EE203 - Electrical Circuits Laboratory

Experiment - 7 Simulation
RC and RL Circuits

Objectives

1. Understand behavior of storage devices.
2. Investigate RC and RL circuit response to switching voltage sources.

Background

Resistors are ***dissipative*** components that convert electrical energy into heat. Capacitors and inductors are ***reactive*** components that store energy to give it back later. As a result of this storage function, voltage and current on these components appear as time derivative or time integral of each other. The following section summarizes the physical principles that establish the link between the natural behavior of capacitors and inductors and the mathematical analysis of **RC** and **RL** circuits.

Exponential Function

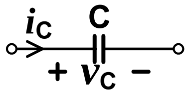
The irrational number ***e*** **= 2.718281828459...** is not just an answer to a mathematical puzzle. In fact, the number ***e*** and the related exponential function ***e***e^x are dictated by the mother nature as a mathematical representation for common behavior of the storage devices. In case of electrical circuits, a capacitor stores electric charge that increases proportionally with the voltage across the capacitor:

$$v_C(t) = \frac{Q(t)}{C}$$

Q = stored charge in Coulombs (coul)
C = capacitance in Farads (F)

Accumulated capacitor charge is simply given by the current (Ampere = coul/s) through the capacitor integrated over time.

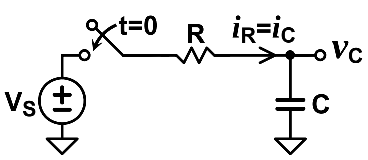
$$v_C(t) = \frac{1}{C}\left(Q(0) + \int_{t'=0}^{t} i_C(t')\,dt'\right) = v_C(0) + \frac{1}{C}\int_{t'=0}^{t} i_C(t')\,dt'$$



Consequently, capacitor current i_C is given by time derivative of v_C:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

The capacitor shown on the right is connected to a constant voltage source V_S when the switch is closed at **t = 0**. Behavior of the capacitor in this circuit can be described as follows.



- Rate of change of capacitor voltage dv_C/dt depends on the current i_C.
- Difference between V_S and v_C determines $i_R = i_C$. Amount of current decreases as v_C gets closer to V_S.

This behavior results in the following first order differential equation.

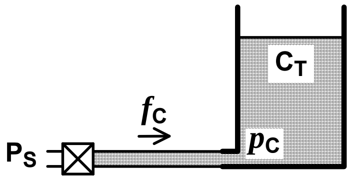
$$i_C(t) = \frac{V_S - v_C(t)}{R} = C\frac{dv_C(t)}{dt}\ , \quad \text{or} \quad V_S - v_C(t) = RC\frac{dv_C(t)}{dt}$$

There are other physical processes that present the same behavior:

Hydraulics:

A water tank with the capacity $\mathbf{C_T}$ is connected to a constant pressure source $\mathbf{P_S}$ through a thin pipe and a valve. The valve at the pressure source opens at **t = 0**.

- Rate of change of tank pressure dp_C/dt is given by the flow rate f_C through the pipe.
- Difference between $\mathbf{P_S}$ and p_C determines the flow rate f_C depending on the pipe resistance $\mathbf{R_{pipe}}$.

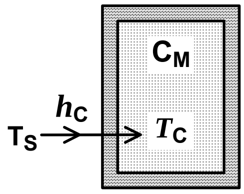


$$f_C = \frac{P_S - p_C}{R_{pipe}} = C_T\frac{dp_C}{dt}$$

Thermal conduction:

A container is filled with a material that has the heat capacity $\mathbf{C_M}$. The container is placed in a medium at constant temperature $\mathbf{T_S}$ at **t = 0**.

- Rate of change of temperature in the container dT_C/dt is given by heat tranfer h_C through the container walls.
- Difference between $\mathbf{T_S}$ and T_C determines the heat tranfer h_C depending on the thermal resistance $\mathbf{\theta_T}$ of the container.

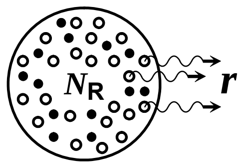


$$h_C = \frac{T_S - T_C}{\theta_T} = C_M\frac{dT_C}{dt}$$

Radioactive decay:

A radioactive nucleus becomes a stable element releasing a photon with the decay probability of $\mathbf{p_d}$ in unit time. Radiation rate r is the number of photons released per unit time. The number of remaining radioactive nuclei is N_R.

- Rate of change of number of radioactive nuclei dN_R/dt is given by the radiation rate r.
- Radiation rate r is proportional to remaining N_R at any time.



$$r = p_d N_R = -\frac{dN_R}{dt}$$

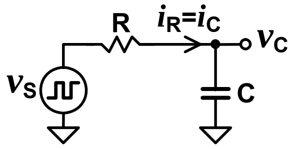
All of these physical processes result in the same form of differential equation that relates a function, v_C, p_C, T_C, or N_R to the time derivative of the function itself multiplied by a constant. <u>The exponential function $e^{-t/\tau}$ is the solution for all of these differential equations, because derivative of e^t is given by e^t itself</u>:

$$\frac{de^t}{dt} = e^t\text{, and following the chain rule of differentiation, } \frac{de^{-t/\tau}}{dt} = \frac{1}{\tau}e^{-t/\tau}.$$

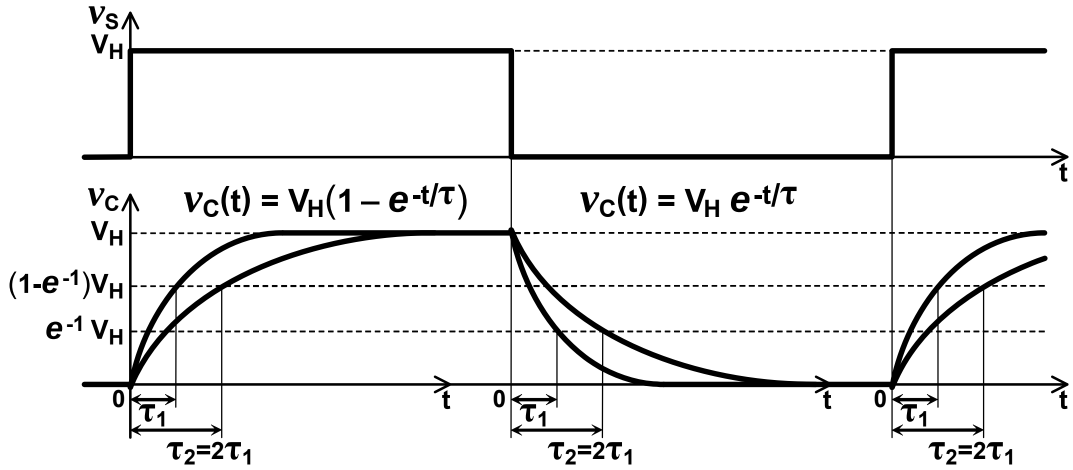
The **time constant τ** determines the decay rate of the exponential functions and it is given by the components involved in the process.

RC Circuits

For a simple **RC** circuit as shown on the right, the time constant is given by $\boldsymbol{\tau} = \mathbf{RC}$. A bigger resistance or a bigger capacitance results in a longer time constant, which means it will take a longer time to charge or discharge the capacitor.



Following figure shows $v_C(t)$ for two different time constants while v_S switches between **0 V** and $\mathbf{V_H}$. The capacitor voltage is given by $v_C(t) = V_H(1 - e^{-t/\tau})$ after v_S switches from **0 V** to $\mathbf{V_H}$, and by $v_C(t) = V_H e^{-t/\tau}$ after v_S switches back to **0 V**.

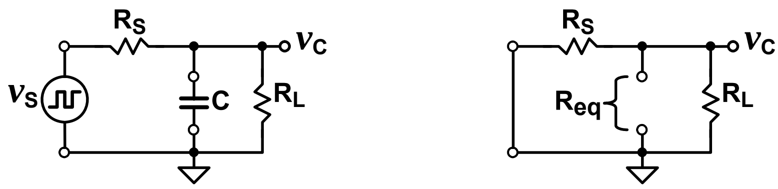


A generalized solution for an **RC** circuit driven by a switching voltage source is

$$v_C(t) = V_\infty - (V_\infty - V_O)e^{-t/\tau}$$

where V_O is the initial voltage on the capacitor at $t = 0$, and V_∞ is the final voltage expected to be seen on the capacitor at $t = \infty$ or when $t \gg \tau$. V_O and V_∞ can be determined easily in a typical application, knowing that the expected circuit response is a single exponential function.

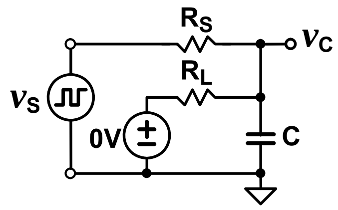
If there are multiple resistors and a single capacitor in an **RC** circuit, then the time constant is given by an equivalent resistor R_{eq}. R_{eq} can be obtained by calculating the resistance seen by the capacitor when the independent sources in the circuit are disabled. In the following example, the time constant is $\tau = (R_S//R_L)C$, where "//" sign means equivalent value of the two resistors connected in parallel.



$R_{eq} = R_S//R_L$

$\tau = R_{eq}C$

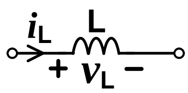
If you have any doubt about including R_L in the time constant calculation, then you should consider the equivalent circuit given on the right. Connecting R_L to ground is not any different then connecting it to a **0 V** source. Although v_S is the only active source in the circuit, as a matter of fact the capacitor is driven by the Thevenin equivalent of v_S and **0 V** voltage sources.



RL Circuits

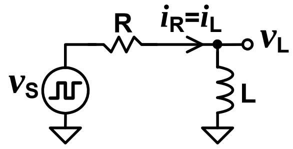
Inductors are the other type of energy storage devices that complement function of capacitors in electrical circuits. Time-varying current through an inductor generates a time-varying magnetic field in the inductor, and in turn, this time-varying magnetic field induces a voltage across the inductor. Consequently, voltage on the inductor is proportional to the time derivative of the current:

$$v_L(t) = L\frac{di_L(t)}{dt}, \qquad i_L(t) = i_L(0) + \frac{1}{L}\int_{t'=0}^{t} v_L(t')\,dt'$$

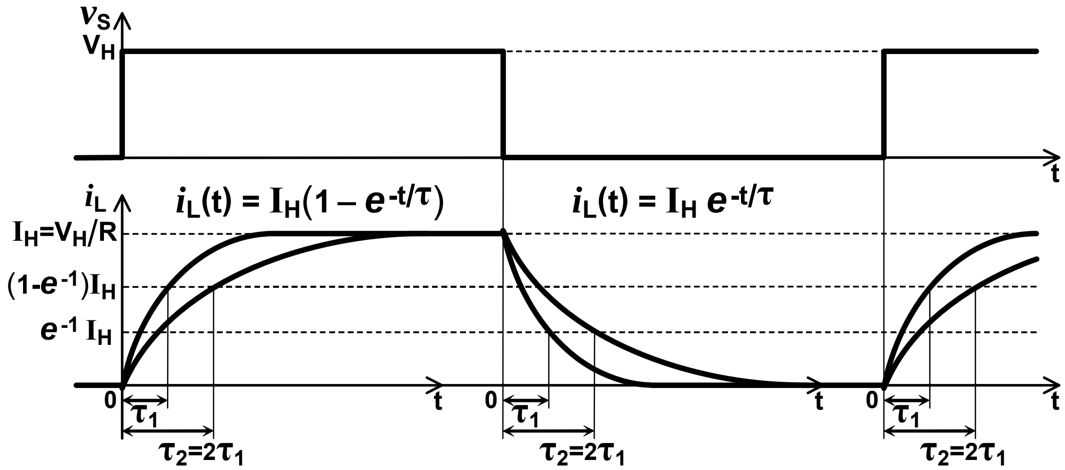


where, **L** is the inductance in henries (**H**). Analysis of simple **RL** circuits is similar to the analysis of **RC** circuits that require solution of first order differential equations.

The **RL** circuit response to a switching voltage source is also given by an exponential function.



For a simple **RL** circuit as shown on the right, the time constant is given by $\boldsymbol{\tau = L/R}$. A smaller resistance or a bigger inductance results in a longer time constant, which means it will take a longer time for i_L to reach its steady state value.



A generalized solution for an **RL** circuit driven by a switching voltage source is

$$i_L(t) = I_\infty - (I_\infty - I_O)e^{-t/\tau}$$

where I_O is the initial current through the inductor at **t = 0**, and I_∞ is the final current value expected at $\mathbf{t = \infty}$ or when $\mathbf{t \gg \tau}$.

Similar to the **RC** circuits, the time constant is given by an equivalent resistor $\mathbf{R_{eq}}$ when there are multiple resistors and a single inductor in an **RL** circuit. $\mathbf{R_{eq}}$ can be obtained by calculating the resistance seen by the inductor when the independent sources in the circuit are disabled.

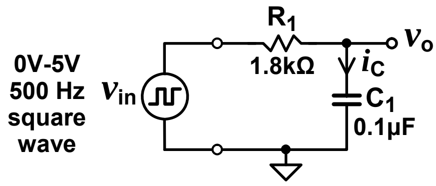
Important characteristics of capacitors and inductors are summarized in the following table.

Capacitor	**Inductor**
$i_C(t) = C\dfrac{dv_C(t)}{dt}$	$v_L(t) = L\dfrac{di_L(t)}{dt}$
If $v_C(t)$ is constant then $i_C(t)$ is zero.	If $i_L(t)$ is constant then $v_L(t)$ is zero.
$v_C(t)$ had to be continuous, because any jump in $v_C(t)$ would require infinite $i_C(t)$.	$i_L(t)$ had to be continuous, because any jump in $i_L(t)$ would require infinite $v_L(t)$.
Single **RC** response for a switching source: $v_C(t) = V_\infty - (V_\infty - V_O)e^{-t/\tau}$ $\tau = RC$	Single **RL** response for a switching source: $i_L(t) = I_\infty - (I_\infty - I_O)e^{-t/\tau}$ $\tau = L/R$
Energy stored in a capacitor: $W_C = \frac{1}{2} C\, v_C^2$	Energy stored in an inductor: $W_L = \frac{1}{2} L\, i_L^2$

Preliminary Work

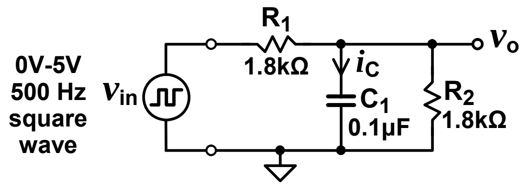
Note: Assume that capacitor voltage and inductor current reach their final steady state values before the next v_{in} transition, since τ is much shorter than period of v_{in} waveform in all questions given below.

1.a) Calculate the time constant $\tau = R_1C_1$ for the circuit given below and write $v_o(t)$ as a function of time after v_{in} switches between **0 V** and **5 V**.

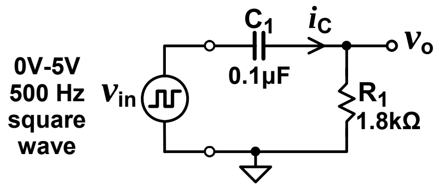


1.b) Draw $v_{in}(t)$ and $v_o(t)$ on the same plot. Calculate the output voltages at $t = \tau$ on the rising and falling v_o waveforms and mark these voltage levels on the plot.

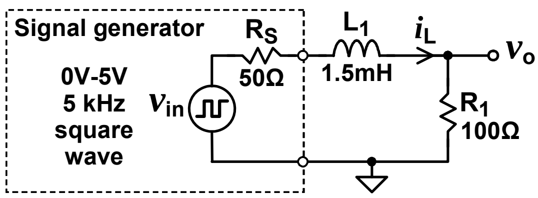
2. Calculate the time constant τ and the output function $v_o(t)$ after v_{in} switches between **0 V** and **5 V** in the following circuit. Find the output voltages at $t = \tau$ on the rising and falling v_o waveforms.



3. Calculate the time constant τ and the output function $v_o(t)$ after v_{in} switches between **0 V** and **5 V** in the following circuit. Find the output voltages at $t = \tau$ on the rising and falling v_o waveforms.

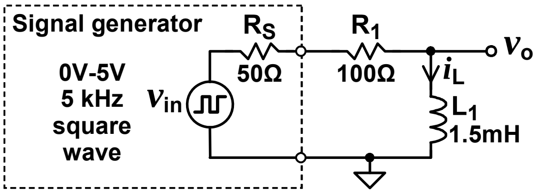


4.a) Calculate the time constant $\tau = L_1/(R_S+R_1)$ for the circuit given below and write $v_o(t)$ as a function of time after v_{in} switches between **0 V** and **5 V**. Note that output resistance R_S **= 50 Ω** of the signal generator should be taken into account since it is comparable to R_1.

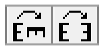
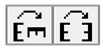


4.b) Draw v_{in}, i_L, and v_o on the same plot. Calculate i_L and v_o at $t = \tau$ after v_{in} switches between **5 V** and **0 V**, and mark these signal levels on the plot.

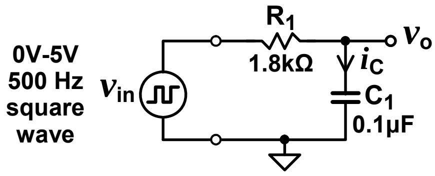
5. Calculate and plot $\boldsymbol{i}_\mathrm{L}$ and $\boldsymbol{v}_\mathrm{o}$ as $\boldsymbol{v}_\mathrm{in}$ switches between **0 V** and **5 V** in the following circuit. Mark the $\boldsymbol{i}_\mathrm{L}$ and $\boldsymbol{v}_\mathrm{o}$ values at **t = $\boldsymbol{\tau}$** on the plot. Simulate the circuit in LTspice and verify your calculations.



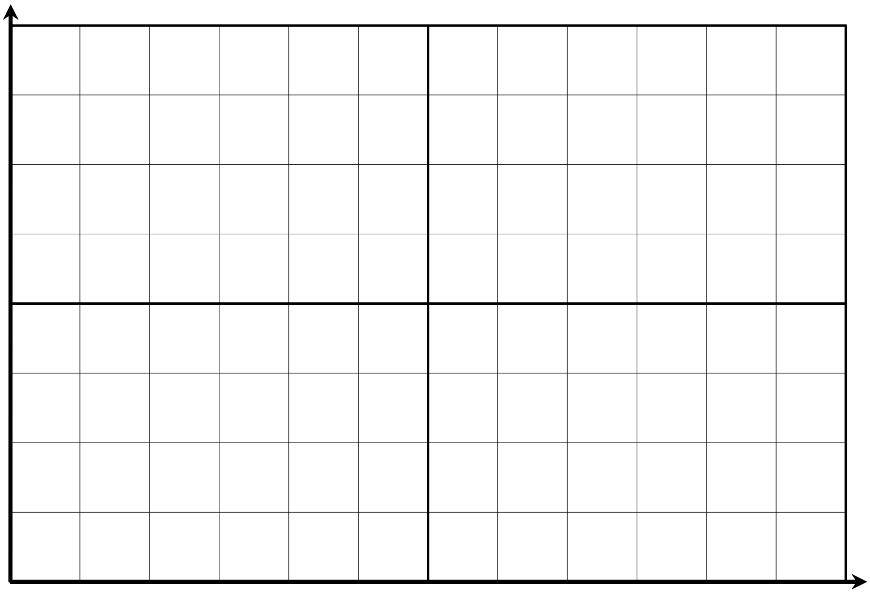
Procedure

Note: Make sure that current waveforms have correct polarities in LTspice. Verify the current direction looking at the current probe  that appears when the cursor moves over a component. If it is necessary to reverse the current direction, then grab the component with  tool and move it over  buttons. Click on  buttons to change the component orientation until you obtain the correct polarity.

1. Build the circuit given on the right and set v_{in} source to obtain **500 Hz** square wave that switches between **0 V** and **5 V**.



1.1 Display v_{in} and v_o waveforms and plot them.



1.2 Record the high and low voltage levels at v_o.

V_H = __________ V_L = __________

1.3 Measure the time constant on rising and falling v_o waveforms. Zoom into the v_o waveform to obtain the best possible timing accuracy.

τ_{rise} = __________ τ_{fall} = __________

On rising exponential: $v_o(\tau_{rise}) = V_L + (V_H - V_L)(1 - e^{-1}) = V_L + 0.632\,(V_H - V_L)$

If $V_L = 0$: $v_o(\tau_{rise}) = V_H(1 - e^{-1}) = 0.632\,V_H$

On falling exponential: $v_o(\tau_{fall}) = V_L + (V_H - V_L)e^{-1} = V_L + 0.368\,(V_H - V_L)$

If $V_L = 0$: $v_o(\tau_{fall}) = V_H\,e^{-1} = 0.368\,V_H$

As an alternative you can measure the time $T_{1/2}$ where,

$v_o(T_{1/2}) = V_L + (V_H - V_L)/2$.

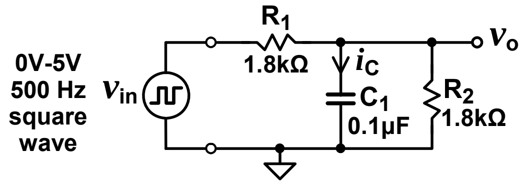
Then $\tau_{rise} = T_{1/2}/ln(2) = 1.443\,T_{1/2}$ and $\tau_{fall} = 1.443\,T_{1/2}$.

1.4 Compare the measured values with those calculated in the preliminary work.

1.5 Apply **5 V**p-p sinusoidal waveform as v_{in} and record the peak-to-peak output voltage at the following frequencies.

v_{in} frequency	v_o (**V**p-p)
500 Hz	
5 kHz	
50 kHz	

2. Add **R**$_2$ **= 1.8 kΩ** in parallel to the capacitor on the circuit used in step **1**.



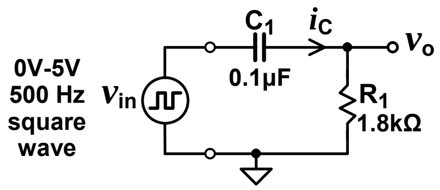
2.1 Record the high and low voltage levels at v_o, and repeat the time constant measurements.

V_H = __________ V_L = __________

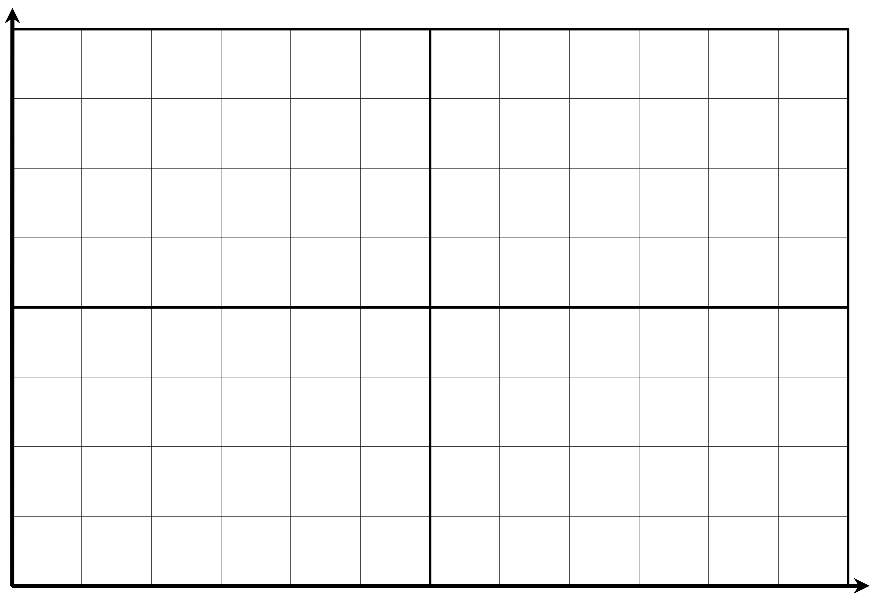
τ_{rise} = __________ τ_{fall} = __________

2.2 Compare the measured values with those calculated in the preliminary work.

3. Set up the circuit given below.



3.1 Display v_{in} and v_o waveforms and plot them. Measure initial voltage, final voltage and time constant of the v_o waveform when v_{in} **= 0 V** and v_{in} **= 5 V**. Indicate the measured values on the plot.

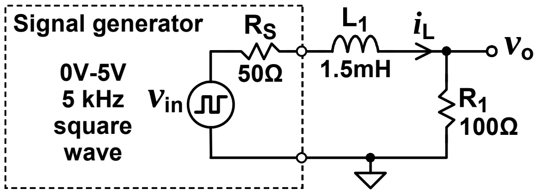


3.2 Compare the measured values with those calculated in the preliminary work.

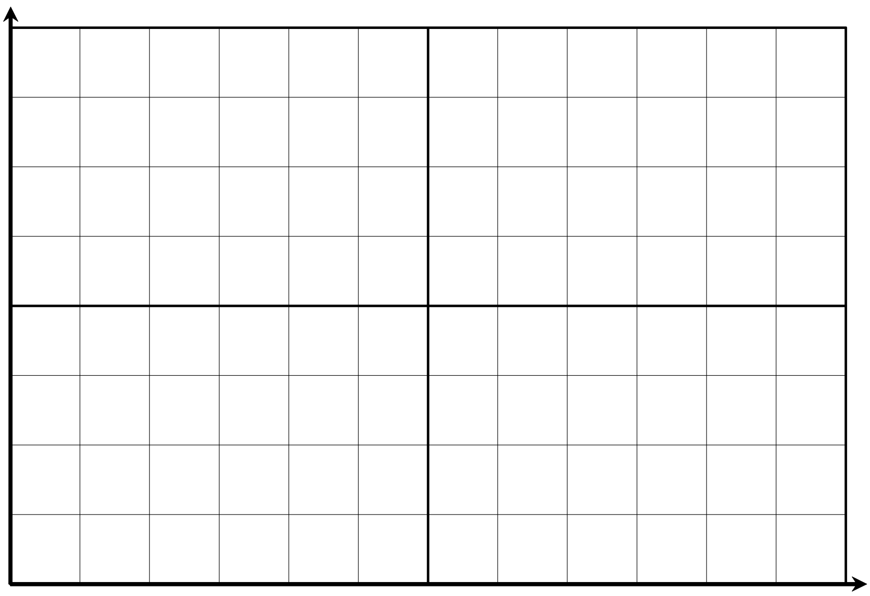
3.3 Apply **5 V**p-p sinusoidal waveform as v_{in} and record the peak-to-peak output voltage at the following frequencies.

v_{in} frequency	v_o (**V**p-p)
500 Hz	
5 kHz	
50 kHz	

4. Set up the circuit given below. Adjust the signal source to obtain **5 kHz**, **0-5 V** square wave. The source resistance **R_S = 50Ω** is added to obtain a realistic model of the laboratory signal generator.



4.1 Display v_{in} and v_o waveforms and plot them. Measure initial voltage, final voltage and time constant of the v_o waveform when v_{in} **= 0 V** and v_{in} **= 5 V**. Indicate the measured values on the plot.

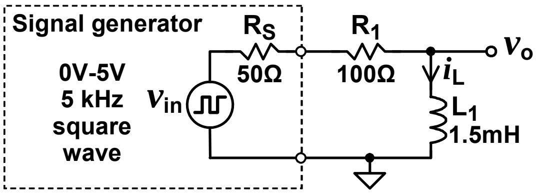


4.2 Compare the measured values with those calculated in the preliminary work.

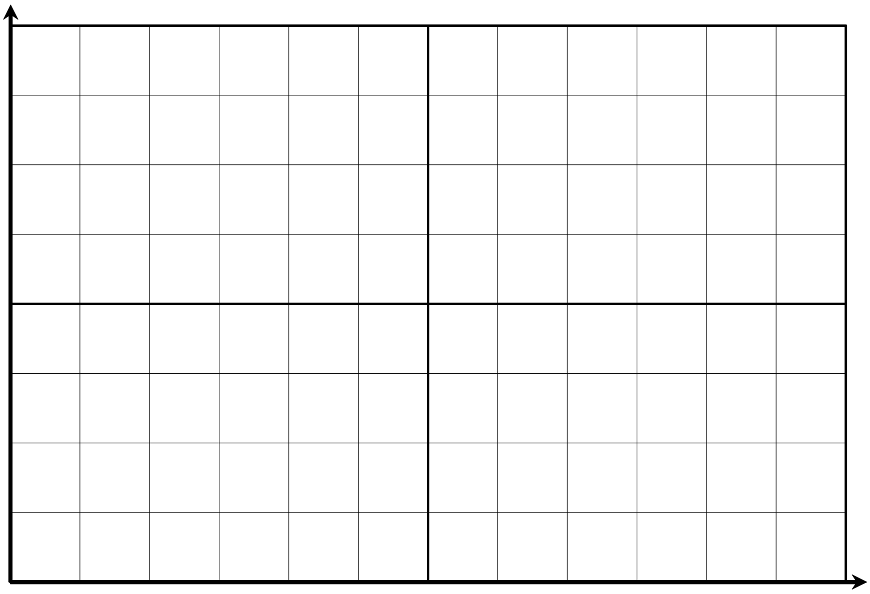
4.3 Set the v_{in} signal source to obtain **5 V**p-p sinusoidal waveform. Record the peak-to-peak v_o output voltage at the following frequencies.

v_{in} frequency	v_o (**V**p-p)
5 kHz	
50 kHz	
500 kHz	

5. Set up the circuit given below. Set the v_{in} signal source to obtain **5 kHz**, **0-5 V** square wave.



5.1 Display v_{in} and v_o waveforms and plot them. Measure initial voltage, final voltage and time constant of the v_o waveform when v_{in} **= 0 V** and v_{in} **= 5 V**. Indicate the measured values on the plot.



5.2 Compare the measured values with those calculated in the preliminary work.

5.3 Apply **5 V**p-p sinusoidal waveform as v_{in} and record the peak-to-peak output voltage at the following frequencies.

v_{in} frequency	v_o (**V**p-p)
5 kHz	
50 kHz	
500 kHz	

Questions

Q1.a) Calculate power on $\mathbf{C_1}$ as a function of time in the circuit used for procedure step **1** when v_{in} **= 0 V** and v_{in} **= 5 V**.

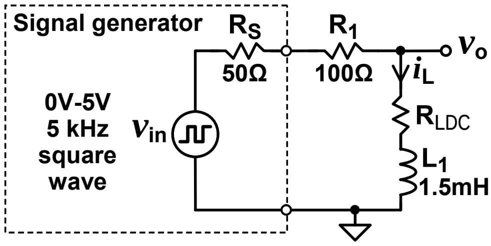
b) Simulate the circuit (preliminary work part **1**) and plot power on $\mathbf{C_1}$ to verify your calculations.
Right-click on the waveform window and select **Add Traces** option. Enter an expression that multiplies output voltage with $\mathbf{C_1}$ current. Make sure that $\mathbf{C_1}$ current has the correct direction.

c) What is the meaning of positive power and negative power on a capacitor?

Q2. Simulate the same circuit in step **1** when v_{in} is **5 V**p-p sinusoidal source with **500 Hz**, **5 kHz**, and **50 kHz** frequency. Compare the simulation results with the measurements in procedure step **1.5**.

Q3. Negative voltage peaks are observed at the output in procedure steps **3** and **5**. How is it possible to obtain negative output voltages when there is no negative voltage source in a circuit?

Q4. In procedure step **5**, you observed an offset at v_o when v_{in} **= 5 V**. Calculate the final value of v_o at the end of exponential decay after including the measured DC resistance $\mathbf{R_{LDC}}$ of the inductor as shown on the right. Compare your calculation with the measured DC level in step **5.1.**

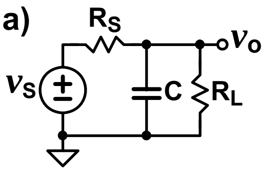


Q5. Based on your observations in the experiment, write the expected steady state output voltage in each of the following circuits for two extreme cases:

Case 1: v_S is a constant, DC voltage source

Case 2: v_S is a high-frequency (**HF**) source such as **sin(2π f t)**, where **f ≫ 1/τ**.

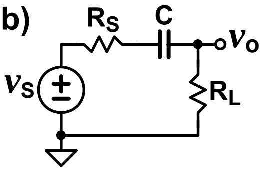
a)



DC v_S ⇒

HF v_S ⇒

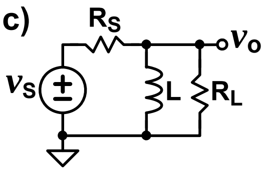
b)



DC v_S ⇒

HF v_S ⇒

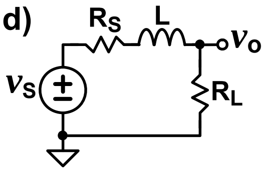
c)



DC v_S ⇒

HF v_S ⇒

d)



DC v_S ⇒

HF v_S ⇒

Hint: Answer is either $v_o = 0$ or $v_o \simeq v_S \frac{R_L}{R_S+R_L}$