Probability Theory
Homework 7

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Due Friday, October 30

1. a.

$$P_{A,B}(a,b) = \begin{cases} \frac{1}{9} & a=0, b=0, \\ \frac{2}{9} & a=0, b=1, \\ \frac{2}{9} & a=1, b=0, \\ \frac{2}{9} & a=1, b=1, \\ \frac{1}{9} & a=2, b=0, \\ \frac{1}{9} & a=0, b=2, \\ 0 & \text{else.} \end{cases}$$

b.

$$\begin{aligned} \mathbb{P}(B < A) &= \frac{2}{9} + \frac{1}{9} \\ &= \frac{1}{3}. \end{aligned}$$

c.

$$\begin{aligned} \mathbb{E}[A^B] &= \sum_{a=0}^{2} \sum_{b=0}^{2} P_{A,B}(a,b) \cdot a^b \\ &= \frac{1}{9} \cdot 0^0 + \frac{2}{9} \cdot 0^1 + \frac{2}{9} \cdot 1^0 + \frac{2}{9} \cdot 1^1 + \frac{1}{9} \cdot 2^0 + \frac{1}{9} \cdot 0^2 \\ &= \frac{1}{9} + \frac{2}{9} + \frac{2}{9} + \frac{1}{9} \\ &= \frac{2}{3}. \end{aligned}$$

d.

$$p_A(a) = \begin{cases} \frac{4}{9} & a = 0, \\ \frac{4}{9} & a = 1, \\ \frac{1}{9} & a = 2, \\ 0 & \text{else.} \end{cases}$$

$$\mathbb{P}(A < 2) = \frac{8}{9}.$$

e.

$$p_{B|A}(b|0) = \begin{cases} \frac{1}{4} & b = 0 \\ \frac{1}{2} & b = 1 \\ \frac{1}{4} & b = 2 \end{cases}$$

$$\mathbb{P}(B \geq 1 | A = 0) = \frac{3}{4}$$

f.

$$\begin{aligned} \mathbb{P}(A = 0)\mathbb{P}(B = 0) &= \frac{4}{9} \cdot \frac{4}{9} \\ &= \frac{16}{81}. \end{aligned}$$

$$\mathbb{P}(A = 0, B = 0) = \frac{1}{9}.$$

Thus, A and B are not independent.

2.

a.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ky\,dy\,dx = 1$$

$$\int_0^1\int_0^1 ky\,dy\,dx + \int_1^2\int_0^2 ky\,dy\,dx = 1$$

$$\int_0^1\int_0^1 y\,dy\,dx + \int_1^2\int_0^2 y\,dy\,dx = \frac{1}{k}$$

$$\frac{1}{2} + 2 = \frac{1}{k}$$

$$k = \frac{2}{5}$$

b.