GENELLEŞTİRİLMİŞ İNTEGRALLER
(Has Olmayan İntegraller)

$\int_a^b f(x).dx$ belirli integralinde eğer;

i) $f(x)$ integral fonksiyonu $a \leq x \leq b$ aralığının bir veya daha fazla noktasında süreksiz ise, $\left(\int_0^1 \frac{dx}{x-1} \right)$

veya

ii) en az bir integral sınırı sonsuz ise

bu integrale "has olmayan integral" adı verilir.

SONSUZ İNTEGRAL SINIRLARI

i) Eğer $f(x)$, $a \leq x \leq u$ aralığında sürekli ise; limitin varolması halinde belirli integral;

$$\boxed{\int_a^{\infty} f(x).dx = \lim_{u \to \infty} \int_a^{u} f(x).dx}$$

şeklinde tanımlanır.

ii) Eğer $f(x)$, $v \leq x \leq b$ aralığında sürekli ise; limitin var olması halinde belirli integral;

$$\boxed{\int_{-\infty}^{b} f(x).dx = \lim_{v \to -\infty} \int_v^{b} f(x).dx}$$

şeklinde tanımlanır.

iii) Eğer $f(x)$, $v \leq x \leq u$ aralığında sürekli ise $\int_{-\infty}^{\infty} f(x).dx$ integrali;

$$\boxed{\int_{-\infty}^{\infty} f(x).dx = \lim_{u \to \infty} \int_a^{u} f(x).dx + \lim_{v \to -\infty} \int_v^{b} f(x).dx}$$

şeklinde tanımlanır.

$$\Rightarrow \int_{-\infty}^{0} e^{2x}.dx = ?$$

$$= \lim_{u\to-\infty} \int_{u}^{0} e^{2x}.dx = \lim_{u\to-\infty}\left[\frac{1}{2}e^{2x}\Big|_{u}^{0}\right] = \lim_{u\to-\infty}\left[\frac{1}{2}.e^{0} - \frac{1}{2}e^{2u}\right] = $$

anlamlı (yakınsak)

$$\Rightarrow \int_{1}^{\infty}\frac{dx}{\sqrt{x}} = \lim_{u\to\infty}\int_{1}^{u}\frac{dx}{\sqrt{x}} = \lim_{u\to\infty}\left[2\sqrt{x}\Big|_{1}^{u}\right] = \lim_{u\to\infty}\left[2\sqrt{u} - 2\sqrt{1}\right] = \infty$$

(yakınsak değil ve anlamsızdır.) ıraksak

$$\Rightarrow \int_{0}^{\infty}\frac{dx}{x^2+4} = \lim_{u\to\infty}\int_{0}^{u}\frac{dx}{x^2+4} = \lim_{u\to\infty}\left[\frac{1}{2}\arctan\frac{x}{2}\Big|_{0}^{u}\right]$$

$$= \lim_{u\to\infty}\left[\underbrace{\frac{1}{2}\arctan u}_{} - \underbrace{\arctan 0^{\circ}}_{}\right] = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}.0 = \frac{\pi}{4}$$

$\arctan 0^{\circ} = A$ olsun

$\tan A = 0$

$\frac{\sin A}{\cos A} = 0 \Rightarrow \boxed{A = 0}$

$\arctan\infty = B$ olsun

$\tan B = \infty$

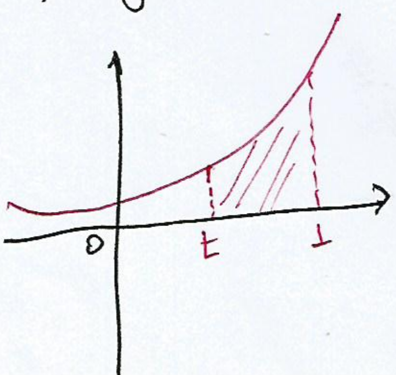
$\frac{\sin A}{\cos A}$

$\boxed{B = \frac{\pi}{2}}$

$$\Rightarrow \int_{1}^{\infty}\frac{dx}{x^{4/3}} = \lim_{u\to\infty}\int_{1}^{u} x^{-4/3}.dx = \lim_{u\to\infty}\left(-\frac{3}{x^{1/3}}\right)\Big|_{1}^{u}$$

$$= \lim_{u\to\infty}\left(-\frac{3}{u^{1/3}} + \frac{3}{1^{1/3}}\right) = -\frac{3}{\infty} + 3 = 3 \ \#$$

$\Rightarrow y = e^x,\ y = 0,\ x = 1$ ve $x = t < 1$ eğrileri ile SB nin alanını bulunu



$$A = \int_{-\infty}^{1} e^x.dx = \lim_{u\to-\infty}\int_{u}^{1} e^x.dx$$

$$= \lim_{u\to-\infty}\left(e^x\Big|_{u}^{1}\right) = \lim_{u\to-\infty}\left[e - e^{u}\right]$$

$$= e - e^{-\infty} = e \ \text{br}^2 \ \#$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{e^{x}+e^{-x}} = ? \qquad \left(\int \frac{dx}{e^{x}+\frac{1}{e^{x}}} = \int \frac{e^{x}}{(e^{x})^{2}+1}\, .dx = \int \frac{\cancel{u}}{u^{2}+1} \cdot \frac{du}{\cancel{e^{x}=u}} = \arctan u \right.$$

$$\left(u = e^{x} \Rightarrow \frac{du}{dx} = e^{x} \right)$$

3

$$= \int_{-\infty}^{0} \frac{dx}{e^{x}+e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^{x}+e^{-x}}$$

$$= \lim_{u \to -\infty} \int_{u}^{0} \frac{dx}{e^{x}+e^{-x}} + \lim_{v \to \infty} \int_{0}^{v} \frac{dx}{e^{x}+e^{-x}}$$

$$= \lim_{u \to -\infty} \left[\arctan e^{x} \Big|_{u}^{0} \right] + \lim_{v \to \infty} \left[\arctan e^{x} \Big|_{0}^{v} \right]$$

$$= \lim_{u \to -\infty} \left[\arctan e^{0} - \arctan u \right] + \lim_{v \to \infty} \left[\arctan e^{\infty} - \arctan 0^{\circ} \right]$$

$$= \lim_{u \to -\infty} \left[\arctan 1 - \arctan u \right] + \lim_{v \to \infty} \left[\arctan e^{v} - \arctan 0^{\circ} \right]$$

$$= \arctan 1 - \arctan(-\infty) + \arctan e^{\infty} - \arctan 0^{0}$$

$$= \arctan 1 - \arctan(-\infty) + \arctan \infty - \arctan 0^{\circ} = \frac{\pi}{2}$$

$\Rightarrow \int_0^2 \frac{dx}{2-x} = ?$ $\qquad \left(\int \frac{dx}{2-x} = -\ln|2-x| + c \right)$

$\star$ Eğer $f(x)$, $a \leq x \leq b$ aralığındaki $a < c < b$ olan c değerinden başka bütün x-değerleri için sürekli ise belirli integral, limitin var olması halinde

$$\int_a^b f(x).dx = \lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x).dx + \lim_{\varepsilon \to 0^+} \int_{c+\varepsilon}^b f(x).dx$$

şeklinde hesaplanır.

$$= \int_0^2 \frac{dx}{2-x} = \lim_{\varepsilon \to 0^+} \int_0^{2-\varepsilon} \frac{dx}{2-x} = \lim_{\varepsilon \to 0^+} \left(-\ln(2-x) \Big|_0^{2-\varepsilon} \right)$$

$$= \lim_{\varepsilon \to 0^+} \Big[-\ln[2-(2-\varepsilon)] + \ln[2-0] \Big]$$

$$= \lim_{\varepsilon \to 0^+} \Big[-\ln\varepsilon + \ln 2 \Big] = \infty$$

($\ln 0 = \infty$, integral yakınsak değil, anlamsızdır.)

$\Rightarrow \int_0^3 \frac{dx}{\sqrt{9-x^2}} = ?$ $\qquad \left(\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{9.\left[1-\left(\frac{x}{3}\right)^2\right]}} = \cancel{\frac{1}{3}}.\cancel{3}.\arcsin\frac{x}{3} + c \right)$

$$= \lim_{\varepsilon \to 0^+} \int_0^{3-\varepsilon} \frac{dx}{\sqrt{9-x^2}} = \lim_{\varepsilon \to 0^+} \left[\arcsin\frac{x}{3} \Big|_0^{3-\varepsilon} \right]$$

$$= \lim_{\varepsilon \to 0^+} \left[\arcsin\left(\frac{3-\varepsilon}{3}\right) - \arcsin 0^\circ \right]$$

$$= \arcsin 1^\circ - \arcsin 0^\circ$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \#$$

$\left\{ \begin{array}{c|c} \arcsin 1 = A & \arcsin 0^\circ = B \\ \sin A = 1 & \sin B = 0 \\ A = \frac{\pi}{2} & B = 0 \end{array} \right\}$

$$\Rightarrow \int_0^4 \frac{dx}{(x-1)^2} = ? \qquad \left(\int \frac{dx}{(x-1)^2} = \int (x-1)^{-2}.dx = 1.\frac{(x-1)^{-1}}{-1} = -\frac{1}{x-1} + c \right)$$

5

$$= \int_0^1 \frac{dx}{(x-1)^2} + \int_1^4 \frac{dx}{(x-1)^2}$$

$$= \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dx}{(x-1)^2} + \lim_{\varepsilon \to 0^+} \int_{1+\varepsilon}^4 \frac{dx}{(x-1)^2}$$

$$= \lim_{\varepsilon \to 0^+} \left[-\frac{1}{x-1} \Big|_0^{1-\varepsilon} \right] + \lim_{\varepsilon \to 0^+} \left[-\frac{1}{x-1} \Big|_{1+\varepsilon}^4 \right]$$

$$= \lim_{\varepsilon \to 0^+} \left\{ \underbrace{-\left[\frac{1}{1-\varepsilon-1} - \frac{1}{0-1} \right]}_{\text{limit yok}} + \underbrace{-\left[\frac{1}{4-1} - \frac{1}{1+\varepsilon-1} \right]}_{\text{limit yok}} \right\}$$

integral anlamsızdır.

NOT: Eğer $x=1$ süreksizlik noktası dikkate alınmasaydı,

$$\int_0^4 \frac{dx}{(x-1)^2} = -\frac{1}{x-1} \Big|_0^4 = -\frac{4}{3}$$ bulunarak BÜYÜK BİR HATA işlemiş olurdu.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1-\sin x}} \quad \left\{ \begin{array}{l} x=\frac{\pi}{2} \text{ için} \\ \sqrt{1-\sin\frac{\pi}{2}}=0 \\ \text{olur, } x=\frac{\pi}{2} \\ \text{süreksizlik} \\ \text{noktasıdır} \end{array} \right\}$$

$$\left(\int \frac{\cos x}{\sqrt{1-\sin x}} \cdot dx \Rightarrow \begin{array}{l} u=1-\sin x \\ \frac{du}{dx}=-\cos x \end{array} \right.$$

$$\left. =\int \frac{\cos x}{\sqrt{u}} \cdot \frac{du}{-\cos x} = -2\sqrt{u}+C = -2.\sqrt{1-\sin x}+C \right)$$

6

$$= \lim_{\varepsilon \to 0^+} \int_0^{\frac{\pi}{2}-\varepsilon} \frac{\cos x}{\sqrt{1-\sin x}} \cdot dx$$

$$= \lim_{\varepsilon \to 0^+} \left[-2.\sqrt{1-\sin x} \Big|_0^{\frac{\pi}{2}-\varepsilon} \right]$$

$$= -2 \lim_{\varepsilon \to 0^+} \left[\sqrt{1-\sin\left(\frac{\pi}{2}-\varepsilon\right)} - \sqrt{1-2.\cancel{\sin 0}^{\,0}} \right]$$

$$= -2 \lim_{\varepsilon \to 0^+} \left[\sqrt{1-\cos\varepsilon} - 1 \right] = -2.\left[\sqrt{1-\cancel{\cos 0^+}^{\,1}} - 1 \right]$$

$$= -2.[-1] = 2$$

$$\Rightarrow \int_{-2}^{7} \frac{dx}{(x+1)^{2/3}} = ? \qquad \left(\int \frac{dx}{(x+1)^{2/3}} = \int (x+1)^{-\frac{2}{3}} dx = 3.(x+1)^{\frac{1}{3}}+C \right)$$

$$= \int_{-2}^{-1} \frac{dx}{(x+1)^{2/3}} + \int_{-1}^{7} \frac{dx}{(x+1)^{2/3}}$$

$$= \lim_{\varepsilon \to 0^+} \int_{-2}^{-1-\varepsilon} \frac{dx}{(x+1)^{2/3}} + \lim_{\varepsilon \to 0^+} \int_{-1+\varepsilon}^{7} \frac{dx}{(x+1)^{2/3}}$$

$$= \lim_{\varepsilon \to 0^+} \left[3.(x+1)^{\frac{1}{3}} \Big|_{-2}^{-1-\varepsilon} \right] + \lim_{\varepsilon \to 0^+} \left[3.(x+1)^{\frac{1}{3}} \Big|_{-1+\varepsilon}^{7} \right]$$

$$\Rightarrow 3.\lim_{\varepsilon \to 0^+} \left[(\cancel{-1}-\varepsilon\cancel{+1})^{\frac{1}{3}} - (-2+1)^{\frac{1}{3}} \right] + 3.\lim_{\varepsilon \to 0^+} \left[(7+1)^{\frac{1}{3}} - (\cancel{-1}+\varepsilon\cancel{+1})^{\frac{1}{3}} \right]$$

$$= 3.\left[0^{\frac{1}{3}} - (-1)^{\frac{1}{3}} \right] + 3.\left[8^{\frac{1}{3}} - 0^{\frac{1}{3}} \right] = 3 + 3.\sqrt[3]{8} = 9$$