Analysis on the critical phenomena of a simple 2D percolation model

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Abstract

This paper presents an analysis of the critical phenomena of a 2D lattice percolation model. The model's characteristic phase transition was successfully simulated, identifying a critical threshold value at which a giant component emerges, $p_c \approx 0.5960$. The spatial properties of the system below, at, and above the critical threshold were characterized, and the scale-invariant properties were verified by analyzing the distribution of cluster sizes at the critical point, which is expected to follow a power-law distribution.

Keywords: percolation, critical point, phase transition

1 Introduction

Percolation is a model in probability theory that aims to explain the notion of critical phenomena [1], or the point at which the system undergoes a phase transition—the emergence of drastic changes from the continuous adjustment of a single parameter. Specifically, percolation theory is a framework for investigating the development of local and global connectivity in a random system [2].

A simple instance that showcases the theory is a two-dimensional (2D) lattice with sites randomly occupied based on a probability p. Conversely, sites are left unoccupied with probability $1 - p$. In this formulation, p is the continuously varying parameter that drives the system. Here, increasing p raises the likelihood of sites to be occupied, and subsequently increases the likelihood of clusters or components, made up of vertically or horizontally adjacent occupied sites, to form. For low values of p, the lattice is characterized by small, isolated clusters, but as it increases towards a critical value p_c, the clusters merge to form bigger ones. Beyond this critical threshold, a single cluster that spans the entire lattice may emerge, called a "giant component". This transition from local to global connectivity has implications on fields such as material science for modeling conductivity [3], polymer chemistry for understanding gelation [4], and epidemiology for disease-spread mapping [5].

This paper aims to investigate the critical phenomena of a simple percolation model. Specifically, a percolation algorithm was implemented on 2D lattice to (a) identify its specific critical threshold value at which a giant component emerges; (b) characterize the spatial properties of the system below, at, and above the critical threshold; and (c) verify the scale-invariant properties of the system at criticality by analyzing the distribution of cluster sizes, which is expected to follow a power-law distribution.

2 Methodology

The lattice and percolation algorithm were simulated in `Python`, using the `NumPy` and `SciPy` libraries for grid manipulation and cluster identification, respectively. A 2D lattice of size $L \times L$ was created where each site was marked as "occupied" if a random number from a uniform distribution was less than a given occupation probability p. Otherwise, the site is left unoccupied or empty.

Clusters were then identified via the `scipy.ndimage.label` function, which identifies all contiguous regions of occupied sites, assigning a unique integer label to each distinct cluster. The size of each cluster was determined by counting the number of sites associated with each unique label. The process at this point constitutes a single trial of the simulation.

For this study, the lattice side length was set to $L = 100$, and p was varied from 0 to 1 in 100 evenly spaced points. For each p, results were averaged over 500 independent trials. The mean sizes of the largest cluster and of all other non-largest clusters across each p was recorded. The critical threshold p_c was subsequently estimated from this data. Snapshots of a single iteration of the lattice were generated for four probabilities: sub-subcritical ($p = 0.2$), subcritical ($p = 0.5$), critical ($p \approx p_c$), and supercritical ($p = 0.7$). For the distribution of cluster sizes at criticality, the number of trials for the simulation at the estimated p_c was increased to 10^3 to obtain a relatively smooth distribution.

3 Results and Discussion

The quantitative results of the simulation are presented in Figure 1, which shows the mean sizes of the clusters as a function of occupation probability

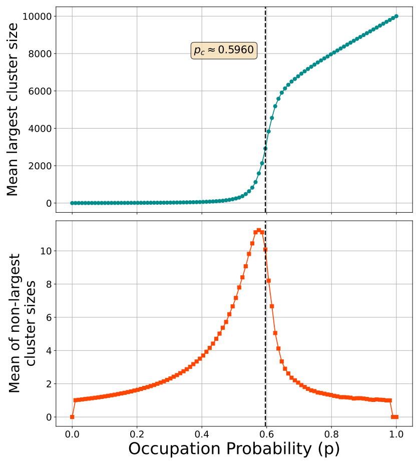


Figure 1: Mean sizes of the largest clusters (top), and all other non-largest ones (bottom). The dashed line indicates the critical threshold at which the system undergoes a phase transition $p_c \approx 0.5960$, taken from the point of sharpest increase of the top plot.

p. The top panel shows that the mean size of the largest cluster is nearly zero at low p, before undergoing a sharp, sigmoidal increase. The distinct behavior of the plot at different regimes of p indicates that the mean size of the largest cluster acts as the order parameter of the system—the measure of the degree of order across boundaries in a phase transition. It clearly distinguishes between a phase where it is nearly zero and one where it is non-zero, divided by a prominent sharp increase, indicative of a phase transition. The point of maximum slope on this curve provides an estimate for the critical probability, calculated here as $p_c \approx 0.5960$.

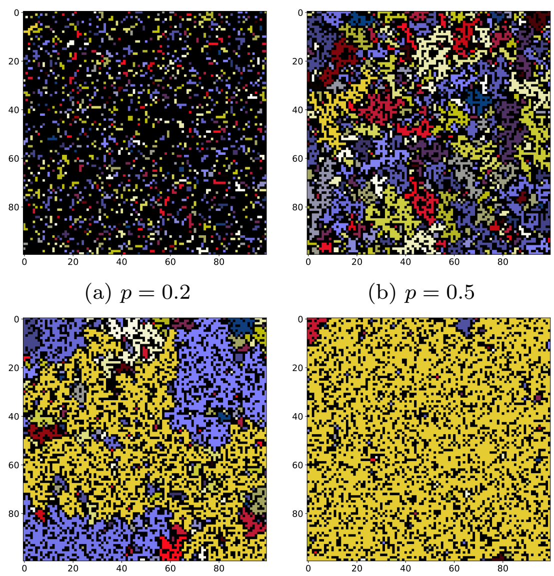
In comparison, the curve for the mean of other non-largest clusters rises from a low-value of p, after which forms a distinct peak, and then decays. Crucially, the peak's location is near p_c, coinciding closely with the critical region identified above. This is because before p approaches p_c, as clusters grow and merge, their average sizes increases. At the critical point $p \approx p_c$, the system contains varying cluster sizes, and the "non-largest" ones from before are at the maximum size they can possibly be right before the point $p > p_c$ and they merge into a giant component spanning the whole system. Beyond this point, the giant component's growth dominates, and the average size of the remaining isolated fragments naturally decays. Therefore, this peak represents the point of maximum diversity in the system.

Notably, the peak of the non-largest clusters is slightly offset to the left of the estimated p_c. This is due to finite-size effects in the system ($L = 100$), resulting in a phase transition that is smeared across a narrow critical region. As explained before, many non-largest clusters merge during criticality ($p \approx p_c$) causing the average size of all clusters to instantaneously start decreasing at the same exact point. However, in this case, that transition happens noticeably after the point of maximum growth, as opposed to an infinitely sharp point expected from an infinitely large lattice.

The qualitative characterizations corresponding to the described phases are shown in the specific lattice snapshots in Figure 2. In the subcritical regime, $p < p_c$, the lattice is fragmented. The top panels (Figures 2a and 2b) are dominated by unoccupied (black) sites, populated by numerous small clusters that are clearly isolated. Especially in deeply subcritical probabilities presented in Figure 2a, where many of these clusters simply consist of a single occupied site. These configurations confirms the presence of a "disconnected" phase in the system as p varies, where the order parameter (mean largest cluster size) is nearly zero due to a lack of global connectivity.

In contrast, the critical configuration in Figure 2c exhibits a single large cluster (yellow) that coexists with clusters of varying sizes. This is the initial formation of the giant component of the system. From here, the supercritical lattice configuration shows a visually dominant giant component (yellow) that spans all sides of the system, shown in Figure 2d. The other clusters end up being few and fragmented, similar to the sub-subcritical lattice. In general, these patterns further further confirm the distinct phases of the system, initially inferred from the mean cluster sizes as p is varied.

To verify the scale-invariant properties of the system at the critical point, the probability distri-



(a) $p = 0.2$ (b) $p = 0.5$

(c) $p = 0.5960$ (d) $p = 0.7$

Figure 2: Snapshots of the percolated lattice at (a) sub-subcritical, (b) subcritical, (c) critical, and (d) supercritical occupation probabilities. The black portions indicate unoccupied sites.

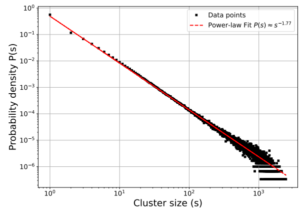


Figure 3: Log-scale distribution of the non-largest cluster sizes s during criticality ($p \approx p_c$). The straight-line fit confirms the power-law behavior of the distribution.

bution of the non-largest cluster sizes were recorded on a logarithmic scale, as shown in Figure 3. The data points form a distinct straight line over several orders of magnitude in cluster size, indicating a power-law distribution of the form $P(s) \propto s^{-\tau}$. Performing a linear fit on the logarithmic data confirms this, with the fitted exponent value estimated to be $\tau \approx 1.77$ and a good coefficient of determination, $R^2 = 0.9765$.

The power-law relationship implies that the system at criticality is scale-invariant. This means there is no characteristic cluster size that would exhibit the critical phenomena. In other words, the point $p_c \approx 0.5960$ will always manifest a complex scenario where the system can either shift towards a globally connected supercritical phase, or further back into a fragmented subcritical phase.

4 Conclusions

This paper present an analysis of the critical phenomena of a 2D lattice percolation model. The simulation successfully simulated the model's characteristic phase transition, identifying a critical threshold at an occupation probability $p_c \approx 0.5960$. This point marks the emergence of a giant component that spans the system, which was confirmed quantitatively through the system's identified order parameter and qualitatively through the visualizations of the system at the subcritical, critical, and supercritical phases. Furthermore, it was found that the distribution of cluster sizes at the critical point follows a power-law distribution, indicating the scale-invariant nature of the system at criticality. Further studies may look into varying lattice geometries other than a simple square; extensions to higher-dimensional lattices; or introducing bond percolation where connections between sites—instead of the sites themselves—are randomly occupied.

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