Problem Set #4

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Assignment: 8.1-8.12, 8.15,8.17,8.18

8.1

see jupyter notebook

8.2

see jupyter notebook

8.3

Let x_{GI} be the number of GI dolls produced in a week and x_J the number of Joey dolls produced in a week.

$max_{x_{GI},x_J} 4x_{GI} + 3x_J$ $s.t.$:

$\frac{1}{4}x_{GI} + \frac{1}{6}x_J \leq 30$
$\frac{1}{30}x_{GI} + \frac{1}{30}x_J \leq 5$
$0 \leq x_J \leq 200$
$x_{GI} \geq 0$

8.4

Let $x_{i,j}$ denote the number units moving from i to j.

$min_{x_{A,B},x_{A,D},x_{B,C},x_{B,E},x_{B,D},x_{B,F},x_{C,F},x_{D,E},x_{E,F}} 2x_{A,B} + 5x_{A,D} + 5x_{B,C} + 7x_{B,E} + 2x_{B,D} + 9x_{B,F} + 2x_{C,F} + 4x_{D,E} + 3x_{E,F}$ $s.t.$:

$x_{A,B} + x_{A,D} = 10$
$x_{B,C} + x_{B,E} + x_{B,D} + x_{B,F} - x_{A,B} = 1$
$x_{C,F} - x_{B,C} = -2$
$x_{D,E} - x_{A,D} - x_{B,D} = -3$
$x_{E,F} - x_{D,E} - x_{E,F} = 4$
$-x_{C,F} - x_{B,F} - x_{E,F} = -10$
$0 \leq x_{A,B} \leq 6$
$0 \leq x_{A,D} \leq 6$
$0 \leq x_{B,C} \leq 6$
$0 \leq x_{B,E} \leq 6$
$0 \leq x_{B,D} \leq 6$
$0 \leq x_{B,F} \leq 6$
$0 \leq x_{C,F} \leq 6$
$0 \leq x_{D,E} \leq 6$
$0 \leq x_{E,F} \leq 6$

8.5

(i)

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 \ s.t.: \\ & x_1 + 3x_2 + w_1 = 15 \\ & 2x_1 + 3x_2 + w_2 = 18 \\ & x_1 - x_2 + w_3 = 4 \\ & x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

$$\begin{array}{ccccccc} \zeta & = & & & 3x_1 & + & x_2 \\ \hline w_1 & = & 15 & - & x_1 & - & 3x_2 \\ w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\ w_3 & = & 4 & - & x_1 & + & x_2 \\ \hline\hline \zeta & = & 12 & + & 4x_2 & - & 3w_3 \\ \hline w_1 & = & 11 & - & 4x_2 & + & w_3 \\ w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\ x_1 & = & 4 & + & x_2 & - & w_3 \\ \hline\hline \zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\ \hline w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\ x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\ x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3 \\ \hline \end{array}$$

Optimizer: $(6, 2)$
Optimum value: 20

(ii)

$$\begin{aligned} \text{maximize} \quad & 4x + 6y \\ \text{subject to} \quad & -x + 3x_2 + w_1 = 11 \\ & x + y + w_2 = 27 \\ & 2x + 5y + w_3 = 90 \\ & x, y, w_1, w_2, w_3 \geq 0 \end{aligned}$$

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4x & + & 6y \\
\hline
w_1 & = & 11 & + & x & - & y \\
w_2 & = & 27 & - & x & - & y \\
w_3 & = & 90 & - & 2x & - & 5y \\
\hline\hline
\zeta & = & 66 & + & 10x & - & 6w_1 \\
\hline
y & = & 11 & + & x & - & w_1 \\
w_2 & = & 16 & - & 2x & + & w_1 \\
w_3 & = & 35 & - & 7x & + & 5w_1 \\
\hline\hline
\zeta & = & 116 & + & \frac{8}{7}w_1 & - & \frac{10}{7}w_3 \\
\hline
y & = & 16 & - & \frac{2}{7}w_1 & - & \frac{1}{7}w_3 \\
w_2 & = & 6 & - & \frac{3}{7}w_1 & + & \frac{2}{7}w_3 \\
x & = & 5 & + & \frac{5}{7}w_1 & - & \frac{1}{7}w_3 \\
\hline\hline
\zeta & = & 132 & - & \frac{8}{3}w_2 & - & \frac{2}{7}w_3 \\
\hline
y & = & 12 & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \\
w_1 & = & 14 & - & \frac{7}{3}w_2 & + & \frac{2}{3}w_3 \\
x & = & 15 & - & \frac{5}{3}w_2 & + & \frac{1}{3}w_3 \\
\hline
\end{array}
$$

Optimizer: $(15, 12)$
Optimum value: 132

8.6

$$
\begin{array}{ll}
\text{maximize} & 4b + 3j \\
\text{subject to} & 3b + 2j + w_1 = 360 \\
& b + j + w_2 = 150 \\
& j + w_3 = 200 \\
& b, j, w_1, w_2, w_3 \geq 0
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4b & + & 3j \\
\hline
w_1 & = & 360 & - & 3b & - & 2j \\
w_2 & = & 150 & - & b & - & j \\
w_3 & = & 200 & - & j & & \\
\hline\hline
\zeta & = & 450 & + & b & - & 3w_2 \\
\hline
w_1 & = & 60 & - & b & + & w_2 \\
j & = & 150 & - & b & - & w_2 \\
w_3 & = & 50 & + & b & + & w_2 \\
\hline\hline
\zeta & = & 510 & - & w_1 & - & 2w_2 \\
\hline
b & = & 60 & - & w_1 & + & w_2 \\
j & = & 90 & + & w_1 & - & 2w_2 \\
w_3 & = & 110 & - & w_1 & + & 2w_2 \\
\hline
\end{array}
$$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls
Max profit: $510

8.7

(i)

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 2x_2 \\
\text{subject to} \quad & -4x_1 - 2x_2 + x_3 = -8 \\
& -2x_1 + 3x_2 + x_4 = 6 \\
& x_1 + x_5 = 3 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Auxiliary problem:

$$
\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{subject to} \quad & -4x_1 - 2x_2 + x_3 - x_0 = -8 \\
& -2x_1 + 3x_2 + x_4 - x_0 = 6 \\
& x_1 + x_5 - x_0 = 3 \\
& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
x_5 & = & 3 & - & x_1 & & & + & x_0 \\
\hline\hline
\zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & x_3 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & x_3 \\
x_4 & = & 14 & - & 2x_1 & - & 5x_2 & + & x_3 \\
x_5 & = & 11 & - & 5x_1 & - & 2x_2 & + & x_3 \\
\hline\hline
\zeta & = & & & & & & - & x_0 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & - & \frac{1}{4}x_0 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & + & \frac{1}{2}x_0 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & + & \frac{5}{4}x_0 \\
\hline
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\hline
\zeta & = & 2 & + & \frac{3}{2}x_2 & + & \frac{1}{4}x_3 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 \\
\hline\hline
\zeta & = & 3 & + & 2x_2 & - & x_5 \\
\hline
x_1 & = & 3 & & & - & x_5 \\
x_4 & = & 12 & - & 3x_2 & - & 2x_5 \\
x_3 & = & 4 & + & 2x_2 & - & 4x_5 \\
\hline\hline
\zeta & = & 11 & - & \frac{2}{3}x_4 & - & \frac{7}{3}x_5 \\
\hline
x_1 & = & 3 & & & - & x_5 \\
x_2 & = & 4 & - & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 \\
x_3 & = & 4 & - & \frac{2}{3}2x_4 & - & \frac{16}{3}x_5 \\
\hline
\end{array}
$$

Optimal point: $(3, 4)$
Optimal value: 11

(ii)

$$
\begin{array}{ll}
\text{maximize} & 5x_1 + 2x_2 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 = 15 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& 4x_1 - 3x_2 + x_5 = -12 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Auxiliary problem:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & 5x_1 + 3x_2 + x_3 - x_0 = 15 \\ & 3x_1 + 5x_2 + x_4 - x_0 = 15 \\ & 4x_1 - 3x_2 + x_5 - x_0 = -12 \\ & x_0, x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

$$\begin{array}{cccccccccc}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
x_4 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
x_5 & = & -12 & - & 4x_1 & + & 3x_2 & + & x_0 \\
\hline\hline
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & x_5 \\
\hline
x_3 & = & 27 & - & x_1 & - & 6x_2 & + & x_5 \\
x_4 & = & 27 & + & x_1 & - & 8x_2 & + & x_5 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & x_5 \\
\hline\hline
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}x_4 & - & \frac{5}{8}x_5 \\
\hline
x_3 & = & \frac{27}{4} & - & \frac{7}{4}x_1 & + & \frac{3}{4}x_4 & + & \frac{1}{4}x_5 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}x_4 & + & \frac{1}{8}x_5 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}x_4 & + & \frac{5}{8}x_5 \\
\hline
\end{array}$$

Coefficients of the objective function of the auxillary problem are negative, but the optimum is negative. Thus, the original problem has no feasible points.

(iii)

$$\begin{aligned} \text{maximize} \quad & -3x_1 + x_2 \\ \text{subject to} \quad & x_2 + x_3 = 4 \\ & -2x_1 + 3x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{array}{cccccccc}
\zeta & = & & - & 3x_1 & + & x_2 \\
\hline
x_3 & = & 4 & & & - & x_2 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2 \\
\hline\hline
\zeta & = & 2 & - & \frac{7}{3}x_1 & - & \frac{1}{3}x_4 \\
\hline
x_3 & = & 2 & - & \frac{2}{3}x_1 & + & \frac{1}{3}x_4 \\
x_2 & = & 2 & + & \frac{2}{3}x_1 & - & \frac{1}{3}x_4 \\
\hline
\end{array}$$

Optimal point: $(0, 2)$
Optimal value: 2

8.8

$$\begin{aligned} \text{maximize} \quad & -x-y-2z \\ \text{subject to} \quad & -x+y \leq 2 \\ & x-y \leq 3 \\ & x-z \leq 4 \\ & -x+z \leq 5 \\ & x,y,z \geq 0 \end{aligned}$$

The unique feasible maximizer is $(0,0,0)$.

8.9

$$\begin{aligned} \text{maximize} \quad & x+y+z \\ \text{subject to} \quad & -x+y \leq 1 \\ & x-y \leq 1 \\ & x-z \leq 1 \\ & -x+z \leq 1 \\ & x,y,z \geq 0 \end{aligned}$$

8.10

$$\begin{aligned} \text{maximize} \quad & -x-y-z \\ \text{subject to} \quad & x+y \leq -1 \\ & x,y,z \geq 0 \end{aligned}$$

8.11

$$
\begin{aligned}
\text{maximize} \quad & x + y + z \\
\text{subject to} \quad & -x + y \leq 1 \\
& x - y \leq 1 \\
& x - z \leq 1 \\
& -x + z \leq 1 \\
& -x - y - z \leq -1 \\
& x + y + z \leq 5 \\
& x, y, z \geq 0
\end{aligned}
$$

The auxillary problem is :

$$
\begin{aligned}
\text{maximize} \quad & -x_0 \\
\text{subject to} \quad & -x + y + x_0 \leq 1 \\
& x - y + x_0 \leq 1 \\
& x - z + x_0 \leq 1 \\
& -x + z + x_0 \leq 1 \\
& -x - y - z + x_0 \leq -1 \\
& x + y + z + x_0 \leq 5 \\
& x, y, z \geq 0
\end{aligned}
$$

8.12

$$
\begin{aligned}
\text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\
\text{subject to} \quad & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\
& 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\
& x_1 + x_7 = 0 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned}
$$

$$
\begin{array}{rcrcrcrcrcr}
\zeta & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
x_5 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_6 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
\hline\hline
\zeta & = & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_5 \\
\hline
x_1 & = & & & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_5 \\
x_6 & = & & & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_5 \\
\hline\hline
\zeta & = & 1 & - & 30x_2 & - & 42x_4 & - & 18x_5 & - & x_7 \\
\hline
x_1 & = & 1 & & & & & & & - & x_7 \\
x_6 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_5 & - & 2x_7 \\
x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_5 & - & x_7 \\
\hline
\end{array}
$$

8.15

Suppose that x is a primal feasible point and y is a dual feasible point. Then we have:

$Ax \preceq b$
$x \succeq 0$
$A^T y \succeq c$
$y \succeq 0$

So using these inequalities we have:
$b^T y \geq (Ax)^T y = x^T A^T y \geq x^T c = (x^T c)^T = c^T x$

8.17

Suppose we have this linear optimization problem:

max $c^T x$ subject to:
$Ax \preceq 0$
$x \succeq 0$

Then the dual of this problem is:

min $b^T y$ subject to:
$A^T y \succeq c$
$y \succeq 0$

The dual of the dual problem therefore is:

max $c^T x$ subject to:
$(A^T)^T x \preceq b$
$x \succeq 0$

which is equivalent to the original problem.

8.18

The dual is:

$$\begin{aligned}
\min\ & 3y_1 + 5y_2 + 4y_3 \\
\text{subject to}\ & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}$$

Putting the dual into standard form, we get:

$$\begin{aligned}
\max\ & -3y_1 - 5y_2 - 4y_3 \\
\text{subject to}\ & -2y_1 - y_2 - 2y_3 \leq -1 \\
& -y_1 - 3y_2 - 3y_3 \leq -1 \\
& y_1, y_2, y_3 \leq 0
\end{aligned}$$

Solving the primal problem yields:

$$\begin{array}{ccccccc}
\zeta & = & & & x_1 & + & x_2 \\
\hline
w_1 & = & 3 & - & 2x_1 & - & x_2 \\
x_2 & = & 5 & - & x_1 & - & 3x_2 \\
x_5 & = & 4 & - & 2x_1 & - & 3x_2 \\
\hline\hline
\zeta & = & \frac{3}{2} & + & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\
\hline
x_1 & = & \frac{3}{2} & - & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\
w_2 & = & \frac{7}{2} & - & \frac{5}{2}x_2 & + & \frac{1}{2}w_1 \\
w_3 & = & 1 & - & 2x_2 & + & w_1 \\
\hline\hline
\zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_1 \\
\hline
x_1 & = & \frac{5}{4} & - & \frac{3}{4}w_1 & + & \frac{1}{4}w_3 \\
w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3 \\
\hline
\end{array}$$

The function achieves its optimum $\frac{7}{4}$ at the point $(\frac{5}{4}, \frac{1}{2})$

To solve the dual we make an auxillary problem:

$$\begin{aligned}
\text{maximize}\ & -y_0 \\
\text{subject to}\ & -2y_1 - y_2 - 3y_3 - y_0 \leq -1 \\
& -y_1 - 3y_2 - 3y_3 - y_0 \leq -1 \\
& y_1, y_2, y_3, y_0 \leq 0
\end{aligned}$$

$$
\begin{array}{ccccccccccc}
\zeta & = & & & & & & & & - & v_0 \\
\hline
v_1 & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & + & v_0 \\
v_2 & = & -1 & + & y_1 & + & 3y_2 & + & 3y_3 & + & v_0 \\
\hline\hline
\zeta & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & - & v_1 \\
\hline
v_0 & = & 1 & - & 2y_1 & - & y_2 & - & 2y_3 & + & v_1 \\
v_2 & = & & - & y_1 & + & 2y_2 & + & y_3 & + & v_1 \\
\hline\hline
\zeta & = & & & & & & & & - & v_0 \\
\hline
y_2 & = & 1 & - & 2y_1 & - & 2y_3 & + & v_1 & - & v_0 \\
v_2 & = & 2 & - & 5y_1 & - & 3y_3 & + & 3v_1 & - & 2v_0 \\
\hline\hline
\end{array}
$$

$$
\begin{array}{ccccccccc}
\zeta & = & -2 & + & y_1 & - & 3y_2 & - & 2v_1 \\
\hline
y_3 & = & \frac{1}{2} & - & y_1 & - & \frac{1}{2}y_2 & + & \frac{1}{2}v_1 \\
v_2 & = & \frac{1}{2} & - & 2y_1 & + & \frac{3}{2}y_2 & + & \frac{3}{2}v_1 \\
\hline\hline
\zeta & = & -\frac{7}{4} & - & \frac{3}{2}y_2 & - & \frac{5}{4}v_1 & - & \frac{1}{2}v_2 \\
\hline
y_3 & = & \frac{1}{4} & - & 2\frac{3}{2}y_2 & - & \frac{1}{4}v_1 & + & \frac{1}{2}v_2 \\
y_1 & = & \frac{1}{4} & + & \frac{3}{2}y_2 & + & \frac{3}{4}v_1 & - & \frac{1}{2}v_2 \\
\hline
\end{array}
$$

The optimum of $\frac{7}{4}$ is achieved at the point $(\frac{1}{4}, 0, \frac{1}{4})$.