_{n_k}\}$ is a subsequence, then

$$\limsup s_{n_k} \leq \limsup s_n.$$

**Theorem 10.1.** If $\{s_n\} \to s > 0$ and $\{t_n\}$ is any sequence, then

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

Here we allow the conventions $s \cdot (\pm\infty) = \pm\infty$ for $s > 0$.

*Proof.* □

**Question.** If $\{s_{n_k} \cdot t_{n_k}\}$ converges, does that imply $\{t_{n_k}\}$ converges?

**Answer.** Yes. (Why?)

**Theorem 10.2.** Let $\{s_n\}$ be any sequence of nonzero real numbers. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|.$$

**Question.** If $\{s_n\}$ is a bounded positive sequence, is $\frac{s_{n+1}}{s_n}$ a bounded sequence?

**Answer.** No. Consider $0 < a, b < 1$, and take $a = \frac{1}{2}$ and $b = \frac{1}{n}$, then $\frac{a}{b} = \frac{n}{2}$.

**Claim.** If $\{s_n\}$ is bounded and monotone, then the ratio $\frac{s_{n+1}}{s_n}$ eventually converges to 1.

*Proof.* Since $\{s_n\}$ is bounded and monotone, it must converge to some limit $s$. Then

$$\lim \frac{s_{n+1}}{s_n} = \frac{\lim s_{n+1}}{s_n} = \frac{s}{s} = 1.$$

□

**Question.** Is it possible to have $s_n$ to be bounded, but $\frac{s_{n+1}}{s_n}$ unbounded?

**Answer.** Yes. Consider

$$s_n = \begin{cases} 1 & n \text{ is even;} \\ \frac{1}{n} & n \text{ is odd.} \end{cases}$$

**Question.** If $\{s_n\}$ is positive and bounded, is it possible that $\frac{s_{n+1}}{s_n} \to 0$?

**Answer.** Yes. Consider $s_n = \frac{1}{n!}$. Then

$$\lim \frac{s_{n+1}}{s_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

## 11 Metric Spaces and Topology

**Definition 11.1** (Metric Space)**.** A **metric space** $S$ is a set $S$ with a metric on it. We denote it by $(S, d)$ where $d$ is a *distance function* or a *metric* on $S$ defined for all pairs $(x, y)$ of elements from $S$ satisfying

(i) $\forall x \in S, d(x, x) = 0$ and $\forall x, y \in S, x \neq y, d(x, y) > 0$.

(ii) $\forall x, y \in S, d(x, y) = d(y, x)$.

(iii) $\forall x, y, z \in S, d(x, z) \leq d(x, y) + d(y, z)$.

**Definition 11.2** (Convergence of Metric Space)**.** A sequence $\{s_n\}$ in a metric space $(S, d)$ **converges** to $s \in S$ if $\lim_{n\to\infty} d(s_n, s) = 0$. The sequence is a *Cauchy sequence* if for each $\epsilon > 0$, there exists an $N$ such that

$$m, n > N \implies d(s_m, s_n) < \epsilon.$$

**Lemma 11.3.** If $\{s_n\}$ converges to $s$, then $s_n$ is Cauchy.

*Proof.* For any $\epsilon > 0$, there exists $N > 0$ such that for all $n > N$

$$d(s_n, s) < \frac{\epsilon}{2}.$$

Thus, for all $n, m > N$, we have

$$\begin{aligned} d(s_n, s_m) &\leq d(s_n, s) + d(s_m, s) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

**Definition 11.4** (Completeness)**.** The metric space $(S, d)$ is **complete** if every Cauchy sequence in $S$ converges to some element in $S$.

**Example 11.5** (Non-complete Metric Spaces)**.**

1. $S = \mathbb{R}\backslash\{0\}$.
2. $S = \mathbb{Q}$.

**Lemma 11.6.** A sequence $\{\boldsymbol{x}^{(n)}\} \in \mathbb{R}^k$ converges iff for each $j = 1, 2, \ldots, k$, the sequence $(x_j^n)$ converges in $\mathbb{R}$. A sequence $\{\boldsymbol{x}^{(n)}\}$ in $\mathbb{R}^k$ is a Cauchy sequence iff each sequence $\{x_j^{(n)}\}$ is a Cauchy sequence in $\mathbb{R}$.

**Theorem 11.7.** Euclidean $k$-space $\mathbb{R}^k$ is complete.

**Theorem 11.8** (Bolzano-Weierstrass Theorem)**.** Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

**Definition 11.9** (Interior)**.** Let $(S, d)$ be a metric space. Let $E \subseteq S$. An element $s_0 \in E$ is **interior** to $E$ if for some $r > 0$ we have

$$\{s \in S : d(s, s_0) < r\} \subseteq E.$$

We write $E^\circ$ for the set of points in $E$ that are interior to $E$.

**Definition 11.10** (Open)**.** The set $E$ is **open** in $S$ if every point in $E$ is interior to $E$, i.e., if $E = E^\circ$.

**Definition 11.11** (Closed)**.** Let $(S, d)$ be a metric space. A subset $E \subseteq S$ is **closed** if its complement $S\backslash E$ is an open set. In other words, $E$ is closed if $E = S\backslash U$ where $U$ is an open set.

**Definition 11.12** (Neighborhood)**.** A **neighborhood** of $p$ is a set $N_r(p)$ consisting of all $q$ such that $d(p, q) < r$, for some $r > 0$, where $r$ is the *radius* of $N_r(p)$.

**Definition 11.13** (Limit Point)**.** A point $p$ is a **limit point** of the set $E$ if every neighborhood of $p$ contains a point $q \neq p$ such that $q \in E$.

**Definition 11.14** (Closure)**.** The **closure** $E^-$ of a set $E$ is the intersection of all closed sets containing $E$. Equivalently, the closure is the union of $E$ and its boundary, which is defined below.

**Definition 11.15** (Boundary Points)**.** The **boundary** of $E$ is the set $E^-\backslash E^\circ$; points in this set are called **boundary points** of $E$.

**Proposition 11.16.** Let $E$ be a subset of a metric space $(S, d)$.

(i) The set $E$ is closed iff $E = E^-$.

(ii) The set $E$ is closed iff it contains the limit of every convergent sequence of points in $E$.

(iii) An element is in $E^-$ iff it is the limit of some sequence of points in $E$.

(iv) A point is in the boundary of $E$ iff it belongs to the closure of both $E$ and its complement.

**Theorem 11.17.** Let $\{F_n\}$ be a decreasing sequence ($F_1 \supseteq F_2 \supseteq \cdots$) of closed bounded nonempty sets in $\mathbb{R}^k$. Then $F = \bigcap_{n=1}^\infty F_n$ is also closed, bounded and nonempty.

## 12 Compact Sets (Rudin)

**Definition 12.1** (Open Cover)**.** Let $E \subset S$. An **open cover** of $E$ is a collection $\{G_\alpha\}$of open subsets of $S$ such that $E \subset \bigcup_\alpha G_\alpha$.

**Definition 12.2** (Compact)**.** Let $K \subset S$. $K$ is **compact** if every open cover of $K$ contains a *finite* subcover. More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of $K$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

**Remark.** Every finite set is compact. $\mathbb{R}$ is not compact.

**Theorem 12.3** (Heine-Borel Theorem)**.** A subset $E$ of $\mathbb{R}^k$ is compact iff it is closed and bounded.

*Proof.* Suppose $E \subset S$ is compact. Then pick some point $p \in S$ and consider $\{B_n(p) \mid n \in \mathbb{N}\}$, which covers $S$ and thus covers $E$ as well:

$$E \subset S = \bigcup_{n \in \mathbb{N}} B_n(p).$$

Since $E$ is compact, there is a finite subcover such that

$$E \subset \bigcup_{i=1}^{M} B_{n_i}(p).$$

We can order the indices such that $n_1 < n_2 < \cdots, n_M$ then

$$E \subset B_{n_M}(p),$$

which implies that $E$ is bounded. In particular, for any points $x, y \in E$,

$$d(x, y) \leq d(x, p) + d(y, p) \leq 2 \cdot n_M.$$

The remaining of the proof is left as an exercise. □

**Theorem 12.4.** Every $k$-cell $F$ in $\mathbb{R}^k$ is compact.

## 13 Series

In this section we are interested in convergence of series, thus we use $\sum a_n$ to denote $\sum_{i=1}^{\infty} a_i$.

**Definition 13.1** (Convergence/Divergence)**.** The $n$-th partial sum of a sequence $\{a_n\}$ is defined as $s_n = \sum_{i=1}^{n} a_i$. We say that $\sum a_n$ **converges** iff the sequence of partial sums $\{s_n\}$ converges to a real number. Otherwise, we say that the series **diverges**.

**Definition 13.2** (Absolute Convergence)**.** The series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

**Definition 13.3** (Geometric Series)**.** A series of the form $\sum_{n=0}^{\infty} ar^n$ for constants $a$ and $r$ is a **geometric series**. For $r \neq 1$,

$$\sum_{k=0}^{n} ar^k = \frac{a(1-r^{n+1})}{1-r}.$$

For $|r| < 1$, since $\lim_{n\to\infty} r^{n+1} = 0$, using the formula above gives

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

If $a \neq 0$ and $|r| \geq 1$, then the sequence $\{ar^n\}$ does not converge to 0, so the series diverges.

**Definition 13.4** (Cauchy Criterion)**.** A series $\sum a_n$ satisfies the **Cauchy criterion** if its sequence $\{s_n\}$ of partial sums is a Cauchy sequence, i.e., for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq m > N \implies \left|\sum_{i=m}^{n} a_i\right| < \epsilon.$$

**Theorem 13.5.** A series converges iff it satisfies the Cauchy criterion.

**Corollary 13.6.** If a series $\sum a_n$ converges, then $\lim a_n = 0$.

*Proof.* By Cauchy criterion, take $n = m$. Then for $\epsilon > 0$, there exists $N$ such that $n > N$ implies $|a_n| < \epsilon$. Thus, $\lim a_n = 0$. $\square$

**Remark.** The converse is not true. Consider $\sum \frac{1}{n} = +\infty$.

**Theorem 13.7** (Comparison Test)**.** Let $\sum a_n$ be a series where $a_n \geq 0$ for all $n$.

(i) If $\sum a_n$ converges and $|b_n| \leq a_n$ for all $n$, then $\sum b_n$ converges.

(ii) If $\sum a_n = +\infty$ and $b_n \geq a_n$ for all $n$, then $\sum b_n = +\infty$.

*Proof of (i).* For $n \geq m$, by the triangle inequality, we have

$$\left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k.$$

Since $\sum a_n$ converges, it satisfies the Cauchy criterion. It follows from the above that $\sum b_n$ also satisfies the Cauchy criterion, and so $\sum b_n$ converges. $\square$

*Proof of (ii).* Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums for $\sum a_n$ and $\sum b_n$ respectively. Since $b_n \geq a_n$ for all $n$, we have $t_n \geq s_n$ for all $n$. Since $\lim s_n = +\infty$, $\lim t_n = +\infty$, and so $\sum b_n = +\infty$. $\square$

**Theorem 13.8** (Ratio Test)**.** A series $\sum a_n$ of nonzero terms

1. converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$;

2. diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$.

3. Otherwise $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$ and the test gives no information.

**Theorem 13.9** (Root Test)**.** Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{\frac{1}{n}}$. The series $\sum a_n$

(i) converges absolutely if $\alpha < 1$;

(ii) diverges if $\alpha > 1$.

(iii) Otherwise, the test gives no information if $\alpha = 1$.

# 14 Alternating Series

**Theorem 14.1.** $\sum \frac{1}{n^p}$ converges iff $p > 1$.

*Proof.* If $p > 1$, then

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq 1 + \int_1^n \frac{1}{x^p} dx = 1 + \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < 1 + \frac{1}{p-1} = \frac{p}{p-1}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \leq \frac{p}{p-1} < +\infty.$$

If $0 < p \leq 1$, then $\frac{1}{n} \leq \frac{1}{n^p}$ for all $n$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n^p}$ diverges as well by the Comparison Test. $\square$

**Theorem 14.2** (Integral Tests)**.**

**Theorem 14.3** (Alternating Series Theorem)**.** If $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$ and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums $s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$ satisfy $|s - s_n| \leq a_n$ for all $n$.

*Proof.* Define $s_n = \sum_{j=1}^{n} a_j$. The subsequence $\{s_{2n}\}$ is increasing because $s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$, Similarly, the subsequence $\{s_{2n-1}\}$ is decreasing. $\square$