Day 6 Spectral methods

Recap of Day 5

5.1 Curve-fitting
Linear fit
Polynomial fit

5.2 Stochastic time-series models
Autoregressive model
Mean, variance and autocovariance

6.1 Power spectrum

Discrete → Continuous probably on a test

6.1.1 Ornstein-Uhlenbeck model

- The autoregressive model is discrete in time.

$$X_k = c + \phi X_{k-1} + \epsilon_k \quad \text{where} \quad \langle \epsilon \rangle = 0 \quad \text{and} \quad \langle \epsilon^2 \rangle = \sigma_\epsilon^2.$$

- Does a continuum limit exist? Assume time-step Δ_t so $t = \Delta_t k$.
  Meaning shorter time steps tend to some limit independent of Δ_t

Question:

not changing ϕ much between time steps

- Let $\overset{\text{scaling}}{\phi} = 1 - \Delta_t/\tau$ where τ is a time constant.
- And call $\langle X \rangle = \mu$ and Var$(X) = \sigma^2$, neither of which are dependent on Δ_t to leading order. To satisfy this restriction...

- Show that $c = \mu \Delta_t/\tau$
- Show that $\sigma_\epsilon^2 = \sigma^2 2\Delta_t/\tau$ to leading order in Δ_t.
- And finally show that in the continuum limit

$$\tau \frac{dX}{dt} = \mu - X + \sigma\sqrt{2\tau}\xi(t)$$

where $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$.

Q:

Show $c = \frac{\mu \Delta_t}{\tau}$

$\langle X \rangle = \mu$ & $\langle X \rangle = \frac{c}{1-\phi}$ (from yesterday)

$\mu = \frac{c}{1-\phi}$

$c = \mu(1-\phi)$

but $\phi = 1 - \frac{\Delta t}{\tau}$, given on previous page

$\Rightarrow$

$c = \mu\left(1 - \left(1 - \frac{\Delta t}{\tau}\right)\right)$

$c = \mu\left(\frac{\Delta t}{\tau}\right)$

Show $\sigma_\varepsilon^2 = \sigma^2 \frac{2\Delta t}{\tau}$

$X_t = c + \phi X_{t-1} + \varepsilon_t$

Show $\tau \frac{dX}{dt} = \mu - X + \sigma\sqrt{2\tau}\, \xi(t)$

Let $\varepsilon_k = \Psi_k \sigma_\varepsilon$, $\Psi_k \sim N(0,1)$

$\xi_k \left(\frac{\Psi_k}{\sqrt{\Delta t}}\right)$

$\sigma^2 = \frac{\sigma_\varepsilon^2}{1-(1-\frac{\Delta t}{\tau})^2} = \frac{\sigma_\varepsilon^2}{2\frac{\Delta t}{\tau} + O(\Delta t^2)}$

$X_t = \mu \frac{\Delta t}{\tau} + \left(1 - \frac{\Delta t}{\tau}\right) X_{t-1} + \Psi_k \sigma_\varepsilon$

$X_t = \mu \frac{\Delta t}{\tau} + \left(\frac{\tau - \Delta t}{\tau}\right) X_{t-1} + \Psi_k \sigma_\varepsilon$

$\tau X_t = \mu \Delta t + (\tau - \Delta t) X_{t-1} + \tau \Psi_k \sigma_\varepsilon$

$X_k = c + \phi X_{k-1} + \varepsilon_k$ $\quad \varepsilon_k = \Psi_k \sigma_\varepsilon$

$X_k = \mu \frac{\Delta t}{\tau} + \left(1 - \frac{\Delta t}{\tau}\right) X_{k-1} + \Psi_k \sigma \sqrt{\frac{2\Delta t}{\tau}}$

$X_k - X_{k-1} = \mu \frac{\Delta t}{\tau} - \frac{\Delta t}{\tau} X_{k-1} + \Psi_k \frac{\sigma \Delta t}{\sqrt{\Delta t}} \sqrt{\frac{2}{\tau}}$ $\quad X_{k-1} = X_k + O(\Delta t)$ so can write it as X_k

$\frac{X_k - X_{k-1}}{\Delta t} = \frac{\mu}{\tau} - \frac{X_k}{\tau} + \frac{\Psi_k}{\sqrt{\Delta t}} \sigma \sqrt{\frac{2}{\tau}}$

$\frac{dX}{dt} = \frac{\mu - X}{\tau} + \sigma \sqrt{\frac{2}{\tau}}\, \xi(t)$ $\quad \xi(t) = \frac{\Psi_k}{\sqrt{\Delta t}}$ process, not a function. goes to $\pm\infty$ for every t

blame physicists. mathematicians would integrate over an infintesinal timestep

$\tau \frac{dX}{dt} = \mu - X + \sigma\sqrt{2\tau}\, \xi(t)$

Show $\langle \xi(t) \rangle = 0$

$\langle \xi(t) \rangle \Rightarrow \frac{\overset{\text{discrete}}{\langle \Psi_k \rangle}}{\sqrt{\Delta t}} = \frac{0}{\sqrt{\Delta t}} = 0$

$\langle \xi(t)\xi(t') \rangle \Rightarrow \frac{\langle \Psi_k \Psi_j \rangle}{\Delta t} = \begin{cases} 0 & j \neq k \\ \frac{1}{\Delta t} & j = k \end{cases}$

Gaussian White Noise

$\delta(t - t')$

dirac delta

WoW from Magnus

Be careful if have form $\frac{dx}{dt} = f(x) + g(x)\,\xi(t)$

matematicians would write $g(x(t-\varepsilon))$

changes violently enough like dirac delta function

$\frac{dx}{dt} = \alpha\, x(t)\, \delta(t)$

$x = 1$ on $(-\infty, 0]$

Itô calculus

Stratonovich calculus

what happens after $x = 0$?

$\frac{dx}{dt} = \alpha x\, \delta(t)$

$\int \frac{1}{x}\, dx = \alpha \int \delta(t)\, dt$

$\ln(x)\Big|_-^+ = \alpha$ $\quad + C$?

$x = e^\alpha$

if wrote $\frac{dx}{dt} = \alpha\, x(t-\varepsilon)\, \delta(t)$

then $x(t>0) = 1 + \alpha$

6.1.2 O-U variance

- Differential equation make sure you can solve

$$\tau \frac{dX}{dt} = \mu - X + \sigma\sqrt{2\tau}\xi(t)$$

- Straightforward to solve $\langle \frac{dX}{dt} \rangle = \frac{d}{dt}\langle X \rangle$

like a dot product

$$X(t) = \mu + \sigma\sqrt{2\tau} \int_{-\infty}^{t} \frac{ds}{\tau} e^{-(t-s)/\tau} \xi(s)$$

convolution

- Mean straightforward. Check the variance... reproduce

$$\langle (X(t) - \mu)^2 \rangle = \sigma^2 2\tau \int_{-\infty}^{t} \frac{ds}{\tau} \int_{-\infty}^{t} \frac{ds'}{\tau} e^{-(t-s)/\tau} e^{-(t'-s')/\tau} \langle \xi(s)\xi(s') \rangle$$

- Use the correlator for white noise $\langle \xi(t)\xi(t') \rangle = \delta(t - t')$ to give

$$\langle (X(t) - \mu)^2 \rangle = \sigma^2 2 \int_{-\infty}^{t} \frac{ds}{\tau} e^{-2(t-s)/\tau} = \sigma^2$$

- What about the autocovariance?

6.1.2 O-U covariance

Know how to do all of this really fast

- The solution to the O-U model

$\tau \frac{dX}{dt} = \mu - X + \sigma\sqrt{2\tau}\xi(t)$ is
$X(t) = \mu + \sigma\sqrt{2\tau} \int_{-\infty}^{t} \frac{ds}{\tau} e^{-(t-s)/\tau} \xi(s)$

- So the covariance can be written as follows (using $x = X - \mu$).

$$\langle x(t)x(t+T)\rangle = \sigma^2 2\tau \int_{-\infty}^{t} \frac{ds}{\tau} \int_{-\infty}^{t+T} \frac{ds'}{\tau} e^{-(t-s)/\tau} e^{-(t+T-s')/\tau} \langle \xi(s)\xi(s'$$

- The calculation is similar to the autoregression model.
- Assume $T \geq 0$, because the $T < 0$ follows by symmetry.
- Non-zero part of integral runs up to T only, leaving...

$$\langle x(t)x(t+T)\rangle = e^{-T/\tau} \sigma^2 2\tau \int_{-\infty}^{t} \frac{ds}{\tau} \int_{-\infty}^{t} \frac{ds'}{\tau} e^{-(t-s)/\tau} e^{-(t-s')/\tau} \langle \xi(s)\xi($$

- So that the autocovariance becomes

$$\langle x(t)x(t+T)\rangle = \sigma^2 e^{-|T|/\tau}.$$

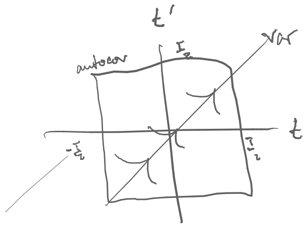
6.1.3 Power spectrum and Wiener-Khinchin theorem

↘ look at frequency components to see what's there

- Consider the Fourier transform of a zero-mean signal over a finite range.

normalization

$$\hat{x}_T(\omega) = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} dt\, x(t) e^{-i\omega t}$$

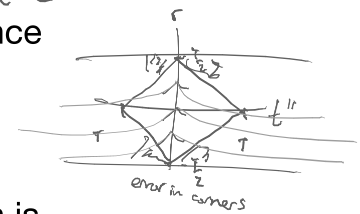


- Power spectrum (expectation of squared amplitude) $S(\omega) = \langle |\hat{x}_T(\omega)|^2 \rangle$ so

$e^{-i\omega t} e^{i\omega t}$

$$S(\omega) = \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} dt dt'\; e^{-i\omega(t-t')} \underbrace{\langle x(t)x(t') \rangle}_{A(t-t')}$$

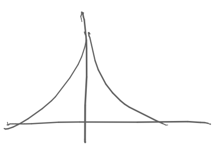
$t - t' = C$



area conserved

- But $\langle x(t)x(t') \rangle = A(t - t')$ is just the autocovariance

auto covar



rotate 45°

- Introduce $r = (t + t')/2$ and $t'' = t - t'$.
- Consider the region of integration for large T.
- If $A(T) \to 0$ for large T then a good approximation is

$$S(\omega) \approx \cancel{\frac{1}{T}} \overbrace{\int_{-T/2}^{T/2} dr}^{T} \underbrace{\int_{-\infty}^{\infty} dt''\; e^{-i\omega t''} A(t'') = \hat{A}(\omega)}_{\text{Fourier transform of the autocov}}$$

$t'' \to \pm\infty$

T large for r

just including more stuff that's negligible, except maybe the error in the corners

- The power-spectrum is the Fourier transform of the autocovariance.
- This is the Wiener-Khinchin theorem.

$S(\omega) = \hat{A}(\omega)$ as $\lim_{t \to \infty}$

6.3 Summary and additional questions

Day 5 Basic time-series analysis

Questions

Make sure you have understood and done all the questions in the lectures.

The questions below are to be handed in for marking by 10am Monday 3rd December 2018.

Q6.1 Power spectrum for the Ornstein-Uhlenbeck process.
Q6.2 Correlated Ornstein-Uhlenbeck processes.

Q6.1 Power-spectrum for the O-U process

- Consider the Ornstein-Uhlenbeck process described in the lecture notes.
  Use it's autocovariance to provide the form of the power spectrum $S(\omega)$.

Q6.2 Correlated O-U processes

- Consider two O-U processes driven by the same Gaussian white noise $\xi(t)$
  But with two different time constants τ_x and τ_y so that

Same var, means = 0

$$\tau_x \frac{dx}{dt} = -x + \sigma\sqrt{2\tau_x}\xi(t) \text{ and } \tau_y \frac{dy}{dt} = -y + \sigma\sqrt{2\tau_y}\xi(t)$$

- What is the same-time correlation $\langle x(t)y(t)\rangle$?

- Calculate the cross-covariance $\langle x(t)y(t+T)\rangle$ and be sure to provide the forms for both positive and negative T.

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