

# Math 185 Notes

## Complex Analysis

Professor: Christopher Ryba  
Scribe: Kelvin Lee

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# Chapter 1

## Complex Numbers

### 1.1 Intro

Suppose we have the a Taylor series as follows:

$$\sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{R}$$

which happens to converge absolutely for  $|x| < r$ , where  $r$  is the *radius of convergence*, i.e.

$$\sum_{n=0}^{\infty} |a_n| |x|^n < \infty \quad \text{when } |x| < r, \quad \text{i.e. } x \in (-r, r).$$

**Example 1.1.1.** The series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

converges absolutely for  $|x| < 1$ .

**Question.** Now what if we replace the real variable  $x$  by the complex variable  $z$ ?

**Answer.** If  $|z| < r$ , then

$$\sum_{n=0}^{\infty} |a_n| |z|^n < \infty.$$

so the sum converges absolutely for  $z \in D(0, r)$  (disc of radius  $r$  centered at zero). This gives a wider range than the real case.

So in this situation, our real-valued series can be extended to a complex-valued series.

**Example 1.1.2.** Let

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

When viewed as a function  $\mathbb{R} \rightarrow \mathbb{R}$ ,  $f(z)$  is infinitely differentiable at  $z = 0$ , and all derivatives of  $f(z)$  are zero at  $z = 0$ . Hence, the Taylor series is

$$f(0) + f'(0) \cdot x + \frac{f''(0)}{2} \cdot x^2 + \cdots = 0 + 0 + 0 + \cdots = 0.$$

So the Taylor series converges to a function different from  $f(z)$ !

**Example 1.1.3.** Consider the same example as above, but with  $z$  as a complex number. Let  $z = it$  where  $t \in \mathbb{R}$ . Then

$$e^{-1/z^2} = e^{1/t^2},$$

and so

$$f(it) = \begin{cases} e^{1/t^2} & t \neq 0, \\ 0 & t = 0 \end{cases}$$

which is not continuous at  $z = 0$  and thus not complex-differentiable at  $z = 0$ .

**Example 1.1.4.** Now let's set  $z = x + iy$  where  $x, y \in \mathbb{R}$ . Consider a suitable power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which we may view as a function  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  (instead of  $\mathbb{C} \rightarrow \mathbb{C}$ ). Let's differentiate with respect to  $x$ :

$$\begin{aligned} \frac{\partial f(z)}{\partial x} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f'(z) \\ \frac{\partial^2 f(z)}{\partial x^2} &= \frac{\partial f'(z)}{\partial x} = \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial x} = f''(z). \end{aligned}$$

Now with respect to  $y$ :

$$\begin{aligned} \frac{\partial f(z)}{\partial y} &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = i f'(z) \\ \frac{\partial^2 f(z)}{\partial y^2} &= i \frac{\partial f'(z)}{\partial y} = i \frac{\partial f'(z)}{\partial z} \cdot \frac{\partial z}{\partial y} = -f''(z). \end{aligned}$$

We observe that

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z) = 0,$$

which means (the real and imaginary parts of)  $f(z)$  satisfy the two-dimensional *Laplace equation*.

Thus, complex analysis is a very powerful tool for solving the 2D Laplace equation.

**Example 1.1.5.** Consider the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \arctan(\infty) - \arctan(-\infty) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

But what about

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2} dx, \quad a \in \mathbb{R}$$

It turns out that this would be quite tricky to compute this with real-valued techniques. But it would be easier using complex techniques (contour integration). Thus, complex analysis is also a powerful too for computing integrals.

## Chapter 2

# Complex Differentiation

### 2.1 Derivatives

**Definition 2.1.1** (Derivative). The *derivative* of a complex-valued function is

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

just as in the real-valued case.

Recall that for real valued  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\lim_{x \rightarrow a} f(x) = L$$

means for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that

$$|f(x) - L| < \epsilon$$

whenever  $0 < |x - a| < \delta$ . (For any "tolerance"  $\epsilon$ , we can guarantee  $f(x)$  is within  $\epsilon$  of  $L$  by forcing  $x$  to be close enough to  $a$ .)

**Remark.** Note that  $x = a$  doesn't satisfy  $0 < |x - a|$ , so the value of  $f$  at  $x = a$  has no bearing on whether  $\lim_{x \rightarrow a} f(x)$  exists.

#### 2.1.1 Continuity

**Definition 2.1.2** (Continuous). If  $\lim_{x \rightarrow a} f(x) = f(a)$ , then we say  $f$  is *continuous* at  $a$ .

**Remark.** Setting  $L = f(a)$  in the limit,  $0 < |x - a| < \delta$  implies  $|f(x) - f(a)| < \epsilon$  (even when  $x = a$ ) when talking about continuity, we leave out the  $0 < |x - a|$  part for convenience because  $x - a = 0$  automatically works.

Now let's consider a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\lim_{z \rightarrow a} f(z) = L$  means for every  $\epsilon > 0$ , there is  $\delta > 0$  such that

$$0 < |z - a| < \delta \implies |f(z) - L| < \epsilon.$$

**Remark.** Now the  $z$ 's that we worry about form an open disc with radius  $\delta$  instead of an interval from the real case.

Similarly, if  $\lim_{z \rightarrow a} f(z) = f(a)$ , we say  $f$  is *continuous* at  $z = a$ .

**Example 2.1.3.**  $f(z) = z$  is continuous at any point  $a \in \mathbb{C}$ .

*Proof.* For  $\epsilon > 0$ , let  $\delta = \epsilon$ , then

$$|z - a| < \delta = \epsilon \implies |f(z) - f(a)| < \epsilon.$$

□

**Example 2.1.4.**  $\lim_{z \rightarrow 0} \bar{z}/z$  (although this is undefined at  $z = 0$ , this has no bearing on whether the limit exists).

*Proof.* Suppose  $\lim_{z \rightarrow 0} \bar{z}/z = L$  for some  $L$ . Let's take  $\epsilon = 1$ . There is a  $\delta > 0$  such that

$$0 < |z - 0| < \delta \implies \left| \frac{\bar{z}}{z} - L \right| < \epsilon = 1.$$

Let  $z = \delta/2$  and so does  $z = i\delta/2$ . Then for  $z = \delta/2$ :

$$\frac{\bar{z}}{z} = \frac{\delta/2}{\delta/2} = 1 \implies |1 - L| < 1,$$

and for  $z = i\delta/2$ :

$$\frac{\bar{z}}{z} = \frac{-i\delta/2}{i\delta/2} = -1 \implies |-1 - L| < 1.$$

Thus, we see that the  $L$  must lie in the intersection of the two open unit discs centered at  $-1$  and  $1$ . However, since they are open discs, these two discs do not overlap and so  $L$  does not exist. □

**Remark.** This implies that there is no way to extend  $\bar{z}/z$  to a continuous function at  $z = 0$ .



### 2.1.2 Properties of Limits

If  $\lim_{x \rightarrow a} f(x) = L_1$ ,  $\lim_{x \rightarrow a} g(x) = L_2$ , then

(i)

$$\lim_{x \rightarrow a} (f(x) + g(x)) = L_1 + L_2.$$

(ii)

$$\lim_{x \rightarrow a} f(x)g(x) = L_1L_2.$$

(iii)

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0.$$

**Remark.** These implies that the sum/product/quotient of continuous functions are continuous.

**Proposition 2.1.5** (Composite function of continuous functions is continuous). *If  $f(x)$  is continuous at  $x = a$ , and  $g(x)$  is continuous at  $x = f(a)$ , then  $g(f(x))$  is continuous at  $x = a$ .*

*Proof.* We want  $|g(f(x)) - g(f(a))| < \epsilon$ . By continuity of  $g$  at  $x = f(a)$ , there exists  $\delta_1 > 0$  such that

$$|w - f(a)| < \delta_1 \implies |g(w) - g(f(a))| < \epsilon.$$

We want to take  $w = f(x)$ , so we need

$$|f(x) - f(a)| < \delta_1.$$

But by continuity of  $f$  at  $x = a$ , we know that  $\delta_1$  will be our  $\epsilon$  when  $|x - a| < \delta_2$  for some  $\delta_2 > 0$ . Then for such  $x$ ,

$$|g(f(x)) - g(f(a))| < \epsilon.$$

□

## 2.2 Derivatives (Cont'd)

**Definition 2.2.1** (Differentiable). We say that  $f(z)$  is differentiable at  $z = a$  iff  $\frac{f(z)-f(a)}{z-a}$  extends to a continuous function at  $z = a$  (the value there is  $f'(a)$ ).

**Example 2.2.2.**  $f(z) = z$  is differentiable with  $f'(z) = 1$ .

*Proof.*

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{z+h-z}{h} = \lim_{h \rightarrow 0} 1 = 1.$$

□

**Example 2.2.3** (Interesting one).  $f(z) = \bar{z}$  is not differentiable but is continuous.

*Proof.*

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} &= \lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\bar{h}}{h} \\ &= \text{DNE} \quad (\text{proved in previous example}) \end{aligned}$$

□

**Proposition 2.2.4** (Differentiability implies continuity).  $f(z)$  differentiable at  $z = a$  implies that  $f(z)$  is continuous at  $z = a$ .

*Proof.* We want to show that  $\lim_{z \rightarrow a} f(z) = f(a)$ .

$$\begin{aligned} \lim_{z \rightarrow a} f(z) - f(a) &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot (z - a) \\ &= \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} \cdot \lim_{z \rightarrow a} (z - a) \quad (\text{assume both limits exist}) \\ &= f'(a) \cdot 0 \\ &= 0. \end{aligned}$$

□

**Remark.** This is a common technique to show continuity by showing the limit of the difference is zero.

### 2.2.1 Properties of complex-derivatives

(i)

$$\frac{d}{dz} cf(z) = cf'(z), \quad \forall c \in \mathbb{C}.$$

(ii)

$$\frac{d}{dz}(f + g) = f'(z) + g'(z).$$

(iii)

$$\frac{d}{dz}(fg) = f'g + fg'.$$

(iv)

$$\frac{d}{dz}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}.$$

(v)

$$\frac{d}{dz}f(g(z)) = g'(z)f'(g(z)).$$

**Proposition 2.2.5** (Power rule).

$$\frac{d}{dz}z^n = nz^{n-1}$$

for all integers  $n$ .*Proof.* We induct on  $n$ . For  $n \geq 0$ , when  $n = 0$ ,

$$\frac{d}{dz}z^0 = \frac{d}{dz}1 = 0 = 0z^{-1}.$$

By the product rule,

$$\begin{aligned} \frac{d}{dz}z^n &= \frac{d}{dz}(z \cdot z^{n-1}) \\ &= 1 \cdot z^{n-1} + z \cdot (n-1)z^{n-2} \quad (\text{inductive hypothesis}) \\ &= nz^{n-1}. \end{aligned}$$

For  $n < 0$ , simply apply quotient rule. □

## Chapter 3

# Holomorphic Functions and Cauchy-Riemann Equations

Recall

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

Being differentiable at a point says little about how "nice" a function is.

**Example 3.0.1.** Consider  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This is nowhere continuous, and thus nowhere differentiable. But if we consider  $x^2 f(x)$ , it is differentiable at  $x = 0$ :

$$\lim_{h \rightarrow 0} \frac{h^2 f(h) - 0^2 f(0)}{h} = \lim_{h \rightarrow 0} h f(h) = 0.$$

Nevertheless, it's still not a very "nice" function.

### 3.1 Holomorphic Functions

**Definition 3.1.1** (Holomorphic). A function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is *holomorphic* at a point  $a$  if it is differentiable at  $z$  for all  $z$  within distance  $r$  of  $a$  for some  $r > 0$ . In other words,  $f(z)$  is differentiable everywhere sufficiently close to  $a$ .

**Definition 3.1.2** (Open/closed disk). The *open disk* of radius  $r$  centered at  $a \in \mathbb{C}$  is

$$D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}.$$

The *closed disk* is

$$\overline{D}(a, r) = \{z \in \mathbb{C} \mid |z - a| \leq r\}.$$

Thus, we can say  $f(z)$  is holomorphic at  $a \in \mathbb{C}$  if  $f(z)$  is differentiable on an open disk centered at  $a$ . (if the point is not specified, it means that  $f$  is holomorphic everywhere.)

**Example 3.1.3** (Polynomials are holomorphic). We saw last time that  $z^n$  is differentiable everywhere for  $n \geq 0$ . Then the linear combinations

$$a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0,$$

which is a polynomial, is differentiable (since multiplying by constants and summing preserves differentiability).

**Example 3.1.4.**  $f(z) = |z|^2 = z\bar{z}$  is differentiable at zero.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h\bar{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} \bar{h} \\ &= 0. \end{aligned}$$

However, this is not differentiable elsewhere (exercise). Thus,  $f$  is not holomorphic.

## 3.2 The Cauchy-Riemann Equations

**Question.** How to tell if a function is complex-differentiable?

**Answer.** We'll reduce this to a question about real derivatives.

Let  $x + iy$ , where  $x, y \in \mathbb{R}$ . If  $f : \mathbb{C} \rightarrow \mathbb{C}$ ,

$$\begin{aligned} \frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h+iy) - f(x+iy)}{h}. \end{aligned}$$

Note that  $h$  is real. Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{f(z+ih) - f(z)}{ih} \\ &= \lim_{h \rightarrow 0} \frac{f(x+i(y+h)) - f(x+iy)}{h}. \end{aligned}$$

**Example 3.2.1.**  $f(z) = z^2$ . Then

$$f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$$\begin{aligned}
 \frac{\partial f}{\partial x}(z) &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - y^2 + 2i(x+h)y - (x^2 - y^2 + 2ixy)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 2ihy}{h} \\
 &= \lim_{h \rightarrow 0} 2x + h + 2iy \\
 &= 2x + 2iy \\
 &= 2z \\
 &= f'(z).
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f}{\partial y}(z) &= \lim_{h \rightarrow 0} \frac{x^2 - (y+h)^2 + 2ix(y+h) - (x^2 - y^2 + 2ixy)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2yh - h^2 + 2ixh}{h} \\
 &= \lim_{h \rightarrow 0} -2y - h + 2ix \\
 &= -2y + 2ix \\
 &= 2i(x + iy) \\
 &= if'(z).
 \end{aligned}$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

**Theorem 3.2.2.**

(i) If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is complex-differentiable, then  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist and they satisfy

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y}.$$

(ii) If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a function and  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  exist and are continuous on some open disk centered at  $z$ , and if

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

then  $f$  is complex-differentiable at  $z$ .

*Proof.*

(i) Since  $f$  is complex-differentiable, we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z).$$

This is equivalent to the statement that for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|h - 0| < \delta \implies \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Suppose  $h$  is real. Then

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

and since  $h$  is real, we get  $\frac{\partial f}{\partial x}$  and thus

$$\frac{\partial f}{\partial x}(z) = f'(z).$$

Now suppose  $h$  is purely imaginary:  $h = ik$  for  $k \in \mathbb{R}$ . Then

$$\frac{f(z+h) - f(z)}{h} = \frac{f(x+iy+ik) - f(x+iy)}{ik}.$$

Then  $h \rightarrow 0$  is equivalent to  $k \rightarrow 0$  since  $|h| = |k|$ . Thus we have

$$\lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

Hence, we have

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = -i \frac{\partial f}{\partial y}.$$

□

Let  $f(z) = u(z) + iv(z)$ . If we choose real values for  $h$ , then the imaginary part  $y$  is kept constant, and the derivative becomes a partial derivative with respect to  $x$ . Thus we have

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values  $ik$  for  $h$ , we obtain

$$f'(z) = \lim_{k \rightarrow 0} \frac{f(z+ik) - f(z)}{ik} = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Since we have

$$\frac{\partial f}{\partial x} = -i \frac{\partial f}{\partial y},$$

this resolves into the following equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or simply

$$u_x = v_y, \quad v_x = -u_y.$$

These are known as the *Cauchy-Riemann* equations.

**Example 3.2.3.** Consider  $f(z) = z^2$ . Then

$$f(x + iy) = x^2 + y^2 + 2ixy.$$

Here  $u(x, y) = x^2 - y^2$  and  $v(x, y) = 2xy$ . We have

$$u_x = 2x = v_y \quad v_x = 2y = -u_y.$$

**Example 3.2.4.** Consider  $f(z) = |z|^2$ . Then  $f(x + iy) = x^2 + y^2$  where  $u(x, y) = x^2 + y^2$  and  $v(x, y) = 0$ . But here we have

$$u_x = 2x \neq v_y = 0 \quad v_x = 0 \neq -u_y = -2y.$$

Thus, the Cauchy-Riemann equations only hold at  $(x, y) = (0, 0)$  and as we saw previously that this function is only differentiable at  $z = 0$  and nowhere else.

We can generalize the Cauchy-Riemann equations further to second order. Suppose we have  $f = u + iv$ . Then

$$u_{xx} = \frac{\partial}{\partial x} u_x = \frac{\partial}{\partial x} v_y = \frac{\partial}{\partial x} \frac{\partial}{\partial y} v = \frac{\partial}{\partial y} \frac{\partial}{\partial x} v = \frac{\partial}{\partial y} v_x = \frac{\partial}{\partial y} (-u_y) = -u_{yy}.$$

Thus, we have

$$u_{xx} + u_{yy} = 0, \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly, we also have

$$v_{xx} = (v_x)_x = (-u_y)_x = -u_{yx} = -u_{xy} = -(u_x)_y = -(v_y)_y = -v_{yy},$$

which gives

$$v_{xx} + v_{yy} = 0, \text{ or } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

These are the *Laplace's equations* in 2D we saw earlier.

For  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we know that  $f'(x) = 0$  implies that  $f$  is constant. But for  $f : \mathbb{C} \rightarrow \mathbb{C}$ , we can use the Cauchy-Riemann equations. Since  $f'(z) = \frac{\partial f}{\partial x}$ ,

$$f'(z) = 0 \implies u_x + iv_x = 0 \implies u_x = 0, v_x = 0.$$

By Cauchy-Riemann, we also have  $u_y = v_y = 0$ . Since  $u_x = 0$ , we know that for fixed  $y$ ,  $u(x, y)$  is some constant that could depend on  $y$ . Thus, we have

$$u(x, y) = g(y).$$

But  $u_y = 0$ , so  $g'(y) = 0$ , which means  $g$  is actually a constant independent of  $y$ . Thus,  $u$  is globally constant. Similar argument applies to  $v$  as well.



## Chapter 4

# Möbius Transformation

**Definition 4.0.1** (Möbius transformation). A *Möbius transformation* is a function of the form

$$f(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d \in \mathbb{C}$  satisfy  $ad - bc \neq 0$ .

**Remark.** If  $ad = bc$ , then  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$ , so rows are linearly dependent:  $\lambda(a, b) + \mu(c, d) = (0, 0)$ , which implies that

$$a = \frac{-\mu}{\lambda}c \quad b = \frac{-\mu}{\lambda}d.$$

Then

$$\begin{aligned} f(z) &= \frac{az + b}{cz + d} \\ &= \frac{-\frac{\mu}{\lambda}(cz + d)}{cz + d} \\ &= -\frac{\mu}{\lambda}, \end{aligned}$$

which is a constant independent of  $z$ .

**Proposition 4.0.2** (Composite Möbius transforms is Möbius ). *If  $f_1(z), f_2(z)$  are Möbius transforms, then  $f_1(f_2(z))$  is also a Möbius transform.*

*Proof.* Suppose

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1} \quad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}.$$

Then

$$\begin{aligned}
 f_1(f_2(z)) &= \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} \\
 &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\
 &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)},
 \end{aligned}$$

which is another Möbius transform. □

**Remark.** Note that

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}$$

and the entries coincide with the composite Möbius transform. If we denote  $f_M(z)$  to be a transform associated with a  $2 \times 2$  matrix  $M$ , then we have just shown that

$$f_M(f_N(z)) = f_{MN}(z).$$

**Remark.** Since  $f_I(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z$ , the inverse of  $f_M$  is  $f_{M^{-1}}$ .

**Remark.** Note that

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies f_M = \frac{az + b}{cz + d}.$$

Meanwhile,

$$\lambda M = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} \implies f_{\lambda M} = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az + b}{cz + d} = f_M.$$

Thus, scaling the matrices doesn't affect the resulting Möbius transformation.

## 4.1 Inverse of Möbius transformation

Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Since the scaling part is redundant, we simply ignore it and obtain the inverse Möbius transform as follows:

$$f(z) = \frac{az + b}{cz + d} \implies f^{-1}(z) = \frac{dz - b}{-cz + a}.$$

**Remark.** Since Möbius transforms have inverses, they should be bijections. However, some details should be noted. If  $c \neq 0$ , then  $\frac{az+b}{cz+d}$  is undefined at  $z = -\frac{d}{c}$ .

Let's consider the value at  $z = -\frac{d}{c}$  to be infinity. It turns out that we can evaluate  $\frac{az+b}{cz+d}$  at  $\infty$ :

$$\begin{aligned}\lim_{z \rightarrow \infty} \frac{az+b}{cz+d} &= \lim_{z \rightarrow \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} \\ &= \frac{a}{c}.\end{aligned}$$

When  $c = 0$ , we view  $\frac{a}{c}$  as  $\infty$ . So now we view Möbius transformations as functions from  $\mathbb{C} \cup \{\infty\}$  to  $\mathbb{C} \cup \{\infty\}$ . This makes all Möbius transformations into bijections. Here, we call  $\mathbb{C} \cup \{\infty\}$  the *extended complex plane* (also called *Riemann sphere*).

**Remark.** For real functions, there are multiple notions of going to infinity:  $x \rightarrow +\infty$  and  $x \rightarrow -\infty$ . But for complex functions, we work with only one infinite point.

**Fact.** If we apply a Möbius transformation to a line or a circle in the complex plane, we would get a line or a circle again (circles can turn into lines and vice versa).

**Example 4.1.1.** Consider  $f(z) = \frac{z-1}{iz+i}$ , let's apply this to the unit circle, i.e. take  $z = e^{i\theta}$ . Then

$$\begin{aligned}f(e^{i\theta}) &= \frac{e^{i\theta} - 1}{i(e^{i\theta} + 1)} = \frac{\cos \theta - 1 + i \sin \theta}{i(\cos \theta + 1 + i \sin \theta)} \\ &= \frac{-2 \sin^2 \frac{\theta}{2} + i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{i(2 \cos^2 \frac{\theta}{2} + i 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})} \\ &= \frac{2i(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \sin \frac{\theta}{2}}{2i(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \cos \frac{\theta}{2}} \\ &= \tan \frac{\theta}{2}.\end{aligned}$$

Note that  $\theta \in (-\pi, \pi)$  and we have  $\tan -\frac{\pi}{2} = -\infty$  and  $\tan \frac{\pi}{2} = +\infty$ . We have mapped a unit circle to a line (real line).

**Fact.**  $f$  sends the interior of the unit disk to the interior of the upper half-plane. If  $g(z)$  is holomorphic on the upper half-plane, then  $g(f(z))$  is a holomorphic function on the unit disk. Taking real and imaginary parts gives a solution to the Laplace equation.

**Remark.** Stereographic projection  $\varphi$  is a bijection that maps a sphere to the extended complex plane. It doesn't preserve distance, but it preserves functions being holomorphic.

**Proposition 4.1.2.** Suppose  $f(z) = \frac{az+b}{cz+d}$  is a Möbius transformation. If  $c = 0$  then

$$f(z) = \frac{a}{d}z + \frac{b}{d},$$

and if  $c \neq 0$ , then

$$f(z) = \frac{bc - ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c}.$$

In particular, every Möbius transformation is a composition of translations, dilations, and inversions.

*Proof.* Simplify. □

**Theorem 4.1.3.** *Möbius transformations map circles and lines into circles and lines.*

*Proof.* Translations and dilations certainly map circles and lines into circles and lines, so by the previous proposition, we only have to prove the statement of the theorem for the inversion  $f(z) = \frac{1}{z}$ .

The equation for a circle centered at  $x_0 + iy_0$  with radius  $r$  is  $(x - x_0)^2 + (y - y_0)^2 = r^2$ , which we can transform to

$$\alpha(x^2 + y^2) + \beta x + \gamma y + \delta = 0$$

for some real numbers  $\alpha, \beta, \gamma$ , and  $\delta$  that satisfy  $\beta^2 + \gamma^2 > 4\alpha\delta$ . The above expression is more convenient for us, because it includes the possibility that the equation describes a line (precisely when  $\alpha = 0$ ).

Suppose  $z = x + iy$  satisfies the above expression; we need to prove that  $u + iv := \frac{1}{z}$  satisfies a similar equation. Since

$$u + iv = \frac{x - iy}{x^2 + y^2},$$

we can rewrite the transformed equation as

$$\begin{aligned} 0 &= \alpha + \beta \frac{x}{x^2 + y^2} + \gamma \frac{y}{x^2 + y^2} + \frac{\delta}{x^2 + y^2} \\ &= \alpha + \beta u - \gamma v + \delta(u^2 + v^2). \end{aligned}$$

But this equation says that  $u + iv$  lies on a circle or line. □

**Fact.** The stereographic projection of a circle on the sphere (intersection of a plane and a sphere) is a circle in the plane. Möbius transformations take circles on the sphere to other circles of the sphere (some of these stereographically project to lines in the plane).

## Chapter 5

# Exponential, Trigonometric, and Logarithmic Functions

### 5.1 Exponential Functions

$$e^{x+iy} = e^x e^{iy} = e^x \cos y + i e^x \sin y \implies e^z = u(x, y) + i v(x, y)$$

where

$$\begin{aligned} u(x, y) &= e^x \cos y \\ v(x, y) &= e^x \sin y \end{aligned}$$

### 5.2 Trigonometric Functions

For  $z \in \mathbb{C}$ ,

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2}. \end{aligned}$$

**Remark.**  $\sin z, \cos z$  are holomorphic since  $e^z$  is holomorphic and so is  $e^{iz}$  and  $e^{-iz}$ .

Trigonometric identities hold for complex numbers.

$$\begin{aligned} \sin^2 z + \cos^2 z &= 1 \\ \sin(2z) &= 2 \sin z \cos z \end{aligned}$$

### 5.3 Logarithmic Functions

We want  $\log z$  to be the unique inverse to the exponential function, i.e. we want  $e^{\log z} = z$ , but then we would also have

$$e^{\log z + 2\pi i k} = z.$$

**Definition 5.3.1** (Principal logarithm). The *principal logarithm* is the function defined by

$$\log(re^{i\theta}) = \log r + i\theta.$$

where  $-\pi < \theta \leq \pi$ .

Let's check if  $\log z$  is differentiable. If

$$z = x + iy = re^{i\theta},$$

then  $r = \sqrt{x^2 + y^2}$  and  $\theta = \tan^{-1}(y/x)$  when  $x \neq 0$ .

$$\begin{aligned} \log x + iy &= \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1}\left(\frac{y}{x}\right) \\ &= u(x, y) + iv(x, y). \end{aligned}$$

Then

$$\begin{aligned} u_x &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2} \\ u_y &= \frac{1}{2} \cdot \frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2} \\ v_x &= \frac{-y}{x^2} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = -\frac{y}{x^2 + y^2} \\ v_y &= \frac{1}{x} \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Thus, we see that the Cauchy-Riemann equations hold for logarithms.

## Chapter 6

# Complex Integration

### 6.1 Definition and Basic Properties

If  $f : \mathbb{R} \rightarrow \mathbb{C}$ , define

$$\int_a^b f(x)dx = \int_a^b \Re f(x)dx + i \int_a^b \Im f(x)dx.$$

**Question.** But how to integrate a function  $f : \mathbb{C} \rightarrow \mathbb{C}$ ?

For real functions, going from a point  $\gamma(a)$  to  $\gamma(b)$  can only happen one way (follow the real axis) but in  $\mathbb{C}$ , we will have to specify the path from  $\gamma(a)$  to  $\gamma(b)$ .

**Definition 6.1.1** (Path/curve). A *path/curve* is the image of a function  $\gamma : [a, b] \rightarrow \mathbb{C}$ .

**Definition 6.1.2** (Integral). The *integral* of the function  $f : \mathbb{C} \rightarrow \mathbb{C}$  along the path parametrized by  $\gamma : [a, b] \rightarrow \mathbb{C}$  is

$$\int_a^b f(\gamma(t))\gamma'(t)dt.$$

This is the integral of a function  $\mathbb{R} \rightarrow \mathbb{C}$ , so we already have a definition for it.

Aside,

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))g'(x)dx.$$

Suppose we have a different parametrization of the image of  $\gamma(t)$ . Write this parametrization as  $\gamma(\theta(t))$  where  $\theta : [a, b] \rightarrow [a, b]$  is a continuous reparametrization of the interval  $[a, b]$  satisfying  $\theta(a) = a$ ,  $\theta(b) = b$  and  $\theta$  is increasing. Then

$$\int_a^b f(\gamma(\theta(t)))\gamma'(\theta(t))\theta'(t)dt = \int_{\theta(a)}^{\theta(b)} f(\gamma(u))\gamma'(u)du$$

where  $u = \theta(t)$  and  $du = \theta'(t)dt$ . So the integrals for  $\gamma(\theta(t))$  and  $\gamma(t)$  are the same, thus the integral depends on the curve in  $\mathbb{C}$ , not how we parametrize it.

We will use

$$\oint_{\gamma} f(z)dz$$

to denote the integral.

**Example 6.1.3.** If  $\gamma(t) = t$ , then  $\gamma'(t) = 1$  and

$$\oint_{\gamma} f(z)dz = \int_a^b f(t)dt.$$

**Example 6.1.4.** If  $\gamma(t) = t + it^2$  and  $f(z) = 1$ , then  $\gamma'(t) = 1 + 2it$  and

$$\begin{aligned} \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b (1 + 2it)dt \\ &= \int_a^b 1dt + i \int_a^b 2tdt \\ &= b - a + i(b^2 - a^2) \\ &= (b + ib^2) - (a + ia^2) \\ &= \gamma(b) - \gamma(a). \end{aligned}$$

**Example 6.1.5** (Very important example). Consider  $\gamma(t) = e^{it}$  where  $0 \leq t \leq 2\pi$ . So  $\gamma(t)$  is the counterclockwise unit circular path. If  $f(z) = z^n$  for some  $n \in \mathbb{Z}$ . Then

$$\begin{aligned} \oint_{\gamma} f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \\ &= \int_0^{2\pi} e^{int} \cdot ie^{it}dt \\ &= i \int_0^{2\pi} e^{it(n+1)}dt. \end{aligned}$$

If  $n \neq -1$ , then  $n + 1 \neq 0$ , the integral evaluates to

$$\begin{aligned} i \left. \frac{e^{it(n+1)}}{i(n+1)} \right|_{t=0}^{2\pi} &= i \left( \frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right) \\ &= 0. \end{aligned}$$



If  $n + 1 = 0$ , then  $n = -1$  and so

$$\begin{aligned} i \int_0^{2\pi} e^{it(n+1)} dt &= i \int_0^{2\pi} 1 dt \\ &= 2\pi i, \end{aligned}$$

which is not zero.

## Chapter 7

# Complex Integration (Cont'd)

### 7.1 Basic Properties

(i) If  $\mu, \lambda \in \mathbb{C}$ , then

$$\begin{aligned}\oint_{\gamma} \lambda f(z) + \mu g(z) dz &= \int_a^b (\lambda f(\gamma(t)) + \mu g(\gamma(t))) \gamma'(t) dt \\ &= \lambda \int_a^b f(\gamma(t)) \gamma'(t) dt + \mu \int_a^b g(\gamma(t)) \gamma'(t) dt \\ &= \lambda \oint_{\gamma} f(z) dz + \mu \oint_{\gamma} g(z) dz.\end{aligned}$$

(ii)

$$\int_{\gamma_1 \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

(iii)

$$\int_{-\gamma} f = - \oint_{\gamma}$$

(iv)

$$\left| \oint_{\gamma} f \right| \leq \max_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma),$$

where

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

View  $|\gamma'(t)|$  as the speed a particle is travelling at and  $\gamma(t)$  as the position of that particle at time  $t$ . Then integrating it gives the total distance.

(v) **(Triangle Inequality)**

$$\left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \leq \int_a^b |f(\gamma(t))\gamma'(t)| dt$$

(vi) **(ML-Lemma)**

$$\begin{aligned} \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \\ &= ML \end{aligned}$$

where  $M = \max_{a \leq t \leq b} |f(\gamma(t))|$  and  $L = \int_a^b |\gamma'(t)| dt$ .

### 7.1.1 Antiderivatives

**Theorem 7.1.1** (Fundamental Theorem of Calculus). *If  $F$  is holomorphic on some subset  $G \subseteq \mathbb{C}$  and  $\frac{d}{dz}F(z) = f(z)$ . Then*

$$\oint_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)).$$

*Proof.* Let  $F(x + iy) = u(x, y) + iv(x, y)$  and  $\gamma(t) = \alpha(t) + i\beta(t)$ . Then

$$F(\gamma(t)) = u(\alpha(t), \beta(t)) + iv(\alpha(t), \beta(t)).$$

By chain rule,

$$\begin{aligned} \frac{d}{dt}F(\gamma(t)) &= u_x\alpha'(t) + u_y\beta'(t) + iv_x\alpha'(t) + iv_y\beta'(t) \\ &= u_x(\alpha'(t) + i\beta'(t)) + iv_x(\alpha'(t) + i\beta'(t)) \quad (u_x = v_y \text{ by CR}) \\ &= F(\gamma(t))\gamma'(t). \end{aligned}$$

□

**Definition 7.1.2** (Closed curve). A *closed curve* is a curve where the start and end points are the same, i.e.  $\gamma(a) = \gamma(b)$ .

So if  $f(z) = \frac{d}{dz}F(z)$ , the integral of  $f(z)$  around a closed curve is zero:

$$\oint_{\gamma} f(z)dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

**Example 7.1.3.** Let  $\gamma$  be the path of unit circle counterclockwise. Then

$$\oint_{\gamma} \frac{1}{z} dz = 2\pi i$$

which is not zero, implying that there is no holomorphic function  $F(z)$  defined on the whole unit circle, having derivative  $\frac{1}{z}$ .

However, consider the principal logarithm  $\log(re^{i\theta}) = \log(r) + i\theta$ , we have

$$\frac{d}{dz} \log(x + iy) = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

So  $\frac{1}{z}$  does have an antiderivative on  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ .

It turns out that if  $f(z)$  is continuous and  $\oint_{\gamma} f(z) dz = 0$  for any closed curve, then  $f(z)$  has an antiderivative, i.e. there's  $F(z)$  such that  $F'(z) = f(z)$ .

We know that by fundamental theorem of calculus

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

By analogy, we want

$$F(w) = \oint_{\gamma} f(z) dz$$

where  $\gamma$  is a curve from a fixed basepoint  $q$  to  $w$ .

First let's check that this doesn't depend on the choice of path from  $q$  to  $w$ . Suppose  $\delta_1, \delta_2$  are two paths from  $q$  to  $w$ . Observe that the reverse of  $\delta_2$  is a curve from  $w$  to  $q$ , and the path obtained by following  $\delta_1$ , then the reverse of  $\delta_2$  goes from  $q$  to  $w$  to  $q$ , so it is a closed curve.

Write  $\delta_1 - \delta_2$  for the closed curve above. Then by assumption, we have

$$\int_{\delta_1 - \delta_2} f(z) dz = 0.$$

This implies that

$$\int_{\delta_1} f(z) dz + \int_{-\delta_2} f(z) dz = \int_{\delta_1} f(z) dz - \int_{\delta_2} f(z) dz = 0.$$

Hence,

$$\int_{\delta_1} f(z) dz = \int_{\delta_2} f(z) dz.$$

Thus, the choice of path doesn't matter and so the formula  $F(w) = \oint_{\gamma} f(z)dz$  where  $\gamma$  is any path from  $q$  to  $w$  makes sense. Let's now check  $\frac{d}{dw}F(w) = f(w)$ .

$$\frac{d}{dw}F(w) = \lim_{h \rightarrow 0} \frac{F(w+h) - F(w)}{h}.$$

To evaluate  $F(w+h)$ , we can choose the path of integration from  $q$  to  $w+h$  arbitrarily. Let's choose one that goes from  $q$  to  $w$  then to  $w+h$  along a line segment (only from  $w$  to  $w+h$ ). This line segment has length  $|h|$ .

If our function is holomorphic at a point  $w$ , it is differentiable on a disk  $D(w, \epsilon)$  for some  $\epsilon > 0$ . So if  $|h| < \epsilon$ , then the line segment  $\ell$  from  $w$  to  $w+h$  is contained in  $D(w, \epsilon)$  and hence in a region where the function is differentiable.

Now  $F(w+h) - F(w)$  is simply the integral of  $f(z)$  from  $w$  to  $w+h$ . We want

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_{\ell} f(z)dz - f(w) = 0.$$

Note that  $f(w)$  is a constant independent of  $z$ . Thus,

$$\int_{\ell} f(w)dz = f(w) \int_{\ell} 1dz = f(w)h.$$

$$\lim_{h \rightarrow 0} \frac{\int_{\ell} f(z)dz - \int_{\ell} f(w)dz}{h} = \lim_{h \rightarrow 0} \frac{\int_{\ell} (f(z) - f(w))dz}{h}.$$

By ML-lemma,

$$\begin{aligned} \left| \frac{\int_{\ell} (f(z) - f(w))dz}{h} \right| &= \frac{|\int_{\ell} (f(z) - f(w))dz|}{|h|} \\ &\leq \max_{z \in \ell} |f(z) - f(w)| \cdot \frac{\text{length}(\ell)}{|h|} \\ &= \max_{z \in \ell} |f(z) - f(w)|. \end{aligned}$$

So it suffices to show that

$$\lim_{h \rightarrow 0} \max_{z \in \ell} |f(z) - f(w)| = 0.$$

Since  $f(z)$  is continuous at  $w$ , for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|z - w| < \delta$  implies  $|f(z) - f(w)| < \epsilon$ . So when  $|h| < \delta$ , any  $z \in \ell$  obeys  $|z - w| < \delta$ . Then also  $|f(z) - f(w)| < \epsilon$ . So for  $|h| < \delta$ ,

$$\left| \frac{F(w+h) - F(w)}{h} - f(w) \right| < \epsilon.$$

Hence,

$$\lim_{h \rightarrow 0} \frac{F(w+h) - F(w)}{h} = f(w).$$

as needed.

## 7.2 Cauchy's Theorem

**Question.** How do we check that  $\oint_{\gamma} f(z)dz = 0$  for any closed curve  $\gamma$ ?

**Theorem 7.2.1** (Cauchy's Theorem). *Suppose  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a closed curve and  $f(z)$  is holomorphic on  $\gamma$  and in the region enclosed by the curve  $\gamma$ . Then*

$$\oint_{\gamma} f(z)dz = 0.$$

*Proof.* Recall that a vector field is a function  $\vec{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  where

$$\vec{F}(x, y) = (F_1(x, y), F_2(x, y)).$$

The line integral of  $\vec{F}$  along a curve  $\gamma$  is

$$\oint_{\gamma} \vec{F} \cdot d\vec{\ell} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

where  $\gamma(t) = (\alpha(t), \beta(t))$  and  $\gamma'(t) = (\alpha'(t), \beta'(t))$  and so

$$\oint_{\gamma} \vec{F} \cdot d\vec{\ell} = \int_a^b F_1(\alpha(t), \beta(t))\alpha'(t) + F_2(\alpha(t), \beta(t))\beta'(t) dt.$$

Now recall the Stokes' theorem

$$\oint_{\gamma} \vec{F} \cdot d\vec{\ell} = \int_{\text{region enclosed by } \gamma} \vec{\nabla} \times \vec{F} dA$$

where

$$\vec{\nabla} \times \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

Then

$$\oint_{\gamma} (\vec{F} \cdot \hat{n}) d\ell = \int_a^b F_1(\alpha(t), \beta(t))\beta'(t) + F_2(\alpha(t), \beta(t))(-\alpha'(t)) dt.$$

The divergence theorem says that

$$\oint_{\gamma} (\vec{F} \cdot \hat{n}) d\ell = \int_{\text{area enclosed by } \gamma} \vec{\nabla} \cdot \vec{F} dA$$

where

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}.$$

Let  $f(x + iy) = u(x, y) + iv(x, y)$  and  $\gamma(t) = \alpha(t) + i\beta(t)$ .

$$\begin{aligned} \oint_{\gamma} f(z)dz &= \int_a^b (u + iv)(\alpha'(t) + i\beta'(t))dt \\ &= \int_a^b u\alpha'(t) - v\beta'(t)dt + i \int_a^b v\alpha'(t) + u\beta'(t)dt. \end{aligned}$$

Note that  $u' - v\beta' = (u, -v) \cdot (\alpha', \beta')$  and  $v\alpha' + u\beta' = (u, -v) \cdot (\beta', \alpha')$ . Now let  $\vec{F}(x, y) = (u(x, y), -v(x, y))$ . Then

$$\oint_{\gamma} f(z)dz = \oint_{\gamma} \vec{F} \cdot d\vec{\ell} + i \oint_{\gamma} (\vec{F} \cdot \hat{n})d\ell.$$

$$\begin{aligned} \vec{\nabla} \times \vec{F} &= \frac{\partial -v}{\partial x} - \frac{\partial u}{\partial y} = -v_x - u_y = 0 \quad (u_y = -v_x \text{ by CR}), \\ \vec{\nabla} \cdot \vec{F} &= \frac{\partial u}{\partial x} + \frac{\partial -v}{\partial y} = u_x - v_y = 0 \quad (u_x = v_y \text{ by CR}). \end{aligned}$$

So by Stokes' theorem and the divergence theorem

$$\oint_{\gamma} f(z)dz = \int_{\text{area enclosed by } \gamma} 0dA + i \int_{\text{area enclosed by } \gamma} 0dA = 0 + i0 = 0.$$

□

**Remark.** The example of  $\oint_{\gamma} \frac{1}{z}dz = 2\pi i$  does not contradict the Cauchy's theorem because  $\frac{1}{z}$  is not holomorphic at  $z = 0$ .

**Remark.** Checking that closed curves have well-defined interior regions is not a triviality: it is the content of the Jordan Curve Theorem (out of scope).

**Remark.** There are several formulations of Cauchy's Theorem. We will assume that  $f'(z)$  is continuous. Some formulations remove the concept of interior region and instead use the notion of a homotopy.

### 7.2.1 Cauchy Integral

**Theorem 7.2.2** (Cauchy integral formula (1st version)). *Suppose  $f(z)$  is holomorphic on the closed disk of radius  $R$  centered at  $a \in \mathbb{C}$ . Then*

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

where  $\gamma$  is the anticlockwise circle of radius  $R$  centered at  $a$ .

*Proof.*  $\frac{f(z)}{z-a}$  may not be holomorphic at  $z = a$ , so to apply Cauchy's theorem we need a curve that doesn't enclose  $a$ . We can create such curve by traversing a donut-like path obtained by traversing a clockwise small circle with radius  $r$  centered at  $a$  after we reached the endpoint of the original curve and then traverse back to the endpoint. Then the enclosed area will not include  $a$ .

This curve of integration has 4 parts:

1.  $\gamma_1$  : big circle of radius  $R$ , anticlockwise,
2.  $\gamma_2$  : line segment connecting from the big circle to the small circle,
3.  $\gamma_3$  : small circle of radius  $r$ , clockwise,
4.  $\gamma_4$  : line segment connecting from the small circle to the big circle.

Note that the integrations of the two line segments cancel out each other. Then Cauchy's theorem tells us that

$$\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0 \implies \int_{\gamma_1} + \int_{\gamma_3} = 0.$$

Thus,

$$\int_{\gamma_1} = - \int_{\gamma_3} = \int_{-\gamma_3},$$

which implies that the integral of the big circle anticlockwise is equal to the integral of the small circle anticlockwise. Hence,

$$\int_{\gamma_1} \frac{f(z)}{z-a} dz = \int_{-\gamma_3} \frac{f(z)}{z-a} dz.$$

for any  $r < R$ , and so we can take  $r \rightarrow 0$ . Now we want to show that

$$\int_{\gamma_3} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$



Let  $\gamma(t) = a + re^{it}$  for  $0 \leq t \leq 2\pi$ . Then

$$\begin{aligned} \oint_{\gamma} \frac{f(a)}{z-a} dz &= f(a) \oint_{\gamma} \frac{1}{z-a} dz \\ &= f(a) \int_0^{2\pi} \frac{1}{\gamma(t)-a} \gamma'(t) dt \\ &= f(a) \int_0^{2\pi} \frac{1}{re^{it}} ire^{it} dt \\ &= 2\pi i f(a). \end{aligned}$$

So we want

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\gamma} \frac{f(a)}{z-a} dz,$$

i.e.,

$$\oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz = 0.$$

To show this, we apply the ML-lemma:

$$\left| \oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| \leq \max_{z \in \gamma} \left| \frac{f(z) - f(a)}{z-a} \right| \cdot \text{length}(\gamma)$$

where  $\gamma$  is all points at distance  $r$  from  $a$ . Then  $z \in \gamma \implies |z-a| = r$  and  $\text{length}(\gamma) = 2\pi r$ . Thus, we get

$$\begin{aligned} \left| \oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| &\leq \max_{z \in \gamma} \frac{|f(z) - f(a)|}{r} \cdot 2\pi r \\ &= 2\pi \cdot \max_{z \in \gamma} |f(z) - f(a)|. \end{aligned}$$

Since  $f(z)$  is differentiable, it is continuous. So for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that  $|z-a| = r < \delta \implies |f(z) - f(a)| < \epsilon$ . So by taking  $r < \delta$ , we get

$$\left| \oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz \right| < 2\pi\epsilon$$

for any  $\epsilon > 0$ , which implies that the absolute value must be zero. Hence,

$$\oint_{\gamma} \frac{f(z) - f(a)}{z-a} dz = 0,$$

and this implies that

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = \oint_{\gamma} \frac{f(a)}{z-a} dz = 2\pi i f(a).$$

□

**Theorem 7.2.3** (Cauchy integral formula (2nd version)). *Let  $\gamma$  be a closed curve that encloses  $a \in \mathbb{C}$  exactly once anticlockwise. Suppose  $f(z)$  is holomorphic inside  $\gamma$ . Then*

$$\oint_{\gamma} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

*Proof.* Similar proof to previous one. □

**Example 7.2.4.** Suppose  $\omega \geq 0$  is a real number. Then

$$\int_{-\infty}^{\infty} \frac{\cos \omega x}{x^2 + 1} dx = \pi e^{-\omega}.$$

Consider

$$f(z) = \frac{e^{i\omega z}}{z^2 + 1} = \frac{e^{i\omega z}}{z+i} = \frac{g(z)}{z-i}$$

where  $g(z) = \frac{e^{i\omega z}}{z+i}$ . Consider the integral of  $f(z)$  over the semi-circle curve anticlockwise with radius  $R$  consisting of a line segment  $\ell$ . On the line segment, let  $\delta : [-R, R] \rightarrow \mathbb{C}$  be defined as  $\delta(t) = t$ . Then

$$\begin{aligned} \int_{\delta} \frac{g(z)}{z-i} dz &= \int_{-R}^R \frac{g(\delta(t))}{\delta(t)-i} \delta'(t) dt \\ &= \int_{-R}^R \frac{g(t)}{t-i} dt \\ &= \int_{-R}^R \frac{e^{i\omega x}}{x^2 + 1} dx \\ &= \int_{-R}^R \frac{\cos(\omega x)}{x^2 + 1} dx + i \int_{-R}^R \frac{\sin(\omega x)}{x^2 + 1} dx. \end{aligned}$$

Our goal is to compute the real part of the above expression. So what we want is

$$\lim_{R \rightarrow \infty} \int_{\ell} = \lim_{R \rightarrow \infty} \oint_{\gamma} - \int_{\text{arc}}.$$

By Cauchy integral formula, we have

$$\begin{aligned} \lim_{R \rightarrow \infty} \oint_{\gamma} \frac{g(z)}{z-i} dz &= 2\pi i g(i) \\ &= 2\pi i \frac{e^{i\omega i}}{i+i} \\ &= \pi e^{-\omega}. \end{aligned}$$

Now we compute

$$\lim_{R \rightarrow \infty} \int_{\text{arc}} \frac{e^{i\omega z}}{z^2 + 1} dz.$$

By ML-inequality, we have

$$\left| \int_{\text{arc}} \right| \leq \max_{z \in \text{arc}} \left| \frac{e^{i\omega z}}{z^2 + 1} \right| \cdot \text{length}(\text{arc}).$$

We needed to work with the upper half plane so that for  $z \in \text{arc}$ ,  $\Im(z) \geq 0$ , so then because  $\omega \geq 0$ ,  $\Im(\omega z) \geq 0$ . If  $\omega z = a + ib$ , then  $b \geq 0$  and  $i\omega z = -b + ia$  has non-positive real part. So

$$|e^{i\omega z}| = e^{\Re(i\omega z)} \leq e^0 = 1.$$

Also  $z \in \text{arc} \implies |z| = R$  and so  $|z^2| = R^2$ . Thus we have  $|z^2 + 1| \geq R^2 - 1$ . Hence,

$$\frac{1}{|z^2 + 1|} \leq \frac{1}{R^2 - 1}.$$

So ML-inequality becomes

$$\left| \int_{\text{arc}} \right| = \frac{1}{R^2 - 1} \pi R \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

So as  $R \rightarrow \infty$ ,

$$\int_{\text{arc}} \rightarrow 0.$$

Hence, we have

$$\begin{aligned} \oint_{\gamma} - \int_{\text{arc}} &= \int_{\ell} \\ \pi e^{-\omega} - 0 &= \int_{-\infty}^{\infty} \frac{\cos(\omega x)}{x^2 + 1} dx + i \int_{-\infty}^{\infty} \frac{\sin(\omega x)}{x^2 + 1} dx. \end{aligned}$$

Note that

$$\oint_{\gamma} \frac{f(z)}{z - a} dz = \begin{cases} 2\pi i f(a) & \text{if } a \text{ is in the region enclosed by } \gamma \\ 0 & \text{else.} \end{cases}$$

So if  $\gamma : [a', b'] \rightarrow \mathbb{C}$  is the curve,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - a} dz \\ &= \frac{1}{2\pi i} \int_{a'}^{b'} \frac{f(\gamma(t))}{\gamma(t) - a} \gamma'(t) dt. \end{aligned}$$

So for example, if  $f(z)$  is zero on the unit circle:  $f(e^{i\theta}) = 0$ , then for  $|a| < 1$ ,

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - a} i e^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} 0 d\theta \\ &= 0. \end{aligned}$$

So if  $f(z)$  is zero on the circle, it is also zero inside. For comparison,

$$f(x + iy) = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

is real differentiable and is zero on the unit circle, but not inside, so it's not holomorphic.

**Claim.**

$$f'(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

*Proof.*

$$\begin{aligned} 2\pi i f'(a) &= \lim_{h \rightarrow 0} 2\pi i \left( \frac{f(a+h) - f(a)}{h} \right) \\ &= \lim_{h \rightarrow 0} \frac{\oint_{\gamma} \frac{f(z)}{z-(a+h)} dz - \oint_{\gamma} \frac{f(z)}{z-a} dz}{h}. \end{aligned}$$

We assumed that  $a$  is inside  $\gamma$ , but we should also check  $a+h$ . Let  $\gamma : [a', b'] \rightarrow \mathbb{C}$  be the curve. Consider  $|\gamma(t) - a|$  (distance from  $\gamma(t)$  to  $a$ ).

The domain of  $|\gamma(t) - a|$  is  $[a', b']$ , a compact set (closed and bounded by Heine-Borel). Continuous image of a compact set is also compact, so in particular closed and so the complement is open. Note that  $|\gamma(t) - a| \neq 0$  (otherwise  $\gamma(t) = a$ , which means that  $a$  would be on the curve  $\gamma$ , which we don't allow). Then 0 is in the complement of the image of  $|\gamma(t) - a|$ . But since the set is open, it must also contain a neighborhood of 0. We can assume it is of the form  $(-\epsilon, \epsilon)$ .

**Conclusion:** not only does  $\gamma(t)$  avoid  $a$ , it never comes within  $\epsilon$  of it:  $|\gamma(t) - a| \geq \epsilon$  (complement contains  $[0, \epsilon)$ ). If  $|h| < \epsilon$ , then also  $a+h$  is inside  $\gamma(t)$ .

So for  $h$  small enough, we can write

$$\begin{aligned}
 2\pi i \frac{f(a+h) - f(a)}{h} &= \oint_{\gamma} \frac{\frac{f(z)}{z-(a+h)} - \frac{f(z)}{z-a}}{h} dz \\
 &= \frac{1}{h} \oint_{\gamma} \frac{(z-a)f(z) - (z-a-h)f(z)}{(z-a)(z-a-h)} dz \\
 &= \frac{1}{h} \oint_{\gamma} \frac{hf(z)}{(z-a)(z-a-h)} dz \\
 &= \oint_{\gamma} \frac{f(z)}{(z-a)(z-a-h)} dz.
 \end{aligned}$$

We want

$$\lim_{h \rightarrow 0} \oint_{\gamma} \frac{f(z)}{(z-a)(z-a-h)} dz = \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

or equivalently,

$$\lim_{h \rightarrow 0} \oint_{\gamma} \left( \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right) dz = 0.$$

$$\begin{aligned}
 \lim_{h \rightarrow 0} \oint_{\gamma} \left( \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right) dz &= \lim_{h \rightarrow 0} \oint_{\gamma} \frac{f(z)(z-a) - f(z)(z-a-h)}{(z-a)^2(z-a-h)} dz \\
 &= \lim_{h \rightarrow 0} h \oint_{\gamma} \frac{f(z)}{(z-a)^2(z-a-h)} dz.
 \end{aligned}$$

The ML-inequality says

$$\left| \oint_{\gamma} \right| \leq \max_{z \in \gamma} \frac{|f(z)|}{|z-a|^2|z-a-h|} \cdot \text{length}(\gamma).$$

Notice that  $|z-a| \geq \epsilon$  so for  $|h| \leq \frac{\epsilon}{2}$ .

$$\begin{aligned}
 |z-a-h| &\geq |z-a| - |h| \\
 &\geq \epsilon - \frac{\epsilon}{2} \\
 &= \frac{\epsilon}{2}.
 \end{aligned}$$

Hence,

$$\left| \oint_{\gamma} \right| \leq \max_{z \in \gamma} \frac{|f(z)|}{\epsilon^2 \cdot \frac{\epsilon}{2}} \cdot \text{length}(\gamma).$$

The bound is independent of  $h$ .

□

### 7.3 Liouville's Theorem

**Theorem 7.3.1** (Liouville's Theorem). *Suppose  $f(z)$  is holomorphic on all of  $\mathbb{C}$ , and  $f(z)$  is bounded, i.e.,  $|f(z)| \leq M$  for some fixed  $M$ . Then  $f(z)$  is constant.*

**Remark.** Note that  $f(x+iy) = \frac{x^2+y^2-1}{x^2+y^2+1} = 1 - \frac{2}{x^2+y^2+1}$  is real differentiable and bounded but it's not constant. So the theorem implies it's not holomorphic.

*Proof.* Let's compute  $f'(a)$  for some  $a \in \mathbb{C}$ . Let  $\gamma$  be the circle of radius  $R$  centered at  $a$ . Then

$$f'(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz.$$

ML-lemma says

$$\begin{aligned} |f'(a)| &\leq \frac{1}{2\pi} \max_{z \in \gamma} \frac{|f(z)|}{|z-a|^2} \cdot \text{length}(\gamma) \\ &= \frac{1}{2\pi} \max_{z \in \gamma} \frac{|f(z)|}{R^2} \cdot 2\pi R \\ &\leq \frac{1}{2\pi} \frac{M}{R^2} 2\pi R \\ &= \frac{M}{R}. \end{aligned}$$

Since  $M$  doesn't depend on  $R$ ,

$$|f'(a)| \leq \frac{M}{R} \quad \text{for any } R > 0.$$

As  $R \rightarrow \infty$ , this gets arbitrarily small. So  $|f'(a)| = 0$  and hence  $f'(a) = 0$ . So it must be true that  $f(a)$  is constant.  $\square$

**Claim.** Suppose  $f(z)$  is holomorphic inside  $\gamma$  and  $a$  is inside  $\gamma$ . Then  $f''(a)$  exists and

$$f''(a) = \frac{1}{\pi i} \oint_{\gamma} \frac{f(z)}{(z-a)^3} dz.$$

*Proof.* For  $w$  inside  $\gamma$ , let

$$f'(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z-w)^2} dz.$$

Then

$$\begin{aligned}
 2\pi i \cdot \frac{f'(a+h) - f'(a)}{h} &= \frac{1}{h} \left( \oint_{\gamma} \frac{f(z)}{(z-a-h)^2} dz - \oint_{\gamma} \frac{f(z)}{(z-a)^2} dz \right) \\
 &= \frac{1}{h} \oint_{\gamma} \frac{(z-a)^2 - (z-a-h)^2}{(z-a)^2(z-a-h)^2} f(z) dz \\
 &= \frac{1}{h} \oint_{\gamma} \frac{h(2z-2a-h)}{(z-a)^2(z-a-h)^2} \cdot f(z) dz.
 \end{aligned}$$

Then to show that

$$\lim_{h \rightarrow 0} \oint_{\gamma} \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2} \cdot f(z) dz = 2 \oint_{\gamma} \frac{f(z)}{(z-a)^3} dz,$$

we show

$$\lim_{h \rightarrow 0} \oint_{\gamma} \frac{(2z-2a-h)}{(z-a)^2(z-a-h)^2} \cdot f(z) dz - \oint_{\gamma} \frac{2f(z)}{(z-a)^3} dz = 0$$

The LHS becomes

$$\begin{aligned}
 &\lim_{h \rightarrow 0} \oint_{\gamma} \frac{(z-a)(2(z-a)-h) - 2(z-a-h)^2}{(z-a)^3(z-a-h)^2} \cdot f(z) dz \\
 &= \lim_{h \rightarrow 0} \oint_{\gamma} \frac{2(z-a)^2 - h(z-a) - 2((z-a)^2 - 2h(z-a) + h^2)}{(z-a)^3(z-a-h)^2} \cdot f(z) dz \\
 &= \lim_{h \rightarrow 0} \oint_{\gamma} \frac{3(z-a)h - 2h^2}{(z-a)^3(z-a-h)^2} \cdot f(z) dz \\
 &= \lim_{h \rightarrow 0} 3h \oint_{\gamma} \frac{(z-a)f(z)}{(z-a)^3(z-a-h)^2} dz - 2h^2 \oint_{\gamma} \frac{f(z)}{(z-a)^3(z-a-h)^2} dz.
 \end{aligned}$$

For the limit as  $h \rightarrow \infty$  to exist and be zero, it's enough that the integral remain bounded.

$$\begin{aligned}
 \left| \oint_{\gamma} \frac{(z-a)}{(z-a)^3(z-a-h)^2} f(z) dz \right| &\leq \max_{z \in \gamma} \left| \frac{f(z)}{(z-a)^2(z-a-h)^2} \right| \text{length}(\gamma) \\
 &\leq \max_{z \in \gamma} \frac{|f(z)|}{\epsilon^2 \cdot \left(\frac{\epsilon}{2}\right)^2} \cdot \text{length}(\gamma).
 \end{aligned}$$

The bound is independent of  $h$ . Similarly for the bound of

$$\left| \oint_{\gamma} \frac{1}{(z-a)^3(z-a-h)^2} f(z) dz \right|.$$

□

Suppose  $f(z)$  is holomorphic at  $a$ , then it is differentiable on a disk centered at  $a$  of some radius  $\epsilon > 0$ . If  $|z - a| < \epsilon$ , then  $f'(z)$  exists. Let  $\gamma$  be the circle centered at  $a$  with radius  $\frac{\epsilon}{2}$ , so that  $f(z)$  is differentiable inside  $\gamma$ . In fact, it's holomorphic inside  $\gamma$ . Since  $f(z)$  is differentiable inside the small disk containing the given point, it's holomorphic at that point. So we can apply the Cauchy integral formula and its corollaries and conclude that  $f''(a)$  exists. In particular,  $f'(z)$  is differentiable at  $z = a$ . Doing this on a disk around  $a$ , we conclude  $f'(z)$  is holomorphic at  $a$ .

Similarly  $f'(z)$  being holomorphic at  $a$  implies that  $f''(z)$  is holomorphic at  $a$ , which then implies  $f'''(z)$  as well, and so on. So  $f(z)$  being holomorphic implies  $f$  is infinitely differentiable.

## 7.4 Fundamental Theorem of Algebra

**Theorem 7.4.1** (Fundamental Theorem of Algebra). *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

*is a polynomial with complex coefficients, then either the polynomial is constant or it can be written as a product of linear factors  $(az + b)$ .*

Here's a weaker version of this theorem:

**Theorem 7.4.2** (Fundamental Theorem of Algebra (Weaker version)). *If  $p(z)$  is a non-constant polynomial, then there is some  $w$  such that  $p(w) = 0$  (any non-constant polynomial has a root).*

**Remark.** If  $az + b$  is a factor of  $p(z)$ , then  $w = -b/a$  makes  $aw + b = 0$ , so  $p(w) = 0$ . If  $p(w) = 0$ , then  $z - w$  divides  $p(z)$ . Then  $\frac{p(z)}{z-w}$  is still a polynomial and we can repeat this until the polynomial becomes a constant.

**Lemma 7.4.3.** *If*

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

*is non-constant ( $a_d \neq 0, d \geq 1$ ), then there is a real number  $R$  such that  $|z| \geq R$  implies*

$$\frac{1}{2}|a_d z^d| \leq |p(z)| \leq \frac{3}{2}|a_d z^d|,$$

*which is equivalent to*

$$\frac{1}{2} \leq \left| \frac{p(z)}{a_d z^d} \right| \leq \frac{3}{2},$$

*which is then equivalent to*

$$\left| \left| \frac{p(z)}{a_d z^d} \right| - 1 \right| \leq \frac{1}{2}.$$



*Proof.* We have

$$\begin{aligned}\frac{p(z)}{a_d z^d} - 1 &= \frac{a_d z^d}{a_d z^d} + \frac{a_{d-1} z^{d-1}}{a_d z^d} + \cdots + \frac{a_0}{a_d z^d} - 1 \\ &= \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots + \frac{a_0}{a_d} z^{-d}.\end{aligned}$$

Note when  $|z| \rightarrow \infty$ ,  $|z|^{-r} \rightarrow 0$  for any  $r > 0$ , i.e., for any  $\epsilon > 0$ , there is an  $R$  such that

$$|z| \geq R \implies ||z|^{-r} - 0| < \epsilon.$$

We want  $R$  such that  $|z| \geq R$  implies  $|z|^{-r} < \epsilon$ . Then

$$\begin{aligned}\log(|z|^{-r}) &< \log(\epsilon) \\ -r \log(|z|) &< \log(\epsilon) \\ \log(|z|) &> -\frac{\log(\epsilon)}{r} \\ |z| &> e^{-\frac{\log(\epsilon)}{r}} = \epsilon^{-1/r}.\end{aligned}$$

We take any  $R > \epsilon^{-1/r}$  to show that the limit exists.

**Conclusion:**

$$\frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots + \frac{a_0}{a_d} z^{-d} \rightarrow 0$$

as  $|z| \rightarrow \infty$ , i.e., for any  $\epsilon > 0$ , there is an such that  $|z| \geq R$  implies

$$\left| \frac{a_{d-1}}{a_d} z^{-1} + \frac{a_{d-2}}{a_d} z^{-2} + \cdots + \frac{a_0}{a_d} z^{-d} \right| < \epsilon.$$

We take  $\epsilon = \frac{1}{2}$  and conclude that for the resulting  $R$ ,  $|z| \geq R$  implies

$$\left| \frac{p(z)}{a_d z^d} - 1 \right| < \epsilon = \frac{1}{2}.$$

□

Now we want to apply Liouville's theorem to  $\frac{1}{p(z)}$ . If  $p(z) \neq 0$  for any  $z$ , then  $\frac{1}{p(z)}$  is holomorphic (composition of  $\frac{1}{z}$  with  $p(z)$ ).

$$\begin{aligned}\frac{1}{2} |a_d z^d| &\leq |p(z)| \\ \left| \frac{1}{p(z)} \right| &\leq \frac{2}{|a_d|} |z|^{-d} \leq \frac{2}{|a_d|} R^{-d} \quad (\text{since } |z| \geq R)\end{aligned}$$

To bound  $\frac{1}{p(z)}$  on the disk of radius  $R$  centered at zero (i.e.  $|z| \leq R$ ), notice that this region is compact (closed and bounded). Since  $\frac{1}{p(z)}$  is holomorphic, it is continuous and  $\left|\frac{1}{p(z)}\right|$  is the composition of the continuous functions  $\frac{1}{p(z)}$  and absolute value, hence it is also continuous. Since the continuous image of a compact set is compact, the image of  $\left|\frac{1}{p(z)}\right|$  on  $|z| \leq R$  is compact (so in particular, it is bounded), i.e.  $\left|\frac{1}{p(z)}\right| \leq M$  when  $|z| \leq R$ .

So now we combine the two bounds

$$\left|\frac{1}{p(z)}\right| \leq \max \left( \underbrace{\frac{2}{|a_d|} R^{-d}}_{\text{valid when } |z| \geq R}, \underbrace{M}_{\text{valid when } |z| \leq R} \right).$$

So for any choice of  $z$ , either  $|z| \geq R$  or  $|z| \leq R$ , and the inequality holds. So we've shown  $\frac{1}{p(z)}$  is a bounded holomorphic function. So by Liouville's theorem, it is constant.

Hence, if  $p(z)$  is never zero,  $\frac{1}{p(z)} = c$  is a constant, so  $p(z) = \frac{1}{c}$ . Since polynomial has no zero implies it is constant, we conclude that a polynomial being non-constant implies that the polynomial has a zero.

**Remark.** Not the only proof but a very typical application of Liouville's theorem (show some condition implies boundedness, deduce it's constant).

## Chapter 8

# Harmonic Functions

Recall the Cauchy-Riemann equations: if

$$f(x + iy) = u(x, y) + iv(x, y)$$

and if  $f$  is holomorphic, we have

$$\begin{aligned}u_x &= v_y \\u_y &= -v_x.\end{aligned}$$

### 8.1 Laplace Equation (2D)

We saw that if we can take a second derivative (and it's continuous, which guarantees  $u_{xx} = u_{yx}$ ) we get

$$u_{xx} = (u_x)_x = (v_y)_x = v_{yx} = v_{xy} = (v_x)_y = (-u_y)_y = -u_{yy}.$$

Then we obtain the  $2D$  Laplace equation:

$$u_{xx} + u_{yy} = 0.$$

Recall that obeying C-R equations (and first derivatives being continuous) implies  $f$  is holomorphic, so it is infinitely differentiable  $\implies$  it has second, third derivatives (second derivative differentiable  $\implies$  it is continuous).

So actually,  $f(x + iy)$  having continuous first derivatives obeying C-R equations is enough to deduce that  $u(x, y), v(x, y)$  are solutions to the Laplace equation.

**Definition 8.1.1** (Harmonic functions). The solutions to the Laplace equation are called *harmonic functions*.

We only consider the 2D Laplace equation.

**Goal:** We want to figure out conditions under which a harmonic function is the real part of a holomorphic function.

For us, a solution of the Laplace equation is a function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that the second derivatives  $u_{xx}, u_{yy}, u_{xy}, u_{yx}$  exist and are continuous and we have  $u_{xx} + u_{yy} = 0$  (this is a normal assumption when solving certain PDEs) (assuming continuity gives us  $u_{xy} = u_{yx}$ ).

**Theorem 8.1.2.** *If  $u(x, y)$  is a harmonic function on an open subset  $G \subseteq \mathbb{R}^2$  with no "holes" (more precisely, for any closed curve  $\gamma \in G$ , the region enclosed by  $\gamma$  is also contained in  $G$ ), then there is a harmonic function  $v : G \rightarrow \mathbb{R}$  such that*

$$f(x + iy) = u(x, y) + iv(x, y)$$

*is holomorphic on  $G$ .*

**Remark.** Such a  $v(x, y)$  is called a *harmonic conjugate* of  $u(x, y)$ .

**Idea:** hard to write down  $v(x, y)$  directly, but  $f'(x + iy) = u_x(x, y) + iv_x(x, y)$  and if  $f$  is holomorphic,  $v_x = -u_y$ , so  $f'(x + iy) = u_x(x, y) - iu_y(x, y)$ . So  $f'(x + iv)$  can be expressed in terms of  $u$ .

*Proof.* Define  $g(x + iy) = u_x(x, y) - iu_y(x, y)$ . Let's check it's holomorphic. Note that it has continuous first derivatives because  $u$  has continuous second derivatives, so it's enough that the C-R equations hold:

$$\begin{aligned}(u_x)_x &= u_{xx} = -u_{yy} = (-u_y)_y. \\ (u_y)_x &= u_{yx} = u_{xy} = -(-u_y)_x.\end{aligned}$$

Now we find an antiderivative of  $g(z)$ , i.e. a function  $f(z)$  such that  $f'(z) = g(z)$ . We saw that this can be done when

- (i)  $g(z)$  is continuous,
- (ii)  $\oint_{\gamma} g(z)dz = 0$  for any closed curve  $\gamma \in G$ .

(The construction was to define  $f(w) = \int_{\delta} g(z)dz$  where  $\delta$  is any path from a fixed basepoint to  $w$ .)

Since  $g(z)$  is holomorphic, it is differentiable and thus continuous. If  $\gamma$  is a closed curve in  $G$ , then by Cauchy's theorem

$$\oint_{\gamma} g(z)dz = 0$$

because  $g(z)$  is holomorphic on  $G$ , in particular inside the region enclosed by  $\gamma$ . So such  $f(z)$  exists and we write

$$f(x + iy) = a(x, y) + ib(x, y)$$

where  $a, b : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then  $f'(z) = g(z)$  implies

$$a_x + ib_y = u_x - iu_y,$$

which means  $a_x = u_x$  and  $-a_y = b_x = -u_y \implies a_y = u_y$  by C-R equations applied to  $f(z)$ , which is holomorphic because it satisfies  $f'(z) = g(z)$ . Now let's integrate:

$$\begin{aligned} a_x = u_x &\implies a(x, y) = u(x, y) + C(y) \\ a_y = u_y &\implies a(x, y) = u(x, y) + D(x). \end{aligned}$$

Taking the difference we have

$$C(y) - D(x) = 0 \implies C(y) = D(x),$$

which implies  $C, D$  are constants that doesn't depend on  $x$  or  $y$ . So we have

$$a(x, y) = u(x, y) + c,$$

where  $c$  is some constant. Then

$$f(x + iy) - c = a(x, y) - c + ib(x, y) = u(x, y) + ib(x, y).$$

Thus,  $f(z)$  is a holomorphic function whose real part is  $u(x, y)$ . □

If  $G = \mathbb{C} \setminus \{0\}$ , the theorem doesn't apply because unit circle encloses  $0 \notin G$ .

$$u(x, y) = \log(r) = \log(\sqrt{x^2 + y^2}).$$

which is a harmonic function but it's not the real part of a holomorphic function on  $G$ . But if we replace  $G$  by  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$  then it's the real part of  $\log(z)$ .

**Corollary 8.1.3.** *Any harmonic function is infinitely differentiable.*

*Proof.* If  $u : G \rightarrow \mathbb{R}$  ( $G \subseteq \mathbb{C}$  open), and  $z \in G$  is the point where we want to check  $u$  is infinitely differentiable. Since  $G$  is open, it contains a open disk centered at  $z$ , radius  $\epsilon > 0$ . So it also contains the closed disk of radius  $\epsilon/2$  centered at  $z$ . This small disk has no holes. Suppose  $\gamma : [a, b] \rightarrow D(z, \epsilon/2)$  is a curve, then  $|\gamma(t) - z| \leq \epsilon/2$ . Any point in the interior of  $\gamma$  is still inside the disk. To see this, note that no point outside the disk can be enclosed by the curve  $\gamma$ . By looking at a small enough part of  $G$  near  $z$ , we can assume our function is the real part of a holomorphic function. Since holomorphic functions are infinitely complex differentiable, they are infinitely real differentiable and so are their real parts, so our function is infinitely differentiable. □

## 8.2 Maximum Modulus Principle

**Proposition 8.2.1.** *Suppose  $u : G \rightarrow \mathbb{R}$  is a harmonic function where  $G$  is an open subset of  $\mathbb{C}$ . Suppose also that the closed disk of radius  $r$  centered at  $w$  is contained inside  $G$ . Then*

$$u(w) = \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta}) d\theta.$$

*This is saying that  $u(w)$  equals the average value of  $u(z)$  on the circle.*

*Proof.* We find  $f(z)$  holomorphic such that  $u(z) = \Re(f(z))$  for  $z$  in the closed disk, then we apply the Cauchy integral formula.

$$f(w) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - w} dz$$

where  $\gamma$  is the circle centered at  $w$  with radius  $r$ , i.e.,  $\gamma(t) = w + re^{it}$  where  $t \in [0, 2\pi]$ .

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - w} \gamma'(t) dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(\gamma(t)) i dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{it}) dt. \end{aligned}$$

Then take real part to recover  $u$ . So if we have a harmonic function defined on a disk, the value at the center is the average of the values on the boundary.  $\square$

**Proposition 8.2.2.** *Suppose  $u$  is a harmonic function on an open set containing the closed disk centered at  $w$  with radius  $r$ , and that  $u(z) \leq u(w)$  for all  $z \in G$ . Then  $u(z) = u(w)$  on the disk centered at  $w$  with radius  $r$ .*

*Proof.* If  $u(w + re^{it}) = u(w)$  for all  $t$ , we have equality. In this case  $u(z) = u(w)$  on the circle of radius  $r$  centered at  $w$ .

To recover the result for points inside the disk, apply the same argument with a smaller radius. So it's enough to show the result holds for  $z$  on the circle. We know  $u(w + re^{it}) \leq u(w)$  by assumption. Suppose  $u(w + re^{it}) \neq u(w)$  for some  $t_0$ . Then

$$u(w + re^{it_0}) < u(w).$$

Since  $u$  is harmonic, it is infinitely differentiable and thus continuous. So the  $u(w + re^{it})$  is a continuous function of  $t$  because it is the composition of continuous functions. Hence, for any  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|t - t_0| < \delta \implies |u(w + re^{it}) - u(w + re^{it_0})| < \epsilon.$$

Then

$$\begin{aligned} u(w + re^{it}) - u(w + re^{it_0}) &\leq |u(w + re^{it}) - u(w + re^{it_0})| \leq \epsilon = \frac{u(w) - u(w + re^{it_0})}{2} \\ 2u(w + re^{it}) - 2u(w + re^{it_0}) &\leq u(w) - u(w + re^{it_0}) \\ u(w) - u(w + re^{it_0}) &\leq 2u(w) - 2u(w + re^{it}) \\ \epsilon &\leq u(w) - u(w + re^{it}). \end{aligned}$$

Then

$$u(w + re^{it}) \leq u(w) - \epsilon.$$

Define  $g(t) = u(w) - u(w + re^{it})$ . Then we have

$$\begin{aligned} \int_0^{2\pi} g(t) dt &\geq \int_{t_0-\delta}^{t_0+\delta} g(t) dt \\ &\geq \int_{t_0-\delta}^{t_0+\delta} \frac{g(t_0)}{2} dt = 2\delta \frac{g(t_0)}{2} > 0, \end{aligned}$$

which is a contradiction, so  $g(t_0) = 0$  as needed.  $\square$

**Definition 8.2.3** (Path-connected). A set  $G$  is *path-connected* if for any  $p, q \in G$ , there is a path with  $p, q$  as endpoints ( $\gamma : [a, b] \rightarrow G$  where  $\gamma(a) = p, \gamma(b) = q$ ).

**Example 8.2.4.** Any disk (open or closed) is path-connected because it is convex (if two points are in a convex set, the line segment joining them is also in the set, which is exactly the path we need). Therefore, any convex set is also path-connected.

**Example 8.2.5** (Non-example). The union of two disjoint open disks of radius 1 centered at 2 and  $-2$  is not path-connected. Suppose  $\gamma : [a, b] \rightarrow G$  with  $\gamma(a) = -2$  and  $\gamma(b) = 2$ . Consider  $\Re(\gamma(t))$ , a real-valued function which is continuous. Then by the Intermediate Value Theorem, there is  $t \in [a, b]$  such that  $\Re(\gamma(t)) = 0$ . But there are not points in  $G$  with real part zero, which implies that  $\gamma(t) \notin G$ , a contradiction.

**Example 8.2.6** (Cont'd). Consider

$$g(p) = \begin{cases} 1 & \text{if } p \text{ is in right circle} \\ 0 & \text{if } p \text{ is in left circle.} \end{cases}$$

Note that  $g(z)$  is holomorphic on  $G$  since

$$\lim_{h \rightarrow 0} \frac{g(p+h) - g(h)}{h} = 0 \quad (g(p+h) = g(h)).$$

$g(p) = \Re(g(p))$  so  $g$  may also be viewed as a harmonic function on  $G$ . The maximal value of  $g$  is 1 (attained on right disk) but  $g$  is not constant (takes different value on left disk).

We want to find a sequence of disks going along the curve joining  $w$  and  $z$  and conclude the function is constant on each disk, eventually covering the whole curve. We will fix a curve  $\gamma$  and then show that for any  $\gamma(t)$ , we can draw a disk of radius  $\epsilon > 0$  (same  $\epsilon$  for all  $\gamma(t)$ ), so it will take roughly  $\text{length}(\gamma)/\epsilon$  steps to go from one end to the other.

So we have  $\gamma[a, b] \rightarrow G$  and we define  $d : [a, b] \rightarrow \mathbb{R}_{\geq 0}$  such that

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

$d(t)$  is the size of the largest disk we can draw at  $\gamma(t)$ . We want  $d(t) \geq \epsilon > 0$  for some  $\epsilon$  and all  $t$ . We will show

- $d(t) > 0$  for all  $t$
- $d(t)$  is continuous.

**Proposition 8.2.7.** *If  $G$  is an open, path-connected region, then a harmonic function  $u : G \rightarrow \mathbb{R}$  such that there is a  $w \in G$  with  $u(w) \geq u(z)$  for all  $z \in G$  must be a constant function.*

*Proof.* We want to show  $u(w) = u(z)$  by stepping along the path from  $w$  to  $z$ , showing  $u$  is constant along each step. If  $u$  is constant on a disk of radius  $\epsilon$  centered at  $w$ , then we can go up to  $\epsilon$  along the curve and  $u$  will still take the value  $u(w)$ .

The argument will be go  $\epsilon/2$  (avoid reaching the boundary since disk is open) along the curve to a point  $\zeta$ . We have  $u(\zeta) = u(w)$ , but the length of the part of  $\gamma$  from  $\zeta$  to  $w$  is  $\text{length}(\gamma) - \epsilon/2$ . We can then repeat the argument to get a path of length  $\text{length}(\gamma) - n\epsilon/2$  for  $n$  steps. Eventually this becomes  $< \epsilon$ , so  $z$  is inside the disk of radius  $\epsilon$  centered at that point, which implies values are equal ( $= u(w)$ ).

**Conclusion:** for any  $z \in G$ ,  $u(z) = u(w)$  so  $u$  is constant on  $G$ .

What we still need to check: we can use disks of the same radius  $\epsilon$  at any point on  $\gamma$ . We need to find  $\epsilon > 0$  and need the disk of radius  $\epsilon$  centered at  $\gamma(t)$  to be contained in  $G$  for all  $t$ . Equivalently, we want  $\epsilon > 0$  such that

$$|\gamma(t) - z| < \epsilon \implies z \in G.$$

We have  $G \subseteq \mathbb{C}$  open,  $\gamma : [a, b] \rightarrow G$  is continuous. Define

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

Note that  $d(t) > 0$  because  $\gamma(t) \in G$ , and  $G$  is open, there is some disk centered at  $\gamma(t)$  contained in  $G$  with radius  $\epsilon(t)$ , so any  $z \in \mathbb{C} \setminus G$  is at least  $\epsilon(t)$  away from  $\gamma(t)$ , i.e.



$|\gamma(t) - z| \geq \epsilon(t) > 0$ , so  $\epsilon(t)$  is a lower bound for the set  $\{|\gamma(t) - z| : z \in \mathbb{C} \setminus G\}$  so it is less than or equal to the greatest lower bound, the infimum:

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z| \geq \epsilon(t) > 0.$$

Now we show  $d(t)$  is continuous, i.e.

$$\lim_{h \rightarrow 0} d(t+h) = d(t).$$

(since  $d(t)$  is a distance (from  $\gamma(t)$  to  $\mathbb{C} \setminus G$ ), we will control it with the triangle inequality).

For  $z \in \mathbb{C} \setminus G$ ,  $|\gamma(t) - z| \geq d(t)$  by definition of  $d(t)$ , which implies  $|\gamma(t+h) - z| \geq d(t+h)$ . Then

$$\begin{aligned} d(t) &\leq |\gamma(t) - z| = |\gamma(t) - \gamma(t+h) + \gamma(t+h) - z| \\ &\leq |\gamma(t) - \gamma(t+h)| + |\gamma(t+h) - z|. \end{aligned}$$

So  $d(t) - |\gamma(t) - \gamma(t+h)| \leq |\gamma(t+h) - z|$ , which means  $d(t) - |\gamma(t) - \gamma(t+h)|$  is a lower bound for  $|\gamma(t+h) - z|$ , which is less than the infimum. Hence,

$$d(t) - |\gamma(t) - \gamma(t+h)| \leq d(t+h).$$

Rewrite this as

$$d(t) - d(t+h) \leq |\gamma(t) - \gamma(t+h)|.$$

To get a lower bound, we use

$$\begin{aligned} d(t+h) &\leq |\gamma(t+h) - z| = |\gamma(t+h) - \gamma(t) + \gamma(t) - z| \\ &\leq |\gamma(t+h) - \gamma(t)| + |\gamma(t) - z|, \end{aligned}$$

so

$$d(t+h) - |\gamma(t+h) - \gamma(t)| \leq |\gamma(t) - z|.$$

Then  $d(t+h) - |\gamma(t+h) - \gamma(t)| \leq d(t)$  and so

$$-|\gamma(t+h) - \gamma(t)| \leq d(t) - d(t+h).$$

**Conclusion:**

$$0 \leq |d(t+h) - d(t)| \leq |\gamma(t+h) - \gamma(t)|.$$

Now use squeeze theorem as  $h \rightarrow 0$ , by continuity of  $\gamma$ ,

$$\lim_{h \rightarrow 0} \gamma(t+h) = \gamma(t).$$

So the RHS  $\rightarrow 0$  and so  $|d(t+h) - d(t)| \rightarrow 0$ . Therefore  $d(t+h) \rightarrow d(t)$ , i.e.  $d(t)$  is continuous. So we have  $d : [a, b] \rightarrow \mathbb{R}$  (domain of  $\gamma : [a, b] \rightarrow G$ ) with  $d(t) > 0$  and  $d$  continuous. Since  $[a, b]$  is compact, the set of values of  $d(t)$  is again compact. By (Weierstrass theorem), a continuous function on a compact set has a minimum which it attains. Then the minimal value of  $d(t)$  for  $t \in [a, b]$  is some positive number  $\epsilon$  (cannot be  $\leq 0$  because  $d$  never takes such values.)

**Conclusion:**  $d(t) \geq \epsilon > 0$  for this  $\epsilon$ . □

**Theorem 8.2.8** (Maximum modulus principle). *Suppose  $G \subseteq \mathbb{C}$  is open and path-connected, then  $f : G \rightarrow \mathbb{C}$  is holomorphic if  $|f(z)|$  attains a maximum in  $G$ , then  $f(z)$  is constant.*

## Chapter 9

# Power Series

**Definition 9.0.1** (Power series).

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

**Example 9.0.2** (Taylor series).

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2!} + \cdots.$$

To understand this, we need to discuss what it means for a sequence of functions to converge.

**Remark.** The convergence of the sequence of partial sums  $\sum_{n=0}^k a_n (z - z_0)^n$  is the same thing as convergence of the series  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ .

## 9.1 Convergence of sequences of functions

Suppose  $G \subseteq \mathbb{C}$  and  $f_n : G \rightarrow \mathbb{C}$  where  $n \in \mathbb{Z}_{\geq 0}$  is a sequence of functions.

### 9.1.1 Pointwise convergence

**Definition 9.1.1** (Pointwise convergence).  $f_n(z)$  converges to  $f(z)$  *pointwise* if for each  $z \in G$ , the sequence  $f_n(z)$  converges to  $f(z)$ , i.e. for each  $z \in G$ , for every  $\epsilon > 0$ , there is a  $N \geq 0$  (can depend on  $z$ ) such that  $n \geq N$  implies

$$|f_n(z) - f(z)| < \epsilon.$$

**Remark.** Evaluating at  $z$  gives a sequence of complex numbers.

**Example 9.1.2.** Consider  $f_n(z) = z^n$ .  $G = [0, 1] \subseteq \mathbb{R}$ . When  $z = 1$ ,  $f_n(z) = 1$ , we get a constant sequence 1 and so  $f_n(z) \rightarrow 1$ . When  $z < 1$ ,  $f_n(z) \rightarrow 0$ . We need to find  $N$  such that

$$\begin{aligned} |f_n(z) - 0| &< \epsilon & \text{for } n \geq N \\ z^n &< \epsilon & \text{for } n \geq N. \end{aligned}$$

This holds when

$$\begin{aligned} n \ln(z) &< \ln(\epsilon) \\ n &> \frac{\ln(\epsilon)}{\ln(z)} & (\text{divide by } \ln(z) < 0 \text{ since } z \in [0, 1]). \end{aligned}$$

Then we choose  $N = \frac{\ln(\epsilon)}{\ln(z)}$ .

**Conclusion:**

$$f_n(z) \rightarrow \begin{cases} 1 & z = 1, \\ 0 & 0 \leq z < 1. \end{cases}$$

which is not continuous.

To remedy this, we define a more stringent notion of convergence.

### 9.1.2 Uniform convergence

**Definition 9.1.3** (Uniform convergence). A sequence of functions  $f_n(z)$  converges to  $f(z)$  *uniformly* on  $G$  if for every  $\epsilon > 0$ , there exists  $N \geq 0$  such that

$$|f_n(z) - f(z)| < \epsilon \quad \text{for all } n > N \text{ and } z \in G.$$

**Remark.** The difference from pointwise convergence is that  $N$  cannot depend on  $z$ .

**Example 9.1.4** (Non-example). Back to the  $f_n(z) = z^n$  example. Now let  $G = [0, r]$  where  $r < 1$  is fixed.

$$\begin{aligned} |f_n(z) - 0| &< \epsilon \\ z^n &< \epsilon \\ n &> \frac{\ln(\epsilon)}{\ln(z)}. \end{aligned}$$

We need to find an  $N$  such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in G = [0, r],$$

which is maximized at  $z = r$  and so we can take  $N = \frac{\ln(\epsilon)}{\ln(r)}$ . Note though that while  $f_n(z) \rightarrow 0$  on  $G = [0, 1)$ , it does not converge uniformly because we would need an  $N$  such that

$$N \geq \frac{\ln(\epsilon)}{\ln(z)} \quad \text{for all } z \in [0, 1).$$

But as  $z$  approaches 1,  $\ln(\epsilon)/\ln(z) \rightarrow \infty$ , so there is no such  $N$  for which all  $z$  obey the inequality.

Two main desirable properties of uniform convergence:

**Proposition 9.1.5.** *if  $f_n(z)$  are continuous and converge uniformly to  $f(z)$ , then  $f(z)$  is continuous.*

*Proof.* Fix  $z_0$ , we need to show that for every  $\epsilon > 0$ , there is a  $\delta > 0$  such that

$$|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Let  $N$  be such that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for  $n \geq N$  (use uniform convergence). Now pick some  $n \geq N$  and let  $\delta$  be such that

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$$

whenever  $|z - z_0| < \delta$ . This follows from the continuity of  $f_n(z)$ . Finally, if  $|z - z_0| < \delta$ ,

$$\begin{aligned} |f(z) - f(z_0)| &= |f(z) - f_n(z) + f_n(z) - f_n(z_0) + f_n(z_0) - f(z_0)| \\ &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus,  $f(z)$  is continuous. □

**Proposition 9.1.6.** *If  $f_n(z)$  converges to  $f(z)$  uniformly on  $G$ , and  $\gamma$  is a curve in  $G$ , then*

$$\lim_{n \rightarrow \infty} \oint_{\gamma} f_n(z) dz = \oint_{\gamma} f(z) dz.$$

(so we can swap the order if taking limits and integration).

*Proof.* For  $\epsilon > 0$ , pick  $N$  such that for all  $n \geq N$  (use uniform convergence), we have

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

$$\begin{aligned} \left| \oint_{\gamma} f_n(z) dz - \oint_{\gamma} f(z) dz \right| &= \left| \oint_{\gamma} f_n(z) - f(z) dz \right| \\ &\leq \max_{z \in \gamma} |f_n(z) - f(z)| \cdot \text{length}(\gamma). \end{aligned}$$

But for all  $z \in G$  (not just  $\gamma$ ),

$$|f_n(z) - f(z)| < \frac{\epsilon}{\text{length}(\gamma)}.$$

Thus,

$$\begin{aligned} \left| \oint_{\gamma} f_n(z) dz - \oint_{\gamma} f(z) dz \right| &\leq \frac{\epsilon}{\text{length}(\gamma)} \cdot \text{length}(\gamma) \\ &= \epsilon. \end{aligned}$$

□

### 9.1.3 Weierstrass $M$ -test for uniform convergence

**Theorem 9.1.7** (Weierstrass  $M$ -test). *Suppose  $G \subseteq \mathbb{C}$ ,  $f_k : G \rightarrow \mathbb{C}$  and  $M_k \in \mathbb{R}$  with  $|f_k(z)| \leq M_k$  for all  $z \in G$ . Then*

$$\sum_{k=0}^{\infty} M_k \text{ converges} \implies \sum_{k=0}^{\infty} f_k(z) \text{ converges uniformly on } G.$$

*Proof.* Suppose  $\sum_{k=0}^{\infty} M_k$  converges to  $M$ . Since  $0 \leq |f_k(z)| \leq M_k$ ,  $M_k \geq 0$ . For any  $\epsilon > 0$ , there exists  $N > 0$  such that for  $n \geq N$ ,

$$\left| \sum_{k=0}^n M_k - M \right| < \epsilon.$$

Then we have

$$\begin{aligned} \left| \sum_{k=0}^n M_k - \sum_{k=0}^{\infty} M_k \right| &< \epsilon \\ \left| - \sum_{k=n+1}^{\infty} M_k \right| &< \epsilon. \end{aligned}$$

Since each  $M_k \geq 0$ , we have

$$\sum_{k=n+1}^{\infty} M_k < \epsilon.$$

Now we need to show that

$$\left| \sum_{k=0}^n f_k(z) - \sum_{k=0}^{\infty} f_k(z) \right| < \epsilon.$$

Since

$$\left| \sum_{k=0}^{\infty} f_k(z) \right| \leq \sum_{k=0}^{\infty} |f_k(z)| \leq \sum_{k=0}^{\infty} M_k < \infty,$$

$\sum_{k=0}^{\infty} f_k(z)$  converges absolutely and so the limit of the sum exists.

$$\begin{aligned} \left| - \sum_{k=n+1}^{\infty} f_k(z) \right| &\leq \sum_{k=n+1}^{\infty} |f_k(z)| \\ &\leq \sum_{k=n+1}^{\infty} M_k \\ &< \epsilon. \end{aligned}$$

This no longer depends on  $z$ , so we just use the  $N$  from  $\sum_k M_k$ . □

**Example 9.1.8.** Let  $f_k(z) = z^k$  where  $|z| < 1$ . Then

$$\sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}.$$

**Question.** Does it converge uniformly?

Let's view  $f_k(z)$  as functions on  $|z| \leq r$  (here  $0 \leq r < 1$  fixed). Convergence on this set is uniform.

*Proof.* Use  $M$ -test. We need  $|f_k(z)| = |z^k| = |z|^k \leq r^k$ . So choose  $M_k = r^k$ .

$$\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \quad (r < 1).$$

**Conclusion:**  $M$ -test applies and we have uniform convergence. □

**Question.** What about  $|z| < 1$ ?

**Answer.** Convergence is not uniform. If it was, then for every  $\epsilon > 0$ , there is a  $N > 0$  such that for  $n \geq N$

$$\begin{aligned} \left| \sum_{k=0}^n z^k - \frac{1}{1-z} \right| &< \epsilon \\ \left| \frac{1-z^{n+1}}{1-z} - \frac{1}{1-z} \right| &< \epsilon \\ \frac{|z|^{n+1}}{|1-z|} &< \epsilon. \end{aligned}$$

However, the problem is that as  $z \rightarrow 1$ , this goes to infinity, so the bound  $\frac{|z|^{n+1}}{|1-z|} < \epsilon$  cannot hold for all  $|z| < 1$ . Thus, the convergence is not uniform.

**Remark.** Even though  $\{z : |z| < 1\}$  is the union of  $\{z : |z| \leq r\}$  for  $r < 1$ , we don't have uniform convergence on the open unit disk.

## 9.2 Power Series

**Lemma 9.2.1.** *If  $\sum_{k=0}^{\infty} a_k(w - z_0)^k$  converges, then  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  converges absolutely when  $|z - z_0| < |w - z_0|$ . Moreover, if  $0 \leq \ell < |w - z_0|$ , then convergence is uniform on  $|z - z_0| \leq \ell$ .*

*Proof.* Since  $\sum_{k=0}^{\infty} a_k(z - z_0)^k$  converges, the terms must go to zero (because if  $\sum_{k=0}^{\infty} c_k$  converges, then we would have

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} \sum_{k=0}^n c_k - \sum_{k=0}^{n-1} c_k = 0.)$$

Therefore,

$$\lim_{k \rightarrow \infty} a_k(w - z_0)^k = 0,$$

i.e., for any  $\epsilon > 0$ , there is  $N > 0$  such that for  $n \geq N$ ,

$$|a_n(w - z_0)^n - 0| < \epsilon.$$

Thus,  $|a_n(w - z_0)^n| < \epsilon$ . Now let

$$M = \max \{|a_0(w - z_0)^0|, \dots, |a_N(w - z_0)^N|, \epsilon\},$$

so  $M \geq |a_k(w - z_0)^k|$  for all  $k$ . Then

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k(z - z_0)^k| &= \sum_{k=0}^{\infty} |a_k(w - z_0)^k| \left| \frac{z - z_0}{w - z_0} \right|^k \\ &\leq \sum_{k=0}^{\infty} M \left| \frac{z - z_0}{w - z_0} \right|^k. \end{aligned}$$



Since this is a geometric series with common ratio  $\left| \frac{z-z_0}{w-z_0} \right| < 1$ , it converges.

**Conclusion:** we have absolute convergence.

For uniform convergence, use  $M$ -test:

$$|a_k(z-z_0)^k| \leq M \left| \frac{z-z_0}{w-z_0} \right|^k = M_k \leq \left| \frac{\ell}{w-z_0} \right|^k \quad (w-z_0 < 1).$$

We need sum of  $M_k$  to converge:

$$\sum_k M_k = \sum_k M \left( \frac{\ell}{|w-z_0|} \right)^k,$$

which is a geometric series with common ratio  $< 1$ , so it converges and is uniformly convergent by  $M$ -test.  $\square$

**Theorem 9.2.2.** *For a power series  $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ , there is a  $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$  such that the series converges absolutely when  $|z-z_0| < R$  and diverges when  $|z-z_0| > R$ . It converges uniformly on  $|z-z_0| \leq \ell$  ( $\ell < R$  is fixed).*

*Proof.* Consider

$$S = \left\{ x \in \mathbb{R}_{\geq 0} \mid \sum_{k=0}^{\infty} a_k x^k \text{ converges} \right\}.$$

Note that  $0 \in S$ , and so  $S$  is nonempty (can ask for supremum, which will be in  $\mathbb{R}_{\geq 0} \cup \{\infty\}$ ). Applying the lemma above, if  $x \in S$ , then  $\sum_{k=0}^{\infty} a_k x^k$  converges and so  $\sum_{k=0}^{\infty} a_k(z-z_0)^k$  converges absolutely for  $|z-z_0| < |x| = x$  ( $w = z_0 + x$ ). So if we let  $z = z_0 + y$ , then

$$|y| < x \implies \sum_{k=0}^{\infty} a_k y^k \text{ converges absolutely} \implies y \in S.$$

$y$  can be in  $[0, x)$ . So  $x \in S \implies [0, x) \subseteq S \implies [0, x] \subseteq S$ . Consider the following cases:

1.  $\sup S = \infty$  ( $S$  is unbounded):  $S$  contains a sequence  $x_i$  with limit  $\infty$ . Then  $S$  contains  $[0, x_i]$  for all  $i$ . For any real number  $t$ , eventually  $x_i \geq t$ . Then  $t \in [0, x_i] \subseteq S$ . Hence,  $t \in S$  and  $S = \mathbb{R}_{\geq 0}$  ( $R = \infty$ ).
2.  $\sup S = R$ . If  $|z-z_0| > R$  and  $\sum_{k=0}^{\infty} a_k(z-z_0)^k$  converges, then  $S$  contains all  $x$  with  $0 \leq x < |z-z_0|$ . But  $R$  is an upper bound for  $S$  so it must be at least as big as any such  $x$ , for example,

$$x = \frac{R + |z-z_0|}{2} > R.$$

This contradicts  $R$  being an upper bound for  $S$ . So  $|z-z_0| > R \implies$  divergence. Now if  $|z-z_0| < R$ , then  $S$  contains an element  $x$  such that  $|z-z_0| < x \leq R$  (otherwise

$|z - z_0|$  would be an upper bound for  $S$  strictly less than  $R$ , the least upper bound). Now  $\sum_{k \geq 0} a_k x^k$  converges as  $x \in S$  and  $|z - z_0| < x \implies$  absolute convergence for  $|z - z_0| < x$  (by lemma). Hence, we have absolute convergence for  $|z - z_0| < R$ .

□

**Definition 9.2.3** (Radius of convergence).  $R$  is the *radius of convergence* of  $\sum_{k \geq 0} a_k (z - z_0)^k$  where  $|z - z_0| < R$  (region on which we have convergence).

**Remark.** We cannot conclude anything about convergence when  $|z - z_0| = R$ .

**Lemma 9.2.4.** *The radius of convergence  $R$  obeys*

$$\frac{1}{R} = \limsup \sqrt[k]{|a_k|}.$$

*Proof.* If  $\limsup \sqrt[k]{|a_k|} = L$ . For every  $\epsilon > 0$ , there exists  $N$  such that  $k \geq N$  implies

$$\sqrt[k]{|a_k|} > L + \epsilon.$$

Then

$$|a_k| > (L + \epsilon)^k.$$

Hence,

$$\begin{aligned} \sum_{k \geq 0} |a_k (z - z_0)|^k &\leq \sum_{k=0}^N |a_k| |z - z_0|^k + \sum_{k > N} |a_k| |z - z_0|^k \\ &= \text{constant} + \sum_{k > N} (L + \epsilon)^k |z - z_0|^k. \end{aligned}$$

This converges when  $(L + \epsilon)|z - z_0| < 1$  for some  $\epsilon > 0$ , equivalently,

$$|z - z_0| < \frac{1}{L + \epsilon},$$

and so we have absolute convergence when  $|z - z_0| < 1/L$ . We conclude  $R \geq 1/L$  because if  $1/L > R$ , choose  $z$  such that  $1/L > |z - z_0| > R$  (first inequality implies absolute convergence but the second inequality implies divergence, contradiction!)

If  $L = \limsup \sqrt[k]{|a_k|}$ , for any  $\epsilon > 0$ , there are infinitely many  $k$  such that  $\sqrt[k]{|a_k|} > L - \epsilon$ . Assume  $L > 0$  (need to show  $L = 0 \iff R = \infty$ ) for  $\epsilon$  small enough such that  $L - \epsilon > 0$ . For infinitely many  $k$ , we have

$$\begin{aligned} |a_k| &> (L - \epsilon)^k \\ |a_k| |z - z_0|^k &\geq (L - \epsilon)^k |z - z_0|^k. \end{aligned}$$

If converged, it would go to zero and then  $(L - \epsilon)|z - z_0| < 1$ . If  $\frac{1}{L - \epsilon} \leq |z - z_0|$ , we have divergence. So  $\frac{1}{L} < |z - z_0|$  implies divergence.

**Claim.**

$$\frac{1}{L} \geq R$$

*Proof.* If not, we have a contradiction because

$$R > |z - z_0| > \frac{1}{L}$$

implies both convergence and divergence. □

Since we have  $\frac{1}{L} \geq R$  and  $\frac{1}{L} \leq R$ , we have  $\frac{1}{L} = R$ . □

**Remark.** If  $L = 0$ , any  $\epsilon > 0$  is an upper bound on all but finitely many terms  $\sqrt[k]{|a_k|}$ .

**Example 9.2.5.** Consider  $a_k = \frac{1}{k!}$ ,  $z_0 = 0$ ,  $\sum_{k \geq 0} \frac{z^k}{k!} = e^z$ . Then we can find the radius of convergence via

$$\frac{1}{R} = \limsup \sqrt[k]{\left| \frac{1}{k!} \right|}.$$

To compute this directly, we can use stirling's formula:

$$n! \approx \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.$$

We can see that

$$\sqrt[n]{n!} \approx \frac{n}{e} + \text{lower terms.}$$

Therefore,

$$\sqrt[n]{\frac{1}{n!}} \approx \frac{e}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

So  $R = \infty$ . We can also use ratio test.

**Example 9.2.6.**  $\sum_k a_k (z - z_0)^k = z - \frac{z^3}{3} + \frac{z^5}{5} - \dots$  where  $z_0 = 0$  and

$$a_k = \begin{cases} 0 & k \text{ even} \\ \frac{(-1)^{(k-1)/2}}{k} & k \text{ odd.} \end{cases}$$

$$\limsup_{k \text{ odd}} \sqrt[k]{\left| \frac{(-1)^{(k-1)/2}}{k} \right|} = \sqrt[k]{\frac{1}{k}}.$$

Taking log gives

$$\ln \left( \sqrt[k]{\frac{1}{k}} \right) = \frac{1}{k} (-\ln(k)) \rightarrow 0.$$

Then

$$\sqrt[k]{\frac{1}{k}} \rightarrow 1.$$

**Proposition 9.2.7.** *If  $\sum a_k(z - z_0)^k$  has radius of convergence  $R$ , then the limit function is continuous on the open disk centered at  $z_0$  with radius  $R$ .*

*Proof.* We want to argue that uniform convergence of continuous things gives something continuous. Let  $z$  be in the disk  $|z - z_0| < R$ .  $z$  is contained in the disk  $|w - z_0| \leq \frac{|z - z_0| + R}{2}$ . By what we know, convergence is uniform here, hence it is continuous on this disk, in particular at  $z$ . So for any  $z$  in the disk, we have continuity (i.e., the power series is continuous).  $\square$

**Proposition 9.2.8.** *If  $\gamma$  is a curve in the open disk  $|z - z_0| < R$  centered at  $z_0$ , then*

$$\oint_{\gamma} \sum_{k \geq 0} a_k(z - z_0)^k dz = \sum_{k \geq 0} a_k \oint_{\gamma} (z - z_0)^k dz.$$

*If  $\gamma$  is a closed curve, then this is zero.*

*Proof.* If we have uniform convergence, then

$$\begin{aligned} \oint_{\gamma} \sum_{k \geq 0} a_k(z - z_0)^k dz &= \lim_{n \rightarrow \infty} \oint_{\gamma} \sum_{k=0}^n a_k(z - z_0)^k dz \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \oint_{\gamma} a_k(z - z_0)^k dz \\ &= \sum_{k=0}^{\infty} \oint_{\gamma} a_k(z - z_0)^k dz. \end{aligned}$$

So it would be enough for the series to converge uniformly on a set containing the curve  $\gamma$ . We want to find a disk of radius  $< R$  still containing  $\gamma$ . When we discussed how harmonic functions, we considered

$$d(t) = \inf_{z \in \mathbb{C} \setminus G} |\gamma(t) - z|.$$

We showed  $d(t) \geq \epsilon > 0$ .

**Claim.** The disk of radius  $R - \epsilon/2$  contains  $\gamma$ .

If there was a point on  $\gamma$  outside this region, then it is within  $\epsilon/2$  of a point outside the disk of radius  $R$ . This would contradict

$$\inf |\gamma(t) - z| \geq \epsilon.$$

We would have produced a value of this that was at most  $\epsilon/2$ . So we can find such a disk and we have uniform convergence.

If  $\gamma$  is closed, by Cauchy's theorem

$$\oint_{\gamma} (z - z_0)^k = 0.$$

Then

$$\oint_{\gamma} \sum_{k \geq 0} a_k (z - z_0)^k dz = \sum_{k=0}^{\infty} a_k \cdot 0 = 0.$$

□

**Theorem 9.2.9** (Morera's Theorem). *Suppose  $f(z)$  is continuous on an open set  $G$  and for any closed curve  $\gamma$  in  $G$ ,*

$$\oint_{\gamma} f(z) dz = 0.$$

*Then  $f(z)$  is holomorphic.*

*Proof.* Recall that if  $f(z)$  is continuous and  $\oint_{\gamma} f(z) dz = 0$  for any closed curve  $\gamma$ , then it has an antiderivative  $F(z)$  such that  $F'(z) = f(z)$ . Note that  $F(z)$  is differentiable on  $G$ , so actually holomorphic on  $G$ . But holomorphic functions are infinitely differentiable. So  $F$  is twice differentiable, and thus  $F'(z) = f(z)$  is differentiable on  $G$ . Thus,  $f(z)$  is holomorphic on  $G$ . □

**Theorem 9.2.10.** *If  $\sum_{k \geq 0} a_k (z - z_0)^k$  has radius of convergence  $R$ , then on  $|z - z_0| < R$ , it defines a holomorphic function.*

*Proof.* We saw that it's continuous, and the integrals along closed curves are zero, so Morera's theorem applies. □

Now let's understand derivatives of power series.

**Lemma 9.2.11.** *If  $f(z) = \sum_{k \geq 0} a_k (z - z_0)^k$  for  $|z - z_0| < R$ , then for such  $z$ ,*

$$f'(z) = \sum_{k \geq 0} k a_k (z - z_0)^{k-1}.$$

*Proof.* Recall that

$$f'(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z)^2} dw,$$

where  $\gamma$  is a closed curve that encloses  $z$ . Then

$$f'(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\sum_{k \geq 0} a_k (w - z_0)^k}{(w - z)^2} dw.$$

We want this series to converge uniformly on a set containing  $\gamma$  (actually we consider the set  $\gamma$ ). Note that  $\gamma(t)$  is continuous, and so  $\left| \frac{1}{(\gamma(t)-z)^2} \right|$  is continuous as  $z$  is not on the curve  $\gamma$ . Therefore,  $\left| \frac{1}{(\gamma(t)-z)^2} \right|$  is a continuous function  $f : [a, b] \rightarrow \mathbb{R}$ . Then its values are bounded since  $[a, b]$  is compact.

**Claim.** If the series  $\sum_{n \geq 0} g_n(z)$  converges uniformly to  $g(z)$  and  $h(z)$  is bounded, then

$$\sum_{n \geq 0} g_n(z)h(z) \rightarrow g(z)h(z) \quad \text{uniformly.}$$

*Proof.* Given

$$\left| \sum_{n=0}^k g_n(z) - g(z) \right| < \epsilon$$

for any  $\epsilon > 0$ , there is  $N$  such that  $k \geq N$  implies this holds for all  $z$ .

$$\begin{aligned} \left| \sum_{n=0}^k g_n(z)h(z) - g(z)h(z) \right| &= \left| \sum_{n=0}^k g_n(z) - g(z) \right| |h(z)| \\ &\leq \epsilon \cdot \sup_z |h(z)|. \end{aligned}$$

This holds uniformly. □

Now back to the original proof, we have  $g_n(w) = a_n(w - z_0)^n$  and  $h(w) = \frac{1}{(w-z)^2}$ . Then

$$\begin{aligned} f'(z) &= \frac{1}{2\pi i} \sum_{k \geq 0} a_k \oint_{\gamma} \frac{(w - z_0)^k}{(w - z)^2} dw \\ &= \frac{1}{2\pi i} \sum_{k \geq 0} a_k 2\pi i \left. \frac{\partial}{\partial w} (w - z_0)^k \right|_{w=z} \\ &= \sum_{k \geq 0} a_k k (z - z_0)^{k-1}. \end{aligned}$$

If  $f(z) = \sum_{k \geq 0} a_k (z - z_0)^k$ , then

$$\begin{aligned} f(z_0) &= a_0 \\ f'(z_0) &= 1 \cdot a_1 \\ f''(z_0) &= 2 \cdot a_2 \\ f^{(n)}(z_0) &= n! a_n. \end{aligned}$$

Thus,

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k = \sum_{k \geq 0} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

□

**Example 9.2.12.** Consider  $z_0 = 0$  and  $a_k = 1/k!$ . Then

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

We will show that this is equal to  $e^z$ . Note that

$$f'(z) = \sum_{k=0}^{\infty} \frac{k}{k!} (z - z_0)^{k-1} = f(z).$$

$f(0) = 1$ . Now consider  $f(z)e^{-z}$ . Then taking the derivative gives

$$f'(z)e^{-z} - f(z)(e^{-z}) = 0.$$

This implies that  $f(z)e^{-z} = C$  is a constant. At  $z = 0$ , we have  $f(0)e^{-0} = C \implies C = 1$ . Thus

$$f(z)e^{-z} = 1 \implies f(z) = e^z.$$

**Theorem 9.2.13.** Suppose  $f(z)$  is holomorphic on  $|z - z_0| < R$ . Then

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

and  $\gamma$  is a curve enclosing  $z_0$  but contained in the disk. The series converges on  $|z - z_0| < R$ .

*Proof.* By Cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} dw.$$

Now write

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

We want to expand the last factor as a geometric series to make sure  $|w - z_0| > |z - z_0|$  on  $\gamma$ . We choose  $\gamma$  to be the circle  $|w - z_0| = \frac{|z - z_0| + R}{2}$ . Then we have uniform convergence on the enclosed region and so

$$\sum_{k \geq 0} \left( \frac{z - z_0}{w - z_0} \right)^k \xrightarrow{\text{uniformly}} \frac{1}{1 - \frac{z - z_0}{w - z_0}}.$$

Consider

$$\begin{aligned} \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z_0} \cdot \sum_{k=0}^{\infty} \left( \frac{z - z_0}{w - z_0} \right)^k dw &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{w - z} dw \\ &= f(z). \end{aligned}$$

We appealed to uniform convergence to interchange the order of integration and summation. Although  $\frac{f(w)}{w - z_0}$  is not defined at  $w = z_0$ , it is defined and bounded on  $\gamma$  (because it is continuous, and  $\gamma$  is a compact set). So we can write

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \oint_{\gamma} \frac{f(w)}{w - z_0} \frac{(z - z_0)^k}{(w - z_0)^k} dw \\ &= \sum_{k=0}^{\infty} \underbrace{\left[ \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw \right]}_{a_k} (z - z_0)^k. \end{aligned}$$

Notice that

$$\oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

is the same for any  $\gamma$  enclosing  $z_0$ . The argument with uniform convergence implies that this series converges for our particular choice of  $z$ . We could do this for any  $z$  with  $|z - z_0| < R$ , so we have convergence for such  $z$ . Hence, the radius of convergence is at least  $R$  (but possibly could be larger).  $\square$

**Theorem 9.2.14.** *The radius of convergence equals to the largest number  $R'$  such that  $f(z)$  can be extended to a holomorphic function on  $|z - z_0| < R'$ .*

*Proof.* Previous result tells us that the radius of convergence  $R$  is at least  $R'$  ( $R \geq R'$ ). On the other hand, if the radius of convergence is  $R$ , the power series defines a holomorphic function extending  $f(z)$  on  $|z - z_0| < R$ . So  $R' \geq R$ . Hence,  $R = R'$ .  $\square$

**Example 9.2.15.** Consider

$$f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots = \arctan(z).$$



Previously we saw that the radius of convergence is 1. Even though  $\arctan(x)$  is infinitely differentiable as a function of a real variable, the Taylor series still only has finite radius of convergence (as opposed to  $e^x$ ). Actually  $\arctan(z)$  does not extend holomorphically to  $z = \pm i$ . If  $\tan(z) = i$ , then  $\sin(z) = i \cos(z)$  and so  $\sin^2(z) + \cos^2(z) = 0$ . But  $\sin^2(z) + \cos^2(z) = 1$  for any  $z$ , so there is no such  $z$ . Therefore, we cannot extend  $\arctan(z)$  outside  $|z| < 1$  holomorphically. Thus,  $R = 1$  as we already checked.

Alternatively,

$$\arctan(z) = \int_0^z \frac{1}{1+w^2} dw,$$

which blows up at  $w = \pm i$ .

**Proposition 9.2.16.**

$$\frac{\partial^k f}{\partial z^k}(z_0) = \frac{k!}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

where  $\gamma$  encloses  $z_0$ . (We already saw this for  $k = 0, 1, 2$ )

*Proof.* Our power series is a Taylor series:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

where

$$a_k = \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(z_0).$$

We get that by differentiating  $k$  times and evaluating at  $z_0$ . Then compare this to the formula for  $a_k$  from the theorem.  $\square$

**Remark.** Functions that can be written as a convergent power series on a disk centered at  $z_0$  are called *analytic* (at  $z_0$ ). In complex analysis, holomorphic and analytic mean the same thing.

Recall that if  $p(z)$  is a non-constant polynomial and if  $p(z_0) = 0$ , then

$$p(z) = (z - z_0) \cdot \frac{p(z)}{z - z_0},$$

where  $\frac{p(z)}{z - z_0}$  is still a polynomial. We can do the same thing for power series and holomorphic functions.

**Theorem 9.2.17.** *If  $f(z)$  is holomorphic at  $z_0$  and  $f(z_0) = 0$ , then either  $f(z)$  is constant (on some neighborhood of  $z_0$ ) or there is  $m \in \mathbb{Z}_{\geq 0}$  such that*

$$f(z) = g(z)(z - z_0)^m$$

and

(i)  $g(z)$  is holomorphic at  $z_0$ .

(ii)  $g(z_0) \neq 0$ .

**Remark.** This  $m$  is called the order of the zero  $z_0$  for  $f(z)$  or the order of vanishing of  $f(z)$  at  $z_0$ .

*Proof.* If

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k,$$

either all  $a_k = 0$ , in which case  $f(z) = \sum_{k \geq 0} 0 = 0$  (constant), or there is  $m$  such that  $a_0, a_1, \dots, a_{m-1}$  are zero and  $a_m \neq 0$ , then

$$f(z) = (z - z_0)^m \sum_{k \geq m} a_k (z - z_0)^{k-m}.$$

or equivalently,

$$f(z) = (z - z_0)^m \underbrace{\sum_{k \geq 0} a_{k+m} (z - z_0)^k}_{g(z)}.$$

We check that  $g(z)$  is holomorphic at  $z_0$  and  $g(z_0) \neq 0$ . Since convergent power series are holomorphic, it suffices to show that this has positive radius of convergence.

$$\frac{1}{R} = \limsup \sqrt[k]{|a_{k+m}|}.$$

Since the  $a_k$  came from the power series

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k,$$

which is holomorphic at  $z_0$ , then this implies that it has a positive radius of convergence. Hence, we have

$$\limsup \sqrt[k]{|a_k|} < \alpha \quad \text{for some } \alpha > 0.$$

So for all  $k$  large enough  $\sqrt[k]{|a_k|} < \alpha$  and hence  $|a_k| < \alpha^k$  and  $|a_{k+m}| < \alpha^{k+m}$ , so

$$\sqrt[k]{|a_{k+m}|} < \alpha^{\frac{k+m}{k}} = \alpha^{1 + \frac{m}{k}} \leq \alpha^{1+m},$$

so

$$\limsup \sqrt[k]{|a_{k+m}|} \leq \alpha^{1+m}.$$

Then we have

$$g(z) = \sum_{k \geq 0} a_{k+m}(z - z_0)^k$$

converges on some disk centered at  $z_0$ , which defines a holomorphic function at  $z_0$ .

$$g(z_0) = a_m \neq 0.$$

Note that if there is no  $a_m$  that is non-zero, then

$$f(z) = \sum_{k \geq 0} a_k(z - z_0)^k = \sum_{k \geq 0} 0 = 0.$$

Therefore,  $f$  is the constant function equal to zero. □

### 9.3 Identity Principle

**Corollary 9.3.1.** *If  $f(z)$  is holomorphic at  $z_0$  and  $f(z_0) = 0$ , then there is an open disk centered at  $z_0$  on which  $f$  takes the value 0 only at  $z_0$ .*

*Proof.* Write  $f(z) = (z - z_0)^m g(z)$  with  $g(z_0) \neq 0$  and  $g(z) \neq 0$  is continuous at  $z_0$  (since it's holomorphic). Then for every  $\epsilon > 0$ , there is  $\delta > 0$  such that if  $|z - z_0| < \delta$ , then  $|g(z) - g(z_0)| < \epsilon$ . Take  $\epsilon = |a_m|$ , then

$$|g(z) - a_m| < |a_m|,$$

which forbids  $g(z) = 0$  for  $|z - z_0| < \delta$ . So zeros of holomorphic functions are isolated: each is contained in an open set containing no other points. □

**Theorem 9.3.2** (Identity principle). *If  $f(z)$  is holomorphic on an open set  $G$ , and  $z_1, z_2, z_3, \dots$  are points in  $G$  with  $f(z_i) = 0$ , if  $(z_i)$  has an accumulation point in  $G$ , then  $f(z)$  is the zero function.*

*Proof.* Let  $w$  be an accumulation point of  $z_j$  in  $G$ , so there is a subsequence  $z_{i_j}$  converging to  $w$ . Since  $f$  is holomorphic in  $G$ , it is continuous at  $w$ .

$$f(w) = f\left(\lim_j z_{i_j}\right) = \lim_j f(z_{i_j}) = \lim_j 0 = 0.$$

Now this contradicts the previous result because  $f(z)$  must be nonzero on some disk centered at  $w$  (except at  $w$  itself) if it is nonconstant. But the subsequence  $z_{i_j}$  must get arbitrarily close to  $w$ , so it would intersect any disk centered at  $w$ , and give a point where  $f$  vanishes. Therefore, the function cannot be nonconstant and so we must have  $f(z) = 0$ . □

**Corollary 9.3.3.** *If  $f(z)$ ,  $g(z)$  are holomorphic on  $G$ , and agree on a set  $\{z_1, z_2, \dots\}$  with an accumulation point ( $f(z_i) = g(z_i)$  for each  $i$ ), then  $f(z) = g(z)$ .*

*Proof.* Apply the previous theorem to  $f(z) - g(z)$  (holomorphic on  $G$ , vanishes at  $z_i$ ), then  $f(z) - g(z) = 0$ . Thus,  $f(z) = g(z)$ .  $\square$

**Remark.** Writing

$$f(z) = \sum_k a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw.$$

Then  $f(z)$  is determined by its power series, hence by  $a_k$ , which is determined by values of  $f(z)$  along  $\gamma$ , a curve around  $z_0$ .

**Conclusion:**  $f(z)$  is determined by its behavior on an open set containing  $z_0$ .

We have shown that  $f(z)$  is determined by its value on a set having  $z_0$  as an accumulation point. This could be a discrete set. Although we know that if  $g(z)$  is another holomorphic function agreeing with  $f(z)$  on this set then  $f(z) = g(z)$  (so the set of values determines  $f$ ). It isn't clear how to find values of  $f(z)$  at other points from this information.

**Example 9.3.4.**  $\sin\left(\frac{1}{z}\right)$  holomorphic for  $z \neq 0$  (composition of holomorphic functions). Then the zeros are  $z_n = \frac{1}{\pi n}$  for  $n = 1, 2, \dots$   $z_n$  converges to 0. But  $\sin\left(\frac{1}{z}\right)$  is not constant. Not a contradiction because the accumulation point is outside the set  $G$  where the function is holomorphic.

## Chapter 10

# Laurent Series

**Definition 10.0.1** (Laurent Series). A *Laurent series* is a series of the form

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

Here

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges when  $|z - z_0| < R_2$  and

$$\sum_{k=-\infty}^{-1} a_k (z - z_0)^k = \sum_{k=1}^{\infty} a_{-k} ((z - z_0)^{-1})^k$$

is a power series in the variable  $(z - z_0)^{-1}$  that converges when  $|(z - z_0)^{-1}| < \frac{1}{R_1}$ , i.e.  $R_1 < |z - z_0|$ . Thus, the Laurent series converges when the points satisfies  $R_1 < |z - z_0| < R_2$ , i.e., they lie in an annulus (flat doughnut shape).

**Remark.** Sometimes  $R_1 \geq R_2$ , in which case the series might never converge. For example, consider  $a_k = 1$  for all  $k$ . Then we have

$$\sum_{k=-\infty}^{\infty} z^k = \sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} z^k.$$

The first summation from the RHS converges only when  $|z^{-1}| < 1$  and the second summation converges only when  $|z| < 1$ . However, we cannot have both  $|z| < 1$  and  $|z^{-1}| < 1$ . Hence, it doesn't converge for any  $z$ .

**Theorem 10.0.2.** Suppose  $f(z)$  is holomorphic in the annulus  $R_1 < |z - z_0| < R_2$ . Then

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{k+1}} dw$$

( $\gamma$  is any curve inside the annulus enclosing  $z_0$ , usually a circle centered at  $z_0$  with radius between  $R_1$  and  $R_2$ ).

*Proof.* Modify the argument for power series. By Cauchy integral formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma'} \frac{f(w)}{w - z} dw$$

where  $\gamma'$  is a curve around  $z$ , inside of which  $f(z)$  is holomorphic. We choose  $\gamma'$  to be

- anticlockwise big circle (encloses  $z$ , smaller radius than  $R_2$ ),
- clockwise small circle (doesn't enclose  $z$ , radius larger than  $R_1$ ).

Then

$$f(z) = \frac{1}{2\pi i} \left[ \int_{\text{big circle}} \frac{f(w)}{w - z} dw - \int_{\text{small circle}} \frac{f(w)}{w - z} dw \right].$$

Here the minus sign accounts for the orientation. (Note that if  $f$  is holomorphic for all  $|z - z_0| < R_2$ , then the second integral is zero by Cauchy's theorem, and we are in the same situation as when we discussed power series.) Instead of rewriting  $\frac{1}{w - z}$  as follows

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{k \geq 0} \left( \frac{z - z_0}{w - z_0} \right)^k,$$

we write

$$\frac{1}{w - z} = \frac{-1}{z - z_0} \frac{1}{1 - \frac{w - z_0}{z - z_0}} = \frac{-1}{z - z_0} \sum_{k \geq 0} \left( \frac{w - z_0}{z - z_0} \right)^k,$$

which is a geometric series if  $|w - z_0| < |z - z_0|$ . Then apply the uniform convergence of geometric series to interchange order of summation and integration. Then we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\text{big circle}} \frac{f(w)}{w - z} dw &= \frac{1}{2\pi i} \int \frac{f(w)}{w - z_0} \sum_{k \geq 0} \left( \frac{z - z_0}{w - z_0} \right)^k dw \\ &= \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\text{big circle}} \frac{f(w)}{(w - z_0)^{k+1}} dw (z - z_0)^k. \end{aligned}$$

$$\begin{aligned} \frac{-1}{2\pi i} \int_{\text{small circle}} \frac{f(w)}{w-z} dw &= \frac{-1}{2\pi i} \int \frac{-f(w)}{z-z_0} \sum_{k \geq 0} \left( \frac{w-z_0}{z-z_0} \right)^k dw \\ &= \sum_{k \geq 0} \frac{1}{2\pi i} \int_{\text{small circle}} \frac{f(w)(w-z_0)^k}{(z-z_0)^{k+1}} dw. \end{aligned}$$

□

**Conclusion:**  $f(z)$  determines the coefficients of its Laurent series so there is a unique Laurent series in a given annulus.

**Remark.** Note though that a given function can have different Laurent series in different annuli. For example, consider  $f(z) = \frac{1}{1-z}$  which has singularity at  $z = 1$ . If  $0 < |z| < 1$ , then

$$f(z) = \sum_{k=0}^{\infty} z^k$$

is a geometric series (Laurent series is just a power series). If  $1 < |z| < \infty$ ,

$$f(z) = \frac{-1}{z} \cdot \frac{1}{1-z^{-1}} = \frac{-1}{z} \sum_{k \geq 0} (z^{-1})^k = - \sum_{k=1}^{\infty} z^{-k}.$$

Recall

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z_0)^{k+1}} dw.$$

Note that

$$2\pi i a_{-1} = \oint_{\gamma} f(w) dw.$$

So if we can find  $a_{-1}$ , we can find integrals of  $f(w)$ . Also note that by Cauchy's theorem, we have

$$\oint_{\gamma} f(z) dz = 0.$$

Hence,

$$2\pi i a_{-1} = 0 \implies a_{-1} = 0.$$

Now consider the outer region where  $\gamma$  is an anticlockwise circle with radius greater than 1. Then

$$\oint_{\gamma} f(z) dz = \oint_{\gamma} \frac{-1}{z-1} dz = 2\pi i(-1) = 2\pi i a_{-1}.$$

So we have  $a_{-1} = -1$ .

Viewing a Laurent series as

$$\underbrace{\sum_{k=0}^{\infty} a_k(z - z_0)^k}_{\text{converges for } |z - z_0| < \text{distance to some singularity}} + \underbrace{\sum_{k=-1}^{-\infty} a_k(z - z_0)^k}_{\text{converges for } \left| \frac{1}{z - z_0} \right| < \text{distance to some singularity}},$$

we find that we can't have convergence outside the annulus.

**Example 10.0.3.** Consider

$$f(z) = \frac{1}{(z - 1)(z - 2)}.$$

Then consider three regions  $0 < |z| < 1$ ,  $1 < |z| < 2$  and  $2 < |z| < \infty$ . First notice that

$$f(z) = \frac{1}{z - 2} - \frac{1}{z - 1}.$$

If  $|z| > 2$ ,

$$\frac{1}{z - 2} = \frac{1}{z} \frac{1}{1 - \frac{z}{2}} = \frac{1}{z} \sum_{k \geq 0} \left( \frac{2}{z} \right)^k.$$

If  $|z| < 2$ ,

$$\frac{1}{z - 2} = \frac{-1}{2} \frac{1}{1 - \frac{z}{2}} = \frac{-1}{2} \sum_{k \geq 0} \left( \frac{z}{2} \right)^k.$$

Similarly there are two expansions for  $\frac{1}{z-1}$ . Add up the series for each term which are valid in the annulus we chose to work in.

## 10.1 Isolated Singularities

Recall that the identity principle states that if the set of zeros of a holomorphic function has an accumulation point (inside the domain) then the function is the zero function.

**Conclusion 1:** the zeros of a non-constant holomorphic function are isolated: for any  $z_0$  with  $f(z_0) = 0$ , there is  $\epsilon > 0$  such that  $f(z) \neq 0$  for  $0 < |z - z_0| < \epsilon$ .

**Definition 10.1.1** (Isolated). A singularity  $z_0$  of a function  $f(z)$  is *isolated* if there is  $\epsilon > 0$  such that  $f(z)$  is holomorphic on  $0 < |z - z_0| < \epsilon$ .

**Remark.** This is an annulus: there is a Laurent series.



**Definition 10.1.2.** If  $z_0$  is an isolated singularity of  $f(z)$ , then

- (a) if  $f(z)$  extends to a holomorphic function on  $|z - z_0| < \epsilon$ , we say  $z_0$  is a *removable singularity*;
- (b) if  $\lim_{z \rightarrow z_0} |f(z)| = \infty$ , then  $f(z)$  has a *pole* at  $z_0$ ;
- (c) otherwise  $z_0$  is called an *essential singularity*.

**Example 10.1.3.**  $f(z) = \frac{e^z - 1}{z}$  at  $z = 0$ . Recall if  $f(z_0) = 0$ , we can write

$$f(z) = (z - z_0)^m g(z)$$

for some  $g(z)$  holomorphic at  $z_0$  and  $g(z_0) \neq 0$ . Let  $f(z) = e^z - 1$ ,  $f(0) = 0$ , so  $f(z) = z^m g(z)$ . Turns out  $m = 1$ ,

$$g(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}.$$

**Example 10.1.4.**  $f(z) = \frac{1}{z^2}$  at  $z = 0$ . Then

$$\lim_{z \rightarrow 0} \left| \frac{1}{z^2} \right| = \infty.$$

Similarly for  $\frac{1}{z}$ ,  $\frac{1}{z^7}$ , etc..

**Example 10.1.5.**  $f(z) = e^{\frac{1}{z}}$ .

$$\lim_{z \rightarrow 0^+} e^{\frac{1}{z}} = \infty$$

$$\lim_{z \rightarrow 0^-} e^{\frac{1}{z}} = 0.$$

Since these limits differ, there is a no continuous extension and thus cannot be a removable singularity. Also  $\lim_{z \rightarrow 0^-} f(z) = 0$  rules out  $\lim_{z \rightarrow 0} |f(z)| = \infty$  and so it is not a pole.

**Proposition 10.1.6.** If  $z_0$  is an isolated singularity of  $f(z)$ ,

- (a)  $z_0$  is removable  $\iff \lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$ .
- (b)  $z_0$  is a pole  $\iff$  it is not removable and  $\lim_{z \rightarrow z_0} (z - z_0)^{n+1} f(z) = 0$  for some  $n \in \mathbb{Z}_{>0}$ .

*Proof.*

- (a) If  $z_0$  is removable, let  $g(z)$  be a holomorphic extension of  $f(z)$  to  $z_0$ .

$$\begin{aligned}\lim_{z \rightarrow z_0} (z - z_0)f(z) &= \lim_{z \rightarrow z_0} (z - z_0)g(z) \\ &= 0g(z_0) \\ &= 0.\end{aligned}$$

Conversely let

$$h(z) = \begin{cases} (z - z_0)^2 f(z) & z \neq z_0 \\ 0 & z = z_0. \end{cases}$$

Let's show that it is differentiable at  $z_0$ :

$$\begin{aligned}h'(z_0) &= \lim_{z \rightarrow z_0} \frac{h(z) - h(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{(z - z_0)^2 f(z) - 0}{z - z_0} \\ &= \lim_{z \rightarrow z_0} (z - z_0)f(z) \\ &= 0 \quad (\text{by assumption}).\end{aligned}$$

So actually  $h(z)$  is holomorphic at  $z_0$  and so we can ask to what order it vanishes.

$$h(z) = (z - z_0)^m k(z)$$

with  $k(z)$  holomorphic at  $z_0$  and  $k(z_0) \neq 0$ .

**Claim.**  $m \geq 2$ .

*Proof.* Since  $h(z_0) = 0$ , we must have  $m \geq 1$  ( $m = 0$  gives  $0 = h(z_0) = k(z_0) \neq 0$ ). Since  $h'(z_0) = 0$ , we get

$$0 = m(z_0 - z_0)^{m-1}k(z_0) + (z_0 - z_0)^m k'(z_0),$$

this is only zero if  $m - 1 \geq 1$ , i.e.  $m \geq 2$ . □

Then  $f(z) = (z - z_0)^{m-2}k(z)$  is a holomorphic extension of  $f(z)$ .

- (b) If  $\lim_{z \rightarrow z_0} |f(z)| = \infty$ , then for some  $\epsilon > 0$  and  $0 < |z - z_0| < \epsilon$ ,  $|f(z)| > 1$ . By working close enough to  $z_0$ , we can assume  $f(z) \neq 0$ , hence  $\frac{1}{f(z)}$  is holomorphic for  $0 < |z - z_0| < \epsilon$ . But then

$$\lim_{z \rightarrow z_0} |f(z)| = \infty \implies \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0.$$

So

$$\lim_{z \rightarrow z_0} (z - z_0) \frac{1}{f(z)} = 0 \cdot 0 = 0.$$

Hence  $z_0$  is a removable singularity of  $\frac{1}{f(z)}$ .

$$\frac{1}{f(z)} = (z - z_0)^n \ell(z)$$

where  $\ell(z_0) \neq 0$  and  $\ell(z)$  is holomorphic at  $z = z_0$ . Then

$$f(z) = (z - z_0)^{-n} \frac{1}{\ell(z)}.$$

So

$$\begin{aligned} \lim_{z \rightarrow z_0} (z - z_0)^{n+1} f(z) &= \lim_{z \rightarrow z_0} (z - z_0) \frac{1}{\ell(z)} \\ &= 0 \cdot \frac{1}{\ell(z_0)} \\ &= 0. \end{aligned}$$

Suppose  $\lim_{z \rightarrow z_0} (z - z_0)^{n+1} f(z) = 0$  and assume  $n$  is as small as possible. Let  $p(z) = (z - z_0)^n f(z)$ , then

$$\lim_{z \rightarrow z_0} (z - z_0) p(z) = 0.$$

So  $z_0$  is a removable singularity of  $p(z)$ . Hence,  $p(z)$  extends to a function holomorphic at  $z_0$ . Therefore,

$$\lim_{z \rightarrow z_0} (z - z_0)^n f(z)$$

exists ( $= p(z_0)$ ) and it is nonzero by minimality of  $n$ . Then

$$\begin{aligned} \lim_{z \rightarrow z_0} |f(z)| &= \lim_{z \rightarrow z_0} |p(z)(z - z_0)^{-n}| \\ &= \lim_{z \rightarrow z_0} |p(z)| \cdot \lim_{z \rightarrow z_0} |(z - z_0)^{-n}| \\ &= |p(z_0)| \cdot \infty \\ &= \infty. \end{aligned}$$

Thus,  $f(z)$  has a pole at  $z_0$ .

**Definition 10.1.7** (Order). This number  $n$  is the order of the pole of  $f(z)$  at  $z_0$ .

□

**Proposition 10.1.8.** *Suppose  $z_0$  is an isolated singularity of  $f(z)$  and*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$$

*near  $z_0$ , then*

- (a)  $z_0$  is removable  $\iff a_k = 0$  for  $k < 0$
- (b)  $z_0$  is a pole of order  $n$   $\iff a_{-n} \neq 0$  and  $a_k = 0$  for  $k < -n$ .
- (c)  $z_0$  is essential  $\iff a_{-n} \neq 0$  for infinitely many  $n > 0$ .

*Proof.* (a) If there are no negative index terms, the Laurent series is a power series which defines a holomorphic function at  $z_0$ . If  $f(z)$  is holomorphic at  $z_0$ , it has a power series about  $z_0$ , which is the same thing as a Laurent series with no negative index terms.

- (b) If  $f(z)$  has a pole of order  $n$ ,  $f(z) = (z - z_0)^{-n}g(z)$  for some  $g(z)$  that is holomorphic at  $z_0$  ( $g(z_0) \neq 0$ ) (since  $\lim_{z \rightarrow z_0} (z - z_0)^{n+1}f(z) = 0$ ,  $(z - z_0)^n f(z)$  has a removable singularity at  $z_0$ , call resulting function  $g(z)$ ). Write

$$g(z) = \sum_{k \geq 0} a_k(z - z_0)^k$$

with  $a_0 = g(z_0) \neq 0$  (power series about  $z_0$ ) so

$$f(z) = (z - z_0)^{-n}g(z) = \sum_{k \geq 0} a_k(z - z_0)^{k-n}.$$

The coefficient of  $(z - z_0)^{-n}$  is  $a_0 \neq 0$  and all lower coefficients are zero. Conversely, if

$$f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^k$$

with  $a_{-n} \neq 0$ , then

$$(z - z_0)^{n+1}f(z) = \sum_{k=-n}^{\infty} a_k(z - z_0)^{n+k+1}.$$

This is a power series with constant term zero. Thus, this defines a holomorphic function where value at  $z_0$  is zero.

- (c) Automatic if neither (a) nor (b) apply. We have an essential singularity, but neither of (a) and (b) applying means that infinitely many negative index terms are nonzeros.  $\square$

## Chapter 11

# Residue Theorem

### 11.1 Residue

Recall if  $f(z)$  has an isolated singularity at  $z_0$ ,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

with

$$a_k = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

In particular, when  $k = -1$ ,

$$2\pi i a_{-1} = \oint_{\gamma} f(w) dw.$$

So if we can find  $a_{-1}$ , we can find the integral.

**Definition 11.1.1** (Residue). The coefficient  $a_{-1}$  in the Laurent series of  $f(z)$  at  $z_0$  is called the *residue* of  $f(z)$  at  $z_0$ . We write this  $\text{Res}_{z=z_0} f(z)$ .

So we can state the best version of the Cauchy integral formula.

### 11.2 Residue Theorem

**Theorem 11.2.1** (Residue Theorem). *Let  $\gamma$  be a closed curve and  $f(z)$  be a holomorphic function on  $\gamma$  and also on the region enclosed by  $\gamma$  except for isolated singularities. Then there are finitely many singularities inside the enclosed region and*

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_k \text{Res}_{z=z_k} f(z).$$

We are summing over singularities enclosed by  $\gamma$ .

*Proof.* Let  $S$  be the set of singularities inside the enclosed region. Note that  $S$  is bounded (it is contained in the enclosed region, which is bounded by the Jordan curve theorem). Note that  $S$  is also closed (since  $S$  is the complement of the set of points where  $f(z)$  is holomorphic. If  $f(z)$  is holomorphic at  $z_0$ , it is differentiable on a disk centered at  $z_0$ , and so it is holomorphic on a disk centered at  $z_0$ . So the set of points where  $f(z)$  is holomorphic is open, and its complement is closed).

Suppose there are infinitely many points in  $S$ . Consider a sequence of distinct points in  $S$ :  $z_1, z_2, z_3, \dots$ . Then by Bolzano-Weierstrass theorem, which states that a bounded sequence has a convergent subsequence, we know that there exists a subsequence

$$z_{i_1}, z_{i_2}, z_{i_3}, \dots \rightarrow w.$$

Because  $S$  is closed, the limit point  $w$  must still be in  $S$ .

**Claim.**  $w$  is not an isolated singularity.

*Proof.* For any  $\epsilon > 0$ , there is some  $z_{i_j}$  within  $\epsilon$  of  $w$  (because  $z_{i_j} \rightarrow w$ , it must get arbitrarily close to  $w$ ), so  $f$  cannot be holomorphic on  $0 < |z - w| < \epsilon$ . Thus, there can be only finitely many points in  $S$ .  $\square$

Now draw little circles around singularities in  $\gamma$  and connect them to the boundary with line segments. Then

$$0 = \oint_{\text{new curve}} = \oint_{\gamma} + \sum \int_{\text{line segments}} + \sum \oint_{\text{little circles clockwise}}.$$

$\square$

**Example 11.2.2.** Compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^n} dx.$$

Note if  $f(z) = \frac{g(z)}{h(z)}$  with  $g(z), h(z)$  holomorphic.  $f(z)$  has singularities when  $h(z) = 0$ . But we showed that the zeros of a non-constant holomorphic function are isolated, which implies singularities of  $f(z)$  are isolated. We take  $g(z) = (z - z_0)^a \alpha(z)$  and  $h(z) = (z - z_0)^b \beta(z)$  such that

$$f(z) = (z - z_0)^{a-b} \frac{\alpha(z)}{\beta(z)}.$$

$\alpha(z_0) \neq 0, \beta(z_0) \neq 0 \implies \frac{\alpha(z_0)}{\beta(z_0)} \neq 0$ .  $f(z)$  has a zero of order  $a - b$  if  $a > b$  and has a pole of order  $b - a$  if  $a < b$ . So the singularities are poles (essential singularities).

**Remark.** If all singularities of  $f(z)$  are isolated poles (not essential singularities). Then  $f(z)$  is called a *meromorphic* function. All meromorphic functions are of the form  $\frac{g(z)}{h(z)}$  with  $g, h$  holomorphic.

**Answer.** Let  $\gamma$  be the upper-plane semicircle anticlockwise with radius  $R$ . The singular points are  $z = \pm i$ , but  $z = i$  is inside  $\gamma$ . Then by Residue theorem,

$$\int_{-R}^R \frac{1}{(1+x^2)^n} dx + \int_{\text{arc}} \frac{1}{(1+x^2)^n} dx = 2\pi i \operatorname{Res}_{z=i} \frac{1}{(z^2+1)^n}.$$

By ML-lemma,

$$\begin{aligned} \left| \int_{\text{arc}} \frac{1}{(z^2+1)^n} dz \right| &\leq \pi R \max \left( \frac{1}{|z^2+1|^n} \right) \\ &= \frac{\pi R}{\min |z^2+1|^n} \\ &\leq \frac{\pi R}{\min (|z|^2-1)^n} \quad (\text{reverse triangle inequality}) \\ &= \frac{\pi R}{(R^2-1)^n} \rightarrow 0. \end{aligned}$$

So we only need to compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} dx = 2\pi i \operatorname{Res}_{z=i} \frac{1}{(z^2+1)^n}.$$

Consider when  $n = 1$ :

$$\begin{aligned} \frac{1}{z^2+1} &= \frac{1}{(z-i)(z+i)}. \\ \frac{1}{z+i} &= \frac{1}{(z-i)+2i} \\ &= \frac{1}{2i} \frac{1}{1 + \frac{z-i}{2i}} \\ &= \frac{1}{2i} \sum_{k \geq 0} \left( -\frac{z-i}{2i} \right)^k. \end{aligned}$$

So

$$\frac{1}{z^2+1} = \frac{1}{2i} \sum_{k \geq 0} \left( -\frac{1}{2i} \right)^k (z-i)^{k-1}.$$

The coefficient of  $(z-i)^{-1}$  (i.e. the residue) comes from the  $k = 0$  term,

$$\frac{1}{2i} \left( -\frac{1}{2i} \right)^0 = \frac{1}{2i}.$$

So the integral is

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx = 2\pi i \frac{1}{2i} = \pi.$$

Now for  $n = 2$ , we have

$$\begin{aligned} \frac{1}{(z^2 + 1)^2} &= \frac{1}{(z - i)^2} \frac{1}{(z + i)^2} \\ \frac{1}{(z + i)^2} &= \left[ \frac{1}{2i} \sum_{k \geq 0} \left( -\frac{z - i}{2i} \right)^k \right] \\ &= \left( \frac{1}{2i} \right)^2 + 2 \frac{1}{2i} \frac{1}{2i} \left( -\frac{z - i}{2i} \right) + \text{higher order terms.} \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{(z^2 + i)^2} &= \frac{1}{(z - i)^2} \left[ \frac{1}{(2i)^2} + 2 \frac{1}{(2i)^2} \left( -\frac{z - i}{2i} + \dots \right) \right] \\ &= \frac{1}{(2i)^2} \frac{1}{(z - i)^2} + \frac{2}{(2i)^2} \left( \frac{1}{-2i} \right) (z - i)^{-1} + \dots \end{aligned}$$

The residue is  $\frac{2}{(2i)^2} \frac{1}{-2i} = \frac{2}{8i} = \frac{1}{4i}$ . Thus,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} dx = 2\pi i \frac{1}{4i} = \frac{\pi}{2}.$$

Recall the Binomial Theorem:

$$(1 + z)^n = \sum_{r=0}^n \binom{n}{r} x^r.$$

If  $n$  is replaced by  $\alpha$  not necessarily positive integer, then

$$(1 + z)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} z^k = \sum_{k \geq 0} \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k(k - 1) \cdots 1} z^k.$$

Write

$$(1 + z)^\alpha = e^{\alpha \log(1+z)},$$

which is holomorphic for  $|z| < 1$ . To compute Taylor series, differentiate  $(1 + z)^\alpha$   $n$  times:

$$\begin{aligned} &\alpha(1 + z)^{\alpha-1} \\ &\alpha(\alpha - 1)(1 + z)^{\alpha-2} \\ &\vdots \\ &\alpha(\alpha - 1) \cdots (\alpha - k + 1)(1 + z)^{\alpha-k} \Big|_{z=0} \end{aligned}$$



$$\sum_{k \geq 0} \frac{f^{(k)}(z)}{k!} z^k = \sum_{k \geq 0} \binom{\alpha}{k} z^k$$

which converges for  $|z| < 1$ .