Topology and Analysis
MATH 202A

Instructor: Marc Rieffel

Kelvin Lee

UC Berkeley

Contents

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1 Metric Spaces

1.1 Fundamentals

Definition 1.1.1. Let X be a set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ that satisfies:

(i) $d(x, y) = d(y, x)\ \forall\ x, y \in X$

(ii) $d(x, y) \leq d(x, z) + d(z, y)\ \forall\ x, y, z \in X$

(iii) $d(x, y) = 0 \iff x = y$

If a function d satisfies (i), (ii) above, and $d(x, x) = 0$ for all $x \in X$, then d is a **semi-metric**.

Example 1.1.2. On $\mathbb{C}^n$, the following are common metrics:

- $d_p(x, y) = \Big(\sum\limits_{j=1}^{n} |x_j - y_j|^p \Big)^{1/p}$ for $p \geq 1$
- $d_\infty(x, y) = \sup \{|x_j - y_j| : 1 \leq j \leq n\}$

(Verify that these are metrics.)

Fact. If $S \subseteq X$, and d is a metric on X, then d is a metric on S.

Definition 1.1.3. (X, d) where d is a metric of X is called a **metric space**.

Remark. If $Y \subseteq X$, restrict d to $Y \times Y \subseteq X \times X$, denoted $d\,|_Y$, then $(Y, d\,|_Y)$ is a metric space.

Definition 1.1.4. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ such that:

(i) $\|cv\| = |c| \cdot \|v\|$ for $c \in$ or and $v \in V$

(ii) $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$

(iii) $\|v\| = 0$ implies $v = 0$

A function that satisfies only (i) and (ii) above is called a **seminorm**.

Remark. Any norm $\|\cdot\|$ on X induces the metric $d(x,y) \coloneqq \|x-y\|$.

Example 1.1.5. Let V be the space of continuous functions on $[0,1]$. Then $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ is a norm on V.
It can also be shown that $\|f\|_p \coloneqq \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ is a norm on V.

Definition 1.1.6. Let (X, d_x) and (Y, d_y) be metric spaces. A function $f : X \to Y$ is **isometric** if $d_y(f(x_1), f(x_2)) = d_x(x_1, x_2)$ for all $x_1, x_2 \in X$.

Remark. All isometries are injective.

Example 1.1.7. If $S \subseteq X$, and $f : S \to X$ is defnined by $f(x) = x$ (inclusion), then f is an isometry. If f is also onto, then f is viewed as an isometric isomorphism between (X, d_x) and (Y, d_y). f^{-1} is also an isomorphism.

Definition 1.1.8. A function $f : X \to Y$ is **Lipschitz** if there is a constant $k \geq 0$ such that $d_y(f(x_1), f(x_2)) \leq k \cdot d_x(x_1, x_2)$. The smallest such constant is the **Lipschitz constant** for f.

Definition 1.1.9. $f : X \to Y$ is **uniformly continuous** if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_y(f(x_1), f(x_2)) < \epsilon$ whenever $d_x(x_1, x_2) < \delta$.

Remark. It is easy to see that if f is Lipschitz, then it is uniformly continuous.

Definition 1.1.10. $f : X \to Y$ is **continuous at** x_0 if $\forall \epsilon > 0$, $\exists \delta(x_0) > 0$ such that $d_y(f(x), f(x_0)) < \epsilon$ whenever $d_x(x, x_0) < \delta(x_0)$. We say f is **continuous** if it is continuous at every $x \in X$.

Definition 1.1.11. A sequence $\{x_n\}$ in X **converges** to $x^* \in X$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n \geq N$, we have $d(x_n, x^*) < \epsilon$.

> **Proposition 1.1.12.** If a function $f : X \to Y$ is continuous and $\{x_n\} \to x^*$, then $f(x_n) \to f(x^*)$.

Proof. Let $\epsilon > 0$. Since f is continuous at x^*, there exists a $\delta > 0$ such that

$$\forall x, d_X(x, x^*) < \delta \implies d_Y(f(x), f(x^*)) < \epsilon$$

Since $\{x_n\} \to x^*$, there is some N such that for all $n \geq N, d_X(x_n, x^*) < \delta$. Then, we can see that $d_Y(f(x_n) - f(x^*)) < \epsilon$ for all $n \geq N$. Thus $\{f(x_n)\} \to f(x^*)$. □

Definition 1.1.13. $S \subseteq X$ is **dense** in X if $\forall x \in X$ and $\epsilon > 0$, $\exists s \in S$ such that $d(x, s) < \epsilon$. That is, for any point $x \in X$, there is a point $s \in S$ which is arbitrarily close to x.

Proposition 1.1.14. Let S be dense in X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that $f(s) = g(s)$ for all $s \in S$. Then $f = g$ on X.

Proof. Because S is dense in X, for any $x \in X$, there exists a sequence $\{s_n\} \subseteq S$ which converges to x (choose any point s_n in S such that $d(s_n, x) < \epsilon$). By the previous proposition, we can conclude that $\{f(s_n) = g(s_n)\} \to f(x) = g(x)$. □

Definition 1.1.15. A sequence $\{x_n\}$ is **Cauchy** if $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. A metric space is **complete** if every Cauchy sequence in it converges.

Example 1.1.16. Consider $(\mathbb{Q}, |\cdot|)$. We know there exists a Cauchy sequence converging to $\sqrt{2} \in \mathbb{R}$, but in this metric space, $\sqrt{2}$ is not an element, so this sequence does not converge, hence this metric space is not complete.

1.2 Completion of a Metric Space

Proposition 1.2.1. If $f : X \to Y$ is uniformly continuous, and $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Proof. Let $\epsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that if $x, x' \in X$ and $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$. Since $\{x_n\}$ is Cauchy, there is an N such that if $m, n \geq N$ then $d(x_m, x_n) < \delta$. Thus

$$d(f(x_m), f(x_n)) < \epsilon \quad \forall m, n \geq N.$$

This proves that $\{f(x_n)\}$ is Cauchy. □

Definition 1.2.2. Let (X, d) be a metric space. A complete metric space $(\widetilde{X}, \widetilde{d})$, together with an isometric function $f : X \to \widetilde{X}$ with dense range is a **completion** of (X, d).

Remark. Completions are unique up to isomorphism.

Proposition 1.2.3. If $((Y_1, d_1), f_1)$ and $((Y_2, d_2), f_2)$ are completions of (X, d), then $\exists$ an onto isometry (metric space isomorphism) $g : Y_1 \to Y_2$ with $f_2 = g \circ f_1$.

This can be visualized by the following commutative diagram:

$$\begin{tikzcd} & Y_1 \arrow[d, "g"] \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & Y_2 \end{tikzcd}$$

Every metric space has a completion, and the proof will be constructive. The completion will be defnined using equivalence classes of Cauchy sequences. We will need the following lemmas to support the construction.

Lemma 1.2.4. If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in X, then the sequence $\{d(s_n, t_n)\}$ in $\mathbb{R}$ converges.

Proof. Let $\epsilon > 0$, and let N such that for every $m, n > N, d\left(s_m, s_n\right), d\left(t_m, t_n\right) < \epsilon/2$. It follows that

$$\left|d\left(s_m, t_m\right) - d\left(s_n, t_n\right)\right| \leq d\left(s_m, s_n\right) + d\left(t_m, t_n\right) < \epsilon$$

and the sequence is Cauchy. Since $\mathbb{R}$ is complete, the sequence converges. □

Lemma 1.2.5. Let $\mathrm{CS}(X)$ denote the set of all Cauchy sequences in X. Then the relation $\{s_n\} \sim \{t_n\} \iff d(s_n, t_n) \to 0$ is an equivalence relation.

Proof. Reflexivity and symmetry are trivial. Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$. Then $d(s_n, t_n) \leq d(s_n, r_n) + d(r_n, t_n)$ for all $n \in \mathbb{N}$. The result follows immediately. □

Lemma 1.2.6. Let $\overline{X}$ be the set of all equivalence classes of $\mathrm{CS}(X)$ under the equivalence relation above. Then $\overline{d} : \overline{X} \to [0, \infty)$ defined by $\overline{d}(\{s_n\}, \{t_n\}) := \lim_{n\to\infty} d(s_n, t_n)$ is a metric on X.

Proof. First, note that by Lemma 1.2.4, $\overline{d}$ is always defined. Since we are dealing with equivalence classes, we must show that $\overline{d}$ is also well-defnined. Let $\xi, \eta \in X$, and let $\{x_n\}, \{s_n\} \in \xi$, and $\{y_n\}, \{t_n\} \in \eta$. We have $\lim d(x_n, s_n) = \lim d(y_n, t_n) = 0$. Thus, $d(s_n, t_n) \leq d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n)$. $\forall\ \epsilon > 0$, we can find $N \in \mathbb{N}$ such that both $d(s_n, x_n) < \epsilon/2$ and $d(y_n, t_n) < \epsilon/2$ for $n \geq N$. Then $|d(s_n, t_n) - d(x_n, y_n)| < \epsilon$. It follows that $d(\xi, \eta) = \lim d(x_n, y_n) = \lim d(s_n, t_n)$, so that d is indeed well-defnined.

Symmetry is trivial. The triangle inequality follows from the proof to Lemma 1.2.5. If $d(\xi, \eta) = 0$, then $\forall\ \{x_n\} \in \xi, \{y_n\} \in \eta$, we have $\lim d(x_n, y_n) = 0$, so in particular, $\{y_n\} \in \xi$, hence $\xi = \eta$. □

Theorem 1.2.7. Let (X, d_x) and (Y, d_y) be metric spaces with Y complete. If $S \subseteq X$ is dense, and $f : S \to Y$ is uniformly continuous, then $\exists$ a unique continuous extension $\overline{f} : X \to Y$ of f. In fact, $\overline{f}$ is uniformly continuous.

Proof. (Existence only) For $x \in X$, choose a Cauchy sequence $\{s_n\}$ in S converging to x. Then $\{f(s_n)\}$ is Cauchy in Y, so it converges to a point $p \in Y$. Set $\overline{f}(x) := p$. We show that $\overline{f}$ is well-defnined. Indeed, if $\{t_n\} \in \mathrm{CS}(S)$ and converges to x, then we have $\lim d_x(s_n, t_n) = 0$, implying that $\lim d_y(f(s_n), f(t_n)) = 0$. Therefore $\lim d_y(f(t_n), p) = 0$, so $\{f(t_n)\}$ converges to p also. It remains to show continuity, which is left as an exercise. $\square$

Theorem 1.2.8. Every metric space (X, d) has a completion.

Proof. As in Lemma 3, (X, d) is a completion of (X, d). We embed X in X by the isometry $\iota : X \to X$ defnined by $\iota(x) := [\{x, x, x, ...\}]$, where $[\cdot]$ denotes the corresponding equivalence class. Note that $d\Big|_X = d$, i.e., $d(\iota(x), \iota(y)) = d(x, y)$.
It remains to show that d has dense range, and that (X, d) is complete.

- Let $\xi \in X, \epsilon > 0, \{x_n\} \in \xi$. $\exists\ N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Then $d(\iota(x_N), \xi) = \lim_{n\to\infty} d(x_N, x_n) < \epsilon$. Therefore d has dense range by considering $\iota(x_N)$.
- Let $\{\xi_n\}$ be a Cauchy sequence in X. For each $m \in \mathbb{N}$, pick $x_m \in X$ such that $d(\iota(x_m), \xi_m) < 1/m$. Then $\{x_m\}$ is a Cauchy sequence, and it follows that $\{\xi_m\}$ converges to the equivalence class of $\{x_m\}$.

$\square$

Remark. Denote $C([0, 1])$ the space of continuous functions on $[0, 1]$. Consider the metric space $C([0, 1])$ induced by the norms $\|\cdot\|_\infty$ or $\|\cdot\|_p$. This space is not complete. It is easy to come up with a sequence of continuous functions converging under these norms to a function that is not continuous.

Remark. Let V be a vector space with norm $\|\cdot\|$. Consider V^∞, the space of all sequences of elements in V. This is also a vector space. It can be shown that $\mathrm{CS}(V)$ is a subspace of V^∞. Now let $\mathcal{N}(V)$ denote the set of all Cauchy sequences in V converging to 0. Then $\mathcal{N}(V)$ is a subspace of $\mathrm{CS}(V)$. If $\{v_n\}$ and $\{w_n\}$ are equivalent Cauchy sequences, then
$\|v_n - w_m\| \to 0$, so $\{v_n - w_n\} \in \mathcal{N}(V)$. Thus V is in fact the quotient space $\mathrm{CS}(V)/\mathcal{N}(V)$.

Fact. Any two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite dimensional vector space are **equivalent**, meaning that there are constants $c, C > 0$ such that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ for all x. If a function is continuous with respect to a particular norm, then it is easily seen that it is continuous with respect to any equivalent norm.

1.3 Openness

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a map between the two metric spaces. Recall that f is continuous at $x_0 \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$.

Definition 1.3.1 (Open ball)**.** Let (X, d_X) be a metric space. The **open ball** around $x_0 \in X$ with radius $r > 0$ is defined as

$$\mathcal{B}_r(x_0) = \{x \in X \mid d_X(x, x_0) < r\}.$$

Remark. For any open ball U in Y, there exists an open ball $\mathcal{O}$ in X such that if $x \in O$, then $f(x) \in U$.

Now we can rephrase continuity using the notion of open balls:

Definition 1.3.2 (Continuity). $f : X \to Y$ is **continuous at** x_0 if $\forall \epsilon > 0, \exists \delta > 0$ such that $f(\mathcal{B}_\delta(x_0)) \subseteq \mathcal{B}_\epsilon(f(x_0))$.

If $y \in \mathcal{B}_\epsilon(f(x_0))$ and $y = f(x)$ for some $x \in X$, let $\epsilon' = \epsilon - d(y, f(x_0)) > 0$. Then $\mathcal{B}_{\epsilon'}(y) \subseteq \mathcal{B}_\epsilon(f(x_0))$, so there exists $\delta' > 0$ such that $f(\mathcal{B}_{\delta'}(x)) \subseteq \mathcal{B}_\epsilon(f(x_0))$ If $x_1 \in f^{-1}(\mathcal{B}_\epsilon(f(x)))$, there is an open ball $\mathcal{B}_{\delta'}(x)$ such that $\mathcal{B}_{\delta'}(x_1) \subseteq f^{-1}(\mathcal{B}_\epsilon(f(x)))$ Thus $f^{-1}(\mathcal{B}_\epsilon(f(x)))$ is a union of open balls in X. Similarly, $f^{-1}(\mathcal{B}_\epsilon(y))$ is a union of open balls in X. This leads to the definition of open sets.

1.3.1 Open Sets

Definition 1.3.3 (Open set). A subset A of X is **open** if A is a union of open balls it contains, i.e. $\forall x \in A, \exists r > 0$ such that $\mathcal{B}_r(x) \subset A$.

Theorem 1.3.4. Let (X, d) be a metric space, and $\mathcal{T}$ be the collection of all open sets. Then

(i) If $\{\mathcal{O}_\alpha\}$ is an arbitrary collection of subsets in $\mathcal{T}$, then $\bigcup_\alpha \mathcal{O}_\alpha$ is open.

(ii) If $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is a finite collection of subsets in $\mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{O}_i$ is open.

(iii) $X \in \mathcal{T}$ (X is open).

Proof of (iii). If $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ are open, and $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, then there exist open balls $\mathcal{B}_{r_1}(x) \subseteq \mathcal{O}_1$, $\mathcal{B}_{r_2}(x) \subseteq \mathcal{O}_2, \ldots, \mathcal{B}_{r_n}(x) \subseteq \mathcal{O}_n$. Let $r = \min_{1 \le i \le n}\{r_i\}$. Then $\mathcal{B}_r(x) \subseteq \bigcap_{i=1}^n \mathcal{O}_i$. □

2 Topology

2.1 Topological Spaces

Definition 2.1.1 (Topology). Let X be a set. The **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying:

(i) $X, \emptyset \in \mathcal{T}$.

(ii) If any arbitrary family $\{\mathcal{O}_\alpha\} \subseteq \mathcal{T}$, then $\bigcup_\alpha \mathcal{O}_\alpha \in \mathcal{T}$.

(iii) If $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{O}_i \in \mathcal{T}$.

Definition 2.1.2 (Topological space). Let $\mathcal{T}$ be a topology on X. Then $(X, \mathcal{T})$ is a **topological space**. The sets in $\mathcal{T}$ are called **open sets** and the complements of the sets in $\mathcal{T}$ are **closed sets**.

Example 2.1.3. Let X be any nonempty set. Then $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are topologies on X. They are called the **discrete topology** and **indiscrete topology** respectively.

Example 2.1.4. Let X be a metric space. The collection of all open sets with respect to the metric is a topology on X.

Definition 2.1.5 (Interior). If $A \subseteq X$, the union of all open sets contained in A is called the **interior** of A, denoted by A°. This is the biggest open set contained in A.

Definition 2.1.6 (Closure). If $A \subseteq X$, the intersection of all closed sets containing A is called a **closure** of A, denoted by $\overline{A}$. This is the smallest closed set containing A.

Definition 2.1.7 (Dense). If $\overline{A} = X$, A is called **dense** in X.

Definition 2.1.8 (Strong/Weak topology). Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on a set X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. We say that $\mathcal{T}_1$ is **weaker** than $\mathcal{T}_2$, or equivalently $\mathcal{T}_2$ is **stronger** than $\mathcal{T}_1$.

2.2 Continuous Maps

Definition 2.2.1 (Continuity). Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is **continuous** if $\forall U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

2.2.1 Bases and Sub-bases

Proposition 2.2.2. Let X be a set and let $\mathcal{C}$ be a collection of topologies on X. Then $\bigcap_{\mathcal{T}\in\mathcal{C}} \mathcal{T}$ is a topology on X.

Then it follows that for any collection S of subsets of X, there is a unique weakest/smallest topology $\mathcal{T}$ on X containing S described as follows.

Definition 2.2.3 (Sub-base)**.** Let $\mathcal{T}(S) = \bigcap_{S\subseteq\mathcal{T}} \mathcal{T}$, the intersection of all topologies on X containing S. It is called the topology **generated** by S and S is the **sub-base** for $\mathcal{T}(S)$.

Definition 2.2.4 (Base)**.** A collection $\mathcal{B} \subseteq \mathcal{T}$ of subsets of a set X is called a **base** for $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Example 2.2.5. Let (X, d) be a metric space. The open balls form a base for the metric topology.

Remark. The intersections of two balls is usually not a ball. If $\mathcal{B}$ is a base, then the intersection of any two elements of $\mathcal{B}$ must be a union of elements of $\mathcal{B}$.

Proposition 2.2.6. If $S \subseteq \mathcal{P}(X)$, the topology $\mathcal{T}(S)$ generated by S consists of $\emptyset, X$, and all unions of finite intersections of members of S.

Proposition 2.2.7. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. If $\mathcal{T}_Y$ is generated by $\mathcal{B}$ (i.e. $\mathcal{B}$ is a sub-base for $\mathcal{T}_Y$), then $f : X \to Y$ is continuous $\iff$ $f^{-1}(U) \in \mathcal{T}_X$ for every $U \in \mathcal{B}$.

Proof. Note that f^{-1} preserves the Boolean operations for any collection of subsets of Y:

- $f^{-1} \bigcap_\alpha A_\alpha = \bigcap_\alpha f^{-1}(A_\alpha)$
- $f^{-1} \bigcup_\alpha A_\alpha = \bigcup_\alpha f^{-1}(A_\alpha)$
- If $A, B \subseteq Y$, then $f^{-1}(A\backslash B) = f^{-1}(A)\backslash f^{-1}(B)$

Then suppose $\{U_n\} \subseteq \mathcal{B}$ is some finite collection of open sets in $\mathcal{B}$, then

$$f^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} f^{-1}(U_i) \in \mathcal{T}_X.$$

Then any finite intersection of elements of $\mathcal{B}$ satisfies the condition as well, i.e. is a base. If $\{U_\alpha\} \subseteq \mathcal{B}$ is a collection (possibly infinite) of open sets in $\mathcal{B}$, then

$$f^{-1}\left(\bigcup_{\alpha} U_\alpha\right) = \bigcup_{\alpha} f^{-1}(U_\alpha) \in \mathcal{T}_X,$$

so $\bigcup_\alpha U_\alpha$ also satisfies the condition. Therefore, all open set U in $\mathcal{T}_Y$ satisfies $f^{-1}(U) \in \mathcal{T}_X$ so f is continuous. $\square$

2.2.2 Homeomorphism

Definition 2.2.8 (Homeomorphism)**.** If $f : X \to Y$ is bijective and f and f^{-1} are both continuous, f is called a **homeomorphism**, and X and Y are said to be homeomorphic.

2.3 Quotient Topologies

Let X be a set and let $(Y_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let $f_\alpha : X \to Y_\alpha$ be any function. Then there is a smallest topology on X for which each f_α is continuous, namely, the smallest topology having as sub-base all sets $f_\alpha^{-1}(U)$, where $U \in \mathcal{T}_\alpha$ for each α.

Definition 2.3.1. Let $(X, \mathcal{T}_X)$ be a topological space. Let Y be a set and $f : X \to Y$ be any function. Then there is a strongest topology on Y for which f is continuous. Namely,

$$\mathcal{T}_Y := \{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\},$$

which is called the **quotient topology** on Y for f.

Remark. Note that if $y \notin f(X)$, then $f^{-1}(\{y\}) = \varnothing$, so $\{y\}$ is open. Also, $f^{-1}(\{y\}^c) = X$, so $\{y\}$ is also closed. Therefore, on $f(X)^c$, the quotient topology is the discrete topology. Thus, we usually require $f : X \to Y$ to be onto.

Let $f : X \to Y$ be onto, and define the equivalence relation on X by $x_1 \sim x_2 \iff f(x_1) = f(x_2)$. f defines a partition, a collection of equivalence classes. Conversely, let $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ be the set of equivalence classes, $x \to [x]$, call it f. Given a topology on X, we call $X/\sim$ with the quotient topology on the projection $X \to X/\sim$ a quotient space.

Definition 2.3.2. Let Y be a set, and $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces and function $f_\alpha : X_\alpha \to Y$ be any function, then there is a strongest topology on Y where all f_α is continuous. Namely

$$\bigcap_\alpha \mathcal{T}_{Y_\alpha}, \text{ where } \mathcal{T}_{Y_\alpha} := \{A_\alpha \subseteq Y_\alpha : f^{-1}(A_\alpha) \in \mathcal{T}_\alpha\}$$

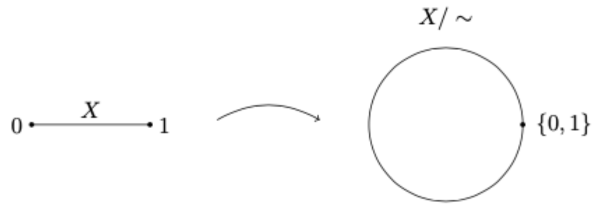
which is the intersection of all quotient topologies for each f_α. This is called a **final topology**.

Definition 2.3.3. Let G be a group. By an **action** of G on $(X, \mathcal{T})$, we mean a group homomorphism $\alpha : G \to \text{Homeo}(X, \mathcal{T})$. For any $x \in X$, its G-orbit is

$$\{\alpha_r(x) : r \in G\}.$$

The orbits form a partition of X. Let Y_α be the set of orbits, we can put on the quotient topology.

Example 2.3.4. Let $X = [0,1]$. Define the equivalence relation $s \sim t \iff s = t$, and have $0 \sim 1$. That is, $\{0,1\}$ is an equivalence class.



Define $f : X \to \{z \in \mathbb{C} : |z| = 1\}$ by $t(t) = e^{2\pi it}$ for $t \in [0,1]$. Note that f is continuous but f^{-1} is not: there is a discontinuity at $1 \in \mathbb{C}$. However, the corresponding function $f : X/\alpha \to \{z \in \mathbb{C} : |z| = 1\}$ is a homeomorphism with the usual topology from $\mathbb{C}$.

Example 2.3.5. Let $X = S^2$ be a sphere on $\mathbb{R}^3 = V$, $v \in S^2$. Let $G = \mathbb{Z}_2$, $\alpha_c(v) = -v$. $S^n \subseteq \mathbb{R}^{n+1}$.

Definition 2.3.6. Let Y be a set and $\{X_\alpha, \mathcal{T}_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and $f_\alpha : Y \to X_\alpha$. We want the weakest topology that make all f_α continuous, namely the **initial topology**. This topology must contain $f_\alpha^{-1}(U)$ for $U \in \mathcal{T}_\alpha$.

Remark. These form a sub-base for the initial topology, whereas the finite intersections of these form a base.

Definition 2.3.7. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ with $f : Y \to X$ defined by $f(y) = y \in X$. The sub-base is $\{f^{-1}(U), U \in \mathcal{T}_X\}$ and so $f^{-1}(U) = U \cap Y$. The initial topology is $\{U \cap Y : U \in \mathcal{T}_X\}$, which is called the **relative topology**.

Definition 2.3.8. Let $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let $Y = \prod_\alpha X_\alpha$ be the product set. Have $\pi_\alpha : Y \to X_\alpha$, $\pi_\alpha(\{x_\beta\}_{\beta \in A}) = x_\alpha$. The **product topology** is the initial topology for the π_α. The sub-base is the $\pi_\alpha^{-1}(U), U \in \mathcal{T}_\alpha$, for all α, U.

Example 2.3.9. If $A = \mathbb{N}$, $(X_n, \mathcal{T}_n)$, $\{x_n\} \in \prod X_n$. If $U \in \mathcal{T}_3$,

$$\pi_3^{-1}(U) = X_1 \times X_2 \times U \times X_4 \times X_5 \times \cdots .$$

The base is the finite intersection of these.

Example 2.3.10. Let $Y = V$ be a vector space over $\mathbb{R}$. Let $\mathcal{L}$ be a collection of linear functionals, $\varphi_\lambda, \lambda \in \mathcal{L}$ and $\varphi_\lambda : V \to \mathbb{R}$. We can ask for the weakest topology on V making all φ_λ continuous.

Example 2.3.11. $V = C([0,1])$ be the continuous function on $[0,1]$ and $\mathcal{L} = C([0,1])$. For $g \in \mathcal{L}$, $\varphi_g(f) = \int_0^1 f(t)g(t)dt$.

> **Proposition 2.3.12.** Consider $f_\alpha : X \to Y_\alpha$ for $\alpha \in A$. Let $\mathcal{T}_x$ be the corresponding weak topology on X. Let $(Z, \mathcal{T}_z)$ be a topological space, and let $g : Z \to X$. Then g is continuous iff $f_\alpha \circ g$ is continuous for all α.

Proof. Suppose $f_\alpha \circ g$ is continuous for all α. It suffices to check on the sub-base. Let $\mathcal{O} \in \mathcal{T}_\alpha$. Then $g^{-1}\left(f_\alpha^{-1}(\mathcal{O})\right) = \left(f_\alpha \circ g\right)^{-1}(\mathcal{O})$ is open, hence g is continuous. Conversely, if g is continuous, then $\left(f_\alpha \circ g\right)^{-1}(\mathcal{O}) = g^{-1}\left(f_\alpha^{-1}(\mathcal{O})\right)$ is open since $f_\alpha^{-1}(\mathcal{O})$ is open in $\mathcal{T}_x$, thus $(f_\alpha \circ g)$ is continuous. □

Question. What topologies play nicely with $\mathbb{R}$?

Let (X, d) be a metric space. Let $x_1, x_2 \in X, x_1 \neq x_2$. Let $r = d(x_1, x_2)$. Consider the two disjoint balls $\mathcal{B}_{r/3}(x_1), \mathcal{B}_{r/3}(x_2)$.

2.4 Special Topological Spaces

2.4.1 Hausdorff topological space

Definition 2.4.1. A topological space is said to be **Hausdorff** if for any $x_1, x_2 \in X, x_1 \neq x_2$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$, with $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2$.

2.4.2 Normal topological space

Definition 2.4.2. $(X, \mathcal{T})$ is **normal** if for disjoint closed sets $\mathcal{C}_1, \mathcal{C}_2$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$, such that $\mathcal{C}_1 \subseteq \mathcal{O}_1, \mathcal{C}_2 \subseteq \mathcal{O}_2$.

Proposition 2.4.3. If (X, d) is a metric space, then its topology is normal.

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be disjoint closed sets. For each $x \in \mathcal{C}_1$, we can choose r_x such that $\mathcal{B}_{r_x}(x) \cap \mathcal{C}_2 = \emptyset$. For each $y \in \mathcal{C}_2$, we choose r_y such that $\mathcal{B}_{r_y}(y) \cap \mathcal{C}_1 = \emptyset$. Let

$$\mathcal{O}_1 = \bigcup_{x \in \mathcal{C}_1} \mathcal{B}_{r_x/3}(x)$$
$$\mathcal{O}_2 = \bigcup_{y \in \mathcal{C}_2} \mathcal{B}_{r_y/3}(y).$$

Then $\mathcal{C}_1 \subseteq \mathcal{O}_1, \mathcal{C}_2 \subseteq \mathcal{O}_2$. Now let $z \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then there exists $x \in \mathcal{C}_1$ with $z \in \mathcal{B}_{r_{x/3}}(x)$. Then

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r_x}{3} + \frac{r_y}{3}.$$

Suppose $r_x \geq r_y$, then $d(x, y) \leq \frac{2}{3} r_x$. So $y \in \mathcal{C}_2$ and $y \in \mathcal{B}_{r_x}(x)$ but $\mathcal{C}_2$ and $\mathcal{B}_{r_x}(x)$ are disjoint. Hence, a contradiction. Therefore, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. □

2.4.3 Urysohn's Lemma

> **Lemma 2.4.4.** Let $(X, \mathcal{T})$ be a normal space, and let $\mathcal{C} \subseteq X$ be a closed subset. Let $\mathcal{O} \subseteq X$ be an open subset such that $\mathcal{C} \subseteq \mathcal{O}$. Then there exists an open set U such that $\mathcal{C} \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$.

Proof. $\mathcal{C}$ and $\mathcal{O}^c$ are disjoint closed sets, so there are disjoint open sets U, V such that $\mathcal{C} \subseteq U$ and $\mathcal{O}^c \subseteq V$. Then $\mathcal{C} \subseteq U \subseteq V^c \subseteq \mathcal{O}$. V^c is a closed set containing U; it therefore contains the closure $\overline{U}$, so that $\mathcal{C} \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$. □

> **Lemma 2.4.5** (Urysohn's Lemma). Let $(X, \mathcal{T})$ be normal, and let $\mathcal{C}_0, \mathcal{C}_1$ be disjoint closed subsets. Then there exists a continuous function $f : X \to [0, 1]$ such that $f(\mathcal{C}_0) = \{0\}, f(\mathcal{C}_1) = \{1\}$.

Proof. Set $\mathcal{O}_1 = \mathcal{C}_1^c$ and $\mathcal{C}_0 \subseteq \mathcal{O}_1$. Then by the lemma there exists an open $\mathcal{O}_{1/2}$ with $\mathcal{C}_0 \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_1$. Applying the lemma again, there exist open sets $\mathcal{O}_{1/4}, \mathcal{O}_{3/4}$. Hence,

$$\mathcal{C}_0 \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4} \subseteq \mathcal{O}_1.$$

Then there exist $\mathcal{O}_{1/8}, \mathcal{O}_{3/8}, \mathcal{O}_{5/8}, \mathcal{O}_{7/8}$ such that

$$\mathcal{C}_0 \subseteq \mathcal{O}_{1/8} \subseteq \overline{\mathcal{O}}_{1/8} \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{3/8} \subseteq \overline{\mathcal{O}}_{3/8} \subseteq \cdots \subseteq \overline{\mathcal{O}}_{7/8} \subseteq \mathcal{C}^c.$$

Then by induction, for each dyadic rational numbers

$$\Delta = \{r = m2^{-n} : 1 \leq m \leq 2^n, m, n \in \mathbb{N}\}.$$

we get an open set $\mathcal{O}_r$ such that if $r, s \in \Delta, r < s$, then $\overline{\mathcal{O}}_r \subseteq \mathcal{O}_s, \mathcal{C}_s \subseteq \mathcal{O}_r^c$.

Define $f : X \to [0, 1]$ by

$$f(x) = \inf \{r \in \Delta : x \in \mathcal{O}_r\}.$$

Clearly, if $x \in \mathcal{C}_0$, then $x \in \mathcal{O}_{2^{-n}}$ for any $n \in \mathbb{N}$, so it follows that $f(x) = 0$. On the other hand, if $x \in \mathcal{C}_1$, then $x \notin \mathcal{O}_r$ for any $r \in \Delta$, hence $f(x) = 1$ on $\mathcal{C}_1$. Thus, it remains to show that f is continuous. Recall that it suffices to consider the sub-base of open rays. Use as sub-base $\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, +\infty) : b \in \mathbb{R}\}$.

Since $f : X \to [0, 1]$, then for $a \leq 0, b \geq 1$, $f^{-1}((-\infty, a)) = f^{-1}((b, +\infty)) = \emptyset$. Suppose $0 < a \leq 1$. If $x \in X$, and $f(x) < a$, then there is a dyadic rational number $r \in \Delta$ such that $f(x) < r < a$, so $x \in \mathcal{O}_r$. Then we have

$$f^{-1}((-\infty, a)) = \bigcup_{r<a} \mathcal{O}_r,$$

which is open. Similarly, suppose $0 \leq b < 1$. If $x \in f^{-1}((b, +\infty))$, i.e. $f(x) > b$, then there exists a dyadic rational $s \in \Delta$ such that $f(x) > s > b$, so $x \notin \mathcal{O}_s$, and there exists a dyadic rational $r \in \Delta$ such that $s > r > b$, so $\overline{\mathcal{O}}_r \subseteq \mathcal{O}_s$, and so $x \notin \overline{\mathcal{O}}_r$, so $x \in \overline{\mathcal{O}}_r^c$, which is open. Then

$$f^{-1}((b, \infty)) = \bigcup_{r>b} \overline{\mathcal{O}}_r^c$$

is open. □

2.5 Banach Spaces

Definition 2.5.1. A **Banach space** is a complete, normed vector space.

Let X be a set, and let V be a normed vector space. Let $B(X, V)$ denote the set of all bounded functions from X to V, that is, functions whose range is contained in an open ball. Then it can easily be checked that $B(X, V)$ is a vector space for pointwise operations, and that $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$ is a norm on $B(X, V)$.

Proposition 2.5.2. If V is a Banach space, then $B(X, V)$ with $\|\cdot\|_\infty$ is a Banach space.

Proof. First, we show that $B(X, V)$ is a normed vector space. If $f, g \in B(X, V)$, there exists some M, N such that $|f(x)| < M, |g(x)| < N$ for each $x \in X$ by boundedness. Then $|(f + g)(x)| \leq |f(x)| + |g(x)| < M + N$, so $f + g$ is also a bounded function. If $c \in \mathbb{R}$, then $|cf(x)| < |c|M$, so cf is also bounded. This shows that the space of bounded functions is a vector space. Furthermore, the norm is indeed a norm because $\|(f + g)(x)\|_\infty \leq \|f\|_\infty + \|g\|_\infty \implies \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$, and all the other norm properties hold.

Now, we must show that $B(X, V)$ is complete. Take some Cauchy sequence $\{f_n\}$ in $B(X, V)$. For each $x \in X$, $\{f_n(x)\}$ is Cauchy in V, so by the completeness of V, such sequence converges to some limit $f(x) = \lim_{n\to\infty} f_n(x)$. Since all f_n's are bounded, the limit f is bounded as well. We need to show that $f_n \to f$ in norm:
Let $\epsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that for $n, m \geq N_1$, we have

$$\|f_n - f_m\|_\infty < \frac{\epsilon}{2}.$$

For a fixed $x \in X$, there exists $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, we have

$$\|f_n(x) - f(x)\| < \frac{\epsilon}{2}.$$

Then for $n \geq \max(N_1, N_2)$, we have

$$\|f_n(x) - f(x)\|_\infty \leq \|f_n - f_{n+1}\|_\infty + \|f_{n+1}(x) - f(x)\| < \epsilon.$$

Thus, $\|f_n - f\| < \epsilon$. □

Proposition 2.5.3. Let $(X, \mathcal{T})$ be a topological space, and let $C_b(X, V)$ be a set of bounded continuous functions from X to V. Then $C_b(X, V)$ is a closed subspace.

Proof. Exercise. □

2.5.1 Tietze Extension Theorem

Theorem 2.5.4. Let $(X, \mathcal{T})$ be a normal topological space, and let $A \subseteq X$ be closed. Let $f : A \to \mathbb{R}$ be continuous. Then f has a continuous extension $\tilde{f} : X \to \mathbb{R}$, i.e. $\tilde{f}|_A = f$. If $f : A \to [a, b]$, then we can arrange the extension $\tilde{f} : X \to [a, b]$.

Proof. First, we prove the case $f : A \to [0, 1]$. For E_0, F_0 disjoint closed sets in X, by Urysohn's lemma, let $h_{E_0,F_0} : X \to [0, 1]$ be a continuous function such that $h_{E_0,F_0}|_{E_0} = 0$ and $h_{E_0,F_0}|_{F_0} = 1$.

Let $f_0 = f$, and let $A_0 = \{x \in A : f_0(x) \leq \frac{1}{3}\}$, $B_0 = \{x \in A : f_0(x) \geq \frac{2}{3}\}$. Clearly A_0 and B_0 are disjoint. Let

$$g_1 = \frac{1}{3} h_{A_0,B_0}.$$

Now let $f_1 = f_0 - g_1|_A$. That is, $f_1 : A \to [0, 2/3]$ and $g_1 : X \to [0, 1/3]$. Inductively, let $f_n : A \to [0, (2/3)^n]$. Let $A_n = \{x \in A : f(x) \leq \frac{1}{3}\left(\frac{2}{3}\right)^n\}$, $B_n = \{x \in A : f(x) \geq \frac{2}{3}\left(\frac{2}{3}\right)^n\}$ with

$$g_{n+1} = \frac{1}{3}\left(\frac{2}{3}\right)^n h_{A_n,B_n},$$

so $g_{n+1} : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^n]$. Let $f_{n+1} = f_n - g_{n+1}|_A$, so $f_{n+1} : A \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^{n+1}]$.

Note that $\|g_n\|_\infty = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Let $\tilde{f} = \sum_{n=1}^\infty g_n$. We will show that the sequence of partial sums is Cauchy in $C_b(X, \mathbb{R})$, thus $\sum_{n=1}^\infty g_n$ converges.

Let $k_n = \sum_{j=1}^n g_j$. For $m < n$, consider $k_n - k_m = \sum_{j=m+1}^n g_j$. Then

$$\|k_n - k_m\|_\infty \leq \sum_{j=m+1}^n \|g_j\|_\infty = \sum_{j=m+1}^n \frac{1}{3}\left(\frac{2}{3}\right)^{j-1}.$$

Clearly, for large enough n, m, we can make this arbitrarily small. Thus $\tilde{f}$ is well-defined and continuous, by the previous proposition. Then

$$f_n = f_{n-1} - g_n = f_{n-2} - g_{n-1} - g_n = \cdots = f_0 - \sum_{j=1}^n g_j,$$

so $\|f_n\|_\infty = \left(\frac{2}{3}\right)^n$, so $\|f_n\|_\infty \to 0$, thus $f - \tilde{f}|_A = 0$, i.e. $\tilde{f}|_A = f$.
Finally, we want to check that the range of $\tilde{f}$ is contained in $[0, 1]$. Note that

$$\tilde{f}(x) = \sum_{n=1}^\infty g_n(x) \leq \frac{1}{3}\sum_{n=1}^\infty \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\sum_{n=0}^\infty \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1.$$

Therefore $0 \leq \tilde{f}(x) \leq 1$ for all $x \in X$.

Now suppose that $f : A \to \mathbb{R}$ is unbounded. Let h be a homeomorphism of $\mathbb{R}$ with $(0, 1)$. Let $g = h \circ f$, so $g : A \to (0, 1) \subset [0, 1]$. By the arguments above, we can find an extension $\tilde{g} : X \to [0, 1]$ such that $\tilde{g}|_A = g$. Let $B = \tilde{g}^{-1}(\{0, 1\})$. Since $\tilde{g}$ is continuous, B is closed in X and is disjoint from A. By Urysohn's Lemma, there exists a continuous function $k : X \to [0, 1]$ such that $k|_B = 0$ and $k|_A = 1$. Define $\hat{g} = \tilde{g}k$ (pointwise product). Then the function $\tilde{f} = h^{-1} \circ \hat{g}$ is a continuous extension of f to X. $\square$

3 Compactness

3.1 Fundamentals

Definition 3.1.1 (Cover/Subcover)**.** Let X be a set. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a collection of subsets of X. We say that $\mathcal{C}$ is a **cover** for X if

$$\bigcup_{A \in \mathcal{C}} A = X.$$

If $\mathcal{D} \subseteq \mathcal{C}$ and $\mathcal{D}$ is also a cover for X, then $\mathcal{D}$ is a **subcover** of X.

Definition 3.1.2 (Open cover)**.** For a topological space $(X, \mathcal{T})$, an **open cover** is a cover of X that is contained in $\mathcal{T}$.

Definition 3.1.3 (Compact)**.** $(X, \mathcal{T})$ is **compact** if every open cover has a finite subcover.

Proposition 3.1.4. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then the following are equivalent:

(i) A is compact in the relative topology.

(ii) Given any $\mathcal{C} \subseteq \mathcal{T}$ such that $A \subseteq \bigcup_{\mathcal{O} \in \mathcal{C}} \mathcal{O}$, there exist $\{\mathcal{O}_i\}_{i=1}^n \in \mathcal{C}$ with $A \subseteq \bigcup_{i=1}^n \mathcal{O}_i$.

Proposition 3.1.5. If $(X, \mathcal{T})$ is compact and $A \subseteq X$ is closed, then A is compact (in the relative topology).

Proof. Let $\mathcal{C} \subseteq \mathcal{T}$ be an open cover of A. Since A is closed, A^c is open. Then $\mathcal{C} \cup \{A^c\}$ is an open cover of X. Since $(X, \mathcal{T})$ is compact, there is a finite subcover of X, so clearly there is a finite subcover for A. Hence, A is compact. □

Remark. A being compact does not imply A closed. For example consider sets with the indiscrete topology.

Proposition 3.1.6. Let $(X, \mathcal{T})$ be Hausdorff. Suppose $A \subseteq X$ is compact. Then for any $y \notin A$, there are disjoint open sets $U, \mathcal{O} \in \mathcal{T}$ with $y \in U$, $A \subseteq \mathcal{O}$.

Proof. By definition of Hausdorff, for each $x \in A$, there exists disjoint open sets $U_x, \mathcal{O}_x \in \mathcal{T}$ with $y \in U_x, x \in \mathcal{O}_x$. The set $\{\mathcal{O}_x : x \in A\}$ is a cover of A. Since A is compact, then there exists $\{\mathcal{O}_{x_i}\}_{i=1}^n$ that covers A. Let $\mathcal{O} = \bigcup_{i=1}^n \mathcal{O}_{x_i} \supseteq A$ be open. Let $U = \bigcup_{i=1}^n U_{x_i}$ be open and $y \in U$. Then we have $U \cap \mathcal{O} = \emptyset$. $\square$

Corollary 3.1.7. If $(X, \mathcal{T})$ is Hausdorff, then any compact subset $A \subseteq X$ is closed.

Definition 3.1.8. $(X, \mathcal{T})$ is **regular** if for any closed set $A \subseteq X$ and any $x \notin A$, there are disjoint open sets $\mathcal{O}, U$ such that $A \subseteq \mathcal{O}$ and $x \in U$

Proposition 3.1.9. If $(X, \mathcal{T})$ is compact Hausdorff, then it is regular.

Proposition 3.1.10. If $(X, \mathcal{T})$ is compact Hausdorff, then it is normal.

Proof. Let A, B be disjoint closed subsets of X. For each $y \in B$, by regularity, there exists disjoint open sets $\mathcal{O}_y, U_y$ with $A \subseteq \mathcal{O}_y, y \in U_y$, The U_y's form an open cover of B, so $\{U_{y_i}\}_{i=1}^n$ cover B, Set

$$U = \bigcup_{i=1}^n U_{y_i} \supseteq B$$
$$\mathcal{O} = \bigcap_{i=1}^n \mathcal{O}_{y_i}.$$

$\square$

Proposition 3.1.11. If $(X, \mathcal{T}_X)$ is compact and $(Y, \mathcal{T}_Y)$ is a topological space, and $f : X \to Y$ be continuous, then $f(X)$ is compact in Y.

Proof. Let $\mathcal{C}$ be an open cover of $f(X)$, $\mathcal{C} \subseteq \mathcal{T}_Y$. Then $\{f^{-1}(U) : U \in \mathcal{C}\}$ is an open cover for X. Since $(X, \mathcal{T})$ is compact, there exists $\{U_i\}_{i=1}^n \in \mathcal{C}$ with

$$\bigcup_{i=1}^n f^{-1}(U_i) = X.$$

Then $\{U_i\}_{i=1}^n$ cover $f(X)$. $\square$

Proposition 3.1.12. If $(X, \mathcal{T}_X)$ is compact and $(Y, \mathcal{T}_Y)$ is Hausdorff, and $f : X \to Y$ is continuous and bijective, then f^{-1} is continuous, so f is a homeomorphism.

Proof. Note that $f^{-1}(A) = f(A)$. To show that f^{-1} is continuous, we need that for any $\mathcal{O} \in \mathcal{T}_X$, $f(\mathcal{O}) \in \mathcal{T}_Y$, i.e. f is an open function.

Given $\mathcal{O} \in \mathcal{T}_X$, $\mathcal{O}^c$ is closed, so $\mathcal{O}^c$ is compact. Hence $f(\mathcal{O}^c)$ is compact. Since $\mathcal{T}_Y$ is Hausdorff, $f(\mathcal{O}^c)$ is closed, which implies $(f^{-1}(\mathcal{O}^c))^c = f(\mathcal{O})$ is open. $\square$

3.1.1 Compactness in terms of Closed Sets

If X is a set, $\mathcal{C}$ is a collection of subsets. Then $\mathcal{C}$ is a cover of X

$$\bigcup_{A\in\mathcal{C}} A = X.$$

Then

$$\bigcap_{\{A^c : A\in\mathcal{C}\}} = \emptyset.$$

So if $(X, \mathcal{T})$ is compact, $\mathcal{C}$ is an open cover, then there exists a finite subcover, i.e. if

$$\bigcap_{\{\mathcal{O}^c : \mathcal{O}\in\mathcal{C}\}} = \emptyset,$$

then there exists a finite closed sets $\mathcal{O}_1^c, \dots, \mathcal{O}_n^c$ with

$$\bigcap_{i=1}^{n} \mathcal{O}_i^c = \emptyset,$$

where $\mathcal{O}_1, \dots, \mathcal{O}_n \in \mathcal{C}$.

Definition 3.1.13. If X is a set and $\mathcal{C}$ is a collection of subsets. We say $\mathcal{C}$ has the **finite intersection property** (**FIP**) if for any $A_1, \dots, A_n \in \mathcal{C}$,

$$\bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Proposition 3.1.14. $(X, \mathcal{T})$ is compact if for any collection $\mathcal{C}$ of closed subsets if $\mathcal{C}$ has the FIP, then $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$. (Used for existence proofs)

3.2 Tychonoff's Theorem

Theorem 3.2.1 (Axiom of Choice)**.** Given any family of non-empty sets, there is a set containing an element from each of these sets.

Theorem 3.2.2 (Tychonoff's Theorem)**.** Let $\{(X_\alpha, \mathcal{T}_\alpha)\}$ be a family of compact spaces indexed by A. Then

$$\prod_{\alpha} X_\alpha$$

with the product topology is compact.

Proof. Let $\mathcal{C}$ be a collection of closed subsets that has the FIP. We must show that $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$.

Let Θ be a collection of of all collections $\mathcal{D}$ of subsets of $\prod_\alpha X_\alpha$ such that $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D}$ has FIP. Then Θ is inductively ordered by inclusion. Then by Zorn's lemma, there is a maximal element $\mathcal{D}^*$. □

Example 3.2.3. Let $A = C([0,1])$. Let $\| \cdot \|_2, \langle , \rangle$. Let $\mathcal{B}$ be an unit ball for $\|.\|_2$ in $C([0,1])$, then

$$\mathcal{B} = \{f \in C([0,1]) : \int_0^1 f(t)^2 dt \leq 1\}.$$

For each $\alpha \in A$ define $\varphi_\alpha : \mathcal{B} \to \mathbb{R}$ by $\varphi_\alpha(f) = \langle f, \alpha \rangle$. By Cauchy-Schwartz, we have

$$|\varphi_\alpha(f)| \leq \|f\|_2 \|\alpha\|_2 = \|\alpha\|_2.$$

Consider $\prod_\alpha [-\|\alpha\|_2, \|\alpha\|_2]$.

3.2.1 Zorn's Lemma

Definition 3.2.4. A **chain** $\mathcal{C}$ in $\mathcal{P}$ is a totally ordered subset of $\mathcal{P}$.

Definition 3.2.5. $\mathcal{P}$ is **inductively ordered** if for any chain $\mathcal{C}$ in $\mathcal{P}$, there is $a \in \mathcal{P}$ (maybe in $\mathcal{C}$) such that $c \leq a$ for all $c \in \mathcal{C}$, i.e. every chain in $\mathcal{P}$ has an upper bound.

Definition 3.2.6. $m \in \mathcal{P}$ is a **maximal element** if $a \geq m \implies a = m$.

Lemma 3.2.7 (Zorn's Lemma)**.** If $\mathcal{P}$ is inductively ordered, then every chain $\mathcal{C}$ has a maximal element m for $\mathcal{C}$ with $a \leq m$ for any $a \in \mathcal{C}$.

Proposition 3.2.8. Let R be a ring. Every two-sided ideal is contained in a maximal two-sided ideal.

Example 3.2.9. Consider $\mathbb{Z}_5$. Let R be the sequences of elements of $\mathbb{Z}_5$, $\prod_{n=1}^\infty \mathbb{Z}_5$. Let I be sequences in R that eventually take value 0 for all entries.

Theorem 3.2.10. Tychonoff's Theorem $\implies$ Axiom of Choice.

Proof. Let $\{X_\alpha\}_{\alpha \in A}$ be a family of non-empty sets. Let ω be some point that is not in $\bigcup_{\alpha \in A} X_\alpha$ (for example $= \{\bigcup X_\alpha\}$). For each α, let $Y_\alpha = X_\alpha \cup \{\omega\}$. Let $\mathcal{T}_{Y_\alpha} = \{\emptyset, X_{Y}\alpha, \{\omega\}\}$. Then $\{Y_\alpha, \mathcal{T}_{Y_\alpha}\}$ is compact. Let $Y = \prod Y_\alpha$. Then Y is compact by Tychonoff. $\pi_\alpha : Y \to Y_\alpha$. Note that X_α is closed since $\{\omega\}$ is open. Let

$$F_\alpha = \pi_\alpha^{-1}(X_\alpha).$$

Then F_α is closed in Y. Now we claim that $\{F_\alpha\}_{\alpha \in A}$ has FIP.
Given $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$, choose $y_{\alpha_j} \in F_{\alpha_j}, j = 1, \ldots, n$. Define $y \in Y$ by

$$y_\alpha = \begin{cases} y_{\alpha_j} & \text{if } \alpha = \alpha_j \\ \omega & \text{if } \alpha \neq \alpha_j, j = 1, \ldots, n \end{cases}.$$

Then $y_\alpha \in \bigcap_{i=1}^n F_{\alpha_i}$. Since Y is compact, there exists $y \in \bigcap_{\alpha \in A} F_\alpha$. Thus $y_\alpha \in F_\alpha$. Let $x_\alpha = \pi_\alpha(y_\alpha)$. $\square$

3.3 Metric Spaces and Compactness

Let (X, d) be a metric space. Let $A \subseteq X$ and suppose that A is compact. Let $\epsilon > 0$ be given. The collection $\mathcal{B}_\epsilon(a)$ for all $a \in A$ covers A, and so there is a finite subcover.

Definition 3.3.1. Let (X, d) be a metric space. A subset $A \subseteq X$ is **totally bounded** if for any $\epsilon > 0$, A can be covered by a finite collection of ϵ-balls.

Remark. Any subset of a totally bounded set is totally bounded.

Remark. If we can cover with balls with center at X, then we can cover with balls with center at A.

Proposition 3.3.2. If $A \subseteq X$ is totally bounded, so is $\overline{A}$.

Proof. Given $\epsilon > 0$, find $\mathcal{B}_{\epsilon/2}(a_j)$ that cover A. If $b \in \overline{A}$, then $\mathcal{B}_{\epsilon/2}(b) \cap A \neq \emptyset$. Let a be in that intersection. Then for some j, $a \in \mathcal{B}_{\epsilon/2}(a_j)$, so $b \in \mathcal{B}_{\epsilon/2}(a_j)$.

□

Definition 3.3.3. Let $\{x_n\}$ be a sequence in $X, \mathcal{T}$. A **cluster point** of $\{x_n\}$ is a point x^* such that for any open set $\mathcal{O}$ with $x^* \in \mathcal{O}$, the sequence $\{x_n\}$ is frequently in $\mathcal{O}$, i.e., given an m, there is $n > m$ with $x_n \in \mathcal{O}$.

Proposition 3.3.4. If $(X, \mathcal{T})$ is compact, then every sequence $\{x_n\}$ has at least one cluster point.

Proof. Let $A_n = \{x_n, x_{n+1}, \dots\}$. The A_n's have FIP. So $\overline{A}_n$'s have FIP. Since X is compact, $\bigcap \overline{A}_n \neq \emptyset$. We claim that any $x^* \in \bigcap \overline{A}_n$ is a cluster point. Given $\mathcal{O}, x^* \in \mathcal{O}$, $\mathcal{O} \cap A_n \neq \emptyset$ for all n. Let $A \subseteq X$ be compact and (X, d) be a metric space. Let $\{x_n\}$ be a Cauchy Sequence in A. By previous proposition, it has a cluster point x^*For $x^* \in \mathcal{B}_{\epsilon/2}(x^*)$, find N such that for $m, n \geq N$, $d(x_m, x_n) < \frac{\epsilon}{2}$ and $x_N \in \mathcal{B}_{\epsilon/2}(x^*)$. Thus for $n \geq N$, $x_n \in \mathcal{B}_\epsilon(x^*)$, so $\{x_n\}$ converges to x^*. □

Theorem 3.3.5. If (X, d) is a complete metric space and totally bounded, then it is compact.

Proof. Let $\mathcal{C}$ be an open cover of X. To show that it has a finite subcover, we prove by contradiction. Assume that it does not have a finite subcover. Cover X by a finite number of closed balls of radius 1, call them $B_n^1, \dots, B_{n_1}^1$. Take the closure of the balls. There must be at least one of these that cannot be finitely covered, call it A^1. Cover A^1 with a finite number of closed balls of radius $\frac{1}{2}$, call them $B_1^2, \dots, B_{n_2}^2$, take closure. Then at least one of these cannot be finitely covered, call it A^2. Cover A^2 by a finite number of closed balls of radius $\frac{1}{4}$, take closure. Then at least one of these cannot be finitely covered, call it A^3, and so on. Then we get a sequence $\{A_n\}$ of closed sets where $A^{n+1} \subseteq A^n$, and each A^n is not finitely covered, and diameter(A^n)$\to 0$ as $n \to \infty$. The for each n, choose a Cauchy sequence $\{x_n\} \in A_n$. Since X is complete, $\{x_n\}$ converges to x^*. Since $\mathcal{C}$ is a cover, there is $\mathcal{O} \in \mathcal{C}$ with $x^* \in \mathcal{O}$. Hence. there is an $\epsilon > 0$ such that $\mathcal{B}_\epsilon(x^*) \subseteq \mathcal{O}$. Choose n so that diameter(A^n)$< \epsilon$. Then $A^n \subseteq \mathcal{B}_\epsilon(x^*) \subseteq \mathcal{O}$. Hence, a contradiction. □

Corollary 3.3.6. If (X, d) is a complete metric space and $A \subseteq X$, to show that A is compact, it suffices to show A is totally bounded and A is closed in X.

Remark. If A is bounded, then A is totally bounded. If A is also closed, then A is compact.

If $(X, \mathcal{T}_X)$ is a topological space and (Y, d) a metric space, consider the set of bounded continuous functions $C_b(X, Y)$ with d_∞ defined by

$$d_\infty(f, g) \coloneqq \sup \{d(f(x), g(x)) \mid x \in X\}.$$

What are the compact subsets of $C_b(X, Y)$ and what are the totally bounded subsets of $C_b(X, Y)$?

Let $\mathcal{F}$ be a totally bounded subset of $C_b(X, Y)$. Then given $\epsilon > 0$, there are $f_1, \ldots, f_n$ such that $\mathcal{B}_\epsilon(f_j)$ cover $\mathcal{F}$. Then if $g \in \mathcal{F}$, there is some j such that $g \in \mathcal{B}_\epsilon(f_j)$. Then for $x^*, x \in X$, we have

$$\begin{aligned} d(g(x), g(x^*)) &\le d(g(x), f_j(x)) + d(f_j(x), f_j(x^*)) + d(f_j(x^*), g(x^*)) \\ &\le \epsilon + {}'' + \epsilon \end{aligned}$$

Since f_j is continuous, there exists $\mathcal{O}_j \in \mathcal{T}$, $x^* \in \mathcal{O}_j$ such that if $x \in \mathcal{O}_j$, $d(f_j(x), f_j(x^*)) < \epsilon$. Thus if $x \in \mathcal{O}_j$, $d(g(x), g(x^*)) < 3\epsilon$. For each j, let $\mathcal{O} = \bigcap_{j=1}^n \mathcal{O}_j$, $x^* \in \mathcal{O}$. We find that for any $g \in \mathcal{F}$ and any $x \in \mathcal{O}$, $d(g(x), g(x^*)) < 3\epsilon$.

Definition 3.3.7. A family $\mathcal{F}$ of continuous functions is **equicontinuous** at x^* if for any $\epsilon > 0$, there exists $\mathcal{O}$ such that $x^* \in \mathcal{O}$ and if $x \in \mathcal{O}$, then $d(f(x), f(x^*)) < \epsilon$ for any $f \in \mathcal{F}$. $\mathcal{F}$ is equicontinuous if it is equicontinuous at each $x \in X$.

Continuing from previous discussion, given x^*, $\epsilon > 0$ for any $g \in \mathcal{F}$, $d(g(x^*), f_j(x^*)) < \epsilon$ for some j. Thus

$$\underbrace{\{g(x^*) : g \in \mathcal{F}\}}_{\text{totally bounded}} \subseteq \bigcup_{j=1}^n \mathcal{B}_\epsilon(f_j(x^*)\}$$

so $\mathcal{F}$ is **pointwise totally bounded.**.

Theorem 3.3.8 (Arzela-Ascoli Theorem)**.** If $(X, \mathcal{T})$ is compact, and if $\mathcal{F} \subseteq C_b(X, Y)$ such that $\mathcal{F}$ is equicontinuous and pointwise totally bounded. Then $\mathcal{F}$ is totally bounded.

Proof. Let $\epsilon > 0$ be given. By equicontinuity for each $x \in X$, there is $\mathcal{O}_x \in \mathcal{T}$ such that if $x' \in \mathcal{O}$, then $d(f(x'), f(x)) < \frac{\epsilon}{4}$ for any $f \in \mathcal{F}$. Since X is compact, there is a finite subcover $\mathcal{O}_{x_1}, \ldots, \mathcal{O}_{x_n}$. For each j, $\{f(x_j) : f \in \mathcal{F}\}$ is totally bounded. Hence, choose a finite subset S_j

of that set that is $\epsilon/4$ dense in that set. Let $S = \bigcup S_j$, a finite set. Let $\Phi = \{\psi : \{1, \dots, n\} \to S\}$ and Φ is finite. For $\psi \in \Phi$, let

$$\mathcal{F}_\psi = \{f \in \mathcal{F} \mid f(x_j) \in \mathcal{B}_{\epsilon/4}(\psi(j)\}.$$

Thus,

$$\mathcal{F} = \bigcup_{\psi \in \Phi} \mathcal{F}_\psi.$$

□