

Math 185 Notes

Complex Analysis

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Lecture 1

Complex Numbers

1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for $|x| < r$, where r is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n| |x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

Example 1.1.1. The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for $|x| < 1$.

Question. Now what if we replace the real variable x by the complex variable z ?

Answer. If $|z| < r$, then

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

so the sum converges absolutely for $z \in D(0, r)$ (disc of radius r centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

Example 1.1.2. Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function $\mathbb{R} \rightarrow \mathbb{R}$, $f(z)$ is infinitely differentiable at $z = 0$, and all derivatives of $f(z)$ are zero at $z = 0$. Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from $f(z)$!

Example 1.1.3. Consider the same example as above, but with z as a complex number. Let $z = it$ where $t \in \mathbb{R}$. Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at $z = 0$ and thus not complex-differentiable at $z = 0$.

Example 1.1.4. Now let's set $z = x + iy$ where $x, y \in \mathbb{R}$. Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ (instead of $\mathbb{C} \rightarrow \mathbb{C}$). Let's differentiate with respect to x :

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to y :

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = i f'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i \frac{\partial f'(z)}{\partial y} = i \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of) $f(z)$ satisfy the two-dimensional *Laplace equation*.

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

Example 1.1.5. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful tool for computing integrals.

Lecture 2

Complex Differentiation

2.1 Derivatives

Definition 2.1.1 (Derivative). The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\lim_{x \rightarrow a} f(x) = L$$

means for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$. (For any "tolerance" ϵ , we can guarantee $f(x)$ is within ϵ of L by forcing x to be close enough to a .)

Remark. Note that $x = a$ doesn't satisfy $0 < |x - a|$, so the value of f at $x = a$ has no bearing on whether $\lim_{x \rightarrow a} f(x)$ exists.

2.1.1 Continuity

Definition 2.1.2 (Continuous). If $\lim_{x \rightarrow a} f(x) = f(a)$, then we say f is *continuous* at a .

Remark. Setting $L = f(a)$ in the limit, $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (even when $x = a$) when talking about continuity, we leave out the $0 < |x - a|$ part for convenience because $x - a = 0$ automatically works.

Now let's consider a function $f : \mathbb{C} \rightarrow \mathbb{C}$, $\lim_{z \rightarrow a} f(z) = L$ means for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

Remark. Now the z 's that we worry about form an open disc with radius δ instead of an interval from the real case.

Similarly, if $\lim_{z \rightarrow a} f(z) = f(a)$, we say f is *continuous* at $z = a$.

Example 2.1.3. $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.

Proof. For $\epsilon > 0$, let $\delta = \epsilon$, then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

Example 2.1.4. $\lim_{z \rightarrow 0} \bar{z}/z$ (although this is undefined at $z = 0$, this has no bearing on whether the limit exists).

Proof. Suppose $\lim_{z \rightarrow 0} \bar{z}/z = L$ for some L . Let's take $\epsilon = 1$. There is a $\delta > 0$ such that

$$0 < |z - 0| < \delta \implies \left| \frac{\bar{z}}{z} - L \right| < \epsilon = 1.$$

Let $z = \delta/2$ and so does $z = i\delta/2$. Then for $z = \delta/2$:

$$\frac{\bar{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for $z = i\delta/2$:

$$\frac{\bar{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the L must lie in the intersection of the two open unit discs centered at -1 and 1 . However, since they are open discs, these two discs do not overlap and so L does not exist. □

Remark. This implies that there is no way to extend \bar{z}/z to a continuous function at $z = 0$.

2.1.2 Properties of Limits

If $\lim_{x \rightarrow a} f(x) = L_1$, $\lim_{x \rightarrow a} g(x) = L_2$, then

(i)

$$\lim_{x \rightarrow a} (f(x) + g(x)) = L_1 + L_2.$$

(ii)

$$\lim_{x \rightarrow a} f(x)g(x) = L_1L_2.$$

(iii)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Remark. These implies that the sum/product/quotient of continuous functions are continuous.

Proposition 2.1.5 (Composite function of continuous functions is continuous). *If $f(x)$ is continuous at $x = a$, and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.*

Proof. We want $|g(f(x)) - g(f(a))| < \epsilon$. By continuity of g at $x = f(a)$, there exists $\delta_1 > 0$ such that

$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$

We want to take $w = f(x)$, so we need

$$|f(x) - f(a)| < \delta_1.$$

But by continuity of f at $x = a$, we know that δ_1 will be our ϵ when $|x - a| < \delta_2$ for some $\delta_2 > 0$. Then for such x ,

$$|g(f(x)) - g(f(a))| < \epsilon.$$

□

2.2 Derivatives (Cont'd)

Definition 2.2.1 (Differentiable). We say that $f(z)$ is differentiable at $z = a$ iff $\frac{f(z)-f(a)}{z-a}$ extends to a continuous function at $z = a$ (the value there is $f'(a)$).

Example 2.2.2. $f(z) = z$ is differentiable with $f'(z) = 1$.

Proof.

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z+h-z}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

□

Example 2.2.3 (Interesting one). $f(z) = \bar{z}$ is not differentiable but is continuous.

Proof.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{h}}{h} \\ &= \text{DNE} \quad (\text{proved in previous example}) \end{aligned}$$

□

Proposition 2.2.4 (Differentiability implies continuity). $f(z)$ differentiable at $z = a$ implies that $f(z)$ is continuous at $z = a$.

Proof. We want to show that $\lim_{z \rightarrow a} f(z) = f(a)$.

$$\begin{aligned} \lim_{z \rightarrow a} f(z) - f(a) &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot (z - a) \\ &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot \lim_{z \rightarrow a} (z - a) \quad (\text{assume both limits exist}) \\ &= f'(a) \cdot 0 \\ &= 0. \end{aligned}$$

□

Remark. This is a common technique to show continuity by showing the limit of the difference is zero.

2.2.1 Properties of complex-derivatives

(i)

$$\frac{d}{dz} cf(z) = cf'(z), \quad \forall c \in \mathbb{C}.$$

(ii)

$$\frac{d}{dz}(f + g) = f'(z) + g'(z).$$

(iii)

$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)

$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)

$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

Proposition 2.2.5 (Power rule).

$$\frac{d}{dz}z^n = nz^{n-1}$$

*for all integers n .**Proof.* We induct on n . For $n \geq 0$, when $n = 0$,

$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$

By the product rule,

$$\begin{aligned} \frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\ &= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \quad (\text{inductive hypothesis}) \\ &= nz^{n-1}. \end{aligned}$$

For $n < 0$, simply apply quotient rule. □

Lecture 3

Holomorphic Functions and Cauchy-Riemann Equations

Recall

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

Being differentiable at a point says little about how "nice" a function is.

Example 3.0.1. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This is nowhere continuous, and thus nowhere differentiable. But if we consider $x^2 f(x)$, it is differentiable at $x = 0$:

$$\lim_{h \rightarrow 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h \rightarrow 0} h f(h) = 0.$$

Nevertheless, it's still not a very "nice" function.

3.1 Holomorphic Functions

Definition 3.1.1 (Holomorphic). A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is *holomorphic* at a point a if it is differentiable at z for all z within distance r of a for some $r > 0$. In other words, $f(z)$ is differentiable everywhere sufficiently close to a .

Definition 3.1.2 (Open/closed disk). The *open disk* of radius r centered at $a \in \mathbb{C}$ is

$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The *closed disk* is

$$\overline{D}(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Thus, we can say $f(z)$ is holomorphic at $a \in \mathbb{C}$ if $f(z)$ is differentiable on an open disk centered at a . (if the point is not specified, it means that f is holomorphic everywhere.)

Example 3.1.3 (Polynomials are holomorphic). We saw last time that z^n is differentiable everywhere for $n \geq 0$. Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

Example 3.1.4. $f(z) = |z|^2 = z\bar{z}$ is differentiable at zero.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h\bar{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} \bar{h} \\ &= 0. \end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus, f is not holomorphic.

3.2 The Cauchy-Riemann Equations

Question. How to tell if a function is complex-differentiable?

Answer. We'll reduce this to a question about real derivatives.

Let $x + iy$, where $x, y \in \mathbb{R}$. If $f : \mathbb{C} \rightarrow \mathbb{C}$,

$$\begin{aligned} \frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h}. \end{aligned}$$

Note that h is real. Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}. \end{aligned}$$

Example 3.2.1. $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}
\frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2ihy}{h} \\
&= \lim_{h \rightarrow 0} 2x + h + 2iy \\
&= 2x + 2iy \\
&= 2z \\
&= f'(z).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{x^2 - (y+h)^2 + 2ix(y+h) - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h \rightarrow 0} \frac{-2yh - h^2 + 2ixh}{h} \\
&= \lim_{h \rightarrow 0} -2y - h + 2ix \\
&= -2y + 2ix \\
&= 2i(x + iy) \\
&= if'(z).
\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

Theorem 3.2.2.

(i) If $f : \mathbb{C} \rightarrow \mathbb{C}$ is complex-differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and they satisfy

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

(ii) If $f : \mathbb{C} \rightarrow \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous on some open disk centered at z , and if

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

then f is complex-differentiable at z .

Proof.

(i) Since f is complex-differentiable, we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose h is real. Then

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since h is real, we get $\frac{\partial f}{\partial x}$ and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose h is purely imaginary: $h = ik$ for $k \in \mathbb{R}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then $h \rightarrow 0$ is equivalent to $k \rightarrow 0$ since $|h| = |k|$. Thus we have

$$\lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

□