Problem Set #1

Measure Theory
Jan Ertl
Kendra Robbins

1.3 $\mathcal{G}_1$ is not an algebra because it is not closed under complements:
Suppose $A \in \mathcal{G}_1$. Then A is open. Then A^c is closed, so $A^c \notin \mathcal{G}_1$.

$\mathcal{G}_2$ is an algebra:

- $\emptyset \in \mathcal{G}_2$
- Suppose $A \in \mathcal{G}_2$. Then A is a finite union of intervals of the form $(a,b], (-\infty,b], (a,\infty)$. Then A^c is a finite union of intervals of the form $(a,b], (-\infty,b], (a,\infty)$, so $A^c \in \mathcal{G}_2$. $\mathcal{G}_2$ is closed under complements.
- If $B \in \mathcal{G}_2$, then B is a finite union of intervals of the form $(c,d], (-\infty,d], (c,\infty)$. Then $A \cup B$ is a finite union of intervals of the form $(a,b], (-\infty,b], (a,\infty)$, so $A \cup B \in \mathcal{G}_2$, so $\mathcal{G}_2$ is closed under finite unions.
- Since $\mathcal{G}_2$ by definition does not contain infinite unions, a countably infinite union is not contained in $\mathcal{G}_2$, so $\mathcal{G}_2$ is not a σ-algebra.

$\mathcal{G}_3$ is a sigma algebra. This can be proved by a modification of the proof for $\mathcal{G}_2$, incorporating that $\mathcal{G}_3$ contains countable unions.

1.7 Suppose A is the smallest σ-algebra of a set X. By definition, $\emptyset \in A$. The complement of $\emptyset$ in X is X. Since A is closed under complements, $X \in A$. Since X is necessarily nonempty, $X \neq \emptyset$, so $A \neq \emptyset$. So $\{\emptyset, X\}$ is the smallest σ-algebra. $A = \{\emptyset, X\}$.
Suppose B is a σ-algebra of X that is larger than the the power set of X, $P(X)$. Then there exists a subset of X, $b \in B$ such that $b \notin P(X)$. But $P(X)$ contains all subsets of X, so $b \in P(X)$. This is a contradiction: B cannot be larger than $P(X)$. $P(X)$ must be the largest σ-algebra.
Suppose C is an arbitrary σ-algebra of a set X. Then $\emptyset \in C$. The complement of $\emptyset$ is X, so $X \in C$. Thus, $\{\emptyset, X\} \subseteq C$. Since all elements of C are subsets of X, every element of C is contained in $P(X)$, so $C \subseteq P(X)$.

1.10 Let $\{S_\alpha\}$ be a family of σ-algebras of X. Since every σ-algebra contains $\emptyset$ by definition, $\emptyset \in S_\alpha \ \forall \alpha$, so $\emptyset \in \cap_\alpha S_\alpha$.
Let $x \in \cap_\alpha S_\alpha$. Then $x \in S_\alpha \ \forall \alpha$. Since σ-algebras are closed under complements, $x^C \in S_\alpha \ \forall \alpha$, so $x^C \in \cap_\alpha S_\alpha$.
Suppose $\{x_i\}_{i=1}^\infty \in \cap_\alpha S_\alpha$. Then $\{x_i\}_{i=1}^\infty \in S_\alpha \ \forall \alpha$. Since σ-algebras are closed under countable unions, the countable union $\bigcup_{i=1}^\infty x_i \in \cap_\alpha S_\alpha$. Then $\bigcup_{i=1}^\infty x_i \in S_\alpha \ \forall \alpha$. Thus, $\bigcup_{i=1}^\infty x_i \in \cap_\alpha S_\alpha$. So $\cap_\alpha \{S_\alpha\}$ is closed under countable unions.

$\Rightarrow \cap_\alpha \{S_\alpha\}$ is a σ-algebra.

1.17 Let (X, S, μ) be a measure space.
Suppose $A, B \in S$ with $A \subset B$.
Let $K = B\ /A$
Then $B = (A \cap B) \cup K = A \cap K$.
Since $(A \cap B) \cap K = \emptyset$, $\mu(B) = \mu(A) + \mu(K)$.
Since $\mu(K) \geq 0, \mu(B) = \mu(A) + \mu(K) \geq \mu(A)$.
$\Rightarrow \mu(A) \leq \mu(B)$.

Suppose $\{A_i\}_{i=1}^{\infty} \in S$.
Let $A_m, A_n \in \{A_i\}_{i=1}^{\infty}$.
Define:
$D = A_m/A_n$
$E = A_n/A_m$
$F = A_n \cap A_m$
Note that D, E, F are disjoint, $A_n = E \cup F$, $A_m = D \cup F$, and $A_n \cup A_m = D \cup E \cup F$.
Then $\mu(A_n \cup A_m) = \mu(D \cup E \cup F) = \mu(D) + \mu(E) + \mu(F)$ since D, E, F are disjoint.
Further, $\mu(A_n) + \mu(A_m) = \mu(E \cup F) + \mu(D \cup F) = \mu(E) + \mu(D) + 2\mu(F)$ since D, E, F are disjoint.
Since $\mu(F) \geq 0, \mu(A_n \cup A_m) = \mu(D) + \mu(E) + \mu(F) \leq \mu(E) + \mu(D) + 2\mu(F) = \mu(A_n) + \mu(A_m)$.
$\Rightarrow \mu(A_n \cup A_m) \leq \mu(A_n) + \mu(A_m)$ for any $A_n, A_m \in \{A_i\}$
$\Rightarrow \mu(\bigcup_{i=1}^{\infty} \{A_i\}) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

1.18 Let (X, S, μ) be a measure space and $B \in S$. Suppose $\lambda : S \to [0, \infty]$ defined by $\lambda(A) = \mu(A \cap B)$.
$\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$ since μ is a measure.
Let $\{A_i\}_{i=1}^{\infty} \in S$ with $A_i \cap A_j = \emptyset$ whenever $i \neq j$.
Then $\lambda(\cup_{i=1}^{\infty} A_i) = \mu((\cup_{i=1}^{\infty} A_i) \cap B) = \mu(\bigcup_{i=1}^{\infty} (A_i \cap B)) = \sum_{i=1}^{\infty} \mu(A_i \cap B) = \sum_{i=1}^{\infty} \lambda(A_i)$ since all A_i are disjoint.
$\Rightarrow \lambda$ is a measure.

1.20 Let μ be a measure on (X, S). and suppose $A_1 \supset A_2 \supset A_3 \supset ...$ for all $A_n \in S$ and $\mu(A_1) < \infty$.
Define $B_n = A_1 \setminus A_n\ \forall n \in \mathbb{N}$.
Then $A_1 = B_n \cup A_n\ \forall n \in \mathbb{N}$ and $B_n \cap A_n = \emptyset\ \forall n \in \mathbb{N}$. Thus, $\mu(A_1) = \mu(B_n) + \mu(A_n)$ and therefore $\mu(A_n) = \mu(A_1) - \mu(B_n)$.

Define $B = \cup_{i=1}^{\infty} B_i$. Then the conditions for part i of 1.19 are met for $\{B_i\}$, so we have: $\mu(B) = lim_{n \to \infty} \mu(B_n)$
Note that:

- $A_1 \setminus \bigcap_i^\infty A_i = A_1 \cap \left(\bigcap_i^\infty A_i\right)^c = \bigcup_{i=1}^\infty \left(A_1 \cap A_i^c\right) = \bigcup_{i=1}^\infty (A_1 \setminus A_i) = B$
- $\mu(A_1) = \mu\big(A_1 \setminus \bigcap_{i=1}^\infty A_i i\big) + \mu\big(\bigcap_{i=1}^\infty A_i\big) = \mu(B) + \mu\big(\bigcap_{i=1}^\infty A_i\big)$
- $\mu\big(\bigcap_{i=1}^\infty A_i\big) = \mu(A_1) - \mu(B)$

So now we have: $\mu\big(\bigcap_{i=1}^\infty A_i\big) = \mu(A_1) - lim_{n\to\infty}\mu(B_n)$
and: $\mu\big(\bigcap_{i=1}^\infty A_i\big) = lim_{n\to\infty}\big(\mu(A_1) - \mu(B_n)\big) = lim_{n\to\infty}\mu(A_n)$

2.10 Let μ^* be an outer measure on a set X and let $\mathcal{M}$ be the set of all $E \in X$ such that $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^C) \ \ \forall B \subset X$.
Then $\mathcal{M}$ is a σ-algebra.
Since μ^* is an outer measure, $\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^C)\ \forall E \in \mathcal{M}$ and $\forall B \subseteq X$ by subadditivity.
By construction we have $\mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^C)\ \forall E \in \mathcal{M}$ and $\forall B \in X$.
So we must have $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^C)\ \forall E \in \mathcal{M}$ and $\forall B \in X$.

2.14 Let O be the set of all open sets of $\mathbb{R}$. Let A be an algebra containing O. A exists because we can form a set containing all open sets, their finite unions, and their complements.
Let $\{S_\alpha\}$ be the set of all σ-algebras containing A.
Then $B(\mathcal{R}) = \sigma(O) \subseteq \sigma(A) = \cap_\alpha\{S_\alpha\}$ because $O \subseteq A \subseteq S_\alpha\ \forall\alpha$.
By 2.12 $\sigma(A) \subset M$, so $B(\mathcal{R}) = \sigma(O) \subset M$.

3.1 Let $A \in M$ be a countable set and $\epsilon > 0$. Since A is countable, it is a null set, so we can construct a set of intervals $\{I_n\}_{n=1}^\infty$ such that $A \subseteq \cup_{n=1}^\infty I_n$ and the sum of the lengths of all intervals in $\{I_n\}_{n=1}^\infty$ is less than ϵ.
Since $\mu = \mu * |_M, \mu$ is monotonic and countably subadditive.
So $\mu(A) \leq \mu(\{I_n\}_{n=1}^\infty \leq \sum_{n=1}^\infty \mu(I_n)$. Since the sum of lengths of intervals $\{I_n\}_{n=1}^\infty$ is less than ϵ and ϵ can be arbitrarily small, we must have $\mu(A) \leq \mu(\{I_n\}_{n=1}^\infty \leq \sum_{n=1}^\infty \mu(I_n) = 0$. Since $\mu(A) \geq 0,\ \mu(A) = 0$.

3.4 Suppose $(X, \mathcal{M}, \mu)$, $f : X \to \mathbb{R}$, and $\forall a \in \mathbb{R}, \{x \in X : f(x) < a\} \in \mathcal{M}$.
Then since $\mathcal{M}$ is closed under complements, $\forall a \in \mathbb{R},\ \{x \in X : f(x) < a\}^C = \{x \in X : f(x) \geq a\} \in \mathcal{M}\ \forall a \in \mathbb{R}$.
Let $b \in \mathbb{R}$. Then $\forall n \in \mathbb{N}, \{x \in X : f(x) < b + \frac{1}{n}\} \in \mathcal{M}$ by assumption.
Since σ-algebras are closed under countable intersections, $\bigcap_{n=1}^\infty \{x \in X : f(x) < b + \frac{1}{n}\} = \{x \in X : f(x) \leq b\} \in \mathcal{M}$.
So $\{x \in X : f(x) < a\} \in \mathcal{M} \Rightarrow \{x \in X : f(x) \leq a\} \in \mathcal{M}$.
Since $\mathcal{M}$ is closed under complements, $\{x \in X : f(x) \leq a\} \in \mathcal{M} \Rightarrow \{x \in X : f(x) < a\}^C = \{x \in X : f(x) \geq a\} \in \mathcal{M}$.
Thus, containment of each of the 4 sets in $\mathcal{M}$ is equivalent.

3.7 Suppose $f, g, \{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}}$ are measurable functions.
Suppose further that $F : (im(f), im(g)) \to \mathbb{R}$ is continuous. Then $f + g$ and fg are continuous and therefore measurable.
Since f and g are measurable on $(X, \mathcal{M})$, $\forall a \in \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$.
$\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$. Thus, $\max(f(x), g(x))$ is measurable. $|f| = max(f, -f)$ so by 1, $|f|$ is measurable.

3.14 Suppose $f : X \to \mathbb{R}$ and $f(x) \leq M$ $\forall x \in X$. Let $\epsilon > 0$.
By the Archimedian property, $\exists N_1 \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$ $\forall n \geq N_1$.
Define $N = \max\{M, N_1\}$. Then $\forall x \in X, \exists i$ such that $x \in E_i^N$. Then $\forall n \geq N$ we have $|s_n(x) - f(x)| < \epsilon$ $\forall x \in X$.

4.13 Suppose f measurable, $||f|| < M$ on $E \in M$, and $\mu(E) < \infty$.
Then $\int_E ||f|| d\mu < \int_E M d\mu = M\mu(E) < \infty$ since $M, \mu(E) < \infty$ by assumption.
By definition, $\int_E ||f|| d\mu < \infty \Rightarrow f \in \mathcal{L}^1(E, \mu)$.

4.14 Proof by contrapositive:
Suppose f is infinite on $E' \subseteq E$ and $\mu(E') > 0$.
Then $f(E') = \infty = \int_{E'} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu$.
$\int_E ||f|| d\mu \leq \infty \Rightarrow f \notin \mathcal{L}^1(E, \mu)$.
So by contrapositive we have shown that $f \in \mathcal{L}^1(E, \mu) \Rightarrow$ if f is infinite on a set $A, then \mu(A) = 0$.

4.15 Suppose $f, g \in \mathcal{L}^1(\mu, E)$ and $f(x) \leq g(x) \forall x \in E$.
Define $h(x) = g(x) - f(x)$ $\forall x \in E$. Then $h(x) \geq 0$ $\forall x \in E$.
By linearity of Lebesque integrals, as is alluded to in the notes and is readily verifiable: $\int_E h d\mu = \int_E (g - f) d\mu = \int_E g d\mu - \int_E f d\mu \geq 0$.
So $\int_E f d\mu \leq \int_E g d\mu$.

4.16 Suppose $f \in \mathcal{L}^1(\mu, E)$, $A \in \mathcal{M}$, and $A \subseteq E$.
$f \in \mathcal{L}^1(\mu, E) \Rightarrow \int_E ||f|| d\mu = \int_E f^+ d\mu + \int_E f^- d\mu < \infty$.
Since $\int_E f^+ d\mu < \infty$ and $A \subseteq E$, $\int_A f^+ d\mu < \infty$.
Since $\int_E f^- d\mu < \infty$ and $A \subseteq E$, $\int_A f^- d\mu < \infty$.
Since $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$, $f \in \mathcal{L}^1(\mu, A)$.

4.21 Suppose $A, B \in \mathcal{M}$, $\mu(A \ /B) = 0$, $B \in A$, and $f \in \mathcal{L}^1$.
Then by 4.19, $\lambda(A) = \int_A f d\mu$ is a measure. Thus it is countably subadditive, $A \ /B \cap B = \emptyset$, and $\int_{A/B} f d\mu = 0$ by 4.6, so:
$\lambda(A) = \int_A f d\mu = \int_{A/B} f d\mu + \int_B f d\mu = \int_B f d\mu = \lambda(A \ /B) + \lambda(B) = \lambda(B)$.
So $\int_A f d\mu = \int_B f d\mu$.