Topology and Analysis

MATH 202A

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Contents

Lecture 1

Metric Spaces

1.1 Fundamentals

Definition 1.1.1. Let X be a set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ that satisfies:

(i) $d(x, y) = d(y, x) \ \forall \ x, y \in X$

(ii) $d(x, y) \leqslant d(x, z) + d(z, y) \ \forall \ x, y, z \in X$

(iii) $d(x, y) = 0 \iff x = y$

If a function d satisfies (i), (ii) above, and $d(x, x) = 0$ for all $x \in X$, then d is a **semi-metric**.

Example 1.1.2. On $\mathbb{C}^n$, the following are common metrics:

- $d_p(x, y) = \left(\sum\limits_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}$ for $p \geqslant 1$
- $d_\infty(x, y) = \sup \{|x_j - y_j| : 1 \leqslant j \leqslant n\}$

(Verify that these are metrics.)

Fact. If $S \subseteq X$, and d is a metric on X, then d is a metric on S.

Definition 1.1.3. (X, d) where d is a metric of X is called a **metric space**.

Remark. If $Y \subseteq X$, restrict d to $Y \times Y \subseteq X \times X$, denoted $d|_Y$, then $(Y, d|_Y)$ is a metric space.

Definition 1.1.4. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ such that:

(i) $\|cv\| = |c| \cdot \|v\|$ for $c \in$ or and $v \in V$

(ii) $\|v + w\| \leqslant \|v\| + \|w\|$ for $v, w \in V$

(iii) $\|v\| = 0$ implies $v = 0$

A function that satisfies only (i) and (ii) above is called a **seminorm**.

Remark. Any norm $\| \cdot \|$ on X induces the metric $d(x, y) := \|x - y\|$.

Example 1.1.5. Let V be the space of continuous functions on $[0,1]$. Then $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ is a norm on V.
It can also be shown that $\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ is a norm on V.

Definition 1.1.6. Let (X, d_x) and (Y, d_y) be metric spaces. A function $f : X \to Y$ is **isometric** if $d_y(f(x_1), f(x_2)) = d_x(x_1, x_2)$ for all $x_1, x_2 \in X$.

Remark. All isometries are injective.

Example 1.1.7. If $S \subseteq X$, and $f : S \to X$ is defined by $f(x) = x$ (inclusion), then f is an isometry. If f is also onto, then f is viewed as an isometric isomorphism between (X, d_x) and (Y, d_y). f^{-1} is also an isomorphism.

Definition 1.1.8. A function $f : X \to Y$ is **Lipschitz** if there is a constant $k \geqslant 0$ such that $d_y(f(x_1), f(x_2)) \leqslant k \cdot d_x(x_1, x_2)$. The smallest such constant is the **Lipschitz constant** for f.

Definition 1.1.9. $f : X \to Y$ is **uniformly continuous** if $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ such that $d_y(f(x_1), f(x_2)) < \epsilon$ whenever $d_x(x_1, x_2) < \delta$.

Remark. It is easy to see that if f is Lipschitz, then it is uniformly continuous.

Definition 1.1.10. $f : X \to Y$ is **continuous at** x_0 if $\forall\ \epsilon > 0$, $\exists\ \delta(x_0) > 0$ such that $d_y(f(x), f(x_0)) < \epsilon$ whenever $d_x(x, x_0) < \delta(x_0)$. We say f is **continuous** if it is continuous at every $x \in X$.

Definition 1.1.11. A sequence $\{x_n\}$ in X **converges** to $x^* \in X$ if $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $d(x_n, x^*) < \epsilon$.

Proposition 1.1.12. A function $f : X \to Y$ is continuous $\iff$ $x_n \to x$ implies $f(x_n) \to f(x)$.

Definition 1.1.13. $S \subseteq X$ is **dense** in X if $\forall\ x \in X$ and $\epsilon > 0$, $\exists\ s \in S$ such that $d(x, s) < \epsilon$.

Proposition 1.1.14. Let S be dense in X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that $f(s) = g(s)$ for all $s \in S$. Then $f = g$ on X.

Proof. Let $x \in X \backslash S$, and let $\epsilon > 0$. Then $\exists\ \delta > 0$ and $s \in S$ such that $d(f(x), f(s)) < \epsilon/2$, and $d(g(x), g(s)) < \epsilon/2$ for $d(x, s) < \delta$, by continuity and density. Then

$$d(f(x), g(x)) \leqslant d(f(x), f(s)) + d(g(s), g(x)) < \epsilon/2 + \epsilon/2 = \epsilon,$$

since $f(s) = g(s)$. Thus $d(f(x), g(x)) = 0$, so $f(x) = g(x)$. $\square$

Definition 1.1.15. A sequence $\{x_n\}$ is **Cauchy** if $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $n, m \geqslant N$ implies $d(x_n, x_m) < \epsilon$. A metric space is **complete** if every Cauchy sequence in it converges.

Example 1.1.16. Consider $(\mathbb{Q}, |\cdot|)$. We know there exists a Cauchy sequence converging to $\sqrt{2} \in \mathbb{R}$, but in this metric space, $\sqrt{2}$ is not an element, so this sequence does not converge, hence this metric space is not complete.

1.2 Completion of a Metric Space

Proposition 1.2.1. If $f : X \to Y$ is uniformly continuous, and $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Definition 1.2.2. Let (X, d) be a metric space. A complete metric space (X, d), together with an isometric function $f : X \to X$ with dense range is a **completion** of (X, d).

Remark. Completions are unique up to isomorphism.

Proposition 1.2.3. If $((Y_1, d_1), f_1)$ and $((Y_2, d_2), f_2)$ are completions of (X, d), then $\exists$ an onto isometry (metric space isomorphism) $g : Y_1 \to Y_2$ with $f_2 = g \circ f_1$. This can be visualized by the following commutative diagram:

$$\begin{tikzcd}
 & Y_1 \arrow[dd, "g"] \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\
 & Y_2
\end{tikzcd}$$

Every metric space has a completion, and the proof will be constructive. The completion will be defined using equivalence classes of Cauchy sequences. We will need the following lemmas to support the construction.

Lemma 1.2.4. If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in X, then the sequence $\{d(s_n, t_n)\}$ in converges.

Proof. Exercise. *Hint:* $\{d(s_n, t_n)\}$ is a Cauchy sequence in a complete metric space. $\square$

Lemma 1.2.5. Let $\mathrm{CS}(X)$ denote the set of all Cauchy sequences in X. Then the relation $\{s_n\} \sim \{t_n\} \iff d(s_n, t_n) \to 0$ is an equivalence relation.

Proof. Reflexivity and symmetry are trivial. Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$. Then $d(s_n, t_n) \leqslant d(s_n, r_n) + d(r_n, t_n)$ for all $n \in \mathbb{N}$. The result follows immediately. $\square$

Lemma 1.2.6. Let X be the set of all equivalence classes of $\mathrm{CS}(X)$ under the equivalence relation above. Then $d : X \to [0, \infty)$ defined by $d(\{s_n\}, \{t_n\}) := \lim_{n\to\infty} d(s_n, t_n)$ is a metric on X.

Proof. First, note that by Lemma 1, d is always defined. Since we are dealing with equivalence classes, we must show that d is also well-defined. Let $\xi, \eta \in X$, and let $\{x_n\}, \{s_n\} \in \xi$, and $\{y_n\}, \{t_n\} \in \eta$. We have $\lim d(x_n, s_n) = \lim d(y_n, t_n) = 0$. Thus,
$d(s_n, t_n) \leqslant d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n)$. $\forall\ \epsilon > 0$, we can find $N \in \mathbb{N}$ such that both $d(s_n, x_n) < \epsilon/2$ and $d(y_n, t_n) < \epsilon/2$ for $n \geqslant N$. Then $|d(s_n, t_n) - d(x_n, y_n)| < \epsilon$. It follows that $d(\xi, \eta) = \lim d(x_n, y_n) = \lim d(s_n, t_n)$, so that d is indeed well-defined.

Symmetry is trivial. The triangle inequality follows from the proof to Lemma 2. If $d(\xi, \eta) = 0$, then $\forall\ \{x_n\} \in \xi, \{y_n\} \in \eta$, we have $\lim d(x_n, y_n) = 0$, so in particular, $\{y_n\} \in \xi$, hence $\xi = \eta$. □

Theorem 1.2.7. *Let (X, d_x) and (Y, d_y) be metric spaces with Y complete. If $S \subseteq X$ is dense, and $f : S \to Y$ is uniformly continuous, then $\exists$ a unique continuous extension $f : X \to Y$ of f. In fact, f is uniformly continuous.*

Proof. (Existence only) For $x \in X$, choose a Cauchy sequence $\{s_n\}$ in S converging to x. Then $\{f(s_n)\}$ is Cauchy in Y, so it converges to a point $p \in Y$. Set $f(x) := p$. We show that f is well-defined. Indeed, if $\{t_n\} \in \mathrm{CS}(S)$ and converges to x, then we have $\lim d_x(s_n, t_n) = 0$, implying that $\lim d_y(f(s_n), f(t_n)) = 0$. Therefore $\lim d_y(f(t_n), p) = 0$, so $\{f(t_n)\}$ converges to p also. It remains to show continuity, which is left as an exercise. □

Theorem 1.2.8. *Every metric space (X, d) has a completion.*

Proof. As in Lemma 3, (X, d) is a completion of (X, d). We embed X in X by the isometry $\iota : X \to X$ defined by $\iota(x) := [\{x, x, x, ...\}]$, where $[\cdot]$ denotes the corresponding equivalence class. Note that $d\Big|_X = d$, i.e., $d(\iota(x), \iota(y)) = d(x, y)$.
It remains to show that d has dense range, and that (X, d) is complete.

- Let $\xi \in X, \epsilon > 0, \{x_n\} \in \xi$. $\exists\ N \in \mathbb{N}$ such that $n, m \geqslant N$ implies $d(x_n, x_m) < \epsilon$. Then $d(\iota(x_N), \xi) = \lim_{n\to\infty} d(x_N, x_n) < \epsilon$. Therefore d has dense range by considering $\iota(x_N)$.
- Let $\{\xi_n\}$ be a Cauchy sequence in X. For each $m \in \mathbb{N}$, pick $x_m \in X$ such that $d(\iota(x_m), \xi_m) < 1/m$. Then $\{x_m\}$ is a Cauchy sequence, and it follows that $\{\xi_m\}$ converges to the equivalence class of $\{x_m\}$.

□

Remark. Denote $C([0,1])$ the space of continuous functions on $[0,1]$. Consider the metric space $C([0,1])$ induced by the norms $\|\cdot\|_\infty$ or $\|\cdot\|_p$. This space is not complete. It is easy to come up with a sequence of continuous functions converging under these norms to a function that is not continuous.

Remark. Let V be a vector space with norm $\|\cdot\|$. Consider V^∞, the space of all sequences of elements in V. This is also a vector space. It can be shown that $\mathrm{CS}(V)$ is a subspace of V^∞.
Now let $\mathcal{N}(V)$ denote the set of all Cauchy sequences in V converging to 0. Then $\mathcal{N}(V)$ is a subspace of $\mathrm{CS}(V)$. If $\{v_n\}$ and $\{w_n\}$ are equivalent Cauchy sequences, then
$\|v_n - w_m\| \to 0$, so $\{v_n - w_n\} \in \mathcal{N}(V)$. Thus V is in fact the quotient space $\mathrm{CS}(V)/\mathcal{N}(V)$.

Fact. Any two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite dimensional vector space are **equivalent**, meaning that there are constants $c, C > 0$ such that $c\|x\|_1 \leqslant \|x\|_2 \leqslant C\|x\|_1$ for all x. If a function is continuous with respect to a particular norm, then it is easily seen that it is continuous with respect to any equivalent norm.

Lecture 2