21

Upper-Bounded Transshipment Problems

The network simplex method can be easily modified to handle problems

$$\text{minimize} \quad \mathbf{cx} \qquad \text{subject to} \quad \mathbf{Ax} = \mathbf{b}, \quad \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}$$

such that $\mathbf{A}$ is the incidence matrix of a network, each component l_{ij} of the vector $\mathbf{l}$ is either a number or the symbol $-\infty$ (meaning that no lower bound is imposed on x_{ij}) and each component u_{ij} of the vector $\mathbf{u}$ is either a number or the symbol $+\infty$ (meaning that no upper bound is imposed on x_{ij}). These modifications are simply a particular instance of the upper-bounding technique described in Chapter 8; we shall develop them from scratch so as to make our presentation of the network simplex method self-contained. To avoid fussy formalism, we shall restrict ourselves to so-called *upper-bounded transshipment problems* (or *minimum cost network flow problems*),

$$\text{minimize} \quad \mathbf{cx} \qquad \text{subject to} \quad \mathbf{Ax} = \mathbf{b}, \quad \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}.$$

The generalization to arbitrary lower bounds is straightforward.

AN EXAMPLE

For illustration, let us consider the data in Figure 21.1. The key to understanding the modifications is the realization that the new constraints $x_{ij} \leq u_{ij}$ should be treated analogously to the constraints $x_{ij} \geq 0$. For instance, a feasible tree solution in a transshipment problem had the property that $x_{ij} = 0$ whenever $ij \notin T$. In the context of upper-bounded transshipment problems, we shall require that

$$x_{ij} = 0 \quad \text{or} \quad x_{ij} = u_{ij} \qquad \text{whenever} \quad ij \notin T.$$

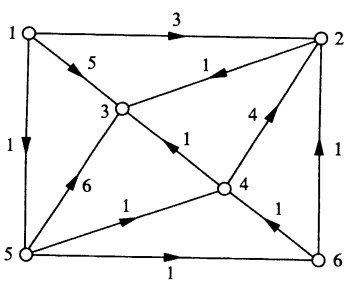
The arcs ij with $x_{ij} = u_{ij}$ are sometimes called *saturated*. An example of a feasible tree solution is shown in Figure 21.2: the in-tree arcs are solid, whereas the saturated out-of-tree arcs are dashed. We shall initialize the network simplex method by this feasible tree solution. The node numbers y_i are defined by the equations

$$y_i + c_{ij} = y_j \qquad \text{for all} \quad ij \in T$$

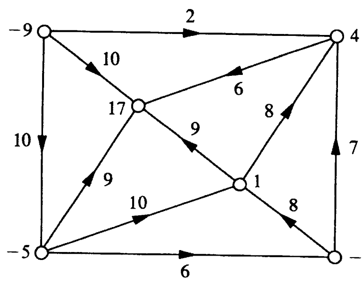
just as they were in Chapter 19 (see Figure 21.3). In transshipment problems, each entering arc $ij \notin T$ has the property that

$$y_i + c_{ij} < y_j \quad \text{and} \quad x_{ij} = 0. \tag{21.1}$$

There is no harm in choosing such an arc in the present context either (although we shall see later in this section that certain arcs $ij \notin T$ with $x_{ij} = u_{ij}$ are eligible as well). For instance, arc 13 will do. As we used to do, we now set $x_{13} = t$ and adjust the values of x_{ij} for $ij \in T$ (see Figure 21.4). The largest value of t for which feasibility is maintained is $t = 8$. The resulting feasible tree solution is shown in Figure 21.5 along with the corresponding node numbers.



Node names and arc costs



Demands and upper bounds

Figure 21.1

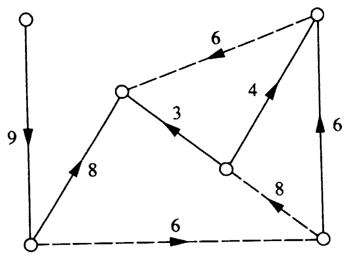


Figure 21.2

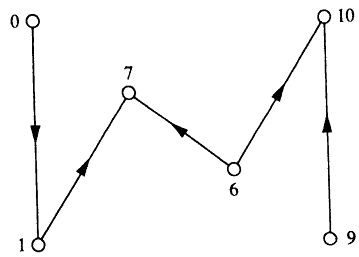


Figure 21.3

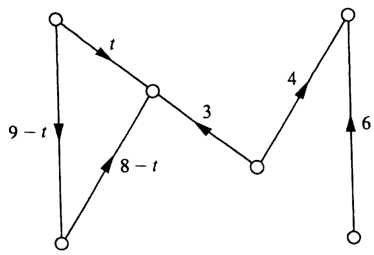


Figure 21.4

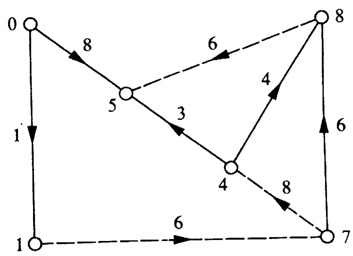


Figure 21.5

Now we choose 54 for the entering arc (note that this arc satisfies (21.1) in place of ij). Setting $x_{54} = t$ and adjusting the values of x_{ij} for $ij \in T$, we obtain Figure 21.6.

Now the constraint $x_{43} \leq u_{43} = 9$ dictates $t \leq 6$. Setting $t = 6$, we obtain the feasible tree solution and new node numbers shown in Figure 21.7. Note that arc 4 3 leaves the tree and becomes saturated.

Now the only arc satisfying (21.1) in place of ij is arc 12. We choose this arc for our next entering arc. With $x_{12} = t$ we are led to consider Figure 21.8. The constraint $x_{12} \leq u_{12} = 2$ forces $t \leq 2$; in fact, the solution corresponding to $t = 2$ remains feasible. Thus the entering arc becomes saturated before any arc is forced to leave the tree. Consequently, the tree does not change even though the solution does. The new solution is shown in Figure 21.9; since the tree does not change, the node numbers do not change.

At this moment, no arc $ij \notin T$ satisfies (21.1) and yet our solution is *not* optimal. Consider, for instance, arc 23. Under the current prices y_i, shipping through this arc is unprofitable ($y_2 + c_{23} > y_3$) and yet we are doing it; in fact, we are using this arc to its full capacity. It seems sensible to *decrease* the value of x_{23}, and that is precisely what we are going to do. Setting $x_{23} = u_{23} - t$, and adjusting the values of x_{ij} with $ij \in T$ accordingly, we obtain Figure 21.10. Since the constraint $x_{42} \geq 0$

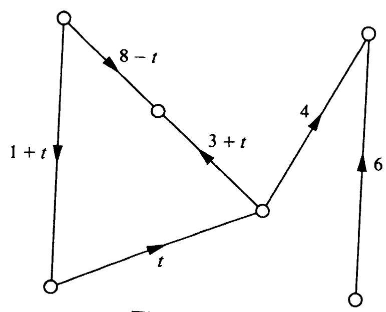


Figure 21.6

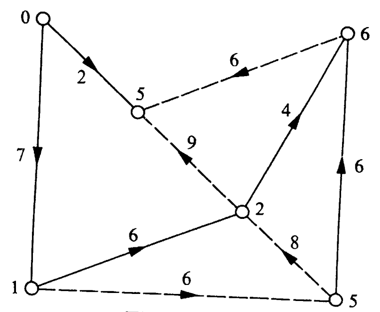


Figure 21.7

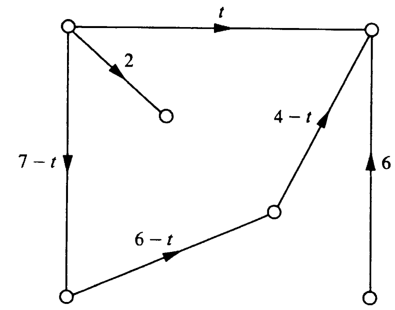


Figure 21.8

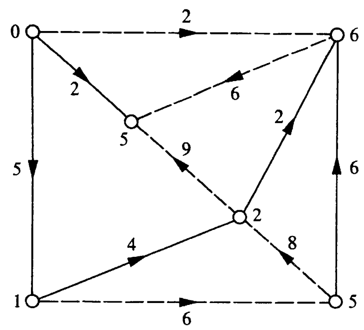


Figure 21.9

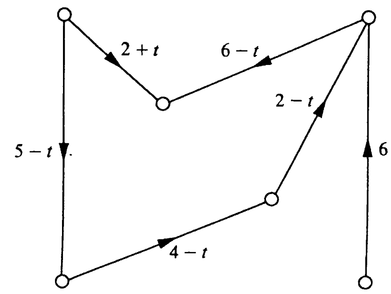


Figure 21.10

forces $t \leq 2$, and since $t = 2$ does yield a feasible solution, we enter the next iteration with the feasible tree solution and node numbers shown in Figure 21.11.

More generally, any arc $ij \notin T$ such that

$$y_i + c_{ij} > y_j \quad \text{and} \quad x_{ij} = u_{ij} \tag{21.2}$$

can be chosen for an entering arc: we shall reduce the total cost by decreasing the value of x_{ij} to $u_{ij} - t$ and adjusting the values of x_{ij} with $ij \in T$ accordingly. At present, the arc 64 satisfies (21.2) in place of ij; choosing this arc for an entering arc we obtain Figure 21.12. As t increases, arc 62 becomes saturated and leaves the tree. The resulting feasible tree solution, along with the corresponding node numbers, is shown in Figure 21.13.

Now arc 56 satisfies (21.2) in place of ij. Choosing this arc for the entering arc we are led to Figure 21.14. As t increases, the entering arc 56 obtains $x_{56} = 0$ before any arc is forced to leave the tree. Hence the tree does not change and the node numbers do not change either. The next feasible tree solution and its node numbers are shown in Figure 21.15.

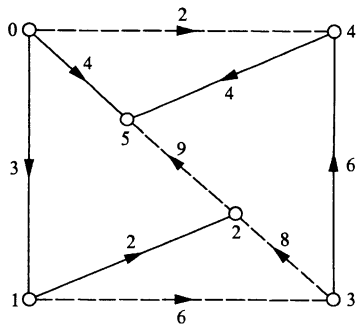


Figure 21.11

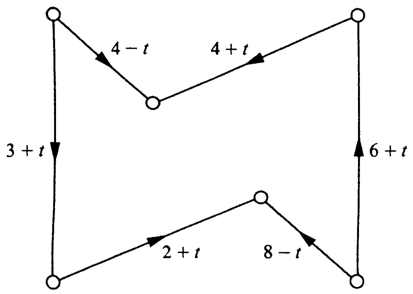


Figure 21.12

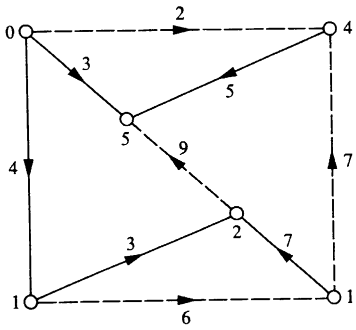


Figure 21.13

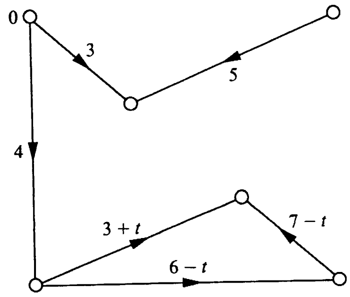


Figure 21.14

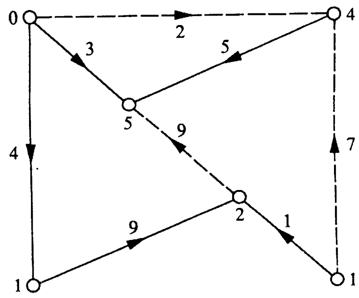


Figure 21.15

Now we have

$$x_{ij} = u_{ij} \qquad \text{whenever} \quad y_i + c_{ij} < y_j \tag{21.3}$$

and

$$x_{ij} = 0 \qquad \text{whenever} \quad y_i + c_{ij} > y_j. \tag{21.4}$$

Thus no arc ij satisfies (21.1) or (21.2). As we are about to prove, this circumstance indicates that the current solution is optimal.

ANALYSIS

Let us show at once that a feasible solution $\mathbf{x}$ of an upper-bounded transshipment problem

$$\text{minimize} \quad \mathbf{cx} \qquad \text{subject to} \quad \mathbf{Ax} = \mathbf{b}, \quad \mathbf{0} \le \mathbf{x} \le \mathbf{u} \tag{21.5}$$

is optimal whenever there are numbers $y_1, y_2, \ldots, y_n$ satisfying (21.3) and (21.4). The proof of this claim is simple: by virtue of (21.3) and (21.4), any feasible solution $\tilde{\mathbf{x}}$ of (21.5) satisfies

$$(c_{ij} + y_i - y_j)\tilde{x}_{ij} \ge (c_{ij} + y_i - y_j)x_{ij}$$

for all arcs ij. If $\bar{\mathbf{c}}$ denotes the vector with components $\bar{c}_{ij} = c_{ij} + y_i - y_j$ then $\mathbf{c} = \bar{\mathbf{c}} + \mathbf{yA}$ and so

$$\mathbf{c}\tilde{\mathbf{x}} = \bar{\mathbf{c}}\tilde{\mathbf{x}} + \mathbf{yA}\tilde{\mathbf{x}} = \bar{\mathbf{c}}\tilde{\mathbf{x}} + \mathbf{yb} \ge \bar{\mathbf{c}}\mathbf{x} + \mathbf{yb} = \bar{\mathbf{c}}\mathbf{x} + \mathbf{yAx} = \mathbf{cx}$$

which is the desired conclusion.

Along similar lines, we can explain the choice of the entering arc e as one satisfying $\bar{c}_e < 0$, $x_e = 0$ or $\bar{c}_e > 0$, $x_e = u_e$. In the corresponding pivot, the current feasible solution $\mathbf{x}$ is replaced by a feasible solution $\tilde{\mathbf{x}}$ such that

$$\tilde{x}_{ij} = x_{ij} \qquad \text{for all arcs } ij \notin T \text{ except the entering arc } e.$$

Since

$$\bar{c}_{ij} = 0 \qquad \text{whenever } ij \in T$$

we have

$$\bar{\mathbf{c}}\tilde{\mathbf{x}} = \bar{\mathbf{c}}\mathbf{x} + \bar{c}_e(\tilde{x}_e - x_e).$$

Now $\mathbf{c} = \bar{\mathbf{c}} + \mathbf{yA}$, $\mathbf{Ax} = \mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}$ imply

$$\mathbf{c}\tilde{\mathbf{x}} = \mathbf{cx} + \bar{c}_e(\tilde{x}_e - x_e).$$

Since we set, for some nonnegative t,