STOCHASTIC PROCESSES

STAT 150

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Contents

1 Probability Review

1.1 Basic Definitions

Definition 1.1.1 (Probability Space). A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a triple consisting of a set Ω called the *sample space*, a set $\mathcal{F} \subseteq \Omega$ satisfying certain closure properties, and a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ that assigns probbailitie to events in a coherent way.

Requirements for $\mathcal{F}$:

(i) $\Omega \in \mathcal{F}$.

(ii) If $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$.

(iii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Requirements for $\mathbb{P}$:

(i) $\mathbb{P}(\Omega) = 1$.

(ii) If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ are pairwise disjoint (meaning $E_i \cap E_j = \varnothing$ for $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Definition 1.1.2 (Random Variable). A *random variable* is a function $X : \Omega \to \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$ whenever B is a "nice" subset of $\mathbb{R}$.

Example 1.1.3. $\Omega = \{H, T\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(\{H\}) = \frac{1}{2}$. $X(H) = 1, \quad X(T) = 0$.

$$\mathbb{P}(X = 1) = \mathbb{P}(\{H\}) = \frac{1}{2}, \quad \mathbb{P}(X = 0) = \mathbb{P}(\{T\}) = \frac{1}{2}.$$

1.2 Overview

Definition 1.2.1 (Stochastic Process). A *stochastic process* is a collection $\{X_t : t \in T\}$ of random variables $X_t : \Omega \to S \subseteq \mathbb{R}$ all defined on the some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here T is some index set (typically representing time) and S is the *state space*. One write this as

$X : \Omega \times T \to S$, $(w, t) \mapsto X_t(\omega)$. For a given outcome $\omega \in \Omega$, we get a sample path trajectory $X(\omega) : T \to S, t \mapsto X_t(\omega)$. A stochastic process can then be thought of as a random function.

The theme of this course is what can we say about the distribution of trajectories?

Example 1.2.2 (Branching Process (DTDS)). $X_0 = 1$, one individual in the 0th generation individuals produce a random number of offspring, i.i.d. $(\xi_i^{(n)})_{i\in\mathbb{N},n\in\mathbb{N}_0}$.

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)}.$$

One interesting question would be what is $\mathbb{P}(X_n = 0 \text{ eventually})$, the probability of dying out?

Example 1.2.3 (Poisson Process (CTDS)). Recall that the Poisson distribution is used to model the number of occurrences of a rare event in some fixed period of time. The Poisson process $(N_t)_{t\geq 0}$ models the number of occurrences throughout time. $N_t = \#$ of occurrences by time t.

2 Probability Review II

2.1 Useful Properties

(i) *(DeMorgan)*
$$(E \cup F)^c = E^c \cap F^c, \quad (E \cap F)^c = E^c \cup F^c.$$

(ii) *(Complementation)*
$$\mathbb{P}(E) = 1 - \mathbb{P}\left(E^c\right).$$

(iii) *(Inclusion-exclusion)*
$$\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$
$$\mathbb{P}\left(\bigcup_{i=1}^n E_i\right) = \sum_{j=1}^n (-1)^{j-1} \sum_{S \in [n] : |S| = j} \mathbb{P}\left(\bigcap_{i \in S} E_i\right).$$

(iv) *(Partitioning)* If $\bigsqcup_{i=1}^\infty E_i = \Omega$, then
$$\mathbb{P}(F) = \mathbb{P}\left(\bigsqcup_{i=1}^\infty (F \cap E_i)\right) = \sum_{i=1}^\infty \mathbb{P}(F \cap E_i)$$

2.2 Conditional Probability

Conditioning: For $\mathbb{P}(F) > 0$,
$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.$$
$\mathbb{P}(\cdot \mid F)$ defines a new probability measure on $(\Omega, \mathcal{F})$.

Multiplication rule:
$$\mathbb{P}(E \cap F) = \mathbb{P}(F)\mathbb{P}(E \mid F).$$

If $\bigsqcup_{i=1}^\infty F_i = \Omega$, then
$$\mathbb{P}(E) = \sum_{i=1}^\infty \mathbb{P}(E \cap F_i) = \sum_{i=1}^\infty \mathbb{P}(F_i)\mathbb{P}(E \mid F_i).$$

Bayes' rule:
$$\mathbb{P}(F_j \mid E) = \frac{\mathbb{P}(F_j)\mathbb{P}(E \mid F_j)}{\sum_{i=1}^\infty \mathbb{P}(F_i)\mathbb{P}(E \mid F_i)}$$

2.3 Discrete Random Variables

If $X : \Omega \to S \subseteq \mathbb{R}$ is discrete,

$$\mathbb{P}(X \in E) = \sum_{x \in E} \mathbb{P}(X = x) = \sum_{x \in E \cap S} \mathbb{P}(X = x).$$

2.3.1 Indicator Random Variable

$$X(\omega) = \mathbb{I}_E(\omega) = \begin{cases} 1 & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

2.3.2 Binomial Random Variable

$$X = \sum_{i=1}^{n} \mathbb{I}_{E_i}, \quad \mathbb{P}(E_i) = p.$$

$$p_X(x) = \mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

2.4 Continuous Random Variables

If X continuous,

$$\mathbb{P}(X \in E) = \int_E f_X(x) dx.$$

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x) dx.$$

2.4.1 Exponential Random Variable

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{x \geq 0}.$$

2.4.2 Gaussian Random Variable

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

2.5 Cumulative Distribution Function (CDF)

$F_X : \mathbb{R} \to [0, 1]$,

$$F_X(r) = \mathbb{P}(X \le r) = \mathbb{P}(X \in (-\infty, r]).$$

If X is discrete,

$$F_X(r) = \sum_{x_i \le r} p_X(x_i).$$

If X is continuous,

$$F_X(r) = \int_{-\infty}^{r} f_X(x)dx.$$

$$\frac{d}{dr} F_X(r) = f_X(r).$$

2.6 Expectation

2.6.1 Discrete case

$$\mathbb{E}[X] = \sum_{x_i \in S} x_i \mathbb{P}(X = x_i).$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \ge n).$$

$$\mathbb{E}[g(X)] = \sum_{x_i \in S} g(x_i)\mathbb{P}(X = x_i).$$

2.6.2 Continuous case

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)dx.$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\mathbb{P}(X \ge x)dx.$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)dx.$$

2.7 Variance

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

2.8 Moments

$$\mathbb{E}[X^m] = \int_{0}^{\infty} m x^{m-1}\mathbb{P}(X \ge x)dx.$$

2.9 Joint Distribution

2.9.1 Discrete

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$$

2.9.2 Continuous

$$\mathbb{P}((X,Y) \in E) = \int\int_E f_{X,Y}(x,y)dxdy$$

2.9.3 Marginal Distribution

$$p_X(x) = \sum_{y \in S_Y} p_{X,Y}(x,y).$$

$$f_X(x) = \int_{y \in S_y} f_{X,Y}(x,y)dy$$

2.10 Independence

$$p_{X,Y}(x,y) = p_X(x)p_Y(y).$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

$$\mathbb{P}(X \leq x, Y \leq y) = F_X(x)F_Y(y).$$