Real Analysis
Homework 17

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Acknowledgements: None.

5.

a. Claim: $\mathbb{Q}$ is not open and not closed.

Proof. $\mathbb{Q}$ is not closed, since $\sqrt{2}$ is a limit point of $\mathbb{Q}$, and $\sqrt{2} \notin \mathbb{Q}$. $\mathbb{Q}$ is not open, since the irrationals are dense in $\mathbb{R}$, so any ϵ-neighborhood of a rational number must contain an irrational number. □

b. Claim: $\mathbb{N}$ is not open and closed.

Proof. $\mathbb{N}$ is closed, since it has no limit points. Let $\epsilon > 0$ be given. Then, the ϵ-neighborhood of 1 contains a number less than 1, which must not be a natural number. Thus, $\mathbb{N}$ is not open. □

c. Claim: $S = \{x \in \mathbb{R} \mid x \neq 0\}$ is open and not closed.

Proof. S is not closed, since 0 is a limit point of S. S is open, since for all $x \in S$, $(x - \left|\frac{x}{2}\right|, x + \left|\frac{x}{2}\right|) \subseteq S$. □

d. Claim: $S = \{1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} \mid n \in \mathbb{N}\}$ is not open and not closed.

Proof. S is not closed, since $\frac{\pi}{2}$ is a limit point of S. S is not open, since S is increasing, S's minimum is 1, and any ϵ-neighborhood of 1 must contain a number less than 1. □

e. Claim: $S = \{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \mid n \in \mathbb{N}\}$ is

Proof. S is closed, since it has no limit points. S is not open, since S is increasing, S's minimum is 1, and any ϵ-neighborhood of 1 must contain a number less than 1. □

6.

a. Example: $A = \mathbb{N}$

b. Example: $A = \mathbb{Q}$

c. Example: $A = \{1\}$

d. Example:

e. Impossible.