BOOLEAN ALGEBRA (Continuing)

❖ **Canonical Forms**

- Sum of Products (SoP) Form

Standard Sum of Products Form

Minterms

- Product of Sums (PoS) Form

Standard Products of Sum Form

Maxterms

- Conversion Between Standard SoP and PoS Forms

Canonical Forms

Logical expressions can be written in Sum of Products (SoP) form and Product of Sums (PoS) form. These forms help us standardize the logical expressions. They also help us understand, simplify, and implement the logical expressions easily and systematically.

Sum of Products Form: An expression which is in SoP form, consists of the sum of products of the variables. We get the product by multiplying the variables or complements of variables. A product does not necessarily have all the variables in it.

The expressions $ABC+A'B+BC'D'$ and $A'+AB+BC'$ are in SoP form. Because we multiply only variables in each term. However, if we multiply something complicated like $(ABC)'$ in a term, we cannot say that the expression is in SoP form.

We can convert any expression into SoP form using Boolean algebra.

Canonical Forms

Example: Let's convert $A'B(C+BD')$ into SoP form.

We apply the distributive law.

$$A'B(C+BD') = A'BC + A'BD'$$

Standard Sum of Products Form: In SoP form, a term may not have all the variables in it. However, in standard SoP form, each term *must have* all the variables in it. To convert an expression in SoP form into Standard SoP form, we must extend all the terms with missing variables.

Example: Let's convert $F(A,B,C) = A.B'+A.C$ into standard SoP form.

We need to extend the first term with $(C+C')$ and the second term with $(B+B')$.

$$\begin{aligned} AB'+AC = AB'(C+C')+A(B+B')C &= \underline{AB'C}+AB'C'+ABC+\underline{AB'C} \\ &= AB'C+AB'C'+ABC \end{aligned}$$

Canonical Forms

Example: Let's convert *F(A,B,C) = A+BC+A'C* into standard SoP form in a more practical way. We create a table for conversion.

A	BC	A'C
ABC	ABC	ABC
100	011	001
101	111	011
110		
111		

$$F(A,B,C) = AB'C'+AB'C+ABC'+\underline{ABC}+\underline{\underline{A'BC}}+\underline{ABC}+A'B'C+\underline{\underline{A'BC}}$$

$$= AB'C'+AB'C+ABC'+ABC+A'BC+A'B'C$$

We will use Standard SoP form to populate truth tables and Karnaugh maps.

Minterms

Each term in an expression in standard SoP form is called a minterm. An expression with *n* variables has 2^n minterms. For example, an expression with 3 variables has 8 minterms. Each minterm in an expression corresponds to a single row (with output 1) in the truth table.

m_{index} denotes the minterm with that index. Index can easily be calculated by the values that make the minterm 1. For example, the term AB'C' will be 1 if and only if A=1, B=0, and C=0 (ABC=100). $(100)_2$ is 4 in decimal. So, AB'C' is m_4.

Variables			Minterms	
A	*B*	*C*	Term	Symbol
0	0	0	*A'B'C'*	m_0
0	0	1	*A'B'C*	m_1
0	1	0	*A'BC'*	m_2
0	1	1	*A'BC*	m_3
1	0	0	*AB'C'*	m_4
1	0	1	*AB'C*	m_5
1	1	0	*ABC'*	m_6
1	1	1	*ABC*	m_7

Minterms

Example: Let's generate the truth table of $F=A'B'C+A'BC+ABC$.

Inputs			Output (F)	Minterms
A	B	C		
0	0	0	0	
0	0	1	1	$A'B'C$
0	1	0	0	
0	1	1	1	$A'BC$
1	0	0	0	
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

In short, we use the $\sum$ symbol to abbreviate the expressions in standard SoP form.

$$F(A,B,C) = m_1 + m_3 + m_7 = \sum(1,3,7)$$

Canonical Forms

Product of Sums (PoS) Form: An expression which is in PoS form, consists of the products of sums of the variables. .

The expressions $(A+B')(A'+B+C)$ and $A(A'+B)(B+C')$ are in PoS form. However, if we add something complicated like $(ABC)'$ in a term, we cannot say that the expression is in PoS form.

Standard Product of Sums Form: In PoS form, a term may not have all the variables in it. However, in standard PoS form, each term *must have* all the variables in it. To convert an expression in PoS form into Standard PoS form, we must extend all the terms with missing variables.

Example: Let's convert $F(A,B,C)=(A+B')(B+C)$ into standard PoS form.

$$(A+B')(B+C) = (\underline{A+B'}+C.C')(A.A'+\underline{B+C})$$

$$= (A+B'+C).(A+B'+C').(A+B+C).(A'+B+C)$$

Maxterms

Each term in an expression in standard PoS form is called a maxterm. An expression with *n* variables has 2^n maxterms. For example, an expression with 3 variables has 8 maxterms. Each maxterm corresponds to a single row (with output 0) in the truth table.

M_{index} denotes the maxterm with that index. Index can easily be calculated by the values that make the maxterm 0. For example, the term A'+B+C will be 0 if and only if A=1, B=0, and C=0 (ABC=100). $(100)_2$ is 4 in decimal. So, A'+B+C is M_4.

Variables			Maxterms		Minterms	
A	*B*	*C*	Term	Symbol	Term	Symbol
0	0	0	$A+B+C$	M_0	$A'B'C'$	m_0
0	0	1	$A+B+C'$	M_1	$A'B'C$	m_1
0	1	0	$A+B'+C$	M_2	$A'BC'$	m_2
0	1	1	$A+B'+C'$	M_3	$A'BC$	m_3
1	0	0	$A'+B+C$	M_4	$AB'C'$	m_4
1	0	1	$A'+B+C'$	M_5	$AB'C$	m_5
1	1	0	$A'+B'+C$	M_6	ABC'	m_6
1	1	1	$A'+B'+C'$	M_7	ABC	m_7

Maxterms and minterms are complements of each other.

$M_0 = m_0'$ and $m_0 = M_0'$

Maxterms

Example: Let's generate the truth table of
$F(A,B,C)=(A+B+C).(A+B'+C).(A'+B+C).(A'+B+C').(A'+B'+C)$

Inputs			Output (F)	Maxterms
A	B	C		
0	0	0	0	$A+B+C$
0	0	1	1	
0	1	0	0	$A+B'+C$
0	1	1	1	
1	0	0	0	$A'+B+C$
1	0	1	0	$A'+B+C'$
1	1	0	0	$A'+B'+C$
1	1	1	1	

In short, we use the $\prod$ symbol to abbreviate the expressions in standard PoS form.

$F = \prod(0,2,4,5,6) = M_0 \cdot M_2 \cdot M_4 \cdot M_5 \cdot M_6$

Conversion Between Standard SoP and PoS Forms

Example: Let's convert $F(A,B,C) = \sum(4,5,6,7)$ into maxterms.

Inputs			F	F'
A	B	C		
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

We will do double negation and find out (F')'.

The complement of function F is;
$F'(A,B,C) = \sum(0,1,2,3) = m_0 + m_1 + m_2 + m_3$

The complement of F' is equal to F. So;
$$\begin{aligned} F(A,B,C) = (m_0 + m_1 + m_2 + m_3)' &= m_0'.\, m_1'.m_2'.m_3' \\ &= M_0.\, M_1.M_2.M_3 \\ &= \prod(0,1,2,3) \end{aligned}$$

Remember that, minterms and maxterms are complements of each other. $M_0 = m_0'$ and $m_0 = M_0'$

Conversion Between Standard SoP and PoS Forms

Example: Let's convert the following expression in standard SoP form to standard PoS form.

$F(A,B,C) = AB'C+A'BC+AB'C'+ABC+A'B'C$

1st way

All the combinations that make F=1 are 101, 011, 100, 111 and 011. It means that, all other combinations 000, 010, and 110 make F=0. Therefore, the maxterms are $(A+B+C)$, $(A+B'+C)$, and $(A'+B'+C)$.

$F(A,B,C)= (A+B+C).(A+B'+C).(A'+B'+C)$

2nd Way

We can write the expression in the following form.

$F(A,B,C) = \sum(1,3,4,5,7)$

Since maxterms are complements of minterms, we can say that;

$F(A,B,C) = \sum (1,3,4,5,7) = \prod (0,2,6)$

Simplification

Example: Let’s find the simplest form of *F* given in the truth table below, using minterms and maxterms.

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

First, let’s write the *F* using minterms, and then simplify.

$$\begin{aligned} F(A,B,C) &= \sum(0,2,3,5,7) = \underline{A'B'C'} + \underline{A'BC'} + A'BC + \underline{\underline{AB'C}} + \underline{\underline{ABC}} \\ &= A'C'(B'+B) + A'BC + AC(B'+B) = A'C' + \underline{A'BC} + \underline{AC} \\ &= A'C' + C(A'B+A) = A'C' + C(A+B) = {\color{red} A'C'+AC+BC} \end{aligned}$$

We could simplify the expression in another way.

$$\begin{aligned} F(A,B,C) &= \sum(0,2,3,5,7) = \underline{A'B'C'} + \underline{A'BC'} + A'BC + \underline{\underline{AB'C}} + \underline{\underline{ABC}} \\ &= A'C'(B'+B) + A'BC + AC(B'+B) = \underline{A'C'} + \underline{A'BC} + AC \\ &= A'(C'+BC) + AC = A'(B+C') + AC = {\color{red} A'C'+AC+A'B} \end{aligned}$$

Both results are correct.

Simplification

A	B	C	F
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

First, let’s write the *F* using maxterms, and then simplify.

$$
\begin{aligned}
F(A,B,C) = \prod(1,4,6) &= (A+B+C').(A'+B+C).(A'+B'+C) \\
&= (AA'+AB+AC+A'B+BB+BC+A'C'+BC'+C'C)(A'+B'+C) \\
&= (\underline{AB}+AC+\underline{A'B}+\underline{B}+\underline{BC}+A'C'+\underline{BC'})(A'+B'+C) \\
&=(B+AC+A'C')(A'+B'+C) \\
&= A'B+\underline{BB'}+BC+\underline{A'AC}+AB'C+ACC+A'A'C'+A'B'C'+\underline{A'C'C} \\
&= A'B+BC+\underline{AB'C}+AC+A'C'+\underline{A'B'C'} \\
&= A'B+B'(\underline{AC+A'C'})+\underline{AC+A'C'} \\
&= {\color{red}A'B+AC+A'C'}
\end{aligned}
$$

As you can see above, this simplified expression is the same as we have found by simplifying minterms. We can also find the other correct result by doing the simplification in another way.