MTH3251 — Assignment 2

Semester 2, 2024

Due date: Friday October 18, 11:55pm

Please submit your assignment electronically on Moodle.

For all questions, you are expected to provide full solutions, with detailed explanation and justification of every step.

1. Consider the following discrete time two-period market model. The savings account is given by $\beta_t = \beta^t$ for $t = 0, 1, 2$. The stock price is given by $S_0 = 1, S_1 = \xi_1$ and $S_2 = \xi_1 \xi_2$ where ξ_1 and ξ_2 are independent random variables, each taking two possible values u and d with positive probabilities. Moreover, assume that $0 < d < \beta < u$.

   (a) Define what is meant by an equivalent martingale measure (EMM). Find the EMM of this model.

   (b) Consider a put option with exercise price $K = \beta$ that expires at time 1. Find the replicating portfolio and the time 0 price of this put option.

   (c) Consider a contract H which pays $H_2 = S_2/S_1$ at time 2. By taking suitable expectations under the EMM, find the time 1 price and the time 0 price of this contract.

   (d) Find the replicating portfolio for the contract from part (c).

   (e) Instead of ξ_1 and ξ_2 being independent, assume that $\xi_2 = u + d - \xi_1$. Show that this model does not have EMMs and find an arbitrage strategy.

2. Let (B_t) denote a Brownian motion under the real-world measure with $B_0 = 0$. Consider the Black-Scholes model for the stock price,

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \quad S_0 = 1,$$

   and the savings account is given by $\beta_t = 1$ for all t.

   (a) Write down the condition for a portfolio in this model to be self-financing. Consider a portfolio given by $a_t = \cos(t)$ (units of the stock) and $b_t = \int_0^t S_u \sin(u) \, du$ (units of the savings account). Determine with proof whether this portfolio is self-financing.

   (b) State the Girsanov theorem. Compute S_t in terms of $\hat{B}_t$, a Brownian motion under the equivalent martingale measure (EMM).

(c) Consider an option which pays $D_T = S_T^{-1}$ at time T. By taking conditional expectations under the EMM, compute D_t, the time t price of this option. (Hint: Express S_T in terms of S_t.)

(d) By writing the option price D_t as a function of (S_t, t) and differentiating this function, verify that it satisfies the Black-Scholes PDE.

(e) Find the replicating portfolio for this option.

3. Consider the following discrete time one-period market model. There is no interest rate. The stock price is given by $S_0 = 2$ and $S_1 = \xi$, where ξ is a random variable taking three possible values 1, 2 and 3, each with positive probability.

   (a) Does this model have arbitrage opportunities?

   (b) Find two different equivalent martingale measures (EMMs) for this model. Using the fundamental theorems of asset pricing, what can you conclude about the model?

   (c) Find a contract that cannot be replicated in this market model.

   (d) Suppose we introduce another tradable asset U into this market model, whose price process is given by $U_0 = 4.5$ and $U_1 = S_1^2$. Prove that this new model is arbitrage-free and complete.

   (e) In this model with S, U and the savings account, price a call option on U with maturity at $T = 1$ and strike price $K = 3$.