CS70: Discrete Mathematics and Probability Theory

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Contents

1 Mathematical Notations

1.1 Sets

- $\{\}$: empty set.
- $A \subset B$: A is a **proper subset** of B, i.e. A is strictly contained in B.
- $A \subseteq B$: A is a **subset** of B, i.e. A is strictly contained in B.
- $|A|$: **cardinality** of A, or the size of A.
- $A \cup B$: the **union** of A and B.
- $A \cap B$: the **intersection** of A and B.
- $A \backslash B$: **relative complement,** elements in A but not in B.
- $A \times B$: **Cartesian product**, $\{(a, b) \mid a \in A, b \in B\}$.
- $\mathscr{P}(S)$: the set of all subsets of S, also called **power set** of S.

1.2 Commonly used sets

- $\mathbb{N}$: the set of all natural numbers: $\{0, 1, 2, 3, \ldots\}$.
- $\mathbb{Z}$: the set of all integer numbers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- $\mathbb{Q}$: the set of all rational numbers: $\left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$.
- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{C}$: the set of all complex numbers.

1.3 Universal and existential quantifiers

- $\forall$: for all.
- $\exists$: there exists.

2 Proofs

2.1 Techniques

- **Direct Proof:** show $P \implies Q$ where P is a given fact and Q is the claim.
- **Contrapositive:** prove $\neg Q \implies \neg P$ if need to show $P \Rightarrow Q$,
- **Contradiction:** to prove claim P, assume $\neg P$ is true and arrive at $R \wedge \neg R$, which is a contradiction. Hence P is true.
- **By cases:** prove P in separate cases, if all cases are true, then P must be true.
- **Induction:** consists of three main components
  1. **Base case:** show that $P(0)$ is true.
  2. **Induction Hypothesis:** Assume $P(k)$ is true for any $k \geq 0$.
  3. **Inductive Step:** prove that $P(k+1)$ is true by showing $P(k) \implies P(k+1)$.

3 Graph Theory

3.1 Basic Terminology

Definition 1 (Graph)**.** A graph G is defined by a set of vertices V and a set of edges E. We write $G = (V, E)$.

- **Directed graph:** with directed edges, i.e., $(u, v) \neq (v, u)$.
- **Undirected graph:** with undirected edges, i.e., $(u, v) = (v, u)$.

Definition 2 (Degree)**.** The degree of a vertex v is defined by the number of edges that are incident to v. A vertex with degree 0 is an *isolated* vertex.

Definition 3 (In-degree)**.** The in-degree of a vertex v is the number of ingoing edges to v.

Definition 4 (Out-degree)**.** The out-degree of a vertex v is the number of outgoing edges from v.

Theorem 5 (Handshaking)**.** Let $G = (V, E)$ be an undirected graph with m edges. Then

$$\sum_{v \in V} \deg(v) = 2m$$

(This applies even if multiple edges and loops are present.)

Theorem 6. An undirected graph has an even number of vertices of odd degree.

Proof. Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)$$

Because $\deg(v)$ is even for $v \in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of the two terms on the right-hand side of the last equality is even, because this sum is $2m$. Hence, the second term in the sum is also even. Because all the terms in this sum are odd, there must be an even number of such terms. Thus, there are an even number of vertices of odd degree. □

Theorem 7 (Euler's Theorem)**.** An undirected graph $G = (V, E)$ has an Eulerian tour iff G is even degree, and connected (except possibly for isolated vertices).

Proof. See notes. □

Definition 8 (Complete graph)**.** A graph G is called **complete** if each pair of its vertices is connected by an edge. We use K_n to denote a complete graph on n vertices.

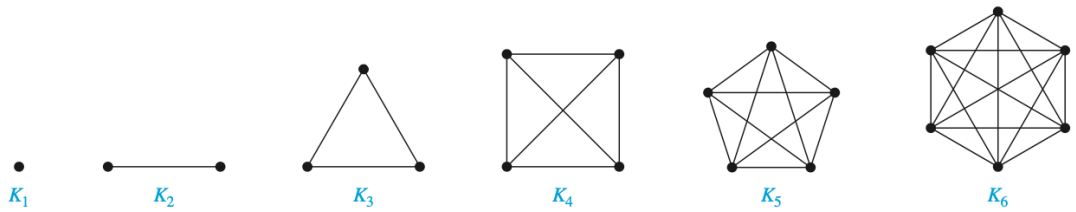


Figure 1: Examples of complete graphs.

3.2 Bipartite Graphs

Definition 9 (Bipartite)**.** A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). We use $K_{n,m}$ to denote a complete bipartite graph partitioned into n and m vertices.

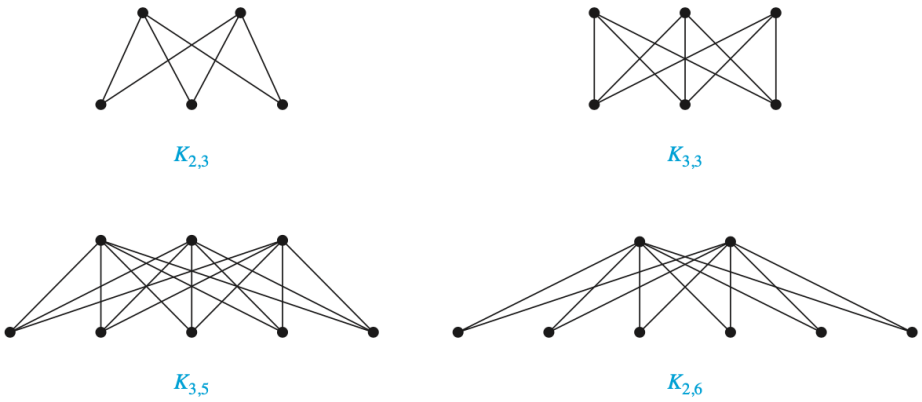


Figure 2: Examples of complete bipartite graphs.

Theorem 10. A simple graph is **bipartite** if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Proof. First, suppose that $G = (V, E)$ is a bipartite simple graph. Then $V = V_1 \cup V_2$, where V_1 and V_2 are disjoint sets and every edge in E connects a vertex in V_1 and a vertex in V_2. If we assign one color to each vertex in V_1 and a second color to each vertex in V_2, then no two adjacent vertices are assigned the same color.

Now suppose that it is possible to assign colors to the vertices of the graph using just two colors so that no two adjacent vertices are assigned the same color. Let V_1 be the set of vertices assigned one color and V_2 be the set of vertices assigned the other color. Then, V_1 and V_2 are disjoint and $V = V_1 \cup V_2$. Furthermore, every edge connects a vertex in V_1 and a vertex in V_2 because no two adjacent vertices are either both in V_1 or both in V_2. Consequently, G is bipartite. □

3.3 Connectivity

Definition 11 (Connected)**.** A graph is said to be **connected** if there is a path between any two distinct vertices.

- A **connected/disconnected** graph always consists collection of **connected components**, i.e., sets $V_1, ..., V_k$ of vertices, such that all vertices in a set V_i are connected.

3.4 Planarity

A graph is called **planar** if it can be drawn in the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint).

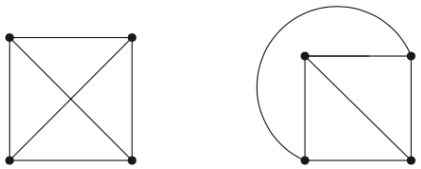


Figure 3: K_4 is planar because it can be drawn without crossing edges.

3.4.1 Euler's Formula

Theorem 12 (Euler's formula)**.** For every **connected planar** graph with v vertices, f faces, and e edges,

$$v + f = e + 2.$$

Proof. By induction on e. It clearly holds when $e = 0$, and $v = f = 1$. Now take any connected planar graph. We consider two cases:

1. If it is a tree, then $f = 1$ (drawing a tree on the plane does not subdivide the plane), and $e = v - 1$ (check homework).
2. If it is not a tree, find a cycle and delete any edge of the cycle. This amounts to reducing both e and f by one. By induction the formula is true in the smaller graph, and so it must be true in the original one.

□

Question. What happens when the graph is disconnected? How does the number of connected components enter the formula?

- Take a planar graph with f faces, and consider one face. It has a number of sides, that is, edges that bound it clockwise.
- Note that an edge may be counted twice if it has the same face on both sides (such edges are called **bridges**).
- Let s_i be the number of sides of face i. If we add the s_i's we are going to get $2e$, because each edge is counted twice.

- We conclude that, in any planar graph,

$$\sum_{i=1}^{f} s_i = 2e$$

- Notice that, since we don't allow parallel edges between the same two nodes, and if we assume that there are at least two edges (so there are at least three vertices), every face has at least three sides, or $s_i \geq 3$ for all i.
- It follows that $3f \leq 2e$.
- Solving for f and plugging into Euler's formula we get

$$e \leq 3v - 6.$$

We have just proved the following corollary:

Corollary 13. If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

Remark. This is an important fact.

- It tells us that planar graphs are sparse, they cannot have too many edges.
- It also tells us that K_5 is not planar.
- $K_{3,3}$ has $v = 6, e = 9$ so it satisfies the Euler's formula. However, using the equation above gives us $4f \leq 2e$, and solving for f and plugging into Euler's formula, $e \leq 2v - 4$, which shows that $K_{3,3}$ is non-planar.

So, we have established that K_5 and $K_{3,3}$ are both non-planar. In some sense, these are the only non-planar graphs. This is made precise in the following famous result, due to the Polish mathematician Kuratowski (this is what "K" stands for).

Theorem 14 (Kuratowski's Theorem)**.** A graph is **non-planar** if and only if it contains K_5 or $K_{3,3}$.

Proof. See notes. □

3.5 Trees

If G is a tree, then

- G is connected and contains no cycles.
- G is connected and has $n-1$ edges (where $n = |V|$).
- G is connected, and the removal of any single edge disconnects G.
- G has no cycles, and the addition of any single edge creates a cycle.

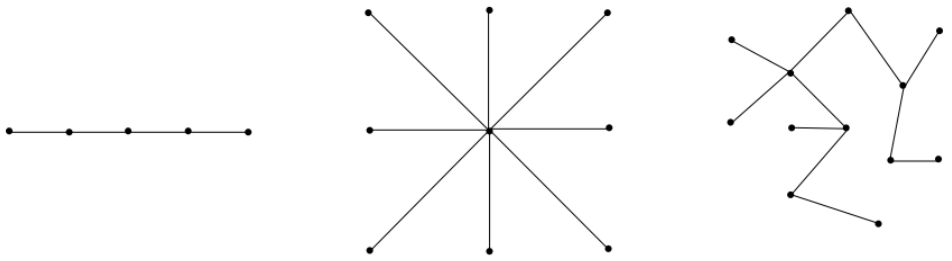


Figure 4: Examples of trees.

Theorem 15. G is connected and contains no cycles is equivalent to G is connected and has $n-1$ edges.

Proof. See notes. □

3.6 Hypercubes

- The vertex set of the n-dimensional hypercube $G = (V, E)$ is given by $V = \{0,1\}^n$, where recall $\{0,1\}^n$ denotes the set of all n-bit strings.
- Each vertex is labeled by a unique n-bit string, such as $00110 \cdots 0100$.
- Two vertices x and y are connected by edge $\{x, y\}$ if and only if x and y differ in exactly one bit position.
- For example, $x = 0000$ and $y = 1000$ are neighbors, but $x = 0000$ and $y = 0011$ are not.
- More formally, $x = x_1x_2 \ldots x_n$ and $y = y_1y_2 \ldots y_n$ are neighbors if and only if there is an $i \in \{1, \ldots, n\}$ such that $x_j = y_j$ for all $j \neq i$, and $x_i \neq y_i$.
- The n-dimensional hypercube has 2^n vertices.

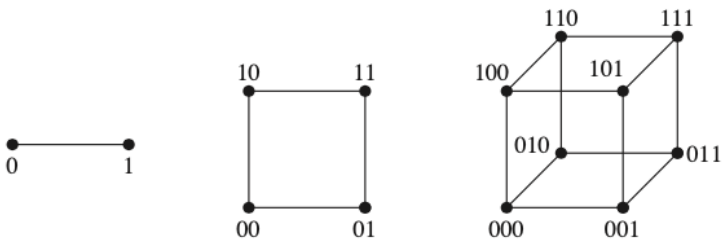


Figure 5: Examples of hypercubes.

Lemma 16. The total number of edges in an n-dimensional hypercube is $n2^{n-1}$.

Proof. The degree of each vertex is n, since n bit positions can be flipped in any $x \in \{0,1\}^n$. since each edge is counted twice, once from each endpoint, this yields a total of $n2^n/2 = n2^{n-1}$ edges. □

4 Modular Arithmetic

4.1 Congruence

Definition 17 (Congruence). x is **congruent** to y modulo m or $x \equiv y \pmod{m}$ if and only if any one of the following is true:

- $(x - y)$ is divisible by m
- x and y have the same remainder w.r.t. m
- $x = y + km$ for some integer k
- In modulo m, only the numbers $\{0, 1, 2, \ldots, m-1\}$ exist.
- Division is not well-defined.

4.2 Multiplicative Inverse

Definition 18 (Multiplicative Inverse). In the modular space, the **multiplicative inverse** of $x \bmod m$ is y if $xy \equiv 1 (\mathrm{mod} m)$.

Theorem 19 (Modular operations). $a \equiv c \bmod m$ and $b \equiv d \bmod m \implies a + b \equiv c + d \pmod{m}$ and $a \cdot b \equiv c \cdot d \pmod{m}$.

Theorem 20 (Existence of multiplicative inverse). $\gcd(x, m) = 1 \implies x$ has a multiplicative inverse modulo m and it is **unique**.

4.3 Euclid's Algorithm

Question. How do we compute gcd of two numbers x and y?

Theorem 21 (Euclid's Algorithm). Let $x \geq y > 0$. Then

$$gcd(x, y) = gcd(y, x \bmod y)$$

Example 4.1. Compute gcd(16,10):

$$\begin{aligned} \gcd(16, 10) &= \gcd(10, 6) \\ &= \gcd(6, 4) \\ &= \gcd(4, 2) \\ &= \gcd(2, 0) \\ &= 2. \end{aligned}$$

4.4 Extended Euclid's algorithm

Question. How to compute the multiplicative inverse?

- Need an algorithm that returns integers a and b such that:

$$\gcd(x, y) = ax + by.$$

Theorem 22 (Bézout's Identity)**.** For nonzero integers x and y, let d be the greatest common divisor such that $d = \gcd(x, y)$. Then, there exist integers a and b such that

$$ax + by = d.$$

- When $\gcd(x, y) = 1$, we can deduce that b is an inverse of $y \bmod x$.
- This uses back substitutions repetitively so that the final expression is in terms of x and y.

4.5 Functions

Definition 23 (Function)**.** Let A and B be nonempty sets. A **function** f from A to B is an assignment of exactly one element of B to each element of A. (vertical line test)

- To denote such a function, we write $f : A \to B$ (f maps A to B).
- A is the **domain** and B is the **co-domain**.
- Pre-image is a **subset** of domain, and the image/range is the **subset** of co-domain.
  - If $f(a) = b$, where $a \in A$ and $b \in B$, then we say that b is the image of a and a is the pre-image of b.

4.6 Bijection

Definition 24 (One-to-one)**.** A function f is said to be **one-to-one** if and only if $f(a) = f(a')$ implies that $a = a'$ for all $a, a' \in A$. A function is said to be **injective** if it is **one-to-one**.

- To show that a function is *one-to-one*, we show that $a \neq a' \implies f(a) \neq f(a')$. (Why?)

Definition 25 (Onto)**.** A function f is called **onto**, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ such that $f(a) = b$. We also say that f is **surjective** if it's onto.

- To show that a function is *onto*, choose $a = f^{-1}(b)$ and so $f(f^{-1}(b)) = b$.

Definition 26 (Bijection)**.** A function f is a **bijection** if and only if it is both *one-to-one* and *onto*. We also say that f is bijective.

- If $f : A \to B$ is a bijection, it will have an **inverse** function (a lemma from notes), and $|A| = |B|$.

4.7 Fermat's Little Theorem

Theorem 27 (Fermat's Little Theorem)**.** For any prime p and any $a \in \{1, 2, ..., p-1\}$, we have

$$a^{p-1} \equiv 1 \pmod{p}.$$

Proof. Consider $S = \{1, 2, \ldots, p-1\}$ and $S' = \{a \bmod p, 2a \bmod p, \ldots, (p-1)a \bmod p\}$. They are the same set under mod p (different order).

$$\prod_{k=1}^{p-1} k \equiv \prod_{k=1}^{p-1} ka \pmod{p}$$

$$(p-1)! \equiv a^{p-1}(p-1)! \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

□

4.8 Chinese Remainder Theorem

Theorem 28 (Chinese Remainder Theorem)**.** Let $n_1, n_2, \ldots, n_k$ be positive integers that are coprime to each other. Then, for any integers a_i, the system of simultaneous congruences

$$x \equiv a_1 \pmod{n_1}, x \equiv a_2 \pmod{n_2}, \ \ldots \ , x \equiv a_k \pmod{n_k}$$

has a unique solution

$$x = \left(\sum_{i=1}^{k} a_i b_i \right) \bmod N$$

where $N = \prod_{i=1}^{k} n_i$ and $b_i = \frac{N}{n_i} \left(\frac{N}{n_i} \right)^{-1}_{n_i}$ where $\left(\frac{N}{n_i} \right)^{-1}_{n_i}$ denotes the multiplicative inverse $(\mathrm{mod} n_i)$ of the integer $\frac{N}{n_i}$.

Proof. To see why x is a solution, notice that for each $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} x &\equiv a_1 y_1 z_1 + a_2 y_2 z_2 + \cdots + a_k y_k z_k \pmod{n_i} \\ &\equiv a_i y_i z_i \pmod{n_i} \\ &\equiv a_i \pmod{n_i}. \end{aligned}$$

- The second line follows since $y_j \equiv 0 \bmod n_i$ for each $j \neq i$.
- The third line follows since $y_i z_i \equiv 1 \bmod n_i$.

Now, to prove uniqueness, suppose there are two solutions x and y.

- Then $n_1 \,|(x-y), n_2|\,(x-y), \ldots, n_k \mid (x-y)$.
- Since $n_1, n_2, \ldots, n_k$ are relatively prime, we have that $n_1 n_2 \cdots n_k$ divides $x - y$, or

$$x \equiv y \pmod{N}.$$

Thus, the solution is unique modulo N. □

General construction:

1. Compute $N = n_1 \times n_2 \times \cdots \times n_k$.
2. For each $i = 1, 2, \ldots, k$, compute
$$y_i = \frac{N}{n_i} = n_1 n_2 \cdots n_{i-1} n_{i+1} \cdots n_k.$$
3. For each $i = 1, 2, \ldots, k$, compute $z_i \equiv y_i^{-1} \mod n_i$ (z_i exists since $n_1, n_2, \ldots, n_k$ are pairwise coprime).
4. Compute
$$x = \sum_{i=1}^{k} a_i y_i z_i$$
and $x \mod N$ is the unique solution modulo N.

Intuitive way to solve for CRT:

1. Begin with the congruence with the largest modulus, $x \equiv a_k \,(\mathrm{mod} n_k)$.
2. Re-write this modulus as an equation, $x = j_k n_k + a_k$, for some positive integer j_k.
3. Substitute the expression for x into the congruence with the next largest modulus, $x \equiv a_k \,(\mathrm{mod} n_k) \implies j_k n_k + a_k \equiv a_{k-1} \,(\mathrm{mod} n_{k-1})$.
4. Solve this congruence for j_k.
5. Write the solved congruence as an equation, and then substitute this expression for j_k into the equation for x.
6. Continue substituting and solving congruences until the equation for x implies the solution to the system of congruences.

Example 4.2. Solve for the following system of congruences

$$\begin{cases} x \equiv 1 & (\mathrm{mod}\ 3) \\ x \equiv 4 & (\mathrm{mod}\ 5) \\ x \equiv 6 & (\mathrm{mod}\ 7) \end{cases}$$

Solution. Start with mod 7.

1. Write $x = 7k + 6$.
2. Then we have $7k + 6 \equiv 4 \pmod 5 \implies k \equiv 4 \pmod 5$.
3. Then solving for k gives $5j + 4$.
4. Now we have $x = 7k + 6 = 7(5j + 4) + 6 = 35j + 34$.
5. Then $35j + 34 \equiv 1 \pmod 3 \implies j \equiv 0 \pmod 3 \implies j = 3t$.
6. Finally, we have $x = 35(3t) + 34 = 105t + 34 \implies x \equiv \boxed{34} \pmod{105}$.

5 RSA

5.1 Basic Ideas

- Alice and Bob wish to communicate secretly over some (insecure) link, and Eve tries to discover what they are saying.
- Alice transmits a message x (in binary) to Bob by applying her **encryption function** E to x and send the encrypted message $E(x)$ over the link.
- Bob, after receiving $E(x)$, applies his **decryption function** D to it and recover the original message: i.e., $D(E(x)) = x$.
- Since the link is insecure, Eve may know what $E(x)$ is.
- We would like to have an encryption function E such that only knowing $E(x)$ cannot reveal anything about x.
- The idea is that each person has a **public key** known to the whole world and a **private key** known only to him- or herself.
- Alice encodes x using Bob's public key. Bob then decrypts it using his private key, thus retrieving x.

5.2 RSA Scheme

- Let p and q be two large primes, and let $N = pq$ (p and q are not public).
- Treat messages to Bob as numbers modulo N, excluding trivial values 0 and 1.
- Let e be any number that is relatively prime to $(p-1)(q-1)$ (Typically e is a small value).
- Then Bob's public key is the pair of numbers (N, e) and his private key is $d = e^{-1} \pmod{(p-1)(q-1)}$.

5.3 RSA Encryption

- **Encryption**: Alice computes the value $E(x) = x^e \bmod N$ and sends this to Bob.
- **Decryption:** Upon receiving the value $y = E(x)$, Bob computes $D(y) = y^d \bmod N$; this will be equal to the original message x.

Theorem 29. Using the encryption and decryption functions E and D, we have $D(E(x)) = x \pmod{N}$ for every possible message $x \in \{0, 1, ..., N-1\}$.

Proof. This can be proved using Chinese Remainder Theorem or Fermat's Little Theorem. For more details, please refer to notes. □

6 Polynomials

6.1 Properies of polynomials

- **Property 1:** A non-zero polynomial of degree d has at most d roots.
- **Property 2:** A polynomial of degree d is **uniquely** determined by $d+1$ distinct points.

6.2 Polynomial Interpolation

Question. Given $d+1$ distinct points, how do we determine the polynomial?

- We use a method called **Lagrange Interpolation**, which works similarly to the **Chinese Remainder Theorem**.
- Suppose the given points are $(x_1, y_1), \ldots, (x_{d+1}, y_{d+1})$. We want to find a polynomial $p(x)$ such that $p(x_i) = y_i$ for $i = 1, \ldots, d+1$.
- In other words, we want to find polynomials $p_1(x), \ldots, p_{d+1}(x)$ such that

$$\begin{aligned}
&p_1(x) = 1 \text{ at } x_1 \text{ and } p_1(x) = 0 \text{ at } x_2, \ldots, x_{d+1};\\
&p_2(x) = 1 \text{ at } x_2 \text{ and } p_2(x) = 0 \text{ at } x_1, x_3 \ldots, x_{d+1};\\
&p_3(x) = 1 \text{ at } x_3 \text{ and } p_3(x) = 0 \text{ at } x_1, x_2, x_4, \ldots, x_{d+1} \text{ and so on...}
\end{aligned}$$

6.3 Lagrange Interpolation

- Let's start by finding $p_1(x)$.
- Since $p_1(x) = 0$ at $x_2, \ldots, x_{d+1}$, $p_1(x)$ must be a multiple of

$$q_1(x) = (x - x_2)(x - x_3)\ldots(x - x_{d+1}).$$

- We also need $p_1(x) = 1$ at x_1. Notice that

$$q_1(x_1) = (x_1 - x_2)(x_1 - x_3)\ldots(x - x_{d+1}).$$

- Then $p_1(x) = \frac{q_1(x)}{q_1(x_1)}$ is the polynomial we are looking for.
- Similarly for $p_i(x)$, we have $p_i(x) = \frac{q_i(x)}{q_i(x_i)}$.
- After finding $p_1(x), \ldots, p_{d+1}(x)$, we can construct $p(x)$ by scaling up each bit by corresponding y_i:

$$p(x) = \sum_{i=1}^{d+1} y_i \cdot p_i(x)$$

  This should remind you of CRT.

- Now let us define $\Delta_i(x)$ in the following way (think of them as a basis):

$$\Delta_i(x) = \frac{\prod_{i \neq j}(x - x_j)}{\prod_{i \neq j}(x_i - x_j)}.$$

- Then we have an **unique** polynomial

$$\boxed{p(x) = \sum_{i=1}^{d+1} y_i \Delta_i(x).}$$

6.4 Finite Fields

- The properties of a polynomial would not hold if the values are restricted to being natural numbers or integers because dividing two integers does not generally result in an integer.
- However, if we work with numbers modulo m where m is a prime number, then we can add, subtract, multiply and divide.
- Then **Property 1** and **Property 2** hold if the coefficients and the variable x are restricted to take on values modulo m. When we work with numbers modulo m, we are working over a **finite field**, denoted by $GF(m)$ (**Galois Field**).

6.5 Secret Sharing

6.5.1 Basic Ideas

- Suppose there are n people. Let s be the secret number and q be a prime number greater than n and s. We will work over $GF(q)$.
- Pick a random polynomial $P(x)$ of degree $k-1$ such that $P(0) = s$.
- Distribute $P(1), \ldots P(n)$ to each person so that each one receives one value.
- Then in order to know what s is, at least k of the n people must work together so that they can perform **Lagrange interpolation** and find P.
- If there are less than k people, they will learn nothing about s!

7 Error Correcting Codes

7.1 Basic Ideas

- **Goal:** Transmit messages across an **unreliable** communication channel.
- The channel may cause **packets**(parts of the message) to be **lost**, or even **corrupted**.
- **Error correcting code** is an encoding scheme to protect messages against these errors by introducing redundancy.

7.2 Erasure Errors

- **Erasure errors** refer to some packets being **lost** during transmission.
- Suppose that the message consists of n packets and at most k packets are lost during transmission.
- To prevent this error, we encode the initial message into a redundant encoding consisting of $n + k$ packets such that the receiver can reconstruct the message from any n received packets using **Lagrange interpolation**.

7.3 General Errors

- Now suppose the packets are **corrupted** during transmission due to channel noise. Such error is called **general errors**.
- Suppose that k out of n characters are corrupted and we have no idea which k these are.
- To guard against k general errors, we must transmit $n + 2k$ characters.
- To reconstruct the polynomial, we need to find a polynomial $P(x)$ of degree $n - 1$ such that $P(i) = r_i$ for at least $n + k$ values of i.

7.4 Error-locator Polynomial

- To efficiently find the polynomial $P(x)$, we need the locations of the k errors.
- Let $e_1, ..., e_k$ be the k locations at which errors occurred. We don't know where these errors are.
- Guessing where the errors are will take exponential time, which is inefficient, so we use the **error-locator polynomial**:

$$E(x) = (x - e_1)(x - e_2) \dots (x - e_k).$$

- Then we have the following:

$$P(i)E(i) = r_i E(i) \quad \text{for } 1 \leq i \leq n + 2k.$$

  This is known as the **Berlekamp–Welch algorithm**.

7.5 Berlekamp–Welch algorithm

- Define $Q(x) = P(x)E(x)$. We have $n + 2k$ equations with $n + 2k$ unknown coefficients:
$$Q(i) = r_i E(i) \quad \text{for } 1 \leq i \leq n + 2k.$$

- We can solve the systems of linear equations and get $E(x)$ and $Q(x)$.
- Finally we compute $\frac{Q(x)}{E(x)}$ to obtain $P(x)$.

8 Counting

8.1 Counting Rules

Theorem 30 (First Rule of Counting)**.** If there are n ways of doing something, and m ways of doing another thing after that, then there are $n \times m$ ways to perform both of these actions.

- Order matters(permutations).
- Sampling k elements from n items:
  - With replacement: n^k.
  - Without replacement: $\frac{n!}{(n-k)!}$.

Theorem 31 (Second Rule of Counting)**.** If order doesn't matter count ordered objects and then divide by number of orderings.

- Without replacement and ordering doesn't matter (combinations).
- Number of ways of choosing k-element subsets out of a set of size n:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

8.2 Stars and Bars

Stars and Bars is a technique used to solve for problems that sample with replacement but order doesn't matter by establishing a bijection between the problem and the stars and bars problem.

Problem 1. Consider the equation $a+b+c+d = 12$ where a, b, c, d are non-negative integers. How many solutions are there to this equation?

- Let's simplify this problem a little bit. Suppose we have 12 and 3 bars.

$$\star\star\,|\,\star\star|\,\star\star\star\,|\,\star\star\star\star\star$$

- How many ways can we arrange them? $\binom{12+3}{3} = \binom{15}{3}$
- This is the answer to our original problem! Do you see the bijection between the two problems?

Theorem 32 (Stars and Bars)**.** The number of ways to distribute n indistinguishable objects into k distinguishable bins is

$$\binom{n+k-1}{k-1}.$$

- Don't memorize the formula! Try to visualize the problem by connecting it to stars and bars. Draw out the stars and the bars!

- Again, this method is useful for with replacement but order doesn't matter type of problems.

Theorem 33 (Zeroth Rule of Counting:)**.** If a set A has a bijection relationship with a set B, then $|A| = |B|$.

The stars and bars method relies on this counting rule and this is the key to many combinatorial arguments as we will explore further later.

8.3 Binomial Theorem

Theorem 34 (Binomial Theorem)**.** For all $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Proof. See notes. □

Corollary 35. For all $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

Proof. Plug in $a = -1$ and $b = 1$ for the binomial theorem. □

8.4 Combinatorial Proofs

- Intuitive counting arguments. No tedious algebraic manipulation.
- Proofs by stories: same story from multiple perspectives.
- Proving an identity by counting the same thing in two different ways.
- Useful identity:

$$\binom{n}{k} = \binom{n}{n-k}.$$

- Choosing k objects to include is equivalent to choosing $n-k$ objects to exclude.

Example 8.1. Using combinatorial arguments, show that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

Proof. We can use binomial theorem by letting $a = b = 1$, however this is not what the question is asking for.
RHS: Total number of subsets of a set of size n.
LHS: The number of ways to choose a subset of size i is $\binom{n}{i}$. To find the total number of subsets, we simply add all the cases when $i = 0, 1, 2, \ldots, n$. □

8.5 Principle of Inclusion-Exclusion

Theorem 36 (Principle of Inclusion-Exclusion(General):)**.** Let $A_1, \ldots, A_n$ be arbitrary subsets of the same finite set A. Then,

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{S \subseteq \{1,\ldots,n\}: |S|=k} |\cap_{i \in S} A_i| \,.$$

Proof. See notes. □

Theorem 37 (Principle of Inclusion-Exclusion(Simplified):)**.**

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

8.6 Summary

	with replacement	w/o replacement
order matters	n^k	$\frac{n!}{(n-k)!}$
order doesn't matter	$\binom{n+k-1}{k-1}$	$\binom{n}{k}$

9 Countability

Question. How do we determine if two sets have the same cardinality, or size?

This is obvious for finite sets, but for infinite sets it becomes quite tricky. We'll see how to formulate the question.

9.1 Bijection

- Two finite sets have the same size if and only if their elements can be paired up, so that each element of one set has a unique partner in the other set, and vice versa.
- We formalize this through the concept of a **bijection**, which is discussed in *section 4.6*.

9.2 Cardinality

- To show that two infinite sets have the same cardinality, we demonstrate a pairing between elements of the two sets, i.e., establish a bijection (one-to-one correspondence) between the two sets.

Problem 2. Are there more natural numbers $\mathbb{N}$ than there are positive integers $\mathbb{Z}^+$?

Answer. It is tempting to answer yes, because every positive integer is also a natural number, and the natural numbers have one extra element 0. However, we can actually define a mapping between the natural numbers and the positive integers as follows:

- Define a function $f : \mathbb{N} \to \mathbb{Z}^+$ such that $f(n) = n+1$. Then we can see there's a one-to-one correspondence in the following figure (try to prove it on your own).

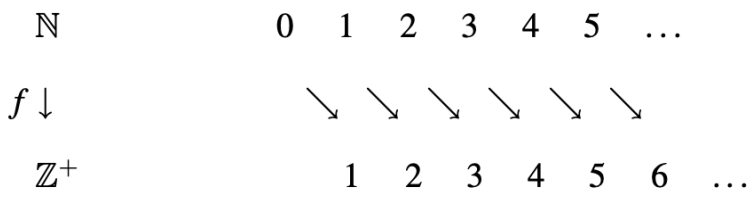


Figure 6: Bijection between $\mathbb{N}$ and $\mathbb{Z}^+$

- Since we have shown a bijection between $\mathbb{N}$ and $\mathbb{Z}^+$, this tells us that there are exactly as many natural numbers as there are positive integers.
- This also implicityly showed the fact that $\infty + 1 = \infty$!

Exercise 1. Show that the cardinality for the set of natural numbers and the set of even natural numbers are the same.

Problem 3. What about $\mathbb{N}$ and $\mathbb{Z}$?

Answer. It may seem obvious that $\mathbb{Z}$ is larger because it includes negative numbers. However, they both actually have the same size! Let's see why, consider the following function f:

$$0 \to 0, 1 \to -1, 2 \to 1, 3 \to -2, 4 \to 2, \ldots, 124 \to 62, \ldots$$

In other words, the function is defined as follows:

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } x \text{ is even} \\ \frac{-(x+1)}{2}, & \text{if } x \text{ is odd} \end{cases}$$

This function $f : \mathbb{N} \to \mathbb{Z}$ is in fact a bijection, refer to the notes for the details. Thus, the two sets have the same size.

Definition 38 (Countable)**.** A set S is **countable** if there is a bijection between S and $\mathbb{N}$ or some subset of $\mathbb{N}$.

- Intuitively, any finite set S is clearly **countable**. But the actual reasoning behind is because there is a bijection between S and the subset $\{0, 1, 2, ..., m-1\}$, where $m = |S|$ is the size of S.
- The examples we did earlier are countable because they are subsets of $\mathbb{N}$.

Problem 4. Now consider the set of rational numbers $\mathbb{Q}$, is it larger than $\mathbb{N}$? Recall that $\mathbb{Q} = \left\{ \frac{x}{y} \mid x, y \in \mathbb{Z}, y \neq 0 \right\}$.

Answer. The two sets actually have the same cardinality! Let's look at this using a different way by introducing some new definitions and an important theorem.

Definition 39. If there is a injective function $f : A \to B$, then $|A| \leq |B|$.

Definition 40. If there is a surjective function $f : A \to B$, then $|A| \geq |B|$.

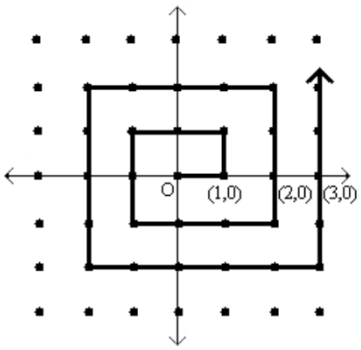
Theorem 41 (Schröder–Bernstein Theorem (Cantor–Bernstein))**.** If A and B are sets with $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$. In other words, if there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijection h between A and B.

Proof. The proof of this theorem is out of scope for this class. We'll skip that for now. □

Remark. This theorem will be very useful when we want to show a set S is countable. We can give separate injections $f : S \to \mathbb{N}$ and $g : \mathbb{N} \to S$, instead of designing a bijection (which is trickier).

- Now back to our problem. First it is obvious that $|\mathbb{N}| \leq |\mathbb{Q}|$ because $\mathbb{N} \subseteq \mathbb{Q}$.
- Now the theorem comes in handy and all we need to do now is to prove $|\mathbb{Q}| \leq |\mathbb{N}|$.
- Recall the definition, we must exhibit an injection $f : \mathbb{Q} \to \mathbb{N}$.
- Notice that each rational number $\frac{a}{b}$ $(\gcd(a, b) = 1)$ can be represented by the point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ (the set of all pairs of integers).
- However, not all points are valid, espeically when the corresponding $\frac{a}{b}$ is undefined (except for $(0, 0)$, which is used to represent rational number 0). The points whose rational representation is an unfactored fraction are also invalid.
- Thus, we can actually tell that $|\mathbb{Z} \times \mathbb{Z}| \geq |\mathbb{Q}|$.

- If we are able to come up with an injection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}$, then this will also be an injection from $\mathbb{Q}$ to $\mathbb{N}$ (why?).



- The idea is to map each pair (a, b) to its position along the spiral, starting at the origin, as shown in the picture above (basically we are indexing throught each point starting from index 0).
- It is clear that this mapping maps every pair of integers injectively to a natural number.
- Thus we have $|\mathbb{Q}| \leq |\mathbb{Z} \times \mathbb{Z}| \leq |\mathbb{N}|$. Remember that $|\mathbb{N}| \leq |\mathbb{Q}|$, then by the Cantor-Bernstein Theorem $|\mathbb{N}| = |\mathbb{Q}|$.

9.3 Cantor's Diagonalization

Now let's consider the set of real numbers, $\mathbb{R}[0, 1]$ specifically. We'll see that it is uncountable using a method called **diagonalization**.

Theorem 42. The real interval $\mathbb{R}[0, 1]$ is uncountable.

Proof. Assume for the sake of contradiction that there is a bijection $f : \mathbb{N} \to \mathbb{R}[0, 1]$. Then, we can enumerate the real numbers in an infinite list $f(0), f(1), f(2), \ldots$ as follows:

$$\begin{aligned} f(0) &= 0.\textcircled{5}2149356\ldots \\ f(1) &= 0.1\textcircled{4}162985\ldots \\ f(2) &= 0.94\textcircled{7}82712\ldots \\ f(3) &= 0.530\textcircled{9}8175\ldots \\ &\vdots \end{aligned}$$

The number circled in the diagonal can be some real number $r = 0.5479\ldots$. Now consider the real number s obtained by modifying every digit of r such that each digit d is replaced with $d + 2 \pmod{10}$, so $s = 0.7691.....$ Then we claim that s is not included in our infinite list of real numbers. Suppose for contradiction that it us, and that it was the n^{th} number in the list, $f(n)$. But by construction s differs from $f(n)$ in the $(n + 1)$th digit, so they cannot be equal! So we have constructed a real number s that is not in the range of f. But this contradicts the assertion that f is a bijection. Thus the real numbers are not countable.

□

Remark. The reason that we modified each digit by adding 2 (mod 10) instead of adding 1 is that the same real number can have two decimal expansions; for example $0.999\ldots = 1.000\ldots$. But if two real numbers differ by more than 1 in any digit they cannot be equal. Thus our modification is safe. (We can also replace each digit by some different digit chosen from the range $\{1, 2, \ldots, 8\}$.)

Theorem 43. If A and B are countable sets, then $A \cup B$ is also countable.

9.4 Power Sets and Higher Orders of Infinity

Definition 44 (Power Set)**.** Recall that the **power set** of S, denoted by $\mathscr{P}(S)$, is the set of all subsets of S. More formally, it is defined as $\mathscr{P}(S) = \{T : T \subseteq S\}$.

Question. What is the cardinality of $\mathscr{P}(S)$?

Answer. If $|S| = k$ is finite, then $|\mathscr{P}(S)| = 2^k$.(why?)

- For finite sets S, the cardinality of the power set of S is exponentially larger than the cardinality of S.
- What about infinite (countable) sets?
- We claim that there is no bijection from S to $\mathscr{P}(S)$ so $\mathscr{P}(S)$ is not countable.
- For example the set of all subsets of natural numbers is not countable, even though the set of natural numbers itself is countable.

Theorem 45. $|\mathscr{P}(\mathbb{N})| > |\mathbb{N}|$.

Proof. See notes. □

10 Probability