# STOCHASTIC PROCESSES

STAT 150

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# Contents

# 1 Probability Review

## 1.1 Basic Definitions

**Definition 1.1.1** (Probability Space). A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple consisting of a set $\Omega$ called the *sample space*, a set $\mathcal{F} \subseteq \Omega$ satisfying certain closure properties, and a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns probbailitie to events in a coherent way.

**Requirements for $\mathcal{F}$:**

(i) $\Omega \in \mathcal{F}$.

(ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$.

(iii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}.$$

**Requirements for $\mathbb{P}$:**

(i) $\mathbb{P}(\Omega) = 1$.

(ii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint (meaning $E_i \cap E_j = \varnothing$ for $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(E_i\right)$$

**Definition 1.1.2** (Random Variable). A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ whenever $B$ is a "nice" subset of $\mathbb{R}$.

**Example 1.1.3.** $\Omega = \{H, T\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(\{H\}) = \frac{1}{2}$. $X(H) = 1, \quad X(T) = 0.$

$$\mathbb{P}(X = 1) = \mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(X = 0) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

## 1.2 Overview

**Definition 1.2.1** (Stochastic Process). A *stochastic process* is a collection $\{X_t : t \in T\}$ of random variables $X_t : \Omega \to S \subseteq \mathbb{R}$ all defined on the some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here $T$ is some index set (typically representing time) and $S$ is the *state space*. One write this as

$X : \Omega \times T \to S$, $(w, t) \mapsto X_t(\omega)$. For a given outcome $\omega \in \Omega$, we get a sample path trajectory $X(\omega) : T \to S, t \mapsto X_t(\omega)$. A stochastic process can then be thought of as a random function.

The theme of this course is what can we say about the distribution of trajectories?

**Example 1.2.2** (Branching Process (DTDS)). $X_0 = 1$, one individual in the 0th generation individuals produce a random number of offspring, i.i.d. $(\xi_i^{(n)})_{i \in \mathbb{N}, n \in \mathbb{N}_0}$.

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

One interesting question would be what is $\mathbb{P}(X_n = 0 \text{ eventually})$, the probability of dying out?

**Example 1.2.3** (Poisson Process (CTDS)). Recall that the Poisson distribution is used to model the number of occurrences of a rare event in some fixed period of time. The Poisson process $(N_t)_{t \geq 0}$ models the number of occurrences throughout time. $N_t = \#$ of occurrences by time $t$.