CS70 Random Variables

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Overview

Definitions

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The **probability mass function**, or **PMF**, of a discrete random variable X is a function mapping X's values to their associated probabilities. It is the function $p :\to [0, 1]$ defined by

$$p_X(x) := P(X = x).$$

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$$\sigma := \sqrt{\mathrm{Var}(X)}.$$

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where the second line follows from independence. □

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- Will be very useful soon for computing expectations.

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This makes use of an important property called the **memoryless property**, which will be covered later in the class.

Poisson Distribution

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  Recall the **Taylor series expansion** from calculus:

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  Recall the **Taylor series expansion** from calculus:
$$e^x = \sum_{i=1}^{\infty} \frac{x^i}{i!}.$$

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Poisson Distribution

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$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=0}^{\infty} i \cdot P(X = i) \\ &= \sum_{i=1}^{\infty} \frac{\lambda^i e^{-\lambda}}{i!} \\ &= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}\end{aligned}$$

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&= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda} e^{\lambda} \qquad \left(e^{\lambda} = \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} \text{ with } j = i - 1\right)
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- **Variance:**

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&= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda} e^{\lambda} && \left(e^{\lambda} = \sum_{j=1}^{\infty} \frac{\lambda^j}{j!} \text{ with } j = i - 1\right) \\
&= \lambda.
\end{aligned}$$

- **Variance:**

$$\mathrm{Var}(X) = \lambda.$$

Problem Time!