Detailed proofs for Game Semantics

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Definition 1. An *arena* A is composed of the following four sets.

- A set M_A.
- A function $\lambda_A : M_A \to \{O, P\} \times \{Q, A\}$
- A relation $\vdash_A$ over $M_A \cup \{\star\}$ satisfying the following;

  (e1) $\star \vdash_A m \Rightarrow [\lambda_A(m) = OQ \ \wedge \ [n \vdash_A m \Leftrightarrow n = \star]]$

  (e2) $[m \vdash_A n \ \wedge \ \lambda_A^{QA} = A] \Rightarrow \lambda_A^{QA}(m) = Q$

  (e3) $[m \vdash_A \ \wedge m \neq \star] \Rightarrow \lambda_A^{OP}(m) \neq \lambda_A^{OP}(n)$

Definition 2. A *justified sequence* in an arena A is a sequence $s = s_1 \dots s_n$ of M_A and a function $p : \{s_2, \dots, s_n\} \to \{s_1, \dots, s_n\}$ called a pointer with,

- if $s_i = p(s_j)$, $i < j \ \wedge \ s_i \vdash_A s_j$.
- $\star \vdash_A s_1$

Definition 3. For each justified sequence s, *player view* $\lceil s \rceil$ and *opponent view* $\lfloor s \rfloor$ are defined by induction as follows.

$$
\begin{aligned}
\lceil \varepsilon \rceil &= \varepsilon \\
\lceil sm \rceil &= \lceil s \rceil m && \text{if } m \text{ is a P-move.} \\
\lceil sm \rceil &= m && \text{if } \star \vdash m. \\
\lceil smtn \rceil &= \lceil s \rceil mn && \text{if } n \text{ is a O-move. } n \text{ is justified by } m.
\end{aligned}
$$

$$
\begin{aligned}
\lfloor \varepsilon \rfloor &= \varepsilon \\
\lfloor sm \rfloor &= \lfloor s \rfloor m && \text{if } m \text{ is a O-move.} \\
\lfloor smtn \rfloor &= \lfloor s \rfloor mn && \text{if } n \text{ is a P-move. } n \text{ is justified by } m.
\end{aligned}
$$

Definition 4. A justified sequence s is called *legal position* when the following conditions are satisfied.

- $s = s'mns'' \Rightarrow \lambda^{OP}(m) \neq \lambda^{OP}(n)$

- $tm \sqsubset s$ (tm is a prefix of s) and $\lambda^{OP}(m) = P \Rightarrow$ the justifier of m occurs in $\lceil t \rceil$.
- $tm \sqsubset s$ and $\lambda^{OP}(m) = O \Rightarrow$ the justifier of m occurs in $\lfloor t \rfloor$.

Definition 5. A move m in legal position s is *hereditary justified* by move n if there is a chain of justification pointer leading from m ends at n. Define $s \upharpoonright_h n$ as the subsequence of s consisting of all moves hereditary justified by n. We also write $s \upharpoonright_h I$ for the subsequence consisting of all moves hereditary justified by any move n in I.

Definition 6. A *game* A is a tuple $\langle M_A, \lambda_A, \vdash_A, P_A \rangle$ with the following conditions.

- $\langle M_A, \lambda_A, \vdash_A \rangle$ is an arena.
- P_A is a nonempty prefix–closed set of legal positions in the arena satisfying if $s \in P_A$ and I is a set of initial moves of s then $s \upharpoonright_h I \in P_A$.

Define games $A \otimes B$, $A \multimap B$ and I as in [1].

$$\begin{aligned} M_{A\otimes B} =\ & M_A + M_B. \\ \lambda_{A\otimes B} =\ & [\lambda_A, \lambda_B]. \\ \star \vdash_{A\otimes B} n \Leftrightarrow\ & \star \vdash_A n \ \vee \ \star \vdash_B n. \\ m \vdash_{A\otimes B} n \Leftrightarrow\ & m \vdash_A n \ \vee \ m \vdash_B n. \\ P_{A\otimes B} =\ & \{s \in L_{A\otimes B} \mid s \upharpoonright A \in P_A \ \wedge \ s \upharpoonright B \in P_B\} \end{aligned}$$

$$\begin{aligned} M_{A\multimap B} =\ & M_A + M_B. \\ \lambda_{A\multimap B} =\ & [\overline{\lambda_A}, \lambda_B]. \\ \star \vdash_{A\multimap B} m \Leftrightarrow\ & \star \vdash_B m. \\ m \vdash_{A\multimap B} n \Leftrightarrow\ & m \vdash_A n \ \vee \ m \vdash_B n \ \vee \\ & [\star \vdash_B m \ \wedge \ \star \vdash_A n] \qquad \text{for } m \neq \star. \\ P_{A\multimap B} =\ & \{s \in L_{A\multimap B} \mid s \upharpoonright A \in P_A \ \wedge \ s \upharpoonright B \in P_B\} \end{aligned}$$

$$I = \langle \emptyset, \emptyset, \emptyset, \{\varepsilon\} \rangle$$

Proposition 1. *$A \otimes B$ is a game.*

Proof. **(i)** $\langle M_{A\otimes B}, \lambda_{A\otimes B}, \vdash_{A\otimes B} \rangle$ is an arena.

(e1)

$$\begin{aligned} & \star \vdash_{A\otimes B} \\ \Rightarrow\ & \star \vdash_A m \ \vee \ \star \vdash_B m \\ \Rightarrow\ & (\lambda_A(m) = OQ \ \wedge \ [n \vdash_A m \Leftrightarrow n = \star]) \vee (\lambda_B(m) = OQ \ \wedge \ [n \vdash_B m \Leftrightarrow n = \star]) \\ \Rightarrow\ & \lambda_{A\otimes B}(m) = OQ, \ \wedge [n \vdash_{A\otimes B} m \iff n \vdash_A m \vee n \vdash_B m \iff n = \star] \end{aligned}$$

(e2)

$$\begin{array}{rl} & m \vdash_{A\otimes B} n \ \wedge \ \lambda^{QA}_{A\otimes B} = A \\ \Rightarrow & (m \vdash_A n \ \wedge \ \lambda^{QA}_{A} = A) \ \vee \ (m \vdash_B n \ \wedge \ \lambda^{QA}_{B} = A) \\ \Rightarrow & (\lambda^{QA}_{A}(m) = Q) \ \vee \ (\lambda^{QA}_{B}(m) = Q) \\ \Longleftrightarrow & \lambda^{QA}_{A\otimes B}(m) = Q \end{array}$$

(e3)

$$\begin{array}{rl} & m \vdash_{A\otimes B} n, \ m \neq \star \\ \Rightarrow & (m \vdash_A n, \ m \neq \star) \ \vee \ (m \vdash_B n, \ m \neq \star) \\ \Rightarrow & \lambda^{OP}_{A}(m) \neq \lambda^{OP}_{A}(n) \ \vee \ \lambda^{OP}_{B}(m) \neq \lambda^{OP}_{B}(n) \\ \Rightarrow & \lambda^{OP}_{A\otimes B}(m) \neq \lambda^{OP}_{A\otimes B}(n) \end{array}$$

(ii) The following shows that $P_{A\otimes B}$ is a nonempty refix-closed subset of L_A.

- $\varepsilon \in P_{A\otimes B}$.
- Assume $sm \in PA \otimes B$ and $m \in M_A$without loss of generality.

$$\begin{array}{rl} & sm \in P_{A\otimes B}, \ m \in M_A \\ \Rightarrow & (s \upharpoonright A)m \in P_A, \ s \upharpoonright B \in P_B \\ \Rightarrow & s \upharpoonright A \in P_A, \ s \upharpoonright B \in P_B \\ \Rightarrow & s \in P_{A\otimes B} \end{array}$$

(iii) Let $s \in P_{A\otimes B}$ and I is a set of initial moves of s. We suffices to prove $s \upharpoonright_h \in P_{A\otimes B}$ to complete the proof. Write $I = I_A + I_B$ where $I_A \subset M_A$, $I_B \subset M_B$. Since if m is hereditary justified by n, either $m, n \in M_A$ or $m, n \in M_B$ is satisfied,

$$\begin{aligned} (a \upharpoonright_h I) \upharpoonright A &= s \upharpoonright_h I_A \\ &= (s \upharpoonright A) \upharpoonright_h I_A. \end{aligned}$$

This is included in P_A since $s \upharpoonright A \in P_A$. In the same way, $(s \upharpoonright_h) \upharpoonright B \in P_B$, and we have $s \upharpoonright_h I \in P_{A\otimes B}$

□

Proposition 2. *$A \multimap B$ is a game.*

Proof. The proof is similar to that of $A \otimes B$. □

Proposition 3. *$I = \langle \emptyset, \emptyset, \emptyset, \{\varepsilon\} \rangle$ is a game.*

Proof. $\langle \emptyset, \emptyset, \emptyset \rangle$ is an arena. $\{\varepsilon\}$ is non-empty prefix-closed. There are no initial moves. $\square$

Proposition 4. *I is the tenser unit.*

Proof. We have $M_{A\otimes I} = M_A$, $\lambda_{A\otimes I} = \lambda_A$, $\lambda_{A\otimes I} = \lambda_A$, $L_{A\otimes I} = L_A$. For all $s \in L_{A\otimes I}$, $s \upharpoonright A = s$ and $s \upharpoonright I = \varepsilon$. These concludes $A \otimes I = A$. (This tenser is symmetric.) $\square$

Definition 7. σ is called a *strategy* for a game A if σ is a set of even-length positions from P_A and

(s1) $\varepsilon \in \sigma$.

(s2) $sab \in \sigma \Rightarrow s \in \sigma$.

(s3) sab, $sac \in \sigma \Rightarrow b = c$ and b,c are justified by the same move in sa.

Proposition 5. $\mathrm{id}_A = \{s \in P^{\mathrm{even}}_{A_1 \multimap A_2} \mid \forall t \sqsubset^{\mathrm{even}} s,\ t \upharpoonright A_1 = t \upharpoonright A_2\}$ *is a strategy for $A \multimap A$.*

Proof. By the definition, id_A is a set of even-length positions in P_A. It suffices to show the following three.

(s1) $\varepsilon \upharpoonright A_1 = \varepsilon \upharpoonright A_2$. So, $\varepsilon \in \mathrm{id}_A$.

(s2) If $sab \in \mathrm{id}_A$ and $t \sqsubset^{\mathrm{even}} s$, $t \sqsubset^{\mathrm{even}} sab$. Since $sab \in \mathrm{id}_A$, $t \upharpoonright A_1 = t \upharpoonright A_2$. This shows $s \in \mathrm{id}_A$.

(s3)

$$\begin{aligned} & sab \in \mathrm{id}_A \\ \Rightarrow\ & s, sab \in \mathrm{id}_A \\ \Rightarrow\ & s \upharpoonright A_1 = s \upharpoonright A_2,\ sab \upharpoonright A_1 = sab \upharpoonright A_2, \\ \Rightarrow\ & ab \upharpoonright A_1 = ab \upharpoonright A_2. \end{aligned}$$

In the each case of $a \in A_1$ and $a \in A_2$, $a = b$ holds. By induction, ignoring whether the moves are in A_1 or A_2, the element s of id_A is presented as

$$m_1 m_1 m_2 m_2 \ldots m_k m_k$$

Because $s \upharpoonright A_1 = s \upharpoonright A_2$, the two m_k points the same move m_i in the arena A.

$\square$

Definition 8. Let A, B, C be games and σ be a strategy of $A \multimap B$ and τ be a strategy of $B \multimap C$. Define *interaction sequences* $\mathrm{int}(A, B, C)$ by setting $u \in \mathrm{int}(A, B, C)$ if and only if $u \in (M_A, M_B, M_C)^*$ and there are justification pointers from all moves except those initial in C.

We also define $\sigma||\tau$ as

$$\{u \in \mathrm{int}(A,B,C) \mid u \upharpoonright A,B \in \sigma,\ u \in B,C \in \tau\}$$

$u \upharpoonright A,B$ is a subsequence of $u \in \mathrm{int}(A,B,C)$ with the pointers in u that are from M_A or M_B and to M_A or M_B.

For each $u \in \mathrm{int}(A,B,C)$, we have a sequence with some pointers $u \upharpoonright A,C$ by restricting the moves in A or C and taking out the pointers in u between the moves of A and B, and adding for each initial moves m in A a pointer to the initial move n in C; Since $u \upharpoonright A,B \in \sigma$, by the definition of the game $A \multimap B$, each initial move has a pointer to an initial move in B. And also an initial move in B has an pointer toward an initial move in C. Therefore there is a initial move in C which is pointed by the initial move in B which is also pointed by n.

We finally define $\sigma;\tau$ by

$$\sigma;\tau = \{u \upharpoonright A,C \mid u \in \sigma||\tau\}$$

Lemma 1. *If $u \in \sigma||\tau$, $u \upharpoonright A,C$ is a justified sequence in the arena $A \multimap C$.*

Proof. By the defition of $\sigma||\tau$, $(u \upharpoonright A,B) \upharpoonright A = u \upharpoonright A = (u \upharpoonright A,C) \upharpoonright A$ is a justified sequence in the arena A, and so is $(u \upharpoonright A,C) \upharpoonright C$ in C. What is left to be checked is about initial moves in A, but by the construction, there is a pointer from each initial move in A to an initial move in C. □

Lemma 2. *If $u \in \sigma||\tau$, $u \upharpoonright A,C$ is also a legal position.*

Proof. Reading the alphabets of u one by one, the five games $A, B, C, A \multimap B, B \multimap C$ must proceed along the automaton below.

$$\begin{array}{c} A, B, C, A \multimap B, B \multimap C \\ \\ [O,O,O,O,O] \\ \text{P-move in C} \uparrow\downarrow \text{O-move in C} \\ [O,O,P,O,P] \\ \text{P-move in B} \uparrow\downarrow \text{O-move in B} \\ [O,P,P,P,O] \\ \text{P-move in A} \uparrow\downarrow \text{O-move in A} \\ [P,P,P,O,O] \end{array}$$

In this automaton, for example, the state $[O,O,P,O,P]$ represents that, at this state, the Player play the next turn in C and $A \multimap C$, and the Opponent plays the next turn in A and B and $A \multimap B$. And this implys that no other state like $[P,P,P,P,P]$ would appear since each restriction of u to five arenas gives a legal positions. Carefully watching this automaton, we observe that, as moves in $A \multimap C$, P-move and O-move appears alternatively. Therefore the first condition of legal position for $u \upharpoonright A,C$ is satisfied.

And the other two are shown in Lemma 2.10. in [3] □

Proposition 6. *$\sigma;\tau$ is a strategy for a game $A \multimap C$.*

Proof. At first we are going to prove that $\sigma;\tau$ is a set of even length positions included in $P_{A\multimap C}$, and then prove (s1) $\sim$ (s3). In this proof, the automaton made in the former lemma is often cited.

1 The length of $u \upharpoonright A, C$ is the sum of that of $u \upharpoonright A$ and $u \upharpoonright C$. We need to show that $u \upharpoonright A$ is even-length iff $u \upharpoonright C$ is even-length, but this follows since these two are equivalent to that $u \upharpoonright B$ is even-length because of the two condition $u \upharpoonright A, B \in \sigma,\ u \upharpoonright B, C \in \tau$.

Assume $s = u \upharpoonright A, C$ where $u \in \sigma||\tau$. Since $s \upharpoonright A = (u \upharpoonright A, C) \upharpoonright A = (u \upharpoonright A, B) \upharpoonright A$ and $u \upharpoonright A, B \in P_{A\multimap B}$, $s \upharpoonright A \in P_A$. In the same way, $s \upharpoonright C \in P_C$. By the previous lemma, s is a legal position in $A \multimap C$, and finally we have $s \in P_{A\multimap C}$.

(s1) $\varepsilon \in \sigma||\tau$. Therefore $\varepsilon \in \sigma;\tau$.

(s2) Assume $sab = u \upharpoonright A, C \in \sigma;\tau$ ($u \in \sigma||\tau$). There is $u' \sqsubset u$ such that $u \upharpoonright A, C = s$. Since s is even-length, after reading u', game $A \multimap C$ is in Opponent's turn, which corresponds to the top state or the bottom state in the automaton in the lemma. In those two states in the automaton, the game $A \multimap B$, $B \multimap C$ is in Opponent's turn. Therefore, $u' \upharpoonright A, B \sqsubset u \upharpoonright A, B \in \sigma$ and $u' \upharpoonright B, C \sqsubset u \upharpoonright B, C \in \tau$ are even-length, and this implies $u' \upharpoonright A, B \in \sigma$ and $u' \upharpoonright B, C \in \tau$. These concludes that $s = u' \upharpoonright A, C \in \sigma;\tau$.

(s3) We prove the following statement first.

Assuming $u \in \sigma||\tau$, $a, m, n \in M_A + M_C$, $v, w \in M_B^*$ and $uavm, uawn \in \sigma||\tau$, it follows that $uavm$ and $uawn$ are identical as justified sequences.

Since $u \in \sigma||\tau$, after reading u, the automaton is in the top or bottom state. Let us assume the top case first, then a is a move in C. We have,

$$\begin{aligned} uavm \upharpoonright A, B =& (u \upharpoonright A, B)v(m \upharpoonright A, B) \in \sigma \\ uawn \upharpoonright A, B =& (u \upharpoonright A, B)w(n \upharpoonright A, B) \in \sigma. \\ uavm \upharpoonright B, C =& (u \upharpoonright A, B)av(m \upharpoonright A, B) \in \tau \\ uawn \upharpoonright B, C =& (u \upharpoonright A, B)aw(n \upharpoonright A, B) \in \tau. \end{aligned}$$

Since $(u \upharpoonright A, B)av_1, (u \upharpoonright A, B)aw_1 \in \tau$, $v_1 = w_1$. And because $(u \upharpoonright A, B)v_1v_2, (u \upharpoonright A, B)w_1w_2 \in \sigma$, $v_2 = w_2$. By induction, we have $vm = wn$. The same proof can be done in the case a is a move in A.

By the induction, the statement shows that for each $s \in \sigma;\tau$, there is a unique $u \in \sigma||\tau$ such that $s = u \upharpoonright A, C$.

Assume $sam, san \in \sigma;\tau$ and $uavm, uawn \in \sigma||\tau$ with $sam = uavm \upharpoonright A, Candsan = uawn \upharpoonright A, C$. By the statement above, $uavm$ and $uawn$ are identical, implying $sam = san$.

$\square$

Definition 9. If $sab, ta \in L_A$, where sab is even-length and $\lceil sa \rceil = \lceil ta \rceil$, there is a unique legal position tab extending ta which satisfies $\lceil sab \rceil = \lceil tab \rceil$. Define this legal position as $\mathrm{match}(sab, ta)$.

A strategy $\sigma : A$ is called *innocent* if

$$sab \in \sigma \ \wedge \ t \in \sigma \ \wedge \ ta \in P_A \ \wedge \ \lceil ta \rceil = \lceil sa \rceil \Rightarrow \mathrm{match}(sab, ta) \in \sigma$$

This means that in an innocent strategy σ, the next move from sa is only determined by by the P-view $\lceil sa \rceil$.

A strategy σ is *well–bracketed* if every time P answers, it must be an answer for the to the most recent unanswered question in the view.

Proposition 7. $\mathrm{id}; \sigma = \sigma = \sigma; id$.

Proof. Before prooving, we take a look at the structure of elements in $\mathrm{id}||\sigma \subset \mathrm{int}(A_1, A_2, B)$. Recalling Proposition.5, in the sequence $u \upharpoonright A_1, A_2 \in \mathrm{id}_A \subset P_{A_1 \multimap A_2}$, P-move in A_1 is always followed just after by the same P-move in A_2 and O-move in A_2 is always followed just after by the same O-move in A_1. Again by the automaton in Lemma.2, this occurs also in u. Therefore, $\mathrm{id}||\sigma \to \sigma$ with $u \mapsto u \upharpoonright A, B$ has the inverse map f defined inductively by

$$\begin{aligned}
f(\varepsilon) =& \varepsilon \\
f(ta) =& f(t)a^{(1)}a^{(2)} & a \in A, \lambda_A^{\mathrm{OP}} = P \\
f(ta) =& f(t)a^{(2)}a^{(1)} & a \in A, \lambda_A^{\mathrm{OP}} = O \\
f(tb) =& f(t)b & b \in B
\end{aligned}$$

where each $a^{(1)}, a^{(2)}$ is the copy of $a \in A$ in A_1, A_2. And this f is also easily shown to be the inverse map of $\mathrm{id}||\sigma \to \sigma$ with $u \mapsto u \upharpoonright A, C$. This shows $\mathrm{id}; \sigma = \sigma$. $\sigma; \mathrm{id} = \sigma$ can be shown in the same way. $\square$

Proposition 8. $(\sigma; \tau); \gamma = \sigma; (\tau; \gamma)$.

Proof. Let σ, τ, γ be a strategy for $A \multimap B, B \multimap C, C \multimap D$.

$$\begin{aligned}
& s \in (\sigma; \tau); \gamma \\
\iff \quad & \exists u \in (\sigma; \tau)||\gamma \text{ s.t. } s = u \upharpoonright A, D \\
\iff \quad & \exists u \in \mathrm{int}(A, C, D) \text{ s.t. } \left\{ \begin{array}{ccc} u \upharpoonright A, C & \in & \sigma; \tau \\ u \upharpoonright C, D & \in & \gamma \\ u \upharpoonright A, D & = & s \end{array} \right. \\
\iff \quad & \exists u \in \mathrm{int}(A, C, D), \exists v \in \mathrm{int}(A, B, C) \text{ s.t. } \left\{ \begin{array}{ccc} v \upharpoonright A, B & \in & \sigma \\ v \upharpoonright B, C & \in & \tau \\ u \upharpoonright C, D & \in & \gamma \\ v \upharpoonright A, C & = & u \upharpoonright A, C \\ u \upharpoonright A, D & = & s \end{array} \right.
\end{aligned}$$

Define $\mathrm{int}(A, B, C, D)$ as the set of the sequences of $M_A + M_B + M_C + M_D$ with justified pointer except the initial move in D. We also define $u \upharpoonright A, B$,

$u \restriction A, C,\ u \restriction A, D,\ \ldots$ for $u \in \mathrm{int}(A, B, C, D)$ by the same way as we did for $\mathrm{int}(A, B, C)$. We show that the condition above is satisfied if and only if

$$\exists u \in \mathrm{int}(A, B, C, D), \begin{cases} u \restriction A, B & \in & \sigma \\ u \restriction B, C & \in & \tau \\ u \restriction C, D & \in & \gamma \\ u \restriction A, D & = & s \end{cases}$$

If this equivalence is proved, it is clear that this is also equivalent to $s \in \sigma; (\tau; \gamma)$, and the proof will be completed.

The "if" part is clear. We have to show the "only if" part by considering insertion of $v \in \mathrm{int}(A, B, C)$ into $u \in \mathrm{int}(A, C, D)$. Now we have two sequences $v \in \mathrm{int}(A, B, C)$, $u \in \mathrm{int}(A, C, D)$, and each of them follows the automaton

$$\begin{array}{ccc} \mathrm{C/P} \uparrow\downarrow \mathrm{C/O} & \qquad & \mathrm{D/P} \uparrow\downarrow \mathrm{D/O} \\ \mathrm{B/P} \uparrow\downarrow \mathrm{B/O} & & \mathrm{C/P} \uparrow\downarrow \mathrm{C/O} \\ \mathrm{A/P} \uparrow\downarrow \mathrm{A/O} & & \mathrm{A/P} \uparrow\downarrow \mathrm{A/O} \end{array}$$

Since $v \restriction A, C = u \restriction A, C$, these two can be put together and make a new sequence which follows to the another automaton

$$\begin{array}{c} \mathrm{D/P} \uparrow\downarrow \mathrm{D/O} \\ \mathrm{C/P} \uparrow\downarrow \mathrm{C/O} \\ \mathrm{B/P} \uparrow\downarrow \mathrm{B/O} \\ \mathrm{A/P} \uparrow\downarrow \mathrm{A/O} \end{array}$$

and the new sequence obviously satisfies the condition we wanted. $\square$

Finnally we have the following corollary.

Corollary 1. *$\mathcal{G}$ is a category.*

Definition 10. Given $\sigma : A \to B$ and $\tau : B \to D$, define $\sigma \otimes \tau : A \otimes C \to B \otimes D$ by

$$\sigma \otimes \tau = \{s \in L_{A \otimes C \multimap B \otimes D} \mid s \restriction A, B \in \sigma,\ s \restriction C, D \in \tau\}.$$

Proposition 9. *$\sigma \otimes \tau$ is a strategy for $A \otimes C \multimap B \otimes D$*

Proof. We first check $\sigma \otimes \tau \subset^{\text{even}} P_{A \otimes C \multimap B \otimes D}$.

$$
\begin{aligned}
& s \in \sigma \otimes \tau \\
\Rightarrow \quad & s \upharpoonright A, B \in \sigma, \quad s \upharpoonright C, D \in \tau \\
\Rightarrow \quad & |s \upharpoonright A, B| : \text{even}, \quad |s \upharpoonright C, D| : \text{even} \\
\Rightarrow \quad & |s| = |s \upharpoonright A, B| + |s \upharpoonright C, D| : \text{even}
\end{aligned}
$$

By the definition, $s \in L_{A \otimes C \multimap B \otimes D}$. It suffices to show $s \upharpoonright A \otimes C \in P_{A \otimes C}$, $s \upharpoonright B \otimes D \in P_{B \otimes D}$. I DO NOT KNOW HOW TO SHOW $s \upharpoonright A \otimes B \in L_{A \otimes C}$, WHICH MIGHT BE IMMEDIATELY SHOWN FROM $s \in L_{A \otimes C \multimap B \otimes D}$. We have $s \upharpoonright A, B \in \sigma \subset P_{A \multimap B}$ and $s \upharpoonright A, B \in \sigma \subset P_{A \multimap B}$ which imply each restriction of s to the game A, B, C, D are valid positions.

We suffices to show that$\sigma \otimes \tau$ satisfies the three conditions for strategy.

(s1) $\varepsilon \in \sigma \otimes \tau$.

(s2) If $s \in \sigma \otimes \tau \subseteq$, we know that for each arena $A \multimap B, C \multimap D, A \otimes C, B \otimes D, A, B, C, D$, players of the moves in s alternates. Therefore, as Propositition.2, we obtain an automaton below.

$$
\begin{tikzcd}
& & (OO|OO|OOOO) \arrow[dl, shift left, "B/O"] \arrow[dr, shift left, "D/O"] & & \\
& (PO|OP|OPOO) \arrow[ur, shift left, "B/P"] \arrow[dl, shift left, "A/O"] & & (OP|OP|OOOP) \arrow[ul, shift left, "D/P"] \arrow[dr, shift left, "C/O"] & \\
(OO|PP|PPOO) \arrow[ur, shift left, "A/P"] & & & & (OO|PP|OOPP) \arrow[ul, shift left, "C/P"]
\end{tikzcd}
$$

So, if $sab \in \sigma \otimes \tau$, $a \in A + B \iff b \in A + B$. Therefore $sab \upharpoonright A, B \in \sigma \Rightarrow s \upharpoonright A, B \in \sigma$, and the same for C, D.

(s3) Immediate from that, if $sab \in \sigma \otimes \tau$, $a \in A + B \iff b \in A + B$.

$\square$

Proposition 10. *$- \otimes B$ is a functor.*

Proof. NOT YET. NEED TO SHOW $(\sigma; \tau) \otimes (\mathrm{id}; \mathrm{id}) = (\sigma; \mathrm{id}) \otimes (\tau; \mathrm{id})$. $\square$

Proposition 11. *$- \otimes B$ has a right adjoint $- \otimes B \dashv B \multimap -$.*

Proof. For each strategy σ for $A \otimes B \multimap C$, we define another strategy $\overline{\sigma}$ for $A \multimap B \multimap C$ and check that $\overline{\sigma}$ is the unique strategy that satisfies

$$
\begin{tikzcd}
(B \multimap C) \otimes B \arrow[r, "ev"] & C \\
A \otimes B. \arrow[u, "\bar{\sigma} \otimes \mathrm{id}_B"] \arrow[ur, "\sigma"'] &
\end{tikzcd}
$$

$\square$

Definition 11. Define $A\&B$ and $!A$ as in [1].

$$
\begin{aligned}
M_{A\&B} &= \ M_A + M_B. \\
\lambda_{A\&B} &= \ [\lambda_A, \lambda_B]. \\
\star \vdash_{A\&B} n \Leftrightarrow &\ \star \vdash_A n \ \vee \ \star \vdash_B n. \\
m \vdash_{A\&B} n \Leftrightarrow &\ m \vdash_A n \ \vee \ m \vdash_B n. \\
P_{A\&B} &= \ \{s \in L_{A\&B} \mid s \upharpoonright A \in P_A \ \wedge \ s \upharpoonright B = \varepsilon\} \\
&\quad \cup \ \{s \in L_{A\&B} \mid s \upharpoonright A = \varepsilon \ \wedge \ s \upharpoonright B \in P_B\}
\end{aligned}
$$

$$
\begin{aligned}
M_{!A} &= \ M_A. \\
\lambda_{!A} &= \ \lambda_A. \\
\vdash_{!A} &= \ \vdash_A . \\
P_{A\&B} &= \ \{s \in L_{!A} \mid \text{for each initial move } m, s \upharpoonright m \in P_A\}
\end{aligned}
$$

Proposition 12. *$A\&B$ is a game.*

Proof. It is clear that $\langle M_{A\&B}, \lambda_{A\&B}, \vdash_{A\&B}\rangle$ defines an arena. Since $\varepsilon \in P_{A\&B}$ and $sa \in P_{A\&B} \Rightarrow s \in P_{A\&B}$, $P_{A\&B}$ is a non-empty prefix-closed set of legal positions. $s \in P_{A\&B} \ \wedge \ I :$ initial moves of $s \Rightarrow s \upharpoonright_h I \in P_{A\&B}$ is also clear. $\square$

Proposition 13. *$A\&B$ is the product of A and B in $\mathcal{G}$.*

Proposition 14. *$!A$ is a game.*

Proof. The arena $!A$ is that of A. $\varepsilon \in P_{!A}$ is clear.

$$
\begin{aligned}
& sa \in P_{!A} \\
& \forall m : \text{initial move}, sa \upharpoonright_h \in P_A \\
& \forall m : \text{initial move}, s \upharpoonright_h \in P_A
\end{aligned}
$$

Therefore, $P_{!A}$ is a non-empty prefix-closed set of legal positions.

To show that $!A$ is a game, it suffices to show

$$s \in P_{!A} \ \wedge \ I : \ \text{initial moves of } s \Rightarrow s \upharpoonright_h I \in P_{!A}$$

If $s \in P_{!A}$ and m is an initial move, $(s \upharpoonright_h I) \upharpoonright_h m = s \upharpoonright_h m \in P_A$. It is left to be shown that $s \upharpoonright_h I \in L_{!A} = L_A$. THIS IS A CONSEQUENCE OF LEMMA 2.6 IN [3]. $\square$

Definition 12. A game A is called *well-opened* if for all $sm \in P_A$, m is not initial when $s \neq \varepsilon$.

For each $\sigma : !A \to B$, define $\sigma^\dagger$ by $\{s \in L_{!A \multimap !B} \mid \text{for all initial } m, \ s \upharpoonright m \in \sigma\}$.

Proposition 15. *If A and B are well-opened and σ is a map of $!A \to B$, $\sigma^\dagger$ is a strategy for $!A \multimap !B$*

Proof. We check that $\sigma^\dagger$ is a set of even-length positions from $P_{!A\multimap !B}$ first. Let s be a sequence included in $\sigma^\dagger$. If I is the set consists of all initial moves in s, $|s| = |s \upharpoonright_h I| = \sum_{m\in I} |s \upharpoonright_h m|$, which is even. To show $\sigma^\dagger \subset P_{!A\multimap !B}$, take $s \in \sigma^\dagger$. We need to show

$$\begin{aligned}
& s \in P_{!A\multimap !B} \\
\iff\quad & s \upharpoonright A \in P_{!A} \quad \wedge \quad s \upharpoonright B \in P_{!B} \\
\iff\quad & s \upharpoonright A \in L_{!A} \quad \wedge \quad \forall m : \text{ initial move in } A, (s \upharpoonright A) \upharpoonright_h m \in P_A \\
& \wedge \quad s \upharpoonright B \in L_{!B} \quad \wedge \quad \forall m : \text{ initial move in } B, (s \upharpoonright B) \upharpoonright_h m \in P_B.
\end{aligned}$$

We have $s \in L_{!A\multimap !B}$, $(s \upharpoonright_h m) \upharpoonright A \in P_{!A}$. BUT, I don't know how to prove $s \upharpoonright A \in L_{!A} = L_A$. Since $s \upharpoonright_h m$ is included in $\sigma \subset P_{!A\multimap B}$, $(s \upharpoonright A) \upharpoonright_h m = (s \upharpoonright m) \upharpoonright \in P_A$. The same consists for B.

We are left to prove the following.

(s1) $\varepsilon \in \sigma^\dagger$.

(s2) Take $sab \in \sigma^\dagger$. Havn't wrote yet.

(s3)

$\square$

Definition 13. Assume A, B, C be well-opened games. When σ and τ are strategies for each game $!A \multimap B$, $!B \multimap C$, define a strategy $\sigma \fatsemi \tau$ for the game $!A \multimap C$ by $\sigma \fatsemi \tau = \sigma^\dagger ; \tau$.

Proposition 16. *For each well-opened game A, set* der_A *as*

$$\mathrm{der}_A = \{s \in P_{!A\multimap A} \mid \forall t \sqsubset^{\text{even}} s, \quad t \upharpoonright !A = t \upharpoonright A\}.$$

Then, der_A *is a strategy for* $!A \to A$ *satisfying* $\mathrm{der} \fatsemi \sigma = \sigma = \sigma \fatsemi \mathrm{der}$ *for any* σ.

Proof. By the definition, $\mathrm{der}_A \subseteq^{\text{even}} P_{!A\multimap A}$. **(s1)** ε is included in der_A. **(s2)** If $sab \in \mathrm{der}_A$, clearly $s \in \mathrm{der}_A$. **(s3)** If $s, sab \in \mathrm{der}_A$, $s \upharpoonright !A = s \upharpoonright A$ and $s \upharpoonright !B = s \upharpoonright B$ are satisfied by the definition. This implies that a and b are the copies of the same move of A but one is a move of A and the other is that of $!A$. Therefore, der_A is a strategy.

By induction, a sequence s included in der_A has a form as

$$s = a_1 a_1 a_2 a_2 \ldots a_k a_k.$$

To prove $\mathrm{der} \fatsemi \sigma = \sigma = \sigma \fatsemi \mathrm{der}$, we show $\mathrm{der}_A^\dagger = \mathrm{id}_{!A}$ and $\sigma^\dagger ; \mathrm{der}_B = \sigma$.

(Proof of $\mathrm{der}_A^\dagger = \mathrm{id}_{!A}$) Assume $s \in \mathrm{id}_{!A} \cap \mathrm{der}_A^\dagger$ and $sa \in L_{!A_1\multimap !A_2}$. It suffices to show that all the three below are equivalent.

$$\begin{gathered}
(i) sab \in \mathrm{id}_{!A} \\
(ii) sab \in \mathrm{der}_A^\dagger \\
(iii) saa = sab \in P_{!A_1\multimap !A_2}
\end{gathered}$$

(s has the form $a_1a_1a_2a_2\dots a_na_n$, and a points some a_i in s. We extend this justified sequence sa to saa by assuming that the latter a belongs to the defferent game from that of the former, and assuming that the latter a points the other a_i.)

$(i) \iff (iii)$ We already know

$$sab \in \mathrm{id}_{!A} \Rightarrow sab = saa.$$

Since s is in $\mathrm{id}_{!A}$, s is an element of $P_{!A_1 \multimap !A_2}$ and for all even-length prefix t of s, $t \upharpoonright !A_1 = t \upharpoonright !A_2$. To show $saa \in \mathrm{id}_{!A}$, we need to check

$$\begin{aligned} &saa \in P_{!A_1 \multimap !A_2} \\ &saa \upharpoonright !A_1 = saa \upharpoonright !A_2. \end{aligned}$$

The latter is trivial from our construction of saa, and therefore the first and the third conditions are equivalent.

$(ii) \Leftarrow (iii)$ By the definition,

$$saa \in \mathrm{der}_A^\dagger \iff saa \in L_{!A_1 \multimap !A_2} \ \wedge \ \text{forall initial } m,\ saa \upharpoonright_h m \in \mathrm{der}_A$$

$saa \in L_{!A_1 \multimap !A_2}$ follows from (iii). Let m be an initial move in saa. Again by the definition,

$$\begin{aligned} & saa \upharpoonright_h \in \mathrm{der}_A \\ \iff & saa \upharpoonright_h m \in P_{!A_1 \multimap A_2} \ \wedge \ \forall t \sqsubset^{\text{even}} saa \upharpoonright_h m,\ t \upharpoonright !A_1 = t \upharpoonright !A_2 \end{aligned}$$

Since $s \upharpoonright m$ is in der_A, $s \upharpoonright_h m$ is even-length. TODO : show both a are hereditary justified by the same initial move. And then $saa \upharpoonright_h m$ is proved to be even-length. Showing this also implies $(saa \upharpoonright_h m) \upharpoonright !A_1 = (saa \upharpoonright_h m) \upharpoonright A_2$.

$(ii) \Rightarrow (iii)$ is easy.

(Proof of $\sigma^\dagger; \mathrm{der} = \sigma$) TODO.

□

Proposition 17. $(\sigma \fatsemi \tau) \fatsemi \gamma = \sigma \fatsemi (\tau \fatsemi \gamma)$.

Proof. We just need to show

$$(\sigma^\dagger; \tau)^\dagger = \sigma^\dagger; \tau^\dagger.$$

See Lemma 2.24. in [3]

□

Corollary 2. *$\mathcal{C}$ is a category.*

Proposition 18. *$A \& B$ is the product in $\mathcal{C}$.*

Lemma 3. *$!(A \& B) = !A \otimes !B$.*

Theorem 1. *$\mathcal{C}$ is CCC.*

References

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