
STOCHASTIC PROCESSES

STAT 150

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Contents

1	Probability Review	4
1.1	Basic Definitions	4
1.2	Overview	4
1.3	Useful Properties	5
1.4	Conditional Probability	5
1.5	Random Variables	6
1.5.1	Discrete Random Variables	6
1.5.2	Indicator Random Variable	6
1.5.3	Continuous Random Variables	6
1.5.4	Cumulative Distribution Function (CDF)	7
1.5.5	Expectation	7
1.5.6	Variance	7
1.5.7	Moments	8
1.5.8	Joint Distribution	8
1.5.9	Independence	8
1.5.10	Linearity of Expectation	8
1.5.11	Convolution	9
1.5.12	Gamma Distribution	10
1.5.13	Moment Generating Function	10
1.6	Conditional Probability (Cont'd)	11
1.6.1	Conditional Expectation	12
2	Random Sums	14
2.1	Mean and Variance of Random Sums	15
3	Markov Chains	16
3.1	Discrete-time Markov Chains	16
3.1.1	n -step transition probabilities	16
3.2	First Step Analysis	17
3.2.1	The General Absorbing Markov Chain	18
3.3	Random Walk	19
3.4	Branching Process	20
3.4.1	Generating Functions	22
4	The Long Run Behavior of Markov Chains	23
4.1	Regular Transition Probability Matrices	23
4.2	Doubly Stochastic Matrices	24
4.3	Interpretation of π	24
4.4	Irreducible Markov Chains	24
4.4.1	Recurrent and Transient States	25
4.5	Periodicity	26

5	Poisson Process	28
5.1	The Law of Rare Events	28
5.2	Poisson Process	28
5.3	Nonhomogeneous Poisson Process	29
5.3.1	Time change	29
5.4	The Law of Rare Events (cont'd)	31
5.5	Waiting time distribution	32

1 Probability Review

1.1 Basic Definitions

Definition 1.1.1 (Probability Space). A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple consisting of a set Ω called the *sample space*, a set $\mathcal{F} \subseteq \Omega$ satisfying certain closure properties, and a function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ that assigns probabilities to events in a coherent way.

Requirements for \mathcal{F} :

- (i) $\Omega \in \mathcal{F}$.
- (ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$.
- (iii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Requirements for \mathbb{P} :

- (i) $\mathbb{P}(\Omega) = 1$.
- (ii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint (meaning $E_i \cap E_j = \emptyset$ for $i \neq j$), then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Definition 1.1.2 (Random Variable). A *random variable* is a function $X : \Omega \rightarrow \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ whenever B is a "nice" subset of \mathbb{R} .

Example 1.1.3. $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, $\mathbb{P}(\{H\}) = \frac{1}{2}$. $X(H) = 1$, $X(T) = 0$.

$$\mathbb{P}(X = 1) = \mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(X = 0) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

1.2 Overview

Definition 1.2.1 (Stochastic Process). A *stochastic process* is a collection $\{X_t : t \in T\}$ of random variables $X_t : \Omega \rightarrow S \subseteq \mathbb{R}$ all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here T is some index set (typically representing time) and S is the *state space*. One write this as

$X : \Omega \times T \rightarrow S$, $(\omega, t) \mapsto X_t(\omega)$. For a given outcome $\omega \in \Omega$, we get a sample path trajectory $X(\omega) : T \rightarrow S, t \mapsto X_t(\omega)$. A stochastic process can then be thought of as a random function.

The theme of this course is what can we say about the distribution of trajectories?

Example 1.2.2 (Branching Process (DTDS)). $X_0 = 1$, one individual in the 0th generation individuals produce a random number of offspring, i.i.d. $(\xi_i^{(n)})_{i \in \mathbb{N}, n \in \mathbb{N}_0}$.

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

One interesting question would be what is $\mathbb{P}(X_n = 0 \text{ eventually})$, the probability of dying out?

Example 1.2.3 (Poisson Process (CTDS)). Recall that the Poisson distribution is used to model the number of occurrences of a rare event in some fixed period of time. The Poisson process $(N_t)_{t \geq 0}$ models the number of occurrences throughout time. $N_t = \#$ of occurrences by time t .

1.3 Useful Properties

(i) (*DeMorgan*)

$$(E \cup F)^c = E^c \cap F^c, \quad (E \cap F)^c = E^c \cup F^c.$$

(ii) (*Complementation*)

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c).$$

(iii) (*Inclusion-exclusion*)

$$\begin{aligned} \mathbb{P}(E \cup F) &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) \\ \mathbb{P}\left(\bigcup_{i=1}^n E_i\right) &= \sum_{j=1}^n (-1)^{j-1} \sum_{S \in [n]: |S|=j} \mathbb{P}\left(\bigcap_{i \in S} E_i\right). \end{aligned}$$

(iv) (*Partitioning*) If $\bigsqcup_{i=1}^{\infty} E_i = \Omega$, then

$$\mathbb{P}(F) = \mathbb{P}\left(\bigsqcup_{i=1}^{\infty} (F \cap E_i)\right) = \sum_{i=1}^{\infty} \mathbb{P}(F \cap E_i)$$

1.4 Conditional Probability

Conditioning: For $\mathbb{P}(F) > 0$,

$$\mathbb{P}(E | F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

$\mathbb{P}(\cdot | F)$ defines a new probability measure on (Ω, \mathcal{F}) .

Multiplication rule:

$$\mathbb{P}(E \cap F) = \mathbb{P}(F) \mathbb{P}(E | F).$$

If $\bigcup_{i=1}^{\infty} F_i = \Omega$, then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E \cap F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i) \mathbb{P}(E | F_i).$$

Bayes' rule:

$$\mathbb{P}(F_j | E) = \frac{\mathbb{P}(F_j) \mathbb{P}(E | F_j)}{\sum_{i=1}^{\infty} \mathbb{P}(F_i) \mathbb{P}(E | F_i)}$$

1.5 Random Variables

1.5.1 Discrete Random Variables

If $X : \Omega \rightarrow S \subseteq \mathbb{R}$ is discrete,

$$\mathbb{P}(X \in E) = \sum_{x \in E} \mathbb{P}(X = x) = \sum_{x \in E \cap S} \mathbb{P}(X = x).$$

1.5.2 Indicator Random Variable

$$X(\omega) = \mathbb{I}_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

1.5.2.1 Binomial Random Variable

$$X = \sum_{i=1}^n \mathbb{I}_{E_i}, \quad \mathbb{P}(E_i) = p.$$

$$p_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

1.5.3 Continuous Random Variables

If X continuous,

$$\mathbb{P}(X \in E) = \int_E f_X(x) dx.$$

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x) dx.$$

1.5.3.1 Exponential Random Variable

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

1.5.3.2 Gaussian Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

1.5.4 Cumulative Distribution Function (CDF)

$$F_X : \mathbb{R} \rightarrow [0, 1],$$

$$F_X(r) = \mathbb{P}(X \leq r) = \mathbb{P}(X \in (-\infty, r]).$$

If X is discrete,

$$F_X(r) = \sum_{x_i \leq r} p_X(x_i).$$

If X is continuous,

$$F_X(r) = \int_{-\infty}^r f_X(x) dx.$$

$$\frac{d}{dr} F_X(r) = f_X(r).$$

1.5.5 Expectation

1.5.5.1 Discrete case

$$\mathbb{E}[X] = \sum_{x_i \in S} x_i \mathbb{P}(X = x_i).$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

$$\mathbb{E}[g(X)] = \sum_{x_i \in S} g(x_i) \mathbb{P}(X = x_i).$$

1.5.5.2 Continuous case

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \mathbb{P}(X \geq x) dx.$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

1.5.6 Variance

$$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

1.5.7 Moments

$$\mathbb{E}[X^m] = \int_0^\infty mx^{m-1}\mathbb{P}(X \geq x)dx.$$

1.5.8 Joint Distribution

1.5.8.1 Discrete

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

1.5.8.2 Continuous

$$\mathbb{P}((X,Y) \in E) = \int \int_E f_{X,Y}(x,y)dx dy$$

1.5.8.3 Marginal Distribution

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y).$$

$$f_X(x) = \int_{y \in S_y} f_{X,Y}(x,y)dy$$

1.5.9 Independence

$$p_{X,Y}(x,y) = p_X(x)p_Y(y).$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x)F_Y(y).$$

1.5.10 Linearity of Expectation

$$\mathbb{E} \left[\sum_{i=1}^n c_i X_i \right] = \sum_{i=1}^n c_i \mathbb{E}[X_i]$$

If $(X_i)_{i=1}^n$ independent,

$$(g(X_i))_{i=1}^n$$

independent.

$$\mathbb{E} \left[\prod_{i=1}^n g(X_i) \right] = \prod_{i=1}^n \mathbb{E}[g(x_i)]$$

$$\text{Var} \left(\sum_{i=1}^n x_i \right) = \sum_{i=1}^n \text{Var}(x_i)$$

In general,

$$\text{Var} \left(\sum_{i=1}^n x_i \right) = \sum_{i,j=1}^n \text{Cov}(x_i, x_j)$$

1.5.11 Convolution

Discrete case: X, Y discrete $X \perp\!\!\!\perp Y$

$$\begin{aligned} \mathbb{P}(X + Y = z) &= \sum_Y \mathbb{P}(X + Y = z, Y = y) \\ &= \sum_y \mathbb{P}(X = z - y, Y = y) \\ &= \sum_y \mathbb{P}(X = z - y) \mathbb{P}(Y = y) \quad (= \sum_x \mathbb{P}(X = x) \mathbb{P}(Y = z - x)). \end{aligned}$$

If X, Y are \mathbb{Z} -valued, this becomes

$$\begin{aligned} \mathbb{P}(X + Y = n) &= \sum_{k=-\infty}^{\infty} \mathbb{P}(X = n - k) \mathbb{P}(Y = k) \\ &= \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k) \mathbb{P}(Y = n - k) \\ &= (\mathbb{P}_X * \mathbb{P}_Y)(n) \end{aligned}$$

Example 1.5.1 (Poisson). $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, $X + Y \sim \text{Poisson}(\lambda + \mu)$

$$\begin{aligned} \mathbb{P}(X + Y = n) &= \sum_{k=0}^n \mathbb{P}(X = k) \mathbb{P}(Y = n - k) \\ &= \sum_{k=0}^n e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{n-k}}{(n-k)!} \\ &= e^{-(\lambda+\mu)} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k} \\ &= \frac{e^{-(\lambda+\mu)}}{n!} (\lambda + \mu)^n \\ &= \mathbb{P}(Z = n) \end{aligned}$$

where $Z \sim \text{Poisson}(\lambda + \mu)$.

Continuous case: X, Y continuous

$$\begin{aligned} \mathbb{P}(X + Y \leq z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^z f_X(x) f_Y(y - x) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_X(x) f_Y(y - x) dx dy \end{aligned}$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y) dy = f_X * f_Y.$$

Example 1.5.2 (Convolution in uniform distributions). $X, Y \sim U[0, 1]$, $X \perp\!\!\!\perp Y$.

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

$$f_X(x) = \mathbb{I}_{[0,1]}(x) \quad f_Y(y) = \mathbb{I}_{[0,1]}(y)$$

so

$$\begin{aligned} f_{X+Y}(z) &= \int_{x \in [0,1], z-x \in [0,1]} 1dx \\ &= \int_{x \in [0,1], x \in [-1+z, z]} 1dx \\ &= \int_{\max(0, -1+z)}^{\min(1, z)} 1dx \\ &= \min(1, z) - \max(0, -1+z). \end{aligned}$$

1.5.12 Gamma Distribution

Definition 1.5.3 (Gamma function). Let $\alpha > 0$. The *gamma function* $\Gamma : (0, \infty) \rightarrow (0, \infty)$ is defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx = \mathbb{E}[X^{\alpha-1}]$$

where $X \sim \text{Exp}(1)$. Let $\alpha, \lambda > 0$. The $\text{Gamma}(\alpha, \lambda)$ distribution is defined by

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

Exercise 1.5.4. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. (Hint: use induction)

1.5.13 Moment Generating Function

Definition 1.5.5 (MGF). For a random variable X , the *moment generating function* (MGF) is the function $M_X : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$,

$$M_X(t) = \mathbb{E}[e^{tX}].$$

If $M_X(t) < +\infty$ for $t \in (-\epsilon, \epsilon)$, then

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k \mathbb{E}[X^k]}{k!} \text{ for } |t| < \epsilon$$

For independent RVs $(X_i)_{i=1}^n$,

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$$

Exercise 1.5.6. If $X \sim \text{Exp}(\lambda)$, then $M_X(t) = \frac{\lambda}{\lambda-t}$ if $t < \lambda$, $+\infty$ otherwise.

If $X \sim \text{Gamma}(n, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda-t} \right)^n.$$

If $X \sim \text{Gamma}(\alpha, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda-t} \right)^{\alpha}.$$

1.6 Conditional Probability (Cont'd)

Exercise 1.6.1 (Generalization). $(X_i)_{i=1}^n, (Y_j)_{j=1}^m$

$$p_{X_1, \dots, X_n | Y_1, \dots, Y_m} (x_1, \dots, x_n | y_1, \dots, y_m) = ?$$

Example 1.6.2. Let $M \in \mathbb{N}$ and $p, q \in (0, 1)$. Consider $N \sim \text{Bin}(M, q)$ and $X \sim \text{Bin}(N, p)$. What is the distribution of X ?

$$\begin{aligned} \mathbb{P}(X = k) &= \sum_{n=0}^M \mathbb{P}(N = n) \mathbb{P}(X = k | N = n) \\ &= \sum_{n=0}^M \binom{M}{n} q^n (1-q)^{M-n} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{p^k}{k!} \sum_{n=k}^M \frac{M!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\ &= \frac{p^k}{k!(M-k)!} \sum_{n=k}^M \frac{M!(M-k)!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\ &= \binom{M}{k} p^k q^k \sum_{n=k}^M \binom{M-k}{n-k} q^{n-k} (1-q)^{M-j} (1-p)^{n-k} \\ &= \binom{M}{k} p^k q^k \sum_{t=0}^{M-k} \binom{M-k}{t} (q(1-p))^t (1-q)^{M-k-t} \\ &= \binom{M}{k} (pq)^k (q(1-p) + (1-q))^{M-k} \\ &= \binom{M}{k} (pq)^k (1-pq)^{M-k}. \end{aligned}$$

Thus, $X \sim \text{Bin}(M, pq)$.

Remark. What if $k > n$ in $\mathbb{P}(X = k | N = n)$ above in the first line? The probability is simply 0.

Question. Why does this answer make sense?

Answer. Think about retesting whenever we succeeded for the first M trials. Then X is simply the number of trials with double successes, thus we have the pq parameter.

Exercise 1.6.3. Consider $N \sim \text{Poisson}(\lambda)$, $X \sim \text{Bin}(N, p)$. What is the distribution of X ?

Answer. $X \sim \text{Poisson}(\lambda p)$.

Question. How can we interpret this?

Answer. We can interpret X as the number of customers visiting a store who purchase something.

1.6.1 Conditional Expectation

For X, Y discrete, $g : \mathbb{R} \rightarrow \mathbb{R}$. Assume $\mathbb{E}[|g(X)|] = \sum_x |g(x)p_X(x)| < \infty$.

Definition 1.6.4 (Conditional expectation). The *conditional expectation* is defined as

$$\mathbb{E}[g(X) | Y = y] = \sum_x g(x)p_{X|Y}(x|y)$$

if $p_Y(y) \neq 0$.

Remark. Note that $\mathbb{E}[g(X) | Y = y]$ is a real number, whereas $\mathbb{E}[g(X) | Y]$ is a random variable.

1.6.1.1 Tower Property

$$\begin{aligned} \mathbb{E}[\mathbb{E}[g(X) | Y]] &= \mathbb{E}\left[\sum_y \mathbb{E}[g(X) | Y = y]p_Y(y)\right] \\ &= \sum_y \mathbb{E}[g(X) | Y = y]p_Y(y) \\ &= \sum_y \sum_x g(x)p_{X|Y}(x|y)p_Y(y) \\ &= \sum_x g(x) \sum_y p_{X|Y}(x|y)p_Y(y) \\ &= \sum_x g(x)p_X(x) \\ &= \mathbb{E}[g(X)]. \end{aligned}$$

Remark. One intuitive example would be considering the averages of heights of students from a classroom. We divide it into several groups and let Y denote the whichever group we select and let $\mathbb{E}[g(X) | Y]$ be the average of those from group Y . Then the average height of the entire classroom $\mathbb{E}[g(X)]$ is equivalent to the average of the average of heights of each group, which is $\mathbb{E}[\mathbb{E}[g(X) | Y]]$.

Properties of conditional expectations:

1. $\mathbb{E}[c_1g(x_1) + c_2h(x_2) | Y = y] = c_1\mathbb{E}[g(X_1) | Y = y] + c_2\mathbb{E}[h(X_2) | Y = y]$
2. If $g \geq 0$, then $\mathbb{E}[g(x) | Y = y] \geq 0$.
3. $\mathbb{E}[f(X, Y) | Y = y] = \mathbb{E}[f(X, y) | Y = y]$.
4. If $X \perp\!\!\!\perp Y$, $\mathbb{E}[g(X) | Y = y] = \mathbb{E}[g(X)]$
5. $\mathbb{E}[g(x)h(y) | Y = y] = h(y)\mathbb{E}[g(x) | Y = y]$
6. $\mathbb{E}[g(x)h(y)] = \sum_y h(y)\mathbb{E}[g(x) | Y = y]p_Y(y) = \mathbb{E}[h(Y)\mathbb{E}[g(X) | Y]]$

Proof of 3.

$$\begin{aligned}\mathbb{E}[f(X, Y) \mid Y = y] &= \sum_{x, z} f(x, z) p_{X, Y \mid Y}(x, z \mid y) \\ &= \sum_{x, z} f(x, z) \frac{p_{X, Y, Y}(x, z, y)}{p_Y(y)} \\ &= \sum_x f(x, y) \frac{p_{X, Y}(x, y)}{p_Y(y)} \\ &= \mathbb{E}[f(X, y) \mid Y = y].\end{aligned}$$

□

Remark. $\mathbb{E}[f(X, y)] \neq \mathbb{E}[f(X, y) \mid Y = y]$.

2 Random Sums

Definition 2.0.1. Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d random variables, N be a \mathbb{N}_0 -valued random variable, $N \perp\!\!\!\perp (\xi_i)_{i=1}^{\infty}$. The *random sum* is defined as

$$X = \sum_{i=1}^N \xi_i = \sum_{n=0}^{\infty} \left(\sum_{i=1}^n \xi_i \right) \mathbf{1}_{N=n} = \begin{cases} \sum_{i=1}^n \xi_i & \text{if } N = n \geq 1 \\ 0 & \text{if } N = 0. \end{cases}$$

Question. What is the distribution of X ?

Let X, N be random variables. N is \mathbb{N}_0 -valued. The condition CDF is

$$F_{X|N}(x|n) = \mathbb{P}(X \leq x \mid N = n)$$

if $\mathbb{P}(N = n) \neq 0$. This is an actual CDF, but for the random variable $X \mid N = n$.

Suppose that X is continuous and $F_{X|N}(x|n)$ is a differentiable function of x for each n such that $p_N(n) > 0$. The conditional PDF is

$$f_{X|N}(x|n) = \frac{d}{dx} F_{X|N}(x|n).$$

$$\begin{aligned} \int_a^b f_{X|N}(x|n) dx &= F_{X|N}(b|n) - F_{X|N}(a|n) \\ &= \mathbb{P}(X \in [a, b] \mid N = n). \end{aligned}$$

Answer.

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) \mathbb{P}(X \leq x \mid N = n).$$

$$f_X(x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) f_{X|N}(x|n).$$

2.1 Mean and Variance of Random Sums

Assume $\mathbb{E}[N] = \nu$ and $\mathbb{E}[\xi_i] = \mu$. Then

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid N]] \\ &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^N \xi_i \mid N\right]\right] \\ &= \mathbb{E}[N\mathbb{E}[\xi_1]] \\ &= \mathbb{E}[N\mu] \\ &= \mu\nu.\end{aligned}$$

3 Markov Chains

3.1 Discrete-time Markov Chains

Definition 3.1.1 (Markov process). A is a stochastic process $(X_t)_{t \in T}$ such that the future, given the present, is independent of the past.

Definition 3.1.2 (Markov property). The *Markov property* for a DTDS stochastic process is

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

Example 3.1.3 (Gambler's ruin). $(X_n)_{n=0}^\infty$, X_n = your wealth after n turns. Stop if $X_n = 0$ or 5. Each play, you win \$1 with probability p and lose \$1 with probability $1 - p$ independently of all previous plays. This process satisfies the markov property.

Example 3.1.4 (Ehrenfest model). Box of N particles. X_n = number of particles on the left side at time n . $N - X_n$ be the number of particles on the other side.

$$\begin{aligned}\mathbb{P}(X_{n+1} = i + 1 \mid X_n = i) &= \frac{N - i}{N} \\ \mathbb{P}(X_{n+1} = i - 1 \mid X_n = i) &= \frac{i}{N}.\end{aligned}$$

Theorem 3.1.5.

Joint PMF of the Markov Chain is determined by initial distribution and $P = (p_{i,j})_{i,j \in S}$.

Proof.

$$\begin{aligned}\mathbb{P}(X_n = i_n, \dots, X_0 = i_0) &= \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \mathbb{P}(X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ &= p_{i_{n-1}, i_n} p_{i_{n-2}, i_{n-1}} \cdots p_{i_0, i_1} \mathbb{P}(X_0 = i_0).\end{aligned}$$

□

3.1.1 n -step transition probabilities

$$p_{i,j} = \mathbb{P}(X_{n+1} = j \mid X_n = i).$$

Theorem 3.1.6.

$$p_{i,j}^{(m)} = \mathbb{P}(X_{n+m} = j \mid X_n = i) = (P^m)_{i,j}.$$

Proof.

$$\begin{aligned} \mathbb{P}(X_{n+m+1} = j \mid X_n = i) &= \sum_{k \in S} \mathbb{P}(X_{n+m+1} = j, X_{n+m} = k \mid X_n = i) \\ &= \sum_{k \in S} \mathbb{P}(X_{n+m+1} = j \mid X_{n+m} = k) \mathbb{P}(X_{n+m} = k \mid X_n = i). \end{aligned}$$

□

Example 3.1.7.

$$\begin{aligned} p_{i,j}^{(2)} &= \mathbb{P}(X_2 = j \mid X_0 = i) \\ &= \sum_{k \in S} \mathbb{P}(X_2 = j, X_1 = k \mid X_0 = i) \\ &= \sum_{k \in S} \mathbb{P}(X_2 = j \mid X_1 = k, X_0 = i) \mathbb{P}(X_1 = k \mid X_0 = i) \\ &= \sum_{k \in S} P_{i,k} P_{k,j} \\ &= (P^2)_{i,j} \end{aligned}$$

Example 3.1.8 (Inventory model). X_n = inventory that you have of this product after the n th business day. If $X_n \leq s$, place an order that brings inventory back to S by next morning. ξ_n = demand on n th day and (ξ_n) are i.i.d..

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} \mathbb{P}(\xi_{n+1} = S - j) & \text{if } i \leq s \\ \mathbb{P}(\xi_{n+1} = i - j) & \text{if } i > s. \end{cases}$$

$\lim_{n \rightarrow \infty} \mathbb{P}(X_n < 0)$ = chance of excess demand.

3.2 First Step Analysis

Consider $(X_n)_{n \geq 0}$ Markov chain on $\{1, \dots, r\} \cup \{r+1, \dots, N\}$ where $\{1, \dots, r\}$ are the *transient states* and $\{r+1, \dots, N\}$ are the *absorbing states* such that

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{i,j}^{(n)} &= 0 & \forall i, j \in \{1, \dots, r\} \\ \lim_{n \rightarrow \infty} p_{i,i}^{(n)} &= 1 & \forall i \in \{r+1, \dots, N\} \end{aligned}$$

Then we can express the transition matrix P as

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix},$$

where Q and R is some transition matrices for the corresponding partitioned states and 0 is the zero matrix and I is the identity matrix.

Let $T = \min \{n \geq 0 : X_n \geq r + 1\}$ be the time of absorption and X_T be the state we get absorbed into. Define $u_{i,k} = \mathbb{P}(X_T = k \mid X_0 = i)$. Then we have

$$\begin{aligned} u_{i,k} &= \sum_{j=1}^N \mathbb{P}(X_T = k, X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^N \mathbb{P}(X_T = k \mid X_1 = j, X_0 = i) \mathbb{P}(X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^N p_{i,j} \mathbb{P}(X_T = k \mid X_1 = j) \\ &= \sum_{j=1}^N \mathbb{P}(X_T = k \mid X_0 = j) \\ &= \sum_{j=1}^r p_{i,j} u_{j,k} + \sum_{j=r+1, j \neq k}^N p_{i,j} u_{j,k} + p_{i,k} u_{k,k}. \end{aligned}$$

Thus,

$$u_{i,k} = \sum_{j=1}^r P_{i,j} u_{j,k} + p_{i,k}$$

Hence, we have

$$U = QU + R \implies (I - Q)U = R \implies U = (I - Q)^{-1}R,$$

where U contains all the $(u_{i,k})_{i \in \{1, \dots, r\}, k \in \{r+1, \dots, N\}}$.

3.2.1 The General Absorbing Markov Chain

Let's suppose that associated with each transient state i is a rate $g(i)$ and that we wish to determine the mean total rate that is accumulated up to absorption. Let v_i be this mean total amount, where the subscript i denotes the starting position $X_0 = i$, i.e.,

$$v_i = \mathbb{E} \left[\sum_{n=0}^{T-1} g(X_n) \mid X_0 = i \right]$$

The choice $g = 1$ will give $v_i = \mathbb{E}[T \mid X_0 = i]$. We can also write for $i \in \{1, \dots, r\}$ that

$$\begin{aligned} v_i &= g(i) + \mathbb{E} \left[\sum_{n=1}^{T-1} g(X_n) \mid X_0 = i \right] \\ &= g(i) + \sum_{j=1}^N p_{i,j} v_j \quad (= \sum_{j=1}^N p_{i,j} (g(i) + v_i)). \end{aligned}$$

Then we can condense this into the following form

$$v = g + Qv \implies v = (I - Q)^{-1}g.$$

where $v = (v_i)_{i \in \{1, \dots, r\}}$ and $g = (g(i))_{i \in \{1, \dots, r\}}$.

3.3 Random Walk

$(\xi_n)_{n=1}^\infty$ i.i.d and \mathbb{Z} -valued. Then

$$X_n = \sum_{i=0}^n \xi_i.$$

$$\begin{aligned} \mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1) &= \mathbb{P}(\xi_{n+1} = j - i \mid \xi_n = i - i_{n-1}, \dots, \xi_1 = i_1) \\ &= \mathbb{P}(\xi_{n+1} = j - i) \\ &= \mathbb{P}(\xi_{n+1} = j - i \mid X_n = i). \end{aligned}$$

Example 3.3.1 (Gambler's Ruin). Win 1 dollar with probability p and lose 1 dollar with probability $q = 1 - p$. Stop when we lose all money or make N dollars. We are interested in $u_k = \mathbb{P}(X_T = 0 \mid X_0 = k)$ and $v_k = \mathbb{E}[T \mid X_0 = k]$. Clearly $u_0 = 1, u_N = 0$. For $k = 1, \dots, N-1$, we have

$$u_k = pu_{k+1} + qu_{k-1} \implies q(u_k - u_{k-1}) = p(u_{k+1} - u_k)$$

Let $\Delta_{k+1} = u_{k+1} - u_k$. Then we have

$$\begin{aligned} q\Delta_k &= p\Delta_{k+1} \\ \Delta_{k+1} &= \frac{q}{p}\Delta_k = \dots = \left(\frac{q}{p}\right)^k \Delta_1. \\ \sum_{i=1}^m \Delta_i &= \Delta_1 \sum_{i=1}^m \left(\frac{q}{p}\right)^{i-1} = \sum_{i=1}^m u_i - u_{i-1} = u_m - u_0 = u_m - 1 \end{aligned}$$

Thus,

$$u_m = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}} \quad m = 1, \dots, N$$

When $m = N$,

$$0 = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \frac{q}{p}} \implies \Delta_1 = -\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N}.$$

Substituting the expression for Δ_1 gives

$$u_m = 1 + \left(-\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N} \right) \left(\frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}} \right) = 1 - \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^m - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}.$$

Note that $p \neq q$. If $p = q$, then

$$\sum_{i=1}^m \Delta_i = \Delta_1 m = u_m - 1 \implies u_m = \frac{N - m}{N}.$$

If we take limit as $N \rightarrow \infty$ for $p \leq q$, then

$$\lim_{N \rightarrow \infty} u_m = 1,$$

which implies that we will be broke at the end no matter how much money we started with. If $p > q$, then

$$\lim_{N \rightarrow \infty} u_m = \left(\frac{q}{p}\right)^m.$$

If m is large, then this quantity becomes small. This implies that if $p > q$ and we started with a lot of money, then the chance of us being broke ultimately becomes smaller.

Now let's compute v_k when $p = q = \frac{1}{2}$. Clearly, $v_0 = 0$ and $v_N = 0$. For $k = 1, \dots, N-1$, we have

$$v_k = 1 + \frac{1}{2}v_{k+1} + \frac{1}{2}v_{k-1}.$$

Let $\Delta_k = v_k - v_{k-1}$. Then we have

$$0 = 1 + \frac{1}{2}(\Delta_{k+1} - \Delta_k).$$

Summing both sides gives

$$\sum_{k=1}^m 0 = m + \sum_{k=1}^m \frac{1}{2}(\Delta_{k+1} - \Delta_k) \implies \Delta_1 = 2m + \Delta_{m+1} \quad m = 0, \dots, N-1.$$

Then

$$\begin{aligned} \sum_{m=0}^k \Delta_1 &= \sum_{m=0}^k (2m + \Delta_{m+1}) \\ (k+1)\Delta_1 &= (k+1)v_1 = \sum_{m=0}^k 2m + \sum_{m=0}^k \Delta_{m+1} \\ (k+1)v_1 &= k(k+1) + (v_{k+1} - v_0) \implies (k+1)v_1 = k(k+1) + v_{k+1}. \end{aligned}$$

Take $k = N-1$ gives

$$Nv_1 = (N-1)N + 0 \implies v_1 = N-1.$$

Then

$$v_{k+1} = (k+1)(v_1 - k) = (k+1)(N-1-k).$$

Hence,

$$v_k = k(N-k).$$

3.4 Branching Process

$(\xi_i^{(n)})_{i=1, n=0}^{\infty, \infty}$ i.i.d. \mathbb{N}_0 -valued random variables where $\xi_i^{(n)}$ is the number of offspring of i th individual in n th generation. $X_0 = 1$. $\mathbb{E}[\xi_i] = \mu$ and $\text{Var}(\xi_i) = \sigma^2$. The population of at time $n+1$ is

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

Our goal is to compute $\mathbb{P}(X_n = 0 \text{ eventually} \mid X_0 = 1)$. But let's first compute $\mathbb{E}[X_{n+1}]$ and $\text{Var}(X_{n+1})$. Recall that

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^N \xi_i\right] &= \mathbb{E}[N]\mathbb{E}[\xi_i] \\ \text{Var}\left(\sum_{i=1}^N \xi_i\right) &= \text{Var}(N)\mathbb{E}[\xi_i]^2 + \text{Var}(\xi_i)\mathbb{E}[N]. \end{aligned}$$

Then we have

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mu \mathbb{E}[X_n] = \mu^{n+1} \\ \text{Var}(X_{n+1}) &= \mu^2 \text{Var}(X_n) + \mu^n \sigma^2.\end{aligned}$$

$$\begin{aligned}c_0 &= \text{Var}(X_0) = 0 \\ c_n &= \text{Var}(X_n) \\ c_{n+1} &= \mu^2 c_n + \mu^n \sigma^2.\end{aligned}$$

Define the generating function $f(x)$ as

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{n+1} x^{n+1} = \mu^2 x \sum_{n=0}^{\infty} c_n x^n + \sigma^2 x \sum_{n=0}^{\infty} \mu^n x^n \\ &= \mu^2 x f(x) + \frac{\sigma^2 x}{1 - \mu x}.\end{aligned}$$

Then

$$f(x) = \frac{\sigma^2 x}{(1 - \mu x)(1 - \mu^2 x)} = \sigma^2 x \frac{1}{1 - \mu x} \frac{1}{1 - \mu^2 x}.$$

Since

$$\sum_{j=1}^{\infty} c_j x^j = \sigma^2 x \sum_{n=0}^{\infty} \mu^n x^n \cdot \sum_{m=0}^{\infty} \mu^{2m} x^m,$$

the coefficient of $x^{j-1} = \sum_{k=0}^{j-1} x^k x^{j-1-k}$ is

$$c_j = \sum_{k=0}^{j-1} \mu^k \mu^{2(j-1-k)}.$$

Thus

$$\text{Var}(X_n) = \sigma^2 \mu^{n-1} \cdot \begin{cases} n & \text{if } \mu = 1 \\ \frac{1 - \mu^{n-1}}{1 - \mu} & \text{if } \mu \neq 1. \end{cases}$$

Remark. When $\mu = 1$, expectation is constant, variance is growing linearly. When $\mu \neq 1$, expectation is increasing/decreasing geometrically, same with variance.

Now let $T = \min \{n \geq 0 : X_n = 0\}$ be the time the population dies out and let $u_n = \mathbb{P}(T \leq n) = \mathbb{P}(X_n = 0)$. Then $\lim_{n \rightarrow \infty} u_n$ is the probability of extinction.

$$u_{n+1} = \sum_{k=0}^{\infty} p_k u_n^k$$

where $p_k = \mathbb{P}(\xi = k)$. We have $u_0 = 0, u_1 = p_0$.

Let $\phi_\xi : [0, 1] \rightarrow [0, 1]$ be the generating function of ξ defined by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

Then we have

$$u_{n+1} = \phi(u_n) \implies u_\infty = \lim_{n \rightarrow \infty} u_{n+1} = \lim_{n \rightarrow \infty} \phi(u_n) \implies u_\infty = \phi(\lim_{n \rightarrow \infty} u_n) = \phi(u_\infty).$$

Thus, u_∞ is a fixed point for ϕ .

3.4.1 Generating Functions

Given any \mathbb{N}_0 -valued random variable ξ with $p_k = \mathbb{P}(\xi = k)$. Then the generating function is given by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

ϕ_ξ completely recovers the distribution of ξ . We have $\phi_\xi(0) = p_0 \phi_\xi(1) = 1$. We can recover p_k via

$$p_k = \frac{\phi^{(k)}(0)}{k!}.$$

Then

$$\mathbb{E}[X] = \phi'(1) = \sum_{k=1}^{\infty} k p_k.$$

In fact, one can check that

$$\begin{aligned} \phi''(1) &= \mathbb{E}[X(X-1)] \\ \phi^{(k)}(1) &= \mathbb{E}[X(X-1)\cdots(X-k+1)]. \end{aligned}$$

Suppose ξ_1, \dots, ξ_n i.i.d has generating function ϕ . Then $Z = \sum_{i=1}^n \xi_i$ has the following generating function:

$$\phi_Z(s) = \mathbb{E}[s^Z] = \mathbb{E}[s^{\sum_{i=1}^n \xi_i}] = \prod_{i=1}^n \mathbb{E}[s^{\xi_i}] = \phi^n(s).$$

But if instead we have $Z = \sum_{i=1}^N \xi_i$ where N is a random variable and N has generating function g_N . Then the generating function would be

$$\begin{aligned} \mathbb{E}[s^{\sum_{i=1}^N \xi_i}] &= \sum_{n=0}^{\infty} \mathbb{P}(N = n) \phi^n(s) \\ &= g_N(\phi(s)). \end{aligned}$$

Now suppose $\phi_n(s)$ is the generating function of X_n defined by

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

Then applying the result from above, we have

$$\phi_{n+1}(s) = \phi_n(\phi(s)) = \phi^{(n+1)}(s).$$

4 The Long Run Behavior of Markov Chains

4.1 Regular Transition Probability Matrices

Suppose $(X_n)_{n=0}^{\infty}$ is a Markov Chain on $\{1, \dots, N\}$.

Definition 4.1.1 (Regular). $(X_n)_{n=0}^{\infty}$ is *regular* if $\exists m \geq 1$ such that P^m has all positive entries.

Theorem 4.1.2.

If $(X_n)_{n=0}^{\infty}$ is regular, there exists a limiting distribution $\hat{\pi} = (\pi_1, \dots, \pi_N)$, where $\pi_i > 0$ and $\sum_{i=1}^N \pi_i = 1$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j, \quad \forall i, j \in \{1, \dots, N\}.$$

This limiting distribution does not depend on initial distribution.

Corollary 4.1.3. Suppose $\mathbb{P}(X_0 = i) = \alpha_i$. Then

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j) = \pi_j > 0.$$

Question. How do we find π ?

Theorem 4.1.4.

π is the unique solution to $\pi P = \pi$ satisfying $\langle \hat{\pi}, \hat{1} \rangle = \sum_{i=1}^N \pi_i = 1$ and $\pi_i \geq 0$ for all i .

Proof. We first check that π is a solution.

$$\begin{aligned} \pi &= \lim_{n \rightarrow \infty} \pi P^n \\ \pi P &= \lim_{n \rightarrow \infty} \pi P^{n+1} = \lim_{m \rightarrow \infty} \pi P^m = \pi. \end{aligned}$$

Now we check for uniqueness. Let τ be any distribution that satisfies $\tau P = \tau$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau P^n &= \pi \\ \lim_{n \rightarrow \infty} \tau &= \pi \\ \tau &= \pi. \end{aligned}$$

□

4.2 Doubly Stochastic Matrices

Definition 4.2.1 (Doubly stochastic). A matrix is *doubly stochastic* if every row and column sums to 1.

Proposition 4.2.2. If (X_n) is doubly stochastic, then

$$\pi = \left(\frac{1}{N}, \dots, \frac{1}{N} \right).$$

Proof.

$$\begin{aligned} \left(\frac{1}{N}, \dots, \frac{1}{N} \right) P &= \left(\frac{1}{N}, \dots, \frac{1}{N} \right) \begin{pmatrix} P_{1,1} & \cdots & \vdots \\ P_{2,1} & \cdots & \vdots \\ \vdots & \ddots & \vdots \\ P_{N,1} & \cdots & \vdots \end{pmatrix} \\ &= \left(\frac{1}{N} \sum_{i=1}^N P_{i,1}, \dots, \frac{1}{N} \sum_{i=1}^N P_{i,m} \right) \\ &= \left(\frac{1}{N}, \dots, \frac{1}{N} \right). \end{aligned}$$

□

4.3 Interpretation of π

- $\pi_j = \lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_0 = i) = \lim_{n \rightarrow \infty} P_{i,j}^n$.
- π_j is the mean fraction of time the process spends in state j .

$$\begin{aligned} \pi_j &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{1}{n+1} \sum_{m=0}^n \mathbf{1}\{X_m = j\} \mid X_0 = i \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{m=0}^n P_{i,j}^m \\ &= \pi_j. \end{aligned}$$

4.4 Irreducible Markov Chains

Definition 4.4.1 (Accessible). State j is *accessible* from state i if there exists n such that $P_{i,j}^{(n)} > 0$.

Definition 4.4.2 (Irreducible). If $\forall i, j \in S$, and $i \leftrightarrow j$ (i and j communicate with each other), we say that $(X_n)_{n \geq 0}$ is *irreducible*.

4.4.1 Recurrent and Transient States

Let $f_{i,i}^{(n)}$ be the probability of first return to i at step n given that we started at i at step 0, i.e.,

$$f_{i,i}^{(n)} = \mathbb{P}(X_n = i, X_{n-1} \neq i, \dots, X_1 \neq i \mid X_0 = i).$$

We have $f_{i,i}^{(0)} = 0$.

Claim. For $n \geq 1$,

$$P_{i,i}^{(n)} = \sum_{k=0}^n f_{i,i}^{(k)} P_{i,i}^{(n-k)} = \sum_{k=1}^n f_{i,i}^{(k)} P_{i,i}^{(n-k)}.$$

Proof. Let E_k be the event that the first return to i is at time k . Then

$$\begin{aligned} P_{i,i}^{(n)} &= \mathbb{P}(X_n = i \mid X_0 = i) \\ &= \sum_{k=1}^n \mathbb{P}(X_n = i, E_k \mid X_0 = i) \\ &= \sum_{k=1}^n \mathbb{P}(X_n = i \mid E_k, X_0 = i) \mathbb{P}(E_k \mid X_0 = i) \\ &= \sum_{k=1}^n \mathbb{P}(X_n = i \mid X_k = i) f_{i,i}^{(k)} \\ &= \sum_{k=1}^n P_{i,i}^{(n-k)} f_{i,i}^{(k)}. \end{aligned}$$

□

Question. What is the chance of returning to i eventually?

Answer. $\sum_{n=0}^{\infty} f_{i,i}^{(n)}$.

Definition 4.4.3 (Recurrent). State i is *recurrent* if and only if $f_{i,i} := \sum_{n=0}^{\infty} f_{i,i}^{(n)} = 1$.

Definition 4.4.4 (Transient). State i is *transient* if and only if $f_{i,i} < 1$.

Let $M = \sum_{n=1}^{\infty} \mathbf{1}\{X_n = i\}$ be the number of returns to i . If i is recurrent, then

$$\mathbb{E}[M \mid X_0 = i] = \infty.$$

If i is transient, then

$$\begin{aligned} \mathbb{E}[M \mid X_0 = i] &= \sum_{m=1}^{\infty} \mathbb{P}(M \geq m \mid X_0 = i) \\ &= \sum_{m=1}^{\infty} f_{i,i}^{(m)} \\ &= \frac{f_{i,i}}{1 - f_{i,i}}. \end{aligned}$$

Theorem 4.4.5.

A state i is recurrent if and only if

$$\sum_{n=1}^{\infty} P_{i,i}^{(n)} = \infty.$$

Equivalently, i is transient if and only if

$$\sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty.$$

Proof. i is transient $\iff \mathbb{E}[M \mid X_0 = i] < \infty \iff \sum_{n=1}^{\infty} P_{i,i}^{(n)} < \infty.$ □

Proposition 4.4.6. If $i \leftrightarrow j$, then i recurrent $\iff j$ recurrent.

Proof. We know that $P_{ij}^{(n)} > 0$ and $P_{ji}^{(m)} > 0$. Note that

$$\begin{aligned} P_{j,j}^{(m+k+n)} &\geq P_{j,i}^{(m)} P_{i,i}^{(k)} P_{i,j}^{(n)} \\ \sum_k P_{j,j}^{(m+k+n)} &\geq \sum_k P_{j,i}^{(m)} P_{i,i}^{(k)} P_{i,j}^{(n)} = P_{j,i}^{(m)} \left(\sum_k P_{i,i}^{(k)} \right) P_{i,j}^{(n)} \geq \infty. \end{aligned}$$

□

4.5 Periodicity

Definition 4.5.1 (Period). For $i \in S$,

$$d(i) = \gcd\{n : P_{i,i}^{(n)} > 0\}$$

is the *period* of i .

Remark. $d(i) \neq \min_n \{n : P_{i,i}^{(n)} > 0\}.$

Fact.

1. $i \leftrightarrow j \implies d(i) = d(j).$
2. $\exists N, \forall n \geq N, P_{i,i}^{(nd(i))} > 0.$
3. $P_{j,i}^{(m)} > 0 \implies P_{j,i}^{(m+nd(i))} > 0$ for $n \geq N.$

Definition 4.5.2 (Aperiodic). Assume a MC is irreducible. If $d(i) = 1$ for some $i \in S$, then the MC is *aperiodic*.

Theorem 4.5.3.

$(X_n)_{n=0}^\infty$ regular $\iff (X_n)_{n=0}^\infty$ irreducible and aperiodic.

Let $R_i = \min \{n \geq 1 : X_n = i\}$. Then

$$\mathbb{P}(R_i = k \mid X_0 = i) = f_{i,i}^{(k)}.$$

If i is recurrent,

$$\mathbb{P}(R_i < \infty) = \sum_k f_{i,i}^{(k)} = 1.$$

Theorem 4.5.4.

Assume (X_n) aperiodic, irreducible, and recurrent, define

$$\mathbb{E}[R_i \mid X_0 = i] = m_i,$$

which is the mean time of first return. Then

$$\lim_{n \rightarrow \infty} P_{i,i}^{(n)} = \lim_{n \rightarrow \infty} P_{j,i}^{(n)} = \frac{1}{m_i}.$$

Definition 4.5.5 (Positive/null recurrent). If $m_i < \infty$, the MC is *positive recurrent*. Otherwise, it is *null recurrent*.

Proposition 4.5.6.

$$\prod_{i=0}^{\infty} (1 - p_i) = 0 \iff \sum_{i=0}^{\infty} p_i = \infty.$$

Theorem 4.5.7.

If $(X_n)_{n=0}^\infty$ is positive recurrent, aperiodic, and irreducible, then π is a limiting distribution that is the unique solution to

$$\pi = \pi P, \quad \sum_i \pi_i = 1, \quad \pi_i \geq 0.$$

5 Poisson Process

Recall that Poisson counts the number of occurrences of a rare event.

5.1 The Law of Rare Events

Consider

$$\text{Bin}\left(n, \frac{\lambda}{n}\right) \stackrel{D}{=} X_n.$$

$$\begin{aligned}\mathbb{E}[X_n] &= \lambda \\ \lim_{n \rightarrow \infty} \mathbb{P}(X_n = k) &= \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{e^{-\lambda} \lambda^k}{k!}.\end{aligned}$$

5.2 Poisson Process

Idea: count the number of occurrences up to a certain time.

Definition 5.2.1 (Poisson Process). The \mathbb{N}_0 -valued process $(N_t)_{t \geq 0}$ is a $PP(\lambda)$ if

- (i) $N_0 = 0$,
- (ii) Increments are independent: for any $t_0 < t_1 < \dots < t_n$,

$$N_{t_n} - N_{t_{n-1}}, \dots, N_{t_1} - N_{t_0}$$

are independent,

- (iii) $N_{t+h} - N_t \sim \text{Poisson}(\lambda h)$.

Example 5.2.2. Customers arriving to a store with rate $\lambda = 10/\text{hour}$. Store opens at 8am. What is the probability that 4 customers arrived by noon and 10 by 4pm?

$$\mathbb{P}(N_4 = 4, N_8 = 10) = \mathbb{P}(N_8 - N_4 = 6, N_4 = 4) = \mathbb{P}(\text{Poisson}(4\lambda) = 6) \mathbb{P}(\text{Poisson}(4\lambda) = 4)$$

Question. Why the $PP(\lambda)$?

Answer. Strong uniqueness and computationally tractable.

$$\begin{aligned}
 \mathbb{P}(N_{t+h} - N_t = 1) &= e^{-\lambda h} \lambda h. \\
 \lim_{h \rightarrow \infty} \frac{\mathbb{P}(N_{t+h} - N_t = 1)}{h} &= \lim_{h \rightarrow \infty} \lambda e^{-\lambda h} = \lambda. \\
 \frac{\mathbb{P}(N_{t+h} - N_t \geq 2)}{h} &= \lambda e^{-\lambda h} \sum_{k=2}^{\infty} \frac{(\lambda h)^{k-1}}{k!} \\
 &= \lambda e^{-\lambda h} \sum_{k=1}^{\infty} \frac{(\lambda h)^k}{(k+1)!} \\
 &\leq \lambda e^{-\lambda h} \sum_{k=1}^{\infty} \frac{(\lambda h)^k}{k!} \\
 &= \lambda e^{-\lambda h} (e^{\lambda h} - 1) \rightarrow 0.
 \end{aligned}$$

Remark. This shows that it is impossible to have more than two arrivals at the exact same time.

Question. What if

$$\lim_{h \rightarrow 0} \frac{\mathbb{P}(N_{t+h} - N_t = 1)}{h} = \lambda(t) \neq \lambda?$$

Answer. This can be done by reducing to a time shift of homogenous Poisson Process.

5.3 Nonhomogeneous Poisson Process

Definition 5.3.1 (Nonhomogeneous Poisson Process). Same assumptions with homogeneous Poisson Process except that we have a rate function $\lambda(t)$ and that

$$N_{t+h} - N_t \sim \text{Poisson} \left(\int_t^{t+h} \lambda(u) du \right).$$

In fact when $\lambda(u)$ is constant, we can recover a homogeneous Poisson Process.

5.3.1 Time change

Suppose we have a continuous Poisson Process $(N_t)_{t \geq 0}$ with $\lambda(t) > 0$. Define

$$\Lambda(t) = \int_0^t \lambda(u) du.$$

Let $Y_s = X_{\Lambda^{-1}(s)}$. Let's check that this PP is homogeneous.

$$\begin{aligned}
 Y_{s+h} - Y_s &= X_{\Lambda^{-1}(s+h)} - X_{\Lambda^{-1}(s)} \\
 &\stackrel{D}{=} PP \left(\int_{\Lambda^{-1}(s)}^{\Lambda^{-1}(s+h)} \lambda(u) du \right) \\
 &= PP \left(\int_0^{\Lambda^{-1}(s+h)} \lambda(u) du - \int_0^{\Lambda^{-1}(s)} \lambda(u) du \right) \\
 &= PP \left(\Lambda(\Lambda^{-1}(s+h)) - \Lambda(\Lambda^{-1}(s)) \right) \\
 &= PP(s+h-s) \\
 &= PP(h).
 \end{aligned}$$

Theorem 5.3.2.

Let $(N_t)_{t \geq 0}$ \mathbb{N}_0 -valued be a stochastic process such that

- (i) $N_0 = 0$,
- (ii) increments are independent,
- (iii) $\mathbb{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$ as $h \downarrow 0$,
- (iv) $\mathbb{P}(N_{t+h} - N_t \geq 2) = o(h)$ as $h \downarrow 0$.

Then $(N_t)_{t \geq 0}$ is $PP(\lambda)$.

Lemma 5.3.3. If $\epsilon \sim \text{Ber}(p_i)$, $\mu = \sum_{i=1}^n p_i$, $S_n = \sum_{i=1}^n \epsilon_i$, $X_n \sim \text{Poisson}(\mu)$, then

$$|\mathbb{P}(S_n = k) - \mathbb{P}(X_n = k)| \leq \sum_{i=1}^n p_i^2$$

Proof.

$$X_n = \sum_{i=1}^n Y_i \quad Y_i \sim \text{Poisson}(p_i).$$

Define $C = \{\epsilon_i = Y_i \text{ for all } i\}$. Then

$$\begin{aligned}
 |\mathbb{P}(S_n = k, C) - \mathbb{P}(X_n = k, C) + \mathbb{P}(S_n = k, C^c) - \mathbb{P}(X_n = k, C^c)| &= |\mathbb{P}(S_n = k, C^c) - \mathbb{P}(X_n = k, C^c)| \\
 &\leq \mathbb{P}(C^c) \\
 &\leq \sum_{i=1}^n \mathbb{P}(\epsilon_i \neq Y_i) \\
 &\leq \sum_{i=1}^n p_i^2.
 \end{aligned}$$

The last line follows because $\mathbb{P}(\epsilon \neq Y) \leq p^2 \implies \mathbb{P}(\epsilon = Y) \geq 1 - p^2$. □

5.4 The Law of Rare Events (cont'd)

$$\text{Bin}\left(n, \frac{\lambda}{n}\right) \xrightarrow{D} \text{Poisson}(\lambda) \quad \text{as } n \rightarrow \infty.$$

What about the error?

Consider $\epsilon_i \sim \text{Ber}(p_i)$. Then

$$\mathbb{P}\left(\sum_{i=1}^n \epsilon_i = k\right) = \sum_{x_1 + \dots + x_n = k, x_i \in \{0,1\}} \prod_{i=1}^n p_i^{x_i} (1 - p_i)^{1-x_i}.$$

Theorem 5.4.1.

Suppose $(M_t)_{t \geq 0}$ is a counting process such that

- (i) $M_0 = 0$,
- (ii) independent increments,
- (iii) distribution of $M_s - M_t$ only depends on $s - t$,
- (iv) $\mathbb{P}(M_{t+h} - M_t = 1) = \lambda h + o(h)$,
- (v) $\mathbb{P}(M_{t+h} - M_t \geq 2) = o(h)$.

Then $(M_t)_{t \geq 0}$ is a $PP(\lambda)$.

Proof. It suffices to show $\mathbb{P}(M_t = k) - \mathbb{P}(\text{Poisson}(\lambda t) = k) = 0$.

Idea:

$$\begin{aligned} M_t &= \sum_{i=1}^n M_{ti/n} - M_{t(i-1)/n} \\ &\approx \sum_{i=1}^n \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = 1} \quad (\text{by (v)}) \\ &\approx \text{Poisson}(\lambda t + o(t)) \quad (\text{by (iv)}) \\ &\rightarrow \text{Poisson}(\lambda t). \end{aligned}$$

$$\begin{aligned} \left| \mathbb{P}\left(\sum_{i=1}^n M_{ti/n} - M_{t(i-1)/n} = k\right) - \mathbb{P}\left(\sum_{i=1}^n \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = 1} = k\right) \right| &\leq \sum_{i=1}^n \mathbb{P}(M_{ti/n} - M_{t(i-1)/n} \neq \mathbf{1}_{M_{ti/n} - M_{t(i-1)/n} = 1}) \\ &= \sum_{i=1}^n o\left(\frac{t}{n}\right) \\ &= o(t) \quad \text{as } n \rightarrow \infty. \end{aligned}$$

□

5.5 Waiting time distribution

Let W_n be the waiting time for the n th arrival. Then

$$\begin{aligned}\mathbb{P}(W_n \geq t) &= \mathbb{P}(N_t \leq n-1) \\ &= \sum_{k=0}^{n-1} \mathbb{P}(N_t = k) \\ &= \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}.\end{aligned}$$

Then taking derivative gives

$$\begin{aligned}-\lambda e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \sum_{k=1}^{n-1} \frac{(\lambda t)^{k-1}}{(k-1)!} &= -\lambda e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} + \lambda e^{-\lambda t} \sum_{k=0}^{n-2} \frac{(\lambda t)^k}{k!} \\ &= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}, \quad t \geq 0,\end{aligned}$$

which is exactly the density of $\text{Gamma}(n, \lambda)$.

Consider $n = 1$. We have $W_1 \sim \text{Exp}(\lambda)$.

Theorem 5.5.1.

Let $S_n = W_{n+1} - W_n$ be the n th interarrival time. Then $S_n \sim \text{Exp}(\lambda)$.

Proof. Let ξ_i be i.i.d. $\text{Exp}(\lambda)$ and let

$$T_n = \sum_{i=1}^n \xi_i, \quad M_t = \max \left\{ n : \sum_{i=1}^n \xi_i \leq t \right\}.$$

Claim. $(M_t)_{t \geq 0}$ is $PP(\lambda)$.

Proof. We need to show the following:

- $M_0 = 0$.

Proof. Trivial. □

- $M_u \sim \text{Poisson}(\lambda u)$.

Proof. $M_h \stackrel{D}{=} M_{t+h} - M_t \stackrel{D}{=} \text{Poisson}(\lambda h)$.

$$\begin{aligned}
 \mathbb{P}(T_n \leq u < T_{n+1}) &= \mathbb{P}(T_n \leq u < T_n + \xi_{n+1}) \\
 &= \int_0^u \int_{u-T}^\infty \lambda e^{-\lambda \xi} \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} d\xi dT \\
 &= \int_0^u \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} \int_{u-T}^\infty \lambda e^{-\lambda \xi} d\xi dT \\
 &= \int_0^u \lambda e^{-\lambda T} \frac{(\lambda T)^{n-1}}{(n-1)!} e^{-\lambda(u-T)} dT \\
 &= \int_0^u \lambda e^{-\lambda u} \frac{(\lambda T)^{n-1}}{(n-1)!} dT \\
 &= e^{-\lambda u} \frac{(\lambda u)^n}{n!} \\
 &= \mathbb{P}(\text{Poisson}(\lambda u) = n).
 \end{aligned}$$

□

- $(M_{t+s} - M_s)_{t \geq 0}$ is independent of $(M_r)_{0 \leq r \leq s}$ and has the same distribution as $(M_t)_{t \geq 0}$.

□

□