MTH 5520 - Assignment 2

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My solution to Question 1

This Question asks to solve Exercise 6.5 (c), (d) and (e) in Filipovic's book.

(c)

Given $f(t,T) = h(T-t) + Z(t)$ is an HJM forward curve evolution by parallel shifts, where $dZ(t) = b(t)dt + \rho(t)dW(t)$.

Recall the HJM frameworks for forward curve with 1 dimensional W.

$$f(t,T) = f(0,T) + \int_0^t \sigma(s,T)\left(\int_u^T \sigma(s,u)du\right) ds + \int_0^t \sigma(s,T)dW(s)$$

As requested, set $b(t) \equiv b$ and $\rho^2(t) \equiv a$.

Starting with given $f(t,T)$ with $Z(t) = Z(0) + \int_0^t b(s)ds + \int_0^t \rho(s)dW(s)$ and $f(0,T) = h(T) + Z(0) = h(T)$

$$f(t,T) = h(T-t) + Z(0) + \int_0^t b(s)ds + \int_0^t \rho(s)dW(s)$$

Consider parital derivatives of $f(t,T)$.

$$\begin{aligned}
\partial_t f(t,T) &= \partial_t \left[h(T-t) + Z(t)\right] \\
&= -\,h'(T-t) + b(t) \\
\partial_t^2 f(t,T) &= \partial_t[-h'(T-t) + b(t)] = h''(T-t) \\
\partial_W f(t,T) &= \rho(t) = \sigma(t,T)
\end{aligned}$$

Putting the results above into HJM framework, we have,

$$f(t,T) = f(0,T) + \int_0^t \rho^2(s)(T-s)ds + \int_0^t \rho(s)dW(s),$$

where $\int_u^T \sigma(s,u)du = \int_u^T \rho(u)du = \rho(t)(T-t)$

With the above new information, we can proceed further,
Consider parital derivatives of $f(t,T)$.

$$\begin{aligned}
\partial_t f(t,T) &= \partial_t \left[h(T-t) + Z(t)\right] \\
&= -h'(T-t) + b(t) \qquad (1) \\
&= \rho(t)^2 (T-t) \qquad (2) \\
\partial_t^2 f(t,T) &= h''(T-t) = \rho^2(t) \qquad (3)
\end{aligned}$$

Set $x = T - t$ and rewrite (1) and (2).

$$\begin{aligned}
(1) = (2) : -h'(x) + b = ax& \\
h'(x) &= -ax + b \\
h(x) &= -a \int x dx + b \\
&= -\frac{a}{2}x^2 + b + C, \qquad (4)
\end{aligned}$$

where C is some arbitrary constant.

(d)

Recall the Ho-Lee short rate model

$$dr(t) = b(t)dt + \sigma dW(t)$$

and its forward curve with ATS solution.

$$\begin{aligned}
f(t,T) &= \partial_T A(t,T) + \partial_T B(t,T) r(t) \\
&= -\frac{\sigma^2}{2}(T-t)^2 + \int_t^T b(s)ds + r(t) \\
&= -\frac{\sigma^2}{2}(T-t)^2 + b(T-t) + r(t) \qquad (5)
\end{aligned}$$

By equating (4) to (5), we can observe that the structure of the equation is the same, where $r(t)$ is an stochastic arbitrary short rate.

(e)

From (3),

$$h''(T-t) \equiv a$$

We can observe that the HJM drift condition forces the second order of the deterministic function $h''(T-t)$ a constant. Then, reversely speaking, the deterministic function f must be a quadratic function to ensure the initial f is a constant.

As such, any non-quadratic initial curve cannot be sustained by mere parallel shifts without creating arbitrage.

Hence, the only arbitrage-free models with arbitrage-free parellel shifts have initial curves $f(0,T)$ of the form

$$-\frac{a}{2}T^2 + bT + c,$$

which is the coincide with my finding in (c) and any non-trivial shapes such as cubbic, piecewise, etc. are excluded.

My solution to Question 2

Given $F(t;T,S)$ is the simple forward rate for $[T,S]$ prevailing at t, that is,

$$F(t;T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$$

Martingale property:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^s}(F(t;T,S)|F_u) &= \mathbb{E}^{\mathbb{Q}^s}\left[\frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}\right) - \frac{1}{S-T}|F_u\right] \\
&= \frac{1}{S-T}\mathbb{E}^{\mathbb{Q}^s}\left[\frac{P(t,T)}{P(t,S)}|F_u\right] - \frac{1}{S-T} \qquad \text{(by linearity)} \\
&= \frac{1}{S-T}\frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{Q}^s}{d\mathbb{Q}} \cdot \frac{P(t,T)}{P(t,S)}\middle| F_u\right]}{\frac{P(u,S)}{P(0,S)B(u)}} - \frac{1}{S-T} \qquad \text{(Bayes' rule)} \\
&= \frac{1}{S-T}\frac{\mathbb{E}^{\mathbb{Q}}\left[\frac{P(t,S)}{P(0,S)B(s)} \cdot \frac{P(t,T)}{P(t,S)}\middle| F_u\right]}{\frac{P(u,S)}{P(0,S)B(u)}} - \frac{1}{S-T} \\
&= \frac{1}{S-T}\left(\frac{1}{P(0,S)} \cdot \frac{P(u,T)}{B(u)}\bigg/\frac{P(u,S)}{P(0,S)B(u)}\right) - \frac{1}{S-T} \\
&= \frac{1}{S-T}\frac{P(u,T)}{P(u,s)} - \frac{1}{S-T} \\
&= \frac{1}{S-T}\left[\frac{P(u,T)}{P(u,S)} - 1\right] \\
&= F(u;T,S)
\end{aligned}$$

Hence, $F(t;T,S)$ is a martingale with respect to $\mathbb{Q}^s$.

My solution to Question 3

(a)

Consider the cost-of-carry model future price below.

$$F_0 = S_0 e^{(r(t)-q(t))T},$$

where:
- S_0 is the current spot price
- F_0 is the futures price
- $r(t)$ is the stochastic short rate, where $r(t) \geq 0$.
- $q(t)$ is the convenience yield (benefits of holding the physical asset)

Theoretically, if $q(t) < r(t)$ such that $F_0 > S_0$, contango exist.

Under normal market condition, for some typical financial assets such as stock underlined in the futures contract, the $q(t)$ can be interpreted as dividend-yield, which is, generally, less than $r(t)$.

(b)

Theoretically, $q(t) > r(t)$, $F_0 < S_0$, backwardation exist.

In reality, $q(t)$ can be greater than $r(t)$ if the market condition is unstable.

The potential reasons for backwardation for oil futures in April 2020 are below:

1. In April 2020, there was a massive supply glut and a collapse in demand, making holding physical oil a major burden due to storage constraints and costs, and thus resulting in a very low convenience yield, or even a negative one.

2. In April 2020, Covid-19 unexpectedly dropped oil demand leading to an extreme oversupply. Under this extreme market condition, backwardation exists.

3. In April 2020, as the poor liquidity in oil market, the demand for storage of oversupplied oil increase causing a sharp increase in storage costs. We can imagine if the consumption of oil is low, the oil storage accumulation increases and the cost to manage oil storage excessively increase to a level higher than usual under Covid-19.

My solution to Question 4

This Question asks to solve Exercise 6.6 in Filipovic's book.

Given the Hull-White extended Vasicek short-rate dynamcis under the EMM $\mathbb{Q}$

$$dr(t) = (b(t) + \beta r(t))dt + \sigma dW^*(t),$$

where W^* is a standard real-valued $\mathbb{Q}$-Brownian motion, β and $\sigma > 0$ are constants, and $b(t)$ is a deterministic continuous function.

By HJM forward rate dynamics, we have,

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW^*(s)$$

(a)

To start with, recall the ATS system of equations:

From the given Hull-White model:

$$|\sigma^2(t,r)| = \sigma^2;\ \bar{b}(t,r) = b(t) + \beta r(t)$$

Substitute the parameters into ATS system of equations, we have:

$$\begin{aligned} \partial_t A(t,T) =& \frac{1}{2}\sigma^2 B^2(t,T) - b(t)B(t,T), && (A(T,T) = 0) \\ \partial_t B(t,T) =& -\beta B(t,T) - 1, && (B(T,T) = 0) \end{aligned}$$

From (1), we have ODE below,

$$\frac{dB(t,T)}{dt} = -\beta B(t,T) - 1$$

By separation of variable,

$$\begin{aligned}
dB(t,T) =& -\beta\left(B(t,T)+\frac{1}{\beta}\right)dt\\
\int_t^T\left(B(s,T)+\frac{1}{\beta}\right)^{-1}dB(s,T) =& \int_t^T -\beta dt\\
\ln\left(\frac{B(T,T)+\frac{1}{\beta}}{B(t,T)+\frac{1}{\beta}}\right) =& -\beta(T-t)\\
\frac{1}{\beta} =& e^{-\beta(T-t)}\left(B(t,T)+\frac{1}{\beta}\right)\\
\frac{1}{\beta}e^{-\beta(T-t)}-\frac{1}{\beta} =& B(t,T)\\
B(t,T) =& \frac{1}{\beta}\left(e^{-\beta(T-t)}-1\right)
\end{aligned}$$

Then, $A(t,T)=\int_t^T\left[\frac{\sigma^2}{2\beta^2}\left(e^{\beta(T-s)}-1\right)^2-\frac{b}{\beta}\left(e^{\beta(T-s)}-1\right)\right]ds$.

Put A, B into P to obtain f

$$\begin{aligned}
P(t,T) =& exp(-A(t,T)-B(t,T)r(t))\\
\ln P(t,T) =& -A(t,T)-B(t,T)r(t)\\
\partial_t \ln P(t,T) =& f(t,T)
\end{aligned}$$

Now, we have done the initial set up from the ATS equation side. Now, let $Y(t,T)=\ln P(t,T)$, by ito's formula,

$$\begin{aligned}
dY(t,T) =& \frac{\partial}{\partial t}(-A(t,T)-B(t,T)r(t))dt+\frac{\partial}{\partial r}(-A(t,T)-B(t,T)r(t))dr(t)+\\
&\frac{1}{2}\frac{\partial}{\partial r^2}(-A(t,T)-B(t,T)r(t))(dr(t))^2\\
=&...\\
=&(-\partial_t A(t,T)-\partial_t(B(t,T)r(t))-B(t,T)(b(t)+\beta r(t)))dt-\underbrace{B(t,T)\sigma(t,r)}_{\text{diffusion coefficient}}dW^*(t)
\end{aligned}$$

Intuitively, forward curve f is the dynamics of log price Y. Hence, the diffusion term of $\ln P(t,T)$ should be inherited into $f(t,T)$ and thus the diffusion coefficient of f (or, Mathematically, recall that $f(t,T)=\partial_T \ln P(t,T)=\partial_T Y(t,T)$).

so, we have,

$$\begin{aligned}
\sigma(t,T) =& -\partial_T B(t,T)\sigma(t,r) \\
=& -\frac{\partial}{\partial T}\left[\frac{1}{\beta}\left(e^{-\beta(T-t)}-1\right)\right]\sigma \\
=& -\frac{\sigma}{\beta}\left[\frac{\partial}{\partial(\beta(T-t))}e^{-\beta(T-t)}\frac{\partial}{\partial T}(-\beta(T-t))\right] \qquad \text{(chain rule)} \\
=& \sigma e^{-\beta(T-t)},
\end{aligned}$$

where $\sigma(t,T)$ is the volatility of forward curve.

$\alpha(t,T)$ is recognized as the drift coefficient of forward curve. Note that this drift is uniquely derived from the volatility structure of the forward curve and has the form

$$-\frac{\sigma^2}{\beta}\left(e^{-2\beta(T-t)}-e^{-\beta(T-t)}\right)$$

In (b), I would prove its explicit solution by using HJM drift condition to verify the abscence of arbitrage.

$f(0,T)$ is the initial forward rate curve. It is an exogenous input to the model, meaning it is not derived but is taken directly from current market observations (e.g., zero-coupon bond prices).

(b)

Recall the dynamic of forward rate (i.e. forward curve) f under HJM framework.

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW^*(s)$$

Under arbitrage-free assumption,

$$f(t,T) = f(0,T) + \int_0^t \underbrace{\sigma(s,t)\left(\int_u^T \sigma(s,u)^T du\right)}_{\text{HJM drift}} ds + \int_0^t \sigma(s,T)dW^*(s)$$

We obtain the arbitrage-free $\alpha(t,T)$.

Hence, the explicit form of drift coefficient under HJM framework is:

$$
\begin{aligned}
\alpha(t,T) &= \sigma(t,T)\int_t^T \sigma(t,s)ds \\
&= \sigma^2 e^{-\beta(t-t)} \int_t^T e^{-\beta(s-t)}ds \\
&= \sigma^2 e^{-\beta(T-t)} \cdot -\frac{1}{\beta}\left[e^{-\beta(T-t)} - 1\right] \\
&= -\frac{\sigma^2}{\beta}\left(e^{-2\beta(T-t)} - e^{-\beta(T-t)}\right)
\end{aligned}
$$

The result aligned with (a).

(c)

Neither $\alpha(s,T)$ nor $\sigma(s,T)$ depends on $b(s)$. $b(s)$ is the calibration tool for the model to capture the real dynamic of the market perfectly. Mathematically, $b(s)$ depends only on initial forward curve (i.e. if the starting point is $t = 0$, then it starts with $t = 0$ and so on). The name "extended" after the "Hull White model" denotes an addition of time-varying drift to fit the market perfectly.

Below is my extra work on derivation of $b(t)$.

Beginning with the pairwise ATS solutions in the context of Hull White extended model:

$$
\begin{aligned}
B(t,T) &= \frac{1}{\beta}\left[e^{\beta(t-T)} - 1\right] \\
A(t,T) &= \int_t^T \frac{\sigma^2}{2} B^2(s,T)ds - \int_t^T b(s)B(s,T)ds \qquad (1)
\end{aligned}
$$

To obtain alternative expression of $A(t,T)$, recall $P(t,T)$:

$$
\begin{aligned}
P(t,T) &= exp(-A(t,T) - B(t,T)r(t)) \\
\ln P(t,T) + B(t,T)r(t) &= -A(t,T) \\
A(t,T) &= -\ln P(t,T) - B(t,T)r(t) \qquad (2)
\end{aligned}
$$

By equating these 2 expressions ((1) and (2)) of $A(t,T)$, we have

$$A(t,T) = \int_t^T \frac{\sigma^2}{2} B^2(s,T)ds - \int_t^T b(s)B(s,T)ds$$

$$\int_t^T b(s)B(s,T)ds = \int_t^T \frac{\sigma^2}{2} B^2(s,T)ds + \ln P(t,T) + B(t,T)r(t) \qquad (3)$$

From (3), take the partial derivative w.r.t. T

$$\frac{\partial}{\partial T}\int_t^T b(s)B(s,T)ds = b(s)B(s,T)\big|_{s=T} + \int_t^T b(s)\frac{\partial}{\partial T}B(s,T)ds \qquad (B(T,T)=0)$$

$$= \int_t^T b(s)e^{\beta(s-T)}ds$$

So, we have:

$$\int_t^T b(s)e^{\beta(s-T)}ds = \frac{\partial}{\partial T}\left[\int_t^T \frac{\sigma^2}{2}\frac{1}{\beta^2}\left(e^{\beta(s-T)}-1\right)^2 ds + \ln P(t,T) + \frac{1}{\beta}\left(e^{\beta(t-T)}-1\right)r(t)\right]$$

$$= -\frac{\sigma^2}{\beta}\int_t^T \left(e^{\beta(s-T)}-1\right)e^{\beta(s-T)}ds + \frac{\partial}{\partial T}\ln P(t,T) + e^{\beta(t-T)}r(t) \qquad (4)$$

Multiply (3) by β and add into (4),

$$\int_t^T b(s)ds = -\frac{\sigma^2}{2\beta}\int_t^T \left(1-e^{2\beta(s-T)}\right)ds + r + \frac{\partial}{\partial T}\ln P(t,T) + \beta \ln P(t,T)$$

$$b(t) = -\frac{\sigma^2}{2\beta}\left(1-e^{2\beta(t-T)}\right) + \frac{\partial^2}{\partial T^2}\ln P(t,T) + \beta\frac{\partial}{\partial T}\ln P(t,T) \qquad (5)$$

from (5), recall that

$$f(t,T) = \partial_T \ln P(t,T)$$

So, we have,

$$b(t) = -\frac{\sigma^2}{2\beta}\left(1-e^{-2\beta(T-t)}\right) + \frac{\partial}{\partial T}f(t,T) + \beta f(t,T) \qquad (6)$$

(d)

For the Vasicek model where $b(s) \equiv b$ (a constant), the entire right hand side of equation (6) must also be a constant. Notice that the extended model provides flexibility on calibration with time-varying b but Vasicek model does not. Vasicek model fixes the drift with constant b providing an "one-size-fit-all" framework for derivative pricing. While misfitting the option price may create arbitrage opportunity, an extension of time-varying $b(t)$ Vasicek model would make the model capture the initial forward curve state perfectly and thus eliminating arbitrage.

(e)

Consider the HJM forward curve

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW(s) \tag{1}$$

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \tag{2}$$

My goal is to show:

$$r(t) = \; r(0) + \int_0^t \zeta(u)du + \int_0^t \sigma(u,u)dW(u) \quad \text{with}$$

$$\zeta(u) = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)ds + \int_0^u \partial_u \sigma(s,u)dW(s)$$

Let denotes $g(t,T) = f(t,T)$ and $T(t) = t$.

By 2 parameters ito formula (a.k.a."Musiela" formula),

$$d(g(t,T)) = \partial_t g dt + \partial_T g dT + \frac{1}{2}\partial^2_{TT} g(dT)^2 + \sigma(t,t)dW(t)$$

Notice that $dT(t) = dt$ and $(d(T))^2 = 0$, this yields,

$$dr(t) = df(t,t) = [\partial_t f(t,T) + \partial_T f(t,T)]_{T=t} dt + \sigma(t,t)dW(t)$$

At this point, $\zeta(t) = [\partial_t f(t,T) + \partial_T f(t,T)]_{T=t}$

From (1), consider the partial derivatives below.

$$\partial_t f(t,T) = \alpha(t,T)$$
$$\partial_t f(t,t) = \alpha(t,t)$$

At this point, $\zeta(t) = \alpha(t,t) + \partial_T f(t,T)|_{T=t} \qquad (3).$

By ito formula again,

$$\begin{aligned} d(\partial_T f(t,T)) &= \partial_T \alpha(t,T)dt + \partial_T \sigma(s,T)dW(t) \\ \partial_T f(u,T) &= \partial_T f(0,T) + \int_0^u \partial_T \alpha(s,T)ds + \int_0^u \partial_T \sigma(s,T)dW(s) \qquad (4) \end{aligned}$$

Put (4) into (3),

$$\begin{aligned} \zeta(u) &= \alpha(u,u) + [\partial_T f(t,T)]_{T=u} \\ &= \alpha(u,u) + [\partial_T f(0,T)]_{T=u} + \int_0^u \partial_T \alpha(s,u)\Big|_{T=u} ds + \int_0^u \partial_T \sigma(s,T)\Big|_{T=u} dW(s) \\ &= \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)ds + \int_0^u \partial_u \sigma(s,u)dW(s) \end{aligned}$$

So, we finally have,

$$r(t) = \; r(0) + \int_0^t \zeta(u)du + \int_0^t \sigma(u,u)dW(u) \quad \text{with}$$
$$\zeta(u) = \alpha(u,u) + \partial_u f(0,u) + \int_0^u \partial_u \alpha(s,u)ds + \int_0^u \partial_u \sigma(s,u)dW(s)$$