Math Problem Set #2

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3.1

i.) $\frac{1}{4}(||x+y||^2 - ||x-y||^2)$
$= \frac{1}{4}(\sqrt{< x+y, x+y >}^2 - \sqrt{< x-y, x-y}^2)$
$= \frac{1}{4}(< x+y, x+y > - < x-y, x-y >)$
$= \frac{1}{4}((< x,x > +2 < x,y > + < y,y >) - (< x,x > -2 < x,y > + < y,y >)$
$= \frac{1}{4}(4 < x,y >) = < x,y >$

ii.) $\frac{1}{2}(||x+y||^2 + ||x-y||^2)$
$= \frac{1}{2}(\sqrt{< x+y, x+y >}^2 + \sqrt{< x-y, x-y}^2)$
$= \frac{1}{2}(< x+y, x+y > + < x-y, x-y >)$
$= \frac{1}{4}((< x,x > +2 < x,y > + < y,y >) + (< x,x > -2 < x,y > + < y,y >)$
$= \frac{1}{2}(2 < x,x > +2 < y,y >)$
$= ||x||^2 + ||y||^2$

3.2 $\frac{1}{4}(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2)$
$= \frac{1}{4}(||x+y||^2 - ||x-y||^2) + \frac{1}{4}(i||x-iy||^2 - i||x+iy||^2)$
$= < x,y > + \frac{1}{4}(i\sqrt{< x-iy, x-iy >}^2 - i\sqrt{< x+iy, x+iy >}^2)$ by 3.1.i
$= < x,y > + \frac{1}{4}(i < x-iy, x-iy > -i < x+iy, x+iy >)$
$= < x,y > + \frac{1}{4}(i(< x, x-iy > + < -iy, x-iy >) - i(< x, x+iy > + < iy, x+iy >))$
$= < x,y > + \frac{1}{4}(i(< x,x > + < x,-iy > + < -iy,x > + < -iy,-iy >)$
$\quad - i(< x,x > + < x,iy > + < iy,x > + < iy,iy >))$
$= < x,y > + \frac{1}{4}(i(< x,x > +i < x,y > +i < y,x > - < y,y >)$
$\quad - i(< x,x > -i < x,y > -i < y,x > - < y,y >))$
$= < x,y > + \frac{1}{4}(i < x,x > - < x,y > - < y,x > -i < y,y >$
$\quad - i < x,x > + < x,y > + < y,x > +i < y,y >)$
$= < x,y > + \frac{1}{4}(- < x,y > + < x,y >)$
$= < x,y >$

3.3

i.) $< x, x^5 > = \int_0^1 x^6 dx = \frac{1}{7}$
$||x|| = \sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}}$
$||x^5|| = \sqrt{\int_0^1 x^5 dx} = \sqrt{\frac{1}{11}}$
$cos\theta = \frac{\frac{1}{7}}{\sqrt{\frac{1}{3}}\sqrt{\frac{1}{11}}}$
$\theta \approx 35°$

ii.) $< x^2, x^4 > = \int_0^1 x^6 dx = \frac{1}{7}$

$||x^2|| = \sqrt{\int_0^1 x^4 dx} = \sqrt{\frac{1}{5}}$
$||x^4|| = \sqrt{\int_0^1 x^8 dx} = \sqrt{\frac{1}{9}}$
$cos\theta = \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}}\sqrt{\frac{1}{9}}}$
$\theta \approx 16°$

3.8

i) $||cos(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(t)dt = \frac{1}{\pi}\frac{cos(x)sin(x)+x}{2}\Big|_{-\pi}^{\pi} = 1$

$||sin(t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(t)dt = \frac{1}{\pi}\frac{-sin(2x)+2x}{4}\Big|_{-\pi}^{\pi} = 1$

$||cos(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} cos^2(2t)dt = \frac{1}{\pi}\frac{sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$

$||sin(2t)||^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^2(2t)dt = \frac{1}{\pi}\frac{-sin(4x)+4x}{8}\Big|_{-\pi}^{\pi} = 1$

$\Rightarrow \forall s \in S,\ ||s|| = \sqrt{1} = 1$

$< \cos(t), \sin(t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(t)dt = \frac{sin^2(x)}{2}\Big|_{-\pi}^{\pi} = sin^2(\pi) - sin^2(-\pi) = 0$

$< \cos(t), \cos(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\cos(2t)dt = \frac{1}{\pi}\ frac3\sin(t) - 2\sin^3(t)3\big|_{-\pi}^{p} i = 0$

$< \cos(t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(t)\sin(2t)dt = \frac{1}{\pi}\ frac{-2}\cos^3(t)3\big|_{-\pi}^{p} i = 0$

$< \cos(2t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)\sin(2t)dt = \frac{1}{\pi}\ \frac{-\cos^2(2t)}{4}\Big|_{-\pi}^{p} i = 0$

Equivalent integrals can demonstrate that $< s_i, s_j >= 0$ whenever $i \neq j$. Therefore the set S is orthonormal.

ii) $||t||^2 = \int_{-\pi}^{\pi} t^2 dt = \frac{t^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^3}{3} - \frac{(-\pi)^3}{3} = \frac{2\pi^3}{3}$
$||t|| = \sqrt{\frac{2\pi^3}{3}}$

iii) $proj_X(cos(3t)) =< sin(t), cos(3t) > sin(t)+ < cos(t), cos(3t) > cos(t)$
$+ < sin(2t), cos(3t) > sin(2t)+ < cos(2t), cos(3t) > cos(2t)$
$= 0 + 0 + 0 + 0 = 0$

iv)$< \sin(t), t >= \sin(t) - t\cos(t)|_{-\pi}^{\pi}= 2\pi,$
$< \cos(t), t >= t\sin(t) - \cos(t)|_{-\pi}^{\pi}= 0,$
$< \cos(2t), t >= (2t\sin(2t) + \cos(2t))/4|_{-\pi}^{\pi}= 0$
$< \sin(2t), t >= \sin(2x) - 2x\cos(2x)/4|_{-\pi}^{\pi}= -\pi$
So $\text{proj}_X(t) = 2\pi\sin(t) - \pi\sin(2t)$

3.9

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^T \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} = \begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & 0\\ 0 & \cos^2(\theta)+\sin^2(\theta)\end{bmatrix} =$$
$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^T = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

3.10

i) Suppose $Q \in M_n(\mathbb{F})$ is orthonormal.
Let $x, y \in \mathbb{F}^n$.
Then $< x, y >=< Qx, Qy >$
$< x, y >= x^H y =< Qx, Qy >= (Qx)^H(Qy) = x^H Q^H Q y$
$x^H y = x^H Q^H Q y \Rightarrow Q^H Q = I_n$
Since Q is orthonormal, Q is invertible. Since inverses are unique, $Q^H = Q^{-1}$, so we must also have $QQ^H = I_n$.

Suppose that $QQ^H = Q^H Q = I_n$.
Let $x, y \in \mathbb{F}^n$.
Then $< x, y >= x^H y = x^H Q^H Q y = (Qx)^H(Qy) =< Qx, Qy >$
$\Rightarrow Q$ is orthonormal.

ii) Suppose $Q \in M_n(\mathbb{F})$ is orthonormal.
Then $||Qx|| = \sqrt{< Qx, Qx >} = \sqrt{x^H Q^H Q x} = \sqrt{< x, x >} = ||x||$

iii) Suppose $Q \in M_n(\mathbb{F})$ is orthonormal.
Then $Q^H Q = QQ^H = I$, so $Q^{-1} = Q^H$ since inverses are unique.
Since $(Q^H)^H = Q$, $QQ^H = (Q^H)^H Q^H = I = Q^H Q = Q^H (Q^H)^H$
$\Rightarrow Q^H$ is orthonormal by 3.10.i

iv) Suppose $Q \in M_n(\mathbb{F})$ is orthonormal.
Then $< x, y >=< Qx, Qy > \ \forall x, y \in \mathbb{F}$.
Let $e_1, e_2, ...e_n$ be the standard basis of $\mathbb{F}$.
Then for $i, j \in 1, 2...n$, $< Qe_i, Qe_j >=< e_i, e_j >= \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$
and Qe_i is the i^{th} column of Q.
$\Rightarrow$ columns of Q are orthonormal.

v) Suppose $Q \in M_n(\mathbb{F})$ is orthonormal.
By 3.10.i we have $QQ^H = I_n$. So $det(Q)det(Q^H) = det(I_n) = 1$.
Recall that $det(A) = det(A^T)\ \forall A \in \mathbb{F}^n$, and the complex conjugate of matrix has the same determinant as the original matrix.
$\Rightarrow det(Q) = det(Q^T) = det(Q^H)$.
So $det(Q)det(Q^H) = (det(Q))^2 = det(I_n) = 1$
$\Rightarrow |det(Q)| = 1$

The converse is not true. Consider $B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. $det(B) = 1$ yet B is not orthonormal.

vi) Suppose $Q_1, Q_2 \in M_n(\mathbb{F})$ are orthonormal.
Then $Q_1^H Q_1 = I_n = Q_2^H Q_2$.
Let $x, y \in \mathbb{F}$. Then:
$< Q_1Q_2x, Q_1Q_2y >= (Q_1Q_2x)^H Q_1Q_2y = x^H Q_2^H Q_1^H Q_1 Q_2 y$
$\quad = x^H Q_2^H Q_2 y = x^H y =< x, y >$
$\Rightarrow Q_1Q_2$ is orthonormal.

3.11

Let V be an inner product space. Suppose $S = \{v_1, v_2, ...v_n\} \subset V$ are linearly dependent. Then $\exists v_i \in S$ such that $v_i \in span(v_1, ..., v_{i-1})$. Therefore $v_i = proj_{Q_{k-1}} v_i$, so we find ourselves dividing by zero while trying to compute q_{i+1}.

3.16

i) Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$
This shows that A can be expressed as a product of an orthonormal matrix and a diagonal matrix in two distinct ways. Thus, QR decomposition is not unique.

ii) Let A be invertible and suppose that QR and $Q'R'$ are QR decompositions of A such that R and R' have only positive diagonal elements.
Then $QR = Q'R'$ and $Q^{'-1}Q = R'R^{-1}$. Name this equality B, so $B = Q^{'-1}Q = R'R^{-1}$.
Since R, R' are upper triangular withs strictly positive diagonal entries, $Q^{'-1}Q = R'R^{-1}$ is upper triangular withs strictly positive diagonal entries.
Since Q' is orthonormal, $Q^{'-1}$ is orthonormal by 3.10.iii. Since $Q^{'-1}$ and Q are orthonormal, $Q^{'-1}Q$ is orthonormal by 3.10.vi.
So B is upper triangular with strictly positive diagonal entries and is also orthonormal. Therefore $B = I_n$.
So $B = Q^{'-1}Q = R'R^{-1} = I_n$. Since orthonormal matrices are invertible and inverses are unique, we must have $Q = Q'$ and $R = R'$. Thus, the QR decomposition is unique.

3.17

Suppose $A \in M_{m\ x\ n}$ and $A = QR$ is a reduced QR decomposition and that we want to solve the system $A^H Ax = A^H b$.
Since Q is orthonormal, we have $Q^H Q = I_n$, so:
$A^H Ax = (QR)^H QRx = R^H Q^H QRx = R^H Rx$
Further, $A^H b = (QR)^H b = R^H Q^H b$.
So solving $A^H Ax = A^H b$ is equivalent to solving $R^H Rx = R^H Q^H b$.
Since R is upper triangular, R is invertible. Therefore R^H is invertible. So $\exists (R^H)^{-1}$ such that $(R^H)^{-1} R^H = R^H (R^H)^{-1} = I_n$.
So $Rx = (R^H)^{-1} R^H Rx = (R^H)^{-1} R^H Q^H b = Q^H b$
Therefore solving $A^H Ax = A^H b$ is equivalent to solving $Rx = Q^H b$.

3.23

By the triangle inequality:

$$|\|x\| - \|y\|| = \begin{cases} \|x\| - \|y\| \leq \|x - y\| + \|y\| - \|y\| = \|x - y\| & \|x\| \geq \|y\| \\ \|y\| - \|x\| \leq \|y - x\| + \|x\| - \|x\| = \|y - x\| = \|x - y\| & \|x\| \geq \|y\| \end{cases}$$

3.24

i) Consider $||f|| = \int_a^b |f(t)|dt$.
For $f \in C([a,b];\mathbb{F})$ with $f \neq 0, ||f|| > 0$. Additionally, if $f = 0$, then $||f|| = 0$. So positivity holds for $||f||$.
Let $c \in \mathbb{F}$. Then:
$||cf|| = \int_a^b |cf(t)|dt = \int_a^b |c||f(t)|dt = |c| \int_a^b |f(t)|dt = |c|||f||$
So preservation of scale holds for $||f||$.
Let $g \in C([a,b];\mathbb{F})$.
Since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$ for all t, then
$||f + g|| = \int_a^b |f(t) + g(t)|dt \leq \int_a^b (|f(t)| + |g(t)|)dt = \int_a^b |f(t)|dt + \int_a^b |g(t)|dt$.
So the triangle inequality holds for $||f||$.

ii) Consider $||f|| = (\int_a^b |f(t)|^2 dt)^{\frac{1}{2}}$.
For $f \in C([a,b];\mathbb{F})$ with $f \neq 0, ||f|| > 0$. Additionally, if $f = 0$, then $||f|| = 0$. So positivity holds for $||f||$.
Let $c \in \mathbb{F}$. Then:
$||cf|| = (\int_a^b |cf(t)|^2 dt)^{\frac{1}{2}} = (\int_a^b |c|^2 |f(t)|^2 dt)^{\frac{1}{2}} = (|c|^2 \int_a^b ||f(t)|^2 dt)^{\frac{1}{2}} = |c|||f||$
So preservation of scale holds for $||f||$.
Let $g \in C([a,b];\mathbb{F})$. Then $||f+g|| = (\int_a^b |f(t)+g(t)|dt)^{\frac{1}{2}} \leq \int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt = ||f|| + ||g||$.
So the triangle inequality holds for $||f||$.

iii) Consider $||f|| = sup_{x \in [a,b]} |f(x)|$.
For $f \in C([a,b];\mathbb{F})$ with $f \neq 0, ||f|| > 0$. Additionally, if $f = 0$, then $||f|| = 0$. So positivity holds for $||f||$.
Let $c \in \mathbb{F}$. Then:

$||cf|| = sup_{x\in[a,b]}|cf(x)| = sup_{x\in[a,b]}|c||f(x)| = |c|sup_{x\in[a,b]}|f(x)|dt = |c|||f||$
So preservation of scale holds for $||f||$.
Let $g \in C([a,b];\mathbb{F})$.
$||f+g|| = sup_{x\in[a,b]}|f(x)+g(x)| \leq sup_{x\in[a,b]}|f(x)| + sup_{x\in[a,b]}|g(x)| = ||f|| + ||g||$.
So the triangle inequality holds for $||f||$.

3.26 Review

Showing that topological equivalence is an equivalence relation:
1) Let $M \geq m$, then $m||x||_a \leq ||x||_a \leq M||x||_1$ for all $x \in X$. So $||\cdot||_a \sim ||\cdot||_a$.
2) Suppose $||\cdot||_a \sim ||\cdot||_b$. Then $m||x||_a \leq ||x||_b \leq M||x||_a$ $\forall x \in X$ for some $0 < m \leq M$. Then $M^{-1}||x||_b \leq ||x||_a \leq m^{-1}||x||_b$. So $||\cdot||_b \sim ||\cdot||_a$
3) Suppose $||\cdot||_a \sim ||\cdot||_b$, and $||\cdot||_b \sim ||\cdot||_c$. Then for some $0 < m \leq M, 0 < k \leq K$ such that $m||x||_a \leq ||x||_b \leq M||x||_a$ and $k||x||_b \leq ||x||_c \leq ||x||_b$ $\forall x \in X$. Then $mn||x||_a \leq ||x||_c \leq MN||x||_a$, so $||x||_a \sim ||x||_c$.

i) $||\cdot||_1 \sim ||\cdot||_2$:
$(||x||_1)^2 = \sum_{i=1}^n \sum_{j=1}^n |x_i||x_j| \geq \sum_{i=1}^n x_i^2 = \langle x, x, \rangle = (||x||_2)^2$
$\Rightarrow ||x||_1 \geq ||x||_2$.
Additionally: $\sum_{i=1}^n |x_i| \cdot 1 \leq \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \left(\sum_{i=1}^n 1^2\right)^{1/2} = \sqrt{n}\left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$
So $||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$.
$\Rightarrow ||\cdot||_1 \sim ||\cdot||_2$:

ii) $||x||_\infty = max_{1\leq i\leq n}\{x_i\} = \sqrt[2]{(max_{1\leq i\leq n}\{x_i\})^2} \leq \sqrt[2]{\sum_{i=1}^n x_i} = ||x||_2$ and $||x||_2^2 = \sum_{i=1}^n |x_i|^2 \leq nmax_i\{x_i\} = (\sqrt{n}||x|_\infty|)^2 \implies ||x||_2 = \sqrt{n}||x||_\infty$
so $||x||\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$
$\Rightarrow ||\cdot||_\infty \sim ||\cdot||_2$

3.28

i)Using 3.26:
$\sup_{x\neq 0} \frac{||Ax||_1}{||x||_1} \leq \sup_{x\neq 0} \frac{||Ax||_1}{||x||_1} \leq \sqrt{n}\sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$
Further:
$\sup_{x\neq 0} \frac{||Ax||_1}{||x||_1} \geq \sup_{x\neq 0} \frac{||Ax||_2}{||x||_1} \geq \frac{1}{\sqrt{n}} \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$
$\Rightarrow \frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq \sqrt{n}||A||_2$.

ii)We have:
$\sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \leq \sup_{x\neq 0} \frac{\sqrt{n}||Ax||_\infty}{||x||_\infty}$
Additionally:
$\sup_{x\neq 0} \frac{||Ax||_2}{||x||_2} \geq \sup_{x\neq 0} \frac{||Ax||_\infty}{\sqrt{n}||x||_\infty}$
$\Rightarrow \frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{n}||A||_\infty$.

3.29

Let Q be an orthonormal matrix. Then:
$||Qx|| = ||x|| \implies sup_{x\neq 0}\frac{||Qx||}{||x||} = ||Q|| = 1$

Define $R_x : M_n(\mathbb{F}) \to \mathbb{F}, R_x(A) = Ax$.
Using 3.28.ii:
$\|R_x\| = sup_{x\neq 0}(\frac{\|Ax\|}{\|A\|}) = sup_{x\neq 0}(\frac{\|Ax\|\|x\|}{\|A\|\|x\|}) \leq (\frac{\|Ax\|\|x\|}{\|Ax\|}) = \|x\|$

3.30

Let $S \in M_n(\mathbb{F})$ be invertible. For $A \in M_n(\mathbb{F})$ define $||A||_S = ||SAS^{-1}||$.
i)Since $||.||$ is a norm, it follows that $||A||_S = ||SAS^{-1}|| \geq 0\ \forall A \in M_n(\mathbb{F})$. Further $SAS^{-1} = 0$ iff $A = 0$, So since $||.||$ is a norm, $||A||_S = ||SAS^{-1}|| = 0$ iff $A = 0$. So $||A||_S$ has positivity.

ii) Let $k \in \mathbb{F}$.
$||kA||_S = ||SkAS^{-1}|| = ||kSAS^{-1}|| = k||SAS^{-1}|| = k||A||_S$
so $||A||_S$ has scalar preservation.

iii) Let $B \in M_n(\mathbb{F})$.
$||(A + B)||_S = ||S(A + B)S^{-1}|| = ||SAS^{-1} + SBS^{-1}|| \leq ||||SAS^{-1}| + ||SBS^{-1}|| = ||A||_S + ||B||_S$
So the triangle inequality holds for $||A||_S$.

iv)$||AB||_S = ||SABS^{-1}|| = ||SAS^{-1}SBS^{-1}|| \leq ||SAS^{-1}|| \cdot ||SBS^{-1}|| = ||A||_S \cdot ||B||_S$
so $||A||_S$ is submultiplicative.

3.37

Take the standard basis $\mathcal{B} = \{1, x, x^2\}$.
We have $L(1) = 0, L(x) = 1, L(x^2) = 2$
So $q = [0\ 1\ 2]$ is the desired vector for the basis $\mathcal{B}$.

3.38

Take $\mathcal{B} = \{1, x, x^2\}$ and let A be the matrix representation of the differentiation operator and the A* the matrix representation of the adjoint. Then:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^* = -A = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

3.39

Let $S, T \in \mathcal{L}(V, W)$ and $S*, T*$ the adjoints of S and T.
i) $\langle (S+T)v, w\rangle = \langle Sv, w\rangle + \langle Tv, w\rangle = \langle v, S^*w\rangle + \langle v, T^*w\rangle = \langle v, (S^*+T^*)w\rangle$
$\langle \alpha T^* v, w\rangle = \alpha\langle Tv, w\rangle = \alpha\langle v, T^*w\rangle = \langle v, \overline{\alpha}T^*w\rangle$

ii) $\langle S^*v, w\rangle = \overline{\langle w, S^*v\rangle} = \overline{\langle Sw, v\rangle} = \langle v, Sw\rangle$

Let $S, T \in \mathcal{L}(V)$ and $S*, T*$ the adjoints of S and T.
iii) $\langle STv, w\rangle = \langle Tv, S^*w\rangle = \langle v, T^*S^*w\rangle$

iv) $\langle T^*(T^{-1})^*x, y\rangle = \langle (T^{-1})^*x, Ty\rangle = \langle x, (T^{-1})Ty\rangle = \langle x, y\rangle$
$\Rightarrow T^*(T^{-1})^* = I$

3.40

i)$< A^H B, C >= tr((A^H B)^H C) = tr(B^H AC) =< B, AC >=< A^*B, C >$

ii) $< B, AC >_F =$ $\mathrm{tr}(B^H AC) = \mathrm{tr}((A^H B)^H C) =< A^H B, C >_F$

iii) Let $B, C \in \mathbb{M}_n(\mathbb{F})$.
Then $< B, AC - CA >=< B, AC > - < B, CA >$.
Using ii: $< B, CA >=< BA^*, C >$
Further: $< B, AC >= \mathrm{tr}(B^H AC) = \mathrm{tr}((A^H B)^H C) =< A^H B, C >=< A^*B, C >$
So we have $T_A^* = T_{A^*}$.

3.44

Suppose $\exists x \in \mathbb{F}^n$ such that $Ax = b$.
Then $\forall\ y \in \mathcal{N}(A^H)$, $< y, b >=< y, Ax >=< A^H y, x >=< 0, x >= 0$.

Next, suppose $\exists y \in \mathcal{N}(A^H)$ such that $< y, b >\neq 0$. Then $b \notin \mathcal{N}(A^H)^\perp = \mathcal{R}(A)$. So $\forall x \in \mathbb{F}^n$, $Ax \neq b$.

3.45

Let $A \in \mathrm{Sym}_n(\mathbb{R})$ and $B \in \mathrm{Skew}_n(\mathbb{R})$.
Then $< B, A >= \mathrm{Tr}(B^T A) = \mathrm{Tr}(AB^T) = Tr(A^T(-B)) = - < A, B >$. $\Rightarrow< A, B >= 0$ and $\mathrm{Skew}_n(\mathbb{R}) \subset \mathrm{Sym}_n(\mathbb{R})^\perp$.

Now suppose $B \in \mathrm{Sym}_n(\mathbb{R})^\perp$. We have $B + B^T \in \mathrm{Sym}_n(\mathbb{R})$. So: $0 =< B + B^T, B >= \mathrm{Tr}((B + B^T)B) = \mathrm{Tr}(BB + B^T B) = \mathrm{Tr}(BB) + \mathrm{Tr}(B^T B)$
$\Rightarrow< B^T, B >=< -B, B >$ and so $B^T = -B$.
$\Rightarrow \mathrm{Sym}_n(\mathbb{R})^\perp = \mathrm{Skew}_n(\mathbb{R})$.

3.47 Suppose $A \in M_{mxn}$ and rank(A)=n. Let $P = A(A^H A)^-1 A^H$.
i) $P^2 = (A(A^H A)^{-1} A^H)(A(A^H A)^{-1} A^H) = A(A^H A)^{-1} A^H A(A^H A)^{-1} A^H$
$= A(A^H A)^{-1} A^H = P$

ii)$P^H = (A(A^H A)^{-1} A^H)^H = (A^H)^H (A^H A)^{-H} A^H = A(A^H A)^{-1} A^H = P$

iii) rank(A) = $\Rightarrow$ rank(P) *leq* n.
Let $y \in im(A)$. Then $\exists x \in \mathbb{F}^n$ such that $y = Ax$.
So $Py = A(A^H A) A^H y = A(A^H A)^{-1} A^H Ax = Ax = y$
$\Rightarrow y \in im(P)$
$\Rightarrow$ rank(P)$\geq$ rank(A)
$\Rightarrow$rank(P) = rank(A) = n.

3.50
The equation for the ellipse is equivalent to $y^2 = 1/s + rx^2/s$.
We can express this as $Ax = b$ where $b_i = [y_1^2, y_2^2, ..., y_n^2]^T$, $A_i = (1\ x_i)$ and $x = (\frac{1}{s}\frac{r}{s})^T$
Then the normal equations are $A^H A\hat{x} = A^H b$, where
$A^H A\hat{x} = \begin{bmatrix} \sum_i 1 & \sum_i x_i^2 \\ \sum_i x_i^2 & \sum_i x_i^4 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} n\hat{\beta}_1 - \hat{\beta}_2 \sum_i x_i^2 \\ \hat{\beta}_1 \sum_i x_i^2 - \hat{\beta}_2 \sum_i x_i^4 \end{bmatrix}$ and $A^H b = \begin{bmatrix} \sum_i y_i^2 \\ \sum_i x_i^2 y_i^2 \end{bmatrix}$.