Math 104
Real Analysis

Instructor: Peng Zhou

Kelvin Lee

UC Berkeley

Contents

LECTURE 1

THE REAL NUMBER SYSTEMS

1 Natural Numbers $\mathbb{N}$

Definition 1 (Peano Axioms (Peano Postulates))**.** The properties of the set of natural numbers, denoted $\mathbb{N}$, are as follows:

(i) 1 belongs to $\mathbb{N}$.

(ii) If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$.

(iii) 1 is not the successor of any element in $\mathbb{N}$.

(iv) If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.

(v) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n, must equal to $\mathbb{N}$.

Remark. The last axiom is the basis of mathematical induction. Let $P_1, P_2, P_3, \ldots$ be a list of propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, \ldots$ are true provided

- P_1 is true. (Basis for induction)
- $P_n \implies P_{n+1}$. (Induction step)

2 Rational Numbers $\mathbb{Q}$

Definition 2 (Rational Numbers)**.** The set of **rational numbers**, denoted $\mathbb{Q}$, is defined by

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid n, m \in \mathbb{Z}, n \neq 0 \right\},$$

which supports addition, multiplication, subtraction, and division.

Remark. $\mathbb{Q}$ is a very nice algebraic system. However, there is no rational solution to equations like $x^2 = 2$.

Definition 3 (Algebraic Number)**.** A number is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $c_0, \ldots, c_n$ are integers, $c_n \neq 0$ and $n \geq 1$.

Remark. Rational numbers are always algebraic numbers.

Theorem 4 (Rational Zeros Theorem)**.** Suppose $c_0, c_1, \ldots, c_n$ are integers and r is a rational number satisfying the polynomial equations

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1, c_n, c_0 \neq 0$. Let $r = \frac{c}{d}$ where $\gcd(c,d) = 1$. Then $c \mid c_0$ and $d \mid c_n$. In simpler terms, the only rational candidates for solutions to the equation have the form $\frac{c}{d}$ where c is a factor of c_0 and d is a factor of c_n.

Proof. Plug in $r = \frac{c}{d}$ to the equation, we get

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0.$$

Then we multiply by d^n on both sides and get

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Solving for $c_0 d^n$, we obtain

$$c_0 d^n = -c \left(c_n c^n + c_{n-1}^{n-2} + \cdots + c_2 c d^{n-2} + c_1 d^{n-1}\right).$$

Then it follows that $c \mid c_0 d^n$. Since $\gcd(c,d) = 1$, c can only divide c_0.
Now let's instead solve for $c_n c^n$, then we have

$$c_n c^n = -d \left(c_{n-1} c^{n-1} + c_{n-2} c^{n-2} d + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}\right).$$

Thus $d \mid c_n c^n$, which implies $d \mid c_n$ because $\gcd(c,d) = 1$. □

Corollary 5. Consider

$$x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. Since the Rational Zeros Theorem states that d must divide c_n, which is 1 in this case, r is an integer and it divides c_0. □

Example 6. $\sqrt{2}$ is not a rational number.

Proof. Using Corollary 5, if $r = \sqrt{2}$ is rational, then $\sqrt{2}$ must be an integer, which is a contradiction. □

3 Real Numbers $\mathbb{R}$

3.1 The Completeness Axiom

Definition 7 (Maximum/minimum)**.** Let S be a nonempty subset of $\mathbb{R}$.

(i) If S contains a largest element s_0(i.e., $s_0 \in S$, $s \leq s_0 \forall s \in S$), then s_0 is the **maximum** of S, denoted $s_0 = \max S$.

(i) If S contains a smallest element, then it is called the **minimum** of S, denoted as $\min S$.

Remark.

- If s_1, s_2 are both maximum of S, then $s_1 \geq s_2, s_2 \geq s_1$, which implies that $s_1 = s_2$. Thus the maximum is **unique** if it exists.
- However, the maximum may not exist (e.g. $S = \mathbb{R}$).
- If $S \subset \mathbb{R}$ is a finite subset, then $\max S$ exists.

Definition 8 (Upper/Lower bound)**.** Let S be a nonempty subset of $\mathbb{R}$.

(i) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is an **upper bound** of S and S is said to be *bounded above*.

(i) If a real number m satisfies $\leq s$ for all $s \in S$, then m is a **lower bound** of S and S is said to be *bounded below*.

(i) S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subset [m, M]$.

Definition 9 (Supremum/Infimum)**.** Let S be a nonempty subset of $\mathbb{R}$.

- If S is bounded above and S has a least upper bound, then it is called the **supremum** of S, denoted by $\sup S$.
- If S is bounded below and S has a greatest lower bound, then it is called the **infimum** of S, denoted by $\inf S$.

Remark. If S has a maximum, then $\max S = \sup S$. Similarly, if S has a minimum, then $\min S = \inf S$. Also note that $\sup S$ and $\inf S$ need not belong to S.

Example 10. Suppose we have $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\max S$ does not exist and $\sup S = 1$.

Proof. Suppose for contradiction that it exists. Then it must be of the form $1 - \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. However,

$$1 - \frac{1}{n_0 + 1} > 1 - \frac{1}{n_0},$$

and $1 - \frac{1}{n_0+1} \in S$. Hence a contradiction. □

Theorem 11 (Completeness Axiom)**.** Every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Corollary 12. Every nonempty subset $S \subset \mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

Proof. Consider the set $-S = \{-s \mid s \in S\}$. Since S is bounded below there exists an $m \in \mathbb{R}$ such that $m \le s$ for all $s \in S$. This implies $-m \ge -s$ for all $s \in S$, so $-m \ge u$ for all $u \in -S$. Thus, $-S$ is bounded above by $-m$. The Completeness Axiom applies to $-S$, so $\sup -S$ exists.
Now we show that $\inf S = -\sup -S$. Let $s_0 = \sup -S$, we need to prove

$$-s_0 \le s \qquad \text{for all } s \in S,$$

and if $t \le s$ for all $s \in S$, then $t \le -s_0$. The first inequality will show that $-s_0$ is a lower bound while the second inequality will show that $-s_0$ is the greatest lower bound, i.e., $-s_0 = \inf S$. The proofs of the two claims are left as an exercise. □

Theorem 13 (Archimedean Property)**.** If $a, b > 0$, then $na > b$ for some positive integer n.

Proof. Suppose the property fails for some pair of $a, b > 0$. That is, for all $n \in \mathbb{N}$, we have $na \le b$, meaning that b is an upper bound for the set $S = \{na \mid n \in \mathbb{N}\}$. Using the Completeness Axiom, we can let $s_0 = \sup S$. Since $a > 0$, we have $s_0 - a < s_0$, so $s_0 - a$ cannot be an upper bound for S. It follows that $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$, which then implies that $s_0 < (n_0 + 1)a$. Since $(n_0 + 1)a$ is in S, s_0 is not an upper bound for S, which is a contradiction. □

Theorem 14 (Denseness of $\mathbb{Q}$)**.** If $a, b \in \mathbb{R}$ and $a < b$, then there is a rational $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. We need to show that $a < \frac{m}{n} < b$ for some integers m and n where $n \neq 0$. Equivalently, we want

$$an < m < bn.$$

Since $b - a > 0$, the Archimedean property shows that there exists an $n \in \mathbb{N}$ such that

$$n(b-a) > 1 \implies bn - an > 1.$$

Now we need to show that there is an integer m between an and bn. □

4 $+\infty$ and $-\infty$

We adjoint $+\infty$ and $-\infty$ to $\mathbb{R}$ and extend our ordering to the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Explicitly, we have $-\infty \le a \le +\infty$ for all $a \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Remark. $+\infty$ and $-\infty$ are not real numbers. Theorems that apply to real numbers would not work.

We define

$$\sup S = +\infty \qquad \text{if } S \text{ is not bounded above}$$

and

$$\inf S = -\infty \qquad \text{if } S \text{ is not bounded below.}$$

5 Reading (Rudin's)

5.1 Ordered Sets

Definition 15 (Order)**.** Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$, then one and only one of the statements

$$s < y, \quad x = y, \quad , y < x$$

is true.

- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 16 (Ordered Set)**.** An **ordered set** is a set S in which an order is defined.

For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

5.2 Fields

Definition 17 (Field)**.** A **field** is a set F with two operations: *addition* and *multiplication*, which satisfy the following **field axioms**:

(A) **Axioms for addition**

(A1) If $x, y \in F$, then $x + y \in F$.

(A2) (Commutativity) $\forall x, y \in F$, $x + y = y + x$.

(A3) (Associativity) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$.

(A4) (Identity) $\forall x \in F$, $0 + x = x$.

(A5) (Inverse) $\forall x \in F$, there exists a corresponding $-x \in F$ such that

$$x + (-x) = 0.$$

(M) **Axioms for multiplication**

(M1) If $x, y \in F$, then $xy \in F$.

(M2) (Commutativity) $\forall x, y \in F$, $xy = yx$.

(M3) (Associativity) $\forall x, y, z \in F$, $(xy)z = x(yz)$.

(M4) (Identity) $\forall x \in F$, $1x = x$.

(M5) (Inverse) $\forall x \in F$, there exists a corresponding $\frac{1}{x} \in F$ such that

$$x\left(\frac{1}{x}\right) = 1.$$

(D) **The distributive law**

$$\forall x, y, z \in F, x(y + z) = xy + xz.$$

Definition 18 (Ordered Field)**.** An **ordered field** is a field F which is also an *ordered set*, such that

(i) if $y < z$ and $x, y, z \in F$, $x + y < x + z$,

(i) if $x, y > 0$ and $x, y \in F$, $xy > 0$.

LECTURE 2

SEQUENCES

6 Limits of Sequences

Definition 19 (Sequence)**.** A **sequence** is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ where m is usually 1 or 0.

One may wonder why do we care about sequence, and the answer is that sequences are useful for *approximation*.

Definition 20. A sequence $\{s_n\}$ of real numbers is said to **converge** to the real number s if $\forall \epsilon > 0$, $\exists N > 0$ such that for all positive integers $n > N$, we have

$$|s_n - s| < \epsilon.$$

If $\{s_n\}$ converges to s, we write $\lim_{n\to\infty} s_n = s$, or simply $s_n \to s$, where s is called the **limit** of the sequence. A sequence that does not converge to some real number is said to **diverge**.

7 Proofs of Limits

Example 21. Prove $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

Scratch. For any $\epsilon > 0$, we want

$$\left|\frac{1}{n^2} - 0\right| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff \frac{1}{\epsilon} < n^2 \iff \frac{1}{\sqrt{\epsilon}} < n.$$

Thus, we can just take $N = \frac{1}{\sqrt{\epsilon}}$. $\square$

Proof. Let $\epsilon > 0$ and $N = \frac{1}{\sqrt{\epsilon}}$. Then $n > N$ implies $n > \frac{1}{\sqrt{\epsilon}}$ which implies $n^2 > \frac{1}{\epsilon}$ and hence $\epsilon > \frac{1}{n^2}$. Thus $n > N$ implies $\left|\frac{1}{n^2} - 0\right| < \epsilon$. This proves our claim. $\square$

Example 22. Prove $\lim_{n\to\infty} \frac{3n+1}{7n-4} = \frac{3}{7}$.

Scratch. $\forall\epsilon > 0$, we need $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon$, which implies that

$$\left|\frac{21n + 7 - 21n + 12}{7(4n - 4)}\right| < \epsilon \implies \left|\frac{19}{7(7n - 4)}\right| < \epsilon.$$

Since $7n - 4 > 0$, we can remove the absolute value sign and have

$$\frac{19}{7\epsilon} < 7n - 4 \implies \frac{19}{49\epsilon} + \frac{4}{7} < n.$$

Thus, we have $N = \frac{19}{49\epsilon} + \frac{4}{7}$. □

Proof. Let $\epsilon > 0$ and let $N = \frac{19}{49\epsilon} + \frac{4}{7}$. Then $n > N$ implies $n > \frac{19}{49\epsilon} + \frac{4}{7}$, hence $7n > \frac{19}{7\epsilon} + 4$, which gives us $\frac{19}{7(7n-4)} < \epsilon$, and thus $\left|\frac{3n+1}{7n-4} - \frac{3}{7}\right| < \epsilon$. Then we are done. □

Example 23. Prove $\lim_{n\to\infty} 1 + \frac{1}{n}(-1)^n = 1$.

Scratch. $\forall\epsilon > 0$, we want n large enough, such that

$$|a_n - 1| < \epsilon \iff \left|1 + \frac{1}{n}(-1)^n - 1\right| < \epsilon \iff \left|\frac{1}{n}(-1)^n\right| < \epsilon \iff \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

Just take $\alpha = \frac{1}{\epsilon}$, then $n > N \to |a_n - 1| < \epsilon$ □

8 Limit Theorems for Sequences

Definition 24 (Bounded)**.** A sequence $\{s_n\}$ of real numbers is said to be **bounded** if the set $\{s_n \mid n \in \mathbb{N}\}$ is a bounded set, i.e., if there exists a constant M such that $|s_n| \le M$ for all n.

Theorem 25. Convergent sequences are bounded.

Proof. Let $\{s_n\}$ be a convergent sequence and let $s = \lim_{n\to\infty} s_n$. Let $\epsilon > 0$ be fixed. Then by convergence of the sequence, there exists an number $N \in \mathbb{N}$ such that

$$n > N \implies |s_n - s| < \epsilon.$$

By the triangle inequality we see that $n > N$ implies $|s_n| < |s| + \epsilon$. Define $M = \max\{|s| + \epsilon, |s_1|, \ldots, |s_N|\}$. Then $|s_n| \le M$ for all $n \in \mathbb{N}$, so $\{s_n\}$ is a bounded sequence. □

Theorem 26. Let $\{s_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ such that $s_n \to s$ adn $t_n \to t$. Let $k \in \mathbb{R}$ be a constant. Then

(i) $ks_n \to ks$.

(ii) $(s_n + t_n) \to s + t$.

(iii) $s_n t_n \to st$.

(iv) If $s_n \neq 0$ for all n, and if $s \neq 0$, then $\frac{1}{s_n} \to \frac{1}{s}$.

(v) If $s_n \neq 0$ and $s \neq 0$ for all n, then $\frac{t_n}{s_n} \to \frac{t}{s}$.

Proof of (i). Since the case where $k = 0$ is trivial, we assume $k \neq 0$. Let $\epsilon > 0$ and we want to show that $|ks_n - ks| < \epsilon$ for large n. Since $\lim_{n\to\infty} = s$, there exists N such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{|k|}.$$

Then

$$n > N \implies |ks_n - ks| < \epsilon.$$

□

Proof of (ii). Let $\epsilon > 0$. We need to show

$$|s_n + t_n - (s+t)| < \epsilon \quad \text{for large } n.$$

Using triangle inequality, we have $|s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t|$. Since $s_n \to s$, there exists N_1 such that

$$n > N_1 \implies |s_n - s| < \frac{\epsilon}{2}.$$

Similarly, there exists N_2 such that

$$n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. Then clearly

$$n > N \implies |s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Proof of (iii). We use the identity

$$s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s).$$

Given $\epsilon > 0$, there are integers N_1, N_2 such that

$$\begin{aligned} n > N_1 &\implies |s_n - s| < \sqrt{\epsilon} \\ n > N_2 &\implies |t_n - t| < \sqrt{\epsilon} \end{aligned}$$

If we take $N = \max\{N_1, N_2\}, n \geq N$ implies

$$|(s_n - s)(t_n - t)| < \epsilon$$

which implies that

$$\lim_{n\to\infty} (s_n - s)(t_n - t) = 0.$$

Applying (i) and (ii), we get

$$\lim_{n\to\infty} (s_n t_n - st) = 0.$$

□

Proof of (iv). Choosing m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \frac{1}{2}|s| \quad (n \geq m).$$

Given $\epsilon > 0$, there is an integer $N > m$ such that $n > N$ implies

$$|s_n - s| < \frac{1}{2}|s|^2\epsilon.$$

Hence, for $n \geq N$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \epsilon.$$

□

Proof of (v). Using (iv), we have $\frac{1}{s_n} \to \frac{1}{s}$, and by (iii), we get

$$\lim_{n\to\infty} \frac{t_n}{s_n} = \lim_{n\to\infty} \frac{1}{s_n} \cdot t_n = \frac{1}{s} \cdot t = \frac{t}{s}.$$

□

Theorem 27.

(i) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.

(ii) $\lim_{n\to\infty} a^n = 0$ if $|a| < 1$.

(iii) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

(iv) $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ for $a > 0$.

Proof of (i). Let $\epsilon > 0$ and let $N = \left(\frac{1}{\epsilon}\right)^{\frac{1}{p}}$. Then $n > N$ implies $n^p > \frac{1}{\epsilon}$ and thus $\epsilon > \frac{1}{n^p}$. Since $\frac{1}{n^p} > 0$, this shows $n > N$ implies $\left|\frac{1}{n^p} - 0\right| < \epsilon$. □

Proof of (ii). The case for $a = 0$ is trivial. Suppose that $a \neq 0$. Since $|a| < 1$, we can write $|a| = \frac{1}{1+b}$ where $b > 0$. By the binomial theorem, we have $(1+b)^n \geq 1 + nb > nb$, then

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n} < \frac{1}{nb}.$$

Consider $\epsilon > 0$ and let $N = \frac{1}{\epsilon b}$. Then $n > N$ implies $n > \frac{1}{\epsilon b}$ and thus $|a^n - 0| < \frac{1}{nb} < \epsilon$. □

Proof of (iii). Let $s_n = n^{\frac{1}{n}} - 1$. Then $s_n \geq 0$ and by the binomial theorem,

$$n = (1 + s_n)^n \geq \frac{n(n-1)}{2} s_n^2.$$

Hence,

$$0 \leq s_n \leq \sqrt{\frac{2}{n-1}} \implies s_n \to 0.$$

□

Proof of (iv). Suppose $a > 1$. Let $s_n = a^{\frac{1}{n}} - 1$. Then $s_n > 0$, and by the binomial theorem,

$$1 + ns_n \leq (1 + s_n)^n = a,$$

so that

$$0 < s_n \leq \frac{p-1}{n}.$$

Hence, $s_n \to 0$. The case for $a = 1$ is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals. □

8.1 Upper and lower limits

Definition 28. Let $\{s_n\}$ be a sequence of real numbers with the property that for every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \to -\infty.$$

9 Monotone Sequences and Cauchy Sequences

Definition 29 (Monotone sequence)**.** A sequence $\{s_n\}$ of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all n, and $\{s_n\}$ is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all n. If $\{s_n\}$ is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing will be called a **monotone sequence** or a **monotonic sequence**.

Theorem 30. All bounded monotone sequences converge.

Proof. Let $\{s_n\}$ be a bounded increasing sequence, Let $= \{s \mid n \in \mathbb{N}\}$ and let $u = \sup S$, Since S is bounded, u represents a real number. We show $s_n \to u$. Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists N such that $s_N > u - \epsilon$.Since $\{s_n\}$ is increasing, $s_N \leq s_n$ for all $n \geq N$. Of course $s_n \leq u$ for all n, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. Hence $s_n \to u$. The proof for bounded decreasing sequences is left as an exercise. □

Theorem 31.

(i) If $\{s_n\}$ is an unbounded increasing sequence, then $s_n \to +\infty$.

(ii) If $\{s_n\}$ is an unbounded decreasing sequence, then $s_n \to -\infty$.

Corollary 32. If $\{s_n\}$ is a monotone sequence, then the sequence either converges, diverges to $+\infty$, or $-\infty$. Thus $\lim s_n$ is always meaningful for monotone sequences.

Proof. Simply apply the previous two theorems. □

Definition 33. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n \mid n > N\}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n \mid n > N\}$$

Theorem 34. Let $\{s_n\}$ be a sequence in $\mathbb{R}$.

(i) If $\lim s_n$ is defined (real, or $\pm\infty$), then

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n$, then $\lim s_n$ is defined and

$$\lim s_n = \liminf s_n = \limsup s_n.$$

Definition 35 (Cauchy sequence)**.** A sequeunce $\{s_n\}$ of real numbers i called a **Cauchy sequeunce** if for each $\epsilon > 0$ there exists a number N such that

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

Lemma 36. Convergent sequences are Cauchy sequences.

Proof. Suppose $\lim s_n = s$. Since the terms s_n are close to s for large n, they must also be close to each other; indeed

$$|s_n - s_m| = |s_n - s + s - s_m| \le |s_n - s| + |s - s_m|.$$

Let $\epsilon > 0$. Then there exists N such that

$$n > N \implies |s_n - s| < \frac{\epsilon}{2}.$$

Clearly we can also write

$$m > N \implies |s_m - s| < \frac{\epsilon}{2},$$

so

$$m, n > N \implies |s_n - s_m| \le |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\{s_n\}$ is a Cauchy sequence. $\square$

Lemma 37. Cauchy sequences are bounded.

Proof. Let $\epsilon = 1$. By definition, we have N in $\mathbb{N}$ such that

$$m, n > N \implies |s_n - s_m| < 1.$$

In particular, $|s_n - s_{N+1}| < 1$ for $n > N$, so $|s_n| < |s_{N+1}| + 1$ for $n > N$. If $M = \max\{|s_{N+1} + 1, |s_1|, |s_2|, \dots, |s_N|\}$, then $|s_n| \le M$ for all $n \in \mathbb{N}$. $\square$

Theorem 38. A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof. Earlier we have shown one direction in a lemma. We now only need to show that Cauchy sequences are convergent sequences. Consider a Cauchy sequence $\{s_n\}$ and it is bounded by previous lemma. We now need to show that

$$\liminf s_n = \limsup s_n.$$

Let $\epsilon > 0$. Since $\{s_n\}$ is a Cauchy sequence, there exists N so that

$$m, n > N \implies |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n \mid n > N\}$, so $v_N = \sup\{s_n \mid n > N\} \le s_m + \epsilon$ for $m > N$. This, in turn, shows $v_N - \epsilon$ is a lower bound for $\{s_m \mid m > N\}$, so $v_N - \epsilon \le \inf\{s_m \mid m > N\} = u_N$. Thus

$$\limsup s_n \le v_N \le u_N + \epsilon \le \liminf s_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\limsup s_n \le \liminf s_n$. Since $\limsup s_n \ge \liminf s_n$ always holds, we are done. $\square$