Topology and Analysis

MATH 202A

Instructor: Marc Rieffel

Kelvin Lee

UC Berkeley

Contents

1 Metric Spaces **3**
1.1 Fundamentals . . . 3
1.2 Completion of a Metric Space . . . 5
1.3 Openness . . . 7
1.3.1 Open Sets . . . 7

2 Topology **9**
2.1 Topological Spaces . . . 9
2.2 Continuous Maps . . . 9
2.2.1 Bases and Sub-bases . . . 9
2.2.2 Homeomorphism . . . 10
2.3 Quotient Topologies . . . 11
2.4 Special Topological Spaces . . . 12
2.4.1 Hausdorff topological space . . . 12
2.4.2 Normal topological space . . . 12
2.4.3 Urysohn's Lemma . . . 13

3 **14**
3.1 . . . 14

Chapter 1

Metric Spaces

1.1 Fundamentals

Definition 1.1.1. Let X be a set. A **metric** on X is a function $d : X \times X \to [0, \infty)$ that satisfies:

(i) $d(x, y) = d(y, x) \ \forall \ x, y \in X$

(ii) $d(x, y) \leq d(x, z) + d(z, y) \ \forall \ x, y, z \in X$

(iii) $d(x, y) = 0 \iff x = y$

If a function d satisfies (i), (ii) above, and $d(x, x) = 0$ for all $x \in X$, then d is a **semi-metric**.

Example 1.1.2. On $\mathbb{C}^n$, the following are common metrics:

- $d_p(x, y) = \left(\sum\limits_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}$ for $p \geq 1$
- $d_\infty(x, y) = \sup \{|x_j - y_j| : 1 \leq j \leq n\}$

(Verify that these are metrics.)

Fact. If $S \subseteq X$, and d is a metric on X, then d is a metric on S.

Definition 1.1.3. (X, d) where d is a metric of X is called a **metric space**.

Remark. If $Y \subseteq X$, restrict d to $Y \times Y \subseteq X \times X$, denoted $d\,|_Y$, then $(Y, d\,|_Y)$ is a metric space.

Definition 1.1.4. Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A **norm** on V is a function $\|\cdot\| : V \to [0, \infty)$ such that:

(i) $\|cv\| = |c| \cdot \|v\|$ for $c \in$ or and $v \in V$

(ii) $\|v + w\| \leq \|v\| + \|w\|$ for $v, w \in V$

(iii) $\|v\| = 0$ implies $v = 0$

A function that satisfies only (i) and (ii) above is called a **seminorm**.

Remark. Any norm $\|\cdot\|$ on X induces the metric $d(x, y) \coloneqq \|x - y\|$.

Example 1.1.5. Let V be the space of continuous functions on $[0,1]$. Then $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ is a norm on V.

It can also be shown that $\|f\|_p := \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}$ is a norm on V.

Definition 1.1.6. Let (X, d_x) and (Y, d_y) be metric spaces. A function $f : X \to Y$ is **isometric** if $d_y(f(x_1), f(x_2)) = d_x(x_1, x_2)$ for all $x_1, x_2 \in X$.

Remark. All isometries are injective.

Example 1.1.7. If $S \subseteq X$, and $f : S \to X$ is defnined by $f(x) = x$ (inclusion), then f is an isometry. If f is also onto, then f is viewed as an isometric isomorphism between (X, d_x) and (Y, d_y). f^{-1} is also an isomorphism.

Definition 1.1.8. A function $f : X \to Y$ is **Lipschitz** if there is a constant $k \geq 0$ such that $d_y(f(x_1), f(x_2)) \leq k \cdot d_x(x_1, x_2)$. The smallest such constant is the **Lipschitz constant** for f.

Definition 1.1.9. $f : X \to Y$ is **uniformly continuous** if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that $d_y(f(x_1), f(x_2)) < \epsilon$ whenever $d_x(x_1, x_2) < \delta$.

Remark. It is easy to see that if f is Lipschitz, then it is uniformly continuous.

Definition 1.1.10. $f : X \to Y$ is **continuous at** x_0 if $\forall\, \epsilon > 0, \exists\, \delta(x_0) > 0$ such that $d_y(f(x), f(x_0)) < \epsilon$ whenever $d_x(x, x_0) < \delta(x_0)$. We say f is **continuous** if it is continuous at every $x \in X$.

Definition 1.1.11. A sequence $\{x_n\}$ in X **converges** to $x^* \in X$ if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that for all $n \geq N$, we have $d(x_n, x^*) < \epsilon$.

Proposition 1.1.12. If a function $f : X \to Y$ is continuous and $\{x_n\} \to x^*$, then $f(x_n) \to f(x^*)$.

Proof. Let $\epsilon > 0$. Since f is continuous at x^*, there exists a $\delta > 0$ such that

$$\forall x, d_X(x, x^*) < \delta \implies d_Y(f(x), f(x^*)) < \epsilon$$

Since $\{x_n\} \to x^*$, there is some N such that for all $n \geq N, d_X(x_n, x^*) < \delta$. Then, we can see that $d_Y(f(x_n) - f(x^*)) < \epsilon$ for all $n \geq N$. Thus $\{f(x_n)\} \to f(x^*)$. ☐

Definition 1.1.13. $S \subseteq X$ is **dense** in X if $\forall\, x \in X$ and $\epsilon > 0$, $\exists\, s \in S$ such that $d(x, s) < \epsilon$. That is, for any point $x \in X$, there is a point $s \in S$ which is arbitrarily close to x.

Proposition 1.1.14. Let S be dense in X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that $f(s) = g(s)$ for all $s \in S$. Then $f = g$ on X.

Proof. Because S is dense in X, for any $x \in X$, there exists a sequence $\{s_n\} \subseteq S$ which converges to x (choose any point s_n in S such that $d(s_n, x) < \epsilon$). By the previous proposition, we can conclude that $\{f(s_n) = g(s_n)\} \to f(x) = g(x)$. ☐

Definition 1.1.15. A sequence $\{x_n\}$ is **Cauchy** if $\forall\, \epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. A metric space is **complete** if every Cauchy sequence in it converges.

Example 1.1.16. Consider $(\mathbb{Q}, |\cdot|)$. We know there exists a Cauchy sequence converging to $\sqrt{2} \in \mathbb{R}$, but in this metric space, $\sqrt{2}$ is not an element, so this sequence does not converge, hence this metric space is not complete.

1.2 Completion of a Metric Space

Proposition 1.2.1. If $f : X \to Y$ is uniformly continuous, and $\{x_n\}$ is Cauchy in X, then $\{f(x_n)\}$ is Cauchy in Y.

Proof. Let $\epsilon > 0$. By uniform continuity, there exists $\delta > 0$ such that if $x, x' \in X$ and $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon$. Since $\{x_n\}$ is Cauchy, there is an N such that if $m, n \geq N$ then $d(x_m, x_n) < \delta$. Thus

$$d(f(x_m), f(x_n)) < \epsilon \quad \forall m, n \geq N.$$

This proves that $\{f(x_n)\}$ is Cauchy. □

Definition 1.2.2. Let (X, d) be a metric space. A complete metric space $(\widetilde{X}, \widetilde{d})$, together with an isometric function $f : X \to \widetilde{X}$ with dense range is a **completion** of (X, d).

Remark. Completions are unique up to isomorphism.

Proposition 1.2.3. If $((Y_1, d_1), f_1)$ and $((Y_2, d_2), f_2)$ are completions of (X, d), then $\exists$ an onto isometry (metric space isomorphism) $g : Y_1 \to Y_2$ with $f_2 = g \circ f_1$.

This can be visualized by the following commutative diagram:

$$\begin{tikzcd}
 & Y_1 \arrow[dd, "g"] \\
X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\
 & Y_2
\end{tikzcd}$$

Every metric space has a completion, and the proof will be constructive. The completion will be defnined using equivalence classes of Cauchy sequences. We will need the following lemmas to support the construction.

Lemma 1.2.4. If $\{s_n\}$ and $\{t_n\}$ are Cauchy sequences in X, then the sequence $\{d(s_n, t_n)\}$ in $\mathbb{R}$ converges.

Proof. Let $\epsilon > 0$, and let N such that for every $m, n > N, d(s_m, s_n), d(t_m, t_n) < \epsilon/2$. It follows that

$$|d(s_m, t_m) - d(s_n, t_n)| \leq d(s_m, s_n) + d(t_m, t_n) < \epsilon$$

and the sequence is Cauchy. Since $\mathbb{R}$ is complete, the sequence converges. □

Lemma 1.2.5. Let $\mathrm{CS}(X)$ denote the set of all Cauchy sequences in X. Then the relation $\{s_n\} \sim \{t_n\} \iff d(s_n, t_n) \to 0$ is an equivalence relation.

Proof. Reflexivity and symmetry are trivial. Suppose $d(s_n, r_n) \to 0$ and $d(r_n, t_n) \to 0$. Then $d(s_n, t_n) \leq d(s_n, r_n) + d(r_n, t_n)$ for all $n \in \mathbb{N}$. The result follows immediately. $\square$

Lemma 1.2.6. Let $\overline{X}$ be the set of all equivalence classes of $\mathrm{CS}(X)$ under the equivalence relation above. Then $\overline{d} : \overline{X} \to [0, \infty)$ defined by $\overline{d}(\{s_n\}, \{t_n\}) := \lim_{n\to\infty} d(s_n, t_n)$ is a metric on X.

Proof. First, note that by Lemma 1.2.4, $\overline{d}$ is always defined. Since we are dealing with equivalence classes, we must show that $\overline{d}$ is also well-defnined. Let $\xi, \eta \in X$, and let $\{x_n\}, \{s_n\} \in \xi$, and $\{y_n\}, \{t_n\} \in \eta$. We have $\lim d(x_n, s_n) = \lim d(y_n, t_n) = 0$. Thus, $d(s_n, t_n) \leq d(s_n, x_n) + d(x_n, y_n) + d(y_n, t_n)$. $\forall\ \epsilon > 0$, we can find $N \in \mathbb{N}$ such that both $d(s_n, x_n) < \epsilon/2$ and $d(y_n, t_n) < \epsilon/2$ for $n \geq N$. Then $|d(s_n, t_n) - d(x_n, y_n)| < \epsilon$. It follows that $d(\xi, \eta) = \lim d(x_n, y_n) = \lim d(s_n, t_n)$, so that d is indeed well-defnined.

Symmetry is trivial. The triangle inequality follows from the proof to Lemma 1.2.5. If $d(\xi, \eta) = 0$, then $\forall\ \{x_n\} \in \xi, \{y_n\} \in \eta$, we have $\lim d(x_n, y_n) = 0$, so in particular, $\{y_n\} \in \xi$, hence $\xi = \eta$. $\square$

Theorem 1.2.7. *Let (X, d_x) and (Y, d_y) be metric spaces with Y complete. If $S \subsetneq X$ is dense, and $f : S \to Y$ is uniformly continuous, then $\exists$ a unique continuous extension $\overline{f} : X \to Y$ of f. In fact, $\overline{f}$ is uniformly continuous.*

Proof. (Existence only) For $x \in X$, choose a Cauchy sequence $\{s_n\}$ in S converging to x. Then $\{f(s_n)\}$ is Cauchy in Y, so it converges to a point $p \in Y$. Set $\overline{f}(x) := p$. We show that $\overline{f}$ is well-defnined. Indeed, if $\{t_n\} \in \mathrm{CS}(S)$ and converges to x, then we have $\lim d_x(s_n, t_n) = 0$, implying that $\lim d_y(f(s_n), f(t_n)) = 0$. Therefore $\lim d_y(f(t_n), p) = 0$, so $\{f(t_n)\}$ converges to p also. It remains to show continuity, which is left as an exercise. $\square$

Theorem 1.2.8. *Every metric space (X, d) has a completion.*

Proof. As in Lemma 3, (X, d) is a completion of (X, d). We embed X in X by the isometry $\iota : X \to X$ defnined by $\iota(x) := [\{x, x, x, ...\}]$, where $[\cdot]$ denotes the corresponding equivalence class. Note that $d\Big|_X = d$, i.e., $d(\iota(x), \iota(y)) = d(x, y)$.
It remains to show that d has dense range, and that (X, d) is complete.

- Let $\xi \in X, \epsilon > 0, \{x_n\} \in \xi$. $\exists\ N \in \mathbb{N}$ such that $n, m \geq N$ implies $d(x_n, x_m) < \epsilon$. Then $d(\iota(x_N), \xi) = \lim_{n\to\infty} d(x_N, x_n) < \epsilon$. Therefore d has dense range by considering $\iota(x_N)$.
- Let $\{\xi_n\}$ be a Cauchy sequence in X. For each $m \in \mathbb{N}$, pick $x_m \in X$ such that $d(\iota(x_m), \xi_m) < 1/m$. Then $\{x_m\}$ is a Cauchy sequence, and it follows that $\{\xi_m\}$ converges to the equivalence class of $\{x_m\}$.

$\square$

Remark. Denote $C([0,1])$ the space of continuous functions on $[0,1]$. Consider the metric space $C([0,1])$ induced by the norms $\|\cdot\|_\infty$ or $\|\cdot\|_p$. This space is not complete. It is easy to come up with a sequence of continuous functions converging under these norms to a function that is not continuous.

Remark. Let V be a vector space with norm $\|\cdot\|$. Consider V^∞, the space of all sequences of elements in V. This is also a vector space. It can be shown that $\mathrm{CS}(V)$ is a subspace of V^∞.
Now let $\mathcal{N}(V)$ denote the set of all Cauchy sequences in V converging to 0. Then $\mathcal{N}(V)$ is a subspace of $\mathrm{CS}(V)$. If $\{v_n\}$ and $\{w_n\}$ are equivalent Cauchy sequences, then
$\|v_n - w_m\| \to 0$, so $\{v_n - w_n\} \in \mathcal{N}(V)$. Thus V is in fact the quotient space $\mathrm{CS}(V)/\mathcal{N}(V)$.

Fact. Any two norms $\|\cdot\|_1, \|\cdot\|_2$ on a finite dimensional vector space are **equivalent**, meaning that there are constants $c, C > 0$ such that $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$ for all x. If a function is continuous with respect to a particular norm, then it is easily seen that it is continuous with respect to any equivalent norm.

1.3 Openness

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a map between the two metric spaces. Recall that f is continuous at $x_0 \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d_X(x, x_0) < \delta$ implies $d_Y(f(x), f(x_0)) < \epsilon$.

Definition 1.3.1 (Open ball)**.** Let (X, d_X) be a metric space. The **open ball** around $x_0 \in X$ with radius $r > 0$ is defined as

$$\mathcal{B}_r(x_0) = \{x \in X \mid d_X(x, x_0) < r\}.$$

Remark. For any open ball U in Y, there exists an open ball $\mathcal{O}$ in X such that if $x \in O$, then $f(x) \in U$.

Now we can rephrase continuity using the notion of open balls:

Definition 1.3.2 (Continuity)**.** $f : X \to Y$ is **continuous at** x_0 if $\forall \epsilon > 0, \exists \delta > 0$ such that $f(\mathcal{B}_\delta(x_0)) \subseteq \mathcal{B}_\epsilon(f(x_0))$.

If $y \in \mathcal{B}_\epsilon(f(x_0))$ and $y = f(x)$ for some $x \in X$, let $\epsilon' = \epsilon - d(y, f(x_0)) > 0$. Then $\mathcal{B}_{\epsilon'}(y) \subseteq \mathcal{B}_\epsilon(f(x_0))$, so there exists $\delta' > 0$ such that $f(\mathcal{B}_{\delta'}(x)) \subseteq \mathcal{B}_\epsilon(f(x_0))$ If $x_1 \in f^{-1}(\mathcal{B}_\epsilon(f(x)))$, there is an open ball $\mathcal{B}_{\delta'}(x)$ such that $\mathcal{B}_{\delta'}(x_1) \subseteq f^{-1}(\mathcal{B}_\epsilon(f(x)))$ Thus $f^{-1}(\mathcal{B}_\epsilon(f(x)))$ is a union of open balls in X. Similarly, $f^{-1}(\mathcal{B}_\epsilon(y))$ is a union of open balls in X. This leads to the definition of open sets.

1.3.1 Open Sets

Definition 1.3.3 (Open set)**.** A subset A of X is **open** if A is a union of open balls it contains, i.e. $\forall x \in A, \exists r > 0$ such that $\mathcal{B}_r(x) \subset A$.

Theorem 1.3.4. *Let (X, d) be a metric space, and $\mathcal{T}$ be the collection of all open sets. Then*

(i) If $\{\mathcal{O}_\alpha\}$ is an arbitrary collection of subsets in $\mathcal{T}$, then $\bigcup_\alpha \mathcal{O}_\alpha$ is open.

(ii) If $\mathcal{O}_1, \ldots, \mathcal{O}_n$ is a finite collection of subsets in $\mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{O}_i$ is open.

> *(iii)* $X \in \mathcal{T}$ *(X is open).*

Proof of (iii). If $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$ are open, and $x \in \mathcal{O}_1 \cap \mathcal{O}_2$, then there exist open balls $\mathcal{B}_{r_1}(x) \subseteq \mathcal{O}_1$, $\mathcal{B}_{r_2}(x) \subseteq \mathcal{O}_2, \ldots, \mathcal{B}_{r_n}(x) \subseteq \mathcal{O}_n$. Let $r = \min_{1 \leq i \leq n}\{r_i\}$. Then $\mathcal{B}_r(x) \subseteq \bigcap_{i=1}^{n} \mathcal{O}_i$. □

Chapter 2

Topology

2.1 Topological Spaces

Definition 2.1.1 (Topology). Let X be a set. The **topology** on X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfying:

(i) $X, \emptyset \in \mathcal{T}$.

(ii) If any arbitrary family $\{\mathcal{O}_\alpha\} \subseteq \mathcal{T}$, then $\bigcup_\alpha \mathcal{O}_\alpha \in \mathcal{T}$.

(iii) If $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{T}$, then $\bigcap_{i=1}^n \mathcal{O}_i \in \mathcal{T}$.

Definition 2.1.2 (Topological space). Let $\mathcal{T}$ be a topology on X. Then $(X, \mathcal{T})$ is a **topological space**. The sets in $\mathcal{T}$ are called **open sets** and the complements of the sets in $\mathcal{T}$ are **closed sets**.

Example 2.1.3. Let X be any nonempty set. Then $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are topologies on X. They are called the **discrete topology** and **indiscrete topology** respectively.

Example 2.1.4. Let X be a metric space. The collection of all open sets with respect to the metric is a topology on X.

Definition 2.1.5 (Interior). If $A \subseteq X$, the union of all open sets contained in A is called the **interior** of A, denoted by A°. This is the biggest open set contained in A.

Definition 2.1.6 (Closure). If $A \subseteq X$, the intersection of all closed sets containing A is called a **closure** of A, denoted by $\overline{A}$. This is the smallest closed set containing A.

Definition 2.1.7 (Dense). If $\overline{A} = X$, A is called **dense** in X.

Definition 2.1.8 (Strong/Weak topology). Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on a set X such that $\mathcal{T}_1 \subset \mathcal{T}_2$. We say that $\mathcal{T}_1$ is **weaker** than $\mathcal{T}_2$, or equivalently $\mathcal{T}_2$ is **stronger** than $\mathcal{T}_1$.

2.2 Continuous Maps

Definition 2.2.1 (Continuity). Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces. A function $f : X \to Y$ is **continuous** if $\forall U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$.

2.2.1 Bases and Sub-bases

Proposition 2.2.2. Let X be a set and let $\mathcal{C}$ be a collection of topologies on X. Then $\bigcap_{\mathcal{T}\in\mathcal{C}}\mathcal{T}$ is a topology on X.

Then it follows that for any collection S of subsets of X, there is a unique weakest/smallest topology $\mathcal{T}$ on X containing S described as follows.

Definition 2.2.3 (Sub-base)**.** Let $\mathcal{T}(S)=\bigcap_{S\subseteq\mathcal{T}}\mathcal{T}$, the intersection of all topologies on X containing S. It is called the topology **generated** by S and S is the **sub-base** for $\mathcal{T}(S)$.

Definition 2.2.4 (Base)**.** A collection $\mathcal{B}\subseteq\mathcal{T}$ of subsets of a set X is called a **base** for $\mathcal{T}$ if every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.

Example 2.2.5. Let (X,d) be a metric space. The open balls form a base for the metric topology.

Remark. The intersections of two balls is usually not a ball. If $\mathcal{B}$ is a base, then the intersection of any two elements of $\mathcal{B}$ must be a union of elements of $\mathcal{B}$.

Proposition 2.2.6. If $S\subseteq\mathcal{P}(X)$, the topology $\mathcal{T}(S)$ generated by S consists of $\emptyset, X$, and all unions of finite intersections of members of S.

Proposition 2.2.7. Let $(X,\mathcal{T}_X),(Y,\mathcal{T}_Y)$ be topological spaces. If $\mathcal{T}_Y$ is generated by $\mathcal{B}$ (i.e. $\mathcal{B}$ is a sub-base for $\mathcal{T}_Y$), then $f:X\to Y$ is continuous $\iff$ $f^{-1}(U)\in\mathcal{T}_X$ for every $U\in\mathcal{B}$.

Proof. Note that f^{-1} preserves the Boolean operations for any collection of subsets of Y:

- $f^{-1}\bigcap_\alpha A_\alpha=\bigcap_\alpha f^{-1}(A_\alpha)$
- $f^{-1}\bigcup_\alpha A_\alpha=\bigcup_\alpha f^{-1}(A_\alpha)$
- If $A,B\subseteq Y$, then $f^{-1}(A\backslash B)=f^{-1}(A)\backslash f^{-1}(B)$

Then suppose $\{U_n\}\subseteq\mathcal{B}$ is some finite collection of open sets in $\mathcal{B}$, then

$$f^{-1}\left(\bigcap_{i=1}^{n}U_i\right)=\bigcap_{i=1}^{n}f^{-1}\left(U_i\right)\in\mathcal{T}_X.$$

Then any finite intersection of elements of $\mathcal{B}$ satisfies the condition as well, i.e. is a base. If $\{U_\alpha\}\subseteq\mathcal{B}$ is a collection (possibly infinite) of open sets in $\mathcal{B}$, then

$$f^{-1}\left(\bigcup_{\alpha}U_\alpha\right)=\bigcup_{\alpha}f^{-1}\left(U_\alpha\right)\in\mathcal{T}_X,$$

so $\bigcup_\alpha U_\alpha$ also satisfies the condition. Therefore, all open set U in $\mathcal{T}_Y$ satisfies $f^{-1}(U)\in\mathcal{T}_X$ so f is continuous. ☐

2.2.2 Homeomorphism

Definition 2.2.8 (Homeomorphism)**.** If $f:X\to Y$ is bijective and f and f^{-1} are both continuous, f is called a **homeomorphism**, and X and Y are said to be homeomorphic.

2.3 Quotient Topologies

Let X be a set and let $(Y_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let $f_\alpha : X \to Y_\alpha$ be any function. Then there is a smallest topology on X for which each f_α is continuous, namely, the smallest topology having as sub-base all sets $f_\alpha^{-1}(U)$, where $U \in \mathcal{T}_\alpha$ for each α.

Definition 2.3.1. Let $(X, \mathcal{T}_X)$ be a topological space. Let Y be a set and $f : X \to Y$ be any function. Then there is a strongest topology on Y for which f is continuous. Namely,

$$\mathcal{T}_Y := \{A \subseteq Y : f^{-1}(A) \in \mathcal{T}_X\},$$

which is called the **quotient topology** on Y for f.

Remark. Note that if $y \notin f(X)$, then $f^{-1}(\{y\}) = \varnothing$, so $\{y\}$ is open. Also, $f^{-1}(\{y\}^c) = X$, so $\{y\}$ is also closed. Therefore, on $f(X)^c$, the quotient topology is the discrete topology. Thus, we usually require $f : X \to Y$ to be onto.

Let $f : X \to Y$ be onto, and define the equivalence relation on X by $x_1 \sim x_2 \iff f(x_1) = f(x_2)$. f defines a partition, a collection of equivalence classes. Conversely, let $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ be the set of equivalence classes, $x \to [x]$, call it f. Given a topology on X, we call $X/\sim$ with the quotient topology on the projection $X \to X/\sim$ a quotient space.

Definition 2.3.2. Let Y be a set, and $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces and function $f_\alpha : X_\alpha \to Y$ be any function, then there is a strongest topology on Y where all f_α is continuous. Namely

$$\bigcap_\alpha \mathcal{T}_{Y_\alpha}, \text{ where } \mathcal{T}_{Y_\alpha} := \{A_\alpha \subseteq Y_\alpha : f^{-1}(A_\alpha) \in \mathcal{T}_\alpha\}$$

which is the intersection of all quotient topologies for each f_α. This is called a **final topology**.

Definition 2.3.3. Let G be a group. By an **action** of G on $(X, \mathcal{T})$, we mean a group homomorphism $\alpha : G \to \text{Homeo}(X, \mathcal{T})$. For any $x \in X$, its G-orbit is

$$\{\alpha_r(x) : r \in G\}.$$

The orbits form a partition of X. Let Y_α be the set of orbits, we can put on the quotient topology.

Example 2.3.4. $G = \mathbb{Z}$, $X = \mathbb{R}$, $\alpha_n(r) = r + n$. Define $f : X \to \{z \in \mathbb{C} : |z| = 1\}$ by $f(r) = e^{2\pi i r}$, for $r \in [0, 1]$ Note that f is continuous but f^{-1} is not: there is a discontinuity at $1 \in \mathbb{C}$. However, the corresponding function $f : X/\alpha \to \{z \in \mathbb{C} : |z| = 1\}$ is a homeomorphism with the usual topology from $\mathbb{C}$.

Example 2.3.5. Let $X = S^2$ be a sphere on $\mathbb{R}^3 = V$, $v \in S^2$. Let $G = \mathbb{Z}_2$, $\alpha_c(v) = -v$. $S^n \subseteq \mathbb{R}^{n+1}$.

Definition 2.3.6. Let Y be a set and $\{X_\alpha, \mathcal{T}_\alpha\}_{\alpha \in A}$ be a collection of topological spaces and $f_\alpha : Y \to X_\alpha$. We want the weakest topology that make all f_α continuous, namely the **initial topology**. This topology must contain $f_\alpha^{-1}(U)$ for $U \in \mathcal{T}_\alpha$.

Remark. These form a sub-base for the initial topology, whereas the finite intersections of these form a base.

Definition 2.3.7. Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$ with $f : Y \to X$ defined by $f(y) = y \in X$. The sub-base is $\{f^{-1}(U), U \in \mathcal{T}_X\}$ and so $f^{-1}(U) = U \cap Y$. The initial topology is $\{U \cap Y : U \in \mathcal{T}_X\}$, which is called the **relative topology**.

Definition 2.3.8. Let $(X_\alpha, \mathcal{T}_\alpha)$ be a collection of topological spaces. Let $Y = \prod_\alpha X_\alpha$ be the product set. Have $\pi_\alpha : Y \to X_\alpha$, $\pi_\alpha(\{x_\beta\}_{\beta \in A}) = x_\alpha$. The **product topology** is the initial topology for the π_α. The sub-base is the $\pi_\alpha^{-1}(U), U \in \mathcal{T}_\alpha$, for all α, U.

Example 2.3.9. If $A = \mathbb{N}$, $(X_n, \mathcal{T}_n)$, $\{x_n\} \in \prod X_n$. If $U \in \mathcal{T}_3$,

$$\pi_3^{-1}(U) = X_1 \times X_2 \times U \times X_4 \times X_5 \times \cdots .$$

The base is the finite intersection of these.

Example 2.3.10. Let $Y = V$ be a vector space over $\mathbb{R}$. Let $\mathcal{L}$ be a collection of linear functionals, $\varphi_\lambda, \lambda \in \mathcal{L}$ and $\varphi_\lambda : V \to \mathbb{R}$. We can ask for the weakest topology on V making all φ_λ continuous.

Example 2.3.11. $V = C([0, 1])$ be the continuous function on $[0, 1]$ and $\mathcal{L} = C([0, 1])$. For $g \in \mathcal{L}$, $\varphi_g(f) = \int_0^1 f(t)g(t)dt$.

Question. What topologies play nicely with $\mathbb{R}$?

Let (X, d) be a metric space. Let $x_1, x_2 \in X, x_1 \neq x_2$. Let $r = d(x_1, x_2)$. Consider the two disjoint balls $\mathcal{B}_{r/3}(x_1), \mathcal{B}_{r/3}(x_2)$.

2.4 Special Topological Spaces

2.4.1 Hausdorff topological space

Definition 2.4.1. A topological space is said to be **Hausdorff** if for any $x_1, x_2 \in X, x_1 \neq x_2$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$, with $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2$.

2.4.2 Normal topological space

Definition 2.4.2. $(X, \mathcal{T})$ is **normal** if for disjoint closed sets $\mathcal{C}_1, \mathcal{C}_2$, there exist disjoint open sets $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{T}$, such that $\mathcal{C}_1 \subseteq \mathcal{O}_1, \mathcal{C}_2 \subseteq \mathcal{O}_2$.

Proposition 2.4.3. If (X, d) is a metric space, then its topology is normal.

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be disjoint closed sets. For each $x \in \mathcal{C}_1$, we can choose r_x such that $\mathcal{B}_{r_x}(x) \cap \mathcal{C}_2 = \emptyset$. For each $y \in \mathcal{C}_2$, we choose r_y such that $\mathcal{B}_{r_y}(y) \cap \mathcal{C}_1 = \emptyset$. Let

$$\mathcal{O}_1 = \bigcup_{x \in \mathcal{C}_1} \mathcal{B}_{r_x/3}(x)$$
$$\mathcal{O}_2 = \bigcup_{y \in \mathcal{C}_2} \mathcal{B}_{r_y/3}(y).$$

Then $\mathcal{C}_1 \subseteq \mathcal{O}_1, \mathcal{C}_2 \subseteq \mathcal{O}_2$. Now let $z \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then there exists $x \in \mathcal{C}_1$ with $z \in \mathcal{B}_{r_{x/3}}(x))$. Then

$$d(x, y) \leq d(x, z) + d(z, y) < \frac{r_x}{3} + \frac{r_y}{3}.$$

Suppose $r_x \geq r_y$, then $d(x, y) \leq \frac{2}{3} r_x$. So $y \in \mathcal{C}_2$ and $y \in \mathcal{B}_{r_x}(x)$ but $\mathcal{C}_2$ and $\mathcal{B}_{r_x}(x)$ are disjoint. Hence, a contradiction. Therefore, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. $\square$

2.4.3 Urysohn's Lemma

Lemma 2.4.4. Let $(X, \mathcal{T})$ be a normal space, and let $\mathcal{C} \subseteq X$ be a closed subset. Let $\mathcal{O} \subseteq X$ be an open subset such that $\mathcal{C} \subseteq \mathcal{O}$. Then there exists an open set U such that $\mathcal{C} \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$.

Proof. $\mathcal{C}$ and $\mathcal{O}^c$ are disjoint closed sets, so there are disjoint open sets U, V such that $\mathcal{C} \subseteq U$ and $\mathcal{O}^c \subseteq V$. Then $\mathcal{C} \subseteq U \subseteq V^c \subseteq \mathcal{O}$. V^c is a closed set containing U; it therefore contains the closure $\overline{U}$, so that $\mathcal{C} \subseteq U \subseteq \overline{U} \subseteq \mathcal{O}$. □

Lemma 2.4.5 (Urysohn's Lemma)**.** Let $(X, \mathcal{T})$ be normal, and let $\mathcal{C}_0, \mathcal{C}_1$ be disjoint closed subsets. Then there exists a continuous function $f : X \to [0,1]$ such that $f(\mathcal{C}_0) = \{0\}, f(\mathcal{C}_1) = \{1\}$.

Proof. Set $\mathcal{O}_1 = \mathcal{C}_1^c$ and $\mathcal{C}_0 \subseteq \mathcal{O}_1$. Then by the lemma there exists an open $\mathcal{O}_{1/2}$ with $\mathcal{C}_0 \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_1$. Applying the lemma again, there exist open sets $\mathcal{O}_{1/4}, \mathcal{O}_{3/4}$. Hence,

$$\mathcal{C}_0 \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4} \subseteq \mathcal{O}_1.$$

Then there exist $\mathcal{O}_{1/8}, \mathcal{O}_{3/8}, \mathcal{O}_{5/8}, \mathcal{O}_{7/8}$ such that

$$\mathcal{C}_0 \subseteq \mathcal{O}_{1/8} \subseteq \overline{\mathcal{O}}_{1/8} \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{3/8} \subseteq \overline{\mathcal{O}}_{3/8} \subseteq \cdots \subseteq \overline{\mathcal{O}}_{7/8} \subseteq \mathcal{C}^c.$$

Then by induction, for each dyadic rational numbers

$$\Delta = \{r = m2^{-n} : 1 \leq m \leq 2^n, m, n \in \mathbb{N}\}.$$

we get an open set $\mathcal{O}_r$ such that if $r, s \in \Delta, r < s$, then $\overline{\mathcal{O}}_r \subseteq \mathcal{O}_s, \mathcal{C}_s \subseteq \mathcal{O}_r^c$.

Define $f : X \to [0,1]$ by

$$f(x) = \inf\{r \in \Delta : x \in \mathcal{O}_r\}.$$

Clearly, if $x \in \mathcal{C}_0$, then $x \in \mathcal{O}_{2^{-n}}$ for any $n \in \mathbb{N}$, so it follows that $f(x) = 0$. On the other hand, if $x \in \mathcal{C}_1$, then $x \notin \mathcal{O}_r$ for any $r \in \Delta$, hence $f(x) = 1$ on $\mathcal{C}_1$. Thus, it remains to show that f is continuous. Recall that it suffices to consider the sub-base of open rays. Use as sub-base $\{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, +\infty) : b \in \mathbb{R}\}$.

Since $f : X \to [0,1]$, then for $a \leq 0, b \geq 1$, $f^{-1}((-\infty, a)) = f^{-1}((b, +\infty)) = \emptyset$. Suppose $0 < a \leq 1$. If $x \in X$, and $f(x) < a$, then there is a dyadic rational number $r \in \Delta$ such that $f(x) < r < a$, so $x \in \mathcal{O}_r$. Then we have

$$f^{-1}((-\infty, a)) = \bigcup_{r<a} \mathcal{O}_r,$$

which is open. Similarly, suppose $0 \leq b < 1$. If $x \in f^{-1}((b, +\infty))$, i.e. $f(x) > b$, then there exists a dyadic rational $s \in \Delta$ such that $f(x) > s > b$, so $x \notin \mathcal{O}_s$, and there exists a dyadic rational $r \in \Delta$ such that $s > r > b$, so $\overline{\mathcal{O}}_r \subseteq \mathcal{O}_s$, and so $x \notin \overline{\mathcal{O}}_r$, so $x \in \overline{\mathcal{O}}_r^c$, which is open. Then

$$f^{-1}((b, \infty)) = \bigcup_{r>b} \overline{\mathcal{O}}_r^c$$

is open. □

Chapter 3

3.1