Math 105 Notes
Real Analysis II

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Contents

Lecture 1

Lebesgue Measure and Integral

1.1 Motivation of Lebesgue Integral

Question. Given some subset $\Omega \subseteq \mathbb{R}^n$, and some real-valued function $f : \Omega \to \mathbb{R}$, is it possible to integrate f on Ω to obtain some number $\int_\Omega f$?

In one dimension we have the notion of a *Riemann integral* $\int_{[a,b]} f$, which answers the question when Ω is an interval $\Omega = [a.b]$ and f is *Riemann integrable.*

Remark. Note that every piecewise continuous function is Riemann integrable, and in particular every piecewise constant function is Riemann integrable. However, not all functions are Riemann integrable.

Although it is possible to extend such notion of a Riemann integral to higher dimensions, one can still only integrate "Riemann integrable" functions, which is a pretty small class of functions. (For example, the pointwise limit of Riemann integrable functions may not be Riemann integrable, but the uniform limits are.)
Thus, we need some truly satisfactory notion of integration that can handle even discontinuous functions, which leads us to the notion of the *Lebesgue integral*, which can handle a very large class of functions.

1.2 Lebesgue Measure

To understand how to compute an integral $\int_\Omega f$, we must know how to measure the length/area/volume of Ω. To understand the connection between the two, observe that if we integrate the function 1 on set Ω, then we obtain the length of Ω (if it's one-dimensional), the area of (if it's two-dimensional), the volume of Ω (if it's three-dimensional).

To avoid considering cases of the dimension, we will use the notion of *measure* $m(\Omega)$ to represent either length, area, volume (or hypervolumes, etc.) of Ω.

Question. Can we "consistently" define measure for any subset $\Omega \subseteq \mathbb{R}^n$?

Here are some desirable properties that we want:

(i) *Monotoncity*: if $A \subseteq B \subseteq \mathbb{R}^n$, then $m(A) \leq m(B)$.

(ii) *Additivity:* if $A \cap B = \emptyset$, then $m(A \cup B) = m(A) + m(B)$.

(iii) *Translational-invariance*: $m(x + A) = m(A)$ for any $x \in \mathbb{R}^n$, and $A \subseteq \mathbb{R}^n$.

Unfortunately, such a measure does not exist. It is impossible to define such a measure for *every* subset of $\mathbb{R}^n$, which goes against one's intuitive concept of volume (one interesting example of this failure of intuition is the *Banach-Tarski paradox*, in which a unit ball in $\mathbb{R}^3$ can be decomposed and reassembled to form two complete and disjoint unit balls via translations and rotations, thus doubling the volume.)

These paradoxes indicate that it is impossible to assign a measure to every single subset of $\mathbb{R}^n$, and so we can solve this issue by just simply consider measuring a certain class of sets in $\mathbb{R}^n$ called the *measurable sets*. By restricting our attention to these sets, we are able to define a measure with above properties.

1.2.1 Measurable Sets

Let $\mathbb{R}^n$ be a Euclidean space. For every measurable set $\Omega \subseteq \mathbb{R}^n$, we will define the *Lebesgue measure* $m(\Omega) \in [0, \infty]$. The measurable set will obey the following properties:

(i) *Borel property*: every open set in $\mathbb{R}^n$ is measurable, as is every closed set.

(ii) *Complementarity*: if Ω is measrable, then $\mathbb{R}^n \backslash \Omega$ is also measurable.

(iii) *Boolean algebra property*: if $(\Omega_j)_{j \in J}$ is any finite collection of measurable sets, then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.

(iv) *σ-algebra property*: if $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets, then the union $\bigcup_{j \in J} \Omega_j$ and intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.

For every measurable set Ω, we assign the *Lebesgue measure* $m(\Omega)$ that will satify the following properties:

(i) *Empty set*: $m(\emptyset) = 0$.

(ii) *Positivity*: $0 \leq m(\Omega) \leq +\infty$ for every measurable set Ω.

(iii) *Monotonicity*: if $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.

(iv) *Finite sub-additivity*: if $(A_j)_{j\in J}$ are a finite collection of measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m(A_j).$$

(v) *Finite additivity*: if $(A_j)_{j\in J}$ are a finite collection of disjoint measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j).$$

(vi) *Countable sub-additivity*: if $(A_j)_{j\in J}$ are a countable collection of measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m(A_j).$$

(vii) *Countable additivity*: if $(A_j)_{j\in J}$ are a countable collection of disjoint measurable sets, then

$$m\left(\bigcup_{j\in J} A_j\right) = \sum_{j\in J} m(A_j).$$

(viii) *Normalization* The unit cube $[0,1]^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : 0 \leq x_j \leq 1 \text{ for all } 1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.

(ix) *Translation invariance*: if Ω is a measurable set, and $x \in \mathbb{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ is also measurable, and $m(x + \Omega) = m(\Omega)$.

Theorem 1.2.1 (Existence of Lebesgue measure)**.** *There exists a concept of a measurable set, and a way to assign a number $m(\Omega)$ to every measurable subset $\Omega \subseteq \mathbb{R}^n$, which obeys all of the properties above.*

Remark. Lebesgue measure is pretty much unique. However, there are other measures which would only obey some of the above axioms, and we may be interested in measures for other domains than Euclidean spaces. This leads to the subject of *measure theory*.

1.2.2 Outer measure

Before we construct Lebesgue measure, we first discuss a naive approach to find the measure of a set by covering the set with boxes and then add up the volume of each box. This leads to the notion of *outer measure* which can be applied to every set and obeys most of the properties except for the additivity properties. We would need to modify it slightly later to recover the additivity property.

Definition 1.2.2 (Open box)**.** An *open box* $B \subseteq \mathbb{R}^n$ is any set of the form

$$B = \prod_{i=1}^{n}(a_i, b_i),$$

where $b_i \geq a_i$ are real numbers. We define the *volume* $\mathrm{vol}(B)$ of such box to be the number

$$\mathrm{vol}(B) \coloneqq \prod_{i=1}^{n}(b_i - a_i).$$

Remark. Open boxes are open in general dimension.

Definition 1.2.3 (Covering by boxes)**.** Let $\Omega \subseteq \mathbb{R}^n$. A collection $(B_j)_{j\in J}$ of boxes *cover* Ω iff $\Omega \subseteq \bigcup_{j\in J} B_j$.

Suppose $\Omega \subseteq \mathbb{R}^n$ can be covered by a finite or countable collection of boxes $(B_j)_{j\in J}$. If we wish Ω to be measurable and have a measure obeying the monotonicity and sub-additivity properties, and $m(B_j) = \mathrm{vol}(B_j)$ for every box j, then we must have

$$m(\Omega) \leq m\left(\bigcup_{j\in J} B_j\right) \leq \sum_{j\in J} m(B_j) = \sum_{j\in J} \mathrm{vol}(B_j).$$

Thus, we conclude

$$m(\Omega) \leq \inf\left\{\sum_{j\in J} \mathrm{vol}(B_j) : \bigcup_{j\in J} B_j \supseteq \Omega; J \text{ at most countable}\right\}.$$

Then we have our formal definition of the outer measure.

Definition 1.2.4 (Outer measure)**.** If Ω is a set, we define the *outer measure* $m^*(\Omega)$ of Ω to be the quantity

$$m^*(\Omega) \coloneqq \inf\left\{\sum_{j\in J} \mathrm{vol}(B_j) : \bigcup_{j\in J} B_j \supseteq \Omega; J \text{ at most countable}\right\}.$$

Remark. Outer measure can be defined for every single set (not just the measurable ones) ebcause we can take the infimum of any non-empty set.

Lemma 1.2.5 (Properties of outer measure)**.**

(i) (Empty set) The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.

(ii) *(Positivity)* $0 \leq m^*(\Omega) \leq +\infty$ *for every measurable set* Ω.

(iii) *(Monotonicity) If* $A \subseteq B \subseteq \mathbb{R}^n$, *then* $m^*(A) \leq m^*(B)$.

(iv) *(Finite sub-additivity) If* $(A_j)_{j\in J}$ *are a finite collection of subsets of* $\mathbb{R}^n$, *then*

$$m^*\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m^*(A_j).$$

(v) *(Countable sub-additivity) If* $(A_j)_{j\in J}$ *are a countable collection of subsets of* $\mathbb{R}^n$, *then*

$$m^*\left(\bigcup_{j\in J} A_j\right) \leq \sum_{j\in J} m^*(A_j).$$

(vi) *(Translation invariance) If* Ω *is a subset of* $\mathbb{R}^n$, *and* $x \in \mathbb{R}^n$, *then* $m^*(x+\Omega) = m^*(\Omega)$.

Proof. Exercise. □

Proposition 1.2.6 (Outer measure of closed box)**.** *For any closed box* $B = \prod_{i=1}^n [a_i, b_i]$, *we have*

$$m^*(B) = \prod_{i=1}^n (b_i - a_i).$$

Proof. □

Corollary 1.2.7 (Outer measure of open box)**.** *For any open box* $B = \prod_{i=1}^n (a_i, b_i)$, *we have*

$$m^*(B) = vol(B) = \prod_{i=1}^n (b_i - a_i).$$

Proof. □