

# Math 185 Notes

## Complex Analysis

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# Contents

<b>1</b>	<b>Complex Numbers</b>	<b>3</b>
1.1	Intro . . . . .	3
<b>2</b>	<b>Complex Differentiation</b>	<b>6</b>
2.1	Derivatives . . . . .	6
2.1.1	Continuity . . . . .	6
2.1.2	Properties of Limits . . . . .	8
2.2	Derivatives (Cont'd) . . . . .	8
2.2.1	Properties of complex-derivatives . . . . .	9
<b>3</b>	<b>Holomorphic Functions and Cauchy-Riemann Equations</b>	<b>11</b>
3.1	Holomorphic Functions . . . . .	11
3.2	The Cauchy-Riemann Equations . . . . .	12
<b>4</b>	<b>Möbius Transformation</b>	<b>16</b>
4.1	Inverse of Möbius transformation . . . . .	17
<b>5</b>	<b>Exponential, Trigonometric, and Logarithmic Functions</b>	<b>20</b>
5.1	Exponential Functions . . . . .	20
5.2	Trigonometric Functions . . . . .	20
5.3	Logarithmic Functions . . . . .	21
<b>6</b>	<b>Complex Integration</b>	<b>22</b>
6.1	Definition and Basic Properties . . . . .	22
<b>7</b>	<b>Complex Integration (Cont'd)</b>	<b>25</b>
7.1	Basic Properties . . . . .	25
7.1.1	Antiderivatives . . . . .	26
7.2	Cauchy's Theorem . . . . .	29
7.2.1	Cauchy Integral . . . . .	31
7.3	Fundamental Theorem of Algebra . . . . .	39

# Lecture 1

## Complex Numbers

### 1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for  $|x| < r$ , where  $r$  is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n| |x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

**Example 1.1.1.** The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for  $|x| < 1$ .

**Question.** Now what if we replace the real variable  $x$  by the complex variable  $z$ ?

**Answer.** If  $|z| < r$ , then

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

so the sum converges absolutely for  $z \in D(0, r)$  (disc of radius  $r$  centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

**Example 1.1.2.** Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function  $\mathbb{R} \rightarrow \mathbb{R}$ ,  $f(z)$  is infinitely differentiable at  $z = 0$ , and all derivatives of  $f(z)$  are zero at  $z = 0$ . Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from  $f(z)$ !

**Example 1.1.3.** Consider the same example as above, but with  $z$  as a complex number. Let  $z = it$  where  $t \in \mathbb{R}$ . Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at  $z = 0$  and thus not complex-differentiable at  $z = 0$ .

**Example 1.1.4.** Now let's set  $z = x + iy$  where  $x, y \in \mathbb{R}$ . Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  (instead of  $\mathbb{C} \rightarrow \mathbb{C}$ ). Let's differentiate with respect to  $x$ :

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to  $y$ :

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = i f'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i \frac{\partial f'(z)}{\partial y} = i \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of)  $f(z)$  satisfy the two-dimensional *Laplace equation*.

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

**Example 1.1.5.** Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful tool for computing integrals.

## Lecture 2

# Complex Differentiation

### 2.1 Derivatives

**Definition 2.1.1** (Derivative). The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\lim_{x \rightarrow a} f(x) = L$$

means for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x) - L| < \epsilon$$

whenever  $0 < |x - a| < \delta$ . (For any "tolerance"  $\epsilon$ , we can guarantee  $f(x)$  is within  $\epsilon$  of  $L$  by forcing  $x$  to be close enough to  $a$ .)

**Remark.** Note that  $x = a$  doesn't satisfy  $0 < |x - a|$ , so the value of  $f$  at  $x = a$  has no bearing on whether  $\lim_{x \rightarrow a} f(x)$  exists.

#### 2.1.1 Continuity

**Definition 2.1.2** (Continuous). If  $\lim_{x \rightarrow a} f(x) = f(a)$ , then we say  $f$  is *continuous* at  $a$ .

**Remark.** Setting  $L = f(a)$  in the limit,  $0 < |x - a| < \delta$  implies  $|f(x) - f(a)| < \epsilon$  (even when  $x = a$ ) when talking about continuity, we leave out the  $0 < |x - a|$  part for convenience because  $x - a = 0$  automatically works.

Now let's consider a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\lim_{z \rightarrow a} f(z) = L$  means for every  $\epsilon > 0$ , there is  $\delta > 0$  such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

**Remark.** Now the  $z$ 's that we worry about form an open disc with radius  $\delta$  instead of an interval from the real case.

Similarly, if  $\lim_{z \rightarrow a} f(z) = f(a)$ , we say  $f$  is *continuous* at  $z = a$ .

**Example 2.1.3.**  $f(z) = z$  is continuous at any point  $a \in \mathbb{C}$ .

*Proof.* For  $\epsilon > 0$ , let  $\delta = \epsilon$ , then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

**Example 2.1.4.**  $\lim_{z \rightarrow 0} \bar{z}/z$  (although this is undefined at  $z = 0$ , this has no bearing on whether the limit exists).

*Proof.* Suppose  $\lim_{z \rightarrow 0} \bar{z}/z = L$  for some  $L$ . Let's take  $\epsilon = 1$ . There is a  $\delta > 0$  such that

$$0 < |z - 0| < \delta \implies \left| \frac{\bar{z}}{z} - L \right| < \epsilon = 1.$$

Let  $z = \delta/2$  and so does  $z = i\delta/2$ . Then for  $z = \delta/2$ :

$$\frac{\bar{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for  $z = i\delta/2$ :

$$\frac{\bar{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the  $L$  must lie in the intersection of the two open unit discs centered at  $-1$  and  $1$ . However, since they are open discs, these two discs do not overlap and so  $L$  does not exist. □

**Remark.** This implies that there is no way to extend  $\bar{z}/z$  to a continuous function at  $z = 0$ .

### 2.1.2 Properties of Limits

If  $\lim_{x \rightarrow a} f(x) = L_1$ ,  $\lim_{x \rightarrow a} g(x) = L_2$ , then

(i)

$$\lim_{x \rightarrow a} (f(x) + g(x)) = L_1 + L_2.$$

(ii)

$$\lim_{x \rightarrow a} f(x)g(x) = L_1L_2.$$

(iii)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

**Remark.** These implies that the sum/product/quotient of continuous functions are continuous.

**Proposition 2.1.5** (Composite function of continuous functions is continuous). *If  $f(x)$  is continuous at  $x = a$ , and  $g(x)$  is continuous at  $x = f(a)$ , then  $g(f(x))$  is continuous at  $x = a$ .*

*Proof.* We want  $|g(f(x)) - g(f(a))| < \epsilon$ . By continuity of  $g$  at  $x = f(a)$ , there exists  $\delta_1 > 0$  such that

$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$

We want to take  $w = f(x)$ , so we need

$$|f(x) - f(a)| < \delta_1.$$

But by continuity of  $f$  at  $x = a$ , we know that  $\delta_1$  will be our  $\epsilon$  when  $|x - a| < \delta_2$  for some  $\delta_2 > 0$ . Then for such  $x$ ,

$$|g(f(x)) - g(f(a))| < \epsilon.$$

□

## 2.2 Derivatives (Cont'd)

**Definition 2.2.1** (Differentiable). We say that  $f(z)$  is differentiable at  $z = a$  iff  $\frac{f(z)-f(a)}{z-a}$  extends to a continuous function at  $z = a$  (the value there is  $f'(a)$ ).

**Example 2.2.2.**  $f(z) = z$  is differentiable with  $f'(z) = 1$ .



*Proof.*

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z+h-z}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

□

**Example 2.2.3** (Interesting one).  $f(z) = \bar{z}$  is not differentiable but is continuous.

*Proof.*

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{h}}{h} \\ &= \text{DNE} \quad (\text{proved in previous example}) \end{aligned}$$

□

**Proposition 2.2.4** (Differentiability implies continuity).  $f(z)$  differentiable at  $z = a$  implies that  $f(z)$  is continuous at  $z = a$ .

*Proof.* We want to show that  $\lim_{z \rightarrow a} f(z) = f(a)$ .

$$\begin{aligned} \lim_{z \rightarrow a} f(z) - f(a) &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot (z - a) \\ &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot \lim_{z \rightarrow a} (z - a) \quad (\text{assume both limits exist}) \\ &= f'(a) \cdot 0 \\ &= 0. \end{aligned}$$

□

**Remark.** This is a common technique to show continuity by showing the limit of the difference is zero.

### 2.2.1 Properties of complex-derivatives

(i)

$$\frac{d}{dz} cf(z) = cf'(z), \quad \forall c \in \mathbb{C}.$$

(ii)

$$\frac{d}{dz}(f + g) = f'(z) + g'(z).$$

(iii)

$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)

$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)

$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

**Proposition 2.2.5** (Power rule).

$$\frac{d}{dz}z^n = nz^{n-1}$$

for all integers  $n$ .*Proof.* We induct on  $n$ . For  $n \geq 0$ , when  $n = 0$ ,

$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$

By the product rule,

$$\begin{aligned} \frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\ &= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \quad (\text{inductive hypothesis}) \\ &= nz^{n-1}. \end{aligned}$$

For  $n < 0$ , simply apply quotient rule. □

## Lecture 3

# Holomorphic Functions and Cauchy-Riemann Equations

Recall

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

Being differentiable at a point says little about how "nice" a function is.

**Example 3.0.1.** Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This is nowhere continuous, and thus nowhere differentiable. But if we consider  $x^2 f(x)$ , it is differentiable at  $x = 0$ :

$$\lim_{h \rightarrow 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h \rightarrow 0} h f(h) = 0.$$

Nevertheless, it's still not a very "nice" function.

### 3.1 Holomorphic Functions

**Definition 3.1.1** (Holomorphic). A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is *holomorphic* at a point  $a$  if it is differentiable at  $z$  for all  $z$  within distance  $r$  of  $a$  for some  $r > 0$ . In other words,  $f(z)$  is differentiable everywhere sufficiently close to  $a$ .

**Definition 3.1.2** (Open/closed disk). The *open disk* of radius  $r$  centered at  $a \in \mathbb{C}$  is

$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The *closed disk* is

$$\overline{D}(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Thus, we can say  $f(z)$  is holomorphic at  $a \in \mathbb{C}$  if  $f(z)$  is differentiable on an open disk centered at  $a$ . (if the point is not specified, it means that  $f$  is holomorphic everywhere.)

**Example 3.1.3** (Polynomials are holomorphic). We saw last time that  $z^n$  is differentiable everywhere for  $n \geq 0$ . Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

**Example 3.1.4.**  $f(z) = |z|^2 = z\bar{z}$  is differentiable at zero.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h\bar{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} \bar{h} \\ &= 0. \end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus,  $f$  is not holomorphic.

## 3.2 The Cauchy-Riemann Equations

**Question.** How to tell if a function is complex-differentiable?

**Answer.** We'll reduce this to a question about real derivatives.

Let  $x + iy$ , where  $x, y \in \mathbb{R}$ . If  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,

$$\begin{aligned} \frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h}. \end{aligned}$$

Note that  $h$  is real. Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}. \end{aligned}$$

**Example 3.2.1.**  $f(z) = z^2$ . Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}
\frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2ihy}{h} \\
&= \lim_{h \rightarrow 0} 2x + h + 2iy \\
&= 2x + 2iy \\
&= 2z \\
&= f'(z).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{x^2 - (y+h)^2 + 2ix(y+h) - (x^2 - y^2 + 2ixy)}{h} \\
&= \lim_{h \rightarrow 0} \frac{-2yh - h^2 + 2ixh}{h} \\
&= \lim_{h \rightarrow 0} -2y - h + 2ix \\
&= -2y + 2ix \\
&= 2i(x + iy) \\
&= if'(z).
\end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

**Theorem 3.2.2.**

(i) If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is complex-differentiable, then  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist and they satisfy

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

(ii) If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a function and  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  exist and are continuous on some open disk centered at  $z$ , and if

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

then  $f$  is complex-differentiable at  $z$ .

*Proof.*

(i) Since  $f$  is complex-differentiable, we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose  $h$  is real. Then

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since  $h$  is real, we get  $\frac{\partial f}{\partial x}$  and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose  $h$  is purely imaginary:  $h = ik$  for  $k \in \mathbb{R}$ . Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then  $h \rightarrow 0$  is equivalent to  $k \rightarrow 0$  since  $|h| = |k|$ . Thus we have

$$\lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

□

Let  $f(z) = u(z) + iv(z)$ . If we choose real values for  $h$ , then the imaginary part  $y$  is kept constant, and the derivative becomes a partial derivative with respect to  $x$ . Thus we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values  $ik$  for  $h$ , we obtain

$$f'(z) = \lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since we have

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

this resolves into the following equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or simply

$$u_x = v_y, \quad v_x = -u_y.$$

These are known as the *Cauchy-Riemann* equations.

**Example 3.2.3.** Consider  $f(z) = z^2$ . Then

$$f(x + iy) = x^2 + y^2 + 2ixy.$$

Here  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy$ . We have

$$u_x = 2x = v_y \quad v_x = 2y = -u_y.$$

**Example 3.2.4.** Consider  $f(z) = |z|^2$ . Then  $f(x + iy) = x^2 + y^2$  where  $u(x, y) = x^2 + y^2$  and  $v(x, y) = 0$ . But here we have

$$u_x = 2x \neq v_y = 0 \quad v_x = 0 \neq -u_y = -2y.$$

Thus, the Cauchy-Riemann equations only hold at  $(x, y) = (0, 0)$  and as we saw previously that this function is only differentiable at  $z = 0$  and nowhere else.

We can generalize the Cauchy-Riemann equations further to second order. Suppose we have  $f = u + iv$ . Then

$$u_{xx} = \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x} v_y = \frac{\partial}{\partial x} \frac{\partial}{\partial y} v = \frac{\partial}{\partial y} \frac{\partial}{\partial x} v = \frac{\partial}{\partial y} v_x = \frac{\partial}{\partial y} (-u_y) = -u_{yy}.$$

Thus, we have

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we also have

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

which gives

$$v_{xx} + v_{yy} = 0, \text{ or } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These are the *Laplace's equations* in 2D we saw earlier.

For  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we know that  $f'(x) = 0$  implies that  $f$  is constant. But for  $f : \mathbb{C} \rightarrow \mathbb{C}$ , we can use the Cauchy-Riemann equations. Since  $f'(z) = \frac{\partial f}{\partial x}$ ,

$$f'(z) = 0 \implies u_x + iv_x = 0 \implies u_x = 0, v_x = 0.$$

By Cauchy-Riemann, we also have  $u_y = v_y = 0$ . Since  $u_x = 0$ , we know that for fixed  $y$ ,  $u(x, y)$  is some constant that could depend on  $y$ . Thus, we have

$$u(x, y) = g(y).$$

But  $u_y = 0$ , so  $g'(y) = 0$ , which means  $g$  is actually a constant independent of  $y$ . Thus,  $u$  is globally constant. Similar argument applies to  $v$  as well.

## Lecture 4

# Möbius Transformation

**Definition 4.0.1** (Möbius transformation). A *Möbius transformation* is a function of the form

$$f(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d \in \mathbb{C}$  satisfy  $ad - bc \neq 0$ .

**Remark.** If  $ad = bc$ , then  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ , so rows are linearly dependent:  $\lambda(a, b) + \mu(c, d) = (0, 0)$ , which implies that

$$a = \frac{-\mu}{\lambda}c \quad b = \frac{-\mu}{\lambda}d.$$

Then

$$\begin{aligned} f(z) &= \frac{az + b}{cz + d} \\ &= \frac{-\frac{\mu}{\lambda}(cz + d)}{cz + d} \\ &= -\frac{\mu}{\lambda}, \end{aligned}$$

which is a constant independent of  $z$ .

**Proposition 4.0.2** (Composite Möbius transforms is Möbius ). *If  $f_1(z), f_2(z)$  are Möbius transforms, then  $f_1(f_2(z))$  is also a Möbius transform.*

*Proof.* Suppose

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1} \quad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}.$$



Then

$$\begin{aligned} f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)}, \end{aligned}$$

which is another Möbius transform. □

**Remark.** Note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and the entries coincide with the composite Möbius transform. If we denote  $f_M(z)$  to be a transform associated with a  $2 \times 2$  matrix  $M$ , then we have just shown that

$$f_M(f_N(z)) = f_{MN}(z).$$

**Remark.** Since  $f_I(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$ , the inverse of  $f_M$  is  $f_{M^{-1}}$ .

**Remark.** Note that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies f_M = \frac{az + b}{cz + d}.$$

Meanwhile,

$$\lambda M = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \implies f_{\lambda M} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az + b}{cz + d} = f_M.$$

Thus, scaling the matrices doesn't affect the resulting Möbius transformation.

## 4.1 Inverse of Möbius transformation

Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the scaling part is redundant, we simply ignore it and obtain the inverse Möbius transform as follows:

$$f(z) = \frac{az + b}{cz + d} \implies f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

**Remark.** Since Möbius transforms have inverses, they should be bijections. However, some details should be noted. If  $c \neq 0$ , then  $\frac{az+b}{cz+d}$  is undefined at  $z = -\frac{d}{c}$ .

Let's consider the value at  $z = -\frac{d}{c}$  to be infinity. It turns out that we can evaluate  $\frac{az+b}{cz+d}$  at  $\infty$ :

$$\begin{aligned}\lim_{z \rightarrow \infty} \frac{az+b}{cz+d} &= \lim_{z \rightarrow \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \\ &= \frac{a}{c}.\end{aligned}$$

When  $c = 0$ , we view  $\frac{a}{c}$  as  $\infty$ . So now we view Möbius transformations as functions from  $\mathbb{C} \cup \{\infty\}$  to  $\mathbb{C} \cup \{\infty\}$ . This makes all Möbius transformations into bijections. Here, we call  $\mathbb{C} \cup \{\infty\}$  the *extended complex plane* (also called *Riemann sphere*).

**Remark.** For real functions, there are multiple notions of going to infinity:  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ . But for complex functions, we work with only one infinite point.

**Fact.** If we apply a Möbius transformation to a line or a circle in the complex plane, we would get a line or a circle again (circles can turn into lines and vice versa).

**Example 4.1.1.** Consider  $f(z) = \frac{z-1}{iz+i}$ , let's apply this to the unit circle, i.e. take  $z = e^{i\theta}$ . Then

$$\begin{aligned}f(e^{i\theta}) &= \frac{e^{i\theta} - 1}{i(e^{i\theta} + 1)} = \frac{\cos \theta - 1 + i \sin \theta}{i(\cos \theta + 1 + i \sin \theta)} \\ &= \frac{-2 \sin^2 \frac{\theta}{2} + i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{i(2 \cos^2 \frac{\theta}{2} + i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})} \\ &= \frac{2i(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \sin \frac{\theta}{2}}{2i(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \cos \frac{\theta}{2}} \\ &= \tan \frac{\theta}{2}.\end{aligned}$$

Note that  $\theta \in (-\pi, \pi)$  and we have  $\tan -\frac{\pi}{2} = -\infty$  and  $\tan \frac{\pi}{2} = +\infty$ . We have mapped a unit circle to a line (real line).

**Fact.**  $f$  sends the interior of the unit disk to the interior of the upper half-plane. If  $g(z)$  is holomorphic on the upper half-plane, then  $g(f(z))$  is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equation.

**Remark.** Stereographic projection  $\varphi$  is a bijection that maps a sphere to the extended complex plane. It doesn't preserve distance, but it preserves functions being holomorphic.

**Proposition 4.1.2.** Suppose  $f(z) = \frac{az+b}{cz+d}$  is a Möbius transformation. If  $c = 0$  then

$$f(z) = \frac{a}{d}z + \frac{b}{d},$$

and if  $c \neq 0$ , then

$$f(z) = \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c}.$$

In particular, every Möbius transformation is a composition of translations, dilations, and inversions.

*Proof.* Simplify. □

**Theorem 4.1.3.** *Möbius transformations map circles and lines into circles and lines.*

*Proof.* Translations and dilations certainly map circles and lines into circles and lines, so by the previous proposition, we only have to prove the statement of the theorem for the inversion  $f(z) = \frac{1}{z}$ .

The equation for a circle centered at  $x_0 + iy_0$  with radius  $r$  is  $(x - x_0)^2 + (y - y_0)^2 = r^2$ , which we can transform to

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

for some real numbers  $\alpha, \beta, \gamma$ , and  $\delta$  that satisfy  $\beta^2 + \gamma^2 > 4\alpha\delta$ . The above expression is more convenient for us, because it includes the possibility that the equation describes a line (precisely when  $\alpha = 0$ ).

Suppose  $z = x + iy$  satisfies the above expression; we need to prove that  $u + iv := \frac{1}{z}$  satisfies a similar equation. Since

$$u + iv = \frac{x - iy}{x^2 + y^2},$$

we can rewrite the transformed equation as

$$\begin{aligned} 0 &= \alpha + \beta \frac{x}{x^2 + y^2} + \gamma \frac{y}{x^2 + y^2} + \frac{\delta}{x^2 + y^2} \\ &= \alpha + \beta u - \gamma v + \delta(u^2 + v^2). \end{aligned}$$

But this equation says that  $u + iv$  lies on a circle or line. □

**Fact.** The stereographic projection of a circle on the sphere (intersection of a plane and a sphere) is a circle in the plane. Möbius transformations take circles on the sphere to other circles of the sphere (some of these stereographically project to lines in the plane).

## Lecture 5

# Exponential, Trigonometric, and Logarithmic Functions

### 5.1 Exponential Functions

$$e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y \implies e^z = u(x, y) + i v(x, y)$$

where

$$\begin{aligned} u(x, y) &= e^x \cos y \\ v(x, y) &= e^x \sin y \end{aligned}$$

### 5.2 Trigonometric Functions

For  $z \in \mathbb{C}$ ,

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2}. \end{aligned}$$

**Remark.**  $\sin z, \cos z$  are holomorphic since  $e^z$  is holomorphic and so is  $e^{iz}$  and  $e^{-iz}$ .

Trigonometric identities hold for complex numbers.

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(2z) = 2 \sin z \cos z$$

### 5.3 Logarithmic Functions

We want  $\log z$  to be the unique inverse to the exponential function, i.e. we want  $e^{\log z} = z$ , but then we would also have

$$e^{\log z + 2\pi i k} = z.$$

**Definition 5.3.1** (Principal logarithm). The *principal logarithm* is the function defined by

$$\log(re^{i\theta}) = \log r + i\theta.$$

where  $-\pi < \theta \leq \pi$ .

Let's check if  $\log z$  is differentiable. If

$$z = x + iy = re^{i\theta},$$

then  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$  when  $x \neq 0$ .

$$\begin{aligned} \log x + iy &= \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right) \\ &= u(x, y) + iv(x, y). \end{aligned}$$

Then

$$\begin{aligned} u_x &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} \\ u_y &= \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2} \\ v_x &= \frac{-y}{x^2} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = -\frac{y}{x^2 + y^2} \\ v_y &= \frac{1}{x} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Thus, we see that the Cauchy-Riemann equations hold for logarithms.

## Lecture 6

# Complex Integration

### 6.1 Definition and Basic Properties

If  $f : \mathbb{R} \rightarrow \mathbb{C}$ , define

$$\int_a^b f(x)dx = \int_a^b \Re f(x)dx + i \int_a^b \Im f(x)dx.$$

**Question.** But how to integrate a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ?

For real functions, going from a point  $\gamma(a)$  to  $\gamma(b)$  can only happen one way (follow the real axis) but in  $\mathbb{C}$ , we will have to specify the path from  $\gamma(a)$  to  $\gamma(b)$ .

**Definition 6.1.1** (Path/curve). A *path/curve* is the image of a function  $\gamma : [a, b] \rightarrow \mathbb{C}$ .

**Definition 6.1.2** (Integral). The *integral* of the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  along the path parametrized by  $\gamma : [a, b] \rightarrow \mathbb{C}$  is

$$\int_a^b f(\gamma(t))\gamma'(t)dt.$$

This is the integral of a function  $\mathbb{R} \rightarrow \mathbb{C}$ , so we already have a definition for it.

Aside,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx.$$

Suppose we have a different parametrization of the image of  $\gamma(t)$ . Write this parametrization as  $\gamma(\theta(t))$  where  $\theta : [a, b] \rightarrow [a, b]$  is a continuous reparametrization of the interval  $[a, b]$  satisfying  $\theta(a) = a$ ,  $\theta(b) = b$  and  $\theta$  is increasing. Then

$$\int_a^b f(\gamma(\theta(t)))\gamma'(\theta(t))\theta'(t)dt = \int_{\theta(a)}^{\theta(b)} f(\gamma(u))\gamma'(u)du$$

where  $u = \theta(t)$  and  $du = \theta'(t)dt$ . So the integrals for  $\gamma(\theta(t))$  and  $\gamma(t)$  are the same, thus the integral depends on the curve in  $\mathbb{C}$ , not how we parametrize it.

We will use

$$\int_{\gamma} f(z)dz$$

to denote the integral.

**Example 6.1.3.** If  $\gamma(t) = t$ , then  $\gamma'(t) = 1$  and

$$\int_{\gamma} f(z)dz = \int_a^b f(t)dt.$$

**Example 6.1.4.** If  $\gamma(t) = t + it^2$  and  $f(z) = 1$ , then  $\gamma'(t) = 1 + 2it$  and

$$\begin{aligned} \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b (1 + 2it)dt \\ &= \int_a^b 1dt + i \int_a^b 2tdt \\ &= b - a + i(b^2 - a^2) \\ &= (b + ib^2) - (a + ia^2) \\ &= \gamma(b) - \gamma(a). \end{aligned}$$

**Example 6.1.5** (Very important example). Consider  $\gamma(t) = e^{it}$  where  $0 \leq t \leq 2\pi$ . So  $\gamma(t)$  is the counterclockwise unit circular path. If  $f(z) = z^n$  for some  $n \in \mathbb{Z}$ . Then

$$\begin{aligned} \int_{\gamma} f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\ &= \int_0^{2\pi} e^{int} \cdot ie^{it}dt \\ &= i \int_0^{2\pi} e^{it(n+1)}dt. \end{aligned}$$

If  $n \neq -1$ , then  $n + 1 \neq 0$ , the integral evaluates to

$$\begin{aligned} i \left. \frac{e^{it(n+1)}}{i(n+1)} \right|_{t=0}^{2\pi} &= i \left( \frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right) \\ &= 0. \end{aligned}$$

If  $n + 1 = 0$ , then  $n = -1$  and so

$$\begin{aligned} i \int_0^{2\pi} e^{it(n+1)} dt &= i \int_0^{2\pi} 1 dt \\ &= 2\pi i, \end{aligned}$$

which is not zero.



## Lecture 7

# Complex Integration (Cont'd)

### 7.1 Basic Properties

(i) If  $\mu, \lambda \in \mathbb{C}$ , then

$$\begin{aligned}\int_{\gamma} \lambda f(z) + \mu g(z) dz &= \int_a^b (\lambda f(\gamma(t)) + \mu g(\gamma(t))) \gamma'(t) dt \\ &= \lambda \int_a^b f(\gamma(t)) \gamma'(t) dt + \mu \int_a^b g(\gamma(t)) \gamma'(t) dt \\ &= \lambda \int_{\gamma} f(z) dz + \mu \int_{\gamma} g(z) dz.\end{aligned}$$

(ii)

$$\int_{\gamma_1 \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

(iii)

$$\int_{-\gamma} f = - \int_{\gamma} f$$

(iv)

$$\left| \int_{\gamma} f \right| \leq \max_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma),$$

where

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

View  $|\gamma'(t)|$  as the speed a particle is travelling at and  $\gamma(t)$  as the position of that particle at time  $t$ . Then integrating it gives the total distance.

(v) **(Triangle Inequality)**

$$\left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| dt$$

(vi) **(ML-Lemma)**

$$\begin{aligned} \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \\ &= ML \end{aligned}$$

where  $M = \max_{a \leq t \leq b} |f(\gamma(t))|$  and  $L = \int_a^b |\gamma'(t)| dt$ .

### 7.1.1 Antiderivatives

**Theorem 7.1.1** (Fundamental Theorem of Calculus). *If  $F$  is holomorphic on some subset  $G \subseteq \mathbb{C}$  and  $\frac{d}{dz}F(z) = f(z)$ . Then*

$$\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

*Proof.* Let  $F(x + iy) = u(x, y) + iv(x, y)$  and  $\gamma(t) = \alpha(t) + i\beta(t)$ . Then

$$F(\gamma(t)) = u(\alpha(t), \beta(t)) + iv(\alpha(t), \beta(t)).$$

By chain rule,

$$\begin{aligned} \frac{d}{dt}F(\gamma(t)) &= u_x\alpha'(t) + u_y\beta'(t) + iv_x\alpha'(t) + iv_y\beta'(t) \\ &= u_x(\alpha'(t) + i\beta'(t)) + iv_x(\alpha'(t) + i\beta'(t)) \quad (u_x = v_y \text{ by CR}) \\ &= F(\gamma(t))\gamma'(t). \end{aligned}$$

□

**Definition 7.1.2** (Closed curve). A *closed curve* is a curve where the start and end points are the same, i.e.  $\gamma(a) = \gamma(b)$ .

So if  $f(z) = \frac{d}{dz}F(z)$ , the integral of  $f(z)$  around a closed curve is zero:

$$\int_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

**Example 7.1.3.** Let  $\gamma$  be the path of unit circle counterclockwise. Then

$$\int_{\gamma} \frac{1}{z} dz = 2\pi i$$

which is not zero, implying that there is no holomorphic function  $F(z)$  defined on the whole unit circle, having derivative  $\frac{1}{z}$ .

However, consider the principal logarithm  $\log(re^{i\theta}) = \log(r) + i\theta$ , we have

$$\frac{d}{dz} \log(x + iy) = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

So  $\frac{1}{z}$  does have an antiderivative on  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ .

It turns out that if  $f(z)$  is continuous and  $\int_{\gamma} f(z) dz = 0$  for any closed curve, then  $f(z)$  has an antiderivative, i.e. there's  $F(z)$  such that  $F'(z) = f(z)$ .

We know that by fundamental theorem of calculus

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

By analogy, we want

$$F(w) = \int_{\gamma} f(z) dz$$

where  $\gamma$  is a curve from a fixed basepoint  $q$  to  $w$ .

First let's check that this doesn't depend on the choice of path from  $q$  to  $w$ . Suppose  $\delta_1, \delta_2$  are two paths from  $q$  to  $w$ . Observe that the reverse of  $\delta_2$  is a curve from  $w$  to  $q$ , and the path obtained by following  $\delta_1$ , then the reverse of  $\delta_2$  goes from  $q$  to  $w$  to  $q$ , so it is a closed curve.

Write  $\delta_1 - \delta_2$  for the closed curve above. Then by assumption, we have

$$\int_{\delta_1 - \delta_2} f(z) dz = 0.$$

This implies that

$$\int_{\delta_1} f(z) dz + \int_{-\delta_2} f(z) dz = \int_{\delta_1} f(z) dz - \int_{\delta_2} f(z) dz = 0.$$

Hence,

$$\int_{\delta_1} f(z) dz = \int_{\delta_2} f(z) dz.$$

Thus, the choice of path doesn't matter and so the formula  $F(w) = \int_{\gamma} f(z)dz$  where  $\gamma$  is any path from  $q$  to  $w$  makes sense. Let's now check  $\frac{d}{dw}F(w) = f(w)$ .

$$\frac{d}{dw}F(w) = \lim_{h \rightarrow 0} \frac{F(w+h) - F(w)}{h}.$$

To evaluate  $F(w+h)$ , we can choose the path of integration from  $q$  to  $w+h$  arbitrarily. Let's choose one that goes from  $q$  to  $w$  then to  $w+h$  along a line segment (only from  $w$  to  $w+h$ ). This line segment has length  $|h|$ .

If our function is holomorphic at a point  $w$ , it is differentiable on a disk  $D(w, \epsilon)$  for some  $\epsilon > 0$ . So if  $|h| < \epsilon$ , then the line segment  $\ell$  from  $w$  to  $w+h$  is contained in  $D(w, \epsilon)$  and hence in a region where the function is differentiable.

Now  $F(w+h) - F(w)$  is simply the integral of  $f(z)$  from  $w$  to  $w+h$ . We want

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\ell} f(z)dz - f(w) = 0.$$

Note that  $f(w)$  is a constant independent of  $z$ . Thus,

$$\int_{\ell} f(w)dz = f(w) \int_{\ell} 1dz = f(w)h.$$

$$\lim_{h \rightarrow 0} \frac{\int_{\ell} f(z)dz - \int_{\ell} f(w)dz}{h} = \lim_{h \rightarrow 0} \frac{\int_{\ell} (f(z) - f(w))dz}{h}.$$

By ML-lemma,

$$\begin{aligned} \left| \frac{\int_{\ell} (f(z) - f(w))dz}{h} \right| &= \frac{|\int_{\ell} (f(z) - f(w))dz|}{|h|} \\ &\leq \max_{z \in \ell} |f(z) - f(w)| \cdot \frac{\text{length}(\ell)}{|h|} \\ &= \max_{z \in \ell} |f(z) - f(w)|. \end{aligned}$$

So it suffices to show that

$$\lim_{h \rightarrow 0} \max_{z \in \ell} |f(z) - f(w)| = 0.$$

Since  $f(z)$  is continuous at  $w$ , for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|z - w| < \delta$  implies  $|f(z) - f(w)| < \epsilon$ . So when  $|h| < \delta$ , any  $z \in \ell$  obeys  $|z - w| < \delta$ . Then also  $|f(z) - f(w)| < \epsilon$ . So for  $|h| < \delta$ ,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| < \epsilon.$$

Hence,

$$\lim_{h \rightarrow 0} \frac{F(w+h) - F(w)}{h} = f(w).$$

as needed.

## 7.2 Cauchy's Theorem

**Question.** How do we check that  $\int_{\gamma} f(z)dz = 0$  for any closed curve  $\gamma$ ?

**Theorem 7.2.1** (Cauchy's Theorem). *Suppose  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a closed curve and  $f(z)$  is holomorphic on  $\gamma$  and in the region enclosed by the curve  $\gamma$ . Then*

$$\int_{\gamma} f(z)dz = 0.$$

*Proof.* Recall that a vector field is a function  $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where

$$\vec{F}(x, y) = (F_1(x, y), F_2(x, y)).$$

The line integral of  $\vec{F}$  along a curve  $\gamma$  is

$$\int_{\gamma} \vec{F} \cdot d\vec{\ell} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

where  $\gamma(t) = (\alpha(t), \beta(t))$  and  $\gamma'(t) = (\alpha'(t), \beta'(t))$  and so

$$\int_{\gamma} \vec{F} \cdot \vec{\ell} = \int_a^b F_1(\alpha(t), \beta(t))\alpha'(t) + F_2(\alpha(t), \beta(t))\beta'(t) dt.$$

Now recall the Stokes' theorem

$$\int_{\gamma} \vec{F} \cdot d\vec{\ell} = \int_{\text{region enclosed by } \gamma} \vec{\nabla} \times \vec{F} dA$$

where

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Then

$$\int_{\gamma} (\vec{F} \cdot \hat{n}) d\ell = \int_a^b F_1(\alpha(t), \beta(t))\beta'(t) + F_2(\alpha(t), \beta(t))(-\alpha'(t)) dt.$$

The divergence theorem says that

$$\int_{\gamma} (\vec{F} \cdot \hat{n}) d\ell = \int_{\text{area enclosed by } \gamma} \vec{\nabla} \cdot \vec{F} dA$$

where

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

Let  $f(x + iy) = u(x, y) + iv(x, y)$  and  $\gamma(t) = \alpha(t) + i\beta(t)$ .

$$\begin{aligned} \int_{\gamma} f(z)dz &= \int_a^b (u + iv)(\alpha'(t) + i\beta'(t))dt \\ &= \int_a^b u\alpha'(t) - v\beta'(t)dt + i \int_a^b v\alpha'(t) + u\beta'(t)dt. \end{aligned}$$

Note that  $u' - v\beta' = (u, -v) \cdot (\alpha', \beta')$  and  $v\alpha' + u\beta' = (u, -v) \cdot (\beta', \alpha')$ . Now let  $\vec{F}(x, y) = (u(x, y), -v(x, y))$ . Then

$$\int_{\gamma} f(z)dz = \int_{\gamma} \vec{F} \cdot d\vec{\ell} + i \int_{\gamma} (\vec{F} \cdot \hat{n})d\ell.$$

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \frac{\partial -v}{\partial x} - \frac{\partial u}{\partial y} = -v_x - u_y = 0 \quad (u_y = -v_x \text{ by CR}), \\ \vec{\nabla} \cdot \vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial -v}{\partial y} = u_x - v_y = 0 \quad (u_x = v_y \text{ by CR}). \end{aligned}$$

So by Stokes' theorem and the divergence theorem

$$\int_{\gamma} f(z)dz = \int_{\text{area enclosed by } \gamma} 0dA + i \int_{\text{area enclosed by } \gamma} 0dA = 0 + i0 = 0.$$

□

**Remark.** The example of  $\int_{\gamma} \frac{1}{z}dz = 2\pi i$  does not contradict the Cauchy's theorem because  $\frac{1}{z}$  is not holomorphic at  $z = 0$ .

**Remark.** Checking that closed curves have well-defined interior regions is not a triviality: it is the content of the Jordan Curve Theorem (out of scope).

**Remark.** There are several formulations of Cauchy's Theorem. We will assume that  $f'(z)$  is continuous. Some formulations remove the concept of interior region and instead use the notion of a homotopy.

### 7.2.1 Cauchy Integral

**Theorem 7.2.2** (Cauchy integral formula (1st version)). *Suppose  $f(z)$  is holomorphic on the closed disk of radius  $R$  centered at  $a \in \mathbb{C}$ . Then*

$$\int_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

where  $\gamma$  is the anticlockwise circle of radius  $R$  centered at  $a$ .

*Proof.*  $\frac{f(z)}{z-a}$  may not be holomorphic at  $z = a$ , so to apply Cauchy's theorem we need a curve that doesn't enclose  $a$ . We can create such curve by traversing a donut-like path obtained by traversing a clockwise small circle with radius  $r$  centered at  $a$  after we reached the endpoint of the original curve and then traverse back to the endpoint. Then the enclosed area will not include  $a$ .

This curve of integration has 4 parts:

1.  $\gamma_1$  : big circle of radius  $R$ , anticlockwise,
2.  $\gamma_2$  : line segment connecting from the big circle to the small circle,
3.  $\gamma_3$  : small circle of radius  $r$ , clockwise,
4.  $\gamma_4$  : line segment connecting from the small circle to the big circle.

Note that the integrations of the two line segments cancel out each other. Then Cauchy's theorem tells us that

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0 \implies \int_{\gamma_1} + \int_{\gamma_3} = 0.$$

Thus,

$$\int_{\gamma_1} = - \int_{\gamma_3} = \int_{-\gamma_3},$$

which implies that the integral of the big circle anticlockwise is equal to the integral of the small circle anticlockwise. Hence,

$$\int_{\gamma_1} \frac{f(z)}{z-a} dz = \int_{-\gamma_3} \frac{f(z)}{z-a} dz.$$

for any  $r < R$ , and so we can take  $r \rightarrow 0$ . Now we want to show that

$$\int_{\gamma_3} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

Let  $\gamma(t) = a + re^{it}$  for  $0 \leq t \leq 2\pi$ . Then

$$\begin{aligned} \int_{\gamma} \frac{f(a)}{z-a} dz &= f(a) \int_{\gamma} \frac{1}{z-a} dz \\ &= f(a) \int_0^{2\pi} \frac{1}{\gamma(t)-a} \gamma'(t) dt \\ &= f(a) \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt \\ &= 2\pi i f(a). \end{aligned}$$

So we want

$$\int_{\gamma} \frac{f(z)}{z-a} dz = \int_{\gamma} \frac{f(a)}{z-a} dz,$$

i.e.,

$$\int_{\gamma} \frac{f(z) - f(a)}{z-a} dz = 0.$$

To show this, we apply the ML-lemma:

$$\left| \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| \leq \max_{z \in \gamma} \left| \frac{f(z) - f(a)}{z-a} \right| \cdot \text{length}(\gamma)$$

where  $\gamma$  is all points at distance  $r$  from  $a$ . Then  $z \in \gamma \implies |z-a| = r$  and  $\text{length}(\gamma) = 2\pi r$ . Thus, we get

$$\begin{aligned} \left| \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| &\leq \max_{z \in \gamma} \frac{|f(z) - f(a)|}{r} \cdot 2\pi r \\ &= 2\pi \cdot \max_{z \in \gamma} |f(z) - f(a)|. \end{aligned}$$

Since  $f(z)$  is differentiable, it is continuous. So for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|z-a| = r < \delta \implies |f(z) - f(a)| < \epsilon$ . So by taking  $r < \delta$ , we get

$$\left| \int_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| < 2\pi\epsilon$$

for any  $\epsilon > 0$ , which implies that the absolute value must be zero. Hence,

$$\int_{\gamma} \frac{f(z) - f(a)}{z-a} dz = 0,$$

and this implies that

$$\int_{\gamma} \frac{f(z)}{z-a} dz = \int_{\gamma} \frac{f(a)}{z-a} dz = 2\pi i f(a).$$

□



**Theorem 7.2.3** (Cauchy integral formula (2nd version)). *Let  $\gamma$  be a closed curve that encloses  $a \in \mathbb{C}$  exactly once anticlockwise. Suppose  $f(z)$  is holomorphic inside  $\gamma$ . Then*

$$\int_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

*Proof.* Similar proof to previous one. □

**Example 7.2.4.** Suppose  $\omega \geq 0$  is a real number. Then

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + 1} dx = \pi e^{-\omega}.$$

Consider

$$f(z) = \frac{e^{i\omega z}}{z^2 + 1} = \frac{e^{i\omega z}}{z+i} = \frac{g(z)}{z-i}$$

where  $g(z) = \frac{e^{i\omega z}}{z+i}$ . Consider the integral of  $f(z)$  over the semi-circle curve anticlockwise with radius  $R$  consisting of a line segment  $\ell$ . On the line segment, let  $\delta : [-R, R] \rightarrow \mathbb{C}$  be defined as  $\delta(t) = t$ . Then

$$\begin{aligned} \int_{\delta} \frac{g(z)}{z-i} dz &= \int_{-R}^R \frac{g(\delta(t))}{\delta(t)-i} \delta'(t) dt \\ &= \int_{-R}^R \frac{g(t)}{t-i} dt \\ &= \int_{-R}^R \frac{e^{i\omega x}}{x^2 + 1} dx \\ &= \int_{-R}^R \frac{\cos(\omega x)}{x^2 + 1} dx + i \int_{-R}^R \frac{\sin(\omega x)}{x^2 + 1} dx. \end{aligned}$$

Our goal is to compute the real part of the above expression. So what we want is

$$\lim_{R \rightarrow \infty} \int_{\ell} = \lim_{R \rightarrow \infty} \int_{\gamma} - \int_{\text{arc}}.$$

By Cauchy integral formula, we have

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{\gamma} \frac{g(z)}{z-i} dz &= 2\pi i g(i) \\ &= 2\pi i \frac{e^{i\omega i}}{i+i} \\ &= \pi e^{-\omega}. \end{aligned}$$

Now we compute

$$\lim_{R \rightarrow \infty} \int_{\text{arc}} \frac{e^{i\omega z}}{z^2 + 1} dz.$$

By ML-inequality, we have

$$\left| \int_{\text{arc}} \right| \leq \max_{z \in \text{arc}} \left| \frac{e^{i\omega z}}{z^2 + 1} \right| \cdot \text{length}(\text{arc}).$$

We needed to work with the upper half plane so that for  $z \in \text{arc}$ ,  $\Im(z) \geq 0$ , so then because  $\omega \geq 0$ ,  $\Im(\omega z) \geq 0$ . If  $\omega z = a + ib$ , then  $b \geq 0$  and  $i\omega z = -b + ia$  has non-positive real part. So

$$|e^{i\omega z}| = e^{\Re(i\omega z)} \leq e^0 = 1.$$

Also  $z \in \text{arc} \implies |z| = R$  and so  $|z^2| = R^2$ . Thus we have  $|z^2 + 1| \geq R^2 - 1$ . Hence,

$$\frac{1}{|z^2 + 1|} \leq \frac{1}{R^2 - 1}.$$

So ML-inequality becomes

$$\left| \int_{\text{arc}} \right| = \frac{1}{R^2 - 1} \pi R \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

So as  $R \rightarrow \infty$ ,

$$\int_{\text{arc}} \rightarrow 0.$$

Hence, we have

$$\begin{aligned} \int_{\gamma} - \int_{\text{arc}} &= \int_{\ell} \\ \pi e^{-\omega} - 0 &= \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 + 1} dx + i \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2 + 1} dx. \end{aligned}$$

Note that

$$\int_{\gamma} \frac{f(z)}{z - a} dz = \begin{cases} 2\pi i f(a) & \text{if } a \text{ is in the region enclosed by } \gamma \\ 0 & \text{else.} \end{cases}$$

So if  $\gamma : [a', b'] \rightarrow \mathbb{C}$  is the curve,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - a} dz \\ &= \frac{1}{2\pi i} \int_{a'}^{b'} \frac{f(\gamma(t))}{\gamma(t) - a} \gamma'(t) dt. \end{aligned}$$

So for example, if  $f(z)$  is zero on the unit circle:  $f(e^{i\theta}) = 0$ , then for  $|a| < 1$ ,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - a} i e^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} 0 d\theta \\ &= 0. \end{aligned}$$

So if  $f(z)$  is zero on the circle, it is also zero inside. For comparison,

$$f(x + iy) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

is real differentiable and is zero on the unit circle, but not inside, so it's not holomorphic.

**Claim.**

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

*Proof.*

$$\begin{aligned} 2\pi i f'(a) &= \lim_{h \rightarrow 0} 2\pi i \left( \frac{f(a+h) - f(a)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{\int_{\gamma} \frac{f(z)}{z-(a+h)} dz - \int_{\gamma} \frac{f(z)}{z-a} dz}{h}. \end{aligned}$$

We assumed that  $a$  is inside  $\gamma$ , but we should also check  $a+h$ . Let  $\gamma : [a', b'] \rightarrow \mathbb{C}$  be the curve. Consider  $|\gamma(t) - a|$  (distance from  $\gamma(t)$  to  $a$ ).

The domain of  $|\gamma(t) - a|$  is  $[a', b']$ , a compact set (closed and bounded by Heine-Borel). Continuous image of a compact set is also compact, so in particular closed and so the complement is open. Note that  $|\gamma(t) - a| \neq 0$  (otherwise  $\gamma(t) = a$ , which means that  $a$  would be on the curve  $\gamma$ , which we don't allow). Then 0 is in the complement of the image of  $|\gamma(t) - a|$ . But since the set is open, it must also contain a neighborhood of 0. We can assume it is of the form  $(-\epsilon, \epsilon)$ .

**Conclusion:** not only does  $\gamma(t)$  avoid  $a$ , it never comes within  $\epsilon$  of it:  $|\gamma(t) - a| \geq \epsilon$  (complement contains  $[0, \epsilon)$ ). If  $|h| < \epsilon$ , then also  $a+h$  is inside  $\gamma(t)$ .

So for  $h$  small enough, we can write

$$\begin{aligned}
 2\pi i \frac{f(a+h) - f(a)}{h} &= \frac{\int_{\gamma} \frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a} dz}{h} \\
 &= \frac{1}{h} \int_{\gamma} \frac{(z-a)f(z) - (z-a-h)f(z)}{(z-a)(z-a-h)} dz \\
 &= \frac{1}{h} \int_{\gamma} \frac{hf(z)}{(z-a)(z-a-h)} dz \\
 &= \int_{\gamma} \frac{f(z)}{(z-a)(z-a-h)} dz.
 \end{aligned}$$

We want

$$\lim_{h \rightarrow 0} \int_{\gamma} \frac{f(z)}{(z-a)(z-a-h)} dz = \int_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

or equivalently,

$$\lim_{h \rightarrow 0} \int_{\gamma} \left( \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right) dz = 0.$$

$$\begin{aligned}
 \lim_{h \rightarrow 0} \int_{\gamma} \left( \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right) dz &= \lim_{h \rightarrow 0} \int_{\gamma} \frac{f(z)(z-a) - f(z)(z-a-h)}{(z-a)^2(z-a-h)} dz \\
 &= \lim_{h \rightarrow 0} h \int_{\gamma} \frac{f(z)}{(z-a)^2(z-a-h)} dz.
 \end{aligned}$$

The ML-inequality says

$$\left| \int_{\gamma} \right| \leq \max_{z \in \gamma} \frac{|f(z)|}{|z-a|^2|z-a-h|} \cdot \text{length}(\gamma).$$

Notice that  $|z-a| \geq \epsilon$  so for  $|h| \leq \frac{\epsilon}{2}$ .

$$\begin{aligned}
 |z-a-h| &\geq |z-a| - |h| \\
 &\geq \epsilon - \frac{\epsilon}{2} \\
 &= \frac{\epsilon}{2}.
 \end{aligned}$$

Hence,

$$\left| \int_{\gamma} \right| \leq \max_{z \in \gamma} \frac{|f(z)|}{\epsilon^2 \cdot \frac{\epsilon}{2}} \cdot \text{length}(\gamma).$$

The bound is independent of  $h$ .

□

**Theorem 7.2.5** (Liouville's Theorem). *Suppose  $f(z)$  is holomorphic on all of  $\mathbb{C}$ , and  $f(z)$  is bounded, i.e.,  $|f(z)| \leq M$  for some fixed  $M$ . Then  $f(z)$  is constant.*

**Remark.** Note that  $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1} = 1 - \frac{2}{x^2+y^2+1}$  is real differentiable and bounded but it's not constant. So the theorem implies it's not holomorphic.

*Proof.* Let's compute  $f'(a)$  for some  $a \in \mathbb{C}$ . Let  $\gamma$  be the circle of radius  $R$  centered at  $a$ . Then

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

ML-lemma says

$$\begin{aligned} |f'(a)| &\leq \frac{1}{2\pi} \max_{z \in \gamma} \frac{|f(z)|}{|z-a|^2} \cdot \text{length}(\gamma) \\ &= \frac{1}{2\pi} \max_{z \in \gamma} \frac{|f(z)|}{R^2} \cdot 2\pi R \\ &\leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R \\ &= \frac{M}{R}. \end{aligned}$$

Since  $M$  doesn't depend on  $R$ ,

$$|f'(a)| \leq \frac{M}{R} \quad \text{for any } R > 0.$$

As  $R \rightarrow \infty$ , this gets arbitrarily small. So  $|f'(a)| = 0$  and hence  $f'(a) = 0$ . So it must be true that  $f(a)$  is constant.  $\square$

**Claim.** Suppose  $f(z)$  is holomorphic inside  $\gamma$  and  $a$  is inside  $\gamma$ . Then  $f''(a)$  exists and

$$f''(a) = \frac{1}{\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^3} dz.$$

*Proof.* For  $w$  inside  $\gamma$ , let

$$f'(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-w)^2} dz.$$

Then

$$\begin{aligned} 2\pi i \cdot \frac{f'(a+h) - f'(a)}{h} &= \frac{1}{h} \left( \int_{\gamma} \frac{f(z)}{(z-a-h)^2} dz - \int_{\gamma} \frac{f(z)}{(z-a)^2} dz \right) \\ &= \frac{1}{h} \int_{\gamma} f(z) \cdot \frac{(z-a)^2 - (z-a-h)^2}{(z-a)^2(z-a-h)^2} dz \\ &= \frac{1}{h} \int_{\gamma} \frac{h(2z-2a-h)}{(z-a)^2(z-a-h)^2} \cdot f(z) dz. \end{aligned}$$

Then to show that

$$\lim_{h \rightarrow 0} \int_{\gamma} \frac{(2z - 2a - h)}{(z - a)^2(z - a - h)^2} \cdot f(z) dz = 2 \int_{\gamma} \frac{f(z)}{(z - a)^3} dz,$$

we show

$$\lim_{h \rightarrow 0} \int_{\gamma} \frac{(2z - 2a - h)}{(z - a)^2(z - a - h)^2} \cdot f(z) dz - \int_{\gamma} \frac{2f(z)}{(z - a)^3} dz = 0$$

The LHS becomes

$$\begin{aligned} & \lim_{h \rightarrow 0} \int_{\gamma} \frac{(z - a)(2(z - a) - h) - 2(z - a - h)^2}{(z - a)^3(z - a - h)^2} \cdot f(z) dz \\ &= \lim_{h \rightarrow 0} \int_{\gamma} \frac{2(z - a)^2 - h(z - a) - 2((z - a)^2 - 2h(z - a) + h^2)}{(z - a)^3(z - a - h)^2} \cdot f(z) dz \\ &= \lim_{h \rightarrow 0} \int_{\gamma} \frac{3(z - a)h - 2h^2}{(z - a)^3(z - a - h)^2} \cdot f(z) dz \\ &= \lim_{h \rightarrow 0} 3h \int_{\gamma} \frac{(z - a)f(z)}{(z - a)^3(z - a - h)^2} dz - 2h^2 \int_{\gamma} \frac{f(z)}{(z - a)^3(z - a - h)^2} dz. \end{aligned}$$

For the limit as  $h \rightarrow \infty$  to exist and be zero, it's enough that the integral remain bounded.

$$\begin{aligned} \left| \int_{\gamma} \frac{(z - a)}{(z - a)^3(z - a - h)^2} f(z) dz \right| &\leq \max_{z \in \gamma} \left| \frac{f(z)}{(z - a)^2(z - a - h)^2} \right| \text{length}(\gamma) \\ &\leq \max_{z \in \gamma} \frac{|f(z)|}{\epsilon^2 \cdot \left(\frac{\epsilon}{2}\right)^2} \cdot \text{length}(\gamma). \end{aligned}$$

The bound is independent of  $h$ . Similarly for the bound of

$$\left| \int_{\gamma} \frac{1}{(z - a)^3(z - a - h)^2} f(z) dz \right|.$$

□

Suppose  $f(z)$  is holomorphic at  $a$ , then it is differentiable on a disk centered at  $a$  of some radius  $\epsilon > 0$ . If  $|z - a| < \epsilon$ , then  $f'(z)$  exists. Let  $\gamma$  be the circle centered at  $a$  with radius  $\frac{\epsilon}{2}$ , so that  $f(z)$  is differentiable inside  $\gamma$ . In fact, it's holomorphic inside  $\gamma$ . Since  $f(z)$  is differentiable inside the small disk containing the given point, it's holomorphic at that point. So we can apply the Cauchy integral formula and its corollaries and conclude that  $f''(a)$  exists. In particular,  $f'(z)$  is differentiable at  $z = a$ . Doing this on a disk around  $a$ , we conclude  $f'(z)$  is holomorphic at  $a$ .

Similarly  $f'(z)$  being holomorphic at  $a$  implies that  $f''(z)$  is holomorphic at  $a$ , which then implies  $f'''(z)$  as well, and so on. So  $f(z)$  being holomorphic implies  $f$  is infinitely differentiable.

### 7.3 Fundamental Theorem of Algebra

**Theorem 7.3.1** (Fundamental Theorem of Algebra). *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

*is a polynomial with complex coefficients, then either the polynomial is constant or it can be written as a product of linear factors  $(az + b)$ .*

Here's a weaker version of this theorem:

**Theorem 7.3.2** (Fundamental Theorem of Algebra (Weaker version)). *If  $p(z)$  is a non-constant polynomial, then there is some  $w$  such that  $p(w) = 0$  (any non-constant polynomial has a root).*

**Remark.** If  $az + b$  is a factor of  $p(z)$ , then  $w = -b/a$  makes  $aw + b = 0$ , so  $p(w) = 0$ . If  $p(w) = 0$ , then  $z - w$  divides  $p(z)$ . Then  $\frac{p(z)}{z-w}$  is still a polynomial and we can repeat this until the polynomial becomes a constant.

**Lemma 7.3.3.** *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

*is non-constant ( $a_d \neq 0, d \geq 1$ ), then there is a real number  $R$  such that  $|z| \geq R$  implies*

$$\frac{1}{2}|a_d z^d| \leq |p(z)| \leq \frac{3}{2}|a_d z^d|,$$

*which is equivalent to*

$$\frac{1}{2} \leq \left| \frac{p(z)}{a_d z^d} \right| \leq \frac{3}{2},$$

*which is then equivalent to*

$$\left| \left| \frac{p(z)}{a_d z^d} \right| - 1 \right| \leq \frac{1}{2}.$$

*Proof.* We have

$$\begin{aligned} \frac{p(z)}{a_d z^d} - 1 &= \frac{a_d z^d}{a_d z^d} + \frac{a_{d-1} z^{d-1}}{a_d z^d} + \cdots + \frac{a_0}{a_d z^d} - 1 \\ &= \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots + \frac{a_0}{a_d} z^{-d}. \end{aligned}$$

Note when  $|z| \rightarrow \infty$ ,  $|z|^{-r} \rightarrow 0$  for any  $r > 0$ , i.e., for any  $\epsilon > 0$ , there is an  $R$  such that

$$|z| \geq R \implies ||z|^{-r} - 0| < \epsilon.$$

We want  $R$  such that  $|z| \geq R$  implies  $|z|^{-r} < \epsilon$ . Then

$$\begin{aligned} \log(|z|^{-r}) &< \log(\epsilon) \\ -r \log(|z|) &< \log(\epsilon) \\ \log(|z|) &> -\frac{\log(\epsilon)}{r} \\ |z| &> e^{-\frac{\log(\epsilon)}{r}} = \epsilon^{-1/r}. \end{aligned}$$

We take any  $R > \epsilon^{-1/r}$  to show that the limit exists.

**Conclusion:**

$$\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \dots + \frac{a_0}{a_d} z^{-d} \rightarrow 0$$

as  $|z| \rightarrow \infty$ , i.e., for any  $\epsilon > 0$ , there is an such that  $|z| \geq R$  implies

$$\left| \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \dots + \frac{a_0}{a_d} z^{-d} \right| < \epsilon.$$

We take  $\epsilon = \frac{1}{2}$  and conclude that for the resulting  $R$ ,  $|z| \geq R$  implies

$$\left| \frac{p(z)}{a_d z^d} - 1 \right| < \epsilon = \frac{1}{2}.$$

□

Now we want to apply Liouville's theorem to  $\frac{1}{p(z)}$ . If  $p(z) \neq 0$  for any  $z$ , then  $\frac{1}{p(z)}$  is holomorphic (composition of  $\frac{1}{z}$  with  $p(z)$ ).

$$\begin{aligned} \frac{1}{2} |a_d z^d| &\leq |p(z)| \\ \left| \frac{1}{p(z)} \right| &\leq \frac{2}{|a_d|} |z|^{-d} \leq \frac{2}{|a_d|} R^{-d} \quad (\text{since } |z| \geq R) \end{aligned}$$

To bound  $\frac{1}{p(z)}$  on the disk of radius  $R$  centered at zero (i.e.  $|z| \leq R$ ), notice that this region is compact (closed and bounded). Since  $\frac{1}{p(z)}$  is holomorphic, it is continuous and  $\left| \frac{1}{p(z)} \right|$  is the composition of the continuous functions  $\frac{1}{p(z)}$  and absolute value, hence it is also continuous. Since the continuous image of a compact set is compact, the image of  $\left| \frac{1}{p(z)} \right|$  on  $|z| \leq R$  is compact (so in particular, it is bounded), i.e.  $\left| \frac{1}{p(z)} \right| \leq M$  when  $|z| \leq R$ .

So now we combine the two bounds

$$\left| \frac{1}{p(z)} \right| \leq \max \left( \underbrace{\frac{2}{|a_d|} R^{-d}}_{\text{valid when } |z| \geq R}, \underbrace{M}_{\text{valid when } |z| \leq R} \right).$$



So for any choice of  $z$ , either  $|z| \geq R$  or  $|z| \leq R$ , and the inequality holds. So we've shown  $\frac{1}{p(z)}$  is a bounded holomorphic function. So by Liouville's theorem, it is constant.

Hence, if  $p(z)$  is never zero,  $\frac{1}{p(z)} = c$  is a constant, so  $p(z) = \frac{1}{c}$ . Since polynomial has no zero implies it is constant, we conclude that a polynomial being non-constant implies that the polynomial has a zero.

**Remark.** Not the only proof but a very typical application of Liouville's theorem (show some condition implies boundedness, deduce it's constant).