
STOCHASTIC PROCESSES

STAT 150

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1 Probability Review

1.1 Basic Definitions

Definition 1.1.1 (Probability Space). A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple consisting of a set Ω called the *sample space*, a set $\mathcal{F} \subseteq \Omega$ satisfying certain closure properties, and a function $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ that assigns probabilities to events in a coherent way.

Requirements for \mathcal{F} :

- (i) $\Omega \in \mathcal{F}$.
- (ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$.
- (iii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Requirements for \mathbb{P} :

- (i) $\mathbb{P}(\Omega) = 1$.
- (ii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint (meaning $E_i \cap E_j = \emptyset$ for $i \neq j$), then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Definition 1.1.2 (Random Variable). A *random variable* is a function $X : \Omega \rightarrow \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ whenever B is a "nice" subset of \mathbb{R} .

Example 1.1.3. $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$, $\mathbb{P}(\{H\}) = \frac{1}{2}$. $X(H) = 1$, $X(T) = 0$.

$$\mathbb{P}(X = 1) = \mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(X = 0) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

1.2 Overview

Definition 1.2.1 (Stochastic Process). A *stochastic process* is a collection $\{X_t : t \in T\}$ of random variables $X_t : \Omega \rightarrow S \subseteq \mathbb{R}$ all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here T is some index set (typically representing time) and S is the *state space*. One write this as

$X : \Omega \times T \rightarrow S$, $(\omega, t) \mapsto X_t(\omega)$. For a given outcome $\omega \in \Omega$, we get a sample path trajectory $X(\omega) : T \rightarrow S, t \mapsto X_t(\omega)$. A stochastic process can then be thought of as a random function.

The theme of this course is what can we say about the distribution of trajectories?

Example 1.2.2 (Branching Process (DTDS)). $X_0 = 1$, one individual in the 0th generation individuals produce a random number of offspring, i.i.d. $(\xi_i^{(n)})_{i \in \mathbb{N}, n \in \mathbb{N}_0}$.

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

One interesting question would be what is $\mathbb{P}(X_n = 0 \text{ eventually})$, the probability of dying out?

Example 1.2.3 (Poisson Process (CTDS)). Recall that the Poisson distribution is used to model the number of occurrences of a rare event in some fixed period of time. The Poisson process $(N_t)_{t \geq 0}$ models the number of occurrences throughout time. $N_t = \#$ of occurrences by time t .

1.3 Useful Properties

(i) (*DeMorgan*)

$$(E \cup F)^c = E^c \cap F^c, \quad (E \cap F)^c = E^c \cup F^c.$$

(ii) (*Complementation*)

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c).$$

(iii) (*Inclusion-exclusion*)

$$\begin{aligned} \mathbb{P}(E \cup F) &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) \\ \mathbb{P}\left(\bigcup_{i=1}^n E_i\right) &= \sum_{j=1}^n (-1)^{j-1} \sum_{S \in [n]: |S|=j} \mathbb{P}\left(\bigcap_{i \in S} E_i\right). \end{aligned}$$

(iv) (*Partitioning*) If $\bigsqcup_{i=1}^{\infty} E_i = \Omega$, then

$$\mathbb{P}(F) = \mathbb{P}\left(\bigsqcup_{i=1}^{\infty} (F \cap E_i)\right) = \sum_{i=1}^{\infty} \mathbb{P}(F \cap E_i)$$

1.4 Conditional Probability

Conditioning: For $\mathbb{P}(F) > 0$,

$$\mathbb{P}(E | F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$

$\mathbb{P}(\cdot | F)$ defines a new probability measure on (Ω, \mathcal{F}) .

Multiplication rule:

$$\mathbb{P}(E \cap F) = \mathbb{P}(F) \mathbb{P}(E | F).$$

If $\bigcup_{i=1}^{\infty} F_i = \Omega$, then

$$\mathbb{P}(E) = \sum_{i=1}^{\infty} \mathbb{P}(E \cap F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i) \mathbb{P}(E | F_i).$$

Bayes' rule:

$$\mathbb{P}(F_j | E) = \frac{\mathbb{P}(F_j) \mathbb{P}(E | F_j)}{\sum_{i=1}^{\infty} \mathbb{P}(F_i) \mathbb{P}(E | F_i)}$$

1.5 Random Variables

1.5.1 Discrete Random Variables

If $X : \Omega \rightarrow S \subseteq \mathbb{R}$ is discrete,

$$\mathbb{P}(X \in E) = \sum_{x \in E} \mathbb{P}(X = x) = \sum_{x \in E \cap S} \mathbb{P}(X = x).$$

1.5.2 Indicator Random Variable

$$X(\omega) = \mathbb{I}_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

1.5.2.1 Binomial Random Variable

$$X = \sum_{i=1}^n \mathbb{I}_{E_i}, \quad \mathbb{P}(E_i) = p.$$

$$p_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

1.5.3 Continuous Random Variables

If X continuous,

$$\mathbb{P}(X \in E) = \int_E f_X(x) dx.$$

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x) dx.$$

1.5.3.1 Exponential Random Variable

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

1.5.3.2 Gaussian Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

1.5.4 Cumulative Distribution Function (CDF)

$$F_X : \mathbb{R} \rightarrow [0, 1],$$

$$F_X(r) = \mathbb{P}(X \leq r) = \mathbb{P}(X \in (-\infty, r]).$$

If X is discrete,

$$F_X(r) = \sum_{x_i \leq r} p_X(x_i).$$

If X is continuous,

$$F_X(r) = \int_{-\infty}^r f_X(x) dx.$$

$$\frac{d}{dr} F_X(r) = f_X(r).$$

1.5.5 Expectation

1.5.5.1 Discrete case

$$\mathbb{E}[X] = \sum_{x_i \in S} x_i \mathbb{P}(X = x_i).$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

$$\mathbb{E}[g(X)] = \sum_{x_i \in S} g(x_i) \mathbb{P}(X = x_i).$$

1.5.5.2 Continuous case

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \mathbb{P}(X \geq x) dx.$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

1.5.6 Variance

$$\text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

1.5.7 Moments

$$\mathbb{E}[X^m] = \int_0^\infty mx^{m-1}\mathbb{P}(X \geq x)dx.$$

1.5.8 Joint Distribution

1.5.8.1 Discrete

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

1.5.8.2 Continuous

$$\mathbb{P}((X,Y) \in E) = \int \int_E f_{X,Y}(x,y)dx dy$$

1.5.8.3 Marginal Distribution

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y).$$

$$f_X(x) = \int_{y \in S_y} f_{X,Y}(x,y)dy$$

1.5.9 Independence

$$p_{X,Y}(x,y) = p_X(x)p_Y(y).$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x)F_Y(y).$$

1.5.10 Linearity of Expectation

$$\mathbb{E} \left[\sum_{i=1}^n c_i X_i \right] = \sum_{i=1}^n c_i \mathbb{E}[X_i]$$

If $(X_i)_{i=1}^n$ independent,

$$(g(X_i))_{i=1}^n$$

independent.

$$\mathbb{E} \left[\prod_{i=1}^n g(X_i) \right] = \prod_{i=1}^n \mathbb{E} [g(x_i)]$$

$$\text{Var} \left(\sum_{i=1}^n x_i \right) = \sum_{i=1}^n \text{Var}(x_i)$$

In general,

$$\text{Var} \left(\sum_{i=1}^n x_i \right) = \sum_{i,j=1}^n \text{Cov}(x_i, x_j)$$

1.5.11 Convolution

Discrete case: X, Y discrete $X \perp\!\!\!\perp Y$

$$\begin{aligned} \mathbb{P}(X + Y = z) &= \sum_Y \mathbb{P}(X + Y = z, Y = y) \\ &= \sum_y \mathbb{P}(X = z - y, Y = y) \\ &= \sum_y \mathbb{P}(X = z - y) \mathbb{P}(Y = y) \quad (= \sum_x \mathbb{P}(X = x) \mathbb{P}(Y = z - x)). \end{aligned}$$

If X, Y are \mathbb{Z} -valued, this becomes

$$\begin{aligned} \mathbb{P}(X + Y = n) &= \sum_{k=-\infty}^{\infty} \mathbb{P}(X = n - k) \mathbb{P}(Y = k) \\ &= \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k) \mathbb{P}(Y = n - k) \\ &= (\mathbb{P}_X * \mathbb{P}_Y)(n) \end{aligned}$$

Example 1.5.1 (Poisson). $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$, $X + Y \sim \text{Poisson}(\lambda + \mu)$

$$\begin{aligned} \mathbb{P}(X + Y = n) &= \sum_{k=0}^n \mathbb{P}(X = k) \mathbb{P}(Y = n - k) \\ &= \sum_{k=0}^n e^{-\lambda} \frac{\lambda^k}{k!} e^{-\mu} \frac{\mu^{n-k}}{(n-k)!} \\ &= e^{-(\lambda+\mu)} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \lambda^k \mu^{n-k} \\ &= \frac{e^{-(\lambda+\mu)}}{n!} (\lambda + \mu)^n \\ &= \mathbb{P}(Z = n) \end{aligned}$$

where $Z \sim \text{Poisson}(\lambda + \mu)$.

Continuous case: X, Y continuous

$$\begin{aligned} \mathbb{P}(X + Y \leq z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_X(x) f_Y(y) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^z f_X(x) f_Y(y - x) dy dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_X(x) f_Y(y - x) dx dy \end{aligned}$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y) dy = f_X * f_Y.$$

Example 1.5.2 (Convolution in uniform distributions). $X, Y \sim U[0, 1]$, $X \perp\!\!\!\perp Y$.

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$

$$f_X(x) = \mathbb{I}_{[0,1]}(x) \quad f_Y(y) = \mathbb{I}_{[0,1]}(y)$$

so

$$\begin{aligned} f_{X+Y}(z) &= \int_{x \in [0,1], z-x \in [0,1]} 1dx \\ &= \int_{x \in [0,1], x \in [-1+z, z]} 1dx \\ &= \int_{\max(0, -1+z)}^{\min(1, z)} 1dx \\ &= \min(1, z) - \max(0, -1+z). \end{aligned}$$

1.5.12 Gamma Distribution

Definition 1.5.3 (Gamma function). Let $\alpha > 0$. The *gamma function* $\Gamma : (0, \infty) \rightarrow (0, \infty)$ is defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx = \mathbb{E}[X^{\alpha-1}]$$

where $X \sim \text{Exp}(1)$. Let $\alpha, \lambda > 0$. The $\text{Gamma}(\alpha, \lambda)$ distribution is defined by

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

Exercise 1.5.4. $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. (Hint: use induction)

1.5.13 Moment Generating Function

Definition 1.5.5 (MGF). For a random variable X , the *moment generating function* (MGF) is the function $M_X : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$,

$$M_X(t) = \mathbb{E}[e^{tX}].$$

If $M_X(t) < +\infty$ for $t \in (-\epsilon, \epsilon)$, then

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k \mathbb{E}[x^k]}{k!} \text{ for } |t| < \epsilon$$

For independent RVs $(X_i)_{i=1}^n$,

$$M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t)$$

Exercise 1.5.6. If $X \sim \text{Exp}(\lambda)$, then $M_X(t) = \frac{\lambda}{\lambda-t}$ if $t < \lambda$, $+\infty$ otherwise.

If $X \sim \text{Gamma}(n, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda-t} \right)^n.$$

If $X \sim \text{Gamma}(\alpha, \lambda)$, then

$$M_X(t) = \left(\frac{\lambda}{\lambda-t} \right)^{\alpha}.$$

1.6 Conditional Probability (Cont'd)

Exercise 1.6.1 (Generalization). $(X_i)_{i=1}^n, (Y_j)_{j=1}^m$

$$p_{X_1, \dots, X_n | Y_1, \dots, Y_m} (x_1, \dots, x_n | y_1, \dots, y_m) = ?$$

Example 1.6.2. Let $M \in \mathbb{N}$ and $p, q \in (0, 1)$. Consider $N \sim \text{Bin}(M, q)$ and $X \sim \text{Bin}(N, p)$. What is the distribution of X ?

$$\begin{aligned} \mathbb{P}(X = k) &= \sum_{n=0}^M \mathbb{P}(N = n) \mathbb{P}(X = k | N = n) \\ &= \sum_{n=0}^M \binom{M}{n} q^n (1-q)^{M-n} \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{p^k}{k!} \sum_{n=k}^M \frac{M!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\ &= \frac{p^k}{k!(M-k)!} \sum_{n=k}^M \frac{M!(M-k)!}{(M-n)!(n-k)!} q^n (1-q)^{M-n} (1-p)^{n-k} \\ &= \binom{M}{k} p^k q^k \sum_{n=k}^M \binom{M-k}{n-k} q^{n-k} (1-q)^{M-j} (1-p)^{n-k} \\ &= \binom{M}{k} p^k q^k \sum_{t=0}^{M-k} \binom{M-k}{t} (q(1-p))^t (1-q)^{M-k-t} \\ &= \binom{M}{k} (pq)^k (q(1-p) + (1-q))^{M-k} \\ &= \binom{M}{k} (pq)^k (1-pq)^{M-k}. \end{aligned}$$

Thus, $X \sim \text{Bin}(M, pq)$.

Remark. What if $k > n$ in $\mathbb{P}(X = k | N = n)$ above in the first line? The probability is simply 0.

Question. Why does this answer make sense?

Answer. Think about retesting whenever we succeeded for the first M trials. Then X is simply the number of trials with double successes, thus we have the pq parameter.

Exercise 1.6.3. Consider $N \sim \text{Poisson}(\lambda)$, $X \sim \text{Bin}(N, p)$. What is the distribution of X ?

Answer. $X \sim \text{Poisson}(\lambda p)$.

Question. How can we interpret this?

Answer. We can interpret X as the number of customers visiting a store who purchase something.

1.6.1 Conditional Expectation

For X, Y discrete, $g : \mathbb{R} \rightarrow \mathbb{R}$. Assume $\mathbb{E}[|g(X)|] = \sum_x |g(x)p_X(x)| < \infty$.

Definition 1.6.4 (Conditional expectation). The *conditional expectation* is defined as

$$\mathbb{E}[g(X) | Y = y] = \sum_x g(x)p_{X|Y}(x|y)$$

if $p_Y(y) \neq 0$.

Remark. Note that $\mathbb{E}[g(X) | Y = y]$ is a real number, whereas $\mathbb{E}[g(X) | Y]$ is a random variable.

1.6.1.1 Tower Property

$$\begin{aligned} \mathbb{E}[\mathbb{E}[g(X) | Y]] &= \mathbb{E}\left[\sum_y \mathbb{E}[g(X) | Y = y]p_Y(y)\right] \\ &= \sum_y \mathbb{E}[g(X) | Y = y]p_Y(y) \\ &= \sum_y \sum_x g(x)p_{X|Y}(x|y)p_Y(y) \\ &= \sum_x g(x) \sum_y p_{X|Y}(x|y)p_Y(y) \\ &= \sum_x g(x)p_X(x) \\ &= \mathbb{E}[g(X)]. \end{aligned}$$

Remark. One intuitive example would be considering the averages of heights of students from a classroom. We divide it into several groups and let Y denote the whichever group we select and let $\mathbb{E}[g(X) | Y]$ be the average of those from group Y . Then the average height of the entire classroom $\mathbb{E}[g(X)]$ is equivalent to the average of the average of heights of each group, which is $\mathbb{E}[\mathbb{E}[g(X) | Y]]$.

Properties of conditional expectations:

1. $\mathbb{E}[c_1g(x_1) + c_2h(x_2) | Y = y] = c_1\mathbb{E}[g(X_1) | Y = y] + c_2\mathbb{E}[h(X_2) | Y = y]$
2. If $g \geq 0$, then $\mathbb{E}[g(x) | Y = y] \geq 0$.
3. $\mathbb{E}[f(X, Y) | Y = y] = \mathbb{E}[f(X, y) | Y = y]$.
4. If $X \perp\!\!\!\perp Y$, $\mathbb{E}[g(X) | Y = y] = \mathbb{E}[g(X)]$
5. $\mathbb{E}[g(x)h(y) | Y = y] = h(y)\mathbb{E}[g(x) | Y = y]$
6. $\mathbb{E}[g(x)h(y)] = \sum_y h(y)\mathbb{E}[g(x) | Y = y]p_Y(y) = \mathbb{E}[h(Y)\mathbb{E}[g(X) | Y]]$

Proof of 3.

$$\begin{aligned}\mathbb{E}[f(X, Y) \mid Y = y] &= \sum_{x, z} f(x, z) p_{X, Y \mid Y}(x, z \mid y) \\ &= \sum_{x, z} f(x, z) \frac{p_{X, Y, Y}(x, z, y)}{p_Y(y)} \\ &= \sum_x f(x, y) \frac{p_{X, Y}(x, y)}{p_Y(y)} \\ &= \mathbb{E}[f(X, y) \mid Y = y].\end{aligned}$$

□

Remark. $\mathbb{E}[f(X, y)] \neq \mathbb{E}[f(X, y) \mid Y = y]$.

2 Random Sums

Definition 2.0.1. Let $(\xi_i)_{i=1}^{\infty}$ be i.i.d random variables, N be a \mathbb{N}_0 -valued random variable, $N \perp\!\!\!\perp (\xi_i)_{i=1}^{\infty}$. The *random sum* is defined as

$$X = \sum_{i=1}^N \xi_i = \sum_{n=0}^{\infty} \left(\sum_{i=1}^n \xi_i \right) \mathbf{1}_{N=n} = \begin{cases} \sum_{i=1}^n \xi_i & \text{if } N = n \geq 1 \\ 0 & \text{if } N = 0. \end{cases}$$

Question. What is the distribution of X ?

Let X, N be random variables. N is \mathbb{N}_0 -valued. The condition CDF is

$$F_{X|N}(x|n) = \mathbb{P}(X \leq x \mid N = n)$$

if $\mathbb{P}(N = n) \neq 0$. This is an actual CDF, but for the random variable $X \mid N = n$.

Suppose that X is continuous and $F_{X|N}(x|n)$ is a differentiable function of x for each n such that $p_N(n) > 0$. The conditional PDF is

$$f_{X|N}(x|n) = \frac{d}{dx} F_{X|N}(x|n).$$

$$\begin{aligned} \int_a^b f_{X|N}(x|n) dx &= F_{X|N}(b|n) - F_{X|N}(a|n) \\ &= \mathbb{P}(X \in [a, b] \mid N = n). \end{aligned}$$

Answer.

$$F_X(x) = \mathbb{P}(X \leq x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) \mathbb{P}(X \leq x \mid N = n).$$

$$f_X(x) = \sum_{n=0}^{\infty} \mathbb{P}(N = n) f_{X|N}(x|n).$$

2.1 Mean and Variance of Random Sums

Assume $\mathbb{E}[N] = \nu$ and $\mathbb{E}[\xi_i] = \mu$. Then

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X \mid N]] \\ &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^N \xi_i \mid N\right]\right] \\ &= \mathbb{E}[N\mathbb{E}[\xi_1]] \\ &= \mathbb{E}[N\mu] \\ &= \mu\nu.\end{aligned}$$

3 Markov Chains

3.1 Discrete-time Markov Chains

Definition 3.1.1 (Markov process). A is a stochastic process $(X_t)_{t \in T}$ such that the future, given the present, is independent of the past.

Definition 3.1.2 (Markov property). The *Markov property* for a DTDS stochastic process is

$$\mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{n+1} = j \mid X_n = i)$$

Example 3.1.3 (Gambler's ruin). $(X_n)_{n=0}^\infty$, X_n = your wealth after n turns. Stop if $X_n = 0$ or 5. Each play, you win \$1 with probability p and lose \$1 with probability $1 - p$ independently of all previous plays. This process satisfies the markov property.

Example 3.1.4 (Ehrenfest model). Box of N particles. X_n = number of particles on the left side at time n . $N - X_n$ be the number of particles on the other side.

$$\begin{aligned}\mathbb{P}(X_{n+1} = i + 1 \mid X_n = i) &= \frac{N - i}{N} \\ \mathbb{P}(X_{n+1} = i - 1 \mid X_n = i) &= \frac{i}{N}.\end{aligned}$$

Theorem 3.1.5.

Joint PMF of the Markov Chain is determined by initial distribution and $P = (p_{i,j})_{i,j \in S}$.

Proof.

$$\begin{aligned}\mathbb{P}(X_n = i_n, \dots, X_0 = i_0) &= \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \mathbb{P}(X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \\ &= p_{i_{n-1}, i_n} p_{i_{n-2}, i_{n-1}} \cdots p_{i_0, i_1} \mathbb{P}(X_0 = i_0).\end{aligned}$$

□

3.1.1 n -step transition probabilities

$$p_{i,j} = \mathbb{P}(X_{n+1} = j \mid X_n = i).$$

Theorem 3.1.6.

$$p_{i,j}^{(m)} = \mathbb{P}(X_{n+m} = j \mid X_n = i) = (P^m)_{i,j}.$$

Proof.

$$\begin{aligned} \mathbb{P}(X_{n+m+1} = j \mid X_n = i) &= \sum_{k \in S} \mathbb{P}(X_{n+m+1} = j, X_{n+m} = k \mid X_n = i) \\ &= \sum_{k \in S} \mathbb{P}(X_{n+m+1} = j \mid X_{n+m} = k) \mathbb{P}(X_{n+m} = k \mid X_n = i). \end{aligned}$$

□

Example 3.1.7.

$$\begin{aligned} p_{i,j}^{(2)} &= \mathbb{P}(X_2 = j \mid X_0 = i) \\ &= \sum_{k \in S} \mathbb{P}(X_2 = j, X_1 = k \mid X_0 = i) \\ &= \sum_{k \in S} \mathbb{P}(X_2 = j \mid X_1 = k, X_0 = i) \mathbb{P}(X_1 = k \mid X_0 = i) \\ &= \sum_{k \in S} P_{i,k} P_{k,j} \\ &= (P^2)_{i,j} \end{aligned}$$

Example 3.1.8 (Inventory model). X_n = inventory that you have of this product after the n th business day. If $X_n \leq s$, place an order that brings inventory back to S by next morning. ξ_n = demand on n th day and (ξ_n) are i.i.d..

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \begin{cases} \mathbb{P}(\xi_{n+1} = S - j) & \text{if } i \leq s \\ \mathbb{P}(\xi_{n+1} = i - j) & \text{if } i > s. \end{cases}$$

$\lim_{n \rightarrow \infty} \mathbb{P}(X_n < 0)$ = chance of excess demand.

3.2 First Step Analysis

Consider $(X_n)_{n \geq 0}$ Markov chain on $\{1, \dots, r\} \cup \{r+1, \dots, N\}$ where $\{1, \dots, r\}$ are the *transient states* and $\{r+1, \dots, N\}$ are the *absorbing states* such that

$$\begin{aligned} \lim_{n \rightarrow \infty} p_{i,j}^{(n)} &= 0 & \forall i, j \in \{1, \dots, r\} \\ \lim_{n \rightarrow \infty} p_{i,i}^{(n)} &= 1 & \forall i \in \{r+1, \dots, N\} \end{aligned}$$

Then we can express the transition matrix P as

$$P = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix},$$

where Q and R is some transition matrices for the corresponding partitioned states and 0 is the zero matrix and I is the identity matrix.

Let $T = \min \{n \geq 0 : X_n \geq r + 1\}$ be the time of absorption and X_T be the state we get absorbed into. Define $u_{i,k} = \mathbb{P}(X_T = k \mid X_0 = i)$. Then we have

$$\begin{aligned} u_{i,k} &= \sum_{j=1}^N \mathbb{P}(X_T = k, X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^N \mathbb{P}(X_T = k \mid X_1 = j, X_0 = i) \mathbb{P}(X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^N p_{i,j} \mathbb{P}(X_T = k \mid X_1 = j) \\ &= \sum_{j=1}^N \mathbb{P}(X_T = k \mid X_0 = j) \\ &= \sum_{j=1}^r p_{i,j} u_{j,k} + \sum_{j=r+1, j \neq k}^N p_{i,j} u_{j,k} + p_{i,k} u_{k,k}. \end{aligned}$$

Thus,

$$u_{i,k} = \sum_{j=1}^r P_{i,j} u_{j,k} + p_{i,k}$$

Hence, we have

$$U = QU + R \implies (I - Q)U = R \implies U = (I - Q)^{-1}R,$$

where U contains all the $(u_{i,k})_{i \in \{1, \dots, r\}, k \in \{r+1, \dots, N\}}$.

3.2.1 The General Absorbing Markov Chain

Let's suppose that associated with each transient state i is a rate $g(i)$ and that we wish to determine the mean total rate that is accumulated up to absorption. Let v_i be this mean total amount, where the subscript i denotes the starting position $X_0 = i$, i.e.,

$$v_i = \mathbb{E} \left[\sum_{n=0}^{T-1} g(X_n) \mid X_0 = i \right]$$

The choice $g = 1$ will give $v_i = \mathbb{E}[T \mid X_0 = i]$. We can also write for $i \in \{1, \dots, r\}$ that

$$\begin{aligned} v_i &= g(i) + \mathbb{E} \left[\sum_{n=1}^{T-1} g(X_n) \mid X_0 = i \right] \\ &= g(i) + \sum_{j=1}^N p_{i,j} v_j \quad (= \sum_{j=1}^N p_{i,j} (g(i) + v_i)). \end{aligned}$$

Then we can condense this into the following form

$$v = g + Qv \implies v = (I - Q)^{-1}g.$$

where $v = (v_i)_{i \in \{1, \dots, r\}}$ and $g = (g(i))_{i \in \{1, \dots, r\}}$.

3.3 Random Walk

$(\xi_n)_{n=1}^\infty$ i.i.d and \mathbb{Z} -valued. Then

$$X_n = \sum_{i=0}^n \xi_i.$$

$$\begin{aligned} \mathbb{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1) &= \mathbb{P}(\xi_{n+1} = j - i \mid \xi_n = i - i_{n-1}, \dots, \xi_1 = i_1) \\ &= \mathbb{P}(\xi_{n+1} = j - i) \\ &= \mathbb{P}(\xi_{n+1} = j - i \mid X_n = i). \end{aligned}$$

Example 3.3.1 (Gambler's Ruin). Win 1 dollar with probability p and lose 1 dollar with probability $q = 1 - p$. Stop when we lose all money or make N dollars. We are interested in $u_k = \mathbb{P}(X_T = 0 \mid X_0 = k)$ and $v_k = \mathbb{E}[T \mid X_0 = k]$. Clearly $u_0 = 1, u_N = 0$. For $k = 1, \dots, N-1$, we have

$$u_k = pu_{k+1} + qu_{k-1} \implies q(u_k - u_{k-1}) = p(u_{k+1} - u_k)$$

Let $\Delta_{k+1} = u_{k+1} - u_k$. Then we have

$$\begin{aligned} q\Delta_k &= p\Delta_{k+1} \\ \Delta_{k+1} &= \frac{q}{p}\Delta_k = \dots = \left(\frac{q}{p}\right)^k \Delta_1. \\ \sum_{i=1}^m \Delta_i &= \Delta_1 \sum_{i=1}^m \left(\frac{q}{p}\right)^{i-1} = \sum_{i=1}^m u_i - u_{i-1} = u_m - u_0 = u_m - 1 \end{aligned}$$

Thus,

$$u_m = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}} \quad m = 1, \dots, N$$

When $m = N$,

$$0 = 1 + \Delta_1 \frac{1 - \left(\frac{q}{p}\right)^N}{1 - \frac{q}{p}} \implies \Delta_1 = -\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N}.$$

Substituting the expression for Δ_1 gives

$$u_m = 1 + \left(-\frac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^N} \right) \left(\frac{1 - \left(\frac{q}{p}\right)^m}{1 - \frac{q}{p}} \right) = 1 - \frac{1 - \left(\frac{q}{p}\right)^m}{1 - \left(\frac{q}{p}\right)^N} = \frac{\left(\frac{q}{p}\right)^m - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N}.$$

Note that $p \neq q$. If $p = q$, then

$$\sum_{i=1}^m \Delta_i = \Delta_1 m = u_m - 1 \implies u_m = \frac{N - m}{N}.$$

If we take limit as $N \rightarrow \infty$ for $p \leq q$, then

$$\lim_{N \rightarrow \infty} u_m = 1,$$

which implies that we will be broke at the end no matter how much money we started with. If $p > q$, then

$$\lim_{N \rightarrow \infty} u_m = \left(\frac{q}{p}\right)^m.$$

If m is large, then this quantity becomes small. This implies that if $p > q$ and we started with a lot of money, then the chance of us being broke ultimately becomes smaller.

Now let's compute v_k when $p = q = \frac{1}{2}$. Clearly, $v_0 = 0$ and $v_N = 0$. For $k = 1, \dots, N-1$, we have

$$v_k = 1 + \frac{1}{2}v_{k+1} + \frac{1}{2}v_{k-1}.$$

Let $\Delta_k = v_k - v_{k-1}$. Then we have

$$0 = 1 + \frac{1}{2}(\Delta_{k+1} - \Delta_k).$$

Summing both sides gives

$$\sum_{k=1}^m 0 = m + \sum_{k=1}^m \frac{1}{2}(\Delta_{k+1} - \Delta_k) \implies \Delta_1 = 2m + \Delta_{m+1} \quad m = 0, \dots, N-1.$$

Then

$$\begin{aligned} \sum_{m=0}^k \Delta_1 &= \sum_{m=0}^k (2m + \Delta_{m+1}) \\ (k+1)\Delta_1 &= (k+1)v_1 = \sum_{m=0}^k 2m + \sum_{m=0}^k \Delta_{m+1} \\ (k+1)v_1 &= k(k+1) + (v_{k+1} - v_0) \implies (k+1)v_1 = k(k+1) + v_{k+1}. \end{aligned}$$

Take $k = N-1$ gives

$$Nv_1 = (N-1)N + 0 \implies v_1 = N-1.$$

Then

$$v_{k+1} = (k+1)(v_1 - k) = (k+1)(N-1-k).$$

Hence,

$$v_k = k(N-k).$$

3.4 Branching Process

$(\xi_i^{(n)})_{i=1, n=0}^{\infty, \infty}$ i.i.d. \mathbb{N}_0 -valued random variables where $\xi_i^{(n)}$ is the number of offspring of i th individual in n th generation. $X_0 = 1$. $\mathbb{E}[\xi_i] = \mu$ and $\text{Var}(\xi_i) = \sigma^2$. The population of at time $n+1$ is

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}$$

Our goal is to compute $\mathbb{P}(X_n = 0 \text{ eventually} \mid X_0 = 1)$. But let's first compute $\mathbb{E}[X_{n+1}]$ and $\text{Var}(X_{n+1})$. Recall that

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^N \xi_i\right] &= \mathbb{E}[N]\mathbb{E}[\xi_i] \\ \text{Var}\left(\sum_{i=1}^N \xi_i\right) &= \text{Var}(N)\mathbb{E}[\xi_i]^2 + \text{Var}(\xi_i)\mathbb{E}[N]. \end{aligned}$$

Then we have

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mu \mathbb{E}[X_n] = \mu^{n+1} \\ \text{Var}(X_{n+1}) &= \mu^2 \text{Var}(X_n) + \mu^n \sigma^2.\end{aligned}$$

$$\begin{aligned}c_0 &= \text{Var}(X_0) = 0 \\ c_n &= \text{Var}(X_n) \\ c_{n+1} &= \mu^2 c_n + \mu^n \sigma^2.\end{aligned}$$

Define the generating function $f(x)$ as

$$\begin{aligned}f(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{n+1} x^{n+1} = \mu^2 x \sum_{n=0}^{\infty} c_n x^n + \sigma^2 x \sum_{n=0}^{\infty} \mu^n x^n \\ &= \mu^2 x f(x) + \frac{\sigma^2 x}{1 - \mu x}.\end{aligned}$$

Then

$$f(x) = \frac{\sigma^2 x}{(1 - \mu x)(1 - \mu^2 x)} = \sigma^2 x \frac{1}{1 - \mu x} \frac{1}{1 - \mu^2 x}.$$

Since

$$\sum_{j=1}^{\infty} c_j x^j = \sigma^2 x \sum_{n=0}^{\infty} \mu^n x^n \cdot \sum_{m=0}^{\infty} \mu^{2m} x^m,$$

the coefficient of $x^{j-1} = \sum_{k=0}^{j-1} x^k x^{j-1-k}$ is

$$c_j = \sum_{k=0}^{j-1} \mu^k \mu^{2(j-1-k)}.$$

Thus

$$\text{Var}(X_n) = \sigma^2 \mu^{n-1} \cdot \begin{cases} n & \text{if } \mu = 1 \\ \frac{1 - \mu^{n-1}}{1 - \mu} & \text{if } \mu \neq 1. \end{cases}$$

Remark. When $\mu = 1$, expectation is constant, variance is growing linearly. When $\mu \neq 1$, expectation is increasing/decreasing geometrically, same with variance.

Now let $T = \min \{n \geq 0 : X_n = 0\}$ be the time the population dies out and let $u_n = \mathbb{P}(T \leq n) = \mathbb{P}(X_n = 0)$. Then $\lim_{n \rightarrow \infty} u_n$ is the probability of extinction.

$$u_{n+1} = \sum_{k=0}^{\infty} p_k u_n^k$$

where $p_k = \mathbb{P}(\xi = k)$. We have $u_0 = 0, u_1 = p_0$.

Let $\phi_\xi : [0, 1] \rightarrow [0, 1]$ be the generating function of ξ defined by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

Then we have

$$u_{n+1} = \phi(u_n) \implies u_\infty = \lim_{n \rightarrow \infty} u_{n+1} = \lim_{n \rightarrow \infty} \phi(u_n) \implies u_\infty = \phi(\lim_{n \rightarrow \infty} u_n) = \phi(u_\infty).$$

Thus, u_∞ is a fixed point for ϕ .

3.4.1 Generating Functions

Given any \mathbb{N}_0 -valued random variable ξ with $p_k = \mathbb{P}(\xi = k)$. Then the generating function is given by

$$\phi_\xi(s) = \mathbb{E}[s^\xi] = \sum_{k=0}^{\infty} p_k s^k.$$

ϕ_ξ completely recovers the distribution of ξ . We have $\phi_\xi(0) = p_0$ and $\phi_\xi(1) = 1$. We can recover p_k via

$$p_k = \frac{\phi^{(k)}(0)}{k!}.$$

Then

$$\mathbb{E}[X] = \phi'(1) = \sum_{k=1}^{\infty} k p_k.$$

In fact, one can check that

$$\begin{aligned}\phi''(1) &= \mathbb{E}[X(X-1)] \\ \phi^{(k)}(1) &= \mathbb{E}[X(X-1)\cdots(X-k+1)].\end{aligned}$$

Suppose ξ_1, \dots, ξ_n i.i.d has generating function ϕ . Then $Z = \sum_{i=1}^n \xi_i$ has the following generating function:

$$\phi_Z(s) = \mathbb{E}[s^Z] = \mathbb{E}[s^{\sum_{i=1}^n \xi_i}] = \prod_{i=1}^n \mathbb{E}[s^{\xi_i}] = \phi^n(s).$$

But if instead we have $Z = \sum_{i=1}^N \xi_i$ where N is a random variable and N has generating function g_N . Then the generating function would be

$$\begin{aligned}\mathbb{E}[s^{\sum_{i=1}^N \xi_i}] &= \sum_{n=0}^{\infty} \mathbb{P}(N=n) \phi^n(s) \\ &= g_N(\phi(s)).\end{aligned}$$

Now suppose $\phi_n(s)$ is the generating function of X_n defined by

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

Then applying the result from above, we have

$$\phi_{n+1}(s) = \phi_n(\phi(s)) = \phi^{(n+1)}(s).$$

3.5 Long-run Behavior (Equilibrium)

Suppose $(X_n)_{n=0}^{\infty}$ is a Markov Chain on $\{1, \dots, N\}$.

Definition 3.5.1 (Regular). $(X_n)_{n=0}^{\infty}$ is *regular* if $\exists m \geq 1$ such that P^m has all positive entries.

If $(X_n)_{n=0}^{\infty}$ is regular, there exists a limiting distribution $\hat{\pi} = (\pi_1, \dots, \pi_N)$, where $\pi_i > 0$ and $\sum_{i=1}^N \pi_i = 1$ such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n = j \mid X_0 = i) = \pi_j, \quad \forall i, j \in \{1, \dots, N\}.$$