# Markov Chain with ASEP polynomial stationary distribution

First Proof - Problem 3

AI Mathematician — AI Reviewer
Prompted by
Scott Armstrong, Julia Kempe and Remi Munos

February 12, 2026

**Problem 3 (restated).** Let $\lambda = (\lambda_1 > \cdots > \lambda_n \geq 0)$ be a partition with distinct parts. Assume moreover that $\lambda$ is *restricted*: it has a unique part of size 0 and no part of size 1. Let $S_n(\lambda)$ denote the $S_n$-orbit of $\lambda$ (i.e. all permutations of its parts). Does there exist a nontrivial Markov chain on $S_n(\lambda)$ whose stationary distribution equals

$$\pi(\mu) \;=\; \frac{F^*_\mu(x_1, \ldots, x_n;\, q = 1, t)}{P^*_\lambda(x_1, \ldots, x_n;\, q = 1, t)} \qquad (\mu \in S_n(\lambda)),$$

where $F^*_\mu$ and $P^*_\lambda$ are the interpolation ASEP and interpolation Macdonald polynomials? If so, prove stationarity. "Nontrivial" means the transition probabilities are not described using the polynomials $F^*_\mu$.

**Main conclusion.** For every $n \geq 2$, every restricted $\lambda$, and every fixed $t \in (0, 1)$, there exist positive parameters $x_1, \ldots, x_n$ for which $F^*_\lambda(x; 1, t)/P^*_\lambda(x; 1, t) < 0$. Hence the proposed formula cannot be a stationary *probability* distribution for all positive $x$, so no such Markov chain exists (trivial or nontrivial).

## 1 Interpretation of the problem

The problem statement leaves the role of the parameters $(x_1, \ldots, x_n, t)$ implicit. Two natural readings are:

- **(A) Universal.** Does there exist a Markov chain (with transition rates that may depend on $x$ and $t$) whose stationary distribution equals the proposed formula for *all* positive $x$ and all $t \in (0, 1)$?
- **(B) Existential.** Does there exist *some* $(x, t)$ and *some* Markov chain whose stationary distribution matches the formula at that specific $(x, t)$?

Reading **(A)** is the more natural interpretation, by analogy with the homogeneous case. In the homogeneous setting, Ayyer–Martin–Williams [1] construct a $t$-PushTASEP whose stationary weights are given by the *homogeneous* ASEP polynomials $F_\mu(x; 1, t)$, with partition function $P_\lambda(x; 1, t)$, and this identity holds for all positive $x$ simultaneously. Problem 3 asks whether the same can be done with the *interpolation* $(*)$ polynomials. Under this reading, the problem asks for a single Markov chain description that works uniformly in the parameters, just as the AMW chain does for the homogeneous polynomials.

Under Reading **(A)**, our result gives a complete negative answer (Theorem 4.1 below): there is no such chain, because the proposed formula fails to be nonnegative for certain positive parameter values.

Under the weaker Reading **(B)**, our sign obstruction shows that the formula can only be a valid probability vector in a restricted parameter regime; it does not by itself resolve the question. We adopt Reading **(A)** in what follows.

## 2 Inputs from the literature

We use two standard facts.

**Lemma 2.1** (Top homogeneous parts)**.** *Fix $t$ and set $q = 1$.*

(a) *The interpolation ASEP polynomial $F^*_\mu(x; 1, t)$ has top homogeneous component $F_\mu(x; 1, t)$.*

(b) *The interpolation Macdonald polynomial $P^*_\lambda(x; 1, t)$ has top homogeneous component $P_\lambda(x; 1, t)$.*

*Proof.* Part (a) is stated explicitly in [2, Def. 1.2] ("the top homogeneous part of $F^*_\mu$ is $F_\mu$"). Part (b) is the defining property of interpolation Macdonald polynomials (top homogeneous part equals the usual Macdonald polynomial), as recalled in [2]. □

**Lemma 2.2** (Factorization for packed compositions)**.** *Let $\mu$ be* packed of type $(k, n)$*, i.e. $\mu_i \neq 0$ for $i \leq k$ and $\mu_i = 0$ for $i > k$. Then at $q = 1$,*

$$F^*_\mu(x_1, \dots, x_n; 1, t) \;=\; \Big(\prod_{i=1}^{k}(x_i - t^{1-n})\Big)\, Q_\mu(x_1, \dots, x_n; 1, t),$$

*for some polynomial $Q_\mu$ of degree $|\mu| - k$.*

*Proof.* This is exactly Step 1 of [2, Theorem 3.3], specialized to $q = 1$ and writing $t^{1-n} = t^{-n+1}$. □

## 3 A nonnegativity lemma for the homogeneous quotient

From now on $\lambda = (\lambda_1 > \dots > \lambda_{n-1} > \lambda_n = 0)$ is *restricted*, so in particular $\lambda_i \geq 2$ for $i \leq n-1$ and $\lambda$ is packed of type $(n-1, n)$.

Set $a := t^{1-n}$. Lemma 2.2 yields

$$F^*_\lambda(x; 1, t) = \Big(\prod_{i=1}^{n-1}(x_i - a)\Big)\, Q_\lambda(x; 1, t), \tag{1}$$

for a polynomial $Q_\lambda$ of degree $d := |\lambda| - (n-1)$.

Let $Q_{\lambda,d}$ denote the top homogeneous component of $Q_\lambda$ (degree $d$).

*Remark* 3.1 (On the "vertical adjacency" constraint in multiline diagrams)*.* Definition 5.2 of [1] contains a minor indexing/wording mismatch: it refers to "vertically adjacent" sites and to "lower/upper" sites, but the printed indices appear as $(i, j)$ and $(i, j+1)$. The intended constraint is *vertical*: if two sites in the *same column* in consecutive rows are both occupied by balls, then the label of the lower ball is at least as large as the label of the upper ball. This interpretation is forced by the generation procedure in [1, §5.2] (where a ball directly below an $h$-labelled ball receives label $h$ via a trivial match), and it is also required for consistency with Figure 3 of [1] (e.g. the displayed bottom row $\rho^{(1)}(D) = (4, 0, 1, 5, 3, 0)$ would violate a horizontal weak-decrease condition along adjacent occupied sites). Throughout we adopt this vertical interpretation.

**Lemma 3.2** (Structure and positivity of $Q_{\lambda,d}$)**.** *With notation above:*

(a) *Taking top homogeneous parts in (1) gives*

$$F_\lambda(x; 1, t) = (x_1 \cdots x_{n-1})\, Q_{\lambda,d}(x; 1, t). \tag{2}$$

(b) *For fixed $t \in (0, 1)$, the polynomial $F_\lambda(x; 1, t)$ has nonnegative coefficients in the monomial basis. Consequently $Q_{\lambda,d}$ has nonnegative coefficients in the monomial basis.*

(c) $Q_{\lambda,d}(0, 1, \ldots, 1; 1, t) > 0$.

*Proof. (a)* By Lemma 2.1(a), the top homogeneous component of $F^*_\lambda$ is $F_\lambda$. The top homogeneous component of $\prod_{i=1}^{n-1}(x_i - a)$ is $x_1 \cdots x_{n-1}$. Taking top components in (1) yields (2).

*(b)* Since $\lambda$ has distinct nonzero parts, Theorem 5.5 of [1] gives a *positive* multiline-diagram expansion

$$F_\lambda(x; 1, t) \;=\; \sum_{D:\, \rho^{(1)}(D)=\lambda} \mathrm{wt}(D),$$

where each weight $\mathrm{wt}(D) = \mathrm{wt}_x(D)\mathrm{wt}_t(D)$ has $\mathrm{wt}_x(D) = \prod_{j=1}^{n} x_j^{c_j(D)}$ with $c_j(D)$ the number of balls in column $j$, and $\mathrm{wt}_t(D) > 0$ for $t \in (0, 1)$ (see [1, Def. 5.3]). Therefore every coefficient of $F_\lambda$ in the monomial basis is a sum of positive real numbers, hence nonnegative. Dividing by $x_1 \cdots x_{n-1}$ in (2) preserves coefficientwise nonnegativity, so $Q_{\lambda,d}$ has nonnegative monomial coefficients.

*(c)* We explicitly construct one multiline diagram $D$ of content $\lambda$ and type $\rho^{(1)}(D) = \lambda$ whose $x$-weight contains $x_1$ to the *first* power.

Let $s := \lambda_1$ (so the diagram has $s$ rows) and let $a_r := \#\{i : \lambda_i \geq r\}$ as in [1, Section 5.1]. For every positive species label $h \in \{\lambda_1, \ldots, \lambda_{n-1}\}$ define a column index

$$c(h) := \begin{cases} n, & h = \lambda_1, \\ i, & h = \lambda_i \text{ for } 2 \leq i \leq n-1. \end{cases}$$

Define a ball system $B$ (in the sense of [1, Def. 5.1]) by placing, for each row $r \geq 2$, balls in the columns $\{c(h) : h \geq r\}$, and for row $r = 1$ balls in columns $\{1, 2, \ldots, n-1\}$ (so column $n$ is empty on row 1). This uses exactly $a_r$ balls in row $r$ because $\{h \geq r\}$ has cardinality $a_r$, and all chosen columns are distinct.

Now label each ball in row $r \geq 2$ in column $c(h)$ by $h$ (for each $h \geq r$), so that for $r \geq 2$ every match from row $r + 1$ to row $r$ is *trivial* (the ball with label $h$ sits directly above the unique ball of the same column). Finally, when matching row 2 to row 1, all labels $h \neq \lambda_1$ are forced to match trivially because row 1 has a ball below their column. The unique remaining label $\lambda_1$ sits in row 2, column $n$, above an *empty* site in row 1, column $n$, so its match is nontrivial. Choosing it to match to the ball in row 1, column 1 is allowed and has positive probability/weight: indeed, in cyclic order starting from column $n$ the first available ball in row 1 is column 1, so this choice corresponds to $k = 1$ in [1, Section 5.2] and contributes the local factor $1/[n-1]_t > 0$ in [1, Def. 5.3].

Hence we obtain a valid multiline diagram $D$ with $\rho^{(1)}(D) = \lambda$ and weight $\mathrm{wt}(D) > 0$. By construction, column 1 contains a ball only in the bottom row, so $c_1(D) = 1$ and therefore $\mathrm{wt}_x(D)$ contains $x_1^1$. Thus $F_\lambda$ contains at least one monomial with exponent of $x_1$ equal to 1.

Now (2) implies that $x_1^2 \mid F_\lambda$ if and only if $x_1 \mid Q_{\lambda,d}$. We have just exhibited a monomial of $F_\lambda$ with $x_1$-exponent 1, so $x_1^2 \nmid F_\lambda$, hence $x_1 \nmid Q_{\lambda,d}$. Since $Q_{\lambda,d}$ has nonnegative coefficients by (b), evaluating at $(0, 1, \ldots, 1)$ gives $Q_{\lambda,d}(0, 1, \ldots, 1; 1, t) > 0$. □

# 4 The sign obstruction and nonexistence of a stationary distribution

**Theorem 4.1** (Negative answer to Problem 3 for all $n \geq 2$)**.** *Let $n \geq 2$ and let $\lambda$ be restricted. Fix $t \in (0, 1)$ and set $a = t^{1-n}$. Then there exist $x_1, \ldots, x_n > 0$ such that*

$$\frac{F^*_\lambda(x_1, \ldots, x_n; 1, t)}{P^*_\lambda(x_1, \ldots, x_n; 1, t)} < 0.$$

*Consequently, the family $\{F^*_\mu(x; 1, t)/P^*_\lambda(x; 1, t)\}_{\mu \in S_n(\lambda)}$ cannot be a stationary probability distribution for all positive $x$, and hence no Markov chain on $S_n(\lambda)$ can have it as stationary distribution (for all positive rate parameters).*

*Proof.* Choose

$$x_1 := \frac{a}{2}, \qquad x_2 = \cdots = x_n := R,$$

with $R > a$ to be taken large.

*Step 1: $F^*_\lambda(x; 1, t) < 0$ for large $R$.* From (1),

$$F^*_\lambda(x; 1, t) = \Big(\prod_{i=1}^{n-1}(x_i - a)\Big)\, Q_\lambda(x; 1, t).$$

Here $x_1 - a = -a/2 < 0$ while $x_i - a = R - a > 0$ for $2 \leq i \leq n-1$, so the prefactor $\prod_{i=1}^{n-1}(x_i - a)$ is negative.

It remains to show $Q_\lambda(x; 1, t) > 0$ for large $R$. Write $Q_\lambda = Q_{\lambda,d} + Q_{<d}$ where $Q_{<d}$ has degree at most $d - 1$. Then

$$Q_{\lambda,d}\Big(\frac{a}{2}, R, \ldots, R\Big) = R^d\, Q_{\lambda,d}\Big(\frac{a}{2R}, 1, \ldots, 1\Big) = R^d\Big(Q_{\lambda,d}(0, 1, \ldots, 1) + O(1/R)\Big),$$

and $Q_{<d}(a/2, R, \ldots, R) = O(R^{d-1})$. By Lemma 3.2(c), $Q_{\lambda,d}(0, 1, \ldots, 1) > 0$, so for all sufficiently large $R$ we have $Q_\lambda(a/2, R, \ldots, R) > 0$. Therefore $F^*_\lambda(x; 1, t)$ is negative for all sufficiently large $R$.

*Step 2: $P^*_\lambda(x; 1, t) > 0$ for large $R$.* By Lemma 2.1(b), $P^*_\lambda$ has top homogeneous part $P_\lambda$. At $q = 1$ we have $P_\lambda(x; 1, t) = e_{\lambda'}(x)$ (see [1, Remark 5.7]), which has nonnegative coefficients. Moreover, since $\lambda$ has exactly one 0 part and no 1 part, we have $\lambda'_1 = n - 1$ and hence every elementary factor $e_{\lambda'_i}$ involves only degrees $\leq n - 1$; thus $e_{\lambda'}(0, 1, \ldots, 1) > 0$. Therefore $P_\lambda(a/2, R, \ldots, R) = R^{|\lambda|}\big(P_\lambda(0, 1, \ldots, 1) + O(1/R)\big) > 0$ for $R$ large. Since $P^*_\lambda = P_\lambda +$ (lower degree terms), it follows that $P^*_\lambda(a/2, R, \ldots, R) > 0$ for all sufficiently large $R$.

Combining Steps 1 and 2, for large $R$ we have

$$F^*_\lambda(a/2, R, \ldots, R; 1, t)/P^*_\lambda(a/2, R, \ldots, R; 1, t) < 0.$$

Finally, if a Markov chain has stationary distribution $\pi(\mu)$, then $\pi(\mu) \geq 0$ for all states. We have produced a parameter choice with $\pi(\lambda) < 0$, so such a chain cannot exist for all positive $x$. □

## References


[1] A. Ayyer, J. Martin, and L. Williams. The inhomogeneous $t$-PushTASEP and Macdonald polynomials. *arXiv:2403.10485* (2024).

[2] H. Ben Dali and L. Williams. A combinatorial formula for interpolation Macdonald polynomials. *arXiv:2510.02587* (2025).