SC42100 Networked and Distributed Control Systems

Assignment 3

TU Delft, 3ME, DCSC, Spring 2022

- The assignments are individual. You may consult and discuss with your colleagues, but the answers you provide, the code you use, and the report you hand in must be independent.
- Provide detailed answers, describing the steps you followed. Only end results of calculations are not sufficient.
- Provide a clear report as a PDF file, typed, preferably using LaTeX.
- Submit the report digitally via Brightspace before the deadline: 9:00, June 24, 2022.

2p **Problem 1**

1p (a) Consider the following convex optimization problem with a complicating constraint:

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & f_1(x_1) + f_2(x_2) \\ \text{subject to} & x_1 \in \mathcal{X}_1, \quad x_2 \in \mathcal{X}_2 \\ & h_1(x_1) + h_2(x_2) \leq 0 \end{array}$$

where $\mathcal{X}_1, \mathcal{X}_2$ are convex sets and all the functions are convex.

Apply primal decomposition to the problem and show the resulting two subproblems. What is the role of the master problem?

1p (b) In order to solve the master problem in question (a), the two subproblems are solved independently to obtain subgradients. It turns out that we can find a subgradient for the optimal value of each subproblem from an optimal dual variable associated with the coupling constraint. This leads to the second question:

Let $p(z)$ be the optimal value of the (possibly non-smooth!) convex optimization problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & f(x) \\ \text{subject to} & x \in \mathcal{X} \\ & h(x) \leq z \end{array}$$

Let λ^* be an optimal dual variable associated with the constraint $h(x) \leq z$. Show that $-\lambda^*$ is a subgradient of p at z.

2p **Problem 2**

Consider the combined consensus/projected incremental subgradient method for N agents shown in the lecture slides:

$$x_{k+1}^i = \mathcal{P}_{\mathcal{X}} \left[\sum_{j=1}^{N} [W^{\varphi}]_{ij} \left(x_k^j - \alpha_k g^j(x_k^j) \right) \right], \quad i = 1, \dots, N$$

The weight matrix $W \in \mathbb{R}^{N \times N}$ fulfills

$$\begin{aligned} &[W]_{ij} = 0, \text{ if } (i,j) \notin \mathcal{E} \text{ and } i \neq j, \\ &W = W^{\intercal}, W\mathbf{1}_N = \mathbf{1}_N, \rho\left(W - \frac{\mathbf{1}_N \mathbf{1}_N^{\intercal}}{N}\right) \leq \gamma < 1, \end{aligned}$$

where $\rho(\cdot)$ is the spectral radius and $\mathbf{1}_N \in \mathbb{R}^N$ is the column vector with all elements equal to one. The matrix W can for example be chosen as the so-called Perron matrix of the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with parameter ε. It is then defined as $W = I - \varepsilon L(\mathcal{G})$, where $L(\mathcal{G})$ represents the Laplacian matrix of the communication graph. Assume that the communication graph $\mathcal{G}$ used for consensus is strongly connected, balanced, with maximum degree Δ, and the matrix W is doubly stochastic with $0 < \varepsilon < 1/\Delta$.

Show that as $\varphi \to \infty$ (i.e., the agents reach consensus in each iteration of the algorithm), the combined consensus / projected incremental subgradient method becomes a standard subgradient method.

3p **Problem 3**

Consider the positive definite quadratic function partitioned into two sets of variables

$$V(u) = \frac{1}{2} u^T H u + c^T u + d$$

$$V(u_1, u_2) = \frac{1}{2} \begin{pmatrix} u_1^T & u_2^T \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1^T & c_2^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + d$$

in which $H > 0$. Imagine we wish to optimize this function by first optimizing over the u_1 variables holding u_2 fixed and then optimizing over the u_2 variables holding u_1 fixed as shown in Figure 1 for a scalar case.

Let us see if this procedure, while not necessarily efficient, is guaranteed to converge to the optimum.

1p (a) Given an initial point (u_1^p, u_2^p), show that the next iteration is

$$\begin{aligned} u_1^{p+1} &= -H_{11}^{-1} \left(H_{12} u_2^p + c_1 \right) \\ u_2^{p+1} &= -H_{22}^{-1} \left(H_{21} u_1^p + c_2 \right) \end{aligned}$$

The procedure can be summarized as

$$u^{p+1} = A u^p + b$$

in which the iteration matrix A and constant b are given by

$$A = \begin{pmatrix} 0 & -H_{11}^{-1} H_{12} \\ -H_{22}^{-1} H_{21} & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -H_{11}^{-1} c_1 \\ -H_{22}^{-1} c_2 \end{pmatrix}$$

1p (b) Establish that the optimization procedure converges by showing the iteration matrix is stable

$$|\,\mathrm{eig}(A)| < 1$$

1p (c) Given that the iteration converges, show that it produces the same solution as

$$u^* = -H^{-1}c$$

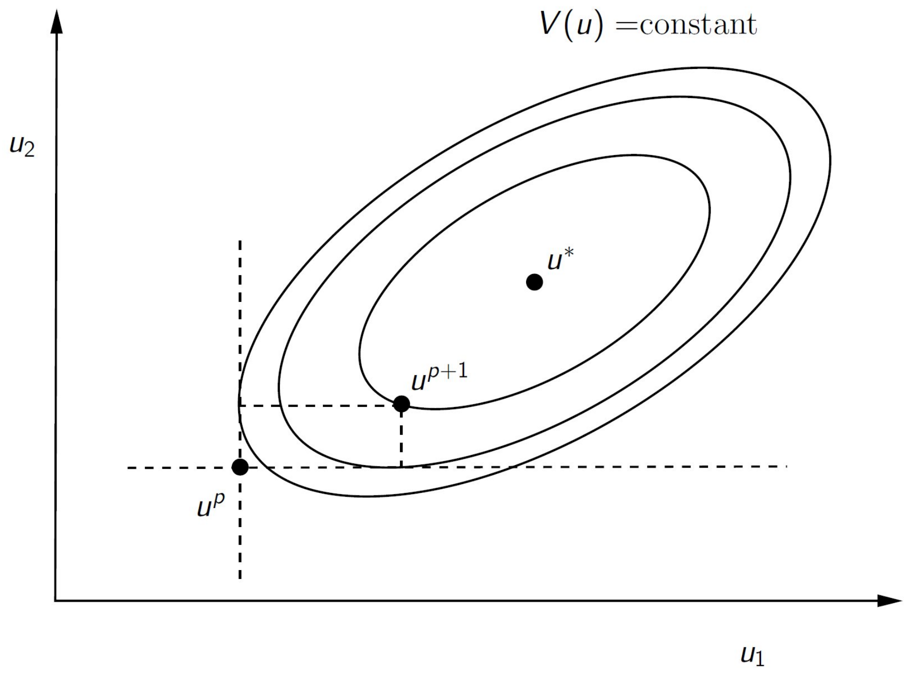

Figure 1: Optimizing a quadratic function in one set of variables at a time.

2p **Problem 4**

Consider again the iteration defined in the previous problem. Prove that the cost function is monotonically decreasing when optimizing one variable at a time

$$V(u^{p+1}) < V(u^p) \quad \forall u^p \neq -H^{-1}c$$

and show that the following expression gives the size of the decrease

$$V(u^{p+1}) - V(u^p) = -\frac{1}{2}(u^p - u^*)^T P(u^p - u^*)$$

in which

$$P = HD^{-1}\tilde{H}D^{-1}H, \quad \tilde{H} = D - N, \quad D = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix}$$

and $u^* = -H^{-1}c$ is the optimum.

Hint: to simplify the algebra, first change coordinates and move the origin of the coordinate system to u^*.

Useful facts:

- If H is a positive definite, symmetric matrix partitioned in the following way, then

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} > 0 \quad \Rightarrow \quad \bar{H} = \begin{pmatrix} H_{11} & -H_{12} \\ -H_{21} & H_{22} \end{pmatrix} > 0.$$

- For any Q real symmetric and R real matrices:

$$Q > 0 \text{ and } R \text{ nonsingular} \quad \Rightarrow \quad R^T Q R > 0.$$