
15-455: UNDERGRADUATE COMPLEXITY THEORY, SPRING 2017, RYAN O'DONNELL

A PREPRINT

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May 2, 2019

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7.1 CIRCUIT-SAT

Definition 7.1. Boolean circuit C is a DAG, in which nodes are called gates, and edges are called wires... usually has n inputs and 1 output... gates are and \wedge , or \vee , negation \neg with fan-in 2, 2, 1 respectively and arbitrary number of fan-out.

A circuit C computes a Boolean function which maps n bits strings into one bit $\{0, 1\}^n \rightarrow \{0, 1\}$.

You can think of circuit as a computational model.

One circuit can only operations on inputs of a fixed length, because number of sources in the DAG is fixed. This is one of the main reason we don't use it as the first choice of our computational model.

When the input to algorithm is a circuit $\langle C \rangle$, we measure run time in terms of $n :=$ number of input gates and $m :=$ the number of total gates. These parameters are polynomially related to the input size, so if we only care if the algorithm can run in P time or not, whether the run time is measured by the size of the input or n, m doesn't matter. $n \leq m \leq |\langle C \rangle| \leq O(m \log m)$.

Definition 7.2 (circuit-eval). given $\langle C, x = \{0, 1\}^n \rangle$, what is $C(x)$? This can also be written into a language $\{\langle C, x \rangle : C(x) = 1\}$.

This can be computed in P time. It's the same as checking an answer of a circuit. Roughly speaking, you can do it in $O(m)$.

Definition 7.3 (circuit-sat). given $\langle C \rangle$, is there an x s.t. $C(x) = 1$? That is, $\{\langle C \rangle : \exists x \text{ s.t. } C(x) = 1\}$.

Brute force, try all 2^n xs, find the one that gives the true answer. This is probably the best we can do. We think that it can be done in P iff $P = NP$.

7.2 FORMULA-SAT

Definition 7.4. A formula is a circuit where all the fan-outs are 1.

eg, $(x_1 \wedge x_2) \wedge (x_2 \vee x_3) \vee (\neg(x_1 \wedge x_2))$. A circuit can be represented by a formula.

Formula-sat is a special case of a circuit-sat. Any formula is an example of a circuit. It's potentially easier, but we still don't have a smarter way of computing it.

7.3 CNF-SAT

Definition 7.5. Special case of formula-sat, where the formula is a conjunction of clauses, and a clause is a disjunction of literals, a literal is a variable or its negation.

For a cnf problem, n is the number of variables, and m is the number of clauses. It's a potential even easier problem. It is in EXP time $O(2^n m)$. It is NP-complete.

Definition 7.6. k-sat is a special case of cnf-sat, where all clauses have at most k literals.

it's potential easier than cnf.

7.4 3-sat

A popular example is 3-sat. WLOG, in 3-sat, $m \leq O(n^3)$, this is over counting a lit, but it's ok. We know it's NP-complete. It's in time 1.34^n .

4-sat is in 1.5^n , 5-sat in 1.6^n , 6-sat 1.667^n , ..., k-sat in time $(2 - 2/k)^n \text{poly}(n)$. 2-sat is in polynomial time.

Theorem 7.1. 2-sat is in P.

e.g. input: $(x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (\bar{x}_3)$

Proof. observation: $(x_1 \vee x_2) = (\bar{x}_1 \rightarrow x_2) = (\bar{x}_2 \rightarrow x_1)$ because of the truth table for disjunction $(x_1 \vee x_2)$ is the same as the truth table for the implication $(\bar{x}_1 \rightarrow x_2)$. See https://en.wikipedia.org/wiki/Truth_table

rewrite the above expression using the \rightarrow formulas, using both directions. ie., $(\bar{x}_1 \rightarrow x_2) \wedge (\bar{x}_2 \rightarrow x_1) \wedge \dots$

draw a directed graph with $2n$ vertices, one for each literal and $2m$ directed edges, one for each implication.

key observation of the directed graph: if $l_i \rightarrow l_j$ is an edge, l is a literal, then there is also an edge $\bar{l}_j \rightarrow \bar{l}_i$.

There is $\bar{x}_2 \rightarrow x_1 \rightarrow x_2$, i.e., there is a path $\bar{x}_2 \rightarrow x_2$. In any satisfiable assignment, x_2 better be TRUE.

Suppose exists a path $x_2 \rightarrow \bar{x}_2$, then x_2 better be FALSE.

If from some i , there exists a path $x_i \rightarrow \bar{x}_i$ and $\bar{x}_i \rightarrow x_i$, then the instance is unsatisfiable. call them contradictory path.

In poly(n) time, can check whether exists contradictory paths.

It remains to show that if there is no such contradictory path, then it is satisfiable.

assume for all i , we don't have contradictory path. □

8 Lecture 8 NP

3-sat is in P iff circuit-sat is in P iff 3-colorable is in P iff k-clique is in P iff every language in NP is in P.

Conjecture: $P \neq NP \iff 3\text{-sat} \notin P$.

Conjecture Exponential time hypothesis (ETH): $\exists \delta > 0$ s.t. $3\text{-sat} \notin \text{Time}((1 + \delta)^n)$.

Conjecture Strong ETH: $\forall \delta > 0, \exists k$ s.t. $k\text{-sat} \notin \text{TIME}((2 - \delta)^n)$. This is almost the same as saying that CNF with no upperbound on k has no better algorithm than just 2^n .

The first conjecture is weaker than the 2nd that is weaker than the last. The last conjecture also implies that $\forall \epsilon > 0$, cannot solve the longest common subsequence problem in time $O(n^{2-\epsilon})$.

8.1 NP

idea: for many problems there exists brute force algorithms enumerate (exponentially many) candidate solutions and in p-time check if each one is a genuine solution.

Two features: candidate solutions are encodable by strings y with polynomial length $|y| \leq \text{poly}(|x|)$. That is, candidate solutions are not extremely long comparing with the input size; there exists an efficient algorithm to check if a candidate solution is a genuine solution.

Roughly: NP is the class of decision problems/languages with those two features.

Un 3-colorable problem is highly believed to not be in NP.

Definition 8.1. An algorithm (TM) V is a verifier for language L if

- V takes as input a pair $\langle x, y \rangle$;
- $\forall x, x \in L \iff \exists y \text{ s.t. } V(\langle x, y \rangle) = \text{accept}$.

Definition 8.2. Verifier V is said to polynomial time if $V \langle x, y \rangle$ runs in time $O(|x|^k)$ for some $k \in \mathbb{N}$

Subtly: V 's running time is measured in terms of $|x|$.

Definition 8.3. $NP = \{L \mid L \text{ has a P-time verifier}\}$.

Example: proving squares = $\{x \mid x = \langle B \rangle, B \in \mathbb{N} \text{ is a perfect square}\}$ is in NP. Proof is omitted by me...

3-colorable is in NP, proof is omitted... two main steps. Firstly, there exists a p-time verifier V . Secondly, if G is 3-col then V accepts. If V accepts then G is 3-col.

Roughly, P is all the decision problems where deciding if there is a genuine solution can be done efficiently. NP is the set of problems where checking if a candidate solution is a genuine solution can be done efficiently.

$P \subseteq NP \subseteq EXP$. Recall $P \neq EXP$ by THT, so $P \neq NP$ and/or $NP \neq EXP$. We can't prove $P \neq NP$ or $NP \neq EXP$, but we can prove $P \neq EXP$.

Proof. Let $L \in P$, wts $L \in NP$. Proof omitted... □

Proof. $P = NP$ iff 3-sat is in P by Cook-Levin theorem... □

9 Lecture 9 Nondeterminism

$NTIME(f(n)) = \{L \mid \exists \text{ a nondeterministic algorithm } N \text{ that decides } L \text{ in time } O(f(n))\}$

$NP = \cup_{k \in \mathbb{N}} NTIME(n^k)$

Theorem 9.1. $Nondeterministic-NP = verifier-NP$

10 Lecture 10 Reductions

\leq_M^P denotes a polynomial mapping from a problem to another problem.

$3COL \leq_M^P 4COL \leq_M^P SAT$

The relation is transitive.

11 Lecture 11 NP-complete and Cook-Levin theorem

12 Lecture 12 NP-completeness reduction

13 Lecture 13 search-to-decision

Find an answer instead of just deciding T or F.

circuit-sat is (downward) self-reducible.

Example, circuit-sat, if $P=NP$, then there is a poly-time algorithm to decide. Imagine a circuit with n inputs, then set the first bit to 0, call the poly-time alg, if yes, then $x_1=0$, otherwise $x_1=1$. Iterate this n times, we find a sat assignment. The total running time is n times the poly-time alg, which is still poly-time.

Theorem 13.1. *Suppose $L \in NP$ with a verifier $V(x, y)$. Assume $P=NP$, then there exists a poly-time algorithm S s.t. $\forall x \in L, S(x) = y$ s.t. $V(x, y)$ accepts.*

Proof. Using Cook-Levin Theorem. Build a search method that is in the form of a circuit, and takes two inputs x and y . Then S wants to find a y such that the circuit outputs 1. This is then reduced to the search-to-decision problem for circuit-sat. \square

padding.

$EXP = \cup_k TIME(2^{n^k})$ deciding a language in exponential time using a deterministic turing machine, and $NEXP = \cup_k NTIME(2^{n^k})$ deciding a language in exponential time using a non-deterministic turing machine.

Similary to $P \subseteq NP$, we have $EXP \subseteq NEXP$.

Theorem 13.2. *If $P = NP$, then $EXP = NEXP$.*

Proof. Prove this using padding. Assume $P=NP$, since $EXP \subseteq NEXP$, it's suffice to show $NEXP \subseteq EXP$.

Let $L \in NEXP$, wts $L \in EXP$... \square

Dichotomy Theorem: Every boolean CSP is either in P or NP .

Graphi Iso best runs in quasi-poly time, it's hard to belive that Graph ISO is in NP , because that implies all NP -complete problems can run in quasi-time.