EXISTENCE IN MULTIDIMENSIONAL SCREENING WITH GENERAL NONLINEAR PREFERENCES

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ABSTRACT. We generalize the approach of Carlier (2001) and provide an existence proof for the multidimensional screening problem with general nonlinear preferences. We first formulate the principal's problem as a maximization problem with G-convexity constraints, and then use G-convex analysis to prove existence.

Keywords: Principal-agent problem; Adverse selection; Bi-level optimization; Incentive-compatibility; Non-quasilinearity

1. INTRODUCTION

This paper provides a general existence for a multidimensional nonlinear pricing model, which is a natural extension of the models studied by Mussa-Rosen [20], Spence [27, 28], Myerson [21], Baron-Myerson [3], Maskin-Riley [15], Wilson [30], Rochet-Choné [23], Monteiro-Page [19] and Carlier [5]. A major distinction lies in whether the private type is one-dimensional (such as [20, 15]), or multidimensional (such as [23, 19, 5]). Another distinction is whether preferences are quasilinear on price (such as [1, 5]) or fully nonlinear (such as [22, 17]), especially for multidimensional models.

This paper proves the existence of a (price menu) solution to a multidimensional multiproduct monopolist problem, by extending Carlier [5] to fully nonlinear preferences. G-convex analysis is employed to deal with the difficulty of non-quasilinear preferences. This method is potentially applicable to other problems under the same principal-agent framework, such as the study of tax policy ([18]) and other regulatory policies ([3]).

Consider the problem for a multiproduct monopolist who sells indivisible products to a population of consumers, who each buy at most one unit. Assume there is neither cooperation, nor competition between agents. Additionally, assume the monopolist is able to produce enough of each product such that there are neither product supply shortages nor economies of scale. Taking into account participation constraints and incentive compatibility, the monopolist would like to find the optimal menu of prices to maximize its total profit.

In this paper, we first identify incentive compatibility with a G-convexity constraint, before rewriting the maximization problem by converting the optimization variables from a product-price pair of mappings to a product-value pair. It can then be shown that the product-value pair converges under the G-convexity constraint. The existence result follows.

Starting from Mirrlees [18] and Spence [27], there are two main types of generalizations. One generalization is in terms of dimension, from 1-dimensional to multi-dimensional. The other generalization is in utility functional form, from quasilinear to non-quasilinear.

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For the quasilinear case, where utility depends linearly on price, theories of existence [4, 24, 5, 19], uniqueness [6, 20, 23] and robustness [3, 11] have been widely studied, among which the equivalence of function space convexity to the non-negative cross-curvature condition revealed by Figalli-Kim-McCann [11] serves as a major milestone orienting our work.

When parameterization of preferences is linear in agent types and price, Rochet and Choné (1998, [23]) not only obtain existence results but also partially characterize optimal solutions and expound their economic interpretations, given that monopolist profits can be characterized by the aggregate difference between selling prices and manufacturing costs.

More generally, Carlier [5]) proves existence results for general quasilinear utility, where agent type and product type are not necessarily of the same dimension and monopolist profit equals selling price minus some linear manufacturing cost.

This paper generalizes the quasilinear case to the non-quasilinear case, which has many applications. For example, some fully nonlinear utilities include scenarios where agents are more sensitive to higher prices and where different agents might have different sensitivities to the same price. See Wilson [30, Chapter 7] for the importance of taking income effects into account. The generalized existence problem is also mentioned as a conjecture by Basov [4, Chapter 8]. However, only a few results are known for the multidimensional non-quasilinear case, and the impact of price on utility could be much more complicated.

Recently, Nöldeke and Samuelson (2015, [22]) provide a general existence result given that the consumer and product space are compact, by implementing a duality argument based on Galois Connections. McCann-Zhang [17] show another general existence result given that the consumer and product space are bounded, using G-convexity, and generalize uniqueness and convexity results of Figalli-Kim-McCann [11] to the non-quasilinear case. In this paper, we also explore existence using G-convex analysis, which will be introduced in Section 3, but with less restriction on boundedness of the product domain and without assuming the generalized single-crossing condition. As a result of lack of natural compactness, the proof of this paper is quite different from that of either of the earlier papers. It should be mentioned here that neither this paper, nor the earlier two, require the monopolist profit to take on a special form, which is a generalization from much of the literature.

The remainder of this paper is organized as follows. Section 2 states the mathematical model and assumptions. Section 3 introduces preliminaries, including G-convexity, G-subdifferentiability, and reformulation of the monopolist's problem. In Section 4, we state the existence theorem as well as the convergence proposition. Section 5 proposes some directions of future work. We leave all the proofs of lemmas, propositions and the existence theorem in Section 6.

2. MODEL

This model is a bilevel optimization. After a monopolist publishes its price menu, each agent maximizes his utility through the purchase of at most one product. The monopolist maximizes aggregate profits based on agents' choices, knowing only the statistical distribution of agent types.

Suppose agents' preferences are given by some parametrized utility function $G(x, y, z)$, where x is a M-dimensional vector of consumer characteristics, y is a N-dimensional vector of attributes of each product, and z represents the price of each product. Denote by X the space of agent types, by Y the space of products, by $cl(Y)$ the closure of Y, by Z the space of prices, and by $cl(Z)$ the closure of Z. In this paper, we only consider the case where both agent types and product attributes are continuous.

The monopolist sells indivisible products to agents. Each agent buys at most one unit of product. We assume no competition, cooperation or communication between agents. For any given price menu

$p : cl(Y) \longrightarrow cl(Z)$, agent $x \in X$ knows his utility $G(x, y, p(y))$ for purchasing each product y at price $p(y)$. It follows that every agent solves the following maximization problem

$$(2.1) \qquad u(x) := \max_{y \in cl(Y)} G(x, y, p(y)),$$

where $u(x)$ represents the maximal utility agent x can obtain, and $u : X \longrightarrow \mathbf{R}$ is also called the value function or indirect utility function. At this point, it is assumed that the maximum in (2.1) is attained for each agent x.

If agent x purchases product y at price $p(y)$, the monopolist would earn from this transaction a profit of $\pi(x, y, p(y))$. For example, monopolist profit can take the form $\pi(x, y, p(y)) = p(y) - c(y)$, where c(y) is a variable manufacturing cost function. Summing over all agents in the distribution $d\mu(x)$, the monopolist's total profit is characterized by

$$(2.2) \qquad \Pi(p, y) := \int_X \pi(x, y(x), p(y(x)))d\mu(x),$$

which depends on her price menu $p : cl(Y) \longrightarrow cl(Z)$ and agents' choices $y : X \longrightarrow cl(Y)$.[1]

Since the monopolist only observes the overall distribution of agent attributes, and is unable to distinguish individual agent characteristics, the monopolist takes into account the following incentive-compatibility constraint when determining product-price pairs $(y, p(y))$.

Definition 2.1 (incentive compatibility). A (product, price) pair of measurable mappings $(y, z) : X \longrightarrow cl(Y) \times cl(Z)$ on agent space X is *incentive compatible* if and only if $G(x, y(x), z(x)) \geq G(x, y(x'), z(x'))$ for all $(x, x') \in X^2$.

Therefore, an incentive-compatible product-price pair $(y, p(y))$ ensures that no agent has incentive to pretend to be another agent type.

In addition, we adopt a participation constraint in order to rule out the possibility of the monopolist charging exorbitant prices and the agents still having to make transactions despite this: each agent $x \in X$ will refuse to participate in the market if the maximum utility he can obtain is less than his reservation value $u_\emptyset(x)$, where the function $u_\emptyset : X \longrightarrow \mathbf{R}$ is given in the form $u_\emptyset(x) := G(x, y_\emptyset, z_\emptyset)$, for some $(y_\emptyset, z_\emptyset) \in cl(Y \times Z)$.

For monopolist profit, some literature assumes $\pi(x, y_\emptyset, z_\emptyset) \geq 0$ for all $x \in X$ to ensure that the outside option is harmless to the monopolist. Here, it is not necessary to adopt such assumption for the sake of generality.

Then the monopolist's problem can be described as follows:

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[1]It is worth mentioning that in some literature, the monopolist's objective is to design a product line $\tilde{Y}$ (i.e. a subset of $cl(Y)$) and a price menu $\tilde{p} : \tilde{Y} \longrightarrow \mathbf{R}$ that jointly maximize overall monopolist profit. Then, given $\tilde{Y}$ and $\tilde{p}$, an agent of type x chooses the product $y(x)$ that solves

$$\max_{y \in \tilde{Y}} G(x, y, \tilde{p}(y)) := u(x).$$

Allowing price to take value $\bar{z}$ (which may be $+\infty$), and assuming Assumption 1 below, the effect of designing product line $\tilde{Y}$ and price menu $\tilde{p} : \tilde{Y} \longrightarrow \mathbf{R}$ is equivalent to that of designing a price menu $p : cl(Y) \longrightarrow (-\infty, +\infty]$, which equals $\tilde{p}$ on $\tilde{Y}$ and maps $cl(Y) \setminus \tilde{Y}$ to $\bar{z}$, such that no agents choose to purchase any product from $cl(Y) \setminus \tilde{Y}$, which is less attractive than the outside option $y_\emptyset$ according to Assumption 1. In this paper, we use the latter as the monopolist's objective.

For any given price menu $p : cl(Y) \longrightarrow (-\infty, +\infty]$, one can construct a mapping $y : X \longrightarrow cl(Y)$ such that each $y(x)$ solves the maximization problem in (2.1). But such mapping is not unique, for some fixed price menu, without the single-crossing type assumptions. Therefore, we adopt in (2.2) the total profit as a functional of both price menu p and its corresponding mapping y.

(2.3) $\qquad$ (P_0) $\begin{cases} \sup \Pi(p, y) = \int_X \pi(x, y(x), p(y(x))) \, d\mu(x) \\ s.t. \ (y, p(y)) \text{ is incentive compatible}; \\ s.t. \ G(x, y(x), p(y(x))) \geq u_\emptyset(x) \text{ for all } x \in X; \\ s.t. \ p \text{ is lower semicontinuous.} \end{cases}$

We assume that p is lower semicontinuous, without which the maximum in (2.1) may not be attained. However, in the equivalence form of (P_0), this restriction will be encoded in G-convexity of the value functions, which will be shown in Proposition 3.8.

This paper makes the followings assumptions. We use $C^0(X)$ to denote the space of all continuous functions on X, and use $C^1(X)$ to denote the space of all differentiable functions on X whose derivative is continuous.

Assumption 1. *Agents' utility $G \in C^1(cl(X \times Y \times Z))$, where the space of agents X is a bounded open convex subset in $\mathbf{R}^M$ with C^1 boundary, the space of products $Y \subset \mathbf{R}^N$, and range of prices $Z = (\underline{z}, \bar{z})$ with $-\infty < \underline{z} < \bar{z} \leq +\infty$. Assume $G(x, y, \bar{z}) := \lim_{z \longrightarrow \bar{z}} G(x, y, z) \leq G(x, y_\emptyset, z_\emptyset)$, for all $(x, y) \in X \times cl(Y)$; and assume this inequality is strict when $\bar{z} = +\infty$.*

Here we do not necessarily assume X, Y, and Z are compact spaces; in particular, Y and Z are potentially unbounded (i.e. we do not set *a priori* bounds for product attributes or an *a priori* upper bound for price). However, we do specify a lower bound for the price range, since the monopolist has no incentive to set price close to negative infinity.

Assumption 2. $G(x, y, z)$ *is strictly decreasing in z for each $(x, y) \in cl(X \times Y)$.*

This is to say that, the higher the price paid to the monopolist, the lower the utility that will be left for the agent, for any given product. For each $(x, y) \in X \times cl(Y)$ and $u \in G(x, y, cl(Z))$, define $H(x, y, u) := z$ whenever $G(x, y, z) = u$, i.e., $H(x, y, \cdot) = G^{-1}(x, y, \cdot)$. Therefore, $H(x, y, u)$ represents the price paid by agent x for product y when receiving value u.

Assumption 3. *G is coordinate-monotone in x. That is, for each $(y, z) \in cl(Y \times Z)$, and for all $(\alpha, \beta) \in X^2$, if $\alpha_i \geq \beta_i$ for all $i = 1, 2, ..., M$, then $G(\alpha, y, z) \geq G(\beta, y, z)$.*

In Assumption 3, we assume that agent utility increases along each consumer attribute coordinate.

In the following, we use $D_x G(x, y, z) := (\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \ldots, \frac{\partial G}{\partial x_M})(x, y, z)$ to denote derivative with respect to x. For any vector in $\mathbf{R}^M$ or $\mathbf{R}^N$, we use $||\cdot||$ and $||\cdot||_\alpha$ to denote its Euclidean 2-norm and α-norm ($\alpha \geq 1$), respectively. For example, for $x \in \mathbf{R}^M$, we have $||x|| = \sqrt{\sum_{i=1}^M x_i^2}$ and $||x||_\alpha = (\sum_{i=1}^M |x_i|^\alpha)^{\frac{1}{\alpha}}$.

In Rochet-Choné's model, $H(x, y, u) = x \cdot y - u$ and $\pi(x, y, z) = z - C(y)$, for some superlinear cost function C. In this case, $\pi(x, y, H(x, y, u)) = x \cdot y - u - C(y)$. Since C is superlinear, it is reasonable to assume the following:

Assumption 4. *$\pi(x, y, H(x, y, u))$ is super-linearly decreasing in y. That is, there exist $\alpha \geq 1$, $a_1, a_2 > 0$ and $b \in \mathbf{R}$, such that $\pi(x, y, H(x, y, u)) \leq -a_1 ||y||_\alpha^\alpha - a_2 u + b$ for all $(x, y, u) \in X \times cl(Y) \times \mathbf{R}$, or equivalently, $\pi(x, y, z) + a_2 G(x, y, z) \leq -a_1 ||y||_\alpha^\alpha + b$ for all $(x, y, z) \in X \times cl(Y) \times \mathbf{R}$.*

As shown in the alternative formulation, Assumption 4 requires the existence of some weighted surplus which is super-linearly decreasing in product.

Assumptions 5-7 are some technical assumptions on $D_x G$.

Assumption 5. *$D_x G(x, y, z)$ is Lipschitz in x, uniformly in (y, z), meaning there exists k such that $||D_x G(x, y, z) - D_x G(x', y, z)|| \leq k||x - x'||$ for all $(x, x', y, z) \in X^2 \times cl(Y) \times cl(Z)$.*

Assumption 6. $||D_x G(x, y, z)||_1$ *increases sub-linearly in* y. *That is, there exist* $\beta \in (0, \alpha], c > 0$, *and* $d \in \mathbf{R}$, *such that* $||D_x G(x, y, z)||_1 \leq c ||y||_\beta^\beta + d$ *for all* $(x, y, z) \in X \times cl(Y) \times cl(Z)$.

Assumption 7. *Coercivity of* 1-*norm of* $(D_x G)$. *For all* $s > 0$, *there exists* $r > 0$, *such that* $\sum_{i=1}^M |D_{x_i} G(x, y, z)| \geq s$, *for all* $(x, y, z) \in X \times cl(Y) \times cl(Z)$, *whenever* $||y|| \geq r$.

Allowing Assumption 3, the derivatives $D_{x_i} G$ are always nonnegative; therefore, we no longer need to take absolute values of $D_{x_i} G$ in the inequality of Assumption 7. And then Assumption 7 says that the marginal utility of agents who select the same product y will increase to infinity as $||y||$ approaches infinity, uniformly for all agents and prices. For instance, when $M = N$, utility $G(x, y, z) = \sum_{i=1}^M x_i y_i^2 - z$ satisfies Assumption 7, because $\sum_{i=1}^M |D_{x_i} G(x, y, z)| = \sum_{i=1}^M D_{x_i} G(x, y, z) = \sum_{i=1}^M y_i^2 \longrightarrow +\infty$ as $||y|| \longrightarrow +\infty$.

Assumptions 8 states constraints on the continuity of principal's profit function π, integrability of participation constraint $u_\emptyset$, and absolutely continuity of measure μ with respect to the Lebesgue measure.

Assumption 8. *Profit function* π *is continuous on* $cl(X \times Y \times Z)$. *The participation constraint* $u_\emptyset$ *is integrable with respect to* $d\mu$, *where the measure* μ *is absolutely continuous with respect to the Lebesgue measure and has* X *as its support.*

For $\alpha \geq 1$, denote $L^\alpha(X)$ as the space of functions for which the α-th power of the absolute value is Lebesgue integrable with respect to the measure $d\mu$. That is, a function $f : X \longrightarrow \mathbf{R}$ is in $L^\alpha(X)$ if and only if $\int_X |f|^\alpha d\mu < +\infty$. For instance, Assumption 8 implies $u_\emptyset \in L^1(X)$.

3. G-convex, G-subdifferentiability, and the Monopolist's Problem

In this section, we introduce some tools from convex analysis and the notion of G-convexity, which is a generalization of ordinary convexity. Several results in this chapter echo those from Section 3 of McCann-Zhang [17] but extended to unbounded domains. The purpose of this section is to fix terminology and prepare the preliminaries for the main results of the next section. We also rewrite the monopolist's problem in Proposition 3.8, which is an equivalent form of (2.3).

Recall that the subdifferential of a (convex) function u at x_0 is defined as the set:

$$\partial u(x_0) = \{y \in cl(Y) | u(x) - u(x_0) \geq \langle x - x_0, y \rangle, \text{ for all } x \in X\}.$$

Here we use $\langle , \rangle$ to denote the Euclidean inner product. This set is nonempty for every x_0 if and only if u is convex. Then for any convex function u on X and any fixed point $x_0 \in X$, there exists $y_0 \in \partial u(x_0)$, satisfying

$$(3.1) \qquad u(x) \geq \langle x, y_0 \rangle - (\langle x_0, y_0 \rangle - u(x_0)), \text{ for all } x \in X,$$

where equality holds at $x = x_0$. On the other hand, if for any $x_0 \in X$, there exists y_0, such that (3.1) holds for all $x \in X$, then u is convex. The following definition is analogous to this property, which is a special case of G-convexity, when $G(x, y, z) = \langle x, y \rangle - z$.

Definition 3.1 (G-convexity). A function $u \in C^0(X)$ is called G-*convex* if for each $x_0 \in X$, there exist $y_0 \in cl(Y)$, and $z_0 \in cl(Z)$ such that $u(x_0) = G(x_0, y_0, z_0)$, and $u(x) \geq G(x, y_0, z_0)$, for all $x \in X$.

Similarly, one can also generalize the definition of subdifferential from (3.1).

Definition 3.2 (G-subdifferentiability). The G-subdifferential of a function $u : X \longrightarrow \mathbf{R}$ is a point-to-set mapping defined by

$$\partial^G u(x) := \{y \in cl(Y) | u(x') \geq G(x', y, H(x, y, u(x))), \text{ for all } x' \in X\}.$$

A function u is said to be *G-subdifferentiable* at x if and only if $\partial^G u(x) \neq \emptyset$.[2]

In particular, if $G(x, y, z) = \langle x, y \rangle - z$, then the G-subdifferential coincides with the subdifferential. There are other generalizations of convexity and subdifferentiability. For instance, h-convexity in Carlier [5], or equivalently, b-convexity in Figalli-Kim-McCann [11], or c-convexity in Gangbo-McCann [12], is a special form of G-convexity, where $G(x, y, z) = h(x, y) - z$, which serves an important role in the quasilinear case. For more references of convexity generalizations, see Kutateladze-Rubinov [13], Elster-Nehse [9], Balder [2], Dolecki-Kurcyusz [7], Singer [26], Rubinov [25], and Martínez-Legaz [14].

As mentioned above, it is known in convex analysis that a function is convex if and only if it is subdifferentiable everywhere. The following lemma adapts this to the notion of G-convexity. The proof can be found in Section 6.

Lemma 3.3. *Given Assumption 2, a function $u : X \longrightarrow \mathbf{R}$ is G-convex if and only if it is G-subdifferentiable everywhere.*

Using Lemma 3.3, one can show the following result, which identifies incentive compatibility using G-convexity and G-subdifferentiability.

Proposition 3.4. *Let (y, z) be a pair of mappings from X to $cl(Y) \times cl(Z)$. Given Assumption 2, this (product, price) pair is incentive compatible if and only if $u(\cdot) := G(\cdot, y(\cdot), z(\cdot))$ is G-convex and $y(x) \in \partial^G u(x)$ for each $x \in X$.*

We introduce implementability here, which is closely related to incentive-compatibility and can also be exhibited by G-convexity and G-subdifferential.

Definition 3.5 (implementability). *A function $y : X \longrightarrow cl(Y)$ is called *implementable* if and only if there exists a function $z : X \longrightarrow \mathbf{R}$ such that the pair (y, z) is incentive compatible.*

Remark 3.6. *Allowing Assumption 2, a function y is implementable if and only if there exists a price menu $p : cl(Y) \longrightarrow \mathbf{R}$ such that the pair $(y, p(y))$ is incentive compatible.*

As a corollary of Proposition 3.4, implementable functions can be characterized as G-subdifferential of G-convex functions.

Corollary 3.7. *Given Assumption 2, a function $y : X \longrightarrow cl(Y)$ is implementable if and only if there exists a G-convex function $u(\cdot)$ such that $y(x) \in \partial^G u(x)$ for each $x \in X$.*

When parameterization of preferences is linear in agent types and price, Corollary 3.7 says that a function is implementable if and only if it is monotone increasing. In general quasilinear cases, this coincides with Proposition 1 of Carlier [5].

From the original monopolist's problem (2.3), we replace product-price pair (p, y) by the value-product pair (u, y), using $u(\cdot) = G(\cdot, y(\cdot), p(y(\cdot)))$. Combining this with Proposition 3.4, the incentive-compatibility constraint $(y, p(y))$ is equivalent to G-convexity of $u(\cdot)$ and $y(x) \in \partial^G u(x)$ for all $x \in X$. Therefore, one can rewrite the monopolist's problem as follows.

Proposition 3.8. *Given Assumptions 1 and 2, the monopolist's problem (P_0) is equivalent to*

$$(3.2) \qquad (P) \begin{cases} \sup \tilde{\Pi}(u, y) := \int_X \pi(x, y(x), H(x, y(x), u(x))) d\mu(x) \\ s.t. \ u \text{ is } G\text{-convex} ; \\ s.t. \ y(x) \in \partial^G u(x) \text{ and } u(x) \geq u_\emptyset(x) \text{ for all } x \in X. \end{cases}$$

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[2]In Trudinger [29], this point-to-set mapping $\partial^G u$ is also called *G-normal* mapping; see paper for more properties related to G-convexity.

In the next section, we will show the existence result of the rewritten monopolist's problem (P) given in (3.2). For preparation of the main result, we introduce the following lemma and propositions.

Proposition 3.9 shows that the inverse function of G is also continuous, because G is continuous and monotonic on the price variable.

Proposition 3.9. *Given Assumption 1 and Assumption 2, function H is continuous.*

Given coordinate monotonicity of G in the first variable, one can show that all the G-convex functions are nondecreasing. Therefore, the value functions are also monotonic with respect to agents' attributes.

Proposition 3.10. *Given Assumption 3, G-convex functions are nondecreasing in coordinates.*

Proposition 3.11 presents that uniform boundedness of agents' value functions on some compact subset implies uniform boundedness of corresponding agents' choices of their favorite products.

Proposition 3.11. *Given Assumptions 1, 2, 3, 7, and let $u(\cdot)$ be a G-convex function on X, ω be a compact subset of X, $\delta > 0, R > 0$, satisfying $\omega + \delta\overline{B(0,1)} \subset X$ and $|u(x)| \leq R$ for all $x \in \omega + \delta\overline{B(0,1)}$ (here, $\overline{B(0,1)}$ denotes the closed unit ball of $\mathbf{R}^M$). Then, there exists $T = T(\omega, \delta, R) > 0$, such that $\|y\| \leq T$ for any $x \in \omega$ and any $y \in \partial^G u(x)$.*

The above two propositions will also be employed in the proof of Proposition 4.3.

4. Main result

In this section, we state the existence theorem, the proof of which is provided in the Section 6.

Theorem 4.1 (Existence). *Under Assumptions 1 - 8, assume μ is equivalent to the Lebesgue measure on X, then the monopolist's problem (P) admits at least one solution.*

Technically, in order to demonstrate existence, we start from a sequence of value-product pairs, whose total profits have a limit that is equal to the supreme of (P). Then we need to show that this sequence converges, up to a subsequence, to a pair of limit mappings. Then we show this limit value-product pair satisfies the constraints of (P); and its corresponding total payoff is better (or no worse) than those of any other admissible pairs.

In the following, we denote $W^{1,1}(X)$ as the Sobolev space of L^1 functions whose first derivatives exist in the weak sense and belong to $L^1(X)$. For more properties of Sobolev spaces and weak derivatives, see Evans [10, Chapter 5]. If ω is some open subset of X, notation $\omega \subset\subset X$ means the closure of ω is also included in X.

Lemma 4.2 provides sequence convergence results of convex functions, which are uniformly bounded in Sobolev spaces on open convex subsets. We state this classical result without proof, which can be found in Carlier [5].

Lemma 4.2. *Let $\{u_n\}$ be a sequence of convex functions in X such that, for every open convex set $\omega \subset\subset X$, the following holds:*

$$\sup_n \|u_n\|_{W^{1,1}(\omega)} < +\infty$$

Then, there exists a function u^* which is convex in X, a measurable subset A of X and a subsequence again labeled $\{u_n\}$ such that
1. $\{u_n\}$ converges to u^* uniformly on compact subsets of X;
2. $\{\nabla u_n\}$ converges to ∇u^* pointwise in A and $dim_H(X \setminus A) \leq M - 1$, where $dim_H(X \setminus A)$ is the Hausdorff dimension of $X \setminus A$.

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We extend the above convergence result to G-convex functions in the following proposition, which is required in the proof of the Existence Theorem, as it extracts a limit function from a converging sequence of value functions.

Proposition 4.3. *Assume Assumptions 1, 2, 3, 5, 7, and let $\{u_n\}$ be a sequence of G-convex functions in X such that for every open convex set $\omega \subset\subset X$, the following holds:*

$$\sup_n \|u_n\|_{W^{1,1}(\omega)} < +\infty$$

Then there exists a function u^* which is G-convex in X, a measurable subset A of X, and a subsequence again labeled $\{u_n\}$ such that
1. $\{u_n\}$ converges to u^* uniformly on compact subsets of X;
2. $\{\nabla u_n\}$ converges to ∇u^* pointwise in A and $\dim_H(X \setminus A) \leq M - 1$.

In the proof of Proposition 4.3, we show that the sequence of G-convex functions is convergent by applying results from Lemma 4.2, then prove that the limit function is also G-convex.

5. Future Work

We have strong interest in giving an explicit solution for the non-quasilinear example on the real line and in high dimension. We also would like to investigate, among other things, the conditions under which the matching map $y : X \longrightarrow cl(Y)$ is continuous and/or differentiable. Given the technical arguments employed in this paper, it may be very fruitful to study possible generalizations of other known results for convex functions to G-convex functions.

6. Proofs

Proof of Lemma 3.3. " $\longrightarrow$ ". Assume u is G-convex, we want to show u is G-subdifferentiable everywhere, i.e., the statement $\partial^G u(x_0) \neq \emptyset$ is true for any $x_0 \in X$.

Since u is G-convex, by definition, for any fixed $x_0 \in X$, there exist y_0 and z_0, such that $u(x_0) = G(x_0, y_0, z_0)$, and for any $x \in X$, $u(x) \geq G(x, y_0, z_0)$. By definition of function H, we know $z_0 = H(x_0, y_0, u(x_0))$. Therefore, combining the above inequality, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$, for any $x \in X$. By definition of G-subdifferentiability, it implies $y_0 \in \partial^G u(x_0)$, that is, $\partial^G u(x_0) \neq \emptyset$.

" $\Leftarrow$ ". Assume u is G-subdifferentiable everywhere. Then for any fixed $x_0 \in X$, there exists $y_0 \in \partial^G u(x_0)$. Define $z_0 := H(x_0, y_0, u(x_0))$, then by definition of function H, one has $u(x_0) = G(x_0, y_0, z_0)$.

Since $y_0 \in \partial^G u(x_0)$, by definitions of G-subdifferentiability, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$ $= G(x, y_0, z_0)$, for any $x \in X$, where the last equality comes from the definition of z_0.

By definition (of G-convexity), u is G-convex. $\qquad\square$

Proof of Proposition 3.4. " $\longrightarrow$ ". Suppose (y, z) is incentive compatible. For any fixed $x_0 \in X$, let $y_0 = y(x_0)$ and $z_0 = z(x_0)$. Then $u(x_0) = G(x_0, y(x_0), z(x_0)) = G(x_0, y_0, z_0)$. By incentive compatibility of the contract (y, z), for any $x \in X$, one has $G(x, y(x), z(x)) \geq G(x, y(x_0), z(x_0))$. This implies $u(x) \geq G(x, y_0, z_0)$, for any $x \in X$, since $u(x) = G(x, y(x), z(x))$, $y_0 = y(x_0)$ and $z_0 = z(x_0)$. By definition, u is G-convex.

Since $u(x_0) = G(x_0, y_0, z_0)$, by definition of function H, one has $z_0 = H(x_0, y_0, u(x_0))$. Combining with $u(x) \geq G(x, y_0, z_0)$, for any $x \in X$, which is concluded from above, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$, for any $x \in X$. By definition of G-subdifferentiability, one has $y_0 \in \partial^G u(x_0)$, and thus $y(x_0) = y_0 \in \partial^G u(x_0)$.

" $\Leftarrow$ ". Assume that $u = G(x, y(x), z(x))$ is G-convex, and $y(x) \in \partial^G u(x)$, for any $x \in X$. For any fixed $x \in X$, since $y(x) \in \partial^G u(x)$, for any $x' \in X$, one has

$$(6.1) \qquad u(x') \geq G(x', y(x), H(x, y(x), u(x)))$$

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Since $u(x) = G(x, y(x), z(x))$, by definition of function H, one has $z(x) = H(x, y(x), u(x))$. Combining with the inequality (6.1), we have $u(x') \geq G(x', y(x), z(x))$. Noticing $u(x') = G(x', y(x'), z(x'))$, one has $G(x', y(x'), z(x')) = u(x') \geq G(x', y(x), z(x))$. By definition, (y,z) is incentive compatible. $\square$

Proof of Remark 3.6. One direction is easier: given p and y, define $z(\cdot) := p(y(\cdot))$. Then the conclusion follows directly.

Given an incentive-compatible pair $(y, z) : X \longrightarrow cl(Y) \times \mathbf{R}$, we need to construct a price menu $p : cl(Y) \longrightarrow \mathbf{R}$. If $y = y(x)$ for some $x \in X$, define $p(y) := z(x)$; for any other $y \in cl(Y)$, define $p(y) := \bar{z}$.

We first show p is well-defined. Suppose $y(x) = y(x')$ with $x \neq x'$, from incentive compatibility of (p, y), we have $G(x, y(x), z(x)) \geq G(x, y(x'), z(x')) = G(x, y(x), z(x'))$. Since G is strictly decreasing on its third variable, the above inequality implies $z(x) \leq z(x')$. Similarly, one has $z(x) \geq z(x')$. Therefore, $z(x) = z(x')$ and thus p is well-defined.

The incentive compatibility of $(y, p(y))$ follows from that of (y, z) and definition of p. $\square$

Proof of Corollary 3.7. One direction is immediately derived from the definition of implementability and Proposition 3.4.

Suppose there exists some convex function u such that $y(x) \in \partial^G u(x)$ for each X. Define $z(\cdot) := H(\cdot, y(\cdot), u(\cdot))$, then $u(x) = G(x, y(x), z(x))$. Proposition 3.4 implies (y, z) is incentive compatible, and thus y is implementable. $\square$

Proof of Proposition 3.8. We need to prove both directions for equivalence of (P_0) and (P).

1. For any incentive-compatible pair $(y, p(y))$, define $u(\cdot) := G(\cdot, y(\cdot), p(y(\cdot)))$. Then by Proposition 3.4, we have $u(\cdot)$ is G-convex and $y(x) \in \partial^G u(x)$ for all $x \in X$. From the participation constraint, $G(x, y(x), p(y(x))) \geq u_\emptyset(x)$ for all $x \in X$. This implies $u(x) \geq u_\emptyset(x)$ for all $x \in X$. Besides, two integrands are equal: $\pi(x, y(x), p(y(x))) = \pi(x, y(x), H(x, y(x), u(x)))$. Therefore, $(P_0) \leq (P)$.

2. On the other hand, assume $u(\cdot)$ is G-convex, $y(x) \in \partial^G u(x)$ and $u(x) \geq u_\emptyset(x)$ for all $x \in X$. From Corollary 3.7 and Remark 3.6, we know y is implementable and there exists a price menu $p : cl(Y) \longrightarrow \mathbf{R}$, such that the pair $(y, p(y))$ is incentive compatible, where $p(y) = H(x, y(x), u(x))$ for $y = y(x) \in y(X) := \{y(x) \in cl(Y) | x \in X\}$; $p(y) = \bar{z}$ for other $y \in cl(Y)$. Firstly, the mapping p is well-defined, using the same argument as that in Remark 3.6. Secondly, the participation constraint holds since $G(x, y(x), p(y(x))) = u(x) \geq u_\emptyset(x)$ for all $x \in X$.

Thirdly, let us show this price menu p is lower semicontinuous. Let $\tilde{p}$ be the restriction of p on $y(X)$. Suppose that $\{y_k\} \subset y(X)$ converges $y_0 \in y(X)$ with $y_k = y(x_k)$ and $y_0 = y(x_0)$, satisfying $\lim_{k \to \infty} \tilde{p}(y_k) = \liminf_{y \to y_0} \tilde{p}(y)$. Let $z_k := \tilde{p}(y_k)$ and $z_\infty := \lim_{k \to \infty} z_k$. To prove lower semicontinuity of $\tilde{p}$, we need to show $\tilde{p}(y_0) \leq z_\infty$. Since $y_k \in \partial^G u(x_k)$, we have $u(x) \geq G(x, y_k, H(x_k, y_k, u(x_k))) = G(x, y_k, z_k)$. Taking $k \to \infty$, we have $u(x) \geq G(x, y_0, z_\infty)$. This implies $G(x_0, y_0, \tilde{p}(y_0)) = u(x_0) \geq G(x_0, y_0, z_\infty)$. By Assumption 2, we know $\tilde{p}(y_0) \leq z_\infty$. Thus $\tilde{p}$ is lower semicontinuous. Since p is an extension of $\tilde{p}$ from $y(X)$ to $cl(Y)$ as its lower semicontinuous hull, satisfying $v(y) = \bar{z}$ for all $y \in cl(Y) \setminus y(X)$, we know p is also lower semicontinuous.

Lastly, two integrands are equal: $\pi(x, y(x), p(y(x))) = \pi(x, y(x), H(x, y(x), u(x)))$. Therefore, $(P_0) \geq (P)$. $\square$

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Proof of Proposition 3.9. (Proof by contradiction). Suppose H is not continuous, then there exists a sequence $(x_n, y_n, z_n) \subset cl(X \times Y \times Z)$ converging to (x, y, z) and $\varepsilon > 0$ such that $|H(x_n, y_n, z_n) - H(x, y, z)| > \varepsilon$ for all $n \in \mathbf{N}$. Without loss of generality, we assume $H(x_n, y_n, z_n) - H(x, y, z) > \varepsilon$ for all $n \in \mathbf{N}$. Therefore, we have $H(x_n, y_n, z_n) > H(x, y, z) + \varepsilon$. By Assumption 2, this implies $z_n < G(x_n, y_n, H(x, y, z) + \varepsilon)$ for all $n \in \mathbf{N}$. Taking limit $n \longrightarrow \infty$ at both sides, since G is continuous from Assumption 1, we have $z \leq G(x, y, H(x, y, z) + \varepsilon)$. It implies $H(x, y, z) \geq H(x, y, z) + \varepsilon$. Contradiction! □

Proof of Proposition 3.11. (Proof by contradiction).By Assumption 3 and Assumption 7, for $s = \frac{4R\sqrt{M}}{\delta}$, there exists $r > 0$, such that for any $(x, y, z) \in X \times cl(Y) \times cl(Z)$, whenever $||y|| \geq r$, we have $\sum_{i=1}^{M} D_{x_i} G(x, y, z) \geq \frac{4R\sqrt{M}}{\delta}$.Assume the boundedness conclusion of this proposition is not true.

Then for this r, there exist $x_0 \in \omega$ and $y_0 \in \partial^G u(x_0)$, such that $||y_0|| \geq r$. Thus,

$$(6.2) \qquad \sum_{i=1}^{M} D_{x_i} G(x, y_0, z) \geq \frac{4R\sqrt{M}}{\delta}, \qquad \text{for all } x \in X, z \in \mathbf{R}.$$

Since $y_0 \in \partial^G u(x_0)$, by definition of G-subdifferential, we have $u(x) \geq G(x, y_0, H(x_0, y_0, u(x_0)))$, for any $x \in X$. Take $x = x_0 + \delta x_{-1}$, where $x_{-1} := (\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}, \cdots, \frac{1}{\sqrt{M}})$ is a unit vector in $\mathbf{R}^M$ with each coordinate equal to $\frac{1}{\sqrt{M}}$. Then

$$(6.3) \qquad u(x_0 + \delta x_{-1}) \geq G(x_0 + \delta x_{-1}, y_0, H(x_0, y_0, u(x_0))).$$

For any $x \in \omega + \delta \overline{B(0, 1)}$, from conditions in the proposition, we have $||u(x)|| \leq R$. Therefore,

$$
\begin{aligned}
2R &\geq |u(x_0 + \delta x_{-1})| + |u(x_0)| \\
&\geq |u(x_0 + \delta x_{-1}) - u(x_0)| &&\text{(By Triangular Inequality)} \\
&\geq u(x_0 + \delta x_{-1}) - u(x_0) \\
&\geq G(x_0 + \delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) &&\text{(By Inequality (6.3)) and (By definition} \\
&\quad - G(x_0, y_0, H(x_0, y_0, u(x_0))) &&\text{of } H, u(x_0) = G(x_0, y_0, H(x_0, y_0, u(x_0))) \\
&= \int_0^1 \delta \langle x_{-1}, D_x G(x_0 + t\delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) \rangle dt &&\text{(By Fundamental Theorem of Calculus)} \\
&= \frac{\delta}{\sqrt{M}} \int_0^1 \sum_{i=1}^{M} D_{x_i} G(x_0 + t\delta x_{-1}, y_0, H(x_0, y_0, u(x_0))) dt \\
&\geq \frac{\delta}{\sqrt{M}} \int_0^1 \frac{4R\sqrt{M}}{\delta} dt &&\text{(By Inequality (6.2))} \\
&= \frac{\delta}{\sqrt{M}} \cdot \frac{4R\sqrt{M}}{\delta} \\
&= 4R
\end{aligned}
$$

Contradiction! Thus, our assumption is wrong. The boundedness conclusion of this proposition is true. That is, there exists $T > 0$, such that for any $x \in \omega$, $y \in \partial^G u(x)$, one has $||y|| \leq T$. In addition, here $T = T(\omega, \delta, R)$ is independent of u. In fact, from the above argument, we can see that $T \leq r$, which does not depend on u. □

Proof of Proposition 3.10. Let u be any G-convex function, and let α, β be any two agent types in X with $\alpha \geq \beta$. By G-convexity of u, for this β, there exist $y \in cl(Y)$ and $z \in cl(Z)$, such that $u(\beta) = G(\beta, y, z)$ and $u(x) \geq G(x, y, z)$, for any $x \in X$. Since $\alpha \geq \beta$, by Assumption 3, we have $G(\alpha, y, z) \geq G(\beta, y, z)$. Combining with $u(\alpha) \geq G(\alpha, y, z)$ and $u(\beta) = G(\beta, y, z)$, one has $u(\alpha) \geq u(\beta)$. Thus, u is nondecreasing. □

Proof of Proposition 4.3. Assume $\{u_n\}$ is a sequence of G-convex functions in X such that for every open convex set $\omega \subset\subset X$, the following holds:

$$\sup_n ||u_n||_{W^{1,1}(\omega)} < +\infty.$$

Step 1: By Assumption 5, there exists $k > 0$, such that for any $(x, x') \in X^2$, $y \in cl(Y)$ and $z \in cl(Z)$, one has $||D_x G(x, y, z) - D_x G(x', y, z)|| \leq k||x - x'||$. Denote $G_\lambda(x, y, z) := G(x, y, z) + \lambda||x||^2$, where $\lambda \geq \frac{1}{2} \text{Lip}(D_x G)$, with $\text{Lip}(D_x G) := \sup_{\{(x, x', y, z) \in X \times X \times cl(Y) \times cl(Z): \ x \neq x'\}} \frac{||D_x G(x,y,z) - D_x G(x',y,z)||}{||x - x'||}$.

Then, for any $(x, x') \in X^2$, by Cauchy–Schwarz inequality, one has

$$\langle D_x G_\lambda(x, y, z) - D_x G_\lambda(x', y, z), x - x' \rangle$$

$$= \langle D_x G(x, y, z) - D_x G(x', y, z), x - x' \rangle + 2\lambda||x - x'||^2 \qquad \text{(By Defition of } G_\lambda(x, y, z))$$

$$\geq -||D_x G(x, y, z) - D_x G(x', y, z)|| \, ||x - x'|| + 2\lambda||x - x'||^2 \qquad \text{(By Cauchy–Schwarz Inequality)}$$

$$\geq [2\lambda - \text{Lip}(D_x G)]||x - x'||^2 \qquad \text{(By Definition of } \text{Lip}(D_x G))$$

$$\geq 0.$$

Thus, $G_\lambda(\cdot, y, z)$ is a convex function on X, for any fixed $(y, z) \in cl(Y) \times cl(Z)$.

Step 2: Since u_n is G-convex, by Lemma 3.3, we know

$$u_n(x) = \max_{x' \in X, y \in \partial^G u_n(x')} G(x, y, H(x', y, u_n(x'))).$$

Define $v_n(x) := u_n(x) + \lambda||x||^2$. Then

$$v_n(x) = \max_{x' \in X, y \in \partial^G u_n(x')} G(x, y, H(x', y, u_n(x'))) + \lambda||x||^2$$

$$= \max_{x' \in X, y \in \partial^G u_n(x')} (G(x, y, H(x', y, u_n(x'))) + \lambda||x||^2)$$

$$= \max_{x' \in X, y \in \partial^G u_n(x')} G_\lambda(x, y, H(x', y, u_n(x'))).$$

Since $G_\lambda(\cdot, y, H(x', y, u_n(x')))$ is convex for each (x', y), we have $v_n(x)$, as supremum of convex functions, is also convex, for all $n \in \mathbf{N}$.

Step 3: Since $v_n := u_n + \lambda||x||^2$ and $\sup_n ||u_n||_{W^{1,1}(\omega)} < +\infty$, one has $\sup_n ||v_n||_{W^{1,1}(\omega)} < +\infty$, for any $\omega \subset\subset X$. Hence $\{v_n\}$ satisfies all the assumptions of Lemma 4.2. So, by conclusion of Lemma 4.2, there exists a convex function v^* in X and a measurable set $A \subset X$, such that $dim(X \setminus A) \leq M - 1$ and up to a subsequence, $\{v_n\}$ converges to v^* uniformly on compact subset of X and (∇v_n) converges to ∇v^* pointwise in A.

Let $u^*(x) := v^*(x) - \lambda||x||^2$, then (u_n) converges to u^* uniformly on compact subset of X and (∇u_n) converges to ∇u^* pointwise in A.

Step 4: Finally, let us prove that u^* is G-convex. Define $\Gamma(x) := \cap_{i \geq 1} \overline{\cup_{n \geq i} \partial^G u_n(x)}$, for all $x \in X$.

Step 4.1. Claim: For any $x' \in X$, we have $\Gamma(x') \neq \emptyset$.

Proof of this Claim:
Step 4.1.1. Let us first show for any $\omega \subset\subset X$, $\sup_n ||u_n||_{L^\infty(\bar\omega)} < +\infty$.

If not, then there exits a sequence $\{x_n\}_{n=1}^{\infty} \subset \bar{\omega}$, such that $\limsup\limits_{n} |u_n(x_n)| = +\infty$.

Since $\bar{\omega}$ is compact, there exists $\bar{x} \in \bar{\omega}$, such that, up to a subsequence, $x_n \longrightarrow \bar{x}$. Again up to a subsequence, we may assume that $u_n(x_n) \longrightarrow +\infty$.

Since $\bar{x} \in \bar{\omega} \subset\subset X$, there exists $\delta > 0$, such that $\bar{x} + \delta x_{-1} \in X$, where $x_{-1} := (\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}, \cdots, \frac{1}{\sqrt{M}})$ is a unit vector in $\mathbf{R}^M$ with each coordinate equal to $\frac{1}{\sqrt{M}}$. For any $x > \bar{x} + \delta x_{-1}$, there exists n_0, such that for any $n > n_0$, we have $x > x_n$. By Proposition 3.10, u_n are nondecreasing, and thus

$$(6.4) \qquad \int_{\{x \in X, x > \bar{x} + \delta x_{-1}\}} u_n(x)dx \geq m\{x \in X, x > \bar{x} + \delta x_{-1}\}u_n(x_n) \longrightarrow +\infty$$

where $m\{x \in X, x > \bar{x} + \delta x_{-1}\}$ denotes Lebesgue measure of $\{x \in X, x > \bar{x} + \delta x_{-1}\}$, which is positive.

Therefore, we have $||u_n||_{W^{1,1}(\omega')} \geq ||u_n||_{L^1(\omega')} \geq \int_{\omega'} u_n(x)dx \longrightarrow +\infty$. This implies $\sup\limits_{n} ||u_n||_{W^{1,1}(\omega')} = +\infty$.

On the other hand, denote $\omega' := \{x \in X| \ x > \bar{x} + \delta x_{-1}\}$, then $\omega' = X \cap \{x \in \mathbf{R}^M| \ x > \bar{x} + \delta x_{-1}\}$. Since both X and $\{x \in \mathbf{R}^M| \ x > \bar{x} + \delta x_{-1}\}$ are open and convex, we have ω' is also open and convex. Therefore, by assumption, we have $\sup\limits_{n} ||u_n||_{W^{1,1}(\omega')} < +\infty$.

Contradiction! Thus, for any $\omega \subset\subset X$, we have $\sup\limits_{n} ||u_n||_{L^{\infty}(\bar{\omega})} < +\infty$.

Step 4.1.2. For any fixed $x' \in X$, there exists an open set $\omega \subset\subset X$ and $\delta > 0$, such that $x' \in \omega$ and $\omega + \delta\overline{B(0,1)} \subset\subset X$.

From Step 4.1.1, we know $\sup\limits_{n} ||u_n||_{L^{\infty}(\omega + \delta\overline{B(0,1)})} < +\infty$. So there exists $R > 0$, such that for all $n \in \mathbf{N}$, we have $|u_n(x)| \leq R$, for all $x \in \omega + \delta\overline{B(0,1)}$. Since u_n are G-convex functions, by Proposition 3.11, there exists $T = T(\omega, \delta, R) > 0$, independent of n, such that $||y|| \leq T$, for any $y \in \partial^G u_n(x')$ and any $n \in \mathbf{N}$. Thus, there exists a sequence $\{y_n\}$, such that $y_n \in \partial^G u_n(x')$ and $||y_n|| \leq T$, for all $n \in \mathbf{N}$.

By compactness theorem for sequence $\{y_n\}$, there exists y', such that, up to a subsequence, $y_n \longrightarrow y'$. Thus, we have $y' \in \overline{\cup_{n \geq i}\partial^G u_n(x')}$, for all $i \in \mathbf{N}$. It implies $y' \in \cap_{i \geq 1}\overline{\cup_{n \geq i}\partial^G u_n(x')} = \Gamma(x')$.

Therefore $\Gamma(x') \neq \emptyset$, for all $x' \in X$.

Step 4.2. Now for any fixed $x \in X$, and any $y \in \Gamma(x)$, by Cantor's diagonal argument, there exists $\{y_{n_k}\}_{k=1}^{\infty}$, such that $y_{n_k} \in \partial^G u_{n_k}(x)$ and $\lim\limits_{k \longrightarrow \infty} y_{n_k} = y$. For any $k \in \mathbf{N}$, by definition of G-subdifferentiability, $u_{n_k}(x') \geq G(x', y_{n_k}, H(x, y_{n_k}, u_{n_k}(x)))$, for any $x' \in X$. Take limit $k \longrightarrow \infty$ at both sides, we get $u^*(x') \geq G(x', y, H(x, y, u^*(x)))$, for any $x' \in X$. Here we use the fact that both functions G and H are continuous by Assumption 1 and Proposition 3.9. Then by definition of G-subdifferentiability, the above inequality implies $y \in \partial^G u^*(x)$.

So $\partial^G u^*(x) \neq \emptyset$, for any $x \in X$, which means u^* is G-subdifferentiable everywhere. By Lemma 3.3, u^* is G-convex. $\qquad \square$

Proof of the Existence Theorem. **Step 1:** Define $\Phi_u : x \longmapsto argmin_{\partial^G u(x)} -\pi(x, \cdot, H(x, \cdot, u(x)))$, then by Proposition 3.11, the measurable section theorem (cf. [8, Theorem 1.2, Chapter VIII]) and Lusin Theorem, one has Φ_u admits measurable selections.

Let $\{(u_n, y_n)\}$ be a maximizing sequence of (P), where maps $u_n : X \longrightarrow \mathbf{R}$ and $y_n : X \longrightarrow cl(Y)$, for all $n \in \mathbf{N}$. Without loss of generality, we may assume that for all n, $y_n(\cdot)$ is measurable and $y_n(x) \in \Phi_{u_n}(x)$, for each $x \in X$. Starting from $\{(u_n, y_n)\}$, we would find an value-product pair (u^*, y^*) satisfying all the constraints in (3.2), and show that it is actually a maximizer.

Step 2: From Assumption 4, there exist $\alpha \geq 1$, $a_1, a_2 > 0$ and $b \in \mathbf{R}$, such that for each $x \in X$ and $n \in \mathbf{N}$,

$$a_1 ||y_n(x)||_{\alpha}^{\alpha} \leq -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) - a_2 u_n(x) + b$$

$$\leq - \pi(x, y_n(x), H(x, y_n(x), u_n(x))) - a_2 u_\emptyset(x) + b,$$

where the second inequality comes from $u_n \geq u_\emptyset$. Together with Assumption 8, this implies $\{y_n\}$ is bounded in $L^\alpha(X)$.

By participation constraint and Assumption 4, we know

$$u_\emptyset(x) \leq u_n(x) = G(x, y_n(x), H(x, y_n(x), u_n(x))) \leq \frac{1}{a_2}(b - \pi(x, y_n(x), H(x, y_n(x), u_n(x)))).$$

Together with Assumption 8, we know $\{u_n\}$ is bounded in $L^1(X)$.

By G-subdifferentiability, $Du_n(x) = D_x G(x, y_n(x), H(x, y_n(x), u_n(x)))$. By Assumption 6, we have $||Du_n||_1 \leq c||y_n||_\beta^\beta + d \leq c(N + ||y_n||_\alpha^\alpha) + d$. The last inequality holds because $\beta \in (0, \alpha]$. Because X is bounded and $\{y_n\}$ is bounded in $L^\alpha(X)$, we know $\{Du_n\}$ is bounded in $L^1(X)$.

Since both $\{u_n\}$ and $\{Du_n\}$ are bounded in $L^1(X)$, one has $\{u_n\}$ is bounded in $W^{1,1}(X)$. By Proposition 4.3, there exists a G-convex function u^* on X, such that, up to a subsequence, $\{u_n\}$ converges to u^* in L^1 and uniformly on compact subset of X, and ∇u_n converges to ∇u^* almost everywhere.

Step 3: Denote $y^*(x)$ as a measurable selection of Φ_{u^*}. Let us show (u^*, y^*) is a maximizer of the principal's program (P).

Step 3.1: By Assumption 4, for all x, $y_n(x)$ and $u_n(x)$, one has

$$
\begin{aligned}
&- \pi(x, y_n(x), H(x, y_n(x), u_n(x))) \\
&\geq a_2 G(x, y_n(x), H(x, y_n(x), u_n(x))) - b \\
&= a_2 u_n(x) - b \\
&\geq a_2 u_\emptyset(x) - b.
\end{aligned}
$$

By Assumption 8, $u_\emptyset$ is measurable, thus one can apply Fatou's Lemma and get

(6.5)
$$
\begin{aligned}
\sup \tilde{\Pi}(u, y) &= \limsup_n \tilde{\Pi}(u_n, y_n) \\
&= -\liminf_n \int_X -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) \, d\mu(x) \\
&\leq -\int_X \liminf_n -\pi(x, y_n(x), H(x, y_n(x), u_n(x))) \, d\mu(x).
\end{aligned}
$$

Let $\gamma(x) := \liminf_n -\pi(x, y_n(x), H(x, y_n(x), u_n(x)))$. For each $x \in X$, by extracting a subsequence of $\{y_n\}$, which is denoted as $\{y_{n_x}\}$, we assume $\gamma(x) = \lim_{n_x} -\pi(x, y_{n_x}(x), H(x, y_{n_x}(x), u_{n_x}(x)))$.

Step 3.2: For any fixed $x \in X$, since u_{n_x} are G-convex functions and $\{u_{n_x}\}$ is bounded in $L^1(X)$, by Proposition 3.10, it is also bounded in $L^\infty_{loc}(X)$. Then by Proposition 3.11, $\{y_{n_x}\}$ is also bounded in $L^\infty_{loc}(X)$. Thus there exists a subsequence of $\{y_{n_x}(x)\}$, again denoted as $\{y_{n_x}(x)\}$, that converges. Denote $\tilde{y}$ a mapping on X such that $y_{n_x}(x) \longrightarrow \tilde{y}(x)$.

Since π and H are continuous, we have $\gamma(x) = -\pi(x, \tilde{y}(x), H(x, \tilde{y}(x), u^*(x)))$.

For each fixed $x \in X$, since u_{n_x} are G-convex and $y_{n_x}(x) \in \partial^G u_{n_x}(x)$, for any $x' \in X$, we have

$$u_{n_x}(x') \geq G(x', y_{n_x}(x), H(x, y_{n_x}(x), u_{n_x}(x))).$$

Take limit $n_x \longrightarrow +\infty$ at both sides, we get $u^*(x') \geq G(x', \tilde{y}(x), H(x, \tilde{y}(x), u^*(x)))$, for any $x' \in X$. By definition of G-subdifferentiability, we have $\tilde{y}(x) \in \partial^G u^*(x)$.

Step 3.3: By definition of y^*, one has

$$-\pi(x, y^*(x), H(x, y^*(x), u^*(x))) \leq -\pi(x, \tilde{y}(x), H(x, \tilde{y}(x), u^*(x))) = \gamma(x)$$

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.

So, together with (6.5), we know

$$(6.6) \qquad \sup \tilde{\Pi}(u, y) \leq - \int_X \gamma(x) d\mu(x) \leq - \int_X -\pi(x, y^*(x), H(x, y^*(x), u^*(x))) d\mu(x) = \tilde{\Pi}(u^*, y^*).$$

Since $\{u_n\}$ converges to u^*, and $u_n(x) \geq u_\emptyset(x)$ for all $n \in \mathbf{N}$ and $x \in X$, we have $u^*(x) \geq u_\emptyset(x)$ for all $x \in X$. In addition, because u^* is G-convex and $y^*(x) \in \partial^G u^*(x)$, we know (u^*, y^*) satisfies all the constraints in (3.2). Together with (6.6), we proved (u^*, y^*) is a solution of the principal's program. $\qquad \square$

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