Real Analysis
Homework 16

Ben Kallus, Noah Barton

Due Thursday, October 22

Acknowledgements: None.

---

29. Claim: If $\sum_{n=1}^{\infty} a_n$ diverges, then for all $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} a_{n+k}$ diverges.

Proof. Since $\sum_{n=1}^{\infty} a_n$ diverges, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n, m \geq N$ with $n > m$ such that $|a_{m+1} + \ldots + a_n| \geq \epsilon$. Then, the same property holds for all $N \in \{n + k \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$. Thus, there exists $\epsilon > 0$ such that for all $N \in \{n+k \mid n \in \mathbb{N}\}$, there exist $n, m \geq N$ with $n > m$ such that $|a_{m+1}+\ldots+a_n| \geq \epsilon$. Therefore, there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n, m \geq N$ with $n > m$ such that $|a_{m+k+1} + \ldots + a_{n+k}| \geq \epsilon$. Thus, $\sum_{n=1}^{\infty} a_{n+k}$ diverges. $\square$

30. Claim: If $\sum\limits_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{k\to\infty}\left(\sum\limits_{n=k}^{\infty} a_n\right)$ converges.

Proof. Let $\epsilon > 0$ be given. Let A be the value to which $\sum\limits_{n=1}^{\infty} a_n$ converges. Then, for all $k \in \mathbb{N}$,

$$\begin{aligned} A &= a_1 + \ldots + a_{k-1} + \sum_{n=k}^{\infty} a_n \\ &= \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{\infty} a_n. \end{aligned}$$

Thus, for all $k \in \mathbb{N}$,

$$\left| A - \sum_{n=1}^{k-1} a_n \right| = \left| \sum_{n=k}^{\infty} a_n - 0 \right|.$$

Since $\sum\limits_{n=1}^{\infty} a_n$ converges to A, there exists $K \in \mathbb{N}$ such that for all $k \geq K$,

$$\left| \sum_{n=1}^{k} a_n - A \right| < \epsilon.$$

Define $K' = K + 1$. Then, for all $k \geq K'$,

$$\left| \sum_{n=1}^{k-1} a_n - A \right| < \epsilon.$$

Thus, for all $k \geq K'$,

$$\left| \sum_{n=k}^{\infty} a_n - 0 \right| < \epsilon.$$

Therefore, $\lim\limits_{k\to\infty}\left(\sum\limits_{n=k}^{\infty} a_n\right) = 0$. $\square$

1. Claim: The set of endpoints in the construction of the Cantor Set, E, is countably infinite.

Proof. Let $f : \mathbb{N} \to \mathbb{Q}$ be defined by

$$f(n) = \frac{1}{3^n}.$$

Note that f is one-to-one, and that $f(n) \in E$ for all $n \in \mathbb{N}$. Thus, $|E| \geq |\mathbb{N}| = \aleph_0$. Note that all elements of E must have the form $\frac{k}{3^n}$ for some $k, n \in \mathbb{N}$. Thus, $E \subseteq \mathbb{Q}$. Thus, $|E| \leq |\mathbb{Q}| = \aleph_0$. Therefore, $|E| = \aleph_0$. $\square$

2. Claim: The base three numbers $[0.\overline{1}]_3$ and $[0.\overline{20}]_3$ are equal to $\frac{1}{2}$ and $\frac{3}{4}$, respectively.

Proof. Observe that

$$\begin{aligned}
[0.\overline{1}]_3 &= \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n \\
&= -1 + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \\
&= -1 + \frac{1}{1-\frac{1}{3}} \\
&= -1 + \frac{3}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[0.\overline{20}]_3 &= 2\left(\frac{1}{3}\right)^1 + 0\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + 0\left(\frac{1}{3}\right)^4 + \dots \\
&= 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^5 + \dots \\
&= 2\left(\left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^5 + \dots\right) \\
&= \frac{2}{3}\left(\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \dots\right) \\
&= \frac{2}{3}\left(\left(\frac{1}{9}\right)^0 + \left(\frac{1}{9}\right)^1 + \left(\frac{1}{9}\right)^2 + \dots\right) \\
&= \frac{2}{3}\sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n \\
&= \frac{2}{3} \cdot \frac{9}{8} \\
&= \frac{3}{4}.
\end{aligned}$$

$\square$

3. Claim: The set of all endpoints in the construction of the Cantor Set, E, is not equal to the Cantor Set.

Proof. Note that all elements of E are of the form $\frac{k}{3^n}$ for some $k, n \in \mathbb{N}$. Thus, $\frac{3}{4} \notin E$. Observe that $\frac{3}{4} = [0.\overline{20}]$ is an element of the Cantor Set, since its ternary representation indicates that it is in the upper third of the interval $[0, 1]$, the lower third of the interval $[\frac{2}{3}, 1]$, the upper third of the interval $[\frac{2}{3}, \frac{7}{9}]$, and so on. Thus, E is not the Cantor Set. $\square$

4. Claim: The Cantor Set contains no intervals.

Proof. Let $a, b \in C$, the Cantor Set. Then, $a = [0.a_1a_2a_3\ldots]_3$ such that $a_i \in \{0, 2\}$ for all $i \in \mathbb{N}$. Similarly, $b = [0.b_1b_2b_3\ldots]_3$ such that $b_i \in \{0, 2\}$ for all $i \in \mathbb{N}$. Let $j \in \mathbb{N}$ be the least number such that $a_j \neq b_j$. Then, since $a < b$, $a_j = 0$ and $b_j = 2$. Define $d = [0.a_1a_2\ldots a_{j-1}1a_{j+1}\ldots]_3$. Then, $d \notin C$, since d's ternary representation contains a 1. Note that $a < d$, since $d = a + \left(\frac{1}{3}\right)^j$. Also note that $b > d$, since the two numbers' ternary representations differ first at index j, at which b contains a 2 and d contains a 1. Thus, C contains no intervals. $\square$