Spectral Theorem

Keng-Yu Chen

June 9, 2025

Theorem (Spectral Theorem)**.** *Every real symmetric matrix can be diagonalized by an orthonormal basis*[1].

We prove this theorem through the following steps.

In the following, we consider a finite n-dimensional vector space V and a real symmetric matrix A for a linear transformation T on V.

Theorem 1. *Eigenvalues of real symmetric matrices are real.*

Proof. Let A be a real symmetric matrix. For any eigenvector $v \in V$ of A with corresponding eigenvalue λ,

$$\begin{aligned}\langle Av, v\rangle &= \langle \lambda v, v\rangle = \lambda\langle v, v\rangle \\ &= \langle v, A^* v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle\end{aligned}$$

Therefore, $\lambda = \bar{\lambda}$, λ is real.

Note that if A is a real symmetric matrix, then its eigenvectors are also real since they are in the kernel of $A - \lambda I$ for some real λ.

□

Theorem 2. *Eigenvectors to distinct eigenvalues of real symmetric matrices are orthogonal*

Proof. Let v, w be two eigenvectors of A, associated with real eigenvalues λ, μ.

$$\begin{aligned}\langle Av, w\rangle &= \lambda\langle v, w\rangle \\ &= \langle v, A^* w\rangle = \langle v, Aw\rangle = \langle v, \mu w\rangle = \bar{\mu}\langle v, w\rangle\end{aligned}$$

Therefore, either $\lambda = \bar{\mu} = \mu$ or $\langle v, w\rangle = 0$.

□

Theorem 3. *Let W be a T-invariant space in V, then its orthogonal complement $W^\perp$ is also a T-invariant space.*

Proof. For any $x \in W, y \in W^\perp$,

$$\langle x, Ay\rangle = \langle A^* x, y\rangle = \langle Ax, y\rangle = \lambda\langle x, y\rangle = 0,$$

which shows Ay also lies in $W^\perp$.

Note that this implies for any eigenvector v of a real symmetric A and $w \perp v$, $Aw \perp v$.

□

Now we can prove the spectral theorem.

Proof. We prove the theorem by induction. The case $n = 1$ is trivial. For a general finite n-dimensional vector space V, by the *fundamental theorem of algebra*, there must exist a root of the characteristic polynomial, which is an eigenvalue. By Theorem 1, it is real, so we can find a 1-unit eigenvector v with a real eigenvalue λ.

Let $W = \operatorname{span}\{v\}$, $W^\perp$ be the orthogonal complement of W. Since $W^\perp$ is $n-1$-dimensional, by induction, we can find an orthonormal basis $\tilde{\mathcal{B}}$ for $W^\perp$. As W and $W^\perp$ are both T-invariant, under the basis $\mathcal{B} = \{v\} \cup \tilde{\mathcal{B}}$, T corresponds to a block matrix,

$$A \sim \begin{bmatrix} \lambda & 0 \\ 0 & \tilde{A} \end{bmatrix},$$

[1] The theorem and the proofs all hold for *self-adjoint* matrices, a more general class of complex matrices.

where $\tilde{A}$ is the matrix representation of T under the basis $\tilde{B}$, restricted on the space $W^\perp$, and $\tilde{A}$ is a diagonal matrix. Moreover, since $W^\perp$ is the orthogonal complement of $W = \{v\}$, v is orthogonal to every vector in $W^\perp$, including all the basis vectors $\tilde{\mathcal{B}}$. Hence, $\mathcal{B}$ is an orthonormal basis for V.

$\square$