Math 185 Notes
Complex Analysis

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Chapter 1

Complex Numbers

1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for $|x| < r$, where r is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n||x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

Example 1.1.1. The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for $|x| < 1$.

Question. Now what if we replace the real variable x by the complex variable z?

Answer. If $|z| < r$, then

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

so the sum converges absolutely for $z \in D(0, r)$ (disc of radius r centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

Example 1.1.2. Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function $\mathbb{R} \to \mathbb{R}$, $f(z)$ is infinitely differentiable at $z = 0$, and all derivatives of $f(z)$ are zero at $z = 0$. Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f"(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from $f(z)$!

Example 1.1.3. Consider the same example as above, but with z as a complex number. Let $z = it$ where $t \in \mathbb{R}$. Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at $z = 0$ and thus not complex-differentiable at $z = 0$.

Example 1.1.4. Now let's set $z = x + iy$ where $x, y \in \mathbb{R}$. Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function $\mathbb{R}^2 \to \mathbb{R}^2$ (instead of $\mathbb{C} \to \mathbb{C}$). Let's differentiate with respect to x:

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to y:

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = if'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i\frac{\partial f'(z)}{\partial y} = i\frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of) $f(z)$ satisfy the two-dimensional *Laplace equation.*

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

Example 1.1.5. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful too for computing integrals.

Chapter 2

Complex Differentiation

2.1 Derivatives

Definition 2.1.1 (Derivative). The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x) = L$$

means for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$. (For any "tolerance" ϵ, we can guarantee $f(x)$ is within ϵ of L by forcing x to be close enough to a.)

Remark. Note that $x = a$ doesn't satisfy $0 < |x - a|$, so the value of f at $x = a$ has no bearing on whether $\lim_{x \to a} f(x)$ exists.

2.1.1 Continuity

Definition 2.1.2 (Continuous). If $\lim_{x \to a} f(x) = f(a)$, then we say f is *continuous* at a.

Remark. Setting $L = f(a)$ in the limit, $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (even when $x = a$) when talking about continuity, we leave out the $0 < |x - a|$ part for convenience because $x - a = 0$ automatically works.

Now let's consider a function $f : \mathbb{C} \to \mathbb{C}$, $\lim_{z \to a} f(z) = L$ means for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

Remark. Now the z's that we worry about form an open disc with radius δ instead of an interval from the real case.

Similarly, if $\lim_{z \to a} f(z) = f(a)$, we say f is *continuous* at $z = a$.

Example 2.1.3. $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.

Proof. For $\epsilon > 0$, let $\delta = \epsilon$, then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

Example 2.1.4. $\lim_{z \to 0} \overline{z}/z$ (although this is undefined at $z = 0$, this has no bearing on whether the limit exists).

Proof. Suppose $\lim_{z \to 0} \overline{z}/z = L$ for some L. Let's take $\epsilon = 1$. There is a $\delta > 0$ such that

$$0 < |z - 0| < \delta \implies \left| \frac{\overline{z}}{z} - L \right| < \epsilon = 1.$$

Let $z = \delta/2$ and so does $z = i\delta/2$. Then for $z = \delta/2$:

$$\frac{\overline{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for $z = i\delta/2$:

$$\frac{\overline{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the L must lie in the intersection of the two open unit discs centered at -1 and 1. However, since they are open discs, these two discs do not overlap and so L does not exist. □

Remark. This implies that there is no way to extend $\overline{z}/z$ to a continuous function at $z = 0$.

2.1.2 Properties of Limits

If $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$, then

(i)
$$\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2.$$

(ii)
$$\lim_{x\to a} f(x)g(x) = L_1 L_2.$$

(iii)
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Remark. These implies that the sum/product/quotient of continuous functions are continuous.

Proposition 2.1.5 (Composite function of continuous functions is continuous)**.** *If $f(x)$ is continuous at $x = a$, and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.*

Proof. We want $|g(f(x)) - g(f(a))| < \epsilon$. By continuity of g at $x = f(a)$, there exists $\delta_1 > 0$ such that
$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$
We want to take $w = f(x)$, so we need
$$|f(x) - f(a)| < \delta_1.$$
But by continuity of f at $x = a$, we know that δ_1 will be our ϵ when $|x - a| < \delta_2$ for some $\delta_2 > 0$. Then for such x,
$$|g(f(x)) - g(f(a))| < \epsilon.$$
□

2.2 Derivatives (Cont'd)

Definition 2.2.1 (Differentiable)**.** We say that $f(z)$ is differentiable at $z = a$ iff $\frac{f(z)-f(a)}{z-a}$ extends to a continuous function at $z = a$ (the value there is $f'(a)$).

Example 2.2.2. $f(z) = z$ is differentiable with $f'(z) = 1$.

Proof.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-z}{h}=\lim_{h\to 0}1=1.$$

□

Example 2.2.3 (Interesting one)**.** $f(z)=\overline{z}$ is not differentiable but is continuous.

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}&=\lim_{h\to 0}\frac{\overline{z+h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{z}+\overline{h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{h}}{h}\\&=\text{DNE}\qquad\text{(proved in previous example)}\end{aligned}$$

□

Proposition 2.2.4 (Differentiability implies continuity)**.** *$f(z)$ differentiable at $z=a$ implies that $f(z)$ is continuous at $z=a$.*

Proof. We want to show that $\lim_{z\to a}f(z)=f(a)$.

$$\begin{aligned}\lim_{z\to a}f(z)-f(a)&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot(z-a)\\&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot\lim_{z\to a}(z-a)\qquad\text{(assume both limits exist)}\\&=f'(a)\cdot 0\\&=0.\end{aligned}$$

□

Remark. This is a common technique to show continuity by showing the limit of the difference is zero.

2.2.1 Properties of complex-derivatives

(i)
$$\frac{d}{dz}cf(z)=cf'(z),\qquad\forall c\in\mathbb{C}.$$

(ii)
$$\frac{d}{dz}(f+g) = f'(z) + g'(z).$$

(iii)
$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)
$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)
$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

Proposition 2.2.5 (Power rule)**.**

$$\frac{d}{dz}z^n = nz^{n-1}$$

for all integers n*.*

Proof. We induct on n. For $n \geq 0$, when $n = 0$,

$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$

By the product rule,

$$\begin{aligned}
\frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\
&= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \qquad \text{(inductive hypothesis)} \\
&= nz^{n-1}.
\end{aligned}$$

For $n < 0$, simply apply quotient rule. □

Chapter 3

Holomorphic Functions and Cauchy-Riemann Equations

Recall
$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}.$$
Being differentiable at a point says little about how "nice" a function is.

Example 3.0.1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$
This is nowhere continuous, and thus nowhere differentiable. But if we consider $x^2 f(x)$, it is differentiable at $x = 0$:
$$\lim_{h\to 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h\to 0} h f(h) = 0.$$
Nevertheless, it's still not a very "nice" function.

3.1 Holomorphic Functions

Definition 3.1.1 (Holomorphic)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at a point a if it is differentiable at z for all z within distance r of a for some $r > 0$. In other words, $f(z)$ is differentiable everywhere sufficiently close to a.

Definition 3.1.2 (Open/closed disk)**.** The *open disk* of radius r centered at $a \in \mathbb{C}$ is
$$D(a,r) = \{z \in \mathbb{C} \mid |z-a| < r\}.$$
The *closed disk* is
$$\overline{D}(a,r) = \{z \in \mathbb{C} \mid |z-a| \leq r\}.$$

Thus, we can say $f(z)$ is holomorphic at $a \in \mathbb{C}$ if $f(z)$ is differentiable on an open disk centered at a. (if the point is not specified, it means that f is holomorphic everywhere.)

Example 3.1.3 (Polynomials are holomorphic)**.** We saw last time that z^n is differentiable everywhere for $n \geq 0$. Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

Example 3.1.4. $f(z) = |z|^2 = z\overline{z}$ is differentiable at zero.

$$\begin{aligned} \lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{h\overline{h} - 0}{h} \\ &= \lim_{h \to 0} \overline{h} \\ &= 0. \end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus, f is not holomorphic.

3.2 The Cauchy-Riemann Equations

Question. How to tell if a function is complex-differentiable?

Answer. We'll reduce this to a question about real derivatives.

Let $x + iy$, where $x, y \in \mathbb{R}$. If $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned} \frac{\partial f}{\partial x}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h}. \end{aligned}$$

Note that h is real. Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \to 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}. \end{aligned}$$

Example 3.2.1. $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}
\frac{\partial f}{\partial x}(z) &= \lim_{h\to 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 2ihy}{h} \\
&= \lim_{h\to 0} 2x + h + 2iy \\
&= 2x + 2iy \\
&= 2z \\
&= f'(z).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y}(z) &= \lim_{h\to 0} \frac{x^2 - (y+h)^2 + 2ix(y+h) - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h\to 0} \frac{-2yh - h^2 + 2ixh}{h} \\
&= \lim_{h\to 0} -2y - h + 2ix \\
&= -2y + 2ix \\
&= 2i(x + iy) \\
&= if'(z).
\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y}.$$

Theorem 3.2.2.

(i) If $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and they satisfy

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

(ii) If $f : \mathbb{C} \to \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous on some open disk centered at z, and if

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y},$$

then f is complex-differentiable at z.

Proof.

(i) Since f is complex-differentiable, we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose h is real. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since h is real, we get $\frac{\partial f}{\partial x}$ and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose h is purely imaginary: $h = ik$ for $k \in \mathbb{R}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then $h \to 0$ is equivalent to $k \to 0$ since $|h| = |k|$. Thus we have

$$\lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

□

Let $f(z) = u(z) + iv(z)$. If we choose real values for h, then the imaginary part y is kept constant, and the derivative becomes a partial derivative with respect to x. Thus we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values ik for h, we obtain

$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since we have

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

this resolves into the following equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or simply

$$u_x = v_y, \qquad v_x = -u_y.$$

These are known as the *Cauchy-Riemann* equations.

Example 3.2.3. Consider $f(z) = z^2$. Then

$$f(x + iy) = x^2 + y^2 + 2ixy.$$

Here $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. We have

$$u_x = 2x = v_y \qquad v_x = 2y = -u_y.$$

Example 3.2.4. Consider $f(z) = |z|^2$. Then $f(x + iy) = x^2 + y^2$ where $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. But here we have

$$u_x = 2x \neq v_y = 0 \qquad v_x = 0 \neq -u_y = -2y.$$

Thus, the Cauchy-Riemann equations only hold at $(x, y) = (0, 0)$ and as we saw previously that this function is only differentiable at $z = 0$ and nowhere else.

We can generalize the Cauchy-Riemann equations further to second order. Suppose we have $f = u + iv$. Then

$$u_{xx} = \frac{\partial}{\partial x}u_x = \frac{\partial}{\partial x}v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}v = \frac{\partial}{\partial y}\frac{\partial}{\partial x}v = \frac{\partial}{\partial y}v_x = \frac{\partial}{\partial y}(-u_y) = -u_{yy}.$$

Thus, we have

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we also have

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

which gives

$$v_{xx} + v_{yy} = 0, \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These are the *Laplace's equations* in 2D we saw earlier.

For $f : \mathbb{R} \to \mathbb{R}$, we know that $f'(x) = 0$ implies that f is constant. But for $f : \mathbb{C} \to \mathbb{C}$, we can use the Cauchy-Riemann equations. Since $f'(z) = \frac{\partial f}{\partial x}$,

$$f'(z) = 0 \implies u_x + iv_x = 0 \implies u_x = 0, v_x = 0.$$

By Cauchy-Riemann, we also have $u_y = v_y = 0$. Since $u_x = 0$, we know that for fixed y, $u(x, y)$ is some constant that could depend on y. Thus, we have

$$u(x, y) = g(y).$$

But $u_y = 0$, so $g'(y) = 0$, which means g is actually a constant independent of y. Thus, u is globally constant. Similar argument applies to v as well.

Chapter 4

Möbius Transformation

Definition 4.0.1 (Möbius transformation)**.** A *Möbius transformation* is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

Remark. If $ad = bc$, then $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, so rows are linearly dependent: $\lambda(a,b) + \mu(c,d) = (0,0)$, which implies that

$$a = \frac{-\mu}{\lambda} c \qquad b = \frac{-\mu}{\lambda} d.$$

Then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{-\frac{\mu}{\lambda}(cz+d)}{cz+d} \\ &= -\frac{\mu}{\lambda}, \end{aligned}$$

which is a constant independent of z.

Proposition 4.0.2 (Composite Möbius transforms is Möbius)**.** *If $f_1(z), f_2(z)$ are Möbius transforms, then then $f_1(f_2(z))$ is also a Möbius transform.*

Proof. Suppose

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then

$$\begin{aligned} f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}, \end{aligned}$$

which is another Möbius transform. □

Remark. Note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and the entries coincide with the composite Möbius transform. If we denote $f_M(z)$ to be a transform associated with a 2×2 matrix M, then we have just shown that

$$f_M(f_N(z)) = f_{MN}(z).$$

Remark. Since $f_I(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$, the inverse of f_M is $f_{M^{-1}}$.

Remark. Note that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies f_M = \frac{az + b}{cz + d}.$$

Meanwhile,

$$\lambda M = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \implies f_{\lambda M} = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az + b}{cz + d} = f_M.$$

Thus, scaling the matrices doesn't affect the resulting Möbius tranformation.

4.1 Inverse of Möbius transformation

Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the scaling part is redundant, we simply ignore it and obtain the inverse Möbius transform as follows:

$$f(z) = \frac{az + b}{cz + d} \implies f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Remark. Since Möbius transforms have inverses, they should be bijections. However, some details should be noted. If $c \neq 0$, then $\frac{az+b}{cz+d}$ is undefined at $z = -\frac{d}{c}$.

Let's consider the value at $z = -\frac{d}{c}$ to be infinity. It turns out that we can evaluate $\frac{az+b}{cz+d}$ at ∞:

$$\begin{aligned}\lim_{z\to\infty} \frac{az+b}{cz+d} &= \lim_{z\to\infty} \frac{a+\frac{b}{z}}{c+\frac{d}{z}} \\ &= \frac{a}{c}.\end{aligned}$$

When $c = 0$, we view $\frac{a}{c}$ as ∞. So now we view Möbius transformations as functions from $\mathbb{C} \cup \{\infty\}$ to $\mathbb{C} \cup \{\infty\}$. This makes all Möbius transformations into bijections. Here, we call $\mathbb{C} \cup \{\infty\}$ the *extended complex plane* (also called *Riemann sphere*).

Remark. For real functions, there are multiple notions of going to infinity: $x \to +\infty$ and $x \to -\infty$. But for complex functions, we work with only one infinite point.

Fact. If we apply a Möbius transformation to a line or a circle in the complex plane, we would get a line or a circle again (circles can turn into lines and vice versa).

Example 4.1.1. Consider $f(z) = \frac{z-1}{iz+i}$, let's apply this to the unit circle, i.e. take $z = e^{i\theta}$. Then

$$\begin{aligned}f(e^{i\theta}) = \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} &= \frac{\cos\theta - 1 + i\sin\theta}{i(\cos\theta + 1 + i\sin\theta)} \\ &= \frac{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{i\left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)} \\ &= \frac{2i(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})\sin\frac{\theta}{2}}{2i\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2}} \\ &= \tan\frac{\theta}{2}.\end{aligned}$$

Note that $\theta \in (-\pi, \pi)$ and we have $\tan -\frac{\pi}{2} = -\infty$ and $\tan\frac{\pi}{2} = -+\infty$. We have mapped a unit circle to a line (real line).

Fact. f sends the interior of the unit disk to the interior of the upper half-plane. If $g(z)$ is holomorphic on the upper half-plane, then $g(f(z))$ is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equation.

Remark. Stereographic projection φ is a bijection that maps a sphere to the extended complex plane. It doesn't preserve distance, but it preserves functions being holomorphic.

Proposition 4.1.2. *Suppose $f(z) = \frac{az+b}{cz+d}$ is a Möbius transformation. If $c = 0$ then*

$$f(z) = \frac{a}{d}z + \frac{b}{d},$$

and if $c \neq 0$, *then*

$$f(z)=\frac{bc-ad}{c^2}\frac{1}{z+\frac{d}{c}}+\frac{a}{c}.$$

In particular, every Möbius transformation is a composition of translations, dilations, and inversions.

Proof. Simplify. □

Theorem 4.1.3. *Möbius transformations map circles and lines into circles and lines.*

Proof. Translations and dilations certainly map circles and lines into circles and lines, so by the previous proposition, we only have to prove the statement of the theorem for the inversion $f(z)=\frac{1}{z}$.

The equation for a circle centered at x_0+iy_0 with radius r is $(x-x_0)^2+(y-y_0)^2=r^2$, which we can transform to

$$\alpha\left(x^2+y^2\right)+\beta x+\gamma y+\delta=0$$

for some real numbers α, β, γ, and δ that satisfy $\beta^2+\gamma^2>4\alpha\delta$. The above expression is more convenient for us, because it includes the possibility that the equation describes a line (precisely when $\alpha=0$).

Suppose $z=x+iy$ satisfies the above expression; we need to prove that $u+iv:=\frac{1}{z}$ satisfies a similar equation. Since

$$u+iv=\frac{x-iy}{x^2+y^2},$$

we can rewrite the transformed equation as

$$\begin{aligned}
0&=\alpha+\beta\frac{x}{x^2+y^2}+\gamma\frac{y}{x^2+y^2}+\frac{\delta}{x^2+y^2}\\
&=\alpha+\beta u-\gamma v+\delta\left(u^2+v^2\right).
\end{aligned}$$

But this equation says that $u+iv$ lies on a circle or line. □

Fact. The stereographic projection of a circle on the sphere (intersection of a plane and a sphere) is a circle in the plane. Möbius transformations take circles on the sphere to other circles of the sphere (some of these stereographically project to lines in the plane).

Chapter 5

Exponential,Trigonometric, and Logarithmic Functions

5.1 Exponential Functions

$$e^{x+iy} = e^x e^{iy} = e^x \cos y + ie^x \sin y \implies e^z = u(x,y) + iv(x,y)$$

where

$$u(x,y) = e^x \cos y$$
$$v(x,y) = e^x \sin y$$

5.2 Trigonometric Functions

For $z \in \mathbb{C}$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Remark. $\sin z, \cos z$ are holomorphic since e^z is holomorphic and so is e^{iz} and e^{-iz}.

Trigonometric identities hold for complex numbers.

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(2z) = 2 \sin z \cos z$$

5.3 Logarithmic Functions

We want $\log z$ to be the unique inverse to the exponential function, i.e. we want $e^{\log z} = z$, but then we would also have

$$e^{\log z + 2\pi i k} = z.$$

Definition 5.3.1 (Principal logarithm)**.** The *principal logarithm* is the function defined by

$$\log\left(re^{i\theta}\right) = \log r + i\theta.$$

where $-\pi < \theta \leq \pi$.

Let's check if $\log z$ is differentiable. If

$$z = x + iy = re^{i\theta},$$

then $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ when $x \neq 0$.

$$\begin{aligned} \log x + iy &= \frac{1}{2}\log\left(x^2+y^2\right) + i\tan^{-1}\left(\frac{y}{x}\right) \\ &= u(x,y) + iv(x,y). \end{aligned}$$

Then

$$\begin{aligned} u_x &= \frac{1}{2}\cdot\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2} \\ u_y &= \frac{1}{2}\cdot\frac{2y}{x^2+y^2} = \frac{y}{x^2+y^2} \\ v_x &= \frac{-y}{x^2}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{x^2+y^2} \\ v_y &= \frac{1}{x}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}. \end{aligned}$$

Thus, we see that the Cauchy-Riemann equations hold for logarithms.

Chapter 6

Complex Integration

6.1 Definition and Basic Properties

If $f : \mathbb{R} \to \mathbb{C}$, define

$$\int_a^b f(x)dx = \int_a^b \Re f(x)dx + i \int_a^b \Im f(x)dx.$$

Question. But how to integrate a function $f : \mathbb{C} \to \mathbb{C}$?

For real functions, going from a point $\gamma(a)$ to $\gamma(b)$ can only happen one way (follow the real axis) but in $\mathbb{C}$, we will have to specify the path from $\gamma(a)$ to $\gamma(b)$.

Definition 6.1.1 (Path/curve)**.** A *path/curve* is the image of a function $\gamma : [a, b] \to \mathbb{C}$.

Definition 6.1.2 (Integral)**.** The *integral* of the function $f : \mathbb{C} \to \mathbb{C}$ along the path parametrized by $\gamma : [a, b] \to \mathbb{C}$ is

$$\int_a^b f(\gamma(t))\gamma'(t)dt.$$

This is the integral of a function $\mathbb{R} \to \mathbb{C}$, so we already have a definition for it.

Aside,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx.$$

Suppose we have a different parametrization of the image of $\gamma(t)$. Write this parametrization as $\gamma(\theta(t))$ where $\theta : [a, b] \to [a, b]$ is a continuous reparametrization of the interval $[a, b]$ satisfying $\theta(a) = a$, $\theta(b) = b$ and θ is increasing. Then

$$\int_a^b f(\gamma(\theta(t)))\gamma'(\theta(t))\theta'(t)dt = \int_{\theta(a)}^{\theta(b)} f(\gamma(u))\gamma'(u)du$$

where $u = \theta(t)$ and $du = \theta'(t)dt$. So the integrals for $\gamma(\theta(t))$ and $\gamma(t)$ are the same, thus the integral depends on the curve in $\mathbb{C}$, not how we parametrize it.

We will use

$$\int_\gamma f(z)dz$$

to denote the integral.

Example 6.1.3. If $\gamma(t) = t$, then $\gamma'(t) = 1$ and

$$\int_\gamma f(z)dz = \int_a^b f(t)dt.$$

Example 6.1.4. If $\gamma(t) = t + it^2$ and $f(z) = 1$, then $\gamma'(t) = 1 + 2it$ and

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b (1+2it)dt \\ &= \int_a^b 1dt + i\int_a^b 2tdt \\ &= b - a + i(b^2 - a^2) \\ &= (b + ib^2) - (a + ia^2) \\ &= \gamma(b) - \gamma(a).\end{aligned}$$

Example 6.1.5 (Very important example)**.** Consider $\gamma(t) = e^{it}$ where $0 \le t \le 2\pi$. So $\gamma(t)$ is the counterclockwise unit circular path. If $f(z) = z^n$ for some $n \in \mathbb{Z}$. Then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\ &= \int_0^{2\pi} e^{int} \cdot ie^{it}dt \\ &= i\int_0^{2\pi} e^{it(n+1)}dt.\end{aligned}$$

If $n \neq -1$, then $n + 1 \neq 0$, the integral evaluates to

$$\begin{aligned}i\left.\frac{e^{it(n+1)}}{i(n+1)}\right|_{t=0}^{2\pi} &= i\left(\frac{1}{i(n+1)} - \frac{1}{i(n+1)}\right) \\ &= 0.\end{aligned}$$

If $n+1=0$, then $n=-1$ and so

$$\begin{aligned} i\int_0^{2\pi} e^{it(n+1)}dt &= i\int_0^{2\pi} 1dt \\ &= 2\pi i, \end{aligned}$$

which is not zero.

Chapter 7

Complex Integration (Cont'd)

7.1 Basic Properties

(i) If $\mu, \lambda \in \mathbb{C}$, then

$$\begin{aligned}\int_\gamma \lambda f(z) + \mu g(z) dz &= \int_a^b (\lambda f(\gamma(t) + \mu g(\gamma(t))\gamma'(t) dt \\ &= \lambda \int_a^b f(\gamma(t)\gamma'(t) dt + \mu \int_a^b g(\gamma(t))\gamma'(t) dt \\ &= \lambda \int_\gamma f(z) dz + \mu \int_\gamma g(z) dz.\end{aligned}$$

(ii)

$$\int_{\gamma_1\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

(iii)

$$\int_{-\gamma} f = -\int_\gamma f$$

(iv)

$$\left| \int_\gamma f \right| \leq \max{}_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma),$$

where

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

View $|\gamma'(t)|$ as the speed a particle is travelling at and $\gamma(t)$ as the position of that particle at time t. Then integrating it gives the total distance.

(v) **(Triangle Inequality)**

$$\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \leq \int_a^b \left|f(\gamma(t))\gamma'(t)\right| dt$$

(vi) **(ML-Lemma)**

$$\begin{aligned}\left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|dt \\ &= ML\end{aligned}$$

where $M = \max_{a\leq t\leq b}|f(\gamma(t))|$ and $L = \int_a^b |\gamma'(t)|dt$.

7.1.1 Antiderivatives

Theorem 7.1.1 (Fundamental Theorem of Calculus)**.** *If F is holomorphic on some subset $G \subseteq \mathbb{C}$ and $\frac{d}{dz}F(z) = f(z)$. Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof. Let $F(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = \alpha(t) + i\beta(t)$. Then

$$F(\gamma(t)) = u(\alpha(t), \beta(t)) + iv(\alpha(t), \beta(t)).$$

By chain rule,

$$\begin{aligned}\frac{d}{dt}F(\gamma(t)) &= u_x\alpha'(t) + u_y\beta'(t) + iv_x\alpha'(t) + iv_y\beta'(t) \\ &= u_x(\alpha'(t) + i\beta'(t)) + iv_x(\alpha'(t) + i\beta'(t)) \qquad (u_x = v_y \text{ by CR}) \\ &= F(\gamma(t))\gamma'(t).\end{aligned}$$

□

Definition 7.1.2 (Closed curve)**.** A *closed curve* is a curve where the start and end points are the same, i.e. $\gamma(a) = \gamma(b)$.

So if $f(z) = \frac{d}{dz}F(z)$, the integral of $f(z)$ around a closed curve is zero:

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

Example 7.1.3. Let γ be the path of unit circle counterclockwise. Then

$$\int_\gamma \frac{1}{z} dz = 2\pi i$$

which is not zero, implying that there is no holomorphic function $F(z)$ defined on the whole unit circle, having derivative $\frac{1}{z}$.

However, consider the principal logarithm $\log(re^{i\theta}) = \log(r) + i\theta$, we have

$$\frac{d}{dz}\log(x+iy) = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2} = \frac{\overline{z}}{z\overline{z}} = \frac{1}{z}.$$

So $\frac{1}{z}$ does have an antiderivative on $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$.

It turns out that if $f(z)$ is continuous and $\int_\gamma f(z)dz = 0$ for any closed curve, then $f(z)$ has an antiderivative, i.e. there's $F(z)$ such that $F'(z) = f(z)$.

We know that by fundamental theorem of calculus

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

By analogy, we want

$$F(w) = \int_\gamma f(z)dz$$

where γ is a curve from a fixed basepoint q to w.

First let's check that this doesn't depend on the choice of path from q to w. Suppose δ_1, δ_2 are two paths from q to w. Observe that the reverse of δ_2 is a curve from w to q, and the path obtained by following δ_1, then the reverse of δ_2 goes from q to w to q, so it is a closed curve.

Write $\delta_1 - \delta_2$ for the closed curve above. Then by assumption, we have

$$\int_{\delta_1-\delta_2} f(z)dz = 0.$$

This implies that

$$\int_{\delta_1} f(z)dz + \int_{-\delta_2} f(z)dz = \int_{\delta_1} f(z)dz - \int_{\delta_2} f(z)dz = 0.$$

Hence,

$$\int_{\delta_1} f(z)dz = \int_{\delta_2} f(z)dz.$$

Thus, the choice of path doesn't matter and so the formula $F(w) = \int_\gamma f(z)dz$ where γ is any path from q to w makes sense. Let's now check $\frac{d}{dw}F(w) = f(w)$.

$$\frac{d}{dw}F(w) = \lim_{h\to 0} \frac{F(w+h) - F(w)}{h}.$$

To evaluate $F(w+h)$, we can choose the path of integration from q to $w+h$ arbitrarily. Let's choose one that goes from q to w then to $w+h$ along a line segment (only from w to $w+h$). This line segment has length $|h|$.

If our function is holomorphic at a point w, it is differentiable on a disk $D(w,\epsilon)$ for some $\epsilon > 0$. So if $|h| < \epsilon$, then the line segment ℓ from w to $w+h$ is contained in $D(w,\epsilon)$ and hence in a region where the function is differentiable.

Now $F(w+h) - F(w)$ is simply the integral of $f(z)$ from w to $w+h$. We want

$$\lim_{h\to\infty} \frac{1}{h} \int_\ell f(z)dz - f(w) = 0.$$

Note that $f(w)$ is a constant independent of z. Thus,

$$\int_\ell f(w)dz = f(w) \int_\ell 1dz = f(w)h.$$

$$\lim_{h\to\infty} \frac{\int_\ell f(z)dz - \int_\ell f(w)dz}{h} = \lim_{h\to\infty} \frac{\int_\ell (f(z) - f(w))dz}{h}.$$

By ML-lemma,

$$\begin{aligned}
\left| \frac{\int_\ell (f(z) - f(w))dz}{h} \right| &= \frac{\left|\int_\ell (f(z) - f(w))dz\right|}{|h|} \\
&\leq \max{}_{z\in\ell} |f(z) - f(w)| \cdot \frac{\text{length}(\ell)}{|h|} \\
&= \max{}_{z\in\ell} |f(z) - f(w)|.
\end{aligned}$$

So it suffices to show that

$$\lim_{h\to 0} \max{}_{z\in\ell} |f(z) - f(w)| = 0.$$

Since $f(z)$ is continuous at w, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon$. So when $|h| < \delta$, any $z \in \ell$ obeys $|z - w| < \delta$. Then also $|f(z) - f(w)| < \epsilon$. So for $|h| < \delta$,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| < \epsilon.$$

Hence,

$$\lim_{h\to 0} \frac{F(w+h)-F(w)}{h} = f(w).$$

as needed.

7.2 Cauchy's Theorem

Question. How do we check that $\int_\gamma f(z)dz = 0$ for any closed curve γ?

Theorem 7.2.1 (Cauchy's Theorem). *Suppose $\gamma : [a,b] \to \mathbb{C}$ is a closed curve and $f(z)$ is holomorphic on γ and in the region enclosed by the curve γ. Then*

$$\int_\gamma f(z)dz = 0.$$

Proof. Recall that a vector field is a function $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\vec{F}(x,y) = (F_1(x,y), F_2(x,y)).$$

The line integral of $\vec{F}$ along a curve γ is

$$\int_\gamma \vec{F} \cdot d\vec{\ell} = \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt$$

where $\gamma(t) = (\alpha(t), \beta(t))$ and $\gamma'(t) = (\alpha'(t), \beta'(t))$ and so

$$\int_\gamma \vec{F} \cdot \vec{\ell} = \int_a^b F_1(\alpha(t), \beta(t))\alpha'(t) + F_2(\alpha(t), \beta(t))\beta'(t)dt.$$

Now recall the Stokes' theorem

$$\int_\gamma \vec{F} \cdot d\vec{\ell} = \int_{\text{region enclosed by } \gamma} \vec{\nabla} \times \vec{F} dA$$

where

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial dx} - \frac{\partial F_1}{\partial y}.$$

Then

$$\int_\gamma (\vec{F} \cdot \hat{n})d\ell = \int_a^b F_1(\alpha(t), \beta(t))\beta'(t) + F_2(\alpha(t), \beta(t))(-\alpha'(t))dt.$$

The divergence theorem says that

$$\int_\gamma (\vec{F} \cdot \hat{n})d\ell = \int_{\text{area enclosed by } \gamma} \vec{\nabla} \cdot \vec{F} dA$$

where
$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$
Let $f(x+iy) = u(x,y) + iv(x,y)$ and $\gamma(t) = \alpha(t) + i\beta(t)$.
$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b (u+iv)(\alpha'(t) + i\beta'(t))dt \\ &= \int_a^b u\alpha'(t) - v\beta'(t)dt + i\int_a^b v\alpha'(t) + u\beta'(t)dt.\end{aligned}$$

Note that $u' - v\beta' = (u,-v)\cdot(\alpha',\beta')$ and $v\alpha' + u\beta' = (u,-v)\cdot(\beta',\alpha')$. Now let $\vec{F}(x,y) = (u(x,y), -v(x,y))$. Then
$$\int_\gamma f(z)dz = \int_\gamma \vec{F}\cdot d\vec{\ell} + i\int_\gamma (\vec{F}\cdot\hat{n})d\ell.$$

$$\begin{aligned}\vec{\nabla}\times\vec{F} &= \frac{\partial - v}{\partial x} - \frac{\partial u}{\partial y} = -v_x - u_y = 0 \qquad (u_y = -v_x \text{ by CR}), \\ \vec{\nabla}\cdot\vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial - v}{\partial y} = u_x - v_y = 0 \qquad (u_x = v_y \text{ by CR}).\end{aligned}$$

So by Stokes' theorem and the divergence theorem
$$\int_\gamma f(z)dz = \int_{\text{area enclosed by }\gamma} 0dA + i\int_{\text{area enclosed by }\gamma} 0dA = 0 + i0 = 0.$$

$\square$

Remark. The example of $\int_\gamma \frac{1}{z}dz = 2\pi i$ does not contradict the Cauchy's theorem because $\frac{1}{z}$ is not holomorphic at $z = 0$.

Remark. Checking that closed curves have well-defined interior regions is not a triviality: it is the content of the Jordan Curve Theorem (out of scope).

Remark. There are several formulations of Cauchy's Theorem. We will assume that $f'(z)$ is continuous. Some formulations remove the concept of interior region and instead use the notion of a homotopy.

7.2.1 Cauchy Integral

Theorem 7.2.2 (Cauchy integral formula (1st version)). *Suppose $f(z)$ is holomorphic on the closed disk of radius R centered at $a \in \mathbb{C}$. Then*

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

where γ is the anticlockwise circle of radius R centered at a.

Proof. $\frac{f(z)}{z-a}$ may not be holomorphic at $z = a$, so to apply Cauchy's theorem we need a curve that doesn't enclose a. We can create such curve by traversing a donut-like path obtained by traversing a clockwise small circle with radius r centered at a after we reached the endpoint of the original curve and then traverse back to the endpoint. Then the enclosed area will not include a.

This curve of integration has 4 parts:

1. γ_1 : big circle of radius R, anticlockwise,
2. γ_2 : line segment connecting from the big circle to the small circle,
3. γ_3 : small circle of radius r, clockwise,
4. γ_4 : line segment connecting from the small circle to the big circle.

Note that the integrations of the two line segments cancel out each other. Then Cauchy's theorem tells us that

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0 \implies \int_{\gamma_1} + \int_{\gamma_3} = 0.$$

Thus,

$$\int_{\gamma_1} = -\int_{\gamma_3} = \int_{-\gamma_3},$$

which implies that the integral of the big circle anticlockwise is equal to the integral of the small circle anticlockwise. Hence,

$$\int_{\gamma_1} \frac{f(z)}{z-a} dz = \int_{-\gamma_3} \frac{f(z)}{z-a} dz.$$

for any $r < R$, and so we can take $r \to 0$. Now we want to show that

$$\int_{\gamma_3} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Let $\gamma(t) = a + re^{it}$ for $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_\gamma \frac{f(a)}{z-a}dz &= f(a)\int_\gamma \frac{1}{z-a}dz \\ &= f(a)\int_0^{2\pi} \frac{1}{\gamma(t)-a}\gamma'(t)dt \\ &= f(a)\int_0^{2\pi} \frac{1}{re^{it}} ire^{it}dt \\ &= 2\pi i f(a).\end{aligned}$$

So we want

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(a)}{z-a}dz,$$

i.e.,

$$\int_\gamma \frac{f(z)-f(a)}{z-a}dz = 0.$$

To show this, we apply the ML-lemma:

$$\left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right| \le \max{}_{z\in\gamma}\left|\frac{f(z)-f(a)}{z-a}\right| \cdot \text{length}(\gamma)$$

where γ is all points at distance r from a. Then $z \in \gamma \implies |z-a| = r$ and $\text{length}(\gamma) = 2\pi r$. Thus, we get

$$\begin{aligned}\left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right| &\le \max{}_{z\in\gamma}\frac{|f(z)-f(a)|}{r} \cdot 2\pi r \\ &= 2\pi \cdot \max{}_{z\in\gamma}|f(z)-f(a)|.\end{aligned}$$

Since $f(z)$ is differentiable, it is continuous. So fro any $\epsilon > 0$. there is a $\delta > 0$ such that $|z-a| = r < \delta \implies |f(z)-f(a)| < \epsilon$. So by taking $r < \delta$, we get

$$\left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right| < 2\pi\epsilon$$

for any $\epsilon > 0$, which implies that the absolute value must be zero. Hence,

$$\int_\gamma \frac{f(z)-f(a)}{z-a}dz = 0,$$

and this implies that

$$\int_\gamma \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(a)}{z-a}dz = 2\pi i f(a).$$

□

Theorem 7.2.3 (Cauchy integral formula (2nd version))**.** *Let γ be a closed curve that encloses $a \in \mathbb{C}$ exactly once anticlockwise. Suppose $f(z)$ is holomorphic inside γ. Then*

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Proof. Similar proof to previous one. $\square$

Example 7.2.4. Suppose $\omega \geq 0$ is a real number. Then

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2+1} dx = \pi e^{-\omega}.$$

Consider

$$f(z) = \frac{e^{i\omega z}}{z^2+1} = \frac{\frac{e^{i\omega z}}{z+i}}{z-i} = \frac{g(z)}{z-i}$$

where $g(z) = \frac{e^{i\omega z}}{z+i}$. Consider the integral of $f(z)$ over the semi-circle curve anticlockwise with radius R consisting of a line segment ℓ. On the line segment, let $\delta : [-R, R] \to \mathbb{C}$ be defined as $\delta(t) = t$. Then

$$\begin{aligned}
\int_\delta \frac{g(z)}{z-i} dz &= \int_{-R}^{R} \frac{g(\delta(t))}{\delta(t)-i} \delta'(t) dt \\
&= \int_{-R}^{R} \frac{g(t)}{t-i} dt \\
&= \int_{-R}^{R} \frac{e^{i\omega x}}{x^2+1} dx \\
&= \int_{-R}^{R} \frac{\cos(\omega x)}{x^2+1} dx + i \int_{-R}^{R} \frac{\sin(\omega x)}{x^2+1} dx.
\end{aligned}$$

Our goal is to compute the real part of the above expression. So what we want is

$$\lim_{R\to\infty} \int_\ell = \lim_{R\to\infty} \int_\gamma - \int_{\text{arc}}.$$

By Cauchy integral formula, we have

$$\begin{aligned}
\lim_{R\to\infty} \int_\gamma \frac{g(z)}{z-i} dz &= 2\pi i g(i) \\
&= 2\pi i \frac{e^{i\omega i}}{i+i} \\
&= \pi e^{-\omega}.
\end{aligned}$$

Now we compute

$$\lim_{R\to\infty}\int_{\text{arc}}\frac{e^{i\omega z}}{z^2+1}dz.$$

By ML-inequality, we have

$$\left|\int_{\text{arc}}\right| \le \max{}_{z\in\text{arc}}\left|\frac{e^{i\omega z}}{z^2+1}\right|\cdot\text{length(arc)}.$$

We needed to work with the upper half plane so that for $z\in$ arc, $\Im(z)\ge 0$, so then because $\omega\ge 0$, $\Im(\omega z)\ge 0$. If $\omega z = a+ib$, then $b\ge 0$ and $i\omega z = -b+ia$ has non-positive real part. So

$$|e^{i\omega z}| = e^{\Re(i\omega z)} \le e^0 = 1.$$

Also $z\in$ arc $\implies$ $|z|=R$ and so $|z^2|=R^2$. Thus we have $|z^2+1|\ge R^2-1$. Hence,

$$\frac{1}{|z^2+1|}\le\frac{1}{R^2-1}.$$

So ML-inequality becomes

$$\left|\int_{\text{arc}}\right| = \frac{1}{R^2-1}\pi R\to 0 \qquad \text{as R}\to\infty.$$

So as $R\to\infty$,

$$\int_{\text{arc}}\to 0.$$

Hence, we have

$$\begin{aligned}\int_\gamma - \int_{\text{arc}} &= \int_\ell\\ \pi e^{-\omega}-0 &= \int_{-\infty}^{\infty}\frac{\cos(\omega x)}{x^2+1}dx + i\int_{-\infty}^{\infty}\frac{\sin(\omega x)}{x^2+1}dx.\end{aligned}$$

Note that

$$\int_\gamma\frac{f(z)}{z-a}dz = \begin{cases}2\pi i f(a) & \text{if } a \text{ is in the region enclosed by } \gamma\\ 0 & \text{else.}\end{cases}$$

So if $\gamma:[a',b']\to\mathbb{C}$ is the curve,

$$\begin{aligned}f(a) &= \frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-a}dz\\ &= \frac{1}{2\pi i}\int_{a'}^{b'}\frac{f(\gamma(t)}{\gamma(t)-a}\gamma'(t)dt.\end{aligned}$$

So for example, if $f(z)$ is zero on the unit circle: $f(e^{i\theta}) = 0$, then for $|a| < 1$,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta}-a} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi i}\int_0^{2\pi} 0 d\theta \\ &= 0. \end{aligned}$$

So if $f(z)$ is zero on the circle, it is also zero inside. For comparison,

$$f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1}$$

is real differentiable and is zero on the unit circle, but not insides, so it's not holomorphic.

Claim.

$$f'(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2} dz.$$

Proof.

$$\begin{aligned} 2\pi i f'(a) &= \lim_{h\to 0} 2\pi i \left(\frac{f(a+h)-f(a)}{h}\right) \\ &= \lim_{h\to 0} \frac{\int_\gamma \frac{f(z)}{z-(a+h)} dz - \int_\gamma \frac{f(z)}{z-a} dz}{h}. \end{aligned}$$

We assumed that a is inside γ, but we should also check $a+h$. Let $\gamma : [a', b'] \to \mathbb{C}$ be the curve. Consider $|\gamma(t) - a|$ (distance from $\gamma(t)$ to a).

The domain of $|\gamma(t) - a|$ is $[a', b']$, a compact set (closed and bounded by Heine-Borel). Continuous image of a compact set is also compact, so in particular closed and so the complement is open. Note that $|\gamma(t) - a| \neq 0$ (otherwise $\gamma(t) = a$, which means that a would be on the curve γ, which we don't allow). Then 0 is in the complement of the image of $|\gamma(t) - a|$. But since the set is open, it must also contain a neighborhood of 0. We can assume it is of the form $(-\epsilon, \epsilon)$.

Conclusion: not only does $\gamma(t)$ avoid a, it never comes within ϵ of it: $|\gamma(t) - a| \geq \epsilon$ (complement contains $[0, \epsilon)$). If $|h| < \epsilon$, then also $a+h$ is inside $\gamma(t)$.

So for h small enough, we can write

$$\begin{aligned}
2\pi i\frac{f(a+h)-f(a)}{h} &= \frac{\int_\gamma \frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a}dz}{h} \\
&= \frac{1}{h}\int_\gamma \frac{(z-a)f(z)-(z-a-h)f(z)}{(z-a)(z-a-h)}dz \\
&= \frac{1}{h}\int_\gamma \frac{hf(z)}{(z-a)(z-a-h)}dz \\
&= \int_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz.
\end{aligned}$$

We want

$$\lim_{h\to 0}\int_\gamma \frac{f(z)}{(z-a)(z-a-h)}dz = \int_\gamma \frac{f(z)}{(z-a)^2}dz.$$

or equivalently,

$$\lim_{h\to 0}\int_\gamma \left(\frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\right)dz = 0.$$

$$\begin{aligned}
\lim_{h\to 0}\int_\gamma \left(\frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2}\right)dz &= \lim_{h\to 0}\int_\gamma \frac{f(z)(z-a)-f(z)(z-a-h)}{(z-a)^2(z-a-h)}dz \\
&= \lim_{h\to 0} h\int_\gamma \frac{f(z)}{(z-a)^2(z-a-h)}dz.
\end{aligned}$$

The ML-inequality says

$$\left|\int_\gamma\right| \leq \max{}_{z\in\gamma}\frac{|f(z)|}{|z-a|^2|z-a-h|}\cdot \text{length}(\gamma).$$

Notice that $|z-a| \geq \epsilon$ so for $|h| \leq \frac{\epsilon}{2}$.

$$\begin{aligned}
|z-a-h| &\geq |z-a| - |h| \\
&\geq \epsilon - \frac{\epsilon}{2} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Hence,

$$\left|\int_\gamma\right| \leq \max{}_{z\in\gamma}\frac{|f(z)|}{\epsilon^2\cdot\frac{\epsilon}{2}}\cdot \text{length}(\gamma).$$

The bound is independent of h. $\square$

7.3 Liouville's Theorem

Theorem 7.3.1 (Liouville's Theorem). *Suppose $f(z)$ is holomorphic on all of $\mathbb{C}$, and $f(z)$ is bounded, i.e., $|f(z)| \leq M$ for some fixed M. Then $f(z)$ is constant.*

Remark. Note that $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1} = 1 - \frac{2}{x^2+y^2+1}$ is real differentiable and bounded but it's not constant. So the theorem implies it's not holomorphic.

Proof. Let's compute $f'(a)$ for some $a \in \mathbb{C}$. Let γ be the circle of radius R centered at a. Then

$$f'(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^2} dz.$$

ML-lemma says

$$\begin{aligned}
|f'(a)| &\leq \frac{1}{2\pi} \max{}_{z\in\gamma} \frac{|f(z)|}{|z-a|^2} \cdot \text{length}(\gamma) \\
&= \frac{1}{2\pi} \max{}_{z\in\gamma} \frac{|f(z)|}{R^2} \cdot 2\pi R \\
&\leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R \\
&= \frac{M}{R}.
\end{aligned}$$

Since M doesn't depend on R,

$$|f'(a)| \leq \frac{M}{R} \qquad \text{for any } R > 0.$$

As $R \to \infty$, this gets arbitrarily small. So $|f'(a)| = 0$ and hence $f'(a) = 0$. So it must be true that $f(a)$ is constant. □

Claim. Suppose $f(z)$ is holomorphic inside γ and a is insides γ. Then $f''(a)$ eixsts and

$$f''(a) = \frac{1}{\pi i} \int_\gamma \frac{f(z)}{(z-a)^3} dz.$$

Proof. For w inside γ, let

$$f'(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^2} dz.$$

Then

$$\begin{aligned}
2\pi i \cdot \frac{f'(a+h)-f'(a)}{h} &= \frac{1}{h}\left(\int_\gamma \frac{f(z)}{(z-a-h)^2}dz - \int_\gamma \frac{f(z)}{(z-a)^2}dz\right) \\
&= \frac{1}{h}\int_\gamma f(z)\cdot \frac{(z-a)^2-(z-a-h)^2}{(z-a)^2(z-a-h)^2}dz \\
&= \frac{1}{h}\int_\gamma \frac{h(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz.
\end{aligned}$$

Then to show that

$$\lim_{h\to 0}\int_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz = 2\int_\gamma \frac{f(z)}{(z-a)^3}dz,$$

we show

$$\lim_{h\to 0}\int_\gamma \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2}\cdot f(z)dz - \int_\gamma \frac{2f(z)}{(z-a)^3}dz = 0$$

The LHS becomes

$$\begin{aligned}
&\lim_{h\to 0}\int_\gamma \frac{(z-a)(2(z-a)-h)-2(z-a-h)^2}{(z-a)^3(z-a-h)^2}\cdot f(z)dz \\
&= \lim_{h\to 0}\int_\gamma \frac{2(z-a)^2-h(z-a)-2((z-a)^2-2h(z-a)+h^2)}{(z-a)^3(z-a-h)^2}\cdot f(z)dz \\
&= \lim_{h\to 0}\int_\gamma \frac{3(z-a)h-2h^2}{(z-a)^3(z-a-h)^2}\cdot f(z)dz \\
&= \lim_{h\to 0} 3h\int_\gamma \frac{(z-a)f(z)}{(z-a)^3(z-a-h)^2}dz - 2h^2\int_\gamma \frac{f(z)}{(z-a)^3(z-a-h)^2}dz.
\end{aligned}$$

For the limit as $h\to\infty$ to exist and be zero, it's enough that the integral remain bounded.

$$\begin{aligned}
\left|\int_\gamma \frac{(z-a)}{(z-a)^3(z-a-h)^2}f(z)dz\right| &\le \max{}_{z\in\gamma}\left|\frac{f(z)}{(z-a)^2(z-a-h)^2}\right| \text{length}(\gamma) \\
&\le \max{}_{z\in\gamma}\frac{|f(z)|}{\epsilon^2\cdot\left(\frac{\epsilon}{2}\right)^2}\cdot \text{length}(\gamma).
\end{aligned}$$

The bound is independent of h. Similarly for the bound of

$$\left|\int_\gamma \frac{1}{(z-a)^3(z-a-h)^2}f(z)dz\right|.$$

□

Suppose $f(z)$ is holomorphic at a, then it is differentiable on a disk centered at a of some radius $\epsilon > 0$. If $|z-a| < \epsilon$, then $f'(z)$ exists. Let γ be the circle centered at a with raidus $\frac{\epsilon}{2}$, so that $f(z)$ is dfferentiable inside γ. In fact, it's holomorphic inside γ. Since $f(z)$ is differentiable inside the small disk containing the given point, it's holomorphic at that point. So we can apply the Cauchy integral formula and its corollaries and conclude that $f''(a)$ exists. In particular, $f'(z)$ is differentiable at $z = a$. Doing this on a disk around a, we conclude $f'(z)$ is holomorphic at a.

Similarly $f'(z)$ being holomorphic at a implies that $f''(z)$ is holomorphic at a, which then implies $f'''(z)$ as well, and so on. So $f(z)$ being holomorphic implies f is infinitely differentiable.

7.4 Fundamental Theorem of Algebra

Theorem 7.4.1 (Fundamental Theorem of Algebra)**.** *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

is a polynomial with complex coefficients, then either the polynomial is constant or it can be written as a product of linear factors $(az+b)$*.*

Here's a weaker version of this theorem:

Theorem 7.4.2 (Fundamental Theorem of Algebra (Weaker version))**.** *If* $p(z)$ *is a non-constant polynomial, then there is some* w *such that* $p(w) = 0$ *(any non-constant polynomial has a root).*

Remark. If $az + b$ is a factor of $p(z)$, then $w = -b/a$ makes $aw + b = 0$, so $p(w) = 0$. If $p(w) = 0$, then $z - w$ divides $p(z)$. Then $\frac{p(z)}{z-w}$ is still a polynomial and we can repeat this until the polynomial becomes a constant.

Lemma 7.4.3. *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

is non-constant $(a_d \neq 0, d \geq 1)$*, then there is a real number* R *such that* $|z| \geq R$ *implies*

$$\frac{1}{2}|a_d z^d| \leq |p(z)| \leq \frac{3}{2}|a_d z^d|,$$

which is equivalent to

$$\frac{1}{2} \leq \left|\frac{p(z)}{a_d z^d}\right| \leq \frac{3}{2},$$

which is then equivalent to

$$\left|\left|\frac{p(z)}{a_d z^d}\right| - 1\right| \leq \frac{1}{2}.$$

Proof. We have

$$\begin{aligned}\frac{p(z)}{a_d z^d} - 1 &= \frac{a_d z^d}{a_d z^d} + \frac{a_{d-1} z^{d-1}}{a_d z^d} + \cdots + \frac{a_0}{a_d z^d} - 1 \\ &= \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d}.\end{aligned}$$

Note when $|z| \to \infty, |z|^{-r} \to 0$ for any $r > 0$, i.e., for any $\epsilon > 0$, there is an R such that

$$|z| \geq R \implies \left||z|^{-r} - 0\right| < \epsilon.$$

We want R such that $|z| \geq R$ implies $|z|^{-r} < \epsilon$. Then

$$\begin{aligned}\log(|z|^{-r}) &< \log(\epsilon) \\ -r\log(|z|) &< \log(\epsilon) \\ \log(|z|) &> -\frac{\log(\epsilon)}{r} \\ |z| &> e^{-\frac{\log(\epsilon}{r}} = \epsilon^{-1/r}.\end{aligned}$$

We take any $R > \epsilon^{-1/r}$ to show that the limit exists.
Conclusion:

$$\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d} \to 0$$

as $|z| \to \infty$, i.e., for any $\epsilon > 0$, there is an such that $|z| \geq R$ implies

$$\left|\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots \frac{a_0}{a_d} z^{-d}\right| < \epsilon.$$

We take $\epsilon = \frac{1}{2}$ and conclude that for the resulting R, $|z| \geq R$ implies

$$\left|\frac{p(z)}{a_d z^d} - 1\right| < \epsilon = \frac{1}{2}.$$

□

Now we want to apply Liouville's theorem to $\frac{1}{p(z)}$. If $p(z) \neq 0$ for any z, then $\frac{1}{p(z)}$ is holomorphic (composition of $\frac{1}{z}$ with $p(z)$).

$$\begin{aligned}&\frac{1}{2}|a_d z^d| \leq |p(z)| \\ &\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_d|}|z|^{-d} \leq \frac{2}{|a_d|} R^{-d} \qquad \text{(since } |z| \geq R\text{)}\end{aligned}$$

To bound $\frac{1}{p(z)}$ on the disk of radius R centered at zero (i.e. $|z| \leq R$), notice that this region is compact (closed and bounded). Since $\frac{1}{p(z)}$ is holomorphic, it is continuous and $\left|\frac{1}{p(z)}\right|$ is the composition of the continuous functions $\frac{1}{p(z)}$ and absolute value, hence it is also continuous. Since the continuous image of a compact set is compact, the image of $\left|\frac{1}{p(z)}\right|$ on $|z| \leq R$ is compact (so in particular, it is bounded), i.e. $\left|\frac{1}{p(z)}\right| \leq M$ when $|z| \leq R$.

So now we combine the two bounds

$$\left|\frac{1}{p(z)}\right| \leq \max\left(\underbrace{\frac{2}{|a_d|}R^{-d}}_{\text{valid when } |z| \geq R}, \underbrace{M}_{\text{valid when } |z| \leq R}\right).$$

So for any choice of z, either $|z| \geq R$ or $|z| \leq R$, and the inequality holds. So we've shown $\frac{1}{p(z)}$ is a bounded holomorphic function. So by Liouville's theorem, it is constant.

Hence, if $p(z)$ is never zero, $\frac{1}{p(z)} = c$ is a constant, so $p(z) = \frac{1}{c}$. Since polynomial has no zero implies it is constant, we conclude that a polynomial being non-constant implies that the polynomial has a zero.

Remark. Not the only proof but a very typical application of Liouville's theorem (show some condition implies boundedness, deduce it's constant).

Chapter 8

Harmonic Functions

Recall the Cauchy-Riemann equations: if

$$f(x+iy) = u(x,y) + iv(x,y)$$

and if f is holomorphic, we have

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

8.1 Laplace Equation (2D)

We saw that if we can take a second derivative (and it's continuous, which guarantees $u_{xx} = u_{yx}$) we get

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}.$$

Then we obtain the $2D$ Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

Recall that obeying C-R equations (and first derivatives being continuous) implies f is holomorphic, so it is infinitely differentiable $\implies$ it has second, third derivatives (second derivative differentiable $\implies$ it is continuous).

So actually, $f(x+iy)$ having continuous first derivatives obeying C-R equations is enough to deduce that $u(x,y), v(x,y)$ are solutions to the Laplace equation.

Definition 8.1.1 (Harmonic functions)**.** The solutions to the Laplace equation are called *harmonic functions.*

We only consider the $2D$ Laplace equation.

Goal: We want to figure out conditions under which a harmonic function is the real part of a holomorphic function.

For us, a solution of the Laplace equation is a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that the second derivatives $u_{xx}, u_{yy}, u_{xy}, u_{yx}$ exist and are continuous and we have $u_{xx} + u_{yy} = 0$ (this is a normal assumption when solving certain PDEs) (assuming continuity gives us $u_{xy} = u_{yx}$).

Theorem 8.1.2. *If $u(x, y)$ is a harmonic function on an open subset $G \subseteq \mathbb{R}^2$ with no "holes" (more precisely, for any closed curve $\gamma \in G$, the region enclosed by γ is also contained in G), then there is a harmonic function $v : G \to \mathbb{R}$ such that*

$$f(x + iy) = u(x, y) + iv(x, y)$$

is holomorphic on G.

Remark. Such a $v(x, y)$ is called a *harmonic conjugate* of $u(x, y)$.

Idea: hard to write down $v(x, y)$ directly, but $f'(x + iy) = u_x(x, y) + iv_x(x, y)$ and if f is holomorphic, $v_x = -u_y$, so $f'(x+iy) = u_x(x, y) - iu_y(x, y)$. So $f'(x+iv)$ can be expressed in terms of u.

Proof. Define $g(x + iy) = u_x(x, y) - iu_y(x, y)$. Let's check it's holomorphic. Note that it has continuous first derivatives because u has continuous second derivatives, so it's enough that the C-R equations hold:

$$(u_x)_x = u_{xx} = -u_{yy} = (-u_y)_y.$$
$$(u_y)_x = u_{yx} = u_{xy} = -(-u_y)_x.$$

Now we find an antiderivative of $g(z)$, i.e. a function $f(z)$ such that $f'(z) = g(z)$. We saw that this can be done when

(i) $g(z)$ is continuous,

(ii) $\int_\gamma g(z)dz = 0$ for any closed curve $\gamma \in G$.

(The construction was to define $f(w) = \int_\delta g(z)dz$ where δ is any path from a fixed basepoint to w.)

Since $g(z)$ is holomorphic, it is differentiable and thus continuous. If γ is a closed curve in G, then by Cauchy's theorem

$$\int_\gamma g(z)dz = 0$$

because $g(z)$ is holomorphic on G, in particular inside the region enclosed by γ. So such $f(z)$ exists and we write

$$f(x+iy) = a(x,y) + ib(x,y)$$

where $a, b : \mathbb{R}^2 \to \mathbb{R}$. Then $f'(z) = g(z)$ implies

$$a_x + ib_y = u_x - iu_y,$$

which means $a_x = u_x$ and $-a_y = b_x = -u_y \implies a_y = u_y$ by C-R equations applied to $f(z)$, which is holomorphic because it satisfies $f'(z) = g(z)$. Now let's integrate:

$$\begin{aligned} a_x = u_x &\implies a(x,y) = u(x,y) + C(y) \\ a_y = u_y &\implies a(x,y) = u(x,t) + D(x). \end{aligned}$$

Taking the difference we have

$$C(y) - D(x) = 0 \implies C(y) = D(x),$$

which implies C, D are constants that doesn't depend on x or y. So we have

$$a(x,y) = u(x,y) + c,$$

where c is some constant. Then

$$f(x+iy) - c = a(x,y) - c + ib(x,y) = u(x,y) + ib(x,y).$$

Thus, $f(z)$ is a holomorphic function whose real part is $u(x,y)$. $\square$

If $G = \mathbb{C}\backslash\{0\}$, the theorem doesn't apply because unit circle encloses $0 \notin G$.

$$u(x,y) = \log(r) = \log(\sqrt{x^2+y^2}).$$

which is a harmonic function but it's not the real part of a holomorphic function on G. But if we replace G by $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$ then it's the real part of $\log(z)$.

Corollary 8.1.3. *Any harmonic function is infinitely differentiable.*

Proof. If $u : G \to \mathbb{R}$ ($G \subseteq \mathbb{C}$ open), and $z \in G$ is the point where we want to check u is infinitely differentiable. Since G is open, it contains a open disk centered at z, radius $\epsilon > 0$. So it also contains the closed disk of radius $\epsilon/2$ centered at z. This small disk has no holes. Suppose $\gamma : [a,b] \to D(z, \epsilon/2)$ is a curve, then $|\gamma(t) - z| \leq \epsilon/2$. Any point in the interior of γ is still inside the disk. To see this, note that no point outside the disk can be enclosed by the curve γ. By looking at a small enough part of G near z, we can assume our function is the real part of a holomorphic function. Since holomorphic functions are infinitely complex differentiable, they are infinitely real differentiable and so are their real parts, so our function is infinitely differentiable. $\square$

8.2 Maximum Modulus Principle

Proposition 8.2.1. *Suppose $u : G \to \mathbb{R}$ is a harmonic function where G is an open subset of $\mathbb{C}$. Suppose also that the closed disk of radius r centered at w is contained inside G. Then*

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta.$$

This is saying that $u(w)$ equals the average value of $u(z)$ on the circle.

Proof. We find $f(z)$ holomorphic such that $u(z) = \Re(f(z))$ for z in the closed disk, then we apply the Cauchy integral formula.

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} dz$$

where γ is the circle centered at w with radius r, i.e., $\gamma(t) = w + re^{it}$ where $t \in [0, 2\pi]$.

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - w} \gamma'(t) dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(\gamma(t)) i dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it}) dt. \end{aligned}$$

Then take real part to recover u. So if we have a harmonic function defined on a disk, the value at the center is the average of the values on the boundary. □

Proposition 8.2.2. *Suppose u is a harmonic function on an open set containing the closed disk centered at w with radius r, and that $u(z) \leq u(w)$ for all $z \in G$. Then $u(z) = u(w)$ on the disk centered at w with radius r.*

Proof. If $u(w + re^{it}) = u(w)$ for all t, we have equality. In this case $u(z) = u(w)$ on the circle of radius r centered at w.

To recover the result for points inside the disk, apply the same argument with a smaller radius. So it's enough to show the result holds for z on the circle. We know $u(w + re^{it}) \leq u(w)$ by assumption. Suppose $u(w + re^{it}) \neq u(w)$ for some t_0. Then

$$u(w + re^{it_0}) < u(w).$$

Since u is harmonic, it is infinitely differentiable and thus continuous. So the $u(w + re^{it})$ is a continuous function of t because it is the composition of continuous functions. Hence, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|t - t_0| < \delta \implies |u(w + re^{it}) - u(w + re^{it_0})| < \epsilon.$$

Then

$$\begin{aligned} u(w+re^{it})-u(w+re^{it_0}) &\le |u(w+re^{it})-u(w+re^{it_0})| \le \epsilon = \frac{u(w)-u(w+re^{it_0})}{2} \\ 2u(w+re^{it})-2u(w+re^{it_0}) &\le u(w)-u(w+re^{it_0}) \\ u(w)-u(w+re^{it_0}) &\le 2u(w)-2u(w+re^{it}) \\ \epsilon &\le u(w)-u(w+re^{it}). \end{aligned}$$

Then

$$u(w+re^{it}) \le u(w)-\epsilon.$$

Define $g(t)=u(w)-u(w+re^{it})$. Then we have

$$\begin{aligned} \int_0^{2\pi} g(t)dt &\ge \int_{t_0-\delta}^{t_0+\delta} g(t)dt \\ &\ge \int_{t_0-\delta}^{t_0+\delta} \frac{g(t_0)}{2}dt = 2\delta\frac{g(t_0)}{2} > 0, \end{aligned}$$

which is a contradiction, so $g(t_0)=0$ as needed. □

Definition 8.2.3 (Path-connected)**.** A set G is *path-connected* if for any $p,q\in G$, there is a path with p,q as endpoints ($\gamma:[a,b]\to G$ where $\gamma(a)=p,\gamma(b)=q$).

Example 8.2.4. Any disk (open or closed) is path-connected because it is convex (if two points are in a convex set, the line segment joining them is also in the set, which is exactly the path we need). Therefore, any convex set is also path-connected.

Example 8.2.5 (Non-example)**.** The union of two disjoint open disks of radius 1 centered at 2 and -2 is not path-connected. Suppose $\gamma:[a,b]\to G$ with $\gamma(a)=-2$ and $\gamma(b)=2$. Consider $\Re(\gamma(t))$, a real-valued function which is continuous. Then by the Intermediate Value Theorem, there is $t\in[a,b]$ such that $\Re(\gamma(t))=0$. But there are not points in G with real part zero, which implies that $\gamma(t)\notin G$, a contradiction.

Example 8.2.6 (Cont'd)**.** Consider

$$g(p)=\begin{cases} 1 & \text{if } p \text{ is in right circle} \\ 0 & \text{if } p \text{ is in left circle.} \end{cases}$$

Note that $g(z)$ is holomorphic on G since

$$\lim_{h\to 0}\frac{g(p+h)-g(h)}{h}=0 \qquad\qquad (g(p+h)=g(h)).$$

$g(p)=\Re(g(p))$ so g may also be viewed as a harmonic function on G. The maximal value of g is 1 (attained on right disk) but g is not constant (takes different value on left disk).

We want to find a sequence of disks going along the curve joining w and z and conclude the function is constant on each disk, eventually covering the whole curve. We will fix a curve γ and then show that for any $\gamma(t)$, we can draw a disk of radius $\epsilon > 0$ (same ϵ for all $\gamma(t)$), so it will take roughly length$(\gamma)/\epsilon$ steps to go from one end to the other.

So we have $\gamma[a, b] \to G$ and we define $d : [a, b] \to \mathbb{R}_{\geq 0}$ such that

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

$d(t)$ is the size of the largest disk we can draw at $\gamma(t)$. We want $d(t) \geq \epsilon > 0$ for some ϵ and all t. We will show

- $d(t) > 0$ for all t
- $d(t)$ is continuous.

Proposition 8.2.7. *If G is an open, path-connected region, then a harmonic function $u : G \to \mathbb{R}$ such that there is a $w \in G$ with $u(w) \geq u(z)$ for all $z \in G$ must be a constant function.*

Proof. We want to show $u(w) = u(z)$ by stepping along the path from w to z, showing u is constant along each step. If u is constant on a disk of radius ϵ centered at w, then we can go up to ϵ along the curve and u will still take the value $u(w)$.

The argument will be go $\epsilon/2$ (avoid reaching the boundary since disk is open) along the curve to a point ζ. We have $u(\zeta) = u(w)$, but the length of the part of γ from ζ to w is length$(\gamma) - \epsilon/2$. We can then repeat the argument to get a path of length length$(\gamma) - n\epsilon/2$ for n steps. Eventually this becomes $< \epsilon$, so z is inside the disk of radius ϵ centered at that point, which implies values are equal ($= u(w)$).

Conclusion: for any $z \in G$, $u(z) = u(w)$ so u is constant on G.

What we still need to check: we can use disks of the same radius ϵ at any point on γ. We need to find $\epsilon > 0$ and need the disk of radius ϵ centered at $\gamma(t)$ to be contained in G for all t. Equivalently, we want $\epsilon > 0$ such that

$$|\gamma(t) - z| < \epsilon \implies z \in G.$$

We have $G \subseteq \mathbb{C}$ open, $\gamma : [a, b] \to G$ is continuous. Define

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

Note that $d(t) > 0$ because $\gamma(t) \in G$, and G is open, there is some disk centered at $\gamma(t)$ contained in G with radius $\epsilon(t)$, so any $z \in \mathbb{C} \setminus G$ is at least $\epsilon(t)$ away from $\gamma(t)$, i.e.

$|\gamma(t) - z| \geq \epsilon(t) > 0$, so $\epsilon(t)$ is a lower bound for the set $\{|\gamma(t) - z| : z \in \mathbb{C}\backslash G\}$ so it is less than or equal to the greatest lower bound, the infimum:

$$d(t) = \inf_{z \in \mathbb{C}\backslash G} |\gamma(t) - z| \geq \epsilon(t) > 0.$$

Now we show $d(t)$ is continuous, i.e.

$$\lim_{h \to 0} d(t+h) = d(t).$$

(since $d(t)$ is a distance (from $\gamma(t)$ to $\mathbb{C}\backslash G$), we will control it with the triangle inequality).

For $z \in \mathbb{C}\backslash G$, $|\gamma(t) - z| \geq d(t)$ by definition of $d(t)$, which implies $|\gamma(t+h) - z| \geq d(t+h)$. Then

$$\begin{aligned} d(t) \leq |\gamma(t) - z| &= |\gamma(t) - \gamma(t+h) + \gamma(t+h) - z| \\ &\leq |\gamma(t) - \gamma(t+h)| + |\gamma(t+h) - z|. \end{aligned}$$

So $d(t) - |\gamma(t) - \gamma(t+h)| \leq |\gamma(t+h) - z|$, which means $d(t) - |\gamma(t) - \gamma(t+h)|$ is a lower bound for $|\gamma(t+h) - z|$, which is less than the infimum. Hence,

$$d(t) - |\gamma(t) - \gamma(t+h)| \leq d(t+h).$$

Rewrite this as

$$d(t) - d(t+h) \leq |\gamma(t) - \gamma(t+h)|.$$

To get a lower bound, we use

$$\begin{aligned} d(t+h) \leq |\gamma(t+h) - z| &= |\gamma(t+h) - \gamma(t) + \gamma(t) - z| \\ &\leq |\gamma(t+h) - \gamma(t)| + |\gamma(t) - z|, \end{aligned}$$

so

$$d(t+h) - |\gamma(t+h) - \gamma(t)| \leq |\gamma(t) - z|.$$

Then $d(t+h) - |\gamma(t+h) - \gamma(t)| \leq d(t)$ and so

$$-|\gamma(t+h) - \gamma(t)| \leq d(t) - d(t+h).$$

Conclusion:

$$0 \leq |d(t+h) - d(t)| \leq |\gamma(t+h) - \gamma(t)|.$$

Now use squeeze theorem as $h \to 0$, by continuity of γ,

$$\lim_{h \to 0} \gamma(t+h) = \gamma(t).$$

So the RHS $\to 0$ and so $|d(t+h) - d(t)| \to 0$. Therefore $d(t+h) \to d(t)$, i.e. $d(t)$ is continuous. So we have $d : [a,b] \to \mathbb{R}$ (domain of $\gamma : [a,b] \to G$) with $d(t) > 0$ and d continuous. Since $[a,b]$ is compact, the set of values of $d(t)$ is again compact. By (Weierstrass theorem), a continuous function on a compact set has a minimum which it attains. Then the minimal value of $d(t)$ for $t \in [a,b]$ is some positive number ϵ (cannot be ≤ 0 because d never takes such values.)

Conclusion: $d(t) \geq \epsilon > 0$ for this ϵ. $\square$

Theorem 8.2.8 (Maximum modulus principle)**.** *Suppose $G \subseteq \mathbb{C}$ is open and path-connected, then $f : G \to \mathbb{C}$ is holomorphic if $|f(z)|$ attains a maximum in G, then $f(z)$ is constant.*

Chapter 9

Power Series

Definition 9.0.1 (Power series)**.**

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

Example 9.0.2 (Taylor series)**.**

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + \cdots .$$

To understand this, we need to discuss what it means for a sequence of functions to converge.

Remark. The convergence of the sequence of partial sums $\sum_{n=0}^{k} a_n(z-z_0)^n$ is the same thing as convergence of the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

9.1 Convergence of sequences of functions

Suppose $G \subseteq \mathbb{C}$ and $f_n : G \to \mathbb{C}$ where $n \in \mathbb{Z}_{\geq 0}$ is a sequence of functions.

9.1.1 Pointwise convergence

Definition 9.1.1 (Pointwise convergence)**.** $f_n(z)$ converges to $f(z)$ *pointwise* if for each $z \in G$, the sequence $f_n(z)$ converges to $f(z)$, i.e. for each $z \in G$, for every $\epsilon > 0$, there is a $N \geq 0$ (can depend on z) such that $n \geq N$ implies

$$|f_n(z) - f(z)| < \epsilon.$$

Remark. Evaluating at z gives a sequence of complex numbers.

Example 9.1.2. Consider $f_n(z) = z^n$. $G = [0, 1] \subseteq \mathbb{R}$. When $z = 1$, $f_n(z) = 1$, we get a constant sequence 1 and so $f_n(z) \to 1$. When $z < 1$, $f_n(z) \to 0$. We need to find N such that

$$\begin{aligned} |f_n(z) - 0| < \epsilon &\quad \text{for } n \geq N \\ z^n < \epsilon &\quad \text{for } n \geq N. \end{aligned}$$

This holds when

$$\begin{aligned} n \ln(z) &< \ln(\epsilon) \\ n &> \frac{\ln(\epsilon)}{\ln(z)} \qquad \text{(divide by } \ln(z) < 0 \text{ since } z \in [0, 1]). \end{aligned}$$

Then we choose $N = \frac{\ln(\epsilon)}{\ln(z)}$.
Conclusion:

$$f_n(z) \to \begin{cases} 1 & z = 1, \\ 0 & 0 \leq z < 1. \end{cases}$$

which is not continuous.

To remedy this, we define a more stringent notion of convergence.

9.1.2 Uniform convergence

Definition 9.1.3 (Uniform convergence)**.** A sequence of functions $f_n(z)$ converges to $f(z)$ *uniformly* on G if for every $\epsilon > 0$, there exists $N \geq 0$ such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all } n > N \text{ and } z \in G.$$

Remark. The difference from pointwise convergence is that N cannot depend on z.

Example 9.1.4 (Non-example)**.** Back to the $f_n(z) = z^n$ example. Now let $G = [0, r]$ where $r < 1$ is fixed.

$$\begin{aligned} |f_n(z) - 0| &< \epsilon \\ z^n &< \epsilon \\ n &> \frac{\ln(\epsilon)}{\ln(z)}. \end{aligned}$$

We need to find an N such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in G = [0, r],$$

which is maximized at $z = r$ and so we can take $N = \frac{\ln(\epsilon)}{\ln(r)}$. Note though that while $f_n(z) \to 0$ on $G = [0, 1)$, it does not converge uniformly because we would need an N such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in [0, 1).$$

But as z approaches 1, $\ln(\epsilon)/\ln(z) \to \infty$, so there is no such N for which all z obey the inequality.

Two main desirable properties of uniform convergence:

Proposition 9.1.5. *if $f_n(z)$ are continuous and converge uniformly to $f(z)$, then $f(z)$ is continuous.*

Proof. Fix z_0, we need to show that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Let N be such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for $n \geq N$ (use uniform convergence). Now pick some $n \geq N$ and let δ be such that

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$$

whenever $|z - z_0| < \delta$. This follows from the continuity of $f_n(z)$. Finally, if $|z - z_0| < \delta$,

$$\begin{aligned}
|f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\
&\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon.
\end{aligned}$$

Thus, $f(z)$ is continuous. □

Proposition 9.1.6. *If $f_n(z)$ converges to $f(z)$ uniformly on G, and γ is a curve in G, then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

(so we can swap the order if taking limits and integration).

Proof. For $\epsilon > 0$, pick N such that for all $n \geq N$ (use uniform convergence), we have

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

$$\begin{aligned}\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &= \left|\int_\gamma f_n(z) - f(z)dz\right| \\ &\le \max_{z\in\gamma} |f_n(z) - f(z)| \cdot \text{length}(\gamma).\end{aligned}$$

But for all $z \in G$ (not just γ),

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

Thus,

$$\begin{aligned}\left|\int_\gamma f_n(z)dz - \int_\gamma f(z)dz\right| &\le \frac{\epsilon}{\text{length}(\gamma)} \cdot \text{length}(\gamma) \\ &= \epsilon.\end{aligned}$$

□

9.1.3 Weierstrass M-test for uniform convergence

Theorem 9.1.7 (Weierstrass M-test)**.** *Suppose $G \subseteq \mathbb{C}, f_k : G \to \mathbb{C}$ and $M_k \in \mathbb{R}$ with $|f_k(z)| \le M_k$ for all $z \in G$. Then*

$$\sum_{k=0}^{\infty} M_k \text{ converges } \implies \sum_{k=0}^{\infty} f_k(z) \text{ converges uniformly on } G.$$

Proof. Suppose $\sum_{k=0}^{\infty} M_k$ converges to M. Since $0 \le |f_k(z)| \le M_k$, $M_k \ge 0$. For any $\epsilon > 0$, there exists $N > 0$ such that for $n \ge N$,

$$\left|\sum_{k=0}^{n} M_k - M\right| < \epsilon.$$

Then we have

$$\begin{aligned}\left|\sum_{k=0}^{n} M_k - \sum_{k=0}^{\infty} M_k\right| &< \epsilon \\ \left|-\sum_{k=n+1}^{\infty} M_k\right| &< \epsilon.\end{aligned}$$

Since each $M_k \ge 0$, we have

$$\sum_{k=n+1}^{\infty} M_k < \epsilon.$$

Now we need to show that

$$\left|\sum_{k=0}^{n} f_k(z) - \sum_{k=0}^{\infty} f_k(z)\right| < \epsilon.$$

Since

$$\left|\sum_{k=0}^{\infty} f_k(z)\right| \leq \sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} M_k < \infty,$$

$\sum_{k=0}^{\infty} f_k(z)$ converges absolutely and so the limit of the sum exists.

$$\begin{aligned}\left|-\sum_{k=n+1}^{\infty} f_k(z)\right| &\leq \sum_{k=n+1}^{\infty} |f_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &< \epsilon.\end{aligned}$$

This no longer depends on z, so we just use the N from $\sum_k M_k$. □

Example 9.1.8. Let $f_k(z) = z^k$ where $|z| < 1$. Then

$$\sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

Question. Does it converge uniformly?

Let's view $f_k(z)$ as functions on $|z| \leq r$ (here $0 \leq r < 1$ fixed). Convergence on this set is uniform.

Proof. Use M-test. We need $|f_k(z)| = |z^k| = |z|^k \leq r^k$. So choose $M_k = r^k$.

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad (r < 1).$$

Conclusion: M-test applies and we have uniform convergence. □

Question. What about $|z| < 1$?

Answer. Convergence is not uniform. If it was, then for every $\epsilon > 0$, there is a $N > 0$ such that for $n \geq N$

$$\left|\sum_{k=0}^{n} z^k - \frac{1}{1-z}\right| < \epsilon$$
$$\left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right| < \epsilon$$
$$\frac{|z|^{n+1}}{|1-z|} < \epsilon.$$

However, the problem is that as $z \to 1$, this goes to infinity, so the bound $\frac{|z|^{n+1}}{|1-z|} < \epsilon$ cannot hold for all $|z| < 1$. Thus, the convergence is not uniform.

Remark. Even though $\{z : |z| < 1\}$ is the union of $\{z : |z| \leq r\}$ for $r < 1$, we don't have uniform convergence on the open unit disk.

9.2 Power Series

Lemma 9.2.1. *If $\sum_{k=0}^{\infty} a_k(w-z_0)^k$ converges, then $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges absolutely when $|z-z_0| < |w-z_0|$. Moreover, if $0 \leq \ell < |w-z_0|$, then convergence is uniform on $|z-z_0| \leq \ell$.*

Proof. Since $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges, the terms must go to zero (because if $\sum_{k=0}^{\infty} c_k$ converges, then we would have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \sum_{k=0}^{n} c_k - \sum_{k=0}^{n-1} c_k = 0.)$$

Therefore,

$$\lim_{k\to\infty} a_k(w-z_0)^k = 0,$$

i.e., for any $\epsilon > 0$, there is $N > 0$ such that for $n \geq N$,

$$|a_n(w-z_0)^n - 0| < \epsilon.$$

Thus, $|a_n(w-z_0)^n| < \epsilon$. Now let

$$M = \max\{|a_0(w-z_0)^0|, \ldots, |a_N(w-z_0)^N|, \epsilon\},$$

so $M \geq |a_k(w-z_0)^k|$ for all k. Then

$$\sum_{k=0}^{\infty} |a_k(z-z_0)^k| = \sum_{k=0}^{\infty} |a_k(w-z_0)^k| \left|\frac{z-z_0}{w-z_0}\right|^k$$
$$\leq \sum_{k=0}^{\infty} M \left|\frac{z-z_0}{w-z_0}\right|^k.$$

Since this is a geometric series with common ratio $\left|\frac{z-z_0}{w-z_0}\right| < 1$, it converges.
Conclusion: we have absolute convergence.

For uniform convergence, use M-test:

$$|a_k(z-z_0)^k| \leq M\left|\frac{z-z_0}{w-z_0}\right|^k = M_k \leq \left|\frac{\ell}{w-z_0}\right|^k \qquad (w-z_0<1).$$

We need sum of M_k to converge:

$$\sum_k M_k = \sum_k M\left(\frac{\ell}{|w-z_0|}\right)^k,$$

which is a geometric series with common ratio <1, so it converges and is uniformly convergent by M-test. □

Theorem 9.2.2. *For a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$, there is a $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that the series converges absolutely when $|z-z_0| < R$ and diverges when $|z-z_0| > R$. It converges uniformly on $|z-z_0| \leq \ell$ ($\ell < R$ is fixed).*

Proof. Consider

$$S = \left\{x \in \mathbb{R}_{\geq 0} \ \mid \sum_{k=0}^{\infty} a_k x^k \text{ converges}\right\}.$$

Note that $0 \in S$, and so S is nonempty (can ask for supremum, which will be in $\mathbb{R}_{\geq 0} \cup \{\infty\}$. Applying the lemma above, if $x \in S$, then $\sum_{k\geq 0} a_k x^k$ converges and so $\sum_{k\geq 0} a_k(z-z_0)^k$ converges absolutely for $|z-z_0| < |x| = x$ ($w = z_0 + x$). So if we let $z = z_0 + y$, then

$$|y| < x \implies \sum_{k\geq 0} a_k y^k \quad \text{converges absolutely} \qquad \implies y \in S.$$

y can be in $[0,x)$. So $x \in S \implies [0,x) \subseteq S \implies [0,x] \subseteq S$. Consider the following cases:

1. $\sup S = \infty$ (S is unbounded): S contains a sequence x_i with limit ∞. Then S contains $[0,x_i]$ for all i. For any real number t, eventually $x_i \geq t$. Then $t \in [0,x_i] \subseteq S$. Hence, $t \in S$ and $S = \mathbb{R}_{\geq 0}$ ($R = \infty$).

2. $\sup S = R$. If $|z-z_0| > R$ and $\sum_{k\geq 0} a_k(z-z_0)^k$ converges, then S contains all x with $0 \leq x < |z-z_0|$. But R is an upper bound for S so it must be at least as big as any such x, for example,
$$x = \frac{R+|z-z_0|}{2} > R.$$
This contradicts R being an upper bound for S. So $|z-z_0| > R \implies$ divergence. Now if $|z-z_0| < R$, then S contains an element x such that $|z-z_0| < x \leq R$ (otherwise

$|z-z_0|$ would be an upper bound for S strictly less than R, the least upper bound). Now $\sum_{k\geq 0} a_k x^k$ converges as $x \in S$ and $|z-z_0| < x \implies$ absolute convergence for $|z-z_0| < x$ (by lemma). Hence, we have absolute convergence for $|z-z_0| < R$.

□

Definition 9.2.3 (Radius of convergence)**.** R is the *radius of convergence* of $\sum_{k\geq 0} a_k(z-z_0)^k$ where $|z-z_0| < R$ (region on which we have convergence).

Remark. We cannot conclude anything about convergence when $|z-z_0| = R$.

Lemma 9.2.4. *The radius of convergence R obeys*

$$\frac{1}{R} = \lim sup \sqrt[k]{|a_k|}.$$

Proof. If $\limsup \sqrt[k]{|a_k|} = L$. For every $\epsilon > 0$, there exists N such that $k \geq N$ implies

$$\sqrt[k]{|a_k|} > L + \epsilon.$$

Then

$$|a_k| > (L+\epsilon)^k.$$

Hence,

$$\begin{aligned}\sum_{k\geq 0} |a_k(z-z_0)|^k &\leq \sum_{k=0}^{N} |a_k||z-z_0|^k + \sum_{k>N} |a_k||z-z_0|^k \\ &= \text{constant} + \sum_{k>N} (L+\epsilon)^k |z-z_0|^k.\end{aligned}$$

This converges when $(L+\epsilon)|z-z_0| < 1$ for some $\epsilon > 0$, equivalently,

$$|z-z_0| < \frac{1}{L+\epsilon},$$

and so we have absolute convergence when $|z-z_0| < 1/L$. We conclude $R \geq 1/L$ because if $1/L > R$, choose z such that $1/L > |z-z_0| > R$ (first inequality implies absolute convergence but the second inequality implies divergence, contradiction!)

If $L = \limsup \sqrt[k]{|a_k|}$, for any $\epsilon > 0$, there are infinitely many k such that $\sqrt[k]{|a_k|} > L - \epsilon$. Assume $L > 0$ (need to show $L = 0 \iff R = \infty$) for ϵ small enough such that $L - \epsilon > 0$. For infinitely many k, we have

$$\begin{gathered}|a_k| > (L-\epsilon)^k \\ |a_k||z-z_0|^k \geq (L-\epsilon)^k |z-z_0|^k.\end{gathered}$$

If converged, it would go to zero and then $(L-\epsilon)|z-z_0| < 1$. If $\frac{1}{L-\epsilon} \leq |z-z_0|$, we have divergence. So $\frac{1}{L} < |z-z_0|$ implies divergence.

Claim.

$$\frac{1}{L} \geq R$$

Proof. If not, we have a contradiction because

$$R > |z - z_0| > \frac{1}{L}$$

implies both convergence and divergence. ☐

Since we have $\frac{1}{L} \geq R$ and $\frac{1}{L} \leq R$, we have $\frac{1}{L} = R$. ☐

Remark. If $L = 0$, any $\epsilon > 0$ is an upper bound on all but finitely many terms $\sqrt[k]{|a_k|}$.

Example 9.2.5. Consider $a_k = \frac{1}{k!}, z_0 = 0, \sum_{k\geq 0} \frac{z^k}{k!} = e^z$. Then we can find the radius of convergence via

$$\frac{1}{R} = \limsup \sqrt[k]{\left|\frac{1}{k!}\right|}.$$

To compute this directly, we can use stirling's formula:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

We can see that

$$\sqrt[n]{n!} \approx \frac{n}{e} + \text{ lower terms.}$$

Therefore,

$$\sqrt[n]{\frac{1}{n!}} \approx \frac{e}{n} \to 0 \quad \text{as } n \to \infty.$$

So $R = \infty$. We can also use ratio test.

Example 9.2.6. $\sum_k a_k(z - z_0)^k = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots$ where $z_0 = 0$ and

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{k} & k \text{ odd.} \end{cases}$$

$$\limsup_{k \text{ odd}} \sqrt[k]{\left|\frac{(-1)^{(k-1)/2}}{k}\right|} = \sqrt[k]{\frac{1}{k}}.$$

Taking log gives

$$\ln\left(\sqrt[k]{\frac{1}{k}}\right) = \frac{1}{k}(-\ln(k)) \to 0.$$

Then

$$\sqrt[k]{\frac{1}{k}} \to 1.$$

Proposition 9.2.7. *If $\sum a_k(z-z_0)^k$ has radius of convergence R, then the limit function is continuous on the open disk centered at z_0 with radius at R.*

Proof. We want to argue that uniform convergence of continuous things gives something continuous. Let z be in the disk $|z-z_0| < R$. z is contained in the disk $|w-z_0| \leq \frac{|z-z_0|+R}{2}$.By what we know, convergence is uniform here, hence it is continuous on this disk, in particular at z. So for any z in the disk, we have continuity (i.e., the power series is continuous). □

Proposition 9.2.8. *If γ is a curve in the open disk $|z-z_0| < R$ centered at z_0, then*

$$\int_\gamma \sum_{k\geq 0} a_k(z-z_0)^k dz = \sum_{k\geq 0} a_k \int_\gamma (z-z_0)^k dz.$$

If γ is a closed curve, then this is zero.

Proof. If we have uniform convergence, then

$$\begin{aligned}\int_\gamma \sum_{k\geq 0} a_k(z-z_0)^k dz &= \lim_{n\to\infty} \int_\gamma \sum_{k=0}^{n} a_k(z-z_0)^k dz \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \int_\gamma a_k(z-z_0)^k dz \\ &= \sum_{k=0}^{\infty} \int_\gamma a_k(z-z_0)^k dz.\end{aligned}$$

So it would be enough for the series to converge uniformly on a set containing the curve γ. We want to find a disk of radius $< R$ still containing γ. When we discussed how harmonic functions, we considered

$$d(t) = \inf_{z\in\mathbb{C}\setminus G} |\gamma(t) - z|.$$

We showed $d(t) \geq \epsilon > 0$.

Claim. The disk of radius $R - \epsilon/2$ contains γ.

If there was a point on γ outside this region, then it is within $\epsilon/2$ of a point outside the disk of radius R. This would contradict

$$\inf |\gamma(t) - z| \geq \epsilon.$$

We would have produced a value of this that was at most $\epsilon/2$. So we can find such a disk and we have uniform convergence.

If γ is closed, by Cauchy's theorem

$$\int_\gamma (z-z_0)^k = 0.$$

Then

$$\int_\gamma \sum_{k\geq 0} a_k (z-z_0)^k dz = \sum_{k=0}^{\infty} a_k \cdot 0 = 0.$$

$\square$

Theorem 9.2.9 (Morera's Theorem). *Suppose $f(z)$ is continuous on an open set G and for any closed curve γ in G,*

$$\int_\gamma f(z)dz = 0.$$

Then $f(z)$ is holomorphic.

Proof. Recall that if $f(z)$ is continuous and $\int_\gamma f(z)dz = 0$ for any closed curve γ, then it has an antiderivative $F(z)$ such that $F'(z) = f(z)$. Note that $F(z)$ is differentiable on G, so actually holomorphic on G. But holomorphic functions are infinitely differentiable. So F is twice differentiable, and thus $F'(z) = f(z)$ is differentiable on G. Thus, $f(z)$ is holomorphic on G. $\square$

Theorem 9.2.10. *If $\sum_{k\geq 0} a_k(z-z_0)^k$ has radius of convergence R, then on $|z-z_0| < R$, it defines a holomorphic function.*

Proof. We saw that it's continuous, and the integrals along closed curves are zero, so Morera's theorem applies. $\square$

Now let's understand derivatives of power series.

Lemma 9.2.11. *If $f(z) = \sum_{k\geq 0} a_k(z-z_0)^k$ for $|z-z_0| < R$, then for such z,*

$$f'(z) = \sum_{k\geq 0} k a_k (z-z_0)^{k-1}.$$

Proof. Recall that

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2} dw,$$

where γ is a closed curve that encloses z. Then

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{\sum_{k\geq 0} a_k (w-z_0)^k}{(w-z)^2} dw.$$

We want this series to converge uniformly on a set containing γ (actually we consider the set γ). Note that $\gamma(t)$ is continuous, and so $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is continuous as z is not on the curve γ. Therefore, $\left|\frac{1}{(\gamma(t)-z)^2}\right|$ is a continuous function $f : [a, b] \to \mathbb{R}$. Then its values are bounded since $[a, b]$ is compact.

Claim. If the series $\sum_{n\geq 0} g_n(z)$ converges uniformly to $g(z)$ and $h(z)$ is bounded, then

$$\sum_{n\geq 0} g_n(z)h(z) \to g(z)h(z) \quad \text{uniformly.}$$

Proof. Given

$$\left|\sum_{n=0}^{k} g_n(z) - g(z)\right| < \epsilon$$

for any $\epsilon > 0$, there is N such that $k \geq N$ implies this holds for all z.

$$\begin{aligned}\left|\sum_{n=0}^{k} g_n(z)h(z) - g(z)h(z)\right| &= \left|\sum_{n=0}^{k} g_n(z) - g(z)\right| |h(z)| \\ &\leq \epsilon \cdot \sup_z |h(z)|.\end{aligned}$$

This holds uniformly. □

Now back to the original proof, we have $g_n(w) = a_n(w - z_0)^n$ and $h(w) = \frac{1}{(w-z)^2}$. Then

$$\begin{aligned}f'(z) &= \frac{1}{2\pi i} \sum_{k\geq 0} a_k \int_\gamma \frac{(w - z_0)^k}{(w - z)^2} dw \\ &= \frac{1}{2\pi i} \sum_{k\geq 0} a_k 2\pi i \left.\frac{\partial}{\partial w}(w - z_0)^k\right|_{w=z} \\ &= \sum_{k\geq 0} a_k k(z - z_0)^{k-1}.\end{aligned}$$

If $f(z) = \sum_{k\geq 0} a_k(z - z_0)^k$, then

$$\begin{aligned}f(z_0) &= a_0 \\ f'(z_0) &= 1 \cdot a_1 \\ f''(z_0) &= 2 \cdot \cdot a_2 \\ f^{(n)}(z_0) &= k! a_k.\end{aligned}$$

Thus,

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k = \sum_{k \geq 0} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

□

Example 9.2.12. Consider $z_0 = 0$ and $a_k = 1/k!$. Then

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

We will show that this is equal to e^z. Note that

$$f'(z) = \sum_{k=0}^{\infty} \frac{k}{k!} (z - z_0)^{k-1} = f(z).$$

$f(0) = 1$. Now consider $f(z)e^{-z}$. Then taking the derivative gives

$$f'(z)e^{-z} - f(z)(e^{-z}) = 0.$$

This implies that $f(z)e^{-z} = C$ is a constant. At $z = 0$, we have $f(0)e^{-0} = C \implies C = 1$. Thus

$$f(z)e^{-z} = 1 \implies f(z) = e^z.$$

Theorem 9.2.13. *Suppose $f(z)$ is holomorphic on $|z - z_0| < R$. Then*

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w - z_0)^{k+1}} dw$$

and γ is a curve enclosing z_0 but contained in the disk. The series converges on $|z - z_0| < R$.

Proof. By Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw.$$

Now write

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

We want to expand the last factor as a geometric series to make sure $|w-z_0|>|z-z_0|$ on γ. We choose γ to be the circle $|w-z_0| = \frac{|z-z_0|+R}{2}$. Then we have uniform convergence on the enclosed region and so

$$\sum_{k\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^k \xrightarrow{\text{uniformly}} \frac{1}{1-\frac{z-z_0}{w-z_0}}.$$

Consider

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0}\cdot\sum_{k=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^k dw &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw \\ &= f(z).\end{aligned}$$

We appealed to uniform convergence to interchange the order of integration and summation. Although $\frac{f(w)}{w-z_0}$ is not defined at $w=z_0$, it is defined and bounded on γ (because it is continuous, and γ is a compact set). So we can write

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\int_\gamma \frac{f(w)}{w-z_0}\frac{(z-z_0)^k}{(w-z_0)^k}dw \\ &= \sum_{k=0}^{\infty}\underbrace{\left[\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z_0)^{k+1}}dw\right]}_{a_k}(z-z_0)^k.\end{aligned}$$

Notice that

$$\int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}dw$$

is the same for any γ enclosing z_0. The argument with uniform convergence implies that this series converges for our particular choice of z. We could do this for any z with $|z-z_0|<R$, so we have convergence for such z. Hence, the radius of convergence is at least R (but possibly could be larger). □

Theorem 9.2.14. *The radius of convergence equals to the largest number R' such that $f(z)$ can be extended to a holomorphic function on $|z-z_0|<R'$.*

Proof. Previous result tells us that the radius of convergence R is at least R' ($R\geq R'$). On the other hand, if the radius of convergence is R, the power series defines a holomorphic function extending $f(z)$ on $|z-z_0|<R$. So $R'\geq R$. Hence, $R=R'$. □

Example 9.2.15. Consider

$$f(z) = z - \frac{z^3}{3}+\frac{z^5}{5}-\frac{z^7}{7}+\cdots = \arctan(z).$$

Previously we saw that the radius of convergence is 1. Even though $\arctan(x)$ is infinitely differentiable as a function of a real variable, the Taylor series still only has finite radius of convergence (as opposed to e^x). Actually $\arctan(z)$ does not extend holomorphically to $z = \pm i$. If $\tan(z) = i$, then $\sin(z) = i\cos(z)$ and so $\sin^2(z) + \cos^2(z) = 0$. But $\sin^2(z)+\cos^2(z) = 1$ for any z, so there is no such z. Therefore, we cannot extend $\arctan(z)$ outside $|z| < 1$ holomorphically. Thus, $R = 1$ as we already checked.

Alternatively,

$$\arctan(z) = \int_0^z \frac{1}{1+w^2} dw,$$

which blows up at $w = \pm i$.

Proposition 9.2.16.

$$\frac{\partial^k f}{\partial z^k}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}} dw$$

where γ encloses z_0. (We already saw this for $k = 0, 1, 2$)

Proof. Our power series is a Taylor series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(z_0).$$

We get that by differentiating k times and evaluating at z_0. Then compare this to the formula for a_k from the theorem. $\square$