Signal processing on graphs

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Signals on graphs

- Consider a dataset with N elements, with a relational structure
- Represent this information by a graph $G = (\mathcal{V}, \mathbf{A})$, where
  $\mathcal{V} = \{v_0, \ldots, v_{N-1}\}$ is the set of nodes and
  $\mathbf{A}$ is the weighted adjacency matrix of the graph.
  Each weight $\mathbf{A}_{n,m}$ is the weight of a *directed* edge from v_m to v_n
  which can take arbitrary real or complex values.
  $\mathcal{N}_n = \{m \mid \mathbf{A}_{n,m} \neq 0\}$ is the *neighborhood* of v_n
- A *graph signal* is a map from the set $\mathcal{V}$ of nodes into the set of complex numbers $\mathbb{C}$, writen as

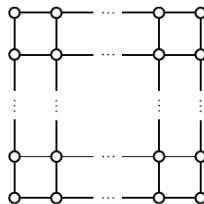$$\mathbf{s} = \begin{pmatrix} s_0 & s_1 & \ldots & s_{N-1} \end{pmatrix}^T$$

  each element s_n being *indexed* by node v_n
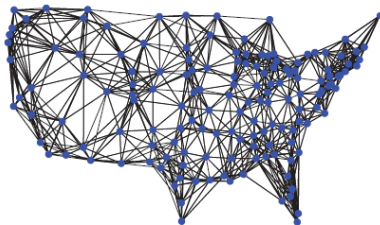- The space $\S$ of graph signals is identical to $\mathbb{C}^N$.
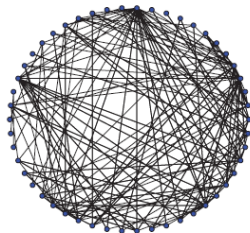
Some graphs and data sets
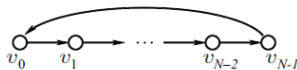
(a) Time series

(b) Digital image

(c) Sensor field

(d) Hyperlinked documents

Some graphs and data sets

- (a) A finite, periodic discrete time series: All edges are directed and have the same weight 1
- (b) A digital image. Each node corresponds to a pixel, and each pixel value (intensity) is related to the values of the four adjacent pixels. All edges are undirected and have the same weight, with possible exceptions of boundary nodes
- (c) The temperature measurements from 150 weather stations (sensors). The relations of temperature measurements are the geodesic distances between sensors, each node being connected to its closest neighbors
- (d) A set of 50 blogs in the World Wide Web connected by hyperlink references. The edges are directed and have the same weights.

Filters on graphs

- **Filters:** systems that take a signal as input and produce another signal as output.
- For *graph filters* one considers only linear, shift-invariant filters, using Jordan normal form and characteristic and minimal polynomials of matrices.
- Use of Jordan decomposition is required since for many real-world datasets the adjacency matrix **A** is not diagonalizable.
- *Graph shift*
  A generalization of the *time shift* or *delay* that delays the input signal **s** by one sample $\widetilde{s}_n = s_{n-1} \,(\mathrm{mod}\,N)$
  For the time series:

$$\mathbf{A}_{n,m} = \begin{cases} 1 & \text{if } n - m = 1 \,(\mathrm{mod}\,N) \\ 0 & \text{otherwise} \end{cases},$$

  the time shift is

$$\widetilde{\mathbf{s}} = \mathbf{A}\mathbf{s}.$$

Filters on graphs

- For a general graph $G = (\mathcal{V}, \mathbf{A})$, the *graph shift* is realized by replacing the sample s_n at node v_n with the weighted linear combination of the signal samples at its neighbors:

$$\tilde{s}_n = \sum_{m=0}^{N-1} \mathbf{A}_{n,m} s_m = \sum_{m \in \mathcal{N}_n} \mathbf{A}_{n,m} s_m.$$

- Any matrix $\mathbf{H} \in \mathbb{C}^{N \times N}$, is a (linear) *graph filter*, that for input $\mathbf{s} \in \S$ produces output $\mathbf{Hs}$

- Is a *linear* system,

$$\mathbf{H}(\alpha \mathbf{s}_1 + \beta \mathbf{s}_2) = \alpha \mathbf{Hs}_1 + \beta \mathbf{Hs}_2$$

- *Shift-invariant graph filters:* applying the graph shift to the output is equivalent to applying the graph shift to the input prior to filtering:

$$\mathbf{A}\left(\mathbf{Hs}\right) = \mathbf{H}\left(\mathbf{As}\right)$$

- All linear, shift-invariant graph filters are given by *polynomials* in the shift $\mathbf{A}$.

Recall on matrices

- Let $\lambda_0, \ldots, \lambda_{M-1}$ denote $M \leq N$ distinct eigenvalues of $\mathbf{A} \in \mathbb{C}^{N \times N}$. Let each eigenvalue λ_m have D_m linearly independent eigenvectors $\mathbf{v}_{m,0}, \ldots, \mathbf{v}_{m,D_m-1}$. D_m is the *geometric multiplicity* of λ_m.

- Each eigenvector $\mathbf{v}_{m,d}$ generates a *Jordan chain* of $R_{m,d} \geq 1$ linearly independent *generalized eigenvectors* $\mathbf{v}_{m,d,r}$, $0 \leq r < R_{m,d}$, where $\mathbf{v}_{m,d,0} = \mathbf{v}_{m,d}$, that satisfy

$$(\mathbf{A} - \lambda_m \mathbf{1})\mathbf{v}_{m,d,r} = \mathbf{v}_{m,d,r-1}.$$

- For each eigenvector $\mathbf{v}_{m,d}$ and its Jordan chain of length $R_{m,d}$, we define a *Jordan block* matrix of dimension $R_{m,d}$ as

$$J_{r_{m,d}}(\lambda_m) = \begin{pmatrix} \lambda_m & 1 & & \\ & \lambda_m & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_m \end{pmatrix} \in \mathbb{C}^{R_{m,d} \times R_{m,d}}.$$

Jordan normal form

Thus, each eigenvalue λ_m is associated with D_m Jordan blocks, each with dimension $R_{m,d}$, $0 \leq d < D_m$. Next, for each eigenvector $\mathbf{v}_{m,d}$, we collect its Jordan chain into a $N \times R_{m,d}$ matrix

$$V_{m,d} = \begin{pmatrix} \mathbf{v}_{m,d,0} & \dots & \mathbf{v}_{m,d,R_{m,d}-1} \end{pmatrix}.$$

We concatenate all blocks $V_{m,d}$, $0 \leq d < D_m$ and $0 \leq m < M$, into one block matrix

$$V = \begin{pmatrix} V_{0,0} & \dots & V_{M-1,D_{M-1}} \end{pmatrix},$$

so that $V_{m,d}$ is at position $\sum_{k=0}^{m-1} D_k + d$ in this matrix. Then, matrix $\mathbf{A}$ has a block-diagonal *Jordan decomposition*

$$\mathbf{A} = VJV^{-1},$$

the *Jordan normal form* of $\mathbf{A}$

$$J = \begin{pmatrix} J_{R_{0,0}}(\lambda_0) & & \\ & \ddots & \\ & & J_{R_{M-1,D_{M-1}}}(\lambda_{M-1}) \end{pmatrix}$$

Jordan normal form: An example

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix} \qquad V = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad VJV^{-1} = \mathbf{A} \qquad VV^{T} \neq I$$

$$\mathbf{A}V\left(:,1\right) = V\left(:,1\right) \qquad \mathbf{A}V\left(:,2\right) = 2V\left(:,2\right)$$
$$\mathbf{A}V\left(:,3\right) = 4V\left(:,3\right) \qquad \left(\mathbf{A} - \mathbf{4I}\right)V\left(:,4\right) = V\left(:,3\right)$$

Minimal and characteristic polynomials

The *minimal polynomial* of matrix $\mathbf{A}$ is the monic polynomial of smallest possible degree that satisfies $m_{\mathbf{A}}(\mathbf{A}) = 0_N$. Let $R_m = \max\{R_{m,0}, \ldots, R_{m,D_m-1}\}$ denote the maximum length of Jordan chains corresponding to eigenvalue λ_m. Then the minimal polynomial $m_{\mathbf{A}}(x)$ is given by

$$m_{\mathbf{A}}(x) = (x - \lambda_0)^{R_1} \ldots (x - \lambda_{M-1})^{R_{M-1}}.$$

The *index* of λ_m is R_m, $1 \leq m < M$. Any polynomial $p(x)$ that satisfies $p(\mathbf{A}) = 0_N$, is a polynomial multiple of $m_{\mathbf{A}}(x)$, i.e., $p(x) = q(x)m_{\mathbf{A}}(x)$. The degree of the minimal polynomial satisfies

$$\deg m_{\mathbf{A}}(x) = N_{\mathbf{A}} = \sum_{m=0}^{M-1} R_m \leq N.$$

The *characteristic polynomial* of the matrix **A** is defined as

$$p_{\mathbf{A}}(x) = \det(\lambda 1 - \mathbf{A}) = (x - \lambda_0)^{A_0} \ldots (x - \lambda_{M-1})^{A_{M-1}}.$$

Here: $A_m = R_{m,0} + \ldots + R_{m,D_m-1}$ for $0 \leq m < M$, is the *algebraic multiplicity* of λ_m; $\deg p_{\mathbf{A}}(x) = N$; $p_{\mathbf{A}}(x)$ is a multiple of $m_{\mathbf{A}}(x)$; and $p_{\mathbf{A}}(x) = m_{\mathbf{A}}(x)$ if and only if the geometric multiplicity of each λ_m, $D_m = 1$, i.e., each eigenvalue λ_m has exactly one eigenvector.

Filters on graphs

- **Theorem:** Let **A** be the graph adjacency matrix and assume that its characteristic and minimal polynomials are equal: $p_\mathbf{A}(x) = m_\mathbf{A}(x)$. Then, a graph filter **H** is linear and shift invariant if and only if (iff) **H** is a *polynomial* in the graph shift **A**, i.e., iff there exists a polynomial

$$h(x) = h_0 + h_1 x + \ldots + h_L x^L$$

  with possibly complex coefficients $h_\ell \in \mathbb{C}$, such that:

$$\mathbf{H} = h(\mathbf{A}) = h_0 \mathbf{1} + h_1 \mathbf{A} + \ldots + h_L \mathbf{A}^L.$$

- The coefficients h_ℓ in the polynomial $h(x)$ are called the graph filter *taps*.

- The theorem requires the equality of the characteristic and minimal polynomials $p_\mathbf{A}(x)$ and $m_\mathbf{A}(x)$. This condition does not always hold, but can be successfully addressed through the concept of *equivalent* graph filters, as defined next.

Filters on graphs. Properties

- **Definition:** Given any shift matrices $\mathbf{A}$ and $\widetilde{\mathbf{A}}$, filters $h(\mathbf{A})$ and $g(\widetilde{\mathbf{A}})$ are called *equivalent* if for all input signals $\mathbf{s} \in \S$ they produce equal outputs: $h(\mathbf{A})\mathbf{s} = g(\widetilde{\mathbf{A}})\mathbf{s}$.

- When no restrictions are placed on the signals, so that $\S = \mathbb{C}^N$, this definition is equivalent to requiring $h(\mathbf{A}) = g(\widetilde{\mathbf{A}})$ as matrices. However, if additional restrictions exist, filters may not necessarily be equal as matrices and still produce the same output for the considered set of signals.

- Given an arbitrary $G = (\mathcal{V}, \mathbf{A})$ with $p_{\mathbf{A}}(x) \neq m_{\mathbf{A}}(x)$, we can consider another graph $\widetilde{G} = (\mathcal{V}, \widetilde{\mathbf{A}})$ with the same set of nodes $\mathcal{V}$ but potentially different edges and edge weights, for which $p_{\widetilde{\mathbf{A}}}(x) = m_{\widetilde{\mathbf{A}}}(x)$ holds true. Then graph filters on G can be expressed as equivalent filters on $\widetilde{G}$

- **Theorem**: For any matrix $\mathbf{A}$ there exists a matrix $\widetilde{\mathbf{A}}$ and polynomial $r(x)$, such that $\mathbf{A} = r(\widetilde{\mathbf{A}})$ and $p_{\widetilde{\mathbf{A}}}(x) = m_{\widetilde{\mathbf{A}}}(x)$.

Filters on graphs. Properties

- As a consequence any filter on the graph $G = (\mathcal{V}, \mathbf{A})$ is equivalent to a filter on the graph $\widetilde{G} = (\mathcal{V}, \widetilde{\mathbf{A}})$, since $h(\mathbf{A}) = h(r(\widetilde{\mathbf{A}})) = (h \circ r)(\widetilde{\mathbf{A}})$, where $h \circ r$ is the composition of polynomials h and r and thus is a polynomial. Thus, the condition $p_{\mathbf{A}}(x) = m_{\mathbf{A}}(x)$ can be assumed to hold for any graph $G = (\mathcal{V}, \mathbf{A})$. Otherwise, we can replace the graph by another $\widetilde{G} = (\mathcal{V}, \widetilde{\mathbf{A}})$ for which the condition holds and assign $\widetilde{\mathbf{A}}$ to $\mathbf{A}$.

- One can limit the number of taps in any graph filter.

- **Theorem**: Any graph filter has a unique equivalent filter on the same graph with at most $\deg m_{\mathbf{A}}(x) = N_{\mathbf{A}}$ taps.

All linear, shift-invariant filters on a graph $G = (\mathcal{V}, \mathbf{A})$ form a vector space

$$\mathcal{F} = \left\{ \mathbf{H} ::: \mathbf{H} = \sum_{\ell=0}^{N_{\mathbf{A}}-1} h_\ell \mathbf{A}^\ell \,\middle|\, h_\ell \in \mathbb{C} \right\}.$$

Filters on graphs. Properties

- Addition and multiplication of filters in $\mathcal{F}$ produce new filters that are equivalent to filters in $\mathcal{F}$. Thus, $\mathcal{F}$ is closed under these operations, and has the structure of an algebra.

- The inverse of a filter on a graph, if it exists, is also a filter on the same graph, i.e., it is a polynomial.

- **Theorem**: A graph filter $\mathbf{H} = h(\mathbf{A}) \in \mathcal{F}$ is invertible iff polynomial $h(x)$ satisfies $h(\lambda_m) \neq 0$ for all distinct eigenvalues $\lambda_0, \ldots, \lambda_{M-1}$, of $\mathbf{A}$. Then, there is a unique polynomial $g(x)$ of degree $\deg g(x) < N_{\mathbf{A}}$ that satisfies
$$h(\mathbf{A})^{-1} = g(\mathbf{A}) \in \mathcal{F}$$

- This theorem implies that instead of inverting the $N \times N$ matrix $h(\mathbf{A})$ directly we only need to construct a polynomial $g(x)$ specified by at most $N_{\mathbf{A}}$ taps.

- Any graph filter $h(\mathbf{A}) \in \mathcal{F}$ is completely specified by its taps $h_0, \cdots, h_{N_{\mathbf{A}}-1}$.

Filters on graphs. Properties

- The filter taps uniquely determine the *impulse response* of the filter, i.e., its output $\mathbf{u} = (g_0, \ldots, g_{N-1})^T$ for unit impulse input $\delta = (1, 0, \ldots, 0)^T$, and vice versa.
- **Theorem**: The filter taps $h_0, \ldots, h_{N_\mathbf{A}-1}$ of the filter $h(\mathbf{A})$ uniquely determine its impulse response $\mathbf{u}$. Conversely, the impulse response $\mathbf{u}$ uniquely determines the filter taps, provided $rank\left(\widehat{\mathbf{A}}\right) = N_\mathbf{A}$, where $\widehat{\mathbf{A}} = \left(\mathbf{A}^0 \delta, \ldots, \mathbf{A}^{N_\mathbf{A}-1} \delta\right)$.
- We can always label nodes so that $rank\left(\widehat{\mathbf{A}}\right) = N_\mathbf{A}$ holds and corresponds to the first column of h. It is invariant to reordering of nodes, in the sense that $\widetilde{h}$ is permuted by the same permutation that reorders the nodes.
- A relabeling of the nodes $v_0, \ldots, v_{N-1}$ does not change the impulse response. If $\mathbf{P}$ is the corresponding permutation matrix, then the unit impulse is $\mathbf{P}\delta$, the adjacency matrix is $\mathbf{PAP}^T$, and the filter becomes $h(\mathbf{PAP}^T) = \mathbf{P}h(\mathbf{A})\mathbf{P}^T$. Hence, the impulse response is simply reordered according to same permutation: $\mathbf{P}h(\mathbf{A})\mathbf{P}^T\mathbf{P}\delta = \mathbf{Pu}$

Signal transforms on graphs

- Social and economic networks, information networks, power grids, biological networks, etc. generate large sets of raw data which require a detailed analysis to extract useful information. The data that is generated in this way is the *vertex picture of the signal.* Like for time series an important step to make signal understandable is the construction of the appropriate signal transforms.

- **Question:** What is the "Fourier transform" for a graph signal ?

Recall the time series: The time series is a signal on a one-dimensional directed graph with the vertices labelled by times $(t_0, t_1, t_2, \cdots)$ and the edges connecting t_{k+1} to t_k. The adjacency matrix $\mathbf{A}$ is, in general

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

Signal transforms on graphs

or, for a time-periodic signal

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The signal transforms for time series are projections of the time signal on a set of eigenvectors of some linear operator. These operators are not arbitrary, but chosen to extract particular features of the signal that is being analyzed. The Fourier transform looks for the periodic features of the signals, wavelets for the multiscale features, etc.

Likewise, useful information from signals on arbitrary graphs may be obtained from projections on set of vectors associated to suitably chosen linear operators. For a time-periodic signal, the discrete Fourier transform is the projection on the eigenvectors of the adjacency matrix $\mathbf{A}$.

Signal transforms on graphs

- Therefore one may define the graph Fourier transform on an arbitrary graph as the projection on the eigenvectors (or on the generalized eigenvectors of the Jordan decomposition) of the adjacency matrix. This was the point of view taken by some authors to develop a theory of discrete signal processing on graphs. However this choice is not unique because, for the time series network, other matrices have the same spectrum, for example the Laplacian matrix

$$\mathbf{L} = \mathbf{G} - \mathbf{A}$$

- Hence the graph Fourier transform might as well be defined as a projection on the generalized eigenvectors of the Laplacian matrix.

Signal transforms on graphs

- Once the eigenvector problem (or the Jordan decomposition) is done these vectors are used to project the signals $\mathbf{s}$, the graph Fourier transform ($\mathbf{A}-$transform) being

$$\widehat{\mathbf{s}} = V^{-1}\mathbf{s}$$

  with inverse transform

$$\mathbf{s} = V\widehat{\mathbf{s}}$$

- For non-symmetric matrices the set of eigenvectors and generalized eigenvectors of the Jordan decomposition do not, in general, form an orthogonal basis. Therefore it might be more convenient to use $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ to generate the expansion basis, leading to what we will call the $\mathbf{A}\mathbf{A}^T-$ or $\mathbf{A}^T\mathbf{A}-$transform. The same procedure may be used for other graph matrices. Recall:

- *The Laplacian matrix:* $\mathbf{L} = \mathbf{G} - \mathbf{A}$
- *The symmetrically normalized Laplacian matrix:* $\mathbf{L}_{sym} = \mathbf{G}^{-\frac{1}{2}}\mathbf{L}\mathbf{G}^{-\frac{1}{2}}$
- *The random walk matrix:* $\mathbf{P} = \mathbf{G}^{-1}\mathbf{A}$

Signal transforms on graphs

- *The random walk Laplacian:* $\mathbf{L}_{RW} = \mathbf{G}^{-1}\mathbf{L}$
- *The lazy random walk matrix:* $\mathbf{P}' = \left(\mathbf{I} + \mathbf{G}^{-1}\mathbf{A}\right)/2$
- *The incidence matrix* ∇: is the $m \times N$ matrix (m=no. edges, N=no. of vertices) given by

$$\nabla_{e,v} = \left\{ \begin{array}{lll} 1 & \text{if} \quad e = (v, w) & \text{and } v < w \\ -1 & \text{if} \quad e = (v, w) & \text{and } v > w \\ 0 & \quad \text{otherwise} \end{array} \right.$$

- *The edge adjacency matrix:* is a $m \times m$ matrix determined by the adjacencies of edges

$$^{e}\mathbf{A}_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if edges i and j are adjacent} \\ 0 & \text{otherwise} \end{array} \right.$$

Recall on tomograms

- Three types of transforms: linear, bilinear and tomograms.

Linear: Fourier, Wavelets, Hilbert, ...

Bilinear: Wigner-Ville, Bertrand, ...

Rigorous simultaneous information on the time-frequency plane is impossible

$$[t, \omega] = \left[t, i\frac{d}{dt}\right] \neq 0$$

A joint probability density can never be defined.

- Time-frequency tomogram: scan the $t - \omega$ plane along lines. Strictly positive probability densities, a full characterization of the signal and robust in the presence of noise.

- May be generalized for other pairs of non-commuting variables.

- Signal $f(t)$ as a vector $|f\rangle$ in a subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$, a family of unitary operators $U(\alpha) = e^{iB(\alpha)}$ and a reference vector h in the dual $\mathcal{N}^*$ of $\mathcal{N}$

Tomograms

- *Linear transform*

$$W_f^{(h)}(\alpha) = \langle U(\alpha) h \mid f \rangle$$

- *Bilinear (quasidistribution)*

$$Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle$$

Let, the unitary operator $U(\alpha) = e^{iB(\alpha)}$, $B(\alpha)$ have the spectral decomposition $B(\alpha) = \int X P(X) \, dX$.

$$P(X) \overset{\circ}{=} |X\rangle \langle X|$$

- *Tomogram*

$$M_f^{(B)}(X) = \langle f | P(X) | f \rangle = \langle f | X \rangle \langle X | f \rangle = |\langle X | f \rangle|^2$$

The problem with bilinear ransforms

$$W(t, \omega) = (U(t, \omega) f, f)$$

Example: The Wigner-Ville transform

$$W(t, \omega) = \int f^* \left(t + \frac{\tau}{2} \right) f \left(t - \frac{\tau}{2} \right) e^{-i\omega\tau} d\tau$$

Wigner-Ville of a chirp: $\exp(it(\omega_0 + t\alpha))$

Wigner-Ville of a Chirp (log10(abs( )))
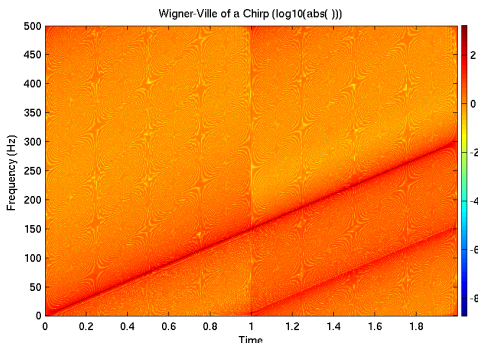
Tomograms

- The tomogram $M_f^{(B)}(X)$ is the squared amplitude of the projection of the signal $|f\rangle \in \mathcal{N}$ on the eigenvector $\langle X| \in \mathcal{N}^*$ of the operator $B(\alpha)$. Therefore it is positive and may be interpreted as a probability distribution on the set of generalized eigenvalues of $B(\alpha)$, that is, as the probability distribution for the random variable X corresponding to the observable defined by the operator $B(\alpha)$.

- Time-frequency:

$$B(\theta) = t\cos(\theta) + \omega\sin(\theta) = t\cos(\theta) + \sin(\theta)\, i\frac{d}{dt}$$

the tomogram is the expectation value of a projection operator with support on a line in the time–frequency plane Therefore, $M_f^{(S)}(X, \cos(\theta), \sin(\theta))$ is the marginal distribution of the variable X along this line in the time–frequency plane.

- Used sucessfully for a robust extraction of compound signal features, used in denoising, component separation and structure identification.

Component separation by tomograms

Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of $y_0(t)$ with a deformed chirp $y_R(t)$ delayed by 3 seconds and with with noise added

$$y_0(t) = e^{i\Phi_0(t)} \qquad\qquad y_R(t) = e^{i\Phi_R(t)}$$

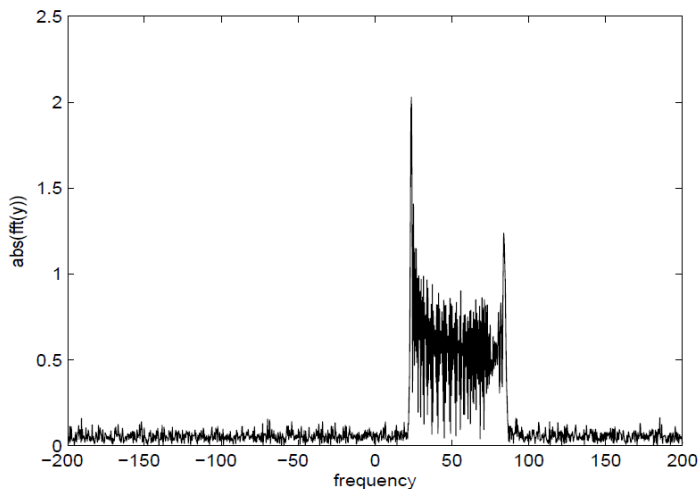$\Phi_0(t) = a_0 t^2 + b_0 t$ e $\Phi_R(t) = a_R(t - t_R)^2 + b_R(t - t_R) + 10(t - t_R)^{\frac{3}{2}}$.
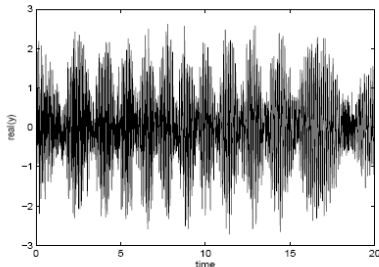
Component separation by tomograms

Comparison of the phase derivatives $\frac{d}{dt}\Phi_0(t)$ and $\frac{d}{dt}\Phi_R(t)$. Except for the first three seconds, the two signals $y_0(t)$ and $y_R(t)$ are superimposed both in time and phase derivative. Frequency separation is impossible.
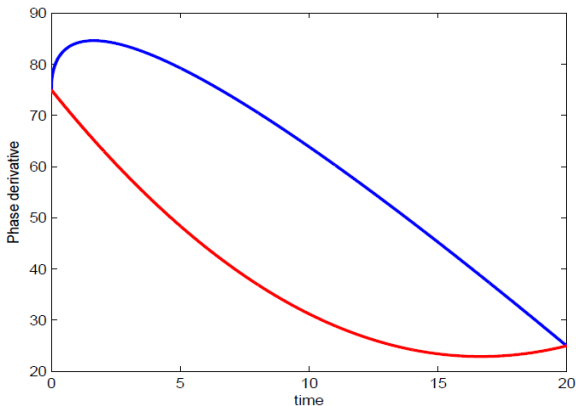
Component separation by tomograms

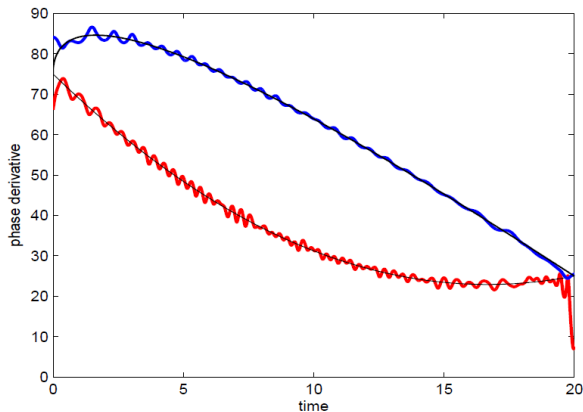- Frequency representation

The signal tomogram (contour plot)
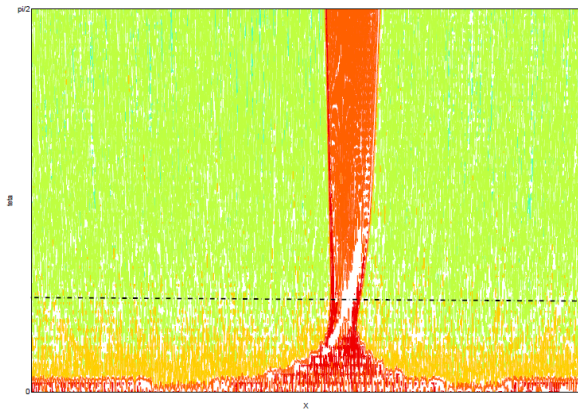
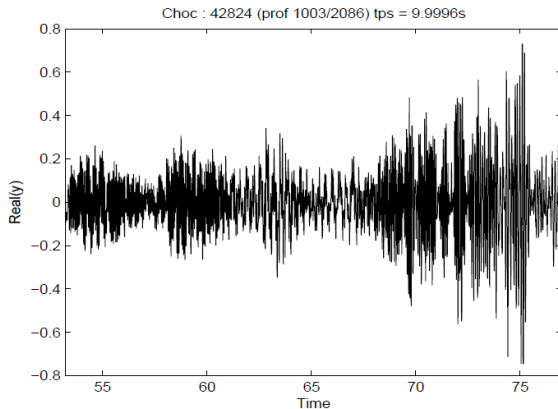# Component separation by tomograms

Cut at $\theta = \frac{\pi}{5}$
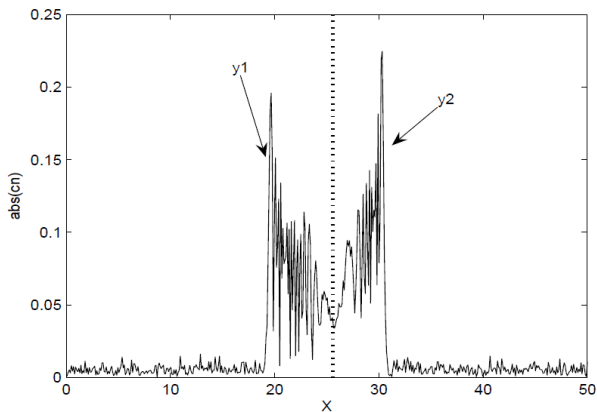
Component separation by tomograms

Comparison of the exact with the reconstructed phase derivatives
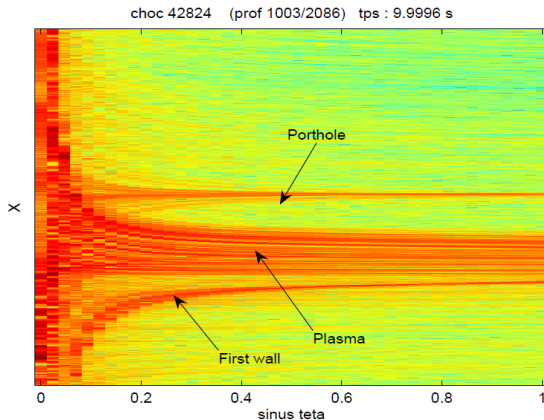
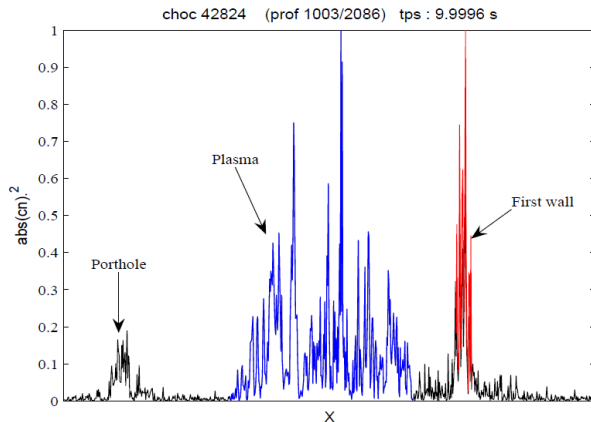# Component separation. Plasma reflectometry

A reflectometry signal

Choc : 42824 (prof 1003/2086) tps = 9.9996s

The tomogram

Cut at $\theta = \pi - \frac{\pi}{5}$

choc 42824   (prof 1003/2086)   tps : 9.9996 s

choc 42824   (prof 1003/2086)   tps : 9.9996 s
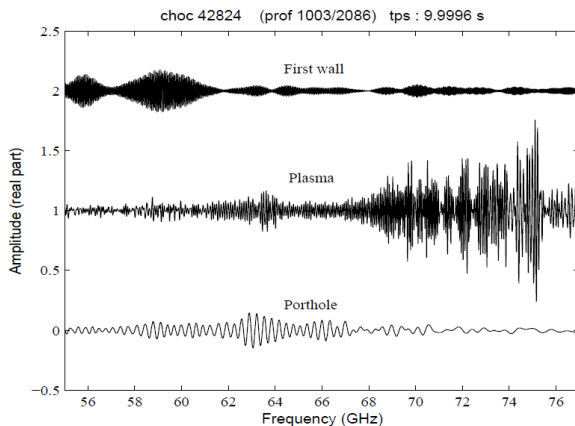
Graph tomograms

- So far signals on graphs have been described either as vectors on vertex space or as projections of these vectors on the generalized eigenvectors of a particular matrix $\mathbf{M}$ (the graph transforms). Each particular matrix emphasizes a specific topological property of the graph. Tomograms attempt to obtain information about more than one property by projecting on the generalized eigenvectors of a matrix that interpolates between two distinct matrices $\mathbf{M}_1$ and $\mathbf{M}_2$.

- When the vertex space has a meaningful physical interpretation it is useful to interpolate between one of the matrices $\mathbf{M}$ listed before and the matrix for which the vertex signal corresponds to a projection on its eigenvectors.

Tomograms on graphs

- For a graph with N vertices, the vectors on vertex space may be considered as projections on the eigenvectors of a *vertex operator*

$$
\mathbf{T} = \begin{pmatrix}
1 & 0 & 0 & \vdots & 0 \\
0 & e^{i\frac{2\pi}{N}} & 0 & \vdots & 0 \\
0 & 0 & e^{i\frac{2\pi}{N}\times 2} & \vdots & 0 \\
\cdots & \cdots & \cdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & e^{i\frac{2\pi}{N}\times(N-1)}
\end{pmatrix}
$$

- Therefore the construction of a tomogram for graph signals would amount to finding an operator that interpolates between T and $\mathbf{A}$. A solution could be the family of operators

$$
B_\alpha = (1-\alpha)\,\mathbf{T} + \alpha\mathbf{M}
$$

with α varying between 0 and 1, the tomogram being obtained by the projections of the signal on the eigenvectors of B_α

Tomograms on graphs

- When the Jordan decomposition of the **M** matrix is used, another operator family B^θ may be defined as follows
$$B_\theta = \mathbf{T}^{\cos\theta} + VJ_\theta V^{-1} - 1$$
where J_θ is similar to the Jordan normal form of **M** with each Jordan block replaced by

$$J_{r_{m,d}}(\lambda_m) = \begin{pmatrix} \lambda_m^{\sin\theta} & \sin\theta & & \\ & \lambda_m^{\sin\theta} & \ddots & \\ & & \ddots & \sin\theta \\ & & & \lambda_m^{\sin\theta} \end{pmatrix}$$

$$\mathbf{T}^{\cos\theta} = \begin{pmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & e^{i\frac{2\pi}{N}\cos\theta} & 0 & \vdots & 0 \\ 0 & 0 & e^{i\frac{2\pi}{N}\times 2\cos\theta} & \vdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i\frac{2\pi}{N}\times(N-1)\cos\theta} \end{pmatrix}$$

Tomograms on graphs

- The next step in the construction of the graph tomogram would be to find the similarity transformation that reduces B_θ to its Jordan form

$$J'_\theta = V_\theta^{-1} B_\theta V_\theta$$

Then the tomogram is obtained by the projections of the signal on the generalized eigenvalues of J'_θ, namely

$$\widehat{\mathbf{s}}_\theta = V_\theta^{-1} \mathbf{s}$$

- If $\mathbf{M}$ is the adjacency matrix $\mathbf{A}$, the above constructions, interpolating between $\mathbf{A}$ and the vertex operator $\mathbf{T}$, are for graphs, the analog of the time-frequency tomograms.

- In graphs, the vertex description and the adjacency matrix projection are incompatible specifications, because in general the $\mathbf{T}$ and $\mathbf{A}$ (or $\mathbf{L}$) matrices do not commute. It is in this sense, that, as recently stated, there is an uncertainty principle for graphs, that is, a fundamental trade-off between a signal's localization on a graph and in its spectral domain.

A market network

An important problem in the design of portfolios or ETF's is the classification of the dynamical behavior of the trading values of market products. Identifying clusters of products with similar behavior allows for the design of simpler portfolios, by the selection of representative elements in each cluster. Here we analyze the daily closing equity prices of 301 companies of the SP500 throughout the 250 trading days of 2012. For the purpose of the calculations the companies are ordered by sectors. From the daily returns

$$r(t) = \log S(t) - \log S(t-1)$$

$S(t)$ being the closing price at day t, one computes a dynamical distance between the company stocks i and j by

$$d_{ij} = \sqrt{\sum_{t=1}^{250} (r_i(t) - r_j(t))^2}$$

the sum being over the 250 trading days in 2012.

Applications

Now one computes the smallest non-zero d_{ij} ($d_{\min}$) and an adjacency matrix A with matrix elements A_{ij} may be defined either by

$$A_{ij}^{\#} = \frac{d_{\min}}{d_{ij}} \left(1 - \delta_{ij}\right)$$

or

$$A_{ij}^{(\beta)} = \left(1 - \delta_{ij}\right) \exp\left(-\beta \left(d_{ij} - d_{\min}\right)\right)$$

The second form is probably the most convenient one because, by varying β, one obtains a multiscale analysis of the dynamical similarities of the companies. In the next figures one shows the color-coded adjacency matrices A and $A^{(\beta=2)}$ that are obtained. One sees that the $A^{(\beta=2)}-$adjacency matrix provides a more detailed picture of the nature of correlations between the return behavior of these equities. From inspection of this matrix one already sees that although the strongest correlations are on the "utilities" sector, many other inter-sectors correlations do exist. The main purpose of the analysis is precisely to identify sets of companies with similar return behavior.
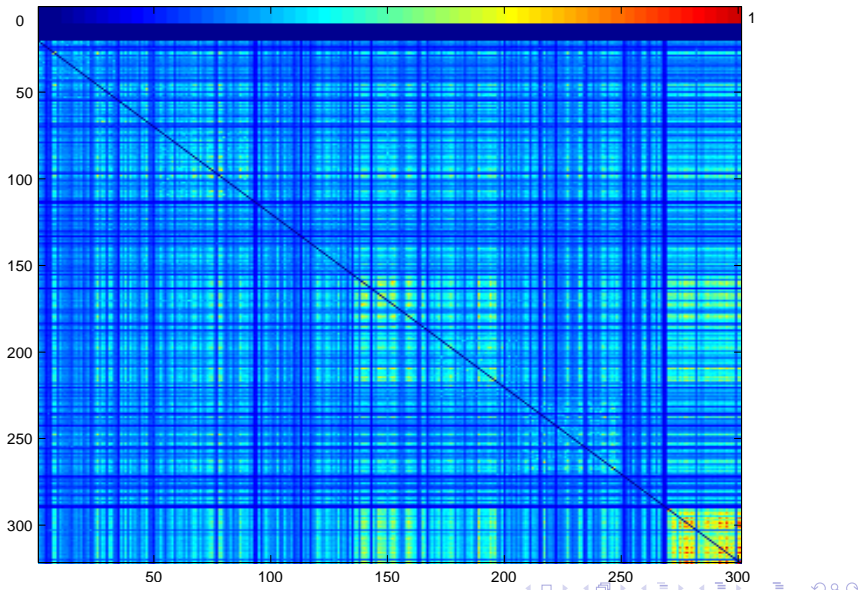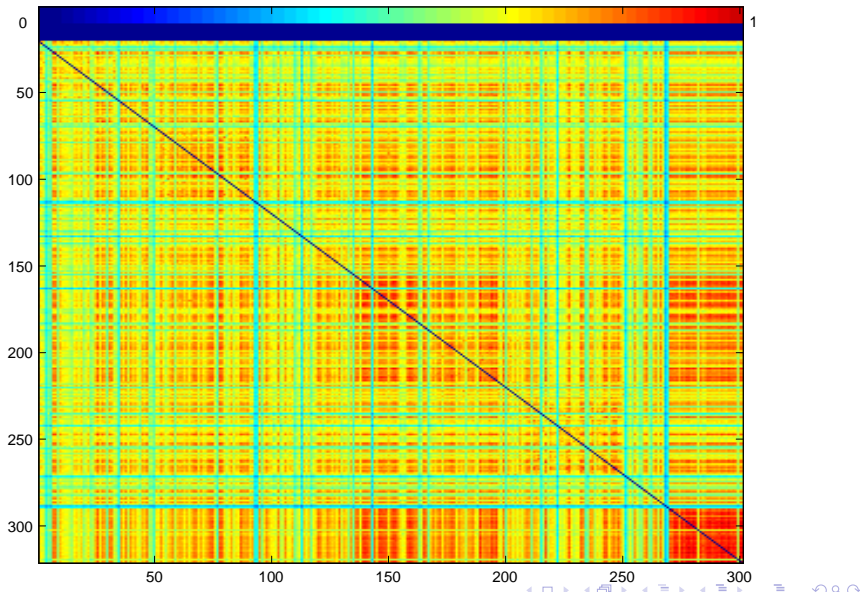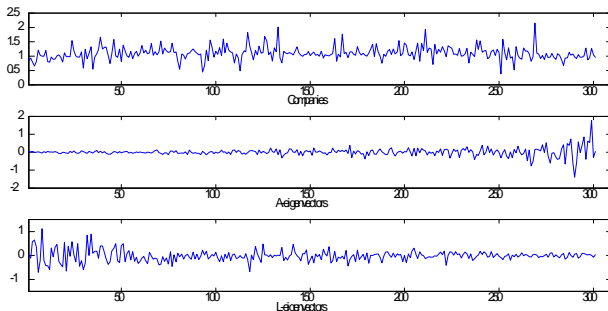
Applications

Now we consider, as the signal on this graph, the yearly compound return

$$R_i = \prod_{t=1}^{250} \left(1 + r_i\left(t \right) \right) \tag{1}$$

In the figure we compare the compound return R_i of the companies with the absolute value of the projection of $R_i - \langle R_i \rangle$ on the eigenvectors of the adjacency A and the Laplacian $L = G - A$ matrices. $\langle R_i \rangle$ is mean value of the compound returns, which in this case was 1.1003.
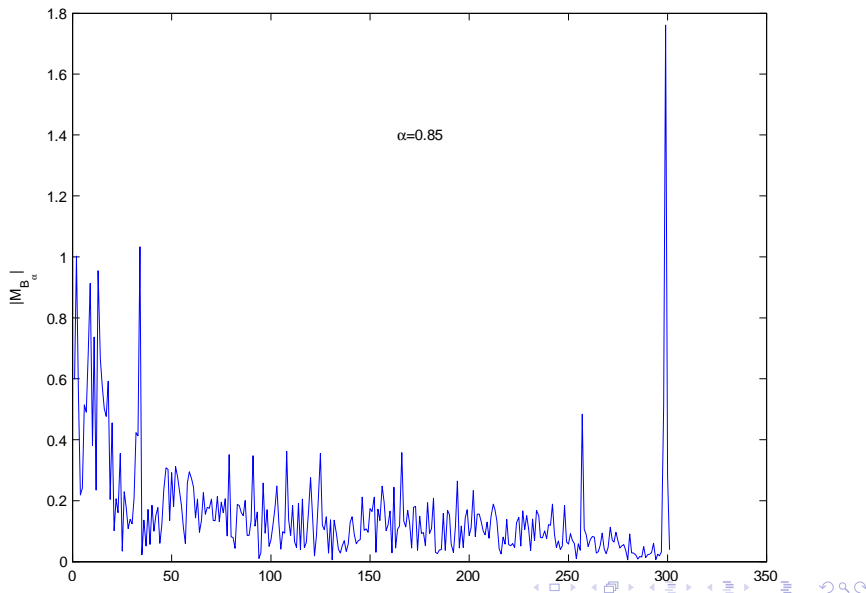
Applications

One sees that the projection on the A−eigenvectors (the A−transform) is the one that provides a better information compression by selecting a smaller number of dominant eigenvectors. How the companies cluster into similar dynamical behaviors compressed in the A−eigenvectors is better understood by the examination of the tomogram.
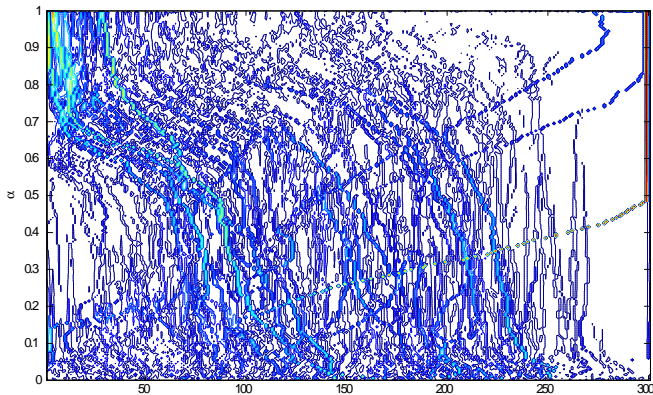
The figure displays a contour plot of the $T - A$ tomogram. That is, it shows the absolute value of the projections of the signal (the compound returns) on the eigenvectors of the linear combination $B_\alpha = (1 - \alpha) T + \alpha A$. One sees how, starting from the compound return signal at $\alpha = 0$, the contributions of each company organize themselves into clusters of similar return behavior. The selection of clusters may be done by cutting the tomogram at the appropriate level and reconstructing the components of the signal. Contrary to the ambiguity of the bilinear transformations, the tomogram has a rigorous probabilistic interpretation and all the signal information is contained at each α level. Therefore the signal components (dynamical clusters) are reconstructed by linear combinations of the eigenvectors around each peak with the coefficients taken from the tomogram. As an example the next figures show a tomogram cut at $\alpha = 0.85$ and then the reconstruction of the signal components around three of its peaks.
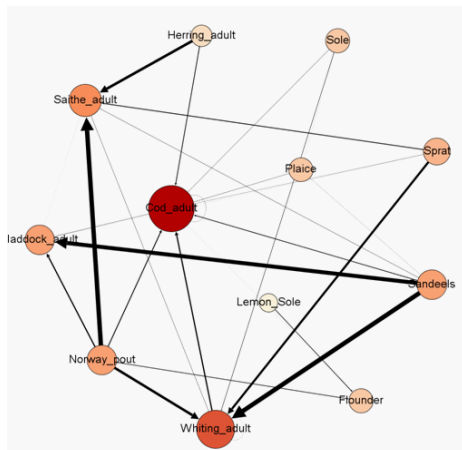
Applications
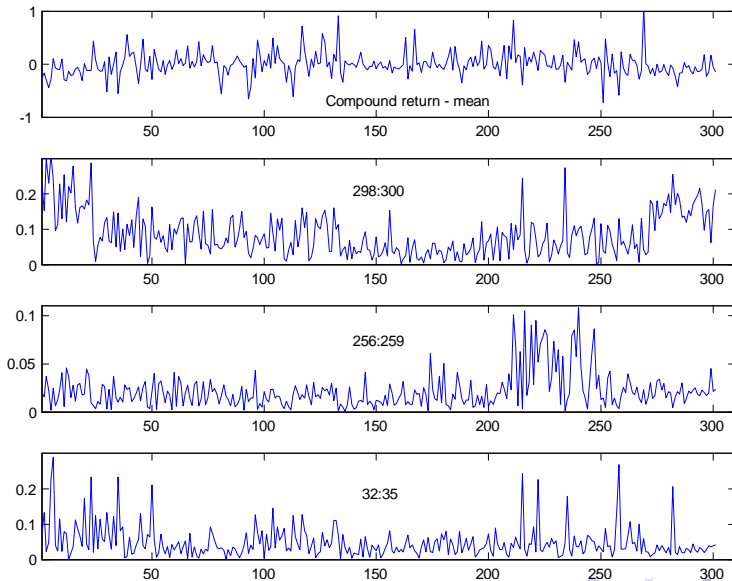
A trophic network

Here we study 12 fish species of the North Atlantic for which we have information both on their trophic relations and their biomass evolution in the period 1976-2013. The trophic relations

The trophic adjacency matrix

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.01											
2	0.091	0.002		0.02								
3	0.019			0.002								
4												
5	0.067	0.183	0.089	0.322						0.044		
6	0.056			0.19								
7	0.013	0.157	0.001	0.057								
8	0.049	0.324	0.288	0.02					0.008			
9	0.018											
10	0.003											
11	0.017	0.03							0.003			
12										0.044		

Cod adult; 2 = Whiting adult; 3 = Haddock adult; 4 = Saithe adult; 5 = Norway pout; 6 = Herring adult; 7 = Sprat; 8 = Sandeels; 9 = Plaice; 10 = Flounder; 11 = Sole; 12 = Lemon Sole.

Applications

On the other hand, considering for each biomass time series $b(t)$ as the most relevant variable the population growth rate

$$r(t) = \log\left(\frac{b(t)}{b(t-1)}\right)$$

define the $\Delta-$ delay distance function

$$d_{ij}^{(\Delta)} = \sqrt{\sum_{t=\Delta}^{38}\left(r_i(t) - r_j(t-\Delta)\right)^2}$$

Consider time-delays for the growth rate distances because in a trophic network the biomass may be related to the offspring of previous years. For each distance matrix, with elements $d_{ij}^{(\Delta)}$ find the smallest nonzero element $(d_{\min}^{(\Delta)})$ and define other adjacency matrices

$$A_{ij}^{(\Delta)} = \frac{d_{\min}^{(\Delta)}}{d_{ij}^{(\Delta)}} \left(1 - \delta_{ij}\right)$$

The question was to find out whether the trophic adjacency matrix A_{troph} provides the same or, at least, similar information as the adjacency matrices derived from the biomass time series. In other words, whether the trophic relations are strongly correlated with the biomass evolution. For this purpose we have normalized to one each column in the trophic matrix, and then compared the 28 nonzero elements with the corresponding elements in the $A_{ij}^{(\Delta)}$ matrices (also normalized to one). The result is shown in the next figure.

Although some partial trends might be similar, the general conclusion is that the biomass evolution seems to depend on many other factors, different from the trophic relations of these 12 species. Etc., work in progress ...

Trophic versus growth rate adjacency matrices
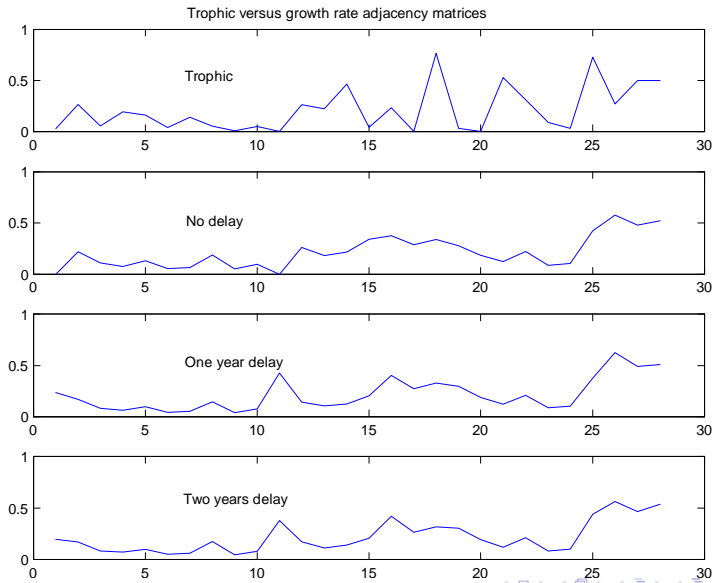
Classification by regularization

- When classifying nodes in a graph, if the classification of some nodes is already known, one may want to extend the classification to the other, yet unclassified nodes, by requiring that the signal in the graph be as smooth as possible.

For this reason this classification scheme is sometimes called "regularization of graphs".

- The total variation of the signal may, for example, be defined by

$$V\left(s\right) = \left\| s - \frac{1}{|\lambda_{\max}|} \mathbf{A}s \right\|$$

Recalling the time series network interpretation, this corresponds to say that the signal is minimally disturbed by a uniform shift.

- At the initial step the signal is fixed where it is known and assigned random values at the other nodes. Then $V\left(s\right)$ is minimized by sucessive iterations.

Linear and nonlinear algebraic signal processing: The linear theory

- Algebraic representation of signals and filters as polynomials
- So far, signals and filters on graphs were vectors and matrices. An alternative representation exists for filters and signals as polynomials. We call this representation the graph $z-$transform, since it generalizes the traditional $z-$transform for discrete time signals that maps signals and filters to polynomials or series in z^{-1}. The graph $z-$transform is defined separately for graph filters and signals.
- Consider a graph $G = (\mathcal{V}, \mathbf{A})$, for which the characteristic and minimal polynomials of the adjacency matrix coincide: $p_{\mathbf{A}}(x) = m_{\mathbf{A}}(x)$. The mapping $\mathbf{A} \mapsto x$ of the adjacency matrix $\mathbf{A}$ to the indeterminate x maps the graph filters $\mathbf{H} = h(\mathbf{A})$ in $\mathcal{F}$ to polynomials $h(x)$.
- Tthe filter space $\mathcal{F}$ becomes a *polynomial algebra*

$$\mathcal{A} = \mathbb{C}[x]/m_{\mathbf{A}}(x).$$

The linear theory

- This is a space of polynomials of degree less than $\deg m_{\mathbf{A}}(x)$ with complex coefficients that is closed under addition and multiplication of polynomials modulo $m_{\mathbf{A}}(x)$. The mapping $\mathcal{F} \to \mathcal{A}$, $h(\mathbf{A}) \mapsto h(x)$, is an isomorphism of $\mathbb{C}$-algebras , which we denote as $\mathcal{F} \cong \mathcal{A}$. We call it the *graph z-transform of filters* on graph $G = (\mathcal{V}, \mathbf{A})$.
- The signal space $\S$ is a vector space that is also closed under filtering, i.e., under multiplication by graph filters from $\mathcal{F}$: for any signal $\mathbf{s} \in \S$ and filter $h(\mathbf{A})$, the output is a signal in the same space: $h(\mathbf{A})\mathbf{s} \in \S$. Thus, $\S$ is an $\mathcal{F}$-module
- The *graph z-transform of signals* is defined as an isomorphism from $\S$ to an $\mathcal{A}$-module.

Theorem: Under the above conditions, the signal space $\S$ is isomorphic to an $\mathcal{A}$-module

$$\mathcal{M} = \mathbb{C}[x]/p_{\mathbf{A}}(x) = \left\{ s(x) = \sum_{n=0}^{N-1} s_n b_n(x) \right\}$$

The linear theory

under the mapping $\mathbf{s} = (s_0, \ldots, s_{N-1})^T \mapsto s(x) = \sum_{n=0}^{N-1} s_n b_n(x)$.

- The polynomials $b_0(x), \ldots, b_{N-1}(x)$ are linearly independent polynomials of degree at most $N - 1$. If we write

$$\mathbf{b}(x) = (b_0(x), \ldots, b_{N-1}(x))^T,$$

then the polynomials satisfy

$$\mathbf{b}^{(r)}(\lambda_m) = \begin{pmatrix} b_0^{(r)}(\lambda_m) & \ldots & b_{N-1}^{(r)}(\lambda_m) \end{pmatrix}^T = r! \widetilde{\mathbf{v}}_{m,0,r}$$

for $0 \leq r < R_{m,0}$ and $0 \leq m < M$, where λ_m and $\widetilde{\mathbf{v}}_{m,0,r}$ are generalized eigenvectors of $\mathbf{A}^T$, and $b_n^{(r)}(\lambda_m)$ denotes the rth derivative of $b_n(x)$ evaluated at $x = \lambda_m$.

The linear theory

Filtering in $\mathcal{M}$ is performed as multiplication modulo $p_{\mathbf{A}}(x)$: if $\widetilde{\mathbf{s}} = h(\mathbf{A})\mathbf{s}$, then

$$\widetilde{\mathbf{s}} \;\mapsto\; \tilde{s}(x) = \sum_{n=0}^{N-1} \tilde{s}_n b_n(x) = h(x)s(x) \bmod_{\mathbf{A}}(x).$$

Extends to the general case $p_{\mathbf{A}}(x) \neq m_{\mathbf{A}}(x)$. By
Theorem: There exists a graph $\widetilde{G} = (\mathcal{V}, \widetilde{\mathbf{A}})$ with $p_{\widetilde{\mathbf{A}}}(x) = m_{\widetilde{\mathbf{A}}}(x)$, such that $\mathbf{A} = r(\widetilde{\mathbf{A}})$. By mapping $\widetilde{\mathbf{A}}$ to x, the filter space has the structure of the polynomial algebra $\mathcal{A} = \mathbb{C}[x]/m_{\mathbf{A}}(r(x)) = \mathbb{C}[x]/(m_{\mathbf{A}} \circ r)(x))$ and the signal space has the structure of the $\mathcal{A}$-module $\mathcal{M} = \mathbb{C}[x]/p_{\widetilde{\mathbf{A}}}(x)$. Multiplication of filters and signals is performed modulo $p_{\widetilde{\mathbf{A}}}(x)$. The basis of $\mathcal{M}$ satisfies, where λ_m and $\mathbf{v}_{m,d,r}$ are eigenvalues and generalized eigenvectors of $\widetilde{\mathbf{A}}$.

Signal transforms as operator symbols

- Tomograms may be described not only as amplitudes of projections on a complete basis of eigenvectors of a family of operators, but also as operator symbols. That is, as a map of operators to a space of functions where the operators non-commutativity is replaced by a modification of the usual product to a star-product.

- Let $\hat{A}$ be an operator in Hilbert space $\mathcal{H}$ and $\hat{U}(\vec{x})$, $\hat{D}(\vec{x})$ two families of operators called *dequantizers* and *quantizers*, respectively, such that

$$\mathrm{Tr}\left\{\hat{U}(\vec{x})\hat{D}(\vec{x}')\right\} = \delta(\vec{x} - \vec{x}')$$

The labels $\vec{x}$ (with components $x_1, x_2, \ldots x_n$) are coordinates in a linear space V where the functions (operator symbols) are defined. Some of the coordinates may take discrete values, then the delta function should be understood as a Kronecker delta. Provided the property above is satisfied, one defines the *symbol of the operator* $\hat{A}$ by the formula

$$f_A(\vec{x}) = \mathrm{Tr}\left\{\hat{U}(\vec{x})\hat{A}\right\},$$

Signal transforms as operator symbols

One has the reconstruction formula

$$\hat{A} = \int f_A(x)\hat{D}(\vec{x})\, d\vec{x}$$

- The role of quantizers and dequantizers may be exchanged. Then

$$f_A^d(\vec{x}) = \text{Tr}\left\{\hat{D}(\vec{x})\,\hat{A}\right\}$$

  is called the dual symbol of $f_A(\vec{x})$ and the reconstruction formula is

$$\hat{A} = \int f_A^d(x)\hat{U}(\vec{x})\, d\vec{x}$$

- Symbols of operators can be multiplied using the star-product kernel as follows

$$f_A(\vec{x}) \star f_B(\vec{x}) = \int f_A(\vec{y})f_B(\vec{z})K(\vec{y},\vec{z},\vec{x})\, d\vec{y}\, d\vec{z}$$

  the kernel being

$$K(\vec{y},\vec{z},\vec{x}) = \text{Tr}\left\{\hat{D}(\vec{y})\hat{D}(\vec{z})\hat{U}(\vec{x})\right\}$$

Signal transforms as operator symbols

The star-product is associative,

$$(f_A(\vec{x}) \star f_B(\vec{x})) \star f_C(\vec{x}) = f_A(\vec{x}) \star (f_B(\vec{x}) \star f_C(\vec{x}))$$

this property corresponding to the associativity of the product of operators in Hilbert space.

With the dual symbols the trace of an operator may be written in integral form

$$\operatorname{Tr}\left\{\hat{A}\hat{B}\right\} = \int f_A^d(\vec{x}) f_B(\vec{x}) \, d\vec{x} = \int f_B^d(\vec{x}) f_A(\vec{x}) \, d\vec{x}.$$

For two different symbols $f_A(\vec{x})$ and $f_A(\vec{y})$ corresponding, respectively, to the pairs $(\hat{U}(\vec{x}), \hat{D}(\vec{x}))$ and $(\hat{U}_1(\vec{y}), \hat{D}_1(\vec{y}))$, one has the relation

$$f_A(\vec{x}) = \int f_A(\vec{y}) K(\vec{x}, \vec{y}) \, d\vec{y},$$

with intertwining kernel

$$K(\vec{x}, \vec{y}) = \operatorname{Tr}\left\{\hat{D}_1(\vec{y}) \hat{U}(\vec{x})\right\}$$

Signal transforms as operator symbols

Let now each signal $f(t)$ be identified with the projection operator Π_f on the function $f(t)$, denoted by

$$\Pi_f = |f\rangle \langle f|$$

Then the tomograms, and also other transforms, are symbols of the projection operators for several choices of quantizers and dequantizers. Some examples:

The *Wigner-Ville function*: is the symbol of $|f\rangle\langle f|$ corresponding to the dequantizer

$$\hat{U}(\vec{x}) = 2\hat{\mathcal{D}}(2\alpha)\hat{P}, \qquad \alpha = \frac{t + i\omega}{\sqrt{2}}$$

where $\hat{P}$ is the inversion operator

$$\hat{P}f(t) = f(-t)$$

Signal transforms as operator symbols

and $\hat{\mathcal{D}}(\gamma)$ is a "displacement" operator

$$\hat{\mathcal{D}}(\gamma) = \exp\left[\frac{1}{\sqrt{2}}\gamma\left(t - \frac{\partial}{\partial t}\right) - \frac{1}{\sqrt{2}}\gamma^*\left(t + \frac{\partial}{\partial t}\right)\right]$$

The quantizer operator is

$$\hat{D}(\vec{x}) := \hat{D}(t, \omega) = \frac{1}{2\pi}\hat{U}(t, \omega),$$

t and ω being time and frequency.
The Wigner–Ville function is

$$W(t, \omega) = 2\text{Tr}\left\{\mid f\rangle\langle f \mid \hat{D}(2\alpha)\hat{D}\right\}$$

or, in integral form

$$W(t, \omega) = 2\int f^*(t)\hat{\mathcal{D}}(2\alpha)f(-t)\,dt$$

Signal transforms as operator symbols

\# The *symplectic tomogram* or time-frequency tomogram of $| f \rangle \langle f |$ corresponds to the dequantizer

$$\hat{U}(\vec{x}) := \hat{U}(X, \mu, \nu) = \delta \left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega} \right),$$

Here the notation $\delta \left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega} \right)$ stands for the projector on the eigenvector of $\mu\hat{t} - \nu\hat{\omega}$ corresponding to the eigenvalue X and

$$\hat{t}f(t) = tf(t), \qquad \hat{\omega}f(t) = -i\frac{\partial}{\partial t}f(t)$$

and $X, \mu, \nu \in R$. The quantizer of the symplectic tomogram is

$$\hat{D}(\vec{x}) := \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp \left[i \left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega} \right) \right]$$

Signal transforms as operator symbols

One important feature of the formulation of tomograms as operator symbols is that one may work with deterministic signals $f(t)$ as easily as with probabilistic ones. In this latter case the projector above would be replaced by

$$\Pi_p = \int p_\mu \left| f_\mu \right\rangle \left\langle f_\mu \right| d\mu$$

with $\int p_\mu d\mu = 1$, the tomogram being the symbol of this new operator. This also provides a framework for an algebraic formulation of signal processing, perhaps more general than the one described before. There, a signal model is a triple $(\mathcal{A}, \mathcal{M}, \Phi)$ $\mathcal{A}$ being an algebra of linear filters, $\mathcal{M}$ a $\mathcal{A}$-module and Φ a map from the vector space of signals to the module. With the operator symbol interpretation both (deterministic or random) signals and (linear or nonlinear) transformations on signals are operators. By the application of the dequantizer they are mapped onto functions, the filter operations becoming star-products.

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