EE203 - Electrical Circuits Laboratory

Experiment - 8 Simulation
RLC Circuits

Objectives

1. Understand basics of resonance mechanisms.
2. Observe RLC circuit response.

Background

In the analysis of **RC** and **RL** circuits, the exponential function $e^{-t/\tau}$ came up as the solution of first order differential equations, because derivative of $e^{-t/\tau}$ is given by $e^{-t/\tau}$ itself multiplied by $1/\tau$. When two energy storage devices, such as a capacitor **C** and an inductor **L** are put together, we end up with a <u>second order differential equation that relates a function to its second order derivative</u>. In this case, the solution is given by a sinusoidal function, because

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t) \qquad \frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)$$

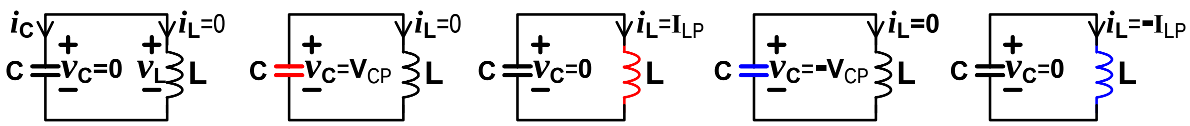
$$\frac{d^2}{dt^2}\sin(\omega t) = \frac{d}{dt}\omega\cos(\omega t) = -\omega^2\sin(\omega t)$$

$$\frac{d^2}{dt^2}\cos(\omega t) = \frac{d}{dt}-\omega\sin(\omega t) = -\omega^2\cos(\omega t)$$

Resonance Mechanisms

All resonance mechanisms transfer energy between two different storage devices. In an **LC** circuit, energy is transferred between an inductor and a capacitor as shown below. The inductor stores energy as a function of current ($W_L = L i_L^2/2$), and the capacitor stores energy as a function of voltage ($W_C = C v_C^2/2$).

Idle state: Resonance state-1: Resonance state-2: Resonance state-3: Resonance state-4:



Idle state	Resonance state-1	Resonance state-2	Resonance state-3	Resonance state-4
$W_C = 0$	$W_C = \frac{1}{2}CV_{CP}^2$	$W_C = 0$	$W_C = \frac{1}{2}CV_{CP}^2$	$W_C = 0$
$W_L = 0$	$W_L = 0$	$W_L = \frac{1}{2}LI_{LP}^2$	$W_L = 0$	$W_L = \frac{1}{2}LI_{LP}^2$

The capacitor voltage can be related to the inductor current by using Kirchhoff's voltage law.

$$v_C(t) = v_L(t) = L\,\frac{di_L(t)}{dt}$$

$$i_C(t) = C\,\frac{dv_C(t)}{dt} = C\,\frac{d}{dt}L\,\frac{di_L(t)}{dt} = LC\,\frac{d^2 i_L(t)}{dt^2}$$

and since $i_C(t) = -i_L(t)$:

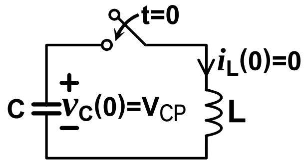
$$\frac{-1}{LC}\,i_L(t) = \frac{d^2 i_L(t)}{dt^2}$$

This is the basic form of differential equation describing ***simple harmonic motion***. A similar differential equation can be obtained for the capacitor voltage:

$$v_C(t) = v_L(t) = L\,\frac{di_L(t)}{dt} = -L\,\frac{di_C(t)}{dt} = -L\,\frac{d}{dt}C\,\frac{dv_C(t)}{dt}$$

$$\frac{-1}{LC}\,v_C(t) = \frac{d^2 v_C(t)}{dt^2}$$

Solution of this differential equation depends on the initial conditions. In the circuit shown on the right, the capacitor voltage is V_{CP} when the switch is closed at **t = 0**, and consequently $v_C(t)$ follows a cosine function.



$$v_C(t) = v_L(t) = V_{CP}\,\cos(\omega_o t)$$

Resonance frequency ω_o is found simply by comparing side by side the differential equation obtained for $v_C(t)$ with the second derivative of $\cos(\omega t)$:

$$\frac{d^2 v_C(t)}{dt^2} = -\frac{1}{LC}\,v_C(t) \qquad \frac{d^2}{dt^2}\cos(\omega_o t) = -\omega_o^2\cos(\omega t)$$

$$\omega_o^2 = \frac{1}{LC}, \qquad \omega_o = \frac{1}{\sqrt{LC}} \text{ in } \mathbf{rad/s}\text{, and}$$

$$f_o = \frac{\omega_o}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \text{ in } \mathbf{Hz}.$$

The energy initially stored in the capacitor is transferred back and forth between the inductor and the capacitor. The oscillations continue forever, since there are no dissipative components in this idealized **LC** circuit. The inductor current $i_L(t)$ can be obtained from $v_C(t)$:

$$i_L(t) = -i_C(t) = -C\,\frac{dv_C(t)}{dt} = -C\,\frac{d}{dt}V_{CP}\cos(\omega_o t) = C\,V_{CP}\,\omega_o\,\sin(\omega_o t)$$

$$= \frac{C\,V_{CP}}{\sqrt{LC}}\sin(\omega_o t) = \sqrt{\frac{C}{L}}\,V_{CP}\sin(\omega_o t)$$

The inductor current can also be obtained based on the conservation of energy:

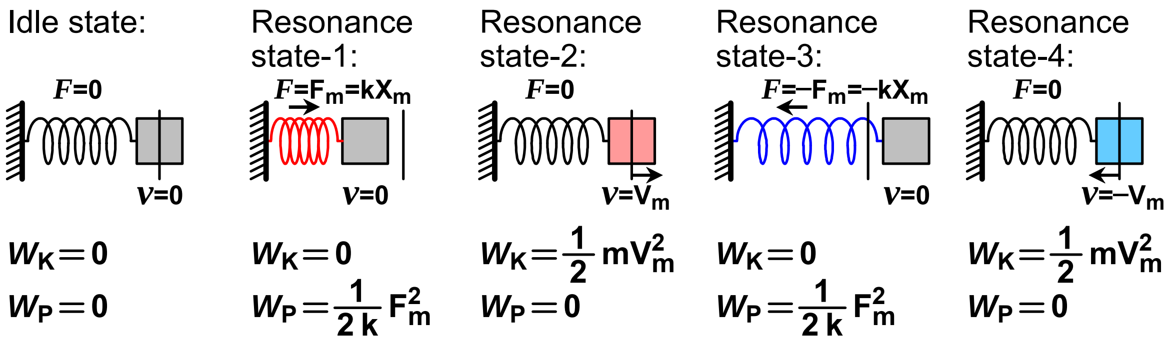
$$\textbf{peak energy stored in } L = \textbf{peak energy stored in } C$$

$$\frac{1}{2}L\,I_{LP}^2 = \frac{1}{2}C\,V_{CP}^2$$

which gives the same result:

$$I_{LP}^2 = \frac{C}{L}V_{CP}^2\,, \text{ or } \quad I_{LP} = \sqrt{\frac{C}{L}}\,V_{CP}$$

The mechanical equivalent of an **LC** resonator is a mass **m** attached to a spring with a spring constant **k**. The moving mass stores kinetic energy $W_K = mv^2/2$, and the spring stores potential energy $W_P = F^2/2k = kx^2/2$, where F is the spring force on the mass, v is the velocity, and x is the position of the mass. Energy is transferred between the two forms as the mass oscillates between $x = -X_m$ and $x = +X_m$ as shown below.



The force $F(t)$ acting on the mass is equal to **m** times the acceleration $a(t)$ which is given by time derivative of the velocity $v(t)$:

$$F(t) = m\,a(t) = m\frac{dv(t)}{dt} \; .$$

$F(t)$ is also related to the position $x(t)$ of the mass through the spring constant **k**:

$$F(t) = -k\,x(t) \quad \text{or} \quad \frac{dF(t)}{dt} = -k\frac{dx(t)}{dt} = -k\,v(t) \; .$$

The equation of simple harmonic motion is obtained when these two equations are combined:

$$-k\,v(t) = \frac{dF(t)}{dt} = \frac{d}{dt}m\frac{dv(t)}{dt} = m\frac{d^2v(t)}{dt^2}$$

$$-\frac{k}{m}v(t) = \frac{d^2v(t)}{dt^2}$$

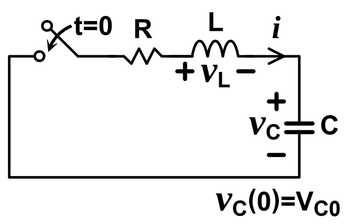
Solution of this equation depends on the initial mass velocity and initial force on the spring. The mechanical resonance frequency appears to be:

$$\omega_o = \sqrt{\frac{k}{m}}$$

Series RLC Circuit

Amplitude of oscillations decay in time or ***dampens*** in the presence of a resistor because of its dissipative nature. The resistor absorbs some of the energy transferred between inductor and capacitor in every oscillation cycle. A differential equation for analysis of the series RLC circuit is obtained by writing voltage across each component in terms of the circuit current to apply Kirchhoff's voltage law.

$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{t'=0}^{t} i(t')\,dt' + V_{C0} = 0$$



After taking derivative and reordering the terms:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

The solution of this differential equation depends on extend of the dissipative behavior in the circuit, which is characterized by a ***damping factor*** defined as $\alpha = R/2L$. Essentially, the time constant τ of the amplitude decay is given by the damping factor **α**, whereas the period T_o of oscillations is given by the resonance frequency ω_o:

$$\tau = \frac{1}{\alpha} = \frac{2L}{R}, \text{ and } \quad T_o = \frac{1}{f_o} = \frac{2\pi}{\omega_o} = 2\pi\sqrt{LC}$$

Comparison between the time constant τ and the period T_o (or **α** and ω_o) determines the series **RLC** circuit response.

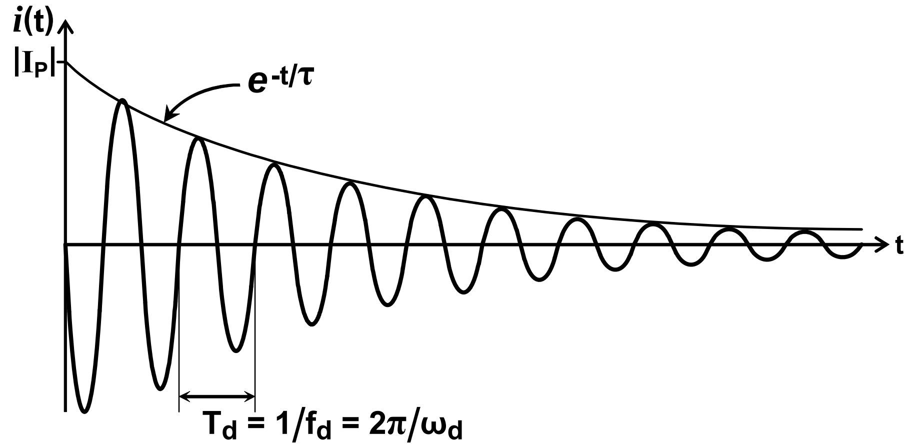
1. $\tau < T_o/2\pi$ (or $\alpha > \omega_o$, or $R > 2\sqrt{L/C}$):
   Amplitude decays in a time shorter than the expected oscillation period since **R** is big enough to dissipate most of the stored energy in one cycle (***overdamped response***). For a spring-mass resonator this corresponds to the case where there is too much friction, so the compressed spring barely pushes the mass to the middle point without any oscillations. In an extreme case, where $R \gg 2\sqrt{L/C}$ the series **RLC** circuit behaves more like a simple **RC** circuit as if **L = 0**.
2. $\tau = T_o/2\pi$ (or $\alpha = \omega_o$, or $R = 2\sqrt{L/C}$): In the step response, the inductor current and the capacitor voltage settle at their steady state values in the shortest possible time without any overshoot or undershoot (***critically damped response***).
3. $\tau > T_o/2\pi$ (or $\alpha < \omega_o$, or $R < 2\sqrt{L/C}$): Amplitude decays in a time longer than the expected oscillation period, and several oscillations are seen until the steady state values are reached (***underdamped response***). The corresponding spring-mass response is obtained when there is too little friction, so that the mass oscillates several times before it settles at the middle point. In each oscillation cycle, some part of the stored energy is dissipated in **R** (or it is lost due to friction), and the magnitude of oscillation decreases at a rate given by the time constant τ.

The underdamped step response of the **RLC** circuit is given by

$$i(t) = I_P \sin(\omega_d t)\, e^{-t/\tau}$$

where ω_d is the ***damped oscillation frequency*** which is slightly lower than the resonance frequency ω_o obtained for the pure **LC** resonator:

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$



The starting peak current I_P can be found according to the initial condition $v_C(0) = V_{C0}$:

- Initially the inductor current is zero because the switch is open until **t = 0**.
- The inductor current is $i_L(0^+) = i(0^+) = 0$ right after the switch is closed at **t = 0**, because current through an inductor had to be continuous.
- Voltage on the resistor is zero since $i(0^+) = 0$ right after the switch is closed. Consequently, $v_L(0^+) = -v_C(0^+)$.

$$v_L(0^+) = L\frac{di(0^+)}{dt} = -v_C(0^+) = V_{C0}\ , \text{ and } \quad I_P = \frac{-1}{\omega_d}\frac{V_{C0}}{L}$$

There are two kinds of common applications that require analysis of **RLC** circuit response. In the first kind of applications, the main purpose is to stabilize signal or power transmission through a cable or PCB trace, where a damped response is required. The second kind of applications involve design of frequency–selective circuits such as tuned radio receivers and reference clock generators. In these applications, ***quality factor Q*** is the most frequently used parameter that determines the frequency selection capability of an **RLC** resonator. Quality factor of a series **RLC** circuit is given by

$$Q = 2\pi\,\frac{\text{maximum energy stored}}{\text{energy lost in one cycle}}$$

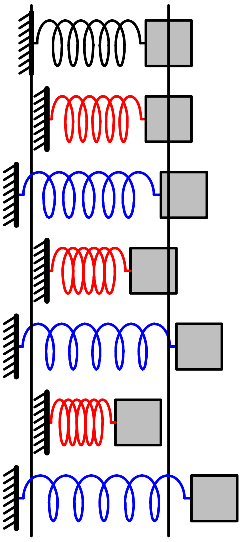
$$Q = \frac{\pi\tau}{T_o} = \frac{\omega_o}{2\alpha} = \frac{\omega_o L}{R} = \frac{1}{\omega_o RC} = \frac{\sqrt{L/C}}{R}$$

Typical **Q** factors that can be obtained by using ordinary components are in the order of **100**. Quartz crystal resonators on the other hand, can provide **Q** factors higher than **10,000**, and they are preferred in precision clock generators.

Ideal inductors and capacitors does not exist in reality, and all reactive components have some loss factors. Following are the major loss factors that result in power dissipation in reactive components.

- All conductors have some finite resistance, and the inductor is made of a conductor wire. The power dissipated on the resistance of the inductor wire is called ***copper loss*** or I^2R ***loss***.
- An inductor with a ferromagnetic core cannot return all of the energy it stores because of the ***core losses***.
- A capacitor cannot return all of the energy it stores because of the ***dielectric losses***.
- All insulators have some finite resistance, and the resistance of dielectric material used in a capacitor results in a ***leakage resistance*** that slowly drains the stored charge.

RLC Circuit Driven by Sinusoidal Source



Consider a special case of the spring-mass resonator where the spring is attached to an **oscillating wall** as shown on the right. Initially the mass is at rest, and the spring is relaxed. The wall first compresses the spring, and it pushes the mass forward as in the second figure. In the next step, the wall expands the spring as the mass reaches its final point before it starts to move backwards. If the frequency of wall oscillations matches the resonance frequency, then the magnitude of mass motion increases in each cycle.

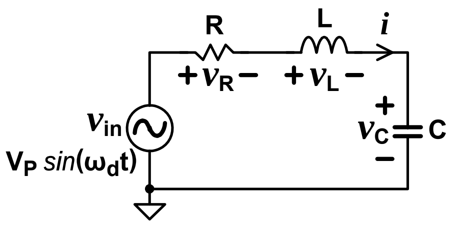
If there is no friction in the system, then magnitude of the oscillations will keep rising until the spring is fully compressed or until it is broken. In the presence of friction, power dissipation in the resonator will increase with the magnitude of the oscillations. The resonator will reach a steady state, where the energy dissipated due to friction is equal to the energy received from the wall in each cycle.

A series RLC circuit driven by a sinusoidal voltage source behaves in a similar way. The sinusoidal source is analogous to the oscillating wall in the spring-mass example given above. The peak capacitor voltage and the peak inductor current keep rising until the power dissipated on the resistor becomes equal to the power received from the voltage source. If the source frequency is exactly the same as the **RLC** oscillation frequency, then at steady state:

$$\begin{array}{c} \text{power received} \\ \text{from source} \end{array} = \begin{array}{c} \text{power dissipated} \\ \text{on resistor} \end{array}$$

$$P_{src}(t) = v_{in}(t)\, i(t) = P_R(t) = v_R(t)\, i(t)$$

$$v_{in}(t) = v_R(t)$$



Kirchhoff's voltage law states:

$$v_R(t) + v_L(t) + v_C(t) = v_{in}(t)$$

$$v_L(t) + v_C(t) = 0$$

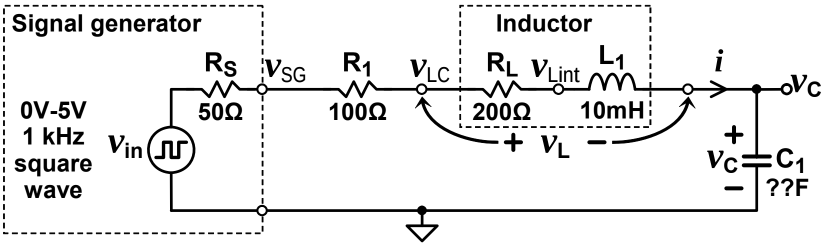
In practice, $v_L(t) + v_C(t)$ drops near **0 V**, but it is not possible to see exactly **0 V** because of the loss factors in real capacitors and inductors. Assuming **L** and **C** are lossless components, the steady state values of $v_L(t)$ and $v_C(t)$ are given by

$$v_L(t) = L\frac{di(t)}{dt} = L\frac{d}{dt}\frac{v_{in}(t)}{R} = L\frac{d}{dt}\frac{V_P}{R}\sin(\omega_d t) = \frac{\omega_d L}{R} V_P \cos(\omega_d t)$$

$$v_C(t) = -v_L(t) = -\frac{\omega_d L}{R} V_P \cos(\omega_d t) = \frac{-1}{\omega_d RC} V_P \cos(\omega_d t)$$

Preliminary Work

1. In the following circuit internal resistance of the inductor is shown as $\mathbf{R_L = 200\ \Omega}$. Output resistance $\mathbf{R_S = 50\ \Omega}$ of the signal generator and $\mathbf{R_L}$ should be included in the calculations when they are comparable to the external resistance $\mathbf{R_1}$.



1.a) Calculate the capacitance $\mathbf{C_1}$ that gives the resonance frequency $\mathbf{f_o = 50\ kHz}$.

1.b) Calculate the damped oscillation frequency $\mathbf{f_d = \omega_d/2\pi}$, and the decay time constant $\boldsymbol{\tau}$ for $\mathbf{R_1 = 0\ \Omega}$, **100 Ω**, and **1 kΩ**.

1.c) Draw $i(t)$ and $v_C(t)$ as a function of time after v_{in} switches from **5 V** to **0 V**, for $\mathbf{R_1 = 100\ \Omega}$.

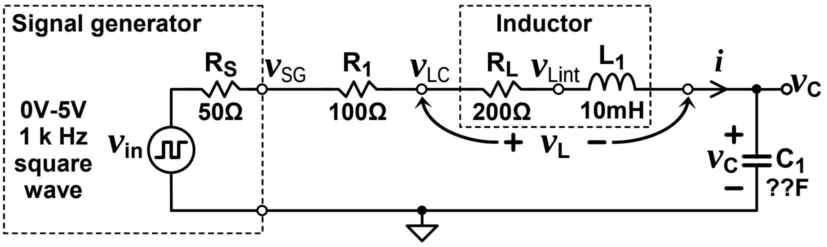
1.d) Calculate the $\mathbf{R_1}$ value that gives the critically damped response.

2. Consider the **RLC** circuit given above when $v_{in} = 2.5\text{V}\ \mathit{sin}(\omega_d t)$.

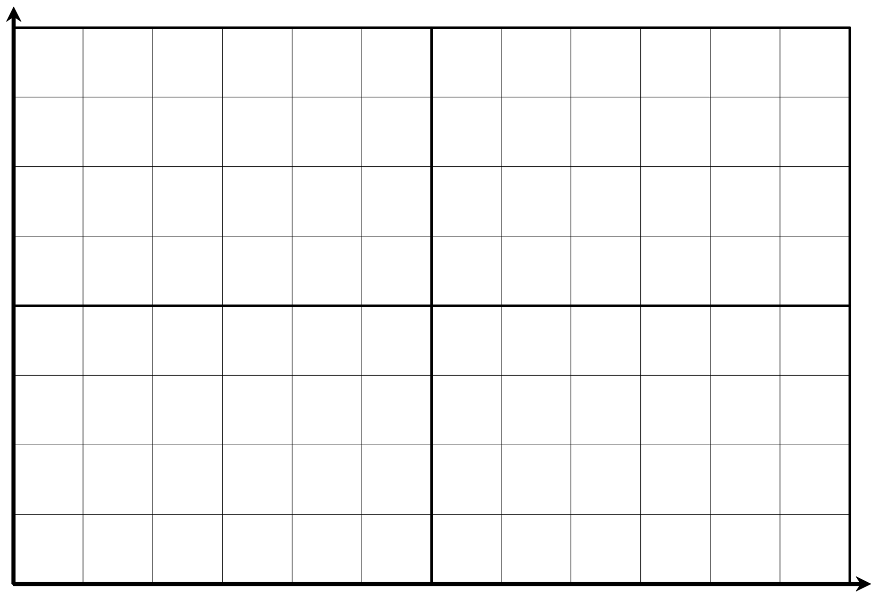
Calculate steady state $i(t)$, $v_C(t)$, and $v_{LC}(t)$ when $\mathbf{R_1 = 0\ \Omega}$, **100 Ω**, and **1 kΩ**.

Procedure

1. Set up the circuit given below using the capacitance **C_1** calculated in the preliminary work and set the v_{in} signal source to obtain **1 kHz** square wave that switches between **0 V** and **5 V** with the timing parameters, `Trise = 1n`, `Tfall = 1n`, `Ton = 500u`, and `Tperiod = 1m`. Set simulation time to **1 ms**.



1.1 Display v_{in}, v_C**(t)** and ***i*****(t)** waveforms and plot v_C**(t)** and ***i*****(t)** waveforms.

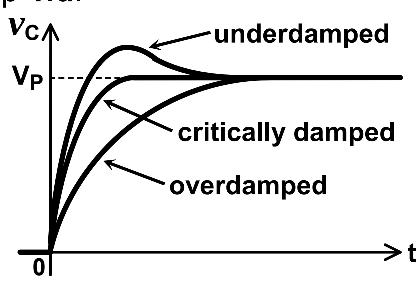


1.2 Measure the v_C oscillation frequency $\mathbf{f_d = 1/T_d}$, and the decay time constant $\boldsymbol{\tau}$ for the **R_1** values given in the following table.

R_1	f_d (Hz)	τ (μs)
0 Ω		
100 Ω		
1 kΩ		

1.3 Observe ***i*****(t)** and v_C**(t)** for several **R_1** values around the resistance calculated for critically damped response in the preliminary work step **1.d**.

Determine the **R_1** resistance that gives the critically damped response by changing **R_1** in **0.1 kΩ** steps. Monitor v_C as you increase the **R_1** resistance until the overshoot at the rising v_C edge disappears. Zoom into the v_C waveform to see the circuit response clearly.



R_1 = __________

R_1 + 250 Ω = __________

2.b) Simulate the circuit in LTspice and verify your calculations.

- Observe $i(t)$, $v_C(t)$, and $v_{LC}(t)$ for **R_1 = 0 Ω**, **100 Ω**, and **1 kΩ**, and compare the steady state peak values with your calculations.

2. Setup the circuit given for step-**1** with **R_1 = 0 Ω**, and set the v_{in} signal source to obtain **5 V**p-p sine wave at the resonance frequency found in step-**1**.

2.1 Observe v_C waveform and change the signal generator frequency in **100 Hz** steps to obtain the maximum peak-to-peak voltage at v_C.

Maximum v_C = __________ at frequency = __________

2.2 Observe $v_{Lint}(t)$ (voltage between R_L and **L** in the inductor), and find the v_{in} frequency that makes steady state amplitude of v_{Lint} **= 0**. Start with the frequency found in **2.1** and change v_{in} frequency in **50 Hz** steps.

Frequency where v_{Lint} **= 0** = __________

2.3 Measure the following peak-to-peak voltage and current values while the signal source voltage and frequency settings are kept the same.

R_1	v_{SG} (Vp-p)	v_{LC} (Vp-p)	i (mAp-p)	v_C (Vp-p)
0 Ω				
100 Ω				
1 kΩ				

Questions

Q1. Compare the oscillation frequency f_d, and the decay time constant τ measured in procedure step-**1.2** with values calculated in the preliminary work.

Q2. Compare the resistance found in procedure step-**1.3** with the resistance calculated for critically damped response in the preliminary work.

Q3. Compare the peak-to-peak voltage and current values measured in procedure step-**2** with the results found in the preliminary work (remember the difference between ***peak-to-peak*** and ***peak*** values).

Q4. How is it possible to obtain a capacitor voltage much higher than the signal generator output in procedure step-**2**?

Q5. The same resonance frequency can be obtained when L_1 and C_1 are replaced by $10\,L_1$ and $0.1\,C_1$, or by $0.1\,L_1$ and $10\,C_1$ in procedure step-**2**. In which of these cases a higher $v_C(t)$ amplitude is obtained? Why?