Alternative Derivation for Change-of-Variable

Suppose we have some RV X, and a (sort of piecewise(?) invertible) function g such that we can express a new RV Y as

$$(*) \qquad Y = g(X). \qquad \text{(I know, he does } X = g(Y) \text{ which is weird)}$$

Assuming we know $f_X(x)$, the PDF of the RV X, then our goal is to determine the PDF of the new RV Y, $f_Y(y)$.

First, lets explore the CDF of Y, $F_Y(y)$:

$$F_Y(y) \overset{\text{def}}{=} P[Y \le y] \overset{(*)}{=} P[g(X) \le y]$$

take g^{-1} of both sides - here we need g to be invertible, you'll see sort of piecewise in the example

$$= P[X \le g^{-1}(y)]$$

$$\overset{\text{def}}{=} F_X(g^{-1}(y)).$$

(if you prefer, since $g^{-1}(y) = x$ we can write $F_X(x)$ instead, but this is unintuitive and tricky because we don't know how g^{-1} behaves.)

So we've got a relation between the CDF of Y and the CDF of X:

$$(**) \qquad F_Y(y) = F_X(g^{-1}(y)).$$

Now we can differentiate the CDF of Y (with respect to Y), to get the PDF of Y, $f_Y(y)$:

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}[F_Y(y)].$$

But we have the relation given by $(**)$, so we can substitute:

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))],$$

and apply the chain rule to differentiate

$$(1) \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot \frac{d}{dy}[g^{-1}(y)].$$

This is slightly different notation to what Magnus uses, but I find it more intuitive and ultimately more useful.

Right so let's do an example!

Example Let X be a RV with some PDF. Define a new RV Y such that

$$Y = X^2.$$

Find the PDF of Y in terms of the PDF of X.

Solution Using what we've done above, we'll first explore the CDF of Y:

$$F_Y(y) = P[Y \le y] = P[g(X) \le y] \longrightarrow = P[X^2 \le y]$$

can go either route

$$= P[X \le g^{-1}(y)] \longrightarrow = P[X \le (\pm\sqrt{y})]$$

$$= P[X \le \sqrt{y}] \cap P[X \ge -\sqrt{y}]$$

$$= P[-\sqrt{y} \le X \le \sqrt{y}]$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Thus we have

$$(\#) \qquad F_Y(y) = F_X(\sqrt{y}) - F(-\sqrt{y}).$$

Now to find the PDF of Y, we differentiate the CDF of Y with respect to Y:

$$f_Y(y) = F_Y'(y) = \frac{d}{dy}[F_Y(y)].$$

But we have the relation $(\#)$ that we derived above, so we can substitute:

$$f_Y(y) = \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(\sqrt{y}) - F_X(-\sqrt{y})].$$

Using the chain rule to differentiate gives

$$f_Y(y) = \frac{d}{dy}[F_X(\sqrt{y}) - F_X(-\sqrt{y})] = \frac{d}{dy}[F_X(\sqrt{y})] - \frac{d}{dy}[F_X(-\sqrt{y})]$$

$$= f_X(\sqrt{y}) \cdot \left(\tfrac{1}{2} y^{-\frac{1}{2}}\right) + f_X(-\sqrt{y}) \cdot \left(+\tfrac{1}{2} y^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left(y^{-\frac{1}{2}}\right)\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right).$$

Hence, we have

$$f_Y(y) = \frac{1}{2}\left(y^{-\frac{1}{2}}\right)\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right).$$

From here, if you know $f_X(x)$, you can just plug in $f_X(x = \sqrt{y})$ and $f_X(x = -\sqrt{y})$ to get the PDF of Y.

Note that since g^{-1} is not a one-to-one (injective) function, we split $f_X(g^{-1}(y))$ into parts so that each part is one-to-one (injective). We're technically defining $g^{-1}(y)$ (very carefully!) piecewise on a partioned domain, and that's how the sort of piecewise invertible requirement for g occurs.

Moreover, notice that we have some symmetry and we can combine some terms in our solution. This arises from g being two-to-one (not injective) and having certain symmetries (i.e. $(-a)^2 = a^2$). In general, we may not be able to combine any terms and end up with a (slightly ugly) summation for the different aforementioned "parts of g".