Chapter 4 Kan extensions

4.1 The definition of Kan extensions ; their expressibility by limits and colimits.

$K : \mathcal{A} \longrightarrow \mathcal{C}$ なる $\mathcal{V}$-functor に対し, $\hat{K} : \mathcal{C}^{op} \longrightarrow [\mathcal{A}, \mathcal{V}]$, $\tilde{K} : \mathcal{C} \longrightarrow [\mathcal{A}^{op}, \mathcal{V}]$

はそれぞれ,

$$\hat{K}? = \mathcal{C}(?, K-) , \qquad \tilde{K}? = \mathcal{C}(K-, ?)$$

で定まる. このとき,

$$F : \mathcal{C} \longrightarrow \mathcal{V} \quad \text{とすると,} \quad (\hat{K} : \mathcal{C}^{op} \longrightarrow [\mathcal{A}, \mathcal{V}] \ \text{なので})$$

$$F \star \hat{K} \in [\mathcal{A}, \mathcal{V}].$$

いま,

$$H : \mathcal{A} \longrightarrow \mathcal{V} \quad \text{とすると,}$$

$$\begin{aligned}
& [\mathcal{A}, \mathcal{V}](F \star \hat{K}, H) \\
\cong\ & [\mathcal{K}^{op}, \mathcal{V}](F, [\mathcal{A}, \mathcal{V}](\hat{K}-, H)) && \text{⋆ の def} \\
\cong\ & \int_A [\mathcal{K}^{op}, \mathcal{V}](F, \mathcal{V}((\hat{K}-)A, HA)) && \text{関手圏の def, } \textstyle\int_A \text{ の移動}
\end{aligned}$$

$$\cong \int_A \mathcal{V}\left(F? \star (\hat{K}?)A \ , \ HA \right)$$

$\star$ の def

$$\cong [\mathcal{A}, \mathcal{V}]\left(F? \star (\hat{K}?)- \ , \ H- \right)$$

yoneda から,

$$(F \star \hat{K})A \cong F? \star (\hat{K}?)A$$

$(\hat{K}?)A : \mathcal{C}^{op} \longrightarrow \mathcal{V}$

$F? : \mathcal{C} \longrightarrow \mathcal{V}$

$$\cong (\hat{K}?)A \star F?$$

$$= \mathcal{C}(?, KA) \star F?$$

$$\cong FKA .$$

$$
\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "T"] & \\
\mathcal{A} \arrow[ur, "K"] \arrow[rr, "G"'] \arrow[rr, phantom, bend left=20, "\Downarrow\psi"] & & \mathcal{B}
\end{tikzcd}
$$

と $F : \mathcal{C} \longrightarrow \mathcal{V}$ に対して,

$$
[\mathcal{C},\mathcal{V}](F, \underset{\textstyle\hat{T}B}{\overset{\shortparallel}{\mathcal{B}(B, T-)}}) \xrightarrow[\text{precomposite}]{[K,-]} [\mathcal{A},\mathcal{B}](FK, \overset{\hat{T}KB}{\overset{\shortparallel}{\mathcal{B}(B, TK-)}})
$$

$$
\xrightarrow{[\mathcal{A},\mathcal{V}](1, \mathcal{B}(1, \psi_-))} [\mathcal{A},\mathcal{B}](FK, \mathcal{B}(B, G-))
$$

いま

$$[\mathcal{C},\mathcal{V}](F, \mathcal{B}(B, T-)) \cong \mathcal{B}(B, \{F,T\})$$

$$[\mathcal{A},\mathcal{B}](FK, \mathcal{B}(B, G-)) \cong \mathcal{B}(B, \{FK,G\})$$

$B = \{F,T\}$ にすれば $I \xrightarrow{j} \mathcal{B}(\{F,T\},\{F,T\}) \longrightarrow \mathcal{B}(\{F,T\},\{FK,G\})$ を得る.

これを,

$$(K,\psi)_* : \{F,T\} \longrightarrow \{FK,G\}$$

とする.