Delft University of Technology

Optimisation for system and Control

SC42056

Assignment 1

Yiting Li 5281873

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Delft
University of
Technology

Setup

The student number for this assignment is 5281873, thus $a = 5, b = 8, c = 3$. The system matrix is described as following:

$$A = \begin{bmatrix} -3 & -2.5 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

The scripts can be found in https://github.com/kemosabe564/NDCS.git

1 Questions 1

1.1 Question 1.1

After the poles placement of $(-1 + j - 1 - j)$, we can get the controller $\bar{K} = [-20]$.

1.2 Question 1.2

With no delay and controller vector, we can give the expression about the exact discrete-time model

$$\begin{aligned} x_{k+1} := \xi\left(s_{k+1}\right) &= e^{Ah}\xi\left(s_k\right) + \int_{s_k}^{s_{k+1}} e^{A(s_{k+1}-s)} Bu_k \mathrm{d}s \\ &= e^{Ah}x_k + \int_0^h e^{A(h)} Bu_k \mathrm{d}s \\ &= F(h)x_k + G(h)u_k = (F(h) - G(h)\bar{K})\bar{x_k} \end{aligned} \tag{2}$$

Then we can write that

$$F(h) = \begin{bmatrix} \mathrm{e}^{-3h} & \frac{5}{8}(\mathrm{e}^{-3h} - \mathrm{e}^{h}) \\ 0 & \mathrm{e}^{h} \end{bmatrix} G(h) = \begin{bmatrix} -\frac{5}{24}\mathrm{e}^{-3h} - \frac{5}{8}\mathrm{e}^{h} + \frac{5}{6} \\ -1 + \mathrm{e}^{h} \end{bmatrix} \tag{3}$$

We can further write the system after the augmentation:

$$A(h) = \begin{bmatrix} \frac{7}{12}\mathrm{e}^{-3h} - \frac{5}{4}\mathrm{e}^{h} + \frac{5}{3} & \frac{5}{8}(\mathrm{e}^{-3h} - \mathrm{e}^{h}) \\ -2 + 2\mathrm{e}^{h} & \mathrm{e}^{h} \end{bmatrix} \tag{4}$$

2 Questions 2

2.1 Question 2.1

After include the delay, we can give the system model as below, with$\tau \in [0, h)$:

$$\begin{aligned} x_{k+1} &= e^{A(h)}x_k + \int_{h-\tau}^{h} e^{As} Bdsu_{k-1} + \int_0^{h-\tau} e^{As} Bdsu_k \\ &= F_x(h)x_k + F_u(h,\tau)u_{k-1} + G_1(h,\tau)u_k \end{aligned} \tag{5}$$

Where,

$$
\begin{aligned}
F_x(h) &= \begin{bmatrix} e^{-3h} & \frac{5}{8}(e^{-3h} - e^{h}) \\ 0 & e^{h} \end{bmatrix} \\
F_u(h,\tau) &= \begin{bmatrix} \frac{5}{24}(e^{-3h+3\tau} - e^{-3h}) + \frac{5}{8}(e^{h-\tau} - e^{h}) \\ -e^{h-\tau} + e^{-h} \end{bmatrix} \\
G_1(h,\tau) &= \begin{bmatrix} -\frac{5}{24}e^{-3h+3\tau} - \frac{5}{8}e^{h-\tau} + \frac{5}{6} \\ -1 + e^{h-\tau} \end{bmatrix}
\end{aligned} \tag{6}
$$

2.2 Question 2.2

We use state augmentation $x_k^e = \begin{bmatrix} x_k^T & u_{k-1}^T \end{bmatrix}$ to transform the model into $x_{k+1}^e = F(h,\tau)x_k^e + G(h,\tau)u_k$, the corresponding components are described as follow:

$$
\begin{aligned}
F(h,\tau) &= \begin{bmatrix} F_x(h) & F_u(h,\tau) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^{-3h} & \frac{5}{8}(e^{-3h} - e^{h}) & \frac{5}{24}(e^{-3h+3\tau} - e^{-3h}) + \frac{5}{8}(e^{h-\tau} - e^{h}) \\ 0 & e^{h} & -e^{h-\tau} + e^{h} \\ 0 & 0 & 0 \end{bmatrix} \\
G(h,\tau) &= \begin{bmatrix} G_1(h,\tau) \\ I \end{bmatrix} = \begin{bmatrix} -\frac{5}{24}e^{-3h+3\tau} - \frac{5}{8}e^{h-\tau} + \frac{5}{6} \\ -1 + e^{h-\tau} \\ 1 \end{bmatrix}
\end{aligned} \tag{7}
$$

Applying the same controller to the system, by checking the maximum eigenvalue of A, with $A = F - G\bar{K}$, we can determine whether the origin of the discrete-time closed-loop is a GES under different sampling period h and time delayτ. The result is shown in the figure1.

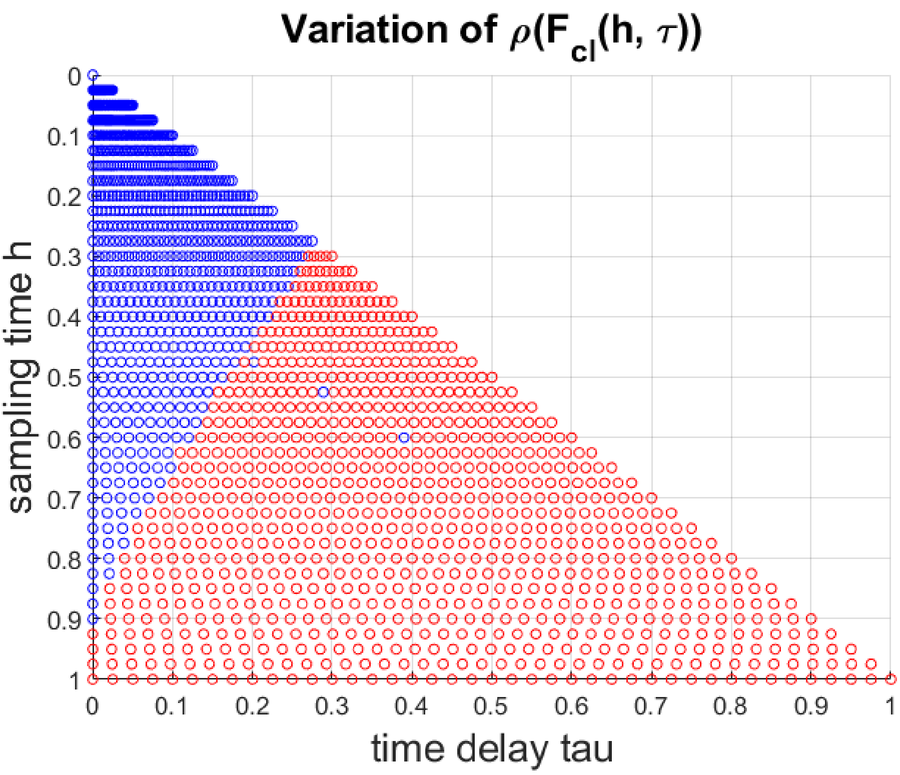
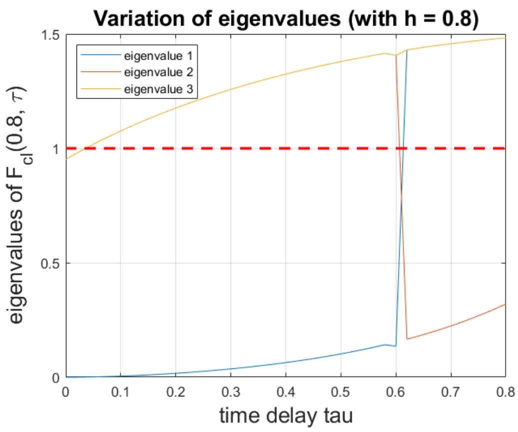


Figure 1: Eigenvalue Comparison under different h and τ

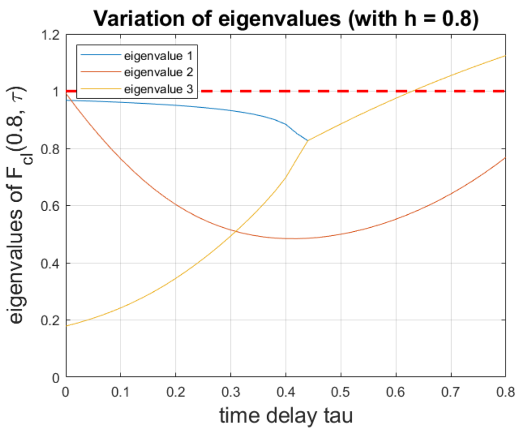
In above figure, the blue dots represent those eigenvalues are smaller then 1, red dots represent unstable system. We only consider those delays don't exceed sampling time. The result indicates that with larger sampling period(larger than 0.9), system is not stable any more. Furthermore, when the sampling period is increasing, the range of tolerable delay is getting smaller, only small enough delay will cause a stable system, e.g. when h is 0.8, only delay is less than 5% h will have a stable system.

2.3 Question 2.3

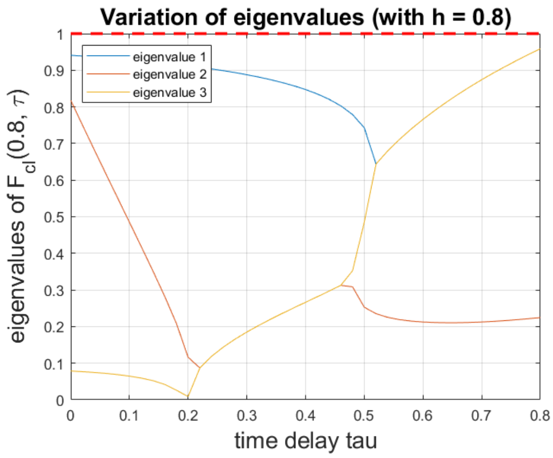
From the figure1, I chose $h = 0.8$ for illustration. The magnitude of three eigenvalues of initial feedback controller based on $h = 0.8$ and different τ is shown in first plot of figure, the maximum eigenvalue stays less then 1 only at the beginning.



(a) Default Controller



(b) Controller(poles(-1+1.3i, -1-1.3i),$K_u = 0.85$)



(c) Controller(poles(-0.36, -3.246),$K_u = 0.3$)

Figure 2: Magnitude of Eigenvalues Under Different Cases

To improve it, I tested different K_u, because the u_{k-1} contain some insight of system at previous timestamp, taking K_u into account might improve the controller, and I found that K_u will "pull" the eigenvalue 3 down but will "raise" the other. Enlarge the distance between the imaginary parts of poles can reduce the "raise" effect, but it's still hard to find a solution that expand range of tolerable delay to whole sampling period. However, I found another result accidentally, when poles have no imaginary parts, the result is quite nice. Then I tested other pairs, and found that if we keep the distance(in real axis) of two poles not so small and poles not too close to imaginary axis, we can improve the controller.

3 Questions 3

3.1 Question 3.1

The time line for large delay is in the figure 3.

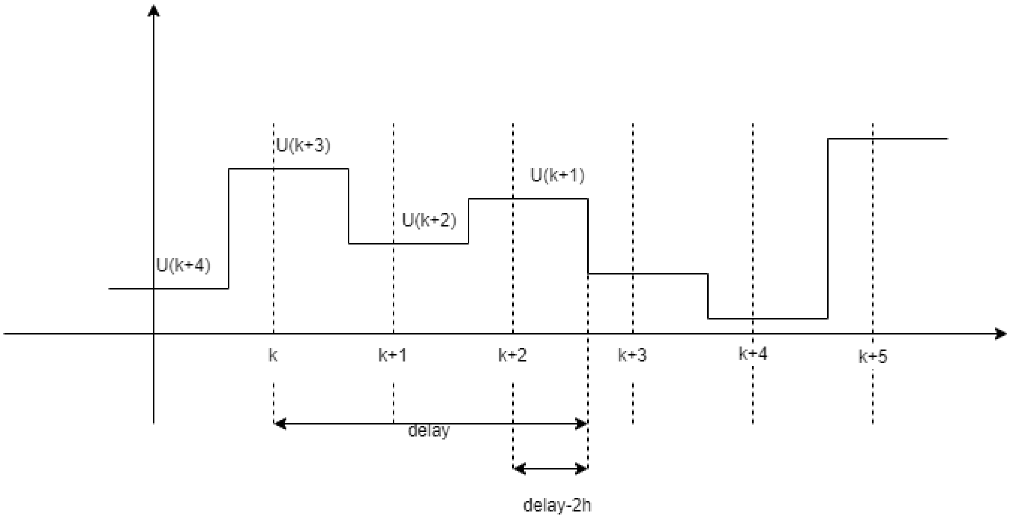


Figure 3: Time Line for Large Delay

Based on this time line we can rewrite the equation as follow:

$$\begin{aligned}
\xi\left(s_{k+1}\right) &= \xi\left(s_k\right) e^{Ah} + \int_{s_k}^{s_{k+1}} e^{A(t-s_k)} B u(k) \mathrm{dt} \\
&= \xi\left(s_k\right) e^{Ah} + \int_{s_k}^{s_k+\delta} e^{A(t-s_k)} B u_{k-3} \mathrm{dt} + \int_{s_k+\delta}^{s_{k+1}} e^{A(t-s_k)} B u_{k-3} \mathrm{dt} \\
&= \xi\left(s_k\right) e^{Ah} + \int_0^{\delta} e^{As} B \mathrm{ds} u_{k-2} + \int_{\delta}^{\tau} e^{As} B \mathrm{ds} u_{k-3}, \text{with } \delta = \tau - 2h
\end{aligned} \tag{8}$$

Then we can rewrite the system model as following:

$$x_{k+1} = e^{Ah} x_k + F_{u_3}(h,\tau) u_{k-3} + F_{u_2}(h,\tau) u_{k-2} + F_{u_1}(h,\tau) u_{k-1} \tag{9}$$

$$\begin{aligned}
F_{u_3}(h,\tau) &= \int_{\tau-2h}^{\tau} e^{As} B \mathrm{ds} \\
F_{u_2}(h,\tau) &= \int_0^{\tau-2h} e^{As} B \mathrm{ds} \\
F_{u_1}(h,\tau) &= 0
\end{aligned} \tag{10}$$

state-augmentation expression $x_{k+1}^e = \begin{bmatrix} x_k^T & u_{k-1}^T & u_{k-2}^T & u_{k-3}^T \end{bmatrix}$:

$$x_{k+1}^e = F(h,\tau) x_k^e + G(h,\tau) u_k \tag{11}$$

$$F(h,\tau) = \begin{bmatrix} e^{Ah} & 0_{2\times1} & F_{u_2}(h,\tau) & F_{u_3}(h,\tau) \\ 0_{1\times2} & 0 & 0 & 0 \\ 0_{1\times2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad G(h,\tau) = \begin{bmatrix} 0_{2\times1} \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

With,

$$
\begin{aligned}
F_{u_3}(h,\tau) &= \begin{bmatrix} \frac{5}{6} - \frac{5}{8}(\mathrm{e}^{\tau-2h}) - \frac{5}{24}(\mathrm{e}^{6h-3\tau}) \\ -1 + \mathrm{e}^{\tau-2h} \end{bmatrix} \\
F_{u_2}(h,\tau) &= \begin{bmatrix} \frac{5}{8}(\mathrm{e}^{\tau-2h} - \mathrm{e}^{\tau}) + \frac{5}{24}(\mathrm{e}^{-3\tau+6h} - \mathrm{e}^{-3\tau}s) \\ \mathrm{e}^{\tau} - \mathrm{e}^{\tau-2h} \end{bmatrix}
\end{aligned} \tag{13}
$$

3.2 Question 3.2

Applying the same controller to the system, and following the same procedure, we can get following result, which is shown in the figure4.

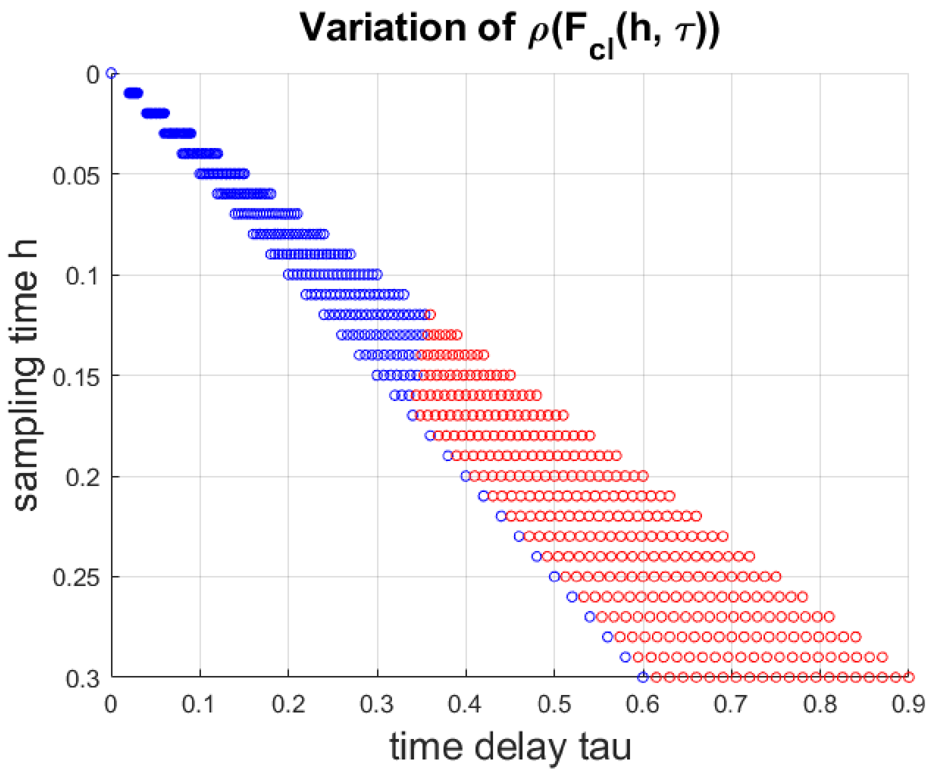
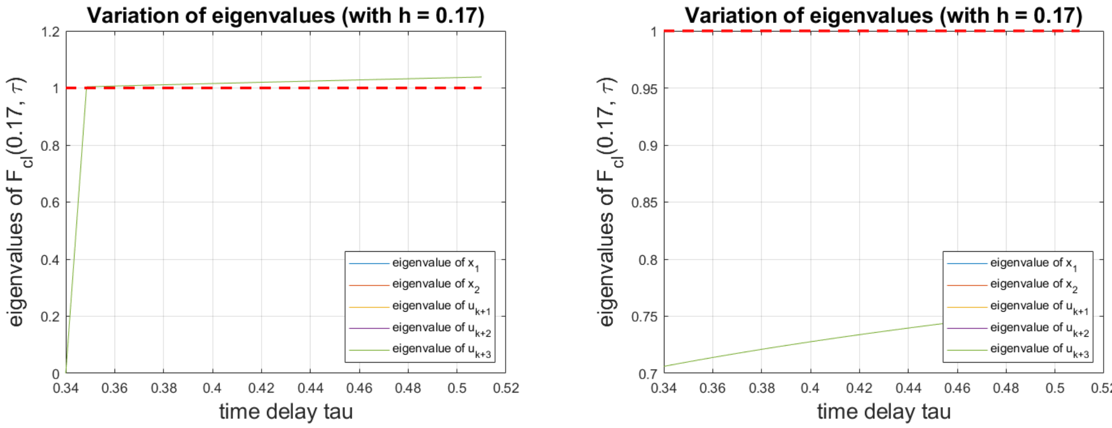


Figure 4: Eigenvalue Comparison under different h and τ with large delay

In above figure, the blue dots represent those eigenvalues are smaller then 1, red dots represent unstable system. We only consider $\tau \in [2h, 3h)$. The result shows the same trend, with larger sampling period(larger than 0.32), system is not stable any more.

3.3 Question 3.3

I chose $h = 0.17$ for illustration.
With too small K_u(below 0.1), the eigenvalues will exceed 1 again, this may because smaller K_u will take too less historical information into account. Therefore, I adjust the K_u to $K_u = [\bar{K}, 0.5, 0.5, 0.5]$so that eigenvalues neither would exceed 1 nor too close to boundary.



(a) Default Controller

(b) Default Controller$(K_u = (0.25, 0.25, 0.25))$

Figure 5: Magnitude of Eigenvalues Under Different Cases

4 Questions 4

4.1 Question 4.1

For a switching system, the system is GES if there exist a matrix $P = P^T$ such that the below LMIs holds for true for any $h \in \mathcal{H}$

$$(F(h) - G(h)\bar{K})^T P (F(h) - G(h)\bar{K}) - P \leq -Q \tag{14}$$

In this switched system, with state definition $x^e_{k+1} = \begin{bmatrix} x_k^T & u_{k-1}^T & u_{k-2}^T \end{bmatrix}$ and period definition $\mathcal{T} = \{0.5h, h, 2h\}$, we should check 3 LMIs:

$$\begin{aligned} \left(F_1(h) - G_1(h)\bar{K}\right)^T P \left(F_1(h) - G_1(h)\bar{K}\right) \mid -P \leq -Q \\ \left(F_2(h) - G_2(h)\bar{K}\right)^T P \left(F_2(h) - G_2(h)\bar{K}\right) - P \leq -Q \\ \left(F_3(h) - G_3(h)\bar{K}\right)^T P \left(F_3(h) - G_3(h)\bar{K}\right) - P \leq -Q \end{aligned} \tag{15}$$

with,

$$F_1(h,\tau)=\begin{bmatrix} e^{-3h} & \frac{5}{8}(e^{-3h}-e^{h}) & \frac{5}{24}(e^{-1.5h}-e^{-3h})+\frac{5}{8}(e^{0.5h}-e^{h}) & 0 & 0 \\ 0 & e^{h} & -e^{0.5h}+e^{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$F_2(h,\tau)=\begin{bmatrix} e^{-3h} & \frac{5}{8}(e^{-3h}-e^{h}) & \frac{5}{6}-\frac{5}{24}(e^{h})-\frac{5}{8}(e^{-3h}) & 0 & 0 \\ 0 & e^{h} & -1+e^{h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$F_3(h,\tau)=\begin{bmatrix} e^{-3h} & \frac{5}{8}(e^{-3h}-e^{h}) & 0 & \frac{5}{6}-\frac{5}{24}(e^{2h})-\frac{5}{8}(e^{-6h}) & 0 \\ 0 & e^{h} & 0 & e^{2h}-1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$
$$G_1(h,\tau)=\begin{bmatrix} -\frac{5}{24}e^{-1.5h}-\frac{5}{8}e^{0.5h}+\frac{5}{6} \\ -1+e^{0.5h} \\ 1 \\ 0 \\ 0 \end{bmatrix} G_2(h,\tau)=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} G_3(h,\tau)=\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} K=\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \tag{16}$$

4.2 Question 4.2

The above system can be checked by using following transformed LMIs. The original LMIs can be rewritten.

$$(F(h)-G(h)\bar{K})^T P (F(h)-G(h)\bar{K}) - P \leq -Q \tag{17}$$

Then applied congruence transformation, with $X = P^{-1} > 0$, $Y = \bar{K}P^{-1}$ and $P^{-1} = (P^{-1})^T$, we have:

$$\left(P^{-1}\right)^T \left(F_i(h)-G_i(h)\bar{K}\right)^T P \left(F_i(h)-G_i(h)\bar{K}\right) P^{-1} - P^{-1}PP^{-1} \leq 0 \tag{18}$$

In practical, we usually transform the right part to $-\epsilon I$. With Schur transformation, we have:

$$\begin{gathered}(F_i(h)X-G_i(h)Y)^T X^{-1} (F_i(h)X-G_i(h)Y) - X \leq -\epsilon I \\ X - (F_i(h)X-G_i(h)Y)^T X^{-1} (F_i(h)X-G_i(h)Y) \geq \epsilon I\end{gathered} \tag{19}$$

$$\begin{bmatrix} X & (F_i(h)X-G_i(h)Y)^T \\ (F_i(h)X-G_i(h)Y) & X \end{bmatrix} > \epsilon \quad X > 0 \tag{20}$$

I tried to solve this problem by doing a search. The possible sampling time selected is $h = [0.2 : 0.005 : 0.3]$, since too large h leads to infeasible solution and too small h is not a optimal option, therefore, I shortened the searching space.

Firstly, I start with different h to generate a possible $\bar{K}$, while satisfying the above LMIs constraints, with $\bar{K} = YX^{-1}$.

Later, for one specific controller $\bar{K}$, the system need to be verified under different h. Finally, the best $\bar{K}$ is chosen if it can cover more h than other controller.
By following this guideline, I was able to find the following parameter would be optimal among other choices. The sampling period is 0.26, the system is validated for $h \in \{0.005, 0.010, ..., 0.260\}$

$$K = [-5.97e - 07, 2.24, 0.56, 0.60, -1.22e - 07] \tag{21}$$

4.3 Question 4.3

The time line for the known sequence system is shown as follow:

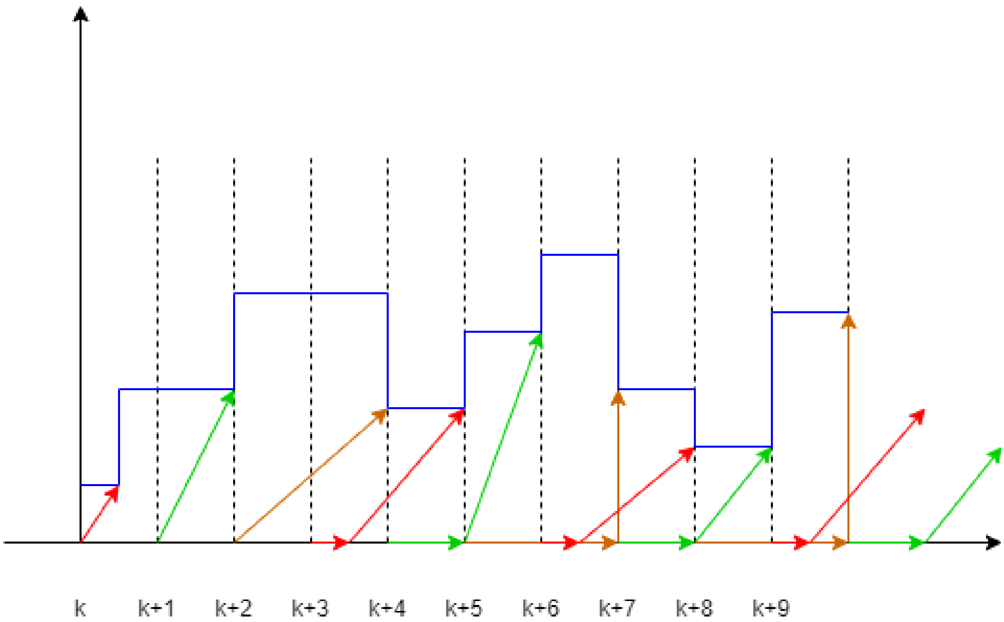


Figure 6: Time Line for Known Sequence System

Assume the system start at k, after first few rounds, we found that all the signal will be send to the system after 2 consecutive periods. Suppose the system evaluates as $x_{k+1} = A(h, \tau)x_k$, we can write the evaluation based on figure6, start at $k + 5$.

$$\begin{aligned} x^e_{s_6} &= A(h, 0.5h)x^e_{s_5} \\ x^e_{s_7} &= A(h, h)x^e_{s_6} \\ x^e_{s_8} &= A(h, 2h)x^e_{s_7} \\ x^e_{s_9} &= A(h, 0.5h)x^e_{s_8} \end{aligned} \tag{22}$$

Rewrite as:

$$x^e_{s_i+3} = A(h, 0.5h)A(h, h)A(h, 2h)x^e_{s_i} \tag{23}$$

Since the system will enter this repetition after finite steps(bounded) then we can ignore them and rewrite the LMI examination as:

$$\bar{A}^T P \bar{A} - P \leq -Q, \bar{A} \quad = A(h, 0.5h)A(h, h)A(h, 2h) \tag{24}$$

4.4 Question 4.4

Since the switched system has a hyper sampling period as 3 intervals, this system can be viewed as a system with constant sampling period and a stitched dynamics, the corresponding matrix is defined as follow:

$$\bar{A} = A(h, 0.5h, K)A(h, h, K)A(h, 2h, K) = g(h, K) \tag{25}$$

And the problem can be converted into a optimization problem as described below:

$$\begin{aligned} \min\nolimits_K \quad & f(h) \\ \text{s.t.} \quad & \rho(g(h, K)) \leq 1 \end{aligned} \tag{26}$$

Since the problem is nonlinear, a searching method is preferable. I applied Simulated Annealing method to search the optimal, and such solution was found.

$$K = [-1.09820.14250.09710.13660.1596], h = 0.6 \tag{27}$$