CS70 Random Variables II

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Overview

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and $-1 \le \rho(X, Y) \le 1$.

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- **Bilinearity**:

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \,\text{Cov}(X_i, Y_j).$$

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Problem Time!