Math 185 Notes
Complex Analysis

Professor: Christopher Ryba
Scribe: Kelvin Lee

Contents

Lecture 1

Complex Numbers

1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for $|x| < r$, where r is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n||x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

Example 1.1.1. The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for $|x| < 1$.

Question. Now what if we replace the real variable x by the complex variable z?

Answer. If $|z| < r$, then

$$\sum_{n=0}^{\infty} |a_n||z|^n < \infty.$$

so the sum converges absolutely for $z \in D(0, r)$ (disc of radius r centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

Example 1.1.2. Let
$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$
When viewed as a function $\mathbb{R} \to \mathbb{R}$, $f(z)$ is infinitely differentiable at $z = 0$, and all derivatives of $f(z)$ are zero at $z = 0$. Hence, the Taylor series is
$$f(0) + f'(0) \cdot x + \frac{f"(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$
So the Taylor series converges to a function different from $f(z)$!

Example 1.1.3. Consider the same example as above, but with z as a complex number. Let $z = it$ where $t \in \mathbb{R}$. Then
$$e^{-1/z^2} = e^{1/t^2},$$
and so
$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$
which is not continuous at $z = 0$ and thus not complex-differentiable at $z = 0$.

Example 1.1.4. Now let's set $z = x + iy$ where $x, y \in \mathbb{R}$. Consider a suitable power series
$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$
which we may view as a function $\mathbb{R}^2 \to \mathbb{R}^2$ (instead of $\mathbb{C} \to \mathbb{C}$). Let's differentiate with respect to x:
$$\frac{\partial f(z)}{\partial x} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z)$$
$$\frac{\partial^2 f(z)}{\partial x^2} = \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z).$$
Now with respect to y:
$$\frac{\partial f(z)}{\partial y} = \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = if'(z)$$
$$\frac{\partial^2 f(z)}{\partial y^2} = i\frac{\partial f'(z)}{\partial y} = i\frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z).$$
We observe that
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$
which means (the real and imaginary parts of) $f(z)$ satisfy the two-dimensional *Laplace equation*.

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

Example 1.1.5. Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful too for computing integrals.

Lecture 2

Complex Differentiation

2.1 Derivatives

Definition 2.1.1 (Derivative)**.** The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x) = L$$

means for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - a| < \delta$. (For any "tolerance" ϵ, we can guarantee $f(x)$ is within ϵ of L by forcing x to be close enough to a.)

Remark. Note that $x = a$ doesn't satisfy $0 < |x - a|$, so the value of f at $x = a$ has no bearing on whether $\lim_{x \to a} f(x)$ exists.

2.1.1 Continuity

Definition 2.1.2 (Continuous)**.** If $\lim_{x \to a} f(x) = f(a)$, then we say f is *continuous* at a.

Remark. Setting $L = f(a)$ in the limit, $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ (even when $x = a$) when talking about continuity, we leave out the $0 < |x - a|$ part for convenience because $x - a = 0$ automatically works.

Now let's consider a function $f : \mathbb{C} \to \mathbb{C}$, $\lim_{z\to a} f(z) = L$ means for every $\epsilon > 0$, there is $\delta > 0$ such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

Remark. Now the z's that we worry about form an open disc with radius δ instead of an interval from the real case.

Similarly, if $\lim_{z\to a} f(z) = f(a)$, we say f is *continuous* at $z = a$.

Example 2.1.3. $f(z) = z$ is continuous at any point $a \in \mathbb{C}$.

Proof. For $\epsilon > 0$, let $\delta = \epsilon$, then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

Example 2.1.4. $\lim_{z\to 0} \overline{z}/z$ (although this is undefined at $z = 0$, this has no bearing on whether the limit exists).

Proof. Suppose $\lim_{z\to 0} \overline{z}/z = L$ for some L. Let's take $\epsilon = 1$. There is a $\delta > 0$ such that

$$0 < |z - 0| < \delta \implies \left|\frac{\overline{z}}{z} - L\right| < \epsilon = 1.$$

Let $z = \delta/2$ and so does $z = i\delta/2$. Then for $z = \delta/2$:

$$\frac{\overline{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for $z = i\delta/2$:

$$\frac{\overline{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the L must lie in the intersection of the two open unit discs centered at -1 and 1. However, since they are open discs, these two discs do not overlap and so L does not exist. □

Remark. This implies that there is no way to extend $\overline{z}/z$ to a continuous function at $z = 0$.

2.1.2 Properties of Limits

If $\lim_{x\to a} f(x) = L_1$, $\lim_{x\to a} g(x) = L_2$, then

(i)
$$\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2.$$

(ii)
$$\lim_{x\to a} f(x)g(x) = L_1 L_2.$$

(iii)
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

Remark. These implies that the sum/product/quotient of continuous functions are continuous.

Proposition 2.1.5 (Composite function of continuous functions is continuous)**.** *If $f(x)$ is continuous at $x = a$, and $g(x)$ is continuous at $x = f(a)$, then $g(f(x))$ is continuous at $x = a$.*

Proof. We want $|g(f(x)) - g(f(a))| < \epsilon$. By continuity of g at $x = f(a)$, there exists $\delta_1 > 0$ such that
$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$
We want to take $w = f(x)$, so we need
$$|f(x) - f(a)| < \delta_1.$$
But by continuity of f at $x = a$, we know that δ_1 will be our ϵ when $|x - a| < \delta_2$ for some $\delta_2 > 0$. Then for such x,
$$|g(f(x)) - g(f(a))| < \epsilon.$$
□

2.2 Derivatives (Cont'd)

Definition 2.2.1 (Differentiable)**.** We say that $f(z)$ is differentiable at $z = a$ iff $\frac{f(z)-f(a)}{z-a}$ extends to a continuous function at $z = a$ (the value there is $f'(a)$).

Example 2.2.2. $f(z) = z$ is differentiable with $f'(z) = 1$.

Proof.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{z+h-z}{h}=\lim_{h\to 0}1=1.$$

□

Example 2.2.3 (Interesting one). $f(z)=\overline{z}$ is not differentiable but is continuous.

Proof.

$$\begin{aligned}\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}&=\lim_{h\to 0}\frac{\overline{z+h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{z}+\overline{h}-\overline{z}}{h}\\&=\lim_{h\to 0}\frac{\overline{h}}{h}\\&=\text{DNE}\qquad\text{(proved in previous example)}\end{aligned}$$

□

Proposition 2.2.4 (Differentiability implies continuity). *$f(z)$ differentiable at $z=a$ implies that $f(z)$ is continuous at $z=a$.*

Proof. We want to show that $\lim_{z\to a}f(z)=f(a)$.

$$\begin{aligned}\lim_{z\to a}f(z)-f(a)&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot(z-a)\\&=\lim_{z\to a}\frac{f(z)-f(a)}{z-a}\cdot\lim_{z\to a}(z-a)\qquad\text{(assume both limits exist)}\\&=f'(a)\cdot 0\\&=0.\end{aligned}$$

□

Remark. This is a common technique to show continuity by showing the limit of the difference is zero.

2.2.1 Properties of complex-derivatives

(i)

$$\frac{d}{dz}cf(z)=cf'(z),\qquad\forall c\in\mathbb{C}.$$

(ii)
$$\frac{d}{dz}(f+g) = f'(z) + g'(z).$$

(iii)
$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)
$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)
$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

Proposition 2.2.5 (Power rule)**.**
$$\frac{d}{dz}z^n = nz^{n-1}$$
for all integers n.

Proof. We induct on n. For $n \geq 0$, when $n = 0$,
$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$
By the product rule,
$$\begin{aligned}
\frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\
&= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \qquad \text{(inductive hypothesis)} \\
&= nz^{n-1}.
\end{aligned}$$
For $n < 0$, simply apply quotient rule. $\square$