1 Chapter 1
The elementary notions

1.1 Monoidal categories

Definition 1. (monoidal category)

$$\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$$

with 2 coherence axioms.

Definition 2. (V, element)

$$V = \mathcal{V}_0(I, -) : \mathcal{V}_0 \to \mathbf{Sets}$$

Let X be an object of $\mathcal{V}_0$. We call $f : I \to X \in VX$ an *element* of X.

1.2 The 2-category $\mathcal{V}$-CAT for a monoidal $\mathcal{V}$

Definition 3. ($\mathcal{V}$-category) *$\mathcal{V}$-category* $\mathcal{A}$ consists of the following;

- a set ob$\mathcal{A}$
- a *hom-object* $\mathcal{A}(A, B) \in \mathcal{V}_0$
- a *composition law* $M_{ABC} : \mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$
- an *identity element* $j_A : I \to \mathcal{A}(A, A)$

which satisfy 3 axioms.

Definition 4. $\mathcal{V}$-functor *$\mathcal{V}$-functor* $T : \mathcal{A} \to \mathcal{B}$ consists of

- a function $T : \text{ob}\mathcal{A} \to \text{ob}\mathcal{B}$
- a map $T_{AB} : \mathcal{A}(A, B) \to \mathcal{B}(TA, TB)$ in $\mathcal{V}$

with 2 comutativities.
When T_{AB} is isomorphism in $\mathcal{V}$, T is called *fully faithful*.

Definition 5. ($\mathcal{V}$-natural transformation) *$\mathcal{V}$-natural transformation* $\alpha : T \to S$ is (ob$\mathcal{A}$)-indexed family

$$\alpha_A : I \to \mathcal{B}(TA, SA)$$

with the commutativity.

1.3 The 2-functor $()_0 : \mathcal{V}$-CAT $\to$ CAT

Definition 6. $(\mathcal{I}, (-)_0)$ The *unit $\mathcal{V}$-category* is a category with one object 0 and with $\mathcal{I}(0,0) = I$.

Define $(-)_0 \equiv \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I}, -)$. The ordinary category $\mathcal{A}_0 = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{I}, \mathcal{A})$ is called the *underlying category* of $\mathcal{A}$.

The ordinary category $\mathcal{A}_0$ is composed of

- objects of $\mathcal{A}$,
- morphisms from A to B are elements of $\mathcal{A}(A, B)$. These are morphisms $I \to \mathcal{A}(A, B)$ in $\mathcal{V}$. We write this by $A \overset{\mathcal{V}}{\longrightarrow} B$ or $A \nrightarrow B$.

the composition is defined like virtical composition of $\mathcal{V}$-natural transformations.

Definition 7. (underlying functor) The ordinary functor $T_0 : \mathcal{A}_0 \to \mathcal{B}_0$ induced by the $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ is called the *underlying functor*.

- T_0 sends A to TA.
- T_0 sends $I \to \mathcal{A}(A, B)$ to $I \to \mathcal{A}(A, B) \xrightarrow{T_{AB}} \mathcal{B}(TA, TB)$.

Definition 8. (underlying natural transformation) The ordinary natural transformation *underlying* $\alpha : T \to S$ is $\alpha_0 : T_0 \to S_0$, which is precisely the same as α, in fact

$$\begin{aligned} \alpha_A &: I \to \mathcal{B}(TA, SA) \\ \alpha_{0A} &: TA \nrightarrow SA. \end{aligned}$$

1.4 Symmetric monoidal categories: the tensor product and duality on $\mathcal{V}$-CAT for a symmetric monoidal $\mathcal{V}$

Definition 9. (symmetric monoidal category) Well known.

Definition 10. (tenser product) A *tenser product* $\mathcal{A} \otimes \mathcal{B}$ of a pair $\mathcal{A}, \mathcal{B}$ of $\mathcal{V}$-categories is a $\mathcal{V}$-category

- with objects $\mathrm{ob}\mathcal{A} \times \mathrm{ob}\mathcal{B}$,
- with $(\mathcal{A} \otimes \mathcal{B})\left((A, B), (A', B')\right) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$,
- with composition-law given by

$$\begin{tikzcd} (\mathcal{A}(A', A'') \otimes \mathcal{B}(B', B'')) \otimes (\mathcal{A}(A, A') \otimes \mathcal{B}(B, B')) \arrow[r, "M"] \arrow[d, "m"] & \mathcal{A}(A, A'') \otimes \mathcal{B}(B, B'') \\ (\mathcal{A}(A', A'') \otimes \mathcal{A}(A, A')) \otimes (\mathcal{B}(B', B'') \otimes \mathcal{B}(B, B')) \arrow[ur, "M \otimes M"'] & \end{tikzcd}$$

- , and identity $I \cong I \otimes I \xrightarrow{j_A \otimes j_B} \mathcal{A}(A,A) \otimes \mathcal{B}(B,B)$.

Proposition 1. $\otimes : \mathcal{V}\text{-}\mathbf{CAT} \times \mathcal{V}\text{-}\mathbf{CAT} \to \mathcal{V}\text{-}\mathbf{CAT}$ *defines a 2-functor.*

Proof. $\otimes$ sends the left diagrams in $\mathcal{V}\text{-}\mathbf{CAT} \times \mathcal{V}\text{-}\mathbf{CAT}$ to the right in $\mathcal{V}\text{-}\mathbf{CAT}$.

$$
\begin{array}{ccc}
\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "T"] \arrow[r, bend right, "T'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{A}' \end{tikzcd} & & \\
& \mapsto & \begin{tikzcd}[column sep=huge] \mathcal{A}\otimes\mathcal{B} \arrow[r, bend left, "T\otimes S"] \arrow[r, bend right, "T'\otimes S'"'] \arrow[r, phantom, "\Downarrow\alpha\otimes\beta"] & \mathcal{A}'\otimes\mathcal{B}' \end{tikzcd} \\
\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "S"] \arrow[r, bend right, "S'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{B}' \end{tikzcd} & &
\end{array}
$$

Here, $T \otimes S$ is composed of

- a function $\mathrm{ob}\mathcal{A} \times \mathrm{ob}\mathcal{B} \xrightarrow{T \times S} \mathrm{ob}\mathcal{A}' \times \mathrm{ob}\mathcal{B}'$
- and $\mathcal{V}$-functors $\mathcal{A}(A,A') \otimes \mathcal{B}(B,B') \xrightarrow{T_{AA'} \otimes S_{BB'}} \mathcal{A}'(TA,TA') \otimes \mathcal{B}'(SB,SB')$,

and $\alpha \otimes \beta$ is a family of

$$(\alpha \otimes \beta)_{(A,B)} : I \cong I \otimes I \xrightarrow{\alpha_A \otimes \beta_B} \mathcal{A}'(TA,T'A) \otimes \mathcal{B}'(SB,S'B).$$

$\square$

Definition 11. (dual) $\mathcal{A}^{\mathrm{op}}$ is a $\mathcal{V}$-category with the same objects as $\mathcal{A}$ and with $\mathcal{A}^{\mathrm{ob}}(A,B) = \mathcal{A}(B,A)$.

$\mathcal{V}$-functor $T^{\mathrm{op}} : \mathcal{A} \to \mathcal{B}$ is defined by the function $T^{\mathrm{op}} = T : \mathrm{ob}\mathcal{A} \to \mathrm{ob}\mathcal{B}$ and

$$
\begin{tikzcd}
\mathcal{A}^{\mathrm{op}}(A,B) \arrow[r, "T^{\mathrm{op}}_{AB}"] \arrow[d, equal] & \mathcal{B}^{\mathrm{op}}(TA,TB) \arrow[d, equal] \\
\mathcal{A}(B,A) \arrow[r, "T_{BA}"] & \mathcal{B}(TB,TA).
\end{tikzcd}
$$

Also, for $\alpha : T \to S$, $\alpha^{\mathrm{op}} : S^{\mathrm{op}} \to T^{\mathrm{op}}$ is defined by

$$\alpha^{op}_A = \alpha_A : I \longrightarrow \mathcal{B}^{\mathrm{op}}(SA,TA) = \mathcal{B}(TA,SA).$$

Note that, $(-)^{\mathrm{op}}$ reverses 2-cells but not 1-cells.

Theorem 1. *Two family of functors*

$$T(A,-) : \mathcal{B} \to \mathcal{C}$$
$$T(-,B) : \mathcal{A} \to \mathcal{C}$$

give rise to a functor

$$T : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$$

of which the two given functors are partial functors iff

$$
\begin{tikzcd}
\mathcal{A}(A,A')\otimes\mathcal{B}(B,B') \arrow[r, "T(-,B')\otimes T(A,-)"] \arrow[dd, "c"] & \mathcal{C}(T(A,B'),T(A',B'))\otimes\mathcal{C}(T(A,B),T(A,B')) \arrow[d, "M"] \\
& \mathcal{C}(T(A,B),T(A',B')) \\
\mathcal{B}(B,B')\otimes\mathcal{A}(A,A') \arrow[r, "T(A',-)\otimes T(-,B)"] & \mathcal{C}(T(A',B),T(A',B'))\otimes\mathcal{C}(T(A,B),T(A',B)) \arrow[u, "M"']
\end{tikzcd}
$$

Corollary 1. *Assume $T, S : \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$. A family $\alpha_{AB} : I \to \mathcal{C}(T(A,B), S(A,B)))$ constitutes a $\mathcal{V}$-natural transformation $T \to S$ iff*

- *for each fixed A, $T(A,-) \to S(A,-)$:$\mathcal{V}$-natural*
- *for each fixed B, $T(-,B) \to S(-,B)$:$\mathcal{V}$-natural.*

We have $(\mathcal{A}^{\mathrm{op}})_0 = (\mathcal{A}_0)^{\mathrm{op}}$ and $(T^{\mathrm{op}})_0 = (T_0)^{\mathrm{op}}$. However, $(\mathcal{A}\otimes\mathcal{B})_0$ is not $\mathcal{A}_0 \times \mathcal{B}_0$; rather there is an evident canonical functor $\mathcal{A}_0 \times \mathcal{B}_0 \to (\mathcal{A}\otimes\mathcal{B})_0$ Here, for $T : \mathcal{A}\otimes\mathcal{B} \to \mathcal{C}$, the ordinary functors $T(A,-)_0$ and $T(-,B)_0$, underlying the partial functor of T, can be expressed by the ordinary partial functors of

$$\mathcal{A}_0 \times \mathcal{B}_0 \to (\mathcal{A}\otimes\mathcal{B})_0 \xrightarrow{T_0} \mathcal{C}_0.$$

1.5 Closed and biclosed monoidal categories

Definition 12. (closed) The monoidal category $\mathcal{V}$ is said to be *closed* if each functor $-\otimes Y : \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[Y,-]$.

This means there is a bijective correspondence

$$\pi : \mathrm{Hom}_{\mathcal{V}_0}(X\otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{V}_0}(X, [Y,Z])$$

with unit and counit

$$d : X \longrightarrow [Y, X\otimes Y], \qquad e : [Y,Z]\otimes Y \longrightarrow Z.$$

The latter e is called *evaluation*. Putting $X = I$ in the isomorphism between Hom, we get a natural isomorphism

$$\mathcal{V}_0(Y,Z) \cong V[Y,Z].$$

$[Y,Z]$ is called the *internal hom* of Y and Z. Also, by putting $Y = I$, again, we get

$$i : Z \cong [I,Z].$$

Replacing X by $W\otimes X$, we deduce a natural isomorphism

$$p : [X\otimes Y, Z] \cong [X,[Y,Z]]$$

Definition 13. (biclosed) The monoidal category $\mathcal{V}$ is said to be *biclosed* if each of $-\otimes Y$ and $X\otimes -$ has a right adjoint $[Y,-]$ and $[\![X,-]\!]$

1.6 $\mathcal{V}$ as a $\mathcal{V}$-category for symmetric monoidal closed $\mathcal{V}$; representable $\mathcal{V}$-functors

From now on we suppose that $\mathcal{V}$ is *symmetric monoidal closed*.

Proposition 2. *$\mathcal{V}$ itself is a $\mathcal{V}$-category by assuming*

- *objects are* ob$\mathcal{V}$*,*
- *hom-object $\mathcal{V}(X,Y)$ is internal hom $[X,Y]$,*
- *composition $M : [Y,Z] \otimes [X,Y] \to [X,Z]$ is the transpose of*

$$([Y,Z] \otimes [X,Y]) \otimes X \xrightarrow{a} [Y,Z] \otimes ([X,Y] \otimes X) \xrightarrow{1\otimes e} [Y,Z] \otimes Y \xrightarrow{e} Z$$

- *identity $I \to [X,X]$ is the transpose of $l : I \otimes X \cong X$*

Now, a morphism $A \xrightarrow{\mathcal{V}} B$ in the underlying category of $\mathcal{V}$ is a morphism $I \to [A,B] = \mathcal{V}(A,B)$, and the transpose is $A \to B$ in $\mathcal{V}$. By this bijective correspondence, we idenfity $A \twoheadrightarrow B$ and $A \to B$.

Definition 14. (representable $\mathcal{V}$-functor) For $\mathcal{V}$-category $\mathcal{A}$ and object $A \in \mathcal{A}$, $\mathcal{V}$-functor $\mathcal{A}(A,-) : \mathcal{A} \to \mathcal{V}$ is composed of

- a function sending B to $\mathcal{A}(A,B) \in \mathcal{V}$
- a map
$$\mathcal{A}(A,-)_{BC} : \mathcal{A}(B,C) \longrightarrow [\mathcal{A}(A,B), \mathcal{A}(A,C)]$$
which is the transpose of $\mathcal{A}(B,C) \otimes \mathcal{A}(A,B) \xrightarrow{M} \mathcal{A}(A,C)$

This functor is called the *representable $\mathcal{V}$-functor*. Replacing $\mathcal{A}$ by $\mathcal{A}^{\mathrm{op}}$ gives the *contraveriant representable functor* $\mathcal{A}^{\mathrm{op}}(-,B)$. And these two give rise to the functor

$$\mathrm{Hom}_{\mathcal{A}} : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \longrightarrow \mathcal{V}$$

That is

$$\begin{array}{rr} \mathrm{Hom}_{\mathcal{A}} : & (A,B) \longmapsto \mathcal{A}(A,B) \\ (\mathrm{Hom}_{\mathcal{A}})_{(A,B)(A',B')} : & \mathcal{A}^{\mathrm{op}}(A,A') \otimes \mathcal{A}(B,B') \longrightarrow [\mathcal{A}(A,B), \mathcal{A}(A',B')] \end{array}$$

Definition 15. (hom) Define ordinary functor $\mathrm{hom}_{\mathcal{A}} : \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \to \mathcal{V}_0$ by

$$\mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \longrightarrow (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})_0 \xrightarrow{(\mathrm{Hom}_{\mathcal{A}})_0} \mathcal{V}_0;$$

it sends (A,B) to $\mathcal{A}(A,B)$ and

$$\begin{aligned} f : A \twoheadrightarrow A' \in \mathcal{A}_0^{\mathrm{op}} \\ g : B \twoheadrightarrow B' \in \mathcal{A}_0 \end{aligned}$$

to

$$I \cong I \otimes I \xrightarrow{f \otimes g} \mathcal{A}(A', A) \otimes \mathcal{A}(B, B') \xrightarrow{(\mathrm{Hom}_{\mathcal{A}})_{(A,B)(A',B')}} [\mathcal{A}(A, B), \mathcal{A}(A', B')].$$

We write this map $\mathcal{A}(f, g)$.

Of course, by taking the transpose, we can assume $\mathcal{A}(f, g)$ as the real morphism $\mathcal{A}(A, B) \to \mathcal{A}(A', B')$ in $\mathcal{V}$. By several calculations, $\mathcal{A}(A, g) := \mathcal{A}(A, -)_0 g$ is the composite

$$\mathcal{A}(A, B) \xrightarrow{l^{-1}} I \otimes \mathcal{A}(A, B) \xrightarrow{g \otimes 1} \mathcal{A}(B, C) \otimes \mathcal{B}(A, B) \xrightarrow{M} \mathcal{A}(A, C).$$

From these it follows that

$$\begin{tikzcd} \mathcal{A}_0^{\mathrm{op}} \times \mathcal{A}_0 \arrow[r, "\mathrm{hom}_{\mathcal{A}}"] \arrow[dr, "\mathrm{Hom}_{\mathcal{A}_0}"'] & \mathcal{V}_0 \arrow[d, "V"] \\ & \mathbf{Sets}. \end{tikzcd}$$

Definition 16. (Ten) These is a $\mathcal{V}$-functor

$$\mathrm{Ten} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

- with sending (X, Y) to $\mathrm{Ten}(X, Y) = X \otimes Y$
- and, with $\mathrm{Ten}_{(X,Y)(X',Y')} : [X, X'] \otimes [Y, Y'] \to [X \otimes Y, X' \otimes Y']$ adjunct to the composite

$$([X, X'] \otimes [Y, Y']) \otimes (X \otimes Y) \xrightarrow{m} ([X, X'] \otimes X) \otimes ([Y, Y'] \otimes Y) \xrightarrow{ev \otimes ev} X' \otimes Y'.$$

The ordinary functor

$$\mathcal{V} \times \mathcal{V} \to (\mathcal{V} \otimes \mathcal{V})_0 \xrightarrow{\mathrm{Ten}_0} \mathcal{V}_0$$

is same as $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$.

For any $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ we have

$$\begin{tikzcd} (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B}^{\mathrm{op}}) \otimes (\mathcal{A} \otimes \mathcal{B}) \arrow[r, "\mathrm{Hom}_{\mathcal{A} \otimes \mathcal{B}}"] \arrow[d, "\cong"] & \mathcal{V} \\ (\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A}) \otimes (\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B}) \arrow[r, "\mathrm{Hom}_{\mathcal{A}} \otimes \mathrm{Hom}_{\mathcal{B}}"] & \mathcal{V} \otimes \mathcal{V} \arrow[u, "\mathrm{Ten}"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{B} \arrow[r, "(\mathcal{A} \otimes \mathcal{B})((A,B),-)"] \arrow[dr, "\mathcal{A}(A,-) \otimes \mathcal{B}(B,-)"'] & \mathcal{V} \\ & \mathcal{V} \otimes \mathcal{V}. \arrow[u, "\mathrm{Ten}"] \end{tikzcd}$$

This gives

$$(\mathcal{A} \otimes \mathcal{B})\left((f, g), (f', g')\right) = \mathcal{A}(f, f') \otimes \mathcal{B}(g, g')$$

1.7 Extraordinary $\mathcal{V}$-naturality

The formar section allow us to rewrite $\mathcal{V}$-naturality in the more compact form

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "T"] \arrow[d, "S"'] & \mathcal{B}(TA,TB) \arrow[d, "\mathcal{B}(1,\alpha_B)"] \\ \mathcal{B}(SA,SB) \arrow[r, "\mathcal{B}(\alpha_A,1)"] & \mathcal{B}(TA,SB), \end{tikzcd}$$

since $\mathcal{B}(1,\alpha_B)$ was defined by

$$\mathcal{A}(TA,TB) \xrightarrow{l^{-1}} I\otimes\mathcal{A}(TA,TB) \xrightarrow{\alpha_B\otimes 1} \mathcal{A}(TB,SB)\otimes\mathcal{B}(TA,TB) \xrightarrow{M} \mathcal{A}(TA,SB).$$

Now, some "canonical" maps like $M : \mathcal{A}(B,C)\otimes\mathcal{A}(A,B) \to \mathcal{A}(A,C)$ or $ev : [Y,Z]\otimes Y \to Z$ are not $\mathcal{V}$-natural transformations. In order to admit that these maps are "natural", we have to extend the notion of "$\mathcal{V}$-naturality", which is called *extraordinary $\mathcal{V}$-naturality*. We see those canonical maps actually have $\mathcal{V}$-naturality in the next section.

Definition 17. (extraordinary $\mathcal{V}$-naturality) $\mathcal{A}$-indexed family of maps $\beta_A : K \nrightarrow T(A,A)$ in $\mathcal{B}_0$ is *extraordinary $\mathcal{V}$-natural*, where $K \in \mathcal{B}$ and $T : \mathcal{A}^{\mathrm{op}}\otimes\mathcal{A} \to \mathcal{B}$, if the diagram

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "T(A,-)"] \arrow[d, "T(-,B)"'] & \mathcal{B}(T(A,A),T(A,B)) \arrow[d, "\mathcal{B}(\beta_A,1)"] \\ \mathcal{B}(T(B,B),T(A,B)) \arrow[r, "\mathcal{B}(\beta_B,1)"] & \mathcal{B}(K,T(A,B)) \end{tikzcd}$$

commutes. Duality gives the notion, for the same T and K, of a $\mathcal{V}$-natural family $\gamma_A : T(A,A) \to K$; namely the commutativity of

$$\begin{tikzcd} \mathcal{A}(A,B) \arrow[r, "T(B,-)"] \arrow[d, "T(-,A)"'] & \mathcal{B}(T(B,A),T(B,B)) \arrow[d, "\mathcal{B}(1,\gamma_B)"] \\ \mathcal{B}(T(B,A),T(A,A)) \arrow[r, "\mathcal{B}(1,\gamma_A)"] & \mathcal{B}(T(B,A),K) \end{tikzcd}$$

1.8 The $\mathcal{V}$-naturality of the canonical maps

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1.9 The (weak) Yoneda lemma for $\mathcal{V}$-CAT

Theorem 2. *(week Yoneda) There is a (set theoric) bijection between*

$$\mathcal{V}\text{-nat}(\mathcal{A}(K,-),F)$$

and

$$\mathcal{V}_0(I,FK),$$

given by the correspondence

$$\frac{\alpha : \mathcal{A}(K,-) \longrightarrow F : \mathcal{A} \longrightarrow \mathcal{V}}{I \xrightarrow{j_k} \mathcal{A}(K,K) \xrightarrow{\alpha_K} FK.}$$

Proof. For each $\eta \in \mathcal{V}_0(I, FK)$, the corresponding $\mathcal{V}$-*natural* transformation α_A is given by

$$\mathcal{A}(K,A) \xrightarrow{F_{KA}} [FK, FA] \xrightarrow{[\eta,1]} [I, FA] \xrightarrow{i^{-1}} FA.$$

□

1.10 Representability of $\mathcal{V}$-functors; the representing object as a $\mathcal{V}$-functor of the passive variables

Proposition 3. *Let $F : \mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ be such that each $F(B,-) : \mathcal{A} \to \mathcal{V}$ admits a representation $\alpha_B : \mathcal{A}(KB,-) \to F(B,-)$. ($K$ is a function between objects.) Then there is exactly one way of defining $K_{BC} : \mathcal{B}(B,C) \to \mathcal{A}(KB,KC)$ that makes K a $\mathcal{V}$-$functor$ for which*

$$\alpha_{BA} : \mathcal{A}(KB,A) \to F(B,A)$$

is $\mathcal{V}$-natural in B as well as A.

Proof. If α_{BA} is $\mathcal{V}$-natural in B, η_B is also $\mathcal{V}$-natural in B, since η_B is defined by the composition

$$I \xrightarrow{j_{KB}} \mathcal{A}(KB,KB) \xrightarrow{\alpha_{B,KB}} F(B,KB).$$

On the other hand, α_{BA} is the composition

$$\mathcal{A}(KB,A) \xrightarrow{F(B,-)_{KB,A}} [F(B,KB), F(B,A)] \xrightarrow{[\eta_B,1]} [I, F(B,A)] \xrightarrow{i^{-1}} F(B,A).$$

Therefore, the $\mathcal{V}$-naturality in B of α_{BA} is equivalent to that of η_B. This is also equivalent to the commutativity of

$$\begin{tikzcd}
\mathcal{B}(B,C) \arrow[r, "K_{BC}"] \arrow[d, "F(-,KC)"'] & \mathcal{A}(KB,KC) \arrow[r, "F(B,-)"] & {[F(B,KB), F(B,KC)]} \arrow[d, "{[\eta_B,1]}"] \\
{[F(C,KC), F(B,KC)]} \arrow[rr, "{[\eta_C,1]}"] & & {[I, F(B,KC)].}
\end{tikzcd}$$

Now the composite $[\eta_B, 1]F(B,-)$ is $i\alpha_{B,KC}$, which is isomorphism. This forces us to define K_{BC} as $\alpha^{-1}i^{-1}[\eta_C, 1]F(-,KC)$. □

1.11 Adjunctions and equivalences in $\mathcal{V}$-CAT

Definition 18. (adjunction) An adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend left, "T"] \end{tikzcd}$$

consists of $\eta : 1 \to TS$ and $\varepsilon : ST \to 1$ with triangular identities

$$T\varepsilon \circ \eta_T = 1, \qquad\qquad \varepsilon_S \circ S\eta = 1.$$

Theorem 3. *There is a bijection between*

$$\{n_{BA} : \mathcal{A}(SB, A) \to \mathcal{B}(B, TA) \mid n \text{ is a } \mathcal{V}\text{-natural isomorphism.}\}$$

*and adjunctions $S \dashv T$ in $\mathcal{V}$-**CAT**.*

Proof. Each $\mathcal{V}$-natural isomorphism $n_B : \mathcal{A}(SB, -) \to \mathcal{B}(B, T-)$, by the Yoneda Lemma, has the form

$$\mathcal{A}(SB, A) \xrightarrow{\mathcal{B}(B,T-)} [\mathcal{B}(B, TSB), \mathcal{B}(B, TA)] \xrightarrow{[\eta_B,1]} [I, \mathcal{B}(A, TB)] \xrightarrow{i^{-1}} \mathcal{B}(A, TB)$$

for a unique $\eta_B : I \to \mathcal{B}(B, TSB)$. For the commutativity of the following,

$$\begin{tikzcd}
\mathcal{A}(SB, A) \otimes I \arrow[d, "T \otimes I"] & & \\
\mathcal{B}(TSB, TA) \otimes I \arrow[r, "1 \otimes \eta_B"] \arrow[d, "\mathcal{B}(B,-) \otimes 1"] & \mathcal{B}(TSB, TA) \otimes \mathcal{B}(B, TSB) \arrow[dr, "m"] & \\
{[\mathcal{B}(B, TSB), \mathcal{B}(B, TA)] \otimes I} \arrow[r, "1 \otimes \eta_B"] \arrow[d, "{[\eta_B,1]}"] & {[\mathcal{B}(B, TSB), \mathcal{B}(B, TA)] \otimes \mathcal{B}(B, TSB)} \arrow[r, "ev"] & \mathcal{B}(B, TA) \\
{[I, \mathcal{B}(B, TA)] \otimes I} \arrow[r, "i^{-1} \otimes 1"] \arrow[urr, "ev"] & \mathcal{B}(B, TA) \otimes I \arrow[ur, "r"'] &
\end{tikzcd}$$

n_{AB} can be rewritten as

$$n_{AB} = \mathcal{B}(\eta_A, 1)T_{SB,A},$$

while $n_A^{-1} : \mathcal{A}(-, TA) \to \mathcal{B}(S-, A)$ has the form

$$n_{AB}^{-1} = \mathcal{A}(1, \varepsilon)S.$$

The equation $nn^{-1} = 1$, $n^{-1}n = 1$ means the triangular identities. ☐

By the previous section, a $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ has a left adjoint exactly when each $\mathcal{B}(B, T-)$ is representable. (Let (SB, η_B) be the representation, and extend S to be the functor in the unique way suggested in 1.10.)

Definition 19. (equivalence) Same as 2-category.

As in the case of 2-category, a $\mathcal{V}$-functor $T : \mathcal{A} \to \mathcal{B}$ is an *equivalence* iff T is fully faithful and essentially surjective.

Definition 20. (full image, replate image) Any $T : \mathcal{A} \to \mathcal{B}$ determines two full subcategories *full image* $\mathcal{A}'$, determined by B of the form TA, and *replate image* $\mathcal{A}''$, determined by those B isomorphic to some TA. These $\mathcal{A}'$ and $\mathcal{A}''$ are equivalent.

Definition 21. (replate) A full subcategory which contails all the isos of its objects is said to be *replate*.

Definition 22. (reflective) A full subcategory $\mathcal{A}$ of $\mathcal{B}$ is called *reflective* if the inclusion $T : \mathcal{A} \to \mathcal{B}$ has a left adjoint.

I don't understand what this means; **This implies that every retract (in B_0) of an object of the reflective $\mathcal{A}$ lies in the replation of $\mathcal{A}$.**

Definition 23. (idempotent monad) Choosing the left adjoint $S : \mathcal{B} \to \mathcal{A}$ so that $\varepsilon : ST \to 1$ is the identity for a reflective subcategory $\mathcal{A}$ of $\mathcal{B}$, $R = TS$ satisfies $R^2 = R$, $\eta R = R\eta = 1$. Such an (R, η) is called an *idempotent monad.*