The Yao Graph Y_5 is a Spanner

Wah Loon Keng* Ge Xia†

Abstract

In this paper we prove that Y_5, the Yao graph with five cones, is a spanner with stretch factor $\rho = 2 + \sqrt{3} \approx 3.74$. Since Y_5 is the only Yao graph whose status of being a spanner or not was open, this completes the picture of the Yao graphs that are spanners: a Yao graph Y_k is a spanner if and only if $k \geq 4$.

We complement the above result with a lower bound of 2.87 on the stretch factor of Y_5. We also show that YY_5, the Yao-Yao graph with five cones, is not a spanner.

1 Introduction

Let S be a set of points in the plane. Fix an ordering $\prec$ on all pairs of points $\{a, b\}$ in S based on their Euclidean distance $||ab||$ where ties are broken arbitrarily, i.e. if $||ab|| < ||cd||$ then $\{a, b\} \prec \{c, d\}$. Given an integer parameter $k > 0$, the *directed Yao graph* [10] with parameter k, denoted $\overrightarrow{Y_k}$, is constructed as follows. For each point p in S, partition the space into k equal-measured cones of angle $2\pi/k$ each whose apex is p (the orientation of the cones is fixed for all points). In each cone, p chooses the closest point q in S (if any) according to the ordering $\prec$ and adds (p, q) to $\overrightarrow{Y_k}$ as a directed edge outgoing from p. The (undirected) Yao graph with parameter k, denoted Y_k, is the underlying undirected graph of $\overrightarrow{Y_k}$.

A geometric graph G on the point set S is called a *ρ-spanner* if for every two points $a, b \in S$, the shortest path distance between a and b in G is at most $\rho \cdot ||ab||$. G is called a *geometric spanner* or simply *spanner* if ρ is a constant.

The Yao graphs have been extensively studied, and in particular many of their spanning properties have been discovered. It is known that Y_2 and Y_3 are not spanners [9], Y_4 is a spanner with stretch factor $8\sqrt{2}(29 + 23\sqrt{2})$ [4], Y_6 is a spanner with stretch factor 17.7 [6], and that for $k \geq 7$, Y_k is a spanner with stretch factor $\frac{1}{1-2\sin(\pi/k))}$ [3]. The question of whether or not Y_5 is a spanner was previously open.

In this paper we prove that Y_5 is a ρ-spanner, where $\rho = 2+\sqrt{3} \approx 3.74$. Combining this with the previous results, we now have a complete picture of the spanners that can be constructed with Yao graphs: any Yao graph Y_k is a spanner if and only if $k \geq 4$. We also give a lower bound of 2.87 on the stretch factor of Y_5.

Recent Developments. An earlier version of this paper [8] proved a stretch factor of $\frac{1}{1-2\sin(3\pi/20)} \approx 10.87$ for Y_5 using a simple approach. In a recent manuscript, Barba et al. [1] independently proved the same bound of 10.87 using the same approach and they also used that approach to improve the stretch factor of Y_k for odd $k \geq 7$ to $\frac{1}{1-2\sin(3\pi/4k)}$. In addition, Barba et al. [1] improved the stretch factor of Y_6 to 5.8.

*Lafayette College, Easton, PA 18042, USA. kengw@lafayette.edu.
†Department of Computer Science, Lafayette College, Easton, PA 18042, USA. xiag@lafayette.edu.

Table 1: Stretch factors of Yao and Yao-Yao graphs

Parameter k	Yao Graph Y_k	Yao-Yao Graph YY_k
$k = 2, 3$	not a spanner [9]	not a spanner [9]
$k = 4$	$8\sqrt{2}(29 + 23\sqrt{2})$ [4]	not a spanner [5]
$k = 5$	$1/(1 - 2\sin(3\pi/20)) \approx 10.87$ [1, 8] $2 + \sqrt{3} \approx 3.74$ [this paper]	not a spanner [this paper]
$k = 6$	17.7 [6], 5.8 [1]	not a spanner [9]
$k \geq 7$	$1/(1 - 2\sin(\pi/k))$ for even k [3] $1/(1 - 2\sin(3\pi/4k))$ for odd k [1]	11.67 for $k = 6k'$, $k' \geq 6$ [2] open for other values of $k \geq 7$

In contrast to our main results, we show that YY_5, the Yao-Yao graph with five cones, is not a spanner. The *directed Yao-Yao graph* with parameter $k > 0$, denoted $\overrightarrow{YY_k}$, is constructed in two stages. The first stage proceeds as in the construction of $\overrightarrow{Y_k}$. In the second stage, for each point $p \in S$, and for each cone defined by p in the first stage, point p keeps *only* the shortest incoming edge (if any) according to the ordering $\prec$ in $\overrightarrow{Y_k}$ in the cone. The directed edges kept by the points in S in the second stage constitute $\overrightarrow{YY_k}$. The (undirected) Yao-Yao graph YY_k denotes the underlying undirected graph of $\overrightarrow{YY_k}$. Clearly, $\overrightarrow{YY_k}$ is a subgraph of $\overrightarrow{Y_k}$, and YY_k is a subgraph of Y_k. The Yao-Yao graphs have an advantage over the Yao graphs in that their maximum degree is bounded: Whereas Y_k can have unbounded degree, the maximum degree of YY_k is at most $2k$. It is known that YY_4 is not a spanner [5] and is not plane [7] and that for any integer $k \geq 6$, YY_{6k} is a spanner [2]. It is still open whether the Yao-Yao graph is a spanner for other values of the parameter k.

Table 1 shows the stretch factors of Yao and Yao-Yao graphs for various values of the parameter k.

The paper is organized as follows. In Section 2, we introduce the notations and terminologies used throughout the paper. In Section 3, we prove that Y_5 is a spanner. In Section 4, we give a lower bound of 2.87 on the stretch factor of Y_5. We show in Section 5 that YY_5 is not a spanner. We conclude the paper in Section 6.

2 Preliminaries

Given a set of points S in the two-dimensional Euclidean plane, the complete Euclidean graph $\mathcal{E}$ on S is defined to be the complete graph whose point-set is S. Each edge ab connecting points a and b is assumed to be embedded in the plane as the straight line segment ab; we define its *length* to be the Euclidean distance $||ab||$.

Let G be a subgraph of $\mathcal{E}$. The length of a simple path $P = m_0, m_1, \ldots, m_r = b$ between two points a, b in G is $|P| = \sum_{j=0}^{r-1} ||m_j m_{j+1}||$. For two points a, b in G, we denote by $d_G(a, b)$ (or simply $d(a, b)$ if G is clear from the context) the length of a shortest path between a and b in G. G is said to be a *spanner* (of $\mathcal{E}$) if there is a constant ρ such that, for every two points $a, b \in G$, $d(a, b) \leq \rho \cdot ||ab||$. The constant ρ is called the *stretch factor* or *spanning ratio* of G (with respect to $\mathcal{E}$).

For each point $p \in S$, label the five cones around it by $C_1^p, C_2^p, \ldots, C_5^p$ in the counterclockwise order. The two rays on the boundary of each cone are referred to as the *start-ray* and the *end-ray*, in the counterclockwise order. Fix an orientation of the cones such that the start-ray of C_1^p for all p is horizontal and points to the right. The *bisector* of a cone is a ray that separates the cone into two equal-sized subcones. See Figure 1 for an illustration. The following is a simple fact:

Fact 1. *Rotating around any point in the plane by $2\pi n/5$, where n is an integer, does not change the orientation of the cones (up to a relabeling). Furthermore, mirror-flipping along the bisector of any cone does not change the orientation of the cones (up to a relabeling).*

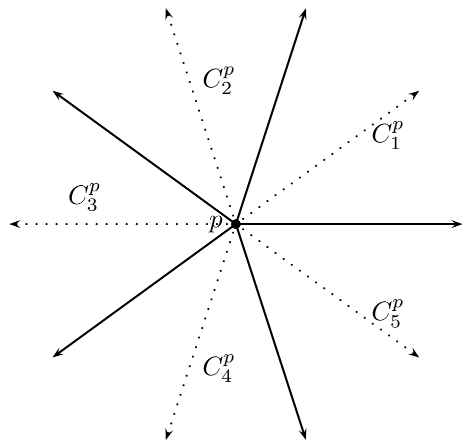


Figure 1: The cones and their bisectors.

In this paper, all the angles labeled as $\angle xyz$ are measured from ray $\overrightarrow{yx}$ to ray $\overrightarrow{yz}$ in counterclockwise direction. $|\angle xyz|$ indicates the (unsigned) magnitude of $\angle xyz$.

Next we give two lemmas that will be useful in our proof.

Lemma 1. *Let a, b, and c be three distinct points in the plane such that $||ac|| \leq ||ab||$ and $|\angle bac| \leq \theta$, where $\theta \in (0, \pi/3)$ is a constant. Then*

$$||ac|| + \lambda||bc|| \leq \lambda||ab||,$$

where $\lambda = \frac{1}{1-2\sin(\theta/2)}$.

Proof. By Lemma 10 of [3], $||bc|| \leq ||ab|| - ||ac||/t$, where $t = \frac{1+\sqrt{2-2\cos\theta}}{2\cos\theta - 1}$. By trigonometric identities, $t = \frac{1+\sqrt{2-2\cos\theta}}{2\cos\theta-1} = \frac{1}{1-\sqrt{2-2\cos\theta}} = \frac{1}{1-\sqrt{4\sin^2\frac{\theta}{2}}} = \frac{1}{1-2\sin\frac{\theta}{2}} = \lambda$. The lemma follows. $\square$

Lemma 2. *Let a, b, c be three points in the plane. Let $\theta = |\angle bac|$ and let $\lambda > 1$ be a constant. Suppose that $\cos\theta > \frac{1}{\lambda}$, $||bc|| < ||ab||$ and $\frac{||ac||}{||ab||} = \frac{2\lambda^2\cos\theta - 2\lambda}{\lambda^2 - 1}$. Then $||ad|| + \lambda||bd|| \leq \lambda||ab||$ for all points d in the line segment ac.*

Proof. Without loss of generality, let $||ab|| = 1$. Let $x = ||ad||$. Then $||bd|| = \sqrt{1 + x^2 - 2x\cos\theta}$. See Figure 2.

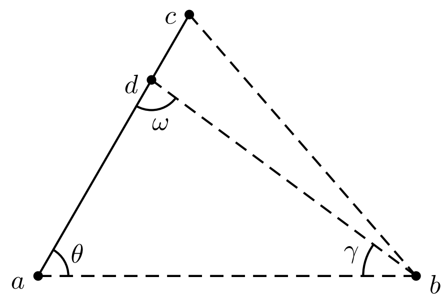


Figure 2: Illustration for the proof of Lemma 2

Note that

$$\begin{aligned}
& \lambda^2 - 2\lambda + 1 \geq 0 \\
\Rightarrow \quad & \lambda^2 - 2\lambda\cos\theta + 1 \geq 0 \\
\Rightarrow \quad & 2\lambda\cos\theta - 2 \leq \lambda^2 - 1 \\
\Rightarrow \quad & \frac{\lambda(2\lambda\cos\theta - 2)}{\lambda^2 - 1} \leq \lambda.
\end{aligned}$$

Therefore $x \leq ||ac|| = \frac{2\lambda^2\cos\theta - 2\lambda}{\lambda^2 - 1} \leq \lambda$. Solve $||ad|| + \lambda||bd|| = x + \lambda\sqrt{1 + x^2 - 2x\cos\theta} = \lambda = \lambda||ab||$ for $x \in (0, \lambda]$, we have

$$\begin{aligned}
& x + \lambda\sqrt{1 + x^2 - 2x\cos\theta} = \lambda \\
\Leftrightarrow \quad & \lambda\sqrt{1 + x^2 - 2x\cos\theta} = \lambda - x \\
\Leftrightarrow \quad & \lambda^2(1 + x^2 - 2x\cos\theta) = (\lambda - x)^2 \\
\Leftrightarrow \quad & \lambda^2(x^2 - 2x\cos\theta) = x^2 - 2\lambda x \\
\Leftrightarrow \quad & \lambda^2(x - 2\cos\theta) = x - 2\lambda \\
\Leftrightarrow \quad & (\lambda^2 - 1)x = 2\lambda^2\cos\theta - 2\lambda \\
\Leftrightarrow \quad & x = \frac{2\lambda^2\cos\theta - 2\lambda}{\lambda^2 - 1} = ||ac||.
\end{aligned}$$

This implies that $||ac|| + \lambda||bc|| = \lambda||ab||$.

Let $\gamma = |\angle dba|$ and $\omega = |\angle adb|$. By the law of sines in the triangle $\triangle abd$, we have

$$\frac{||bd||}{\sin\theta} = \frac{||ad||}{\sin\gamma} = \frac{||ab||}{\sin\omega}. \tag{1}$$

Therefore

$$\frac{||ad||}{||ab|| - ||bd||} = \frac{\sin\gamma}{\sin\omega - \sin\theta} = \frac{\sin\gamma}{\sin(\pi - \theta - \gamma) - \sin\theta} = \frac{\sin\gamma}{\sin(\theta + \gamma) - \sin\theta}.$$

Define a function

$$f = \frac{\sin\gamma}{\sin(\theta + \gamma) - \sin\theta}.$$

We will show $\frac{\partial f}{\partial \gamma} \geq 0$. This is sufficient for the lemma because we can transform the triangle $\triangle abd$ to triangle $\triangle abc$ by moving d toward c (i.e., by increasing γ).

By a standard calculation,

$$\begin{aligned}
\frac{\partial f}{\partial \gamma} &= \frac{\cos\gamma(\sin(\theta+\gamma)-\sin\theta)-\sin\gamma\cos(\theta+\gamma)}{(\sin(\theta+\gamma)-\sin\theta)^2} \\
&= \frac{\cos\gamma\sin(\theta+\gamma)-\cos\gamma\sin\theta-\sin\gamma\cos(\theta+\gamma)}{(\sin(\theta+\gamma)-\sin\theta)^2} \\
&= \frac{\sin\theta-\cos\gamma\sin\theta}{(\sin(\theta+\gamma)-\sin\theta)^2} \\
&= \frac{\sin\theta(1-\cos\gamma)}{(\sin(\theta+\gamma)-\sin\theta)^2}. \qquad (2)
\end{aligned}$$

We have $\frac{\partial f}{\partial \gamma} \geq 0$ because $\sin\theta > 0$, $\cos\gamma \leq 1$, and $||bc|| < ||ab||$ (and hence $\sin(\theta+\gamma) > \sin\theta$). This proves the lemma.

□

3 Y_5 is a Spanner

Let $\rho = 2+\sqrt{3} \approx 3.74$. Fix a constant $\overline{\theta} = \arccos(1-\frac{1}{\rho}) = \arccos(\sqrt{3}-1) \approx 0.75$. It is easy to verify that

$$\rho = \frac{1}{1-\cos\overline{\theta}} = \frac{1}{1-2\sin(\overline{\theta}/2)}.$$

This section contains a proof for the following main theorem.

Theorem 1. *Y_5 is a ρ-spanner, where $\rho = 2+\sqrt{3} \approx 3.74$.*

Let G be a Y_5 graph with point set S. We will prove that for any pair of points $u, v \in S$, $d(u,v) \leq \rho \cdot ||uv||$. We proceed by induction on the ordering $\prec$ of the pairs of points in S (which is based on the Euclidean distance $||uv||$). For the base case where $\{u,v\}$ is the first pair in the ordering $\prec$, u, v is connected in G, and hence $d(u,v) = ||uv|| \leq \rho \cdot ||uv||$.

For the inductive step, we will prove $d(u,v) \leq \rho \cdot ||uv||$ based on the inductive hypothesis that $d(x,y) \leq \rho \cdot ||xy||$ for all pairs of points $x, y \in S$ with $\{x,y\} \prec \{u,v\}$. Without loss of generality, assume $||uv|| = 1$.

Because of Fact 1, we can assume that v is in the first cone of u, i.e., $v \in C_1^u$. Furthermore, we can assume that v is on or below the bisector of C_1^u because otherwise by Fact 1 we can mirror-flip the geometry along the bisector of C_1^u. Let $A_1^u(v)$ be the arc centered at u with radius $||uv||$ that spans cone C_1^u. Let a and b be the start and end of the arc $A_1^u(v)$ (i.e., a is the intersection of $A_1^u(v)$ and the start-ray of C_1^u and b is the intersection of $A_1^u(v)$ and the end-ray of C_1^u). Let $F_1^u(v)$ be the fan-shaped region enclosed by ua, ub and $A_1^u(v)$. See Figure 3 for an illustration. It is easy to verify that u is in the third cone of v, i.e., $u \in C_3^v$. Similarly, let $A_3^v(u)$ be the arc centered at v with radius $||uv||$ that spans cone C_3^v. Let c and d be the start and end of the arc $A_3^v(u)$. Let $F_3^v(u)$ be the fan-shaped region enclosed by vc, vd and $A_3^v(u)$.

We can assume that u, v is not connected in G because otherwise $d(u,v) = ||uv|| \leq \rho \cdot ||uv||$. Therefore, there exists a point $w \in F_1^u(v)$ such that $uw \in G$ and a point $z \in F_3^v(u)$ such that $zv \in G$. Let

$$\alpha = |\angle vuw| \quad \text{and} \quad \beta = |\angle zvu|.$$

Let s be the intersection of the rays $\overrightarrow{ub}$ and $\overrightarrow{vc}$ and let t be the intersection of the rays $\overrightarrow{uw}$ and $\overrightarrow{vz}$. See Figure 3 for an illustration. It is easy to see that $|\angle usv| = 2\pi/5$ because $\overrightarrow{us}$ and $\overrightarrow{vs}$ are the boundaries of

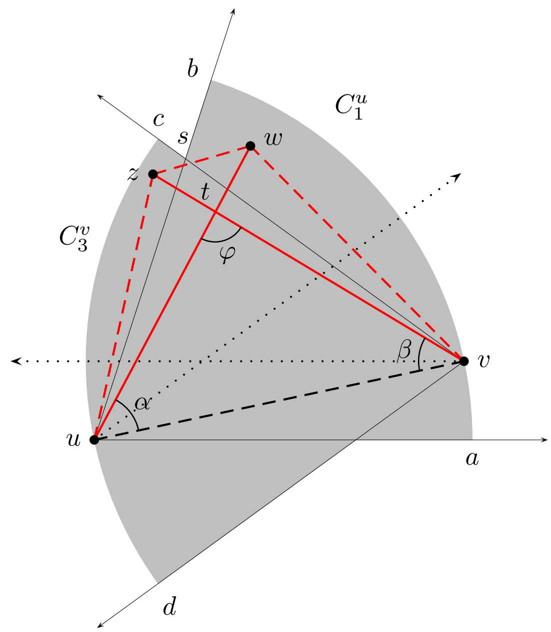


Figure 3: Illustration for the proof of Theorem 1.

the cones C_1^u and C_3^v respectively. Let $\varphi = |\angle utv|$. Then

$$\varphi = |\angle utv| = \pi - \alpha - \beta \geq \pi - |\angle vus| - |\angle svu| = |\angle usv| = 2\pi/5. \tag{3}$$

Since $\alpha + \beta = \pi - \varphi \leq \pi - 2\pi/5 = 3\pi/5$, we have

$$\min(\alpha, \beta) \leq 3\pi/10. \tag{4}$$

Based on the simple observation of (4), one can apply Lemma 1 to easily prove that the stretch factor of Y_5 is at most $\frac{1}{1-2\sin(3\pi/20)} \approx 10.87$, which is the same result obtained in an earlier version of this paper [8] and, independently, in [1]. Here we apply a more careful analysis to obtain a tighter upper bound on the stretch factor of Y_5.

We consider three paths between u and v:

1. P_1 consists of the edge $(u, w) \in G$ and the shortest path from w to v. The length of P_1 is $|P_1| = ||uw|| + d(v, w)$.

2. P_2 consists of the edge $(v, z) \in G$ and the shortest path from z to u. The length of P_2 is $|P_2| = ||vz|| + d(u, z)$.

3. P_3 consists of the edge $(u, w) \in G$, the shortest path from w to z, and the edge $(z, v) \in G$. The length of P_3 is $|P_3| = ||uw|| + ||vz|| + d(z, w)$.

Clearly, $d(u, v) \leq \min(|P_1|, |P_2|, |P_3|)$.

Define three values

$$g_1 = ||uw|| + \rho||vw||, \tag{5}$$

$$g_2 = ||vz|| + \rho||uz||, \tag{6}$$

$$g_3 = ||uw|| + ||vz|| + \rho||zw||. \tag{7}$$

In order to prove the theorem, it suffices to prove that

$$\min(g_1, g_2, g_3) \leq \rho||uv||. \tag{8}$$

Here is why: if $g_1 = ||uw|| + \rho||vw|| \leq \rho||uv||$, then $||vw|| < ||uv||$ and by the inductive hypothesis $d(v, w) \leq \rho||vw||$, which gives us

$$|P_1| = ||uw|| + d(v, w) \leq ||uw|| + \rho||vw|| \leq \rho||uv||.$$

Similarly, if $g_2 \leq \rho||uv||$ then $|P_2| \leq \rho||uv||$ and if $g_3 \leq \rho||uv||$ then $|P_3| \leq \rho||uv||$. In any of the these cases, we have $d(u, v) \leq \min(|P_1|, |P_2|, |P_3|) \leq \rho||uv||$ and the theorem is proven.

In the following, we will prove (8) using analysis and geometric observations. We start by bounding the values of α and β.

If $\alpha \leq \overline{\theta}$, then by Lemma 1,

$$||uw|| + \frac{1}{1 - 2\sin(\overline{\theta}/2)} \cdot ||vw|| \leq \frac{1}{1 - 2\sin(\overline{\theta}/2)} \cdot ||uv||.$$

Since $\rho = \frac{1}{1-2\sin(\overline{\theta}/2)}$, this implies

$$g_1 = ||uw|| + \rho||vw|| \leq \rho||uv||, \tag{9}$$

and we are done. Similarly, if $\beta \leq \overline{\theta}$, then $g_2 = ||vz|| + \rho||uz|| \leq \rho||uv||$ and we are done.

Therefore we can assume $\alpha > \overline{\theta}$ and $\beta > \overline{\theta}$. Since v is on or below the bisector of C_1^u, we have $|\angle auv| \leq \pi/5 < \overline{\theta}$ and $|\angle uvd| \leq \pi/5 < \overline{\theta}$. This implies that neither z or w is below the line uv. So we can assume that both z and w are above the line uv, as illustrated by Figure 3.

The following proposition plays a key role in this proof.

Proposition 1. *If $g_1 > \rho||uv||$ and $g_2 > \rho||uv||$, then $||wz|| \leq 2\cos\overline{\theta} - 1$.*

Proof. Let w', w'' be two points in the ray $\overrightarrow{uw}$ such that

$$||uw'|| = \frac{2\rho^2\cos\alpha - 2\rho}{\rho^2 - 1} \quad \text{and} \quad ||uw''| = 1.$$

By Lemma 2, if $||uw|| \leq ||uw'||$ then $g_1 = ||uw|| + \rho||vw|| \leq \rho||uv||$. So we can assume w is in the line segment $w'w''$. See Figure 4.

Similarly, let z', z'' be two points in the ray $\overrightarrow{vz}$ such that

$$||vz'|| = \frac{2\rho^2\cos\beta - 2\rho}{\rho^2 - 1} \quad \text{and} \quad ||vz''| = 1.$$

Since $g_2 > \rho||uv||$, we can assume z is in the line segment $z'z''$.

By linearity, we have

$$||wz|| \leq \max(||w'z'||, ||w'z''||, ||w''z'||, ||w''z''||).$$

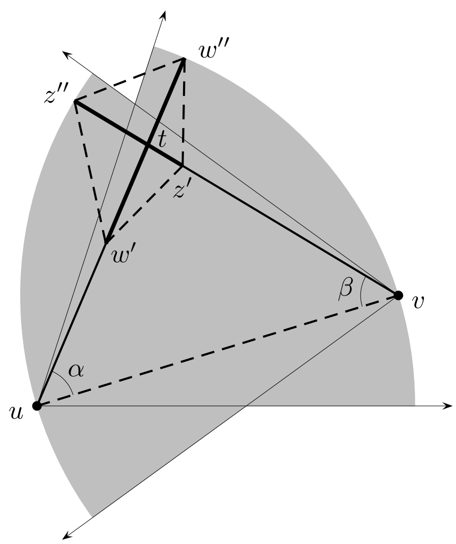


Figure 4: Illustration for the proof of Proposition 1. w and z are in the line segments $w'w''$ and $z'z''$, respectively.

By the law of sines in the triangle $\triangle uvt$, we have (recall that we assume $||uv|| = 1$):

$$||ut|| = \frac{\sin\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad ||vt|| = \frac{\sin\alpha}{\sin(\alpha+\beta)}.$$

See Figure 4 for illustration.

We continue by distinguishing two cases.

Case 1. First consider the case where uw and vs cross each other. See Figure 5 (a). In this case, since wz is a line segment in the triangle $\triangle tw''z''$, we have $||wz|| \leq \max(||tw''||, ||tz''||, ||w''z''||)$. Since $\alpha \geq \overline{\theta}$, $\beta \geq \overline{\theta}$ and $\sin(\alpha+\beta) \leq 1$, we have $||tw''|| = ||uw''| - ||ut|| = 1 - \frac{\sin\beta}{\sin(\alpha+\beta)} \leq 1 - \sin\overline{\theta} < 2\cos\overline{\theta} - 1$ and $||tz''|| = 1 - \frac{\sin\alpha}{\sin(\alpha+\beta)} \leq 1 - \sin\overline{\theta} < 2\cos\overline{\theta} - 1$. Now consider $||w''z''||$. It is easy to see that $||w''z''||$ increases when we fix vz'' and rotate uw'' clockwise around u until $\alpha = \overline{\theta}$. Similarly, $||w''z''||$ increases when we fix uw'' and rotate vz'' counterclockwise around v until $\beta = \overline{\theta}$. Therefore $||w''z''||$ is maximized when $\alpha = \beta = \overline{\theta}$. See Figure 5 (b). In this case it is a simple calculation based on the geometry to verify that $||w''z''|| = 2\cos\overline{\theta} - 1$.

Case 2. Now assume that uw and vs do not cross each other. See Figure 6 (a). In this case either $||uw|| < ||ut||$ or $||vz|| < ||vt||$ or both. If $||vz|| < ||vt||$, then $||wz||$ increases when we fix vz and rotate $\overrightarrow{uw}$ counterclockwise around u until $\alpha = 3\pi/5 - \beta$. Otherwise we have $||uw|| < ||ut||$; then $||wz||$ increases when we fix uw and rotate $\overrightarrow{vz}$ clockwise around v until $\beta = 3\pi/5 - \alpha$. Note that in the above rotating process, it is possible for $\overrightarrow{uw}$ or $\overrightarrow{vz}$ to go beyond the boundaries of the cones C_1^u or C_3^v respectively, but this is not a problem because we only need to bound $||wz||$ in this proposition and going beyond the boundaries of the cones does not affect the discussion that follows. So in either case, we can assume $\alpha + \beta = 3\pi/5$.

Since $||uw'|| = \frac{2\rho^2\cos\alpha - 2\rho}{\rho^2 - 1}$ decreases when α increases, w is still in the line segment $w'w''$ after rotation.

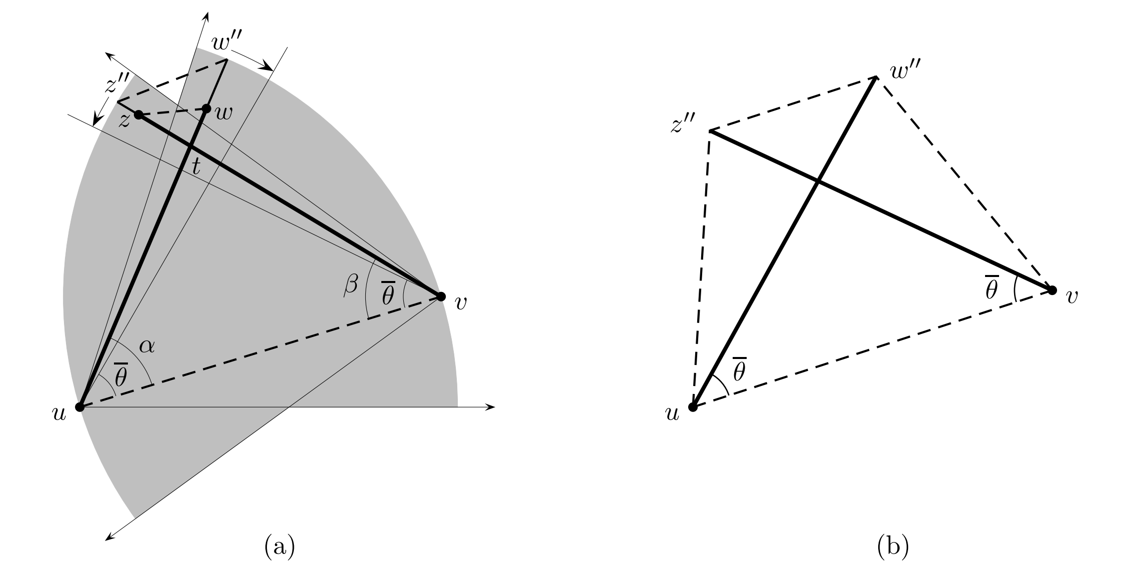


Figure 5: Illustration for the case 1 of the proof of Proposition 1. (a) illustrates the rotation. (b) shows that $||w''z''||$ is maximized when $\alpha = \beta = \overline{\theta}$.

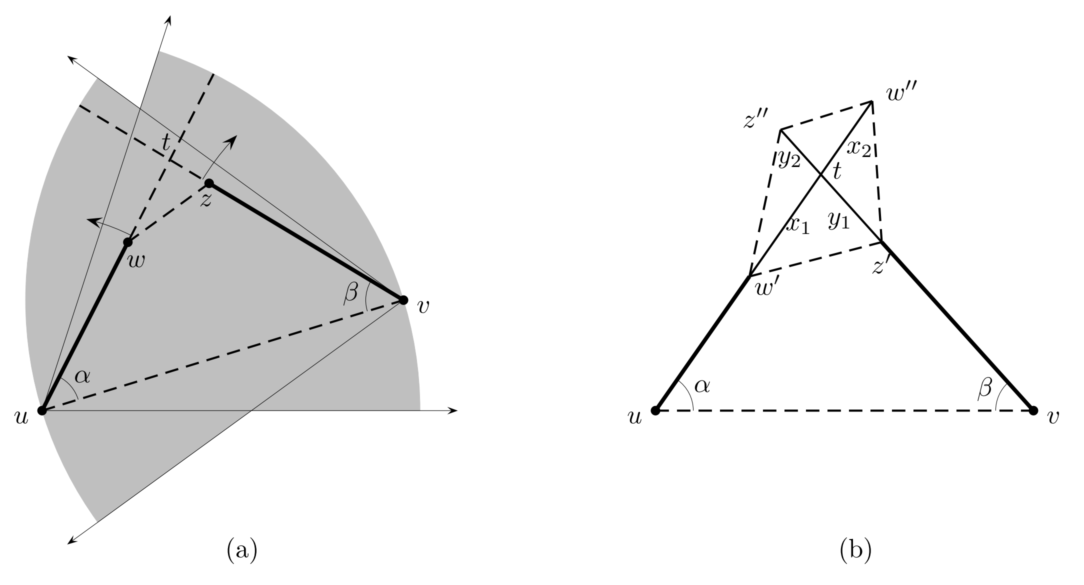


Figure 6: Illustration for case 2 of Proposition 1

Similarly, z is in the line segment $z'z''$ after rotation. This means that

$$||wz|| \leq \max(||w'z'||, ||w'z''||, ||w''z'||, ||w''z''||)$$

still holds after the rotation. See Figure 6 (b). Without loss of generally, assume that $\alpha \geq \beta$. Therefore $3\pi/10 \leq \alpha \leq 3\pi/5 - \overline{\theta}$ and $\overline{\theta} \leq \beta \leq 3\pi/10$. Let $c_1 = \frac{2\rho^2}{\rho^2-1}$ and $c_2 = \frac{1}{\sin(3\pi/5)}$. We have

$$\frac{d||uw'||}{d\alpha} = \frac{d(\frac{2\rho^2\cos\alpha - 2\rho}{\rho^2-1})}{d\alpha} = \frac{-2\rho^2\sin\alpha}{\rho^2-1} = -c_1\sin\alpha, \tag{10}$$

$$\frac{d||vz'||}{d\alpha} = \frac{d(\frac{2\rho^2\cos\beta - 2\rho}{\rho^2-1})}{d\alpha} = \frac{d(\frac{2\rho^2\cos(3\pi/5-\alpha) - 2\rho}{\rho^2-1})}{d\alpha} = \frac{2\rho^2\sin(3\pi/5-\alpha)}{\rho^2-1} = c_1\sin(3\pi/5-\alpha), \tag{11}$$

$$\frac{d||ut||}{d\alpha} = \frac{d(\frac{\sin\beta}{\sin(\alpha+\beta)})}{d\alpha} = \frac{d(\frac{\sin(3\pi/5-\alpha)}{\sin(3\pi/5)})}{d\alpha} = \frac{-\cos(3\pi/5-\alpha)}{\sin(3\pi/5)} = -c_2\cos(3\pi/5-\alpha), \tag{12}$$

$$\frac{d||vt||}{d\alpha} = \frac{d(\frac{\sin\alpha}{\sin(\alpha+\beta)})}{d\alpha} = \frac{d(\frac{\sin\alpha}{\sin(3\pi/5)})}{d\alpha} = \frac{\cos\alpha}{\sin(3\pi/5)} = c_2\cos\alpha. \tag{13}$$

Let

$$x_1 = ||ut|| - ||uw'|| = \frac{\sin\beta}{\sin(\alpha+\beta)} - \frac{2\rho^2\cos\alpha - 2\rho}{\rho^2-1} = \frac{\sin(3\pi/5-\alpha)}{\sin(3\pi/5)} - \frac{2\rho^2\cos\alpha - 2\rho}{\rho^2-1}, \tag{14}$$

$$x_2 = ||uw''|| - ||ut|| = 1 - \frac{\sin\beta}{\sin(\alpha+\beta)} = 1 - \frac{\sin(3\pi/5-\alpha)}{\sin(3\pi/5)}, \tag{15}$$

$$y_1 = ||vt|| - ||vz'|| = \frac{\sin\alpha}{\sin(\alpha+\beta)} - \frac{2\rho^2\cos\beta - 2\rho}{\rho^2-1} = \frac{\sin\alpha}{\sin(3\pi/5)} - \frac{2\rho^2\cos\beta - 2\rho}{\rho^2-1}, \tag{16}$$

$$y_2 = ||vz''|| - ||vt|| = 1 - \frac{\sin\alpha}{\sin(\alpha+\beta)} = 1 - \frac{\sin\alpha}{\sin(3\pi/5)}. \tag{17}$$

Note that the values of x_1 and y_1 can be positive or negative. From (10) - (13), we have

$$\frac{dx_1}{d\alpha} = \frac{d(||ut|| - ||uw'||)}{d\alpha} = -c_2\cos(3\pi/5-\alpha) + c_1\sin\alpha, \tag{18}$$

$$\frac{dx_2}{d\alpha} = \frac{d(||uw''|| - ||ut||)}{d\alpha} = \frac{d(1 - ||ut||)}{d\alpha} = c_2\cos(3\pi/5-\alpha), \tag{19}$$

$$\frac{dy_1}{d\alpha} = \frac{d(||vt|| - ||vz'||)}{d\alpha} = c_2\cos\alpha - c_1\sin(3\pi/5-\alpha), \tag{20}$$

$$\frac{dy_2}{d\alpha} = \frac{d(||vz''|| - ||vt||)}{d\alpha} = \frac{d(1 - ||vt||)}{d\alpha} = -c_2\cos\alpha. \tag{21}$$

Recall that $c_1 = \frac{2\rho^2}{\rho^2-1}$, $c_2 = \frac{1}{\sin(3\pi/5)}$, and $3\pi/10 \leq \alpha \leq 3\pi/5 - \overline{\theta}$, we verify the following:

$$\frac{d^2x_1}{d\alpha^2} = -c_2\sin(3\pi/5-\alpha) + c_1\cos\alpha > -1.1\cdot\sin(3\pi/10) + 2.1\cdot\cos(3\pi/5-\overline{\theta}) > 0, \tag{22}$$

$$\frac{d^2x_2}{d\alpha^2} = c_2\sin(3\pi/5-\alpha) > 0, \tag{23}$$

$$\frac{d^2y_1}{d\alpha^2} = -c_2\sin\alpha + c_1\cos(3\pi/5-\alpha) > -1.1\cdot\sin(3\pi/5-\overline{\theta}) + 2.1\cdot\cos(3\pi/10) > 0, \tag{24}$$

$$\frac{d^2y_2}{d\alpha^2} = c_2\sin\alpha > 0. \tag{25}$$

$$\tag{26}$$

Therefore, by plugging $\alpha = 3\pi/10$ or $\alpha = 3\pi/5 - \overline{\theta}$ as the lower- or upper-bound of α into (18)-(21), we can verify the following ranges:

$$-c_2 \cos(3\pi/10) + c_1 \sin(3\pi/10) \leq \frac{dx_1}{d\alpha} \leq -c_2 \cos\overline{\theta} + c_1 \sin(3\pi/5 - \overline{\theta}), \tag{27}$$

$$c_2 \cos(3\pi/10) \leq \frac{dx_2}{d\alpha} \leq c_2 \cos\overline{\theta}, \tag{28}$$

$$c_2 \cos(3\pi/10) - c_1 \sin(3\pi/10) \leq \frac{dy_1}{d\alpha} \leq c_2 \cos(3\pi/5 - \overline{\theta}) - c_1 \sin\overline{\theta}, \tag{29}$$

$$-c_2 \cos(3\pi/10) \leq \frac{dy_2}{d\alpha} \leq -c_2 \cos(3\pi/5 - \overline{\theta}). \tag{30}$$

Specifically, we can verify that

$$\frac{dx_1}{d\alpha} \geq \max(\frac{dx_2}{d\alpha}, \left|\frac{dy_1}{d\alpha}\right|, \left|\frac{dy_2}{d\alpha}\right|), \tag{31}$$

which implies $\frac{d(x_1 - x_2)}{d\alpha} = \frac{dx_1}{d\alpha} - \frac{dx_2}{d\alpha} > 0$. By simply plugging $\alpha = 3\pi/10$ into (14) and (15), we verify that $(x_1 - x_2) > 0$ when $\alpha = 3\pi/10$ and hence $x_1 > x_2$ for all $\alpha \in [3\pi/10, 3\pi/5 - \overline{\theta}]$. Similarly, we have $x_2 > 0$ when $\alpha = 3\pi/10$, and hence by (28), $x_2 > 0$ for all $\alpha \in [3\pi/10, 3\pi/5 - \overline{\theta}]$. Now we have $x_1 > x_2 > 0$.

By the triangle inequality,

$$||w'z'|| \leq ||tw'|| + ||tz'|| = |x_1| + |y_1| = x_1 + |y_1|, \tag{32}$$

$$||w'z''|| \leq ||tw'|| + ||tz''|| = |x_1| + |y_2| = x_1 + |y_2|, \tag{33}$$

$$||w''z'|| \leq ||tw''|| + ||tz'|| = |x_2| + |y_1| = x_2 + |y_1| \leq x_1 + |y_1|, \tag{34}$$

$$||w''z''|| \leq ||tw''|| + ||tz''|| = |x_2| + |y_2| = x_2 + |y_2| \leq x_1 + |y_2|. \tag{35}$$

By (31),

$$\frac{d(x_1 + |y_1|)}{d\alpha} \geq \frac{d(x_1)}{d\alpha} - \left|\frac{d(y_1)}{d\alpha}\right| \geq 0, \tag{36}$$

$$\frac{d(x_1 + |y_2|)}{d\alpha} \geq \frac{d(x_1)}{d\alpha} - \left|\frac{d(y_2)}{d\alpha}\right| \geq 0. \tag{37}$$

By plugging $\alpha = 3\pi/5 - \overline{\theta}$ into (14), (16), and (17), one can easily verify that $x_1 + |y_1| \leq 2\cos\overline{\theta} - 1$ and $x_1 + |y_2| \leq 2\cos\overline{\theta} - 1$ when $\alpha = 3\pi/5 - \overline{\theta}$ (i.e., when α is maximized). Therefore $\max(x_1 + |y_1|, x_1 + |y_2|) \leq 2\cos\overline{\theta} - 1$ for all $\alpha \in [3\pi/10, 3\pi/5 - \overline{\theta}]$, and hence $||wz|| \leq \max(||w'z'||, ||w'z''||, ||w''z'||, ||w''z''||) \leq 2\cos\overline{\theta} - 1$ as required.

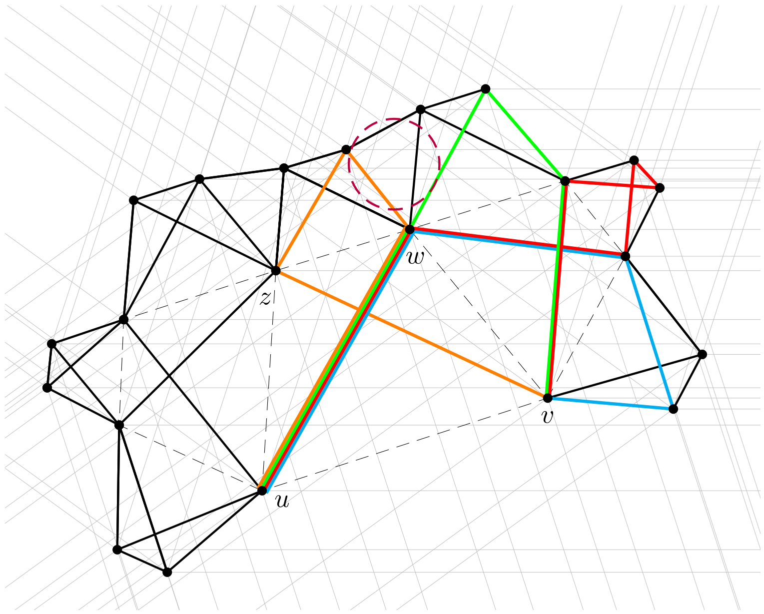
This proves that $||wz|| \leq 2\cos\overline{\theta} - 1$.

□

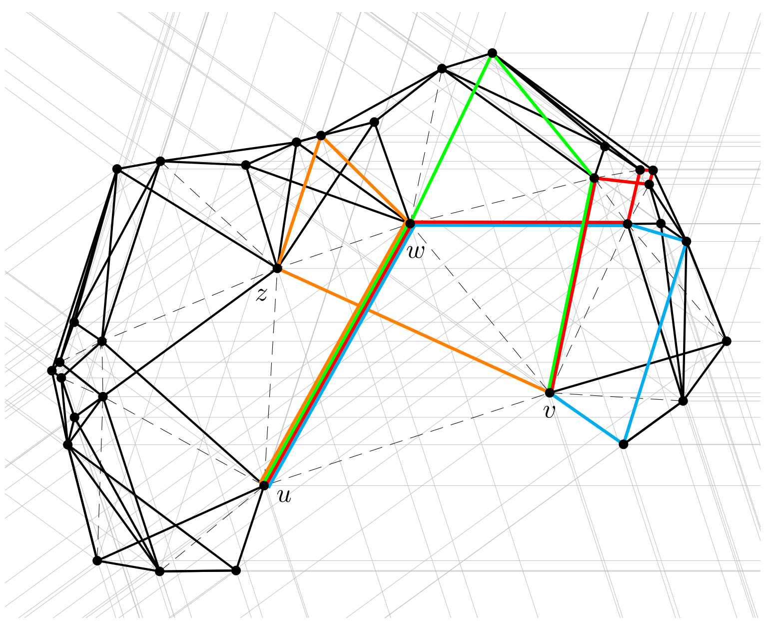
The theorem follows immediately from Proposition 1: If $g_1 \leq \rho||uv||$ or $g_2 \leq \rho||uv||$, then we are done; otherwise by Proposition 1

$$g_3 = ||uw|| + ||vz|| + \rho||zw|| \leq 1 + 1 + \rho(2\cos\overline{\theta} - 1) = \rho,$$

since $\cos\overline{\theta} = 1 - \frac{1}{\rho}$. Therefore we have $\min(g_1, g_2, g_3) \leq \rho$, as required. This completes the proof of the main theorem.



(a)



(b)

Figure 7: (a) shows that the fractal growth is limited by collision of branches (in the circled area), lowering the stretch factor to 2.66. In (b), the stretch factor is increased to 2.87 by adjusting the shape of fractal to equalize the lengths of the shortest paths between u and v. The shortest paths between u and v in the left side of the figure are shown as colors paths.

4 A lower bound on the stretch factor of Y_5

The preceding inductive proof of the upper bound on the stretch factor of Y_5 suggests a possible construction that gives a lower bound of the stretch factor of Y_5. It is based on recursively attaching the "lattice" as shown in Figure 5 (b) to pairs of non-adjacent points (e.g., pairs $\{u, z\}, \{z, w\}, \{w, v\}$ in Figure 5 (b)). This recursion-based construction results in a "fractal" starting from the pair $\{u, v\}$. See Figure 7 (a). However, the growth of fractal is limited because neighboring fractal branches collide into each other, thereby creating shortcuts to the paths, as shown in the circled area of Figure 7 (a). This lowers the stretch factor of the fractal to 2.66.

We adjust the shape of the fractal to increase the stretch factor. In Figure 7 (b), we obtained a stretch factor of more than 2.87 by equalizing the length of all shortest paths between u and v, as shown in Figure 7 (b). The exact locations of the points are given in the appendix.

5 YY_5 is not a Spanner

We give a construction of a YY_5 graph whose stretch factor is unbounded. Figure 8 shows the initial steps of constructing such a YY_5 graph, where the path between a and b can grow horizontally to the right by adding more points following the pattern, exceeding any bound on the stretch factor.

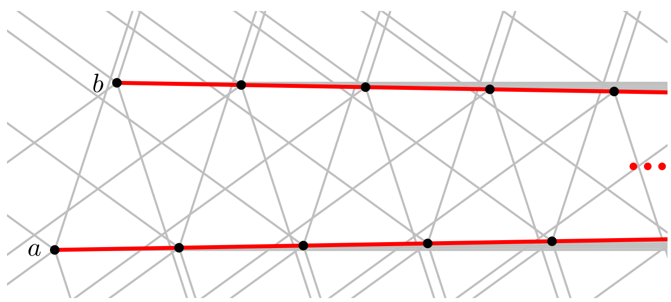


Figure 8: The initial steps of constructing a YY_5 graph with unbounded stretch factor. The pattern continues to the right. The gray lines are the boundaries of the cones.

6 Concluding Remarks

In this paper we prove that the stretch factor of Y_5 is in the interval (2.87, 3.74). While the gap between the upper bound of 3.74 and the lower bound of 2.87 proved in this paper is small, the tight bound of the stretch factor of Y_5 remains unknown. Similarly, it will be interesting to study the tight bounds of other Yao graphs Y_k for $k \geq 4$.

Clearly, the Yao-Yao graphs are less well understood than the Yao graphs. While we know some partial results on the stretch factors of Yao-Yao graphs, many questions about the spanning properties of Yao-Yao graphs remain unresolved. For example, are the Yao-Yao graphs spanners for *all* $k > 6$?

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7 Appendix

The following are the locations of the points in the Y_5 graph shown in Figure 7 (b) whose stretch factor is more than 2.87.

Locations of the points in Figure 7 (b)	
(0, 0)	u
(252, 82)	v
(130, 230)	z
(12, 193)	w
(30, 302)	
(293, 269)	
(321, 229)	
(-143, 130)	
(-143, 80)	
(193, 384)	
(158, 367)	
(-135, 272)	
(-91, 287)	
(-153, -55)	
(371, 75)	
(410, 115)	
(334, 276)	
(341, 264)	
(-179, 97)	
(-180, 112)	
(-91, -75)	
(316, 36)	
(352, 229)	
(303, 297)	
(-167, 63)	
(-167, 147)	
(-26, -75)	
(371, 213)	
(51, 310)	
(-176, 37)	
(344, 274)	
(-189, 105)	
(99, 320)	
(-15, 284)	