6.1

Consider the O-U process (continuous)

$$\tau \frac{dx}{dt} = \mu - x + \sigma\sqrt{2\tau}\ \xi(t), \text{ where } \langle \xi(t) \rangle = 0 \text{ and } \langle \xi(t)\xi(t') \rangle = \delta(t-t').$$

Use its autocovariance to provide the form of the power spectrum $S(\omega)$.

Let $A(T)$ be the autocovariance function of the O-U process (in the time domain). We have

$$A(T) = \langle x(t)\, x(t+T) \rangle = \sigma^2 \exp\left(-\frac{|T|}{\tau}\right).$$

Note that $A(T) \to 0$ as $t \to \infty$. Then the Wiener-Kinchin Theorem states that

$$S(\omega) = \hat{A}(\omega) \text{ in the limit as } T \to \infty,$$

where $\hat{A}(\omega)$ is the Fourier transform of $A(T)$. Thus we have

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma^2 e^{-\frac{|T|}{\tau}} e^{-i\omega T}\, dT.$$

The absolute value in the exponential means we have to split the integral above and below 0.

$$S(\omega) = \frac{\sigma^2}{2\pi}\left[\int_{-\infty}^{0} e^{-\frac{1}{\tau}(-T)} e^{-i\omega T}\, dT + \int_{0}^{\infty} e^{-\frac{1}{\tau}(T)} e^{-i\omega T}\, dT\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\int_{-\infty}^{0} e^{\frac{1}{\tau}T} e^{-i\omega T}\, dT + \int_{0}^{\infty} e^{-\frac{1}{\tau}T} e^{-i\omega T}\, dT\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\int_{-\infty}^{0} e^{(\frac{1}{\tau} - i\omega)T}\, dT + \int_{0}^{\infty} e^{-(\frac{1}{\tau} + i\omega)T}\, dT\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\left(\frac{1}{\tau} - i\omega\right)^{-1}\left[e^{(\frac{1}{\tau} - i\omega)T}\right]_{-\infty}^{0} - \left(\frac{1}{\tau} + i\omega\right)^{-1}\left[e^{-(\frac{1}{\tau} + i\omega)T}\right]_{0}^{\infty}\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\left(\frac{1}{\tau} - i\omega\right)^{-1}[1 - 0] - \left(\frac{1}{\tau} + i\omega\right)^{-1}[0 - 1]\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\left(\frac{1}{\tau} - i\omega\right)^{-1} + \left(\frac{1}{\tau} + i\omega\right)^{-1}\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\frac{1}{(\frac{1}{\tau} - i\omega)} \cdot \frac{(\frac{1}{\tau} + i\omega)}{(\frac{1}{\tau} + i\omega)} + \frac{1}{(\frac{1}{\tau} + i\omega)} \cdot \frac{(\frac{1}{\tau} - i\omega)}{(\frac{1}{\tau} - i\omega)}\right]$$

$$= \frac{\sigma^2}{2\pi}\left[\frac{\frac{1}{\tau} \cancel{+ i\omega} + \frac{1}{\tau} \cancel{- i\omega}}{\tau^{-2} \cancel{- i^2}\, \omega^2}\right] \quad +1$$

$$= \frac{\sigma^2}{\cancel{2}\pi}\left(\frac{\frac{\cancel{2}}{\tau}}{\tau^{-2} + \omega^2}\right) = \frac{\sigma^2}{\pi\tau} \cdot \frac{1}{\tau^{-2} + \omega^2}$$

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if $\tau = \frac{1}{\omega_0}$ and $\sigma^2 = \frac{\varsigma^2 \tau}{2}$, then

$$S(\omega) = \frac{\left(\frac{\varsigma^2 \cancel{\tau}}{2}\right)}{\pi \cancel{\tau}} \cdot \frac{1}{\omega_0^2 + \omega^2}$$

$$= \frac{\varsigma^2}{2\pi} \cdot \frac{1}{\omega_0^2 + \omega^2}$$