Miscellaneous Notes

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Warning:

These notes are on topics that I have not (yet) studied in class. It is likely that some of them are wrong. This file is for accumulating definitions that I would otherwise forget, but may be useful to me in the future.

Metric

A metric f is a function that defines a concept of distance between any two members of a set S. A metric satisfies the following properties for all $a, b \in S$:

$$\begin{aligned} f(a,b) = 0 &\implies a = b, \\ f(a,b) &= f(b,a), \\ |f(a,b)| &= f(a,b), \\ f(a,b) &\leq f(a,c) + f(c,b). \end{aligned}$$

Metric Space

A metric space is a set S together with a metric on S.

Compactness

A space is considered compact if every infinite subsequence of points sampled from the space has an infinite subsequence that converges to some point of the space. Bolzano-Weierstrass tells us that $\mathbb{R}$ has this property, so $\mathbb{R}$ is compact. There are other notions of compactness, but I think this is the one that I will care about for now.

Neighborhood (Topology)

Let $p \in S$, a set. A neighborhood N of p is a subset of S containing an open subset of S containing p. For example, $[1, 5]$ is a neighborhood of $3 \in \mathbb{R}$.

Topological Space

Let S be (potentially empty) set. Let $\mathbf{N}$ be a function mapping each $p \in S$ to a set of subsets of S, which we'll call neighborhoods. S is a topological space if the following all hold:

- $p \in N$ for all $N \in \mathbf{N}(p)$.
- If $M \subseteq S$ and $N \subseteq M$ for some $N \in \mathbf{N}(p)$, then M is a neighborhood of p.
- For all $N_1, N_2 \in \mathbf{N}(p)$, $N_1 \cap N_2 \in \mathbf{N}(p)$.
- For all $N \in \mathbf{N}(p)$, there exists $M \in \mathbf{N}(p)$ such that $M \subseteq N$ and $N \in \mathbf{N}(m)$ for all $m \in M$.

Discrete Space

A discrete space is a topological space in which all subsets are open.

Category

A category C consists of a class $\mathrm{ob}(C)$ of objects and a class of arrows $\hom(C)$ between the objects such that arrows can be composed associatively, and there exists an arrow id_X from X to X for all $X \in \mathrm{ob}(C)$.

Examples:

- The category of sets, in which the objects are sets and the arrows are unary functions.
- The category of rings, in which the objects are rings and the arrows are ring homomorphisms.

Functor

Let C, D be categories. A functor F from C to D is a mapping such that

- For each object $X \in \mathrm{ob}(C)$, $F(X) \in \mathrm{ob}(D)$.
- For each morphism $f : X \to Y$ for $X, Y \in \mathrm{ob}(C)$, $F(f) : F(X) \to F(Y)$.
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{ob}(C)$.
- $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y, g : Y \to Z \in \mathrm{ob}(C)$.

In other words, a functor is a mapping from one category to another that preserves composition of arrows and plays nice with identity arrows.

Endofunctor

An endofunctor is a functor from a category C to itself.