Math 185 Notes
Complex Analysis

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Lecture 1

Complex Numbers

1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for $|x| < r$, where r is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n||x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

Example 1.1.1. The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for $|x| < 1$.

Question. Now what if we replace the real variable x by the complex variable z?

Answer. If $|z| < r$, then

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

so the sum converges absolutely for $z \in D(0, r)$ (disc of radius r centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

Example 1.1.2. Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function $\mathbb{R} \to \mathbb{R}$, $f(z)$ is infinitely differentiable at $z = 0$, and all derivatives of $f(z)$ are zero at $z = 0$. Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f"(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from $f(z)$!

Example 1.1.3. Consider the same example as above, but with z as a complex number. Let $z = it$ where $t \in \mathbb{R}$. Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at $z = 0$ and thus not complex-differentiable at $z = 0$.

Example 1.1.4. Now let's set $z = x + iy$ where $x, y \in \mathbb{R}$. Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function $\mathbb{R}^2 \to \mathbb{R}^2$ (instead of $\mathbb{C} \to \mathbb{C}$). Let's differentiate with respect to x:

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to y:

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = if'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i\frac{\partial f'(z)}{\partial y} = i\frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of) $f(z)$ satisfy the two-dimensional *Laplace equation.*

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

Example 1.1.5. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful too for computing integrals.

Lecture 2

Complex Differentiation

2.1 Derivatives

Definition 2.1.1 (Derivative)**.** The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x) = L$$

means for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$. (For any "tolerance" ϵ, we can guarantee $f(x)$ is within ϵ of L by forcing x to be close enough to a.)

Remark. Note that $x = a$ doesn't satisfy $0 < |x - a|$, so the value of f at $x = a$ has no bearing on whether $\lim_{x \to a} f(x)$ exists.

2.1.1 Continuity

Definition 2.1.2 (Continuous)**.** If $\lim_{x \to a} f(x) = f(a)$, then we say f is *continuous* at a.

Remark. Setting $L = f(a)$ in the limit, $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (even when $x = a$) when talking about continuity, we leave out the $0 < |x - a|$ part for convenience because $x - a = 0$ automatically works.

Now let's consider a function $f : \mathbb{C} \to \mathbb{C}$, $\lim_{z\to a} f(z) = L$ means for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

Remark. Now the z's that we worry about form an open disc with radius δ instead of an interval from the real case.

Similarly, if $\lim_{z\to a} f(z) = f(a)$, we say f is *continuous* at $z = a$.

Example 2.1.3. $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.

Proof. For $\epsilon > 0$, let $\delta = \epsilon$, then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

Example 2.1.4. $\lim_{z\to 0} \overline{z}/z$ (although this is undefined at $z = 0$, this has no bearing on whether the limit exists).

Proof. Suppose $\lim_{z\to 0} \overline{z}/z = L$ for some L. Let's take $\epsilon = 1$. There is a $\delta > 0$ such that

$$0 < |z - 0| < \delta \implies \left|\frac{\overline{z}}{z} - L\right| < \epsilon = 1.$$

Let $z = \delta/2$ and so does $z = i\delta/2$. Then for $z = \delta/2$:

$$\frac{\overline{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for $z = i\delta/2$:

$$\frac{\overline{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the L must lie in the intersection of the two open unit discs centered at -1 and 1. However, since they are open discs, these two discs do not overlap and so L does not exist. □

Remark. This implies that there is no way to extend $\overline{z}/z$ to a continuous function at $z = 0$.

2.1.2 Properties of Limits

If $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$, then

(i)
$$\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2.$$

(ii)
$$\lim_{x\to a} f(x)g(x) = L_1 L_2.$$

(iii)
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Remark. These implies that the sum/product/quotient of continuous functions are continuous.

Proposition 2.1.5 (Composite function of continuous functions is continuous)**.** *If $f(x)$ is continuous at $x = a$, and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.*

Proof. We want $|g(f(x)) - g(f(a))| < \epsilon$. By continuity of g at $x = f(a)$, there exists $\delta_1 > 0$ such that
$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$
We want to take $w = f(x)$, so we need
$$|f(x) - f(a)| < \delta_1.$$
But by continuity of f at $x = a$, we know that δ_1 will be our ϵ when $|x - a| < \delta_2$ for some $\delta_2 > 0$. Then for such x,
$$|g(f(x)) - g(f(a))| < \epsilon.$$
$\square$

2.2 Derivatives (Cont'd)

Definition 2.2.1 (Differentiable)**.** We say that $f(z)$ is differentiable at $z = a$ iff $\frac{f(z)-f(a)}{z-a}$ extends to a continuous function at $z = a$ (the value there is $f'(a)$).

Example 2.2.2. $f(z) = z$ is differentiable with $f'(z) = 1$.

Proof.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-z}{h}=\lim_{h\to 0}1=1.$$

□

Example 2.2.3 (Interesting one)**.** $f(z)=\overline{z}$ is not differentiable but is continuous.

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}&=\lim_{h\to 0}\frac{\overline{z+h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{z}+\overline{h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{h}}{h}\\&=\text{DNE}\qquad\text{(proved in previous example)}\end{aligned}$$

□

Proposition 2.2.4 (Differentiability implies continuity)**.** *$f(z)$ differentiable at $z=a$ implies that $f(z)$ is continuous at $z=a$.*

Proof. We want to show that $\lim_{z\to a}f(z)=f(a)$.

$$\begin{aligned}\lim_{z\to a}f(z)-f(a)&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot(z-a)\\&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot\lim_{z\to a}(z-a)\qquad\text{(assume both limits exist)}\\&=f'(a)\cdot 0\\&=0.\end{aligned}$$

□

Remark. This is a common technique to show continuity by showing the limit of the difference is zero.

2.2.1 Properties of complex-derivatives

(i)

$$\frac{d}{dz}cf(z)=cf'(z),\qquad\forall c\in\mathbb{C}.$$

(ii)
$$\frac{d}{dz}(f+g) = f'(z) + g'(z).$$

(iii)
$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)
$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)
$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

Proposition 2.2.5 (Power rule)**.**
$$\frac{d}{dz}z^n = nz^{n-1}$$
for all integers n.

Proof. We induct on n. For $n \geq 0$, when $n = 0$,
$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$
By the product rule,
$$\begin{aligned}\frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\ &= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \qquad \text{(inductive hypothesis)} \\ &= nz^{n-1}.\end{aligned}$$
For $n < 0$, simply apply quotient rule. $\square$

Lecture 3

Holomorphic Functions and Cauchy-Riemann Equations

Recall

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Being differentiable at a point says little about how "nice" a function is.

Example 3.0.1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

This is nowhere continuous, and thus nowhere differentiable. But if we consider $x^2 f(x)$, it is differentiable at $x = 0$:

$$\lim_{h \to 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h \to 0} h f(h) = 0.$$

Nevertheless, it's still not a very "nice" function.

3.1 Holomorphic Functions

Definition 3.1.1 (Holomorphic)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at a point a if it is differentiable at z for all z within distance r of a for some $r > 0$. In other words, $f(z)$ is differentiable everywhere sufficiently close to a.

Definition 3.1.2 (Open/closed disk)**.** The *open disk* of radius r centered at $a \in \mathbb{C}$ is

$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The *closed disk* is

$$\overline{D}(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Thus, we can say $f(z)$ is holomorphic at $a \in \mathbb{C}$ if $f(z)$ is differentiable on an open disk centered at a. (if the point is not specified, it means that f is holomorphic everywhere.)

Example 3.1.3 (Polynomials are holomorphic)**.** We saw last time that z^n is differentiable everywhere for $n \geq 0$. Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

Example 3.1.4. $f(z) = |z|^2 = z\overline{z}$ is differentiable at zero.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0} \frac{h\overline{h} - 0}{h} \\
&= \lim_{h \to 0} \overline{h} \\
&= 0.
\end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus, f is not holomorphic.

3.2 The Cauchy-Riemann Equations

Question. How to tell if a function is complex-differentiable?

Answer. We'll reduce this to a question about real derivatives.

Let $x + iy$, where $x, y \in \mathbb{R}$. If $f : \mathbb{C} \to \mathbb{C}$,

$$\begin{aligned}
\frac{\partial f}{\partial x}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h \to 0} \frac{f(x+h+iy) - f(x+iy)}{h}.
\end{aligned}$$

Note that h is real. Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y}(z) &= \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h \to 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}.
\end{aligned}$$

Example 3.2.1. $f(z) = z^2$. Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}
\frac{\partial f}{\partial x}(z) &= \lim_{h\to 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2 + 2ihy}{h} \\
&= \lim_{h\to 0} 2x + h + 2iy \\
&= 2x + 2iy \\
&= 2z \\
&= f'(z).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y}(z) &= \lim_{h\to 0} \frac{x^2 - (y+h)^2 + 2ix(y+h) - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h\to 0} \frac{-2yh - h^2 + 2ixh}{h} \\
&= \lim_{h\to 0} -2y - h + 2ix \\
&= -2y + 2ix \\
&= 2i(x+iy) \\
&= if'(z).
\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y}.$$

Theorem 3.2.2.

(i) If $f : \mathbb{C} \to \mathbb{C}$ is complex-differentiable, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and they satisfy

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

(ii) If $f : \mathbb{C} \to \mathbb{C}$ is a function and $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist and are continuous on some open disk centered at z, and if

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y},$$

then f is complex-differentiable at z.

Proof.

(i) Since f is complex-differentiable, we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose h is real. Then

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since h is real, we get $\frac{\partial f}{\partial x}$ and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose h is purely imaginary: $h = ik$ for $k \in \mathbb{R}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then $h \to 0$ is equivalent to $k \to 0$ since $|h| = |k|$. Thus we have

$$\lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

$\square$

Let $f(z) = u(z) + iv(z)$. If we choose real values for h, then the imaginary part y is kept constant, and the derivative becomes a partial derivative with respect to x. Thus we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values ik for h, we obtain

$$f'(z) = \lim_{k \to 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since we have

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

this resolves into the following equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or simply

$$u_x = v_y, \qquad v_x = -u_y.$$

These are known as the *Cauchy-Riemann* equations.

Example 3.2.3. Consider $f(z) = z^2$. Then

$$f(x+iy) = x^2 + y^2 + 2ixy.$$

Here $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We have

$$u_x = 2x = v_y \qquad v_x = 2y = -u_y.$$

Example 3.2.4. Consider $f(z) = |z|^2$. Then $f(x+iy) = x^2+y^2$ where $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. But here we have

$$u_x = 2x \neq v_y = 0 \qquad v_x = 0 \neq -u_y = -2y.$$

Thus, the Cauchy-Riemann equations only hold at $(x,y) = (0,0)$ and as we saw previously that this function is only differentiable at $z = 0$ and nowhere else.

We can generalize the Cauchy-Riemann equations further to second order. Suppose we have $f = u + iv$. Then

$$u_{xx} = \frac{\partial}{\partial x}u_x = \frac{\partial}{\partial x}v_y = \frac{\partial}{\partial x}\frac{\partial}{\partial y}v = \frac{\partial}{\partial y}\frac{\partial}{\partial x}v = \frac{\partial}{\partial y}v_x = \frac{\partial}{\partial y}(-u_y) = -u_{yy}.$$

Thus, we have

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we also have

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

which gives

$$v_{xx} + v_{yy} = 0, \text{or} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These are the *Laplace's equations* in 2D we saw earlier.

For $f : \mathbb{R} \to \mathbb{R}$, we know that $f'(x) = 0$ implies that f is constant. But for $f : \mathbb{C} \to \mathbb{C}$, we can use the Cauchy-Riemann equations. Since $f'(z) = \frac{\partial f}{\partial x}$,

$$f'(z) = 0 \implies u_x + iv_x = 0 \implies u_x = 0, v_x = 0.$$

By Cauchy-Riemann, we also have $u_y = v_y = 0$. Since $u_x = 0$, we know that for fixed y, $u(x,y)$ is some constant that could depend on y. Thus, we have

$$u(x,y) = g(y).$$

But $u_y = 0$, so $g'(y) = 0$, which means g is actually a constant independent of y. Thus, u is globally constant. Similar argument applies to v as well.

Lecture 4

Möbius Transformation

Definition 4.0.1 (Möbius transformation)**.** A *Möbius transformation* is a function of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$.

Remark. If $ad = bc$, then $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, so rows are linearly dependent: $\lambda(a, b) + \mu(c, d) = (0, 0)$, which implies that

$$a = \frac{-\mu}{\lambda} c \qquad b = \frac{-\mu}{\lambda} d.$$

Then

$$\begin{aligned} f(z) &= \frac{az+b}{cz+d} \\ &= \frac{-\frac{\mu}{\lambda}(cz+d)}{cz+d} \\ &= -\frac{\mu}{\lambda}, \end{aligned}$$

which is a constant independent of z.

Proposition 4.0.2 (Composite Möbius transforms is Möbius)**.** *If $f_1(z), f_2(z)$ are Möbius transforms, then then $f_1(f_2(z))$ is also a Möbius transform.*

Proof. Suppose

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \qquad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}.$$

Then

$$\begin{aligned} f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}, \end{aligned}$$

which is another Möbius transform. □

Remark. Note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and the entries coincide with the composite Möbius transform. If we denote $f_M(z)$ to be a transform associated with a 2×2 matrix M, then we have just shown that

$$f_M(f_N(z)) = f_{MN}(z).$$

Remark. Since $f_I(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$, the inverse of f_M is $f_{M^{-1}}$.

Remark. Note that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies f_M = \frac{az + b}{cz + d}.$$

Meanwhile,

$$\lambda M = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \implies f_{\lambda M} = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az + b}{cz + d} = f_M.$$

Thus, scaling the matrices doesn't affect the resulting Möbius tranformation.

4.1 Inverse of Möbius transformation

Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the scaling part is redundant, we simply ignore it and obtain the inverse Möbius transform as follows:

$$f(z) = \frac{az + b}{cz + d} \implies f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

Remark. Since Möbius transforms have inverses, they should be bijections. However, some details should be noted. If $c \neq 0$, then $\frac{az+b}{cz+d}$ is undefined at $z = -\frac{d}{c}$.

Let's consider the value at $z = -\frac{d}{c}$ to be infinity. It turns out that we can evaluate $\frac{az+b}{cz+d}$ at ∞:

$$\begin{aligned}\lim_{z\to\infty}\frac{az+b}{cz+d} &= \lim_{z\to\infty}\frac{a+\frac{b}{z}}{c+\frac{d}{z}}\\ &= \frac{a}{c}.\end{aligned}$$

When $c = 0$, we view $\frac{a}{c}$ as ∞. So now we view Möbius transformations as functions from $\mathbb{C}\cup\{\infty\}$ to $\mathbb{C}\cup\{\infty\}$. This makes all Möbius transformations into bijections. Here, we call $\mathbb{C}\cup\{\infty\}$ the *extended complex plane* (also called *Riemann sphere*).

Remark. For real functions, there are multiple notions of going to infinity: $x \to +\infty$ and $x \to -\infty$. But for complex functions, we work with only one infinite point.

Fact. If we apply a Möbius transformation to a line or a circle in the complex plane, we would get a line or a circle again (circles can turn into lines and vice versa).

Example 4.1.1. Consider $f(z) = \frac{z-1}{iz+i}$, let's apply this to the unit circle, i.e. take $z = e^{i\theta}$. Then

$$\begin{aligned}f(e^{i\theta}) = \frac{e^{i\theta}-1}{i(e^{i\theta}+1)} &= \frac{\cos\theta - 1 + i\sin\theta}{i(\cos\theta + 1 + i\sin\theta)}\\ &= \frac{-2\sin^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{i\left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}\\ &= \frac{2i(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})\sin\frac{\theta}{2}}{2i\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2}}\\ &= \tan\frac{\theta}{2}.\end{aligned}$$

Note that $\theta \in (-\pi, \pi)$ and we have $\tan -\frac{\pi}{2} = -\infty$ and $\tan\frac{\pi}{2} = -+\infty$. We have mapped a unit circle to a line (real line).

Fact. f sends the interior of the unit disk to the interior of the upper half-plane. If $g(z)$ is holomorphic on the upper half-plane, then $g(f(z))$ is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equation.