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Weak recursively simplicial graphs

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1 Preliminary

Definition 1 Let $F = (V, E)$ and $F' = (V', E')$ be two graphs. If $V' \subseteq V$ and $E' \subseteq E$, then F' is a **subgraph** of F (and F is a **supergraph** of F'), written as $F' \subseteq F$. If $F' \subseteq F$ and F' contains all the edges $xy \in E$ with $x, y \in V'$, then F' is an **induced subgraph** of F, written as $F' = F[V']$.

Definition 2 A **clique** is a subset of nodes in an undirected graph where every two distinct nodes are adjacent.

Definition 3 A **simplicial node** in an undirected graph is a node whose neighbours form a clique.

Definition 4 A graph $F = (V, E)$ is **recursively simplicial** if it contains a simplicial node v_i and the induced subgraph $F[V \setminus \{v_i\}]$ is recursively simplicial.

Definition 5 A **simplicial node ordering** of a recursively simplicial graph $F = (V, E)$ is a sequence of nodes $\{v_1, \dots, v_n\}$, where v_i is a simplicial node in the induced subgraph $F[V \setminus \{v_1, \dots, v_{i-1}\}]$.

Definition 6 An **m-cycle** in an undirected graph is a sequence of nodes $\{v_1, \dots, v_{m+1}\}$ where $v_1 = v_{m+1}$ and all the other nodes are distinct.

Definition 7 A graph is **chordal** if each m-cycle for $m \geq 4$ has a chord.

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Definition 8 Let $< G, P >$ be a graphical model over a set $V = \{v_1, \ldots, v_n\}$ of n random variables. The **Markvov blanket** of a variable v_i in G, denoted by B_i^G, is a subset of variables such that $v_i \perp\!\!\!\perp_P V \setminus \{B_i^G \cup \{v_i\}\} \mid B_i^G$.

In a DAG, B_i consists of v_i's parents, children and spouses (i.e., children's other parents). In a undirected graph (UG), B_i consists of the (distance 1) neighbours of v_i. We use B_V to denote the family of Markov blankets $\{B_1, \ldots, B_n\}$ over V.

Definition 9 A **hybrid graph** is a graph consisting of directed and undirected edges.

Definition 10 The **skeleton** of a hybrid graph is the undirected graph obtained by dropping directions of all directed edges.

Definition 11 The **moral graph** of a DAG is the skeleton of the hybrid graph obtained by adding undirected edges between all pairs of non-adjacent parents that have a common child in the DAG.

The following are some well-known properties of chordal graphs (citation!!!).

Proposition 1 *The following properties of an undirected graph F are equivalent:*

1. *F is chordal.*
2. *F is recursively simplicial.*
3. *F can be oriented into a DAG G s.t. the moral graph of G is F.*
4. *F can be oriented into a DAG G s.t. F and G imply the same conditional independencies.*

As a consequence of 1 and 4, F can be oriented into a DAG G s.t. $B_X^F = B_X^G$. The following proposition states that there exists non-chordal graphs, who can be oriented into DAGs with the same Markov blanket families.

Proposition 2 *Let $\mathcal{F}$ be the set of chordal graphs $F = (V, E)$ and $\mathcal{B}$ be the set of Markov blanket families of DAGs over V. Then there exists an injective but non-surjective function $\phi : \mathcal{F} \rightarrow \mathcal{B}$.*

Proof Let F_1, F_2 be two identical chordal graphs. It implies $B_V^{F_1} = B_V^{F_2}$. Since F_i is a chordal, it can be oriented into a DAG G_i s.t. $B_V^{F_i} = B_V^{G_i}, \forall i \in \{1, 2\}$. It follows that $B_V^{G_1} = B_V^{G_2}$, so ϕ is an injective function.

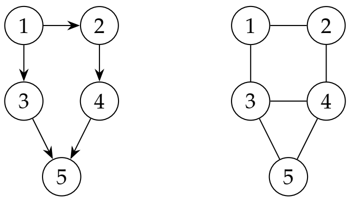


Fig. 1: An example of a DAG whose moral graph is non-chordal.

For any given DAG G, it has a unique moral graph F s.t. $B_V^G = B_V^F$. Figure **??** is an example of a DAG and its moral graph, which is non-choral. Hence, ϕ is non-surjective. □

2 Weak recursively simplicial graphs

Definition 12 A **collider** in a hybrid graph is a node with at least two parents.

Definition 13 A DAG G is a **consistent extension** of a hybrid graph H if G and H have the same skeleton and the same set of colliders.

Let $F = (V, E)$ be a graph over n nodes. Let $S = \{S_1, \ldots, S_m\}$ be a set of disjoint subsets of V s.t. $\cup_{i=1}^m S_i = V$. Let $U = \{U_1, \ldots, U_m\}$ be a set of disjoint subsets of E, where $U_i \subseteq \{v_j v_k \mid \forall v_j, v_k \in N_l, \forall v_l \in S_i\}$. Two extremes of U_i are the empty set and the edges between all pairs of neighbours of v_l for all $v_l \in S_i$. We use $O = \{(S_1, U_1), \ldots, (S_m, U_m)\}$ to denote the pairs. Next we introduce a subset of undirected graphs which are not as strong as recusrively simplicial.

Definition 14 A graph $F = (V, E)$ is **weak recursively simplicial (WRS)** if there exits $O = \{(S_1, U_1), \ldots, (S_m, U_m)\}$ such that S_1 is the set of simplicial nodes in F and S_i is the set of simplicial nodes in the subgraph obtained by removing from F the pairs $\{O_1, \ldots, O_{i-1}\}$ for $i \in [2, m]$.

The reason such a graph is named weak recursively simplicial is because if we only remove the simplicial nodes from F, the induced subgraph over the remaining nodes may or may not have any simplicial nodes. So none or some edges between neighbours of simplicial nodes may need to be removed as well to retain the recursion. By definition, a chordal graph is also weak recursively simplicial by letting $U_i = \emptyset$. The converse, however, is not true. The graph on the right of Figure **??** is a WRS graph. It has $O_1 = \{S_1 = \{5\}, U_1 = \{3 - 4\}\}$. The subgraph obtained by removing O_1 is a chordal graph that is also WRS. Therefore, the original graph is WRS.

Theorem 1 *Let $G = (V, E)$ be a DAG and F be the moral graph of G. Then F is weak recursively simplicial.*

Proof (Base case) $G(1) = F(1)$ is a single node graph, so $F(1)$ is WRS. (Inductive hypothesis) Assuming the moral graph $F(n)$ of any DAG $G(n)$ with n nodes is WRS. (Inductive step) Adding the node v_{n+1} in $G(n)$ as a new leaf produces another DAG $G(n+1)$ with $n+1$ nodes. Since v_{n+1} is a leaf in $G(n+1)$, it becomes a simplicial node in $F(n+1)$ after moralization. In addition, extra edges may be introduced to connect parents of v_{n+1}. Since v_{n+1} is a simplicial node, it can be removed from $F(n+1)$ together with the extra edges introduced. Hence, we get back to the same moral graph $F(n)$ that is assumed to be WRS. Therefore, $F(n+1)$ is also WRS. □

Theorem 1 suggests that the moral graph of any DAG is WRS. To prove there is a one-to-one correspondance between moral graphs and WRS graphs, we want to show that the converse is also true. That is, a WRS graph F can be oriented into a DAG G (possibly with some edge removal) such that the moral graph of G is identical to F.

Theorem 2 *Let $F = (V, E)$ be a weak recursively simplicial graph. Then F is the moral graph of a DAG.*

Proof (Base case) $F(1)$ is a single node WRS graph, and also the moral graph of the single node DAG $G(1)$. (Inductive hypothesis) Assuming any WRS graph $F(n)$ with n nodes is the moral graph of a DAG $G(n)$. (Inductive step) Adding v_{n+1} into $F(n)$ as a simplical node, so the resulting graph $F(n+1)$ is WRS. Since v_{n+1} is a simplicial node, there exists a DAG $G(n+1)$, in which v_{n+1} is a leaf. Therefore, the moral graph of $G(n+1)$ is $F(n+1)$. □

Corollary 1 *An undirected graph is weak recursively simplicial if and only if it is the moral graph of a DAG.*

Proof The corollary follows from the Theorem 1 and Theorem 2. □

Corollary 2 *Let $\mathcal{F} = \{F = (V, E) \mid F$ is weak recursively simplicial$\}$ be the set of weak recursively simplicial graphs over V and $\mathcal{B} = \{B_V^G \mid$ for all DAG G over $V\}$ be the set of Markov blanket families of any DAG over V. Then $|\mathcal{F}| = |\mathcal{B}|$.*

Proof It is straightforward that there is a one-to-one correspondance between $\mathcal{B}$ and moral graphs. Hence, Corollary 1 implies that $\mathcal{F}$ has a one-to-one correspondance with $\mathcal{B}$, so equal cardinality. □

Next, we present a backtracking algorithm for checking whether or not a given graph $F = (V, E)$ is WRS. If it is, the algorithm will find a DAG G, the moral graph of whom is F. Notice that this algorithm may produce a hybrid graph, which always has a consistent DAG extension according to Proposition 2. Hence, we use the algorithm presented in [Dor and Tarsi, 1992] to get it.

Algorithm 1 Backtracking algorithm to test WRS of graphs

Require: F is an undirected graph obtained by connecting all $v_j \in B_i$ with $v_j, \forall v_j \in V$

```
function WRS(F)
    if F is chordal then return TRUE
    end if
    Find the set S of all simplicial nodes in F
    if S ≠ ∅ then
        Find E = {v_j v_k | ∀v_j, v_k ∈ N_i, ∀v_i ∈ S}
        backup = F
        for each U ⊆ E do                          ▷ for each subset of E
            F = backup
            F = F − S − U                          ▷ Remove simplicial nodes S and edges U
            if WRS(F) = TRUE then return TRUE
            end if
        end for
    end if
    return FALSE
end function
```

3 Polytree

Proposition 3 *Let $T = (V, E)$ be a polytree and F be the moral graph of T. Then F is a chordal graph.*

Proof Assuming F is not a chordal graph, there must exist a chordless $C_m \subset F$ for $m \geq 4$. F being moral implies that the C_m shares an edge with a simplicial clique K_n for $n \geq 3$. Hence, there are multiple paths between a node in the C_m and the simplicial node in the K_n via different neighbours of the simplicial node. Henc, the assumption leads to a contradiction to T being singly connected. □

The converse of Proposition 3 is not true. For example, the chordal moral graph in Figure 2 comes from a non-singly connected DAG.

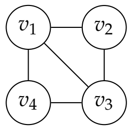


Fig. 2: A chordal graph that comes from moralizing a non-singly connected DAG.

Corollary 3 *Let B_X be a set of Markov blankets over a variable set X. The problem of testing if B_X is consistent with a polytree tree is in polynomial time.*

Proof Chordality can be verified in polynomial time (citation!!!). □

Idea: Since it is in polynomial time to check if a set of learned B_X is consistent with a DAG or not, we could start the structure discovery process by learning a polytree over all variables in X. Then gradually build up a DAG from a polytree. In addition, because a polytree is a subset of DAGs, so there would be less number of consistent polytrees to a chordal graph than consistent DAGs to a wrs graph.

4 Complexity

Definition 15 Let $F = (V, E)$ be an undirected graph. A clique over a subset of variables $S \subset V$ is called a **simplicial clique** if $v_i \in S$ is a simplicial node in F. A **maximal simplicial clique** is a simplicial clique that cannot be extended by including one more vertex.

For example, $5 - 3$ and $5 - 4$ in Figure **??** are two simplicial cliques of size 2. The maximal simplicial clique is over nodes $\{3, 4, 5\}$.

WRS involves indefinite edge removal, depending on whether or not the induced subgraph obtained by removing a simplicial node is again WRS. Recursively simplicial is an extreme case of WRS where no edge is removed. In this section, we show that another extreme of WRS where a simplicial clique is always removed can be adopted to check morality for maximum degree 3 graphs in polynomial time.

Theorem 3 *Let $F = (V, E)$ be a connected moral graph with maximum degree $\Delta(F) = 3$. Then the morality of F can be checked by recursively removing a simplicial clique.*

Proof Assuming F is a moral graph with $\Delta(F) = 3$ and its morality cannot be checked by recursively removing a simplicial clique. It follows that the recursion must stop at a subgraph $F'' \subset F$ that has no simplicial nodes. F is WRS and $\Delta(F) = 3$ imply that F'' must share an edge with a 3-clique K_3 (Figure 3) that exists outside of F'' and will become simplicial during the recursive WRS process. F'' has no simplicial nodes implies that at least one of the two edges $\{v_5 - v_3, v_5 - v_4\}$ in K_3 must be removed during the recursive process of removing simplicial cliques. This requires K_3 to share an edge with another k-clique for $k \geq 3$. However, the nodes v_3 and v_4 have degree 3 implies K_3 cannot share an edge with anything else. Hence, we reach a contradiction. □

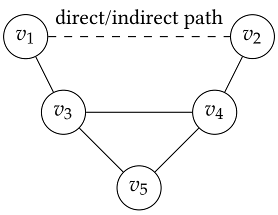


Fig. 3: The subgraph F'' is over $\{v_1, v_2, v_3, v_4, \ldots\}$, its supergraph F' is over $\{v_1, v_2, v_3, v_4, v_5, \ldots\}$.

Corollary 4 *The process of checking whether a given undirected graph F with maximum degree $\Delta(F) \leq 3$ is weak recursively simplicial is in polynomial time.*

Proof The definite process of removing a simplicial clique takes the same time as just removing a simplicial node, hence it is in polynomial time. □

We first prove the following two lemmas that will be used to prove Theorem 4.

Lemma 1 *Let $F = (V, E)$ be an undirected graph with maximum degree $\Delta(F) = 4$. Let $S \subset F$ be an induced subgraph such that S is a stack of at least 4 K_3s in a row (Figure 4). Assuming v_1 is a simplicial node in F, then F is WRS if and only if $G = (V \setminus \{v_1\}, E \setminus \{v_1v_2, v_1v_3, v_2v_3\})$ is WRS.*

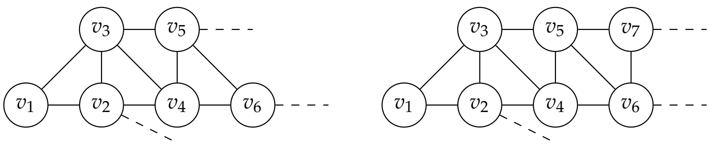


Fig. 4: A stack of 4 and 5 K_3s is contained in a maximum degree 4 graph F.

Proof Figure 4 shows a stack of four and five K_3s. In either case, the two nodes on each end have degrees less than 4. Assuming v_1 is a simplicial node, the other three nodes can be adjacent to others nodes in F. Removing v_1 and the three edges in the simplicial clique, v_3 has degree 2 hence is a simplicial node in G. Because v_1 is simplicial in F and $d_F(v_3) = d_F(v_4) = 4$, none of $\{v_1, v_3, v_4\}$ is adjancent to any nodes in F other than their neighbours in S. Hence, the two cliques over $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ are not necessary breakers for any cycle in F. In addition, the process of removing the simplical K_3 does not break more cliques. Hence, if F is WRS then G is also WRS. The argument can be extend to S with arbitrary (even or odd) number of K_3s.

The only if condition is obvious. If G is WRS, the new graph F obtained by attaching a simplcial K_3 to G is still WRS. □

Lemma 2 *Let $F = (V, E)$ be an undirected graph with maximum degree $\Delta(F) = 4$. Let $S \subset F$ be an induced subgraph such that S is a stack of two K_3s (Figure 5). Assuming v_1 is a simplicial node, then F is WRS i$G = (V \setminus \{v_1\}, E \setminus \{v_1v_2, v_1v_3\})$ is WRS.*

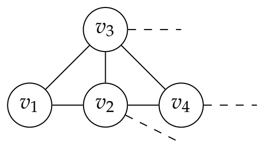


Fig. 5: A stack of 2 K_3s is contained in a maximum degree 4 graph F.

Proof Since v_1 is a simplicial node with degree 2, it is not adjacent to any nodes other than $\{v_2, v_3\}$. Hence, the simplicial clique is not a necessary breaker for any cycle in F. In addition, the process of removing $\{v_1, v_1v_2, v_1v_3\}$ does not reduce any simplicial clique in F. Hence, G is still WRS. If G is WRS, F is different from G by a simplcial node v_1, so F is also WRS. □

It is worth noticing that if S is a stack of 3 K_3s, there is no universal answer for all conditions. Hence, the following theorem states a set of rules for different sinarios and proves that the rules are legitimate for checking morality of maximum degree 4 graphs.

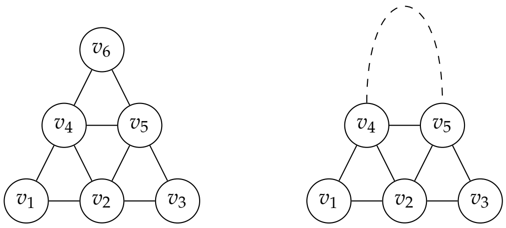


Fig. 6: Two graphs G_1 (left) and G_2 (right) of maximum degree 4. The dashed line represents a path of length at least 3.

all graphs are assumed to be connected. Denote a simplicial K_4 with the simplicial node s by K_4^s.

Algorithm 2 Backtracking algorithm to test WRS of graphs

Require: an undirected graph $F = (V, E)$ s.t. $\Delta(F) = 4$

function $\phi(F)$
  if F is chordal **then return** TRUE
  end if
  Find the set S of all simplicial nodes in F
  Order S by nodes degree $\{0, 1, 3, 2\}$
  For degree 2 nodes $x, y \in S$, if $y \in G_2$ whilest $x \notin G_2$, $I(x) < I(y)$
  $s = S[1]$
  if $d(s) \leq 1$ **then** ▷ 0. Prune all leaves
    $F = F[V \setminus \{s\}]$
    if $\phi(F)$ = TRUE **then return** TRUE
    end if
    return FALSE
  end if
  if $d(s) = 3$ **then** ▷ 1. Remove simplicial K_4
    $F = F - K_4^s$
    if $\phi(F)$ = TRUE **then return** TRUE
    end if
    return FALSE
  end if
  if $d(s) = 2$ **then** ▷ 2. For a simplicial K_3
    if $\phi(F)$ = TRUE **then return** TRUE
    end if
    return FALSE
  end if
  return FALSE
end function

Theorem 4 *Let $F = (V, E)$ be an undirected graph with maximum degree $\Delta(F) = 4$. Let O_1 and O_2 denote the operations of removing a simplicial node and a simplicial clique respectively. Then the morality of F can be checked by the following steps:*

1. *If seeing a simplicial K_4 or K_5, apply O_2.*

2. *If a simplicial K_3,*
   (a) *shares an edge with a K_4, apply O_1;*
   (b) *forms a subgraph with three other K_3s as shown in Figure 6 left, apply O_2;*
   (c) *is in a stack of one or more than three K_3s, apply O_2;*
   (d) *is in a stack of two K_3s, apply O_1;*
   (e) *is in a stack of three K_3s,*
      i. *if it is not in a subgraph as shown in Figure 6 right, apply O_2;*
      ii. *if all simplicial nodes are in subgraphs as shown in Figure 6 right,*
         A. *if two simplicial nodes are in the same stack of K_3s, apply O_1 on both;*
         B. *if no two simplicial nodes are in the same stack of K_3s, apply O_2 on a random simplicial node.*

Proof The maximum degree $\Delta(F) = 4$ implies that a simplicial clique $S = K_m$ where $1 \leq m \leq 5$. For $m = \{1, 2\}$, the cases are trivial so we simply prune the simplicial node of S. For $m = 5$, each node of a K_5 has degree 4 so a K_5 is either a (connect) component of F or it is F. In either case, a K_5 can be removed completely.

When $m = 4$, each node in a K_4 has degree 3. One node is reserved for a simplicial node, so only one edge can be shared with a complete subgraph K_3 (Figure 7). Since two nodes $\{v_2, v_4\}$ reach the maximum degree, none of the three edges of the K_3 appears in a **simple cycle** that shares no edge with the K_4. Equivalently, if the simplicial node v_1 and the two edges $\{v_2v_3, v_3v_4\}$ are removed, the K_3 shares no edges with a simple cycle in the remaining graph. Hence, the simplicail K_4 can be removed completely.

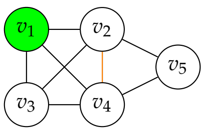
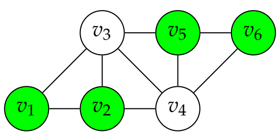


Fig. 7: A simplicial K_4 over $\{v_1, v_2, v_3, v_4\}$ shares an edge v_2v_4 with a K_3 over $\{v_2, v_4, v_5\}$.

When $m = 3$, each node in a K_3 has degree 2. Researve one node as the simplicial node leaves only one edge free to be shared with subgraphs of F. Denote the free edge by e. If $e = K_3 \cap K_4$, remove the simplicial node of the K_3 leads to the case of seeing a simplicial K_4 that has been proved above. If $e = K_3 \cap K_3$, each of e's end node has degree 3 hence opens more possibilities. By Lemma 2, ...

Assuming there is a stack of even number of K_3s, then only four nodes $\{v_1, v_2, v_5, v_6\}$ have degree less than 4. If one of the end node is reserved as a simplicial node,



$K_3 + \{K_3, K_3\}$: The edge $v_2 - v_3$ is shared by three K_3s (Figure 8 bottom right). If the simplicial K_3 is removed, the remaining two K_3s form a C_4. Since the nodes $\{v_2, v_3\}$ reach the maximum degree and are not adjacent to each other, the C_4 cannot share an edge with another K_3. The graph will then be falsely classified as non-WRS. Hence, only the simplicial node is removed in this case.

$K_3 + \{K_3, C_m\}$: If the simplicial node v_1 is removed (Figure 8 bottom left), the remaining K_3 over $\{v_2, v_3, v_4\}$ becomes a potential breaker for the C_m, although other breakers may exist for the same cycle.

[Verma and Pearl, 1993] proved that the problem of deciding whether an undirected graph is WRS is NP-complete. We modifed the prove so that the NP-completeness hodes for undirected graphs with maximum degree 5.

Proposition 4 *The problem of testing whether a given undirected graph with maximum degree 5 is weak recursively simplicial is NP-complete.*

Proof Replace the arbitrarily high degree nodes with chains. Figures!!! □

5 Some random notes

Proposition 5 *Let F be a moral*

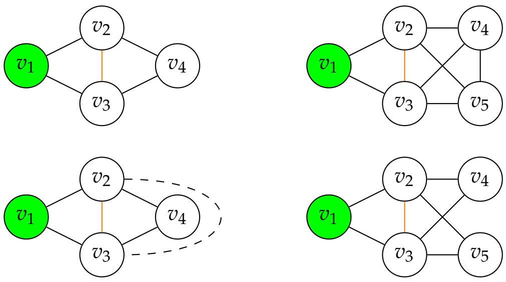


Fig. 8: A simplicial K_3 over $\{v_1, v_2, v_3\}$ shares an edge with a K_3 (top left), K_4 (top right), $\{K_3, C_m\}$ (bottom left) or $\{K_3, K_3\}$ (bottom right).

Lloyd's conjecture: knowing a graph is wrs, adding an edge to obtain a supergraph, perhaps it is efficient to check if the supergraph is wrs.

Proof because remove the added edge, then we obtain the original graph which we know is wrs. however, we don't remove a random edge when checking wrs unless the edge connects to a simplicial node. what if the edge is not connected with a sim node? if we know the original graph is wrs and we know the set of sim nodes in the recursion and the set of edges to delete, then we could easily check if the new edge is connected with any one of the sim node, if it is then good. if not, then we could check if the new edge is connected with two neighbours of a sim node, if it is

Table 1: Comparison between the number of labelled connected chordal graphs, the number of weak recursively simplicial graphs, the number of undirected graphs and the number of Markov equivalent classes.

# nodes	# con-C.G.	# C.G.	# WRS	# U.G.	# MEC
1	1	1	1	1	1
2	1	2	2	2	2
3	4	8	8	8	11
4	35	61	61	64	185
5	541	822	882	1024	8782
6	13302	18154		32768	1067825
7	489287	617675		2097152	312510571
8	25864897	30888596		268435456	212133402500
9	1910753782	2192816760		68719476736	326266056291213
10	1.93×10^{11}	2.15×10^{11}		3.52×10^{13}	1.19×10^{17}

then good. if a graph is wrs, then it eventually will diminish, so the new edge must appear somewhere in the recursion to either stop the recursion or don't stop it.

Corollary 5 *DAGs in the same Markov equivalent class produce the same Markov blanket sets B_X.*

Proof If two DAGs G_1 and G_2 are Markov equivalent, they have the same skeleton and the same set of colliders. This implies $B_i^{G_1} = B_i^{G_2}, \forall X_i \in X$. □

Notice that two Markov equivalent classes could entail the same B_X. For example...

Corollary 6 $|\{$*chordal graphs*$\}| \leq |B_X| \leq |\{$*Markov equivalent classes*$\}|$.

Counting labelled chordal graphs [Wormald, 1985], counting Markov equivalent classes (assymptotic ratio of around 0.27 to DAGs) [Gillispie and Perlman, 2001].

Proposition 6 *Let G be a DAG and F be the moral graph of G. If a node x is a leaf in G, then it must be a simplicial node in F.*

Proof If x is a leaf in G, it has only parents, which form a clique after moralization. By definition, x is a simplicial node in F. □

Corollary 7 *Let G be a DAG and F be the moral graph of G. Then F must have at least one simplicial node.*

Proof Since each DAG has at least one leaf, by Proposition 6 F have at least one simplicial node. □

Corollary 8 *Let G be a DAG and F be the moral graph of G. If a node x is not a simplicial node in F, then it must not be a leaf in G.*

Proposition 7 *Let G be a DAG and F be the moral graph of G. Let S^1 be the set of simplicial nodes in F and F_1 be the induced subgraph of F over $X \setminus S^1$. Then there must exist at least one simplicial node after removing from F' all the edges between $N(X_i), \forall X_i \in S^1$.*

Table 2: Notations

F	a undirected graph
G	a DAG
X	a set of random variables (nodes)
X_i	a random variable (or node) in X
X_{-i}	$X \setminus X_i$
$X_{-[1,\ldots,i]}$	$X \setminus \{X_1, \ldots, X_i\}$
B_i^G	the Markov blanket of a variable X_i in G
B_X^G	$\{B_i \mid \forall X_i \in X\}$

Proof Let F_1' be the result of removing from F_1 all the edges between $N(X_i), \forall X_i \in S^1$. The corresponding directed graph G' of F_1' must be a subgraph of the DAG G, so also acyclic. Assuming F_1' has no simplicial nodes, by Corollary 8 G' has no leaf, which is a contradiction. □

Here are some issues worth discussing:

1. Application: the backtracking algorithm can now be applied when learning MBs in paralle. What if there are conflicts between two MBs, which one should give up? Need to estimate uncertainty?
2. Simplicial nodes in the first step always contain the leaves.
3. Those nodes that become simplicial in the next step without having to delete any edges contain the leaves in the next step.
4. So wrs can be used to test if a MB family is consistent with a DAG, it would be good if we can also find out how many consistent DAGs or essential graphs are there for this MB family.
5. also it would be good if we can explore wrs into details, such as what dag nodes become simplicial nodes in wrs recursion, and if no edges need to be deleted from a simplicial node's neighbours then what's this simplicial node?
6. maybe there is a path from s.t. every step is a moral graph of a dag, perhaps can be proved by delete an edge from a dag.

Questions: If a graph F is known to be wrs, does is help to decide if a subgraph/-supergraph different by one edge from F is wrs or not?

Answer: Probably not. If it is, then we know a base case, any graph can be reached from this base case, hence any graph can be efficiently tested.

6 Notations

6.1 Notations

References

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