
Math 104

Real Analysis

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Contents

1	The Real Number Systems	3
1	Natural Numbers \mathbb{N}	3
2	Rational Numbers \mathbb{Q}	3
3	Real Numbers \mathbb{R}	5
4	$+\infty$ and $-\infty$	6
5	Reading (Rudin's)	6
2	Sequences	9
6	Limits of Sequences	9

LECTURE 1

THE REAL NUMBER SYSTEMS

1 Natural Numbers \mathbb{N}

Definition 1 (Peano Axioms (Peano Postulates)). The properties of the set of natural numbers, denoted \mathbb{N} , are as follows:

- (i) 1 belongs to \mathbb{N} .
- (ii) If n belongs to \mathbb{N} , then its successor $n + 1$ belongs to \mathbb{N} .
- (iii) 1 is not the successor of any element in \mathbb{N} .
- (iv) If $n, m \in \mathbb{N}$ have the same successor, then $n = m$.
- (v) A subset of \mathbb{N} which contains 1, and which contains $n + 1$ whenever it contains n , must equal to \mathbb{N} .

Remark. The last axiom is the basis of mathematical induction. Let P_1, P_2, P_3, \dots be a list of propositions that may or may not be true. The principle of mathematical induction asserts all the statements P_1, P_2, \dots are true provided

- P_1 is true. (Basis for induction)
- $P_n \implies P_{n+1}$. (Induction step)

2 Rational Numbers \mathbb{Q}

Definition 2 (Rational Numbers). The set of **rational numbers**, denoted \mathbb{Q} , is defined by

$$\mathbb{Q} = \left\{ \frac{m}{n} \mid n, m \in \mathbb{Z}, n \neq 0 \right\},$$

which supports addition, multiplication, subtraction, and division.

Remark. \mathbb{Q} is a very nice algebraic system. However, there is no rational solution to equations like $x^2 = 2$.

Definition 3 (Algebraic Number). A number is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where c_0, \dots, c_n are integers, $c_n \neq 0$ and $n \geq 1$.

Remark. Rational numbers are always algebraic numbers.

Theorem 4 (Rational Zeros Theorem). Suppose c_0, c_1, \dots, c_n are integers and r is a rational number satisfying the polynomial equations

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1, c_n, c_0 \neq 0$. Let $r = \frac{c}{d}$ where $\gcd(c, d) = 1$. Then $c \mid c_0$ and $d \mid c_n$. In simpler terms, the only rational candidates for solutions to the equation have the form $\frac{c}{d}$ where c is a factor of c_0 and d is a factor of c_n .

Proof. Plug in $r = \frac{c}{d}$ to the equation, we get

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \left(\frac{c}{d}\right) + c_0 = 0.$$

Then we multiply by d^n on both sides and get

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0.$$

Solving for $c_0 d^n$, we obtain

$$c_0 d^n = -c \left(c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} \right).$$

Then it follows that $c \mid c_0 d^n$. Since $\gcd(c, d) = 1$, c can only divide c_0 .

Now let's instead solve for $c_n c^n$, then we have

$$c_n c^n = -d \left(c_{n-1} c^{n-1} + c_{n-2} c^{n-2} d + \cdots + c_1 c d^{n-2} + c_0 d^{n-1} \right).$$

Thus $d \mid c_n c^n$, which implies $d \mid c_n$ because $\gcd(c, d) = 1$. □

Corollary 5. Consider

$$x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0,$$

where c_0, c_1, \dots, c_{n-1} are integers and $c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0 .

Proof. Since the Rational Zeros Theorem states that d must divide c_n , which is 1 in this case, r is an integer and it divides c_0 . □

Example 6. $\sqrt{2}$ is not a rational number.

Proof. Using Corollary 5, if $r = \sqrt{2}$ is rational, then $\sqrt{2}$ must be an integer, which is a contradiction. □

3 Real Numbers \mathbb{R}

3.1 The Completeness Axiom

Definition 7 (Maximum/minimum). Let S be a nonempty subset of \mathbb{R} .

- (i) If S contains a largest element s_0 (i.e., $s_0 \in S$, $s \leq s_0 \forall s \in S$), then s_0 is the **maximum** of S , denoted $s_0 = \max S$.
- (i) If S contains a smallest element, then it is called the **minimum** of S , denoted as $\min S$.

Remark.

- If s_1, s_2 are both maximum of S , then $s_1 \geq s_2, s_2 \geq s_1$, which implies that $s_1 = s_2$. Thus the maximum is **unique** if it exists.
- However, the maximum may not exist (e.g. $S = \mathbb{R}$).
- If $S \subset \mathbb{R}$ is a finite subset, then $\max S$ exists.

Definition 8 (Upper/Lower bound). Let S be a nonempty subset of \mathbb{R} .

- (i) If a real number M satisfies $s \leq M$ for all $s \in S$, then M is an **upper bound** of S and S is said to be *bounded above*.
- (i) If a real number m satisfies $m \leq s$ for all $s \in S$, then m is a **lower bound** of S and S is said to be *bounded below*.
- (i) S is said to be *bounded* if it is bounded above and bounded below. Thus S is bounded if there exist real numbers m and M such that $S \subset [m, M]$.

Definition 9 (Supremum/Infimum). Let S be a nonempty subset of \mathbb{R} .

- If S is bounded above and S has a least upper bound, then it is called the **supremum** of S , denoted by $\sup S$.
- If S is bounded below and S has a greatest lower bound, then it is called the **infimum** of S , denoted by $\inf S$.

Remark. If S has a maximum, then $\max S = \sup S$. Similarly, if S has a minimum, then $\min S = \inf S$. Also note that $\sup S$ and $\inf S$ need not belong to S .

Example 10. Suppose we have $S = \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$. Then $\max S$ does not exist and $\sup S = 1$.

Proof. Suppose for contradiction that it exists. Then it must be of the form $1 - \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. However,

$$1 - \frac{1}{n_0 + 1} > 1 - \frac{1}{n_0},$$

and $1 - \frac{1}{n_0 + 1} \in S$. Hence a contradiction. □

Theorem 11 (Completeness Axiom). Every nonempty subset $S \subset \mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Corollary 12. Every nonempty subset $S \subset \mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

Proof. Consider the set $-S = \{-s \mid s \in S\}$. Since S is bounded below there exists an $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$. This implies $-m \geq -s$ for all $s \in S$, so $-m \geq u$ for all $u \in -S$. Thus, $-S$ is bounded above by $-m$. The Completeness Axiom applies to $-S$, so $\sup -S$ exists.

Now we show that $\inf S = -\sup -S$. Let $s_0 = \sup -S$, we need to prove

$$-s_0 \leq s \quad \text{for all } s \in S,$$

and if $t \leq s$ for all $s \in S$, then $t \leq -s_0$. The first inequality will show that $-s_0$ is a lower bound while the second inequality will show that $-s_0$ is the greatest lower bound, i.e., $-s_0 = \inf S$. The proofs of the two claims are left as an exercise. \square

Theorem 13 (Archimedean Property). If $a, b > 0$, then $na > b$ for some positive integer n .

Proof. Suppose the property fails for some pair of $a, b > 0$. That is, for all $n \in \mathbb{N}$, we have $na \leq b$, meaning that b is an upper bound for the set $S = \{na \mid n \in \mathbb{N}\}$. Using the Completeness Axiom, we can let $s_0 = \sup S$. Since $a > 0$, we have $s_0 - a < s_0$, so $s_0 - a$ cannot be an upper bound for S . It follows that $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$, which then implies that $s_0 < (n_0 + 1)a$. Since $(n_0 + 1)a$ is in S , s_0 is not an upper bound for S , which is a contradiction. \square

4 $+\infty$ and $-\infty$

We adjoin $+\infty$ and $-\infty$ to \mathbb{R} and extend our ordering to the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Explicitly, we have $-\infty \leq a \leq +\infty$ for all $a \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Remark. $+\infty$ and $-\infty$ are not real numbers. Theorems that apply to real numbers would not work.

We define

$$\sup S = +\infty \quad \text{if } S \text{ is not bounded above}$$

and

$$\inf S = -\infty \quad \text{if } S \text{ is not bounded below.}$$

5 Reading (Rudin's)

5.1 Ordered Sets

Definition 14 (Order). Let S be a set. An **order** on S is a relation, denoted by $<$, with the following two properties:

- If $x \in S$ and $y \in S$, then one and only one of the statements

$$s < y, \quad x = y, \quad , y < x$$

is true.

- If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Definition 15 (Ordered Set). An **ordered set** is a set S in which an order is defined.

For example, \mathbb{Q} is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

5.2 Fields

Definition 16 (Field). A **field** is a set F with two operations: *addition* and *multiplication*, which satisfy the following **field axioms**:

(A) **Axioms for addition**

(A1) If $x, y \in F$, then $x + y \in F$.

(A2) (Commutativity) $\forall x, y \in F, x + y = y + x$.

(A3) (Associativity) $\forall x, y, z \in F, (x + y) + z = x + (y + z)$.

(A4) (Identity) $\forall x \in F, 0 + x = x$.

(A5) (Inverse) $\forall x \in F$, there exists a corresponding $-x \in F$ such that

$$x + (-x) = 0.$$

(M) **Axioms for multiplication**

(M1) If $x, y \in F$, then $xy \in F$.

(M2) (Commutativity) $\forall x, y \in F, xy = yx$.

(M3) (Associativity) $\forall x, y, z \in F, (xy)z = x(yz)$.

(M4) (Identity) $\forall x \in F, 1x = x$.

(M5) (Inverse) $\forall x \in F$, there exists a corresponding $\frac{1}{x} \in F$ such that

$$x \left(\frac{1}{x} \right) = 1.$$

(D) **The distributive law**

$$\forall x, y, z \in F, x(y + z) = xy + xz.$$

Definition 17 (Ordered Field). An **ordered field** is a field F which is also an *ordered set*, such that

(i) if $y < z$ and $x, y, z \in F, x + y < x + z$,

(i) if $x, y > 0$ and $x, y \in F, xy > 0$.

LECTURE 2

SEQUENCES

6 Limits of Sequences

Definition 18 (Sequence). A **sequence** is a function whose domain is a set of the form $\{n \in \mathbb{Z} \mid n \geq m\}$ where m is usually 1 or 0.

One may wonder why do we care about sequence, and the answer is that sequences are useful for *approximation*.

Definition 19. A sequence (s_n) of real numbers is said to **converge** to the real number s if $\forall \epsilon > 0$, $\exists N > 0$ such that for all positive integers $n > N$, we have

$$|s_n - s| < \epsilon.$$

If (s_n) converges to s , we write $\lim_{n \rightarrow \infty} s_n = s$, or simply $s_n \rightarrow s$, where s is called the **limit** of the sequence. A sequence that does not converge to some real number is said to **diverge**.