ProbSet 3
Spectral Theory
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4.2
D is the matrix representation of the derivative operator:
$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$
Since D is upper triangular, its eigenvalues are its diagonal entries, so 0 is an eigenvalue with algebraic multiplicity 3. Geometric multiplicity of 0 is 1 because only constants, the space spanned by $\{1\}$, have a derivative of 0.

4.4
i) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A = A^H$.
$A = A^H \Rightarrow a = \overline{a}$, $d = \overline{d}$, $b = \overline{c}, c = \overline{b} \Rightarrow a, d \in \mathbb{R}$, and $bc = \overline{c}c = \|c\|^2 \in \mathbb{R}$.
By 4.3 and using $det(A) = ad - bc$ the characteristic polynomial has the form:
$p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - (a+d)\lambda + ad - \|c\|^2$
Solutions to $p(\lambda) = 0$ are:
$\lambda_{\pm} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - \|c\|^2)}}{2} = \frac{(a+d) \pm \sqrt{(a-d)^2 + \|c\|^2)}}{2}$
$(a-d)^2 + \|c\|^2) \geq 0 \Rightarrow \lambda_{\pm} \in \mathbb{R}$.

ii) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $A^H = -A$.
$A^H = -A \Rightarrow a = -\overline{a}$, $d = -\overline{d}$, $b = -\overline{c} \Rightarrow a, d \in \mathbb{C}$ and $bc = -\overline{c}c = -\|c\|^2 < 0$ and $ad < 0$.
Using 4.3 the characteristic polynomial has the form:
$p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - (a+d)\lambda + ad + \|c\|^2$
Solutions to $p(\lambda) = 0$ are:
$\lambda_{\pm} = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad + \|c\|^2)}}{2} = \frac{(a+d) \pm \sqrt{(a-d)^2 + \|c\|^2)}}{2}$
$(a-d)^2 < 0$ and $\|c\|^2) < 0$ so $(a-d)^2 + \|c\|^2) < 0$
$\Rightarrow \lambda_{\pm} = \frac{(a+d) \pm \sqrt{(a-d)^2 + \|c\|^2)}}{2}$ is imaginary.

4.6
Let $R \in \mathbb{M}_n(\mathbb{F})$ be an upper-triangular matrix with diagonal entries r_{ii}.
Then $\lambda I - R$ is also upper-triangular and so $detR = \Pi_{i=1}^{n}(\lambda_i - r_{ii})$. Since r_{ii} are the roots of the characteristic polynomials, $\lambda_i = r_{ii}$.

4.8

i) Define $S = \{sin(x), cos(x), sin(2x), cos(2x)\}$ in the vector space $C^\infty(\mathbb{R}, \mathbb{R})$
Recall that we proved that S forms an orthonormal basis in math ProbSet2 under an inner product. Thus they are linearly independent and therefore are a basis of the space that they span.

ii) $D_S = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

iii) $span\{cosx, sinx\}$ and $span\{cos(2x), sin(2x)\}$

4.13

$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

4.15

Suppose $A \in M_n(\mathbb{F})$ is semisimple, $(\lambda_i)_{i=1}^n$ are the eigenvalues of A, and $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial.
By 4.3.7, A is diagonalizable, so $\exists D$ diagonal with eigenvalues of A along the diagonal and $\exists P$ invertible such that $A = PDP^{-1}$.
$f(A) = a_0I + a_1A + \cdots + a_nA^n$
$= a_0PP^{-1} + a_1PDP^{-1} + a_2PD^2P^{-1} + \cdots + a_nPD^nP^{-1} = Pf(D)P^{-1}$
Since D is diagonal with $(\lambda_i)_{i=1}^n$ along its diagonal, $f(D)$ is a diagonal with diagonal entries $(f(\lambda_i))_{i=1}^n$.
$f(A) = Pf(D)P^{-1} \Rightarrow f(D)$ is similar to $f(A)$
$\Rightarrow f(A)$ and $f(D)$ have the same eigenvalues
$\Rightarrow$ eigenvalues of $f(A)$ are $(f(\lambda_i))_{i=1}^n$.

4.16

i) Define $A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$ and $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$ For $n \in \mathbb{N}$, $A^n = PD^nP^{-1} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.4^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 + 0.4^n & 2 - 2 * 0.4^n \\ 1 - 0.4^n & 1 + 2 * 0.4^n \end{bmatrix}$
So $lim_{n\to\infty} A^n = \frac{1}{3} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

ii) With proofs similar to the one above, B is also the limit of A^n for the ∞-norm and Frobenius norm.

iii) Eigenvalues of A are 1 and .4. Using 4.3.12, the eigenvalues of f(A) where $f(x) = 3 + 5x + x^3$ are $f(1) = 9$ and $f(.4) = 5.064$.

4.18

Suppose λ is an eigenvalue of $A \in M_n(\mathbb{F})$.
Then for some eigenvector x, $Ax = \lambda x$.
By taking the transpose of both sides, $(Ax)^T = x^T A^T = (\lambda x)^T = \lambda x^T$.

4.20

Suppose $A = A^H$ and A is orthonormally similar to B.
Then $B = U^H AU$ for some orthonormal matrix U.
By taking the transpose of both sides, $B^H = (U^H AU)^H = U^H A^H U = U^H AU = B$.
$\Rightarrow B = B^H$.

4.24

Suppose $A = A^H$.
Then $< x, Ax >= x^H Ax = (A^H x)^H x =< A^H x, x >=< Ax, x >= \overline{< x, Ax >}$
$\Rightarrow < x, Ax > \in \mathbb{R}$.
Since necessarily $||x||^2 \in \mathbb{R}$, $\rho(x) \in \mathbb{R}$ for a Hermitian matrix A.

Suppose $B^H = -B$.
$< x, Bx >= x^H Bx = (B^H x)^H x = (-Bx)^H x =< -Bx, x >= (-1) < Bx, x >= -1\overline{< x, Bx >}$.
$\Rightarrow < x, Bx >$ is imaginary.
Since necessarily $||x||^2 \in \mathbb{R}$, $\rho(x)$ is imaginary for a skew Hermitian matrix B.

4.25

i) Let $x \in \mathbb{C}^n$. Then there exist $\{a_i\}_{i=1}^n$ such that $x = \sum_i a_i x_i$, since $\{x_i\}_{i=1}^n$ is a basis. Then:$\left(\sum_j x_j x_j^H\right) \sum_i a_i x_i = \sum_j x_j x_j^H a_j x_j + \sum_j \sum_{i \neq j} x_j x_j^H a_i x_i = \sum_j a_j x_j$ because $x_j^H x_j = 1$ for any j and $x_j^H x_i = 0$ for any $i \neq j$.
So $(\sum_j x_j x_j^H)x = x$ for any x in $\mathbb{C}^n$.
$\Rightarrow \sum_j x_j x_j^H = I$.

ii) We have: $Ax = \sum_j A a_j x_j = \sum_j a_j \lambda_j x_j$
and $\left(\sum_j \lambda_j x_j x_j^H\right)(\sum_i a_i x_i) = \sum_j \lambda_j x_j x_j^H a_j x_j + \sum_j \sum_{i \neq j} \lambda_j x_j x_j^H a_i x_i = \sum_j a_j \lambda_j x_j$,
$\Rightarrow A = \sum_j \lambda_j x_j x_j^H$.

4.27

Suppose $A \in M_n(\mathbb{F})$ is positive definite.
Since $A = A^H, < x, Ax > \in \mathbb{R}\ \forall x$ by my proof for 4.24.

Let $\{e_i\}_{i=1}^{n}$ be the standard basis of $\mathbb{F}^n$.
Then $< e_i, Ae_i >$ is the diagonal entry in the i^{th} column of A. Since A is positive definite, $< e_i, Ae_i > > 0$.
So each diagonal entry of A is positive and real.

4.36

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$