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THE EXPONENTIAL PARITY THEOREM FOR TPπ2 NUMBERS: STRUCTURAL CONSTRAINTS IN TWIN PRIME FORMATION

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Abstract
We introduce TPπ2 numbers, defined as $N = 2^{e_2} \times 3^{e_3}$, for positive integers e_2 and e_3, where N is bracketed by twin primes, and present the Exponential Parity Theorem, which establishes that for $N > 6$, the exponents e_2 and e_3 must sum to an odd number. Through analysis of 81 known TPπ2 examples with exponents ranging up to $(e_2, e_3) = (13895, 6868)$, we demonstrate how this parity constraint provides new structural insights into twin prime distribution. The theorem eliminates 50% of potential candidates through modular arithmetic restrictions, offering computational advantages for twin prime searches. Remarkably, while individual TPπ2 numbers become exponentially rarer at larger scales, the underlying mathematical structure suggests unbounded continuation—a beautiful paradox that illuminates fundamental principles of prime distribution and the tension between local sparsity and global unboundedness.

1. Introduction

The study of twin primes—pairs of primes $(p, p + 2)$—represents one of number theory's most enduring challenges [1]. While the Twin Prime Conjecture remains unresolved, recent advances have established bounded gaps between consecutive primes [2, 3], reinvigorating research into the distribution and structure of prime pairs.

We introduce a novel approach by studying numbers of the form $N = 2^{e_2} \times 3^{e_3}$ that are bracketed by twin primes, which we term "TPπ2 numbers." We designate "π-complete numbers" as those integers with a contiguous set of prime factors down to 2, and "Order-2" of these have exactly the two prime factors, 2 and 3, so TPπ2 numbers are Order-2 π-complete numbers that are the average of pairs of twin primes. These 3-smooth numbers provide a structured subset for investigating twin prime formation, analogous to how smooth numbers have proven valuable in factorization algorithms and cryptographic applications.

Our main contribution is the Exponential Parity Theorem, establishing fundamental constraints on the exponents e_2 and e_3 for TPπ2 numbers. Combined with empirical analysis of 81 known examples, this work reveals a striking mathematical paradox: while TPπ2 numbers become exponentially rarer as we explore larger scales, they appear to continue indefinitely. This "sparsity-unboundedness paradox" provides both theoretical insights into prime distribution and practical computational advantages for twin prime research, while illuminating deep principles about how mathematical objects can be simultaneously vanishing and inexhaustible.

2. Definitions and Main Results

Definition 1. A positive integer N is called a *TPπ2 number* if $N = 2^{e_2} \times 3^{e_3}$ for positive integers e_2, e_3, and both $N - 1$ and $N + 1$ are prime.

Theorem 1 (Exponential Parity Theorem)**.** *Let $N = 2^{e_2} \times 3^{e_3}$ with $N > 6$. If N is a TPπ2 number, then $e_2 + e_3$ must be odd.*

Proof. We demonstrate that e_2 and e_3 cannot both be even or both be odd by analyzing those two cases.
Case 1: e_2 and e_3 are both even. If $e_2 = 2k$ and $e_3 = 2m$ for integers $k, m \geq 1$, then $N = (2^k \times 3^m)^2$, making N a perfect square. Consequently, $N - 1$ can be expressed as:

$$N - 1 = (2^k \times 3^m)^2 - 1 = (2^k \times 3^m - 1)(2^k \times 3^m + 1).$$

Both factors are integers greater than 1 for $N > 6$, so $N - 1$ is composite, contradicting the requirement that $N - 1$ be prime.
Case 2: e_2 and e_3 are both odd. We analyze the last digit of N modulo 10. Powers of 2 modulo 10 cycle every 4: $2, 4, 8, 6$. For odd e_2, the last digit is 2 when $e_2 \equiv 1 \pmod 4$ and 8 when $e_2 \equiv 3 \pmod 4$. Powers of 3 modulo 10 cycle every 4: $3, 9, 7, 1$. For odd e_3, the last digit is 3 when $e_3 \equiv 1 \pmod 4$ and 7 when $e_3 \equiv 3 \pmod 4$.

Multiplying these patterns:

$$(2 \text{ or } 8) \times (3 \text{ or } 7) \in \{6, 14, 24, 56\} \pmod{10}$$

$$2 \times 3 = 6,\ 2 \times 7 = 14,\ \text{which is } 4\ (mod 10)$$

$$8 \times 3 = 24 \text{ which is } 4\ (mod 10),\ 8 \times 7 = 56 \text{ which is } 6\ (mod 10)$$

Thus N ends in either 4 or 6 when both exponents are odd.

If N ends in 4, then $N + 1$ ends in 5 and is divisible by 5. If N ends in 6, then $N - 1$ ends in 5 and is divisible by 5. Since $N > 6$, both $N \pm 1 > 5$, so divisibility by 5 implies compositeness, contradicting the twin prime condition (except in the special case where $N = 6$ causing $N - 1 = 5$ and 5 is the only number divisible by 5 that is also prime).

Therefore, for $N > 6$, $e_2 + e_3$ must be odd for N to be the average of consecutive primes. □

Remark 1 (The Computational Rarity Paradox)**.** While Theorem 1 eliminates exactly half of all potential TPπ2 candidates through the parity constraint, the remaining candidates become exponentially rarer as their magnitude increases. Yet our dataset spanning over 7,000 orders of magnitude suggests no mathematical terminus. This creates a fascinating tension: each new discovery becomes exponentially harder to achieve, yet infinitely many discoveries remain theoretically possible.

3. Empirical Data and Analysis

The following table presents our dataset of 81 known TPπ2 numbers, organized by magnitude. This dataset was generated by the Python program listed in Appendix A, which generates test values of N by all partitions of a given sum $e_2 + e_3$. To get to 81 ordered pairs, all partitions of sums through 25,500 were tested. Although generated by exponent sum, the values were subsequently sorted by $log(N) = e_2 \times log(2) + e_3 \times log(3)$ to arrange the list by magnitude of N. The dataset spans from small examples like $(e_2, e_3) = 2^1 \times 3^1$, corresponding to $N = 6$, up to $2^{13895} \times 3^{6868}$ representing numbers with over 7000 decimal digits.

Table 1. Complete dataset of known TPπ2 exponent pairs (e_2, e_3)

Small Examples ($\log N < 12$): (1, 1), (2, 1), (1, 2), (3, 2), (2, 3), (6, 1), (4, 3), (7, 2), (5, 4), (6, 7)

Medium Examples ($12 < \log N < 32$): (3, 10), (18, 1), (12, 5), (2, 15), (18, 5), (21, 4), (24, 5), (27, 4), (30, 7), (33, 8)

Large Examples ($32 < \log N < 248$): (43, 2), (32, 9), (36, 7), (11, 24), (31, 12), (43, 8), (32, 15), (50, 9), (63, 2), (66, 25), (79, 20), (99, 10), (57, 64), (82, 63), (148, 27), (63, 88), (56, 99), (211, 2), (275, 16), (287, 10)

Very Large Examples ($248 < \log N < 1900$): (90, 169), (148, 135), (298, 51), (160, 141), (363, 52), (134, 231), (49, 320), (529, 44), (264, 419), (960, 143), (541, 476), (988, 207), (1015, 332), (1440, 97), (1295, 324), (979, 738), (258, 1493), (637, 1320)

Exceptional Examples ($\log N > 1900$): (2320, 333), (1036, 1167), (2815, 188), (1063, 1440), (180, 2251), (888, 2033), (300, 2819), (2176, 2175), (4014, 1879), (280, 4311), (6228, 571), (6981, 560), (3505, 2892), (5899, 2570), (1538, 5505), (9646, 2829), (5211, 5794), (10031, 2956), (2560, 9423), (2821, 9960), (17480, 4317), (22156, 1545), (13895, 6868)

3.1. Verification of Parity Constraint

Every exponent pair (after the first) in our dataset satisfies the parity constraint $e_2+e_3 \equiv 1 \pmod 2$. As an empirical test, the generator program was run up to the exponential sum of 2000 checking both even and odd sums. In line with the proof above, no even numbered sums were found to produce a TPπ2 number. While, $(1,1)$ gives $1+1=2$ (even), this corresponds to $N=6$, which is the exceptional case; all other pairs have odd sums.

3.2. Growth Patterns

The dataset reveals several interesting growth patterns:
Unbounded growth: Both e_2 and e_3 achieve arbitrarily large values, with the largest being $e_2 = 22156$ and $e_3 = 6868$.
Ping-pong dynamics: Examples alternate between 45 e_2-dominated cases ($e_2 \times \log 2 > e_3 \times \log 3$) and 36 e_3-dominated cases, suggesting a moderate systematic preference for $e_2 \times \log 2$.
Nonexclusive Pairs: A given value of either e_2 or e_3 may appear more than once with another partner. Examples include: (1, 1), (1, 2), (2, 1), (2, 3), (2, 15), (3, 2), (3, 10), (6, 1), (6, 7), (18, 1), (18, 5), (32, 9), (32, 15), (43, 2), (43, 8), (63, 2), (63, 88), (148, 27), (148, 135), (1, 1), (2, 1), (6, 1), (18, 1), (1, 2), (3, 2), (7, 2), (43, 2), (63, 2), (211, 2), (2, 3), (4, 3), (5, 4), (21, 4), (27, 4), (12, 5), (18, 5), (24, 5), (6, 7), (30, 7), (36, 7), (33, 8), (43, 8), (32, 9), (50, 9), (3, 10), (99, 10), (287, 10), (2, 15), (32, 15)

Although we only have a few examples, it appears that a given value of either e_2 or e_3 may form multiple TPπ2 numbers with different partners, perhaps without bound.

Analysis of Multiple Partners:

The table reveals several interesting patterns in the partner structure:

- The exponent $e_3 = 2$ appears most frequently with 6 different partners, suggesting it may be particularly "fertile" for twin prime formation.
- Small values of e_2 and e_3 tend to have more partners, which may reflect the greater density of twin primes in smaller ranges.
- The largest partnership span occurs for $e_3 = 2$ with partners ranging from $e_2 = 1$ to $e_2 = 211$, representing a span of over 200 in exponent values.

Table 1: TPπ2 exponents with multiple partners

Fixed Exponent	**Partner Exponents**	**Count**
Fixed e_2 values		
$e_2 = 1$	$e_3 \in \{1, 2\}$	2
$e_2 = 2$	$e_3 \in \{1, 3, 15\}$	3
$e_2 = 3$	$e_3 \in \{2, 10\}$	2
$e_2 = 6$	$e_3 \in \{1, 7\}$	2
$e_2 = 18$	$e_3 \in \{1, 5\}$	2
$e_2 = 32$	$e_3 \in \{9, 15\}$	2
$e_2 = 43$	$e_3 \in \{2, 8\}$	2
$e_2 = 63$	$e_3 \in \{2, 88\}$	2
$e_2 = 148$	$e_3 \in \{27, 135\}$	2
Fixed e_3 values		
$e_3 = 1$	$e_2 \in \{1, 2, 6, 18\}$	4
$e_3 = 2$	$e_2 \in \{1, 3, 7, 43, 63, 211\}$	6
$e_3 = 3$	$e_2 \in \{2, 4\}$	2
$e_3 = 4$	$e_2 \in \{5, 21, 27\}$	3
$e_3 = 5$	$e_2 \in \{12, 18, 24\}$	3
$e_3 = 7$	$e_2 \in \{6, 30, 36\}$	3
$e_3 = 8$	$e_2 \in \{33, 43\}$	2
$e_3 = 9$	$e_2 \in \{32, 50\}$	2
$e_3 = 10$	$e_2 \in \{3, 99, 287\}$	3
$e_3 = 15$	$e_2 \in \{2, 32\}$	2

- Partnership patterns appear to persist across different magnitude scales, with both small exponents (like $e_2 = 2$) and large exponents (like $e_2 = 148$) exhibiting multiple partners.

These partnership patterns suggest that certain exponent values may possess special properties that enhance twin prime formation when combined with various complementary exponents. This phenomenon warrants further investigation as it may reveal deeper structural constraints beyond the basic parity requirement.

4. Implications for Twin Prime Research

4.1. Computational Advantages

The Exponential Parity Theorem provides immediate computational benefits:
Search space reduction: Eliminates 50% of candidate (e_2, e_3) pairs through parity filtering.
Structured generation: Enables systematic generation of TPπ2 candidates by

alternating exponent parities.
Parallel processing: The constraint structure facilitates efficient multiple CPU-core primality testing.

For searches above $N > 10^{1000}$, this represents a substantial reduction in computational workload.

4.2. Theoretical Contributions

4.2.1. Connection to Hardy-Littlewood Conjecture

The TPπ2 structure provides a concrete testing ground for the Hardy-Littlewood conjecture [1]. If $\pi_2(x)$ denotes the number of twin primes up to x, the conjecture predicts:

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2}$$

where $C_2 \approx 0.6602$ is the twin prime constant. To modify this for TPπ2 numbers, a factor would be needed that represents the probability that N would have prime factors, 2 and 3, and only those prime factors, and that the sum of the exponents of those factors would be an odd number (and an overall $+1$ to include the special case of $N = 6$).

The TPπ2 structure provides a concrete testing ground for refined Hardy-Littlewood estimates that incorporate smooth number density and sieve-theoretic corrections, though developing precise asymptotic formulas remains an open problem. Our dataset of 81 examples provides initial validation points for such refined estimates.

4.2.2. Sieve Theory Applications

The parity constraint aligns with modern sieve theory developments. Following [3], the bounded gap results can potentially be strengthened for smooth number settings. Only having two prime factors to manage, and the parity restriction on their exponents, avoids problems in sieve techniques experienced with multiple prime factors.

4.3. Growth Patterns and the Infinity Question

The dataset reveals several patterns that illuminate the sparsity-unboundedness paradox:
Exponential Gap Growth: On average, the intervals between consecutive TPπ2 discoveries grow exponentially. The four steps from $(e_2, e_3) = (2560, 9423)$ to $(13895, 6868)$ represents a jump of over 2,000 orders of magnitude.
Scale Invariance: The "ping-pong" pattern between e_2-dominated and e_3-dominated cases persists across all scales, suggesting no preferred termination mode.

These patterns support our unboundedness conjecture while illustrating why each successive discovery pushes deeper into uncharted computational territory.

5. Open Questions and Future Directions

Our work raises several questions for future investigation:
Unboundedness: Are there infinitely many TPπ2 numbers? The empirical evidence suggests yes, but a proof would likely require adapting Zhang-Maynard techniques to the smooth number setting. However, from a heuristic standpoint, we see the continuing race between the powers of 2 and 3 for the lead in the product N. If this sequence were to have an end, it would have a final value in which either 2 to its power or 3 to its power is greatest, just as one exponent must be odd while the other even. How could one win this race and the other lose? Without a breaker of the symmetry, it stands to reason that the race simply goes on without limit.
Density estimates: Can we prove precise asymptotic formulas for the count of TPπ2 numbers up to x?
Generalization: Do similar parity constraints hold for TPπn numbers (using primes up to the n-th prime)? Initial investigations of TPπ3 numbers do not show an obvious exponential parity relationship. Might other more subtle relationships exit?
Algorithmic optimization: Can the parity constraint be combined with other structural properties to achieve further computational speedups? Not having to search all even sums of exponents cut search time in half; might other patterns be found to alleviate having to search some of the vast spaces of odd sums as the gaps between TPπ2 numbers grow ever more extensive?

6. The Sparsity-unboundedness Paradox

The TPπ2 framework reveals a profound mathematical paradox that illuminates deep principles of prime distribution theory. While individual TPπ2 numbers become exponentially rarer as we explore larger magnitudes, compelling evidence suggests they continue to exist indefinitely.

6.1. The Density Catastrophe

As demonstrated in Section 3, the ratio of TPπ2 numbers to all twin primes decreases dramatically:

$$\frac{\text{TP}\pi 2 \text{ count up to } x}{\text{All twin primes up to } x} \approx 0.25 \to 0.0019 \to 0 \text{ as } x \to \infty$$

This "density catastrophe" might suggest finite cardinality. Yet our computational evidence reveals the opposite: TPπ2 numbers continue appearing at a scale spanning over 7000 decimal digits, with no indication of termination.

6.2. Resolution Through Growth Rate Analysis

The paradox resolves through careful analysis of competing growth rates:
Individual Probability Decay: Each candidate $N = 2^{e_2} \times 3^{e_3}$ has probability approximately $1/(\log N)^2$ of satisfying the twin prime condition.
The Balance: These rates precisely balance:

$$\text{Expected TP}\pi 2 \text{ count} \sim O((\log X)^2) \times O(1/(\log X)^2) \sim O(1)$$

Candidate Pool Growth: The number of TPπ2 candidates up to scale X can be estimated by counting lattice points (e_2, e_3) satisfying the constraints $e_2, e_3 \geq 1$, $e_2 \log 2 + e_3 \log 3 \leq \log X$, and $e_2 + e_3$ odd. The inequality $e_2 \log 2 + e_3 \log 3 \leq \log X$ defines a triangular region in the first quadrant with vertices approximately at $(1, 1)$, $(\log X/\log 2, 1)$, and $(1, \log X/\log 3)$. The area of this triangle is asymptotically $(\log X)^2/(2\log 2 \log 3)$, and since lattice points have unit density, this gives the count of all candidate pairs. The parity constraint $e_2 + e_3 \equiv 1 \pmod 2$ eliminates approximately half of these candidates, yielding:

$$\text{TP}\pi 2 \text{ candidate count up to } X \sim \frac{(\log X)^2}{4\log 2 \log 3} \approx 0.361(\log X)^2$$

This quadratic growth in $\log X$ provides the foundation for our unboundedness analysis.

6.3. The Symmetry Argument

Our heuristic argument for unboundedness rests on the fundamental asymmetry between powers of 2 and powers of 3. The sequences $\{2^n\}$ and $\{3^n\}$ interleave indefinitely due to the logarithmic incommensurability of $\log 2$ and $\log 3$. Products of the form $2^{e_2} \times 3^{e_3}$ systematically explore this infinite landscape without natural termination.

As noted in our unboundedness conjecture: "How could one (power) win this race and the other lose? Without a breaker of the symmetry, it stands to reason that the race simply goes on without limit."

6.4. The Computational Horizon Problem

The sparsity-unboundedness paradox creates what we term the "computational horizon problem": each successive TPπ2 discovery requires exponentially more computational resources, yet no mathematical barrier prevents infinite continuation.

Our largest example, $(e_2, e_3) = (13895, 6868)$, represents a number with over 7000 decimal digits. The next discovery might require searching numbers with 10000+ digits, pushing against the boundaries of current computational feasibility while remaining theoretically achievable.

This creates a fascinating tension between mathematical possibility and practical limitation, reminiscent of other famous computational challenges in number theory.

7. Philosophical Implications

The TPπ2 paradox exemplifies a broader phenomenon in mathematics where local sparsity coexists with global unboundedness. This pattern appears throughout number theory—from Mersenne primes to perfect numbers to twin primes themselves—suggesting fundamental principles about how mathematical objects can be simultaneously rare and inexhaustible.

Our work demonstrates that structured subsets of classical problems can reveal new perspectives on ancient questions. The parity constraint, while eliminating half of all potential candidates, simultaneously provides the computational leverage necessary to explore previously inaccessible regions of the twin prime landscape.

In this sense, TPπ2 numbers serve as "mathematical telescopes," allowing us to probe the deep structure of prime distribution through highly focused, computationally tractable examples that would otherwise remain beyond reach.

8. Conclusion

The Exponential Parity Theorem establishes fundamental constraints on the structure of numbers of the form $2^{e_2} \times 3^{e_3}$ that are bracketed by twin primes. Through analysis of 81 known examples, we demonstrate how this parity constraint provides both theoretical insights into twin prime distribution and practical computational advantages for large-scale searches.

The TPπ2 framework bridges classical number theory and modern computational approaches, offering a structured pathway for investigating one of mathematics' most famous unsolved problems. Our results suggest that smooth numbers provide a particularly tractable setting for twin prime research, potentially leading to new theoretical breakthroughs.

The intersection of sieve theory, smooth numbers, and twin primes represents a fertile area for continued investigation, with the potential to advance our understanding of prime distribution while providing concrete computational tools for researchers.

The TPπ2 framework thus contributes to mathematics not merely through the discovery of new twin prime examples, but through the revelation of a fundamental principle: mathematical unboundedness can coexist with practical evanescence. Our work demonstrates that some of the most profound mathematical truths emerge precisely at the intersection of theoretical possibility and computational limitation—where each new discovery simultaneously becomes harder to achieve and more precious to obtain.

In this light, TPπ2 numbers represent more than a curious subset of twin primes; they embody a paradigm for how mathematical exploration can continue indefinitely even as it becomes exponentially more challenging. The parity constraint, rather than merely providing computational convenience, becomes essential for maintaining contact with this retreating frontier of mathematical knowledge.

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A. Computer Code

Listing 1: Program to Search for TPπ2 Numbers

```

"""
Program to generate ordered pairs (e2, e3) for N = 2**e2 * 3**e3, where
N-1 and N+1 are both prime numbers (i.e. twin primes).
The algorithm proceeds by trying all partitions of each successive
possible sum of e2+e3 where that sum is an odd number, with the
exception of the starting pair: (1, 1). This process continues
for all odd sums up to a pre-set limit, after which the pairs
are sorted based on the value of e2*log(2)+e3*log(3), which is
log(N).
Initial tests use gmpy2.is_prime(n) which catches a composite quickly
and then with gmpy2.is_prime(n, 75) to increase verification
```

```
probability.

Getting to sum_limit = 25,000 takes about a week on a good laptop.
"""
from math import log
from gmpy2 import is_prime

sum_limit = 25_500


def is_TP_pi_2(j, k): # Are the given exponents for 2 and 3 those
    N = 2**j * 3**k      # of a TPpi2 number?
    if is_prime(N-1) and is_prime(N+1): # First, reject composits quickly, then
        return is_prime(N-1, 75) and is_prime(N+1, 75) # use extra cycles to be sure.
    else:
        return False

def TP_pi_2_upto_sum(limit): # Search all partitions up to the
    l2, l3 = log(2), log(3)      # given exponent sum.
    unsorted_result = [(1, 1, log(6))]
    for exponent_sum in range(3, limit+1, 2):
        for j in range(1, exponent_sum):
             k = exponent_sum - j
             log_N = j*l2 + k*l3
             if is_TP_pi_2(j, k):
                  unsorted_result.append((j, k, log_N))
    sorted_result = sorted(unsorted_result, key=lambda x: x[2])
    sequence = []
    for j, k, _ in sorted_result:
        sequence.extend([(j, k)])
    return sequence

def main():
    final_results = TP_pi_2_upto_sum(sum_limit)
    print(f"{final_results}")

if __name__ == "__main__":
    main()
```