Bernoulli Diferensiyel Denklemi

$\frac{dy}{dx} + P(x)y = Q(x)y^n$ biçimindeki diferensiyel denklemlere "Bernoulli Dif. Denk." denir.

Burada ($n \neq 0$ ve $n \neq 1$) dir.

Denklemi çözmek için $u = y^{1-n}$ dönüşümü yapılır. Denklem lineer denk. indirgenir. Buradan u bulunur ve $u = y^{1-n}$ de yerine konularak y çözümü elde edilir.

SORULAR

Aş. Dif. Denklemlerin çözümlerini bulunuz.

1) $y(6y^2-1)\,dx = 2x\,dy$

2) $6y^2\,dx - x(2x^3+y)\,dy = 0$

3) $x(y+4)y' - y^2 - 2y - 2x = 0$; $\frac{y}{x} = z$

4) $\frac{dy}{dx} + y = xy^3$

5) $\frac{dy}{dx} + \frac{1}{3}y = \frac{1}{3}(1-2x)y^4$

6) $x\,dy - \{y + xy^3(1+\ln x)\}\,dx = 0$

7) $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$

8) $x\frac{dy}{dx} + y = -2x^6y^4$

9) $y^3y' + \frac{1}{x}y^4 = \frac{\sin x}{x^4}$

10) $y' + y = y^2e^x$

Çözümleri

1) $y(6y^2-1)\,dx = 2x\,dy$

$2x\,dy - y(6y^2-1)\,dx = 0$

$\frac{dy}{dx} - \frac{y(6y^2-1)}{2x} = 0$

$\frac{dy}{dx} + \frac{1}{2x}y = \frac{3}{x}y^3$ (Bernoulli)

$u = y^{1-3} \Rightarrow u = y^{-2} \Rightarrow -2y^{-3}\frac{dy}{dx} = \frac{du}{dx}$

Buradan $\frac{dy}{dx} = -\frac{1}{2}y^3\frac{du}{dx}$

bu değeri denklemde yerine koyalım.

$-\frac{1}{2}y^3\frac{du}{dx} + \frac{1}{2x}y = \frac{3}{x}y^3$

$\Rightarrow \frac{du}{dx} - \frac{1}{x}y^{-2} = -\frac{6}{x}$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = -\frac{6}{x}$$ (lineer)

$$\lambda = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

$$\frac{1}{x}u = \int \frac{1}{x}\left(-\frac{6}{x}\right)dx + c$$

$$\Rightarrow \frac{u}{x} = \frac{6}{x} + c \Rightarrow u = cx + 6$$

$$\Rightarrow y^{-2} = cx + 6 \Rightarrow \boxed{y^2 = \frac{1}{cx+6}}$$

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2) $6y^2\,dx - x(2x^3+y)\,dy = 0$

$$6y^2\frac{dx}{dy} - x(2x^3+y) = 0$$

$$\Rightarrow \frac{dx}{dy} - \frac{2x^4+yx}{6y^2} = 0$$

$$\Rightarrow \frac{dx}{dy} - \frac{1}{6y}x = \frac{1}{3y^2}x^4$$

$$u = x^{1-4} = x^{-3} \Rightarrow \frac{du}{dy} = -3x^{-4}\frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = -\frac{1}{3}x^4\frac{du}{dy}$$

$$-\frac{1}{3}x^4\frac{du}{dy} - \frac{1}{6y}x = \frac{1}{3y^2}x^4$$

$$\Rightarrow \frac{du}{dy} + \frac{1}{2y}x^3 = -\frac{1}{y^2}$$

$$\Rightarrow \frac{du}{dy} + \frac{1}{2y} u = -\frac{1}{y^2} \qquad (\text{lineer})$$

$$\lambda = e^{\int \frac{1}{2y} dy} = y^{\frac{1}{2}}$$

$$y^{\frac{1}{2}} \cdot u = \int y^{\frac{1}{2}} \left(-\frac{1}{y^2}\right) dy + c$$

$$\Rightarrow \boxed{y^{\frac{1}{2}} x^{-3} = 2y^{-\frac{1}{2}} + c}$$

3) $x(y+4)y' - y^2 - 2y - 2x = 0 \;;\; \frac{y}{x} = z$

$$\frac{y}{x} = z \Rightarrow y = xz \Rightarrow y' = z + xz'$$

Bunu denklemde yerine yazalım.

$$x(xz+4)(\cancel{z}+xz') - x^2z^2 - 2\cancel{x}z - 2x = 0$$

$$(x^2z+4x)(z+xz') - x^2z^2 - 2xz - 2x = 0$$

$$\cancel{x^2z^2} + x^3zz' + 4xz + 4x^2z' - \cancel{x^2z^2} - 2xz - 2x = 0$$

$$(x^3z+4x^2)z' + 2xz - 2x = 0$$

$$(x^3z+4x^2)\frac{dz}{dx} + 2xz - 2x = 0$$

$$(x^3z+4x^2) + (2xz-2x)\frac{dx}{dz} = 0$$

$$\Rightarrow 2x(z-1)\frac{dx}{dz} + x^3z + 4x^2 = 0$$

$$\Rightarrow \frac{dx}{dz} + \frac{x^2z}{2(z-1)} + \frac{2x}{z-1} = 0$$

Buradan $z = y/x$ konulmasıyla

$$\boxed{\frac{1}{\left(\frac{y}{x}-1\right)^2} x^{-1} = -\frac{1}{2\left(\frac{y}{x}-1\right)} - \frac{1}{4\left(\frac{y}{x}-1\right)^2} + c}$$

4) $\frac{dy}{dx} + y = xy^3$ (Bernoulli)

$u = y^{1-3} = y^{-2}$ $\quad \frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}y^3\frac{du}{dx}$

$-\frac{1}{2}y^3\frac{du}{dx} + y = xy^3$

$\Rightarrow \frac{du}{dx} - 2y^{-2} = -2x \Rightarrow \frac{du}{dx} - 2u = -2x$ (lineer)

$\lambda = e^{-\int 2dx} = e^{-2x}$

$e^{-2x}u = \int e^{-2x}(-2x)\,dx$

$e^{-2x}u = xe^{-2x} + \frac{1}{2}e^{-2x} + c$

$\Rightarrow u = x + \frac{1}{2} + ce^{2x}$

$u = y^{-2}$ için $\boxed{\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}}$

5) $\frac{dy}{dx} + \frac{y}{3} = \frac{1}{3}(1-2x)y^4$ (Bernoulli)

$u = y^{1-4} = y^{-3}$ $\qquad \frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{3}y^4\frac{du}{dx}$

$-\frac{1}{\cancel{3}}y^4\frac{du}{dx} + \frac{y}{\cancel{3}} = \frac{1}{\cancel{3}}(1-2x)y^4$

$\Rightarrow \frac{du}{dx} - y^3 = 2x-1$ (lineer) $\frac{du}{dx} - u = (2x-1)$

$\lambda = e^{-\int dx} = e^{-x}$

$e^{-x}.u = \int e^{-x}(2x-1)\,dx + c$

$= 2\int xe^{-x}\,dx - \int e^{-x}\,dx + c$

$= 2\left[-xe^{-x} - e^{-x}\right] + e^{-x} + c$

$= -2xe^{-x} - e^{-x} + c$

$\Rightarrow u = -2x - 1 + ce^{x}$ $\qquad u = y^{-3}$ için

$$\boxed{\frac{1}{y^3} = -2x - 1 + ce^{x}}$$