6.2

Consider two O-U processes driven by the same Gaussian white noise $\xi(t)$, but with two different time constants τ_x and τ_y so that

$$\tau_x \frac{dx}{dt} = -x + \sigma\sqrt{2\tau_x}\,\xi(t), \quad (1)$$

$$\tau_y \frac{dy}{dt} = -y + \sigma\sqrt{2\tau_y}\,\xi(t). \quad (2)$$

a) What is the same-time correlation $\langle x(t)y(t)\rangle$?

b) Calculate the cross-variance $\langle x(t)y(t+T)\rangle$, and be sure to provide the forms for both positive and negative T.

a) Solving equation (1) yields

$$x(t) = \sigma\sqrt{2\tau_x}\int_{-\infty}^{t}\frac{ds}{\tau_x}\exp\left(\frac{-(t-s)}{\tau_x}\right)\xi(s).$$

Similarly, solving equation (2) yields

$$y(t) = \sigma\sqrt{2\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_y}\right)\xi(s).$$

Then we have

$$\langle x(t)y(t)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s')}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle.$$

Note $\langle\xi(s)\xi(s')\rangle = \delta(s-s')$ is the correlator for white noise. So we continue our calculation:

$$\langle x(t)y(t)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s')}{\tau_y}\right)\delta(s-s')$$

And using the properties of $\delta(s-s')$ yields

$$\langle x(t)y(t)\rangle = \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t}ds\,\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s)}{\tau_y}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t}\exp\left[-\frac{(\tau_y(t-s)+\tau_x(t-s))}{\tau_x\tau_y}\right]ds$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t}\exp\left[\frac{-(\tau_x+\tau_y)}{\tau_x\tau_y}(t-s)\right]ds$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\cdot\frac{\tau_x\tau_y}{\tau_x+\tau_y}\left[\exp\left[\frac{-(\tau_x+\tau_y)}{\tau_x\tau_y}(t-s)\right]\right]_{-\infty}^{t}$$

$$= \frac{2\sigma^2\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\left[1-0\right]$$

$$= 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}$$

Note if $\tau_x=\tau_y$, then $\langle x(t)y(t)\rangle = 2\sigma^2\frac{\sqrt{\tau^2}}{\tau+\tau}$

$$= \frac{\cancel{2}\sigma^2\cancel{\tau}}{\cancel{2\tau}}$$

$$= \sigma^2$$

b) First assume $T>0$ ($T=0$ is part (a)). Then we have

$$\langle x(t)y(t+T)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t+T}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t+T-s')}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle.$$

Note that the non-zero part of the integral runs up to T only, thus we have

$$\langle x(t)y(t+T)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s')}{\tau_y}-\frac{T}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle$$

$$= 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s')}{\tau_y}\right)\exp\left(-\frac{T}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle$$

$$= 2\sigma^2\sqrt{\tau_x\tau_y}\exp\left(-\frac{T}{\tau_y}\right)\left[\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-s')}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle\right].$$

We can use our solution from part (a) to calculate the expression in brackets to get

$$\langle x(t)y(t+T)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\exp\left(-\frac{T}{\tau_y}\right)\left[\frac{1}{\tau_x+\tau_y}\right]$$

$$= \frac{2\sigma^2 e^{-\frac{T}{\tau_y}}\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y} = 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\exp\left[-\frac{T}{\tau_y}\right]$$

Now let $T<0$. Then

$$\langle x(t)y(t+T)\rangle = 2\sigma^2\sqrt{\tau_x\tau_y}\int_{-\infty}^{t}\frac{ds}{\tau_x}\int_{-\infty}^{t-T}\frac{ds'}{\tau_y}\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-T-s')}{\tau_y}\right)\langle\xi(s)\xi(s')\rangle$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t}ds\int_{-\infty}^{t-T}ds'\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-T-s')}{\tau_y}\right)\delta(s-s')$$

Using the properties of $\delta(s-s')$, we get

$$\langle x(t)y(t+T)\rangle = \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left(-\frac{(t-s)}{\tau_x}\right)\exp\left(-\frac{(t-T-s)}{\tau_y}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left(-\frac{(t-s)}{\tau_x}-\frac{(t-T-s)}{\tau_y}\right)$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left[\frac{-\tau_y(t-s)-\tau_x(t-T-s)}{\tau_x\tau_y}\right]$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left[\frac{-\tau_y t+\tau_y s-\tau_x t+\tau_x T+\tau_x s}{\tau_x\tau_y}\right]$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left[\frac{-\tau_y t-\tau_x t+\tau_x T}{\tau_x\tau_y}+\frac{\tau_x+\tau_y}{\tau_x\tau_y}s\right]$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\int_{-\infty}^{t-T}ds\,\exp\left[\frac{-\tau_y t-\tau_x t+\tau_x T}{\tau_x\tau_y}\right]\exp\left[\frac{\tau_x+\tau_y}{\tau_x\tau_y}s\right]$$

$$= \frac{2\sigma^2}{\sqrt{\tau_x\tau_y}}\exp\left[\frac{-\tau_y t-\tau_x t+\tau_x T}{\tau_x\tau_y}\right]\int_{-\infty}^{t-T}ds\,\exp\left[\frac{\tau_x+\tau_y}{\tau_x\tau_y}s\right]$$

$$= \frac{2\sigma^2}{\cancel{\sqrt{\tau_x\tau_y}}}\exp\left[\frac{-\tau_y t-\tau_x t+\tau_x T}{\tau_x\tau_y}\right]\left[\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\right]\left[\exp\left[\frac{\tau_x+\tau_y}{\tau_x\tau_y}s\right]\right]_{-\infty}^{t-T}$$

$$= 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\exp\left[\frac{-\tau_y t-\tau_x t+\tau_x T}{\tau_x\tau_y}\right]\left[\exp\left[\frac{\tau_x+\tau_y}{\tau_x\tau_y}(t-T)\right]-\cancel{0}\right]$$

$$= 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\exp\left[\frac{\cancel{-\tau_y t}\cancel{-\tau_x t}\cancel{+\tau_x T}}{\tau_x\tau_y}+\frac{\cancel{\tau_x t}\cancel{-\tau_x T}\cancel{+\tau_y t}-\tau_y T}{\tau_x\tau_y}\right]$$

$$= 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\exp\left[-\frac{\cancel{\tau_y}T}{\tau_x\cancel{\tau_y}}\right] = 2\sigma^2\frac{\sqrt{\tau_x\tau_y}}{\tau_x+\tau_y}\exp\left[-\frac{T}{\tau_x}\right]$$