Optimizing $f(x_1, x_2) = 2x_2^2 + 3x_2^4 + 5x_1^2 + 3$

subject to $2x_1 - x_1^2 = 5$

- Assuming $g(x_1, x_2) = 2x_1 - x_1^2$

then $\nabla g(x_1, x_2) = \nabla(2x_1 - x_1^2) = \begin{bmatrix} -2x_1 + 2 \\ 0 \end{bmatrix}$

$$\Delta f(x_1, x_2) = \begin{bmatrix} 10x_1 \\ 4x_2 + 12x_2^3 \end{bmatrix}$$

Now using tangency to solve the constrained problem:

$$\nabla f(x_1, x_2) = \lambda \nabla g(x_1, x_2)$$

$$\Rightarrow \begin{bmatrix} 10x_1 \\ 4x_2 + 12x_2^3 \end{bmatrix} = \lambda \begin{bmatrix} -2x_1 + 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 10x_1 = -2\lambda x_1 + 2\lambda \Rightarrow (10 + 2\lambda)x_1 - 2\lambda = 0 \quad \dots(1) \\ 4x_2 + 12x_2^3 = 0 \Rightarrow x_2 = 0 \quad \dots(2) \\ \text{but} \Rightarrow 2x_1 - x_1^2 = 5 \Rightarrow x_1^2 - 2x_1 + 5 = 0 \quad \dots(3) \end{cases}$$

we should solve the above 3 equations

From (3): $x_1 = 1 \mp 2i$ we apply that to check equation(1)

First $(10 + 2\lambda)(1 + 2i) - 2\lambda = 0$

$10 + 20i + 2\lambda + 4\lambda i - 2\lambda = 0$

$(20 + 4\lambda)i + 10 = 0 \Rightarrow 20 + 4\lambda = 0 \Rightarrow \lambda = -5$

second $(10 + 2\lambda)(1 - 2i) - 2\lambda = 0 \Rightarrow -20 - 4\lambda = 0 \Rightarrow \lambda = -5$

~~Solution is $(1+2i, 0)$ and at that tangent~~

(2) $f(x_1, x_2) = x_1^3 + x_1^4 + x_2 + x_2^2$

subject to $x_1^2 + x_2^2 = 0$

Using tangency to solve a contstrained problem

- Assuming $g(x_1, x_2) = x_1^2 + x_2^2$

$$\nabla g(x_1, x_2) = \nabla(x_1^2 + x_2^2) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} 3x_1^2 + 4x_1^3 \\ 1 + 2x_2 \end{bmatrix}$$

Assuming that x_{1m}, x_{2m} are the solution to the problem, then the gradients of $f(x_1, x_2)$ and $g(x_1, x_2)$ are the same victor with constant multplier λ; oR: $f(x_1, x_2) = \lambda g(x_1, x_2)$ λ is a constant

$$\begin{bmatrix} 3x_1^2 + 4x_1^3 \\ 1 + 2x_2 \end{bmatrix} = \lambda \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

OR
$$\begin{cases} 3x_1^2 + 4x_1^3 = \lambda(2x_1) \quad \cdots (1) \\ 1 + 2x_2 = \lambda(2x_2) \quad \cdots (2) \end{cases}$$

but $\Rightarrow x_1^2 + x_2^2 = 0 \quad \cdots\cdots (3)$

From (1) $4x_1^2 + 3x_1 - 2\lambda = 0 \quad \cdots\cdots (1')$

From (2) $1 + x_2(2 - 2\lambda) = 0 \quad \cdots\cdots (2')$

From (1'): $\Delta = 9 - 4(4)(-2\lambda) = \sqrt{9 + 32\lambda}$

$$x_1 = \frac{-3 \pm \sqrt{9 + 32\lambda}}{2(4)}$$

From (2'): $x_2 = \dfrac{-1}{2 - 2\lambda}$

(1) If $x_1 = 0$ then $x_2 = 0$ and λ can take any value and equation (2) won't satisfy

(2) If $x_1 \neq 0$ then $\left(\dfrac{-3 \mp \sqrt{9 + 32\lambda}}{8}\right)^2 + \left(\dfrac{-1}{2 - 2\lambda}\right)^2 = 0$

(3) OR $$\frac{9}{64}+\frac{9+32\lambda}{64}+\frac{-3+\sqrt{9+32\lambda}}{4}+\frac{1}{4+4\lambda^2-8\lambda}=0$$

and $$\frac{9}{64}+\frac{9+32\lambda}{64}+\frac{-3-\sqrt{9+32\lambda}}{4}+\frac{1}{4+4\lambda^2-8\lambda}=0$$

The obove equations must be solved for λ:

$$\frac{18}{64}+\frac{32}{64}\lambda-\frac{3}{4}+\frac{\sqrt{9+32\lambda}}{4}+\frac{1}{4+4\lambda^2-8\lambda}=0$$

$$\frac{\lambda}{2}+\frac{\sqrt{9+32\lambda}}{4}+\frac{1}{4+4\lambda^2-8\lambda}=\frac{-9}{32}+\frac{3}{4}$$

we multiply all by (2):

$$\lambda+\frac{1}{2}\sqrt{9+32\lambda}+\frac{1}{2+2\lambda^2-4\lambda}=\frac{-9}{16}+\frac{3}{2}$$

$$\lambda+\frac{1}{2}\sqrt{9+32\lambda}+\frac{1}{2+2\lambda^2-4\lambda}=\frac{15}{16}$$

- Using Matlab to find λ value we get:
  Solutions should include parameters and Conditions., specify the 'Return Conditions' value as 'true'.
- I used python to solve the problem with two different already-prepared methods in scipy.optimize library.