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MTH 3251

ASM 2

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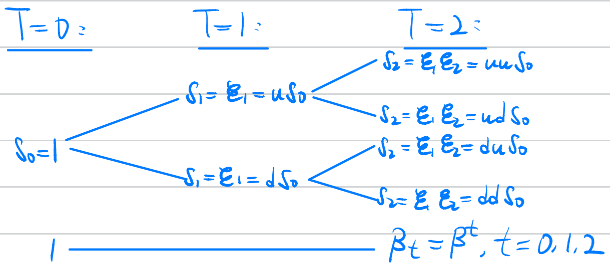
Q1

Given:



, where ① ε_1 & ε_2 are 2 independent random variables.
② u & d are the increasing & decreasing factors.

(a). EMM (a.k.a. risk-neutral probability measure) is a measure under which the discounted stock price is a martingale.

Denote $\mathbb{Q}$ as the risk-neutral probability measure of the given discounted stock price process $\{S_t, t = 0, 1, 2\}$.

To find EMM $\mathbb{Q}$, denotes the probability distribution below:

$$\mathbb{Q}(\varepsilon_i = u) = p,\ \mathbb{Q}(\varepsilon_i = d) = (1-p),\ i = 1, 2$$
$$\mathbb{Q}(\varepsilon_1 = u, \varepsilon_2 = u) = \mathbb{Q}(\varepsilon_1 = u)\mathbb{Q}(\varepsilon_2 = u) = p^2$$
$$\mathbb{Q}(\varepsilon_1 = u, \varepsilon_2 = d) = \mathbb{Q}(\varepsilon_1 = u)\mathbb{Q}(\varepsilon_2 = d) = p(1-p)$$
$$\mathbb{Q}(\varepsilon_1 = d, \varepsilon_2 = u) = \mathbb{Q}(\varepsilon_1 = d)\mathbb{Q}(\varepsilon_2 = u) = (1-p)p$$
$$\mathbb{Q}(\varepsilon_1 = d, \varepsilon_2 = d) = \mathbb{Q}(\varepsilon_1 = d)\mathbb{Q}(\varepsilon_2 = d) = (1-p)^2$$

(by independence)

By definition of EMM, we have the following martingale property of S under EMM $\mathbb{Q}$:

$$E^{\mathbb{Q}}(S_1/\beta \mid S_0) = S_0$$
$$\frac{1}{\beta}E^{\mathbb{Q}}(S_1 \mid S_0) = S_0$$
$$E^{\mathbb{Q}}(S_1) = S_0\beta \quad \text{(by independence)}$$
$$E^{\mathbb{Q}}(\varepsilon_1) = \beta$$
$$u\mathbb{Q}(\varepsilon_1 = u) + d\mathbb{Q}(\varepsilon_1 = d) = \beta$$
$$up + d(1-p) = \beta$$

Solve for p, we have: $p = \dfrac{\beta - d}{u - d}$

Likewise,

$$E^{\mathbb{Q}}(S_2/\beta \mid S_1) = S_1$$
$$\frac{1}{\beta}E^{\mathbb{Q}}(S_2 \mid S_1) = S_1$$
$$E^{\mathbb{Q}}(S_2) = S_1\beta \quad \text{(by independence)}$$
$$E^{\mathbb{Q}}(\varepsilon_2)E^{\mathbb{Q}}(\varepsilon_1) = S_1\beta \quad \text{(by independence)}$$
$$(up + d(1-p))^2 = \beta^2$$
$$up + d(1-p) = \beta$$
$$p = \frac{\beta - d}{u - d}$$

(for $\Delta t = 1$)

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(b). Denotes P_t be the price process of put option where:

$$P_t = (k - S_t)^+ = \max(k - S_t, 0) = \begin{cases} (k - S_u)^+ = 0 \\ (k - S_d)^+ = k - d \end{cases}$$

① Replicating portfolio.

$$P_t = aS_t + b\beta^t = \begin{cases} auS_{t-1} + b\beta^t, \text{ if } S_t = uS_{t-1} \\ adS_{t-1} + b\beta^t, \text{ if } S_t = dS_{t-1} \end{cases}$$, where a is the number of Shares and b is the number of saving account.

we have:

$$\begin{cases} auS_{t-1} + b\beta^t = P_u \text{ ———— ①} \\ adS_{t-1} + b\beta^t = P_d \text{ ———— ②} \end{cases}$$

To solve for a and b,

① − ②:

$$auS_{t-1} + b\beta^t - adS_{t-1} - b\beta^t = P_u - P_d = 0 - (\beta - d) = d - \beta$$

$$a = \frac{d-\beta}{u-d}$$

Put ③ into ①,

$$\frac{d-\beta}{u-d} d(1) + b\beta = \beta - d$$

$$b = \frac{1}{\beta}\left[\frac{(\beta-d)(u-d)}{u-d} - \frac{d^2 - d\beta}{u-d}\right]$$

$$b = \frac{1}{\beta}\left[\frac{\beta u - \beta d - du + d^2 - d^2 + d\beta}{u-d}\right]$$

$$= \frac{1}{\beta}\left[\frac{\beta u - du}{u-d}\right]$$

$$= \frac{u(\beta-d)}{\beta(u-d)}$$

② Find P_0:

from ①, we have,

$$P_0 = aS_0 + b\beta^0 = a + b$$

$$= \frac{d-\beta}{u-d} + \frac{u(\beta-d)}{\beta(u-d)}$$

$$= \frac{d\beta - \beta^2 + u\beta - ud}{\beta(u-d)}$$

$$= \frac{\beta(d-\beta) + u(\beta-d)}{\beta(u-d)}$$

$$= \frac{(d-\beta)(\beta-u)}{\beta(u-d)}$$

✓ 3

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(c). Consider a self-financing portfolio : $H_t = a_t S_t + b_t \beta^t$, $t = 0, 1, 2$

Given $H_2 = S_2/S_1$, we have :

$$H_2 = a_2 S_2 + b_2\beta^2 = S_2/S_1 = \begin{cases} a_2 S_1 u + b_2 \beta^2 = u & \text{①} \\ a_2 S_1 d + b_2 \beta^2 = d & \text{②} \end{cases}$$

(a_2, b_2 are F_1-measurable s.t. $S_2 = S_1 u$ or $S_2 = S_1 d$)

① − ② :

$a_2 S_1 u - a_2 S_1 d = u - d$

$$a_2 = \frac{1}{S_1} \quad \text{③}$$

Put ③ into ②,

$d + b_2\beta^2 = d$

$b_2 = 0$

$\therefore H_1 = \dfrac{H_2}{\beta}$ ($0<d<\beta<u$ assumed s.t. no arbitrage exist and thus $H_2 = H_1\beta$)

$= a_2 \dfrac{S_2}{\beta} + b_2 \dfrac{\beta^2}{\beta}$

$= a_2 S_1 + b_2 \beta$

$= \dfrac{S_1}{S_1} + 0 = 1$

Iterate the above to solve for H_0, we have :

$$H_1 = a_1 S_1 + b_1 \beta = 1 = \begin{cases} a_1 S_0 u + b_1 \beta = 1 & \text{Ⓐ} \\ a_1 S_0 d + b_1 \beta = 1 & \text{Ⓑ} \end{cases}$$

(a_1, b_1 are F_0-measurable s.t. $S_1 = S_0 u$ & $S_1 = S_0 d$)

Ⓐ − Ⓑ :

$a_1 u - a_1 d = 0$

$a_1 = 0$

Put Ⓒ into Ⓑ,

$b_1 \beta = 1$

$b_1 = 1/\beta$

$\therefore H_0 = \dfrac{H_1}{\beta}$ ($0<d<\beta<u$, assumed s.t. no arbitrage exist and thus $H_1 = H_0\beta$)

$= a_1 \dfrac{S_1}{\beta} + b_1 \dfrac{\beta}{\beta}$

$= \dfrac{1}{\beta}$

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(d). from (c), recall that :

$H_t = a_t S_t + b_t \beta^t$, $t = 0, 1, 2$ where the replicating portfolio is a pair of predictable process.

$a_1 = 0$, $b_1 = \dfrac{1}{\beta}$ and $a_2 = \dfrac{1}{S_1}$, $b_2 = 0$

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(e). Note that $H_2 = S_2/S_1 = (\varepsilon_2\varepsilon_1)/\varepsilon_1 = \varepsilon_2$

from (d), we have:

$$\varepsilon_2 = \frac{1}{S_1} S_2 + 0 = u$$

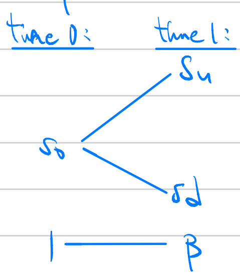
if $\varepsilon_2 = u + d - \varepsilon_1$,

$$u = u + d - \varepsilon_1$$
$$d = \varepsilon_1$$
$$= S_0\beta$$
$$= \beta$$

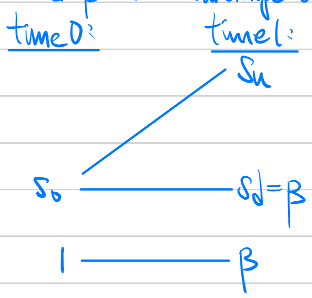
As such, arbitrage exists where $0<d=\beta<u$ (violated $0<d<\beta<u$)

Under $0<d=\beta<u$, we can short saving account to finance the long position for stock at time 0. Sell stock at time t to recover the short position of saving account and gain on the stock price increment.

$d<\beta<u$: (no arbitrage)



$0<d=\beta<u$: (arbitrage exist)



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Q2

Given a B.S. model for stock price,

$dS_t = \mu S_t dt + \sigma S_t dB_t$, $S_0 = 1$ where B_t is a Brownian Motion, $B_0 = 0$ and saving account is given by $\beta_t = 1, \forall t$.

(a). Denote V_t as the self-financing portfolio.

$V_t = a_t S_t + b_t \beta_t$, where (a_t, b_t) are a pair of predictable process that represents number of stock and saving account respectively.

Note that any changes to the value of a self-financing portfolio purely comes from the changes in the values of assets. So, we can rewrite the equation as follows:

$$dV_t = a_t dS_t + b_t d\beta_t$$

such that, in integral form, we can observe that the value of the self-financing portfolio at time t is equal to the initial value plus any gain to the time t.

$$V_t = V_0 + \int_0^t a_t \, dS_t + \int_0^t b_t \, \underline{d\beta_t}^{\,= d(1) = 0}$$

$$= V_0 + \int_0^t a_t (\mu S_t dt + \sigma S_t dB_t) + 0$$

$$= \underset{\uparrow \text{ initial value}}{V_0} + \underbrace{\int_0^t \mu a_t S_t dt + \int_0^t \sigma S_t a_t dB_t}_{\text{gain on stocks from time 0 to time t.}} \quad \text{———————} \quad (?)$$

When $a_t = \cos(t)$ and $b_t = \int_0^t S_u \sin(u) du$.

$$dV_t = \cos(t) dS_t + \left(\int_0^t S_u \sin(u) du\right) \underline{d\beta_t}^{\,= d(1) = 0}$$

$$= \cos(t) dS_t$$

$$= \cos(t)(\mu S_t dt + \sigma S_t dB_t) \qquad \text{(given)}$$

$$= \cos(t)\mu S_t dt + \cos(t)\sigma S_t dB_t$$

$$V_t = V_0 + \int_0^t \mu S_t \cos(t) dt + \int_0^t \sigma S_t \cos(t) dB_t \quad \text{———————} \quad (☆)$$

$$= V_0 + \int_0^t \mu S_t a_t dt + \int_0^t \sigma S_t a_t dB_t \quad \text{———————} \quad (☆) = (?)$$

By equating (☆) to (?) where (?) is the definition of replicating portfolio, we note that (☆) = (?) s.t. V_t is a self-financing portfolio when $a_t = \cos(t)$ & $b_t = \int_0^t S_u \sin(u) du$.

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(b). By Girsanov theorem, there exists an equivalent measure $\mathbb{Q}$ s.t. the process defined by $\hat{B}_t = B_t + ct$ is a $\mathbb{Q}$-B.M.
So, we have:

$\hat{B}_t = B_t + ct = B_t + \frac{\mu}{\sigma}t$, where $r=0$ (given $\beta^t = 1, \forall t$). So, $d\hat{B}_t = dB_t + \frac{\mu}{\sigma}dt$

To solve c, denote $\hat{S}_t$ as a discounted stock price process under $\mathbb{Q}$ with BM. $\hat{B}$ where $\hat{S}_t = S_t e^{-rt}$.

$$
\begin{aligned}
d\hat{S}_t &= e^{-rt} dS_t + S_t d(e^{-rt}) \\
&= e^{-rt}(\mu S_t dt + \sigma S_t dB_t) - re^{-rt} S_t dt \\
&= \mu \hat{S}_t dt + \sigma \hat{S}_t dB_t - \cancel{(0)\hat{S}_t dt} \qquad (\text{Given } \beta^t = 1, \forall t \text{ s.t. } \beta^t = e^{rt} = 1 \text{ where } e^r = 1 \text{ and } r = 0) \\
&= \mu \hat{S}_t dt + \sigma \hat{S}_t dB_t \\
&= \sigma \hat{S}_t \underbrace{\left[\frac{\mu}{\sigma} dt + dB_t\right]}_{d\hat{B}_t} \\
&= \sigma \hat{S}_t d\hat{B}_t
\end{aligned}
$$

Recall the Black-Scholes SDE,

$S_t = \mu S_t dt + \sigma S_t dB_t$, $dB_t \Leftrightarrow d\hat{B}_t$, $= \left(\mu S_t - \cancel{\sigma S_t \frac{\mu}{\sigma}}\right) dt + \sigma S_t d\hat{B}_t = \sigma S_t d\hat{B}_t$ (As such, $\mu = 0$)

To solve $d\hat{S}_t$ under SDE from B.S. model, denote $S_t = f(t, S_t) = \ln S_t$
By itô formula, we have:

$$
\begin{aligned}
d\hat{S}_t = d\ln S_t &= f_t'(t,\hat{S}_t)dt + f_{s_t}'(t,\hat{S}_t)d\hat{S}_t + \frac{1}{2} f_{s_t}''(t,S_t)(d\hat{S}_t)^2 \\
&= 0 + \frac{1}{\hat{S}_t} d\hat{S}_t + \frac{1}{2} \cdot \frac{-1}{\hat{S}_t^2}(d\hat{S}_t)^2 \\
&= \frac{1}{\hat{S}_t}(\mu \hat{S}_t dt + \sigma \hat{S}_t d\hat{B}_t) - \frac{1}{2\hat{S}_t^2}(\mu \hat{S}_t dt + \sigma \hat{S}_t d\hat{B}_t)^2 \\
&= \mu dt + \sigma d\hat{B}_t - \frac{1}{2\hat{S}_t^2}\sigma^2 \hat{S}_t^2 dt \\
&= (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma d\hat{B}_t
\end{aligned}
$$

$$
\begin{aligned}
\ln S_t &= \underbrace{\ln \overset{=1}{S_0}}_{=0} + \int_0^t (\mu - \tfrac{1}{2}\sigma^2) dt + \int_0^t \sigma d\hat{B}_t \\
&= 0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma \hat{B}_t \\
S_t &= \underbrace{S_0}_{=1} e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \hat{B}_t} \\
&= S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \hat{B}_t} \qquad \circledast \qquad (\mu = 0)
\end{aligned}
$$

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(c). By EMM $\mathbb{Q}$,

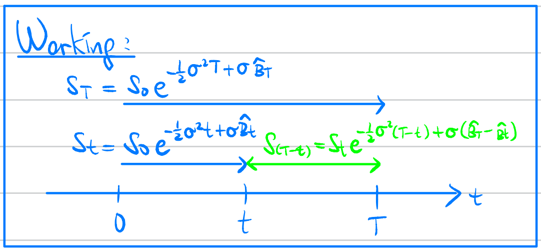
$$D_t = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{\beta^T} D_T\right) \cdot \beta^t$$

from ⊛, we have the following solution for S_t.

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t + \sigma \hat{B}_t}$$

So, express S_T in terms of S_t, we have:

$$S_T = S_t e^{-\frac{1}{2}\sigma^2 (T-t) + \sigma(\hat{B}_T - \hat{B}_t)}$$



Substitute S_T^{-1} to the LHS, we have:

$$S_T^{-1} = S_t^{-1} e^{\frac{1}{2}\sigma^2 (T-t) - \sigma(\hat{B}_T - \hat{B}_t)}$$

Consider the conditional expectation under EMM $\mathbb{Q}$,

$$D_t = \mathbb{E}^{\mathbb{Q}}\left(\frac{\beta^t}{\beta^T} D_T \middle| \mathcal{F}_t\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{1} S_T^{-1} \middle| \mathcal{F}_t\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(S_t^{-1} e^{\sigma^2 (T-t)\frac{1}{2} - \sigma(\hat{B}_T - \hat{B}_t)} \middle| \mathcal{F}_t\right)$$

$$= \mathbb{E}^{\mathbb{Q}}\left(S_t^{-1} e^{\sigma^2 (T-t)\frac{1}{2}} \middle| \mathcal{F}_t\right) \mathbb{E}^{\mathbb{Q}}\left(e^{-\sigma(\hat{B}_T - \hat{B}_t)} \middle| \mathcal{F}_t\right)$$ (by independence property of $\hat{B}_T - \hat{B}_t$)

$$= \underline{S_t^{-1}}\, \underline{e^{\sigma^2(T-t)\frac{1}{2}}}\, \mathbb{E}^{\mathbb{Q}}(1 | \mathcal{F}_t) \cdot \mathbb{E}^{\mathbb{Q}}\left(\underline{e^{-\sigma(\hat{B}_T - \hat{B}_t)}}\right)$$ (S_t^{-1} is $\mathcal{F}_t$-measurable) (linearity) ($\hat{B}_T - \hat{B}_t$ is independent of $\mathcal{F}_t$)

$$= S_t^{-1} e^{\sigma^2(T-t)\frac{1}{2}} (1) \cdot e^{\frac{1}{2}\sigma^2(T-t)}$$ (by m.g.f. of $\mathbb{E}(e^{-\sigma(\hat{B}_T - \hat{B}_t)})$, we obtain $e^{\frac{1}{2}\sigma^2(T-t)}$)

$$= S_t^{-1} e^{\sigma^2(T-t)}$$ ✓ 3

(d). Denote $D_t = f(S_t, t)$. By Itô formula,

$$dD_t = df(S_t, t) = \frac{\partial f(S_t,t)}{\partial t} dt + \frac{\partial f(S_t,t)}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2} (dS_t)^2$$

$$= \frac{\partial f(S_t,t)}{\partial t} dt + \frac{\partial f(S_t,t)}{\partial S_t} dS_t + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2} \sigma^2 S_t^2\, dt$$

$$= \left(\frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\right) dt + \frac{\partial f(S_t,t)}{\partial S_t} dS_t \quad \text{———— (E)}$$

Recall the definition of self-financing from (a), we have:

$$D_t = a_t S_t + b_t \beta_t,\ dD_t = a_t\, dS_t + b_t\, d\beta_t = a_t\, dS_t + b_t r \beta_t\, dt \quad \text{———— (F)}$$

By equating (E), (F), we have: $a_t = \frac{\partial f(S_t,t)}{\partial S_t}$, $b_t \beta_t = r^{-1}\left(\frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\sigma^2 S_t^2\right)$

Putting a_t, b_t back to the self-financing portfolio,

$$D_t = \frac{\partial f(S_t,t)}{\partial S_t} S_t + r^{-1}\left(\frac{\partial f(S_t,t)}{\partial t} + \frac{1}{2}\frac{\partial^2 f(S_t,t)}{\partial S_t^2}\sigma^2 S_t^2\right) \quad \text{———— (G)}$$

Replace S_t by x, we have $f(S_t, t) = f(x,t)$. Rearrange (G) and set (G) $= 0$ to verify whether D_t satisfies the BS-PDE.

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$$\frac{\partial f(x,t)}{\partial t}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 f(x,t)}{\partial x^2}+rx\frac{\partial f(x,t)}{\partial x}-rf(x,t)=0$$

$$\frac{\partial f(x,t)}{\partial t}+\frac{1}{2}\sigma^2 x^2\frac{\partial^2 f(x,t)}{\partial x^2}=0 \qquad (\text{Given } r=0)$$

from (c), note that $D_t = x^{-1}e^{\sigma^2(T-t)}$

$$\frac{\partial f(x,t)}{\partial t}=-\sigma^2 x^{-1}e^{\sigma^2(T-t)},\quad \frac{\partial f(x,t)}{\partial x}=-x^{-2}e^{\sigma^2(T-t)},\quad \frac{\partial^2 f(x,t)}{\partial x^2}=2x^{-3}e^{\sigma^2(T-t)}$$

So, we have,

$$-\sigma^2 x^{-1}e^{\sigma^2(T-t)}+(1/\cancel{2})(\sigma^2x^2)(\cancel{2}x^{-3}e^{\sigma^2(T-t)})=0$$ ✓ 3

$\therefore$ LHS = RHS = 0

$\therefore$ D_t satisfied the BS-PDE.

(e). from (d), $a_t = \frac{\partial f(x,t)}{\partial x} = -x^{-2}e^{\sigma^2(T-t)}$

Substitute back to S_t from x,

$a_t = -S_t^{-2}e^{\sigma^2(T-t)}$ ✓

For b_t, consider the self-financing portfolio, we have:

$D_t = a_tS_t + b_t\beta_t$,

$b_t = \beta_t^{-1}(D_t - a_tS_t)$

$b_t = (1)(D_t + S_t^{-1}e^{\sigma^2(T-t)})$

$= S_t^{-1}e^{\sigma^2(T-t)} + S_t^{-1}e^{\sigma^2(T-t)}$ (from (c), $D_t = S_t^{-1}e^{\sigma^2(T-t)}$)

$= 2S_t^{-1}e^{\sigma^2(T-t)}$ ✓ //

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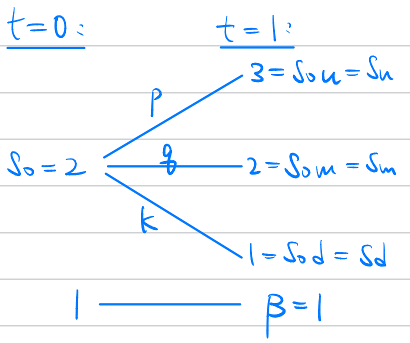
Q3

Given: ① $r=0$ s.t. $\beta^t = 1, \forall t$. $S_0 = 2$, $S_1 = \mathcal{E}$. where $\mathcal{E} = \{1, 2, 3\}$.
Let u, m and d be the growth rate of movement for S in time 1.
Let p, q and k be the probability of S in time 1.



, where $\begin{cases} u = 3/2 = 1.5 \\ m = 2/2 = 1 \\ d = 1/2 = 0.5 \end{cases}$

✓ 2

(a). No. The value of stock always offset the value of saving account in this portfolio.

(b). To find EMMs, firstly, consider the self-financing portfolios below:
Denotes C as a call option. To replicate its value, we have:

$a_u(2) + b(1) = C_u$, if $S_1 = S_0 u$
$a_m(2) + b(1) = C_m$, if $S_1 = S_0 m$
$a_d(2) + b(1) = C_d$, if $S_1 = S_0 d$

Rewrite the above system of equations into matrix form, we have:

$$M \begin{bmatrix} a(2) \\ b(1) \end{bmatrix} = \begin{bmatrix} C_u \\ C_m \\ C_d \end{bmatrix}, \text{ where } M = \begin{bmatrix} 1.5 & 1 \\ 1 & 1 \\ 0.5 & 1 \end{bmatrix}$$

To find EMMs, we have equations below:

$$(1.5p(2) + q(2) + 0.5k(2))/(1) = 2$$

$$2(1.5p + q + 0.5k) = 2$$

$$1.5p + q + 0.5k = 1 \quad \text{———} \quad ①$$

and
$$p + q + k = 1 \quad \text{———} \quad ②$$

Rewrite into matrix form, we have:

$$M^T \begin{bmatrix} p \\ q \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } M^T = \begin{bmatrix} 1.5 & 1 & 0.5 \\ 1 & 1 & 1 \end{bmatrix}$$

We have a non-squared 2×3 matrix M^T s.t. we always have a free parameter in the solution for $(p, q, k)^T$.

Pick k as the free parameter. Set $k = 1/2$ s.t. $p, q \;\&\; k > 0$.
(See next page!)

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$K = 1/2$: from ②

$p = 1 - q - 1/2 = 1/2 - q$

$q = 1 - p - 1/2 = 1/2 - p$

put ② into ①, solve for q,

$$\left(\tfrac{1}{2} - q\right)u + \left(\tfrac{1}{2} - \left(\tfrac{1}{2} - q\right)\right)m + \tfrac{1}{2}d = \beta$$

$$\tfrac{1}{2}u - qu + qm + \tfrac{1}{2}d = \beta$$

$$q = \frac{\beta - \frac{1}{2}(u+d)}{m-u} = \frac{1 - \frac{1}{2}(1.5 + 0.5)}{1 - 1.5} = 0$$

To solve for p,

$$\left(\tfrac{1}{2} - \left(\tfrac{1}{2} - p\right)\right)u + \left(\tfrac{1}{2} - p\right)m + \tfrac{1}{2}d = \beta$$

$$pu + \tfrac{1}{2}m - pm + \tfrac{1}{2}d = \beta$$

$$p = \frac{\beta - \frac{1}{2}(m+d)}{u-m} = \frac{1 - \frac{1}{2}(1 + 1/2)}{1.5 - 1} = 1/2$$

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FTOAP

By the 1st Fundamental Theorem of Asset Pricing, we can conclude that there's no arbitrage for this market model due to the existence of the EMMs.

While, by the 2nd FTOAP, we can conclude that the market is incomplete as nullity of M is not 0. In this case, the EMM is not unique where we can have infinitely many solution for (p, q, k). ✓

(c). Let C_t be an European Call option with Payoff $(S_t - K)^+$ expires at time 1. Set $K > 3$, where 3 is the highest possible value of $\mathcal{E}$.
We can never replicate this call option because the possible value of S_t can never be greater than the exercise price $K > 3$ s.t. the option will never be exercised and have a 0 payoff at time 1.

2 Trivial but true

(d). By introducing a risky asset U into this market model, we have a new self-financing portfolio:

$$C_1 = a_t S_t + c_t U_t + b_t \beta_t = \begin{cases} a_t S_u + c_t U_1 + b_t \beta_t \\ a_t S_m + c_t U_1 + b_t \beta_t \\ a_t S_d + c_t U_1 + b_t \beta_t \end{cases}$$

Denotes (x, y, z) as the growth rate for U

New binomial tree:

$t=0$: $S_0 = 2$

$t=1$: (branches p, q, k)

$3 = S_0 u = S_u$

$2 = S_0 m = S_m$

$1 = S_0 d = S_d$

where, $u = 3/2 = 1.5$

$m = 2/2 = 1$

$d = 1/2 = 0.5$

$1 \;\text{———}\; \beta$

$t=0$: $U_0 = 4.5$

$t=1$: (branches p, q, k)

$S_u^2 = U_1 = U_0 x = 3^2 = 9$

$S_m^2 = U_1 = U_0 y = 2^2 = 4$

$S_d^2 = U_1 = U_0 z = 1^2 = 1$

(Note that, U & S share same set of probability while the growth rate are different)

where, $x = 9/4.5 = 2$

$y = 4/4.5 = 8/9$

$z = 1/4.5 = 2/9$

(cont'd)

(cont'd)

In matrix form, we have:

$$M^T\begin{bmatrix}p\\q\\k\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}, \text{ where } M^T=\begin{bmatrix}1.5 & 1 & 0.5\\ 2 & 8/9 & 2/9\\ 1 & 1 & 1\end{bmatrix}.$$

Then, we obtained an augmented matrix: $\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 1.5 & 1 & 1/2 & 1\\ 2 & 8/9 & 2/9 & 1\end{array}\right]$,

By Gaussian elimination:

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 1.5 & 1 & 1/2 & 1\\ 2 & 8/9 & 2/9 & 1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 1 & 1 & 1\\ 0 & -1/2 & -1 & -1/2\\ 0 & 0 & 4/9 & 1/9\end{array}\right] = \begin{cases}p+q+k=1\\ -1/2q-k=-1/2\\ 4/9k=1/9\end{cases} = \begin{cases}k=1/4\\ q=1/2\\ p=1/4\end{cases}$$

✓

Rank$(M^T)=3$, Nullity$(M^T)=0$

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$\because$ Rank$(M^T)=3=m$ (full rank) while nullity$(M^T)=0$.

$\therefore$ There exists only 1 solution for $(p,q,k)^T=(1/4,1/2,1/4)^T$

$\therefore$ EMM exists

$\therefore$ There's no arbitrage for this market model, by 1st. FTOAP

$\therefore$ As such, (p,q,k) has unique solution by rank-nullity theorem and the market is completed by 2nd FTOAP.

(e) Construct a self-financing portfolio for the option: $(U_1-K)^+$

$$(U_1-3)^+ = \begin{cases}a_1 S_0 u + c_1 U_0 x + b_1\beta\\ a_1 S_0 m + c_1 U_0 y + b_1\beta\\ a_1 S_0 d + c_1 U_0 z + b_1\beta\end{cases}$$

Given a set of parameters from the previous parts: $\begin{Bmatrix}u=1.5, & m=1, & d=0.5, & S_0=2, & \beta=1, \forall t\\ x=2, & y=\frac{8}{9}, & z=\frac{2}{9}, & U_0=4.5, & K=3\end{Bmatrix}$

Substituting into the replicating portfolio, we have the system of linear equations below:

$$\begin{cases}3a_1+9c_1+b_1=6 & \text{——— } (1)\\ 2a_1+4c_1+b_1=1 & \text{——— } (2)\\ a_1+c_1+b_1=0 & \text{——— } (3)\end{cases}$$

Rewrite into matrix form, we have an augmented matrix below:

$$\begin{bmatrix}6\\1\\0\end{bmatrix}=\begin{bmatrix}3 & 9 & 1\\ 2 & 4 & 1\\ 1 & 1 & 1\end{bmatrix} \sim \left[\begin{array}{ccc|c}3 & 9 & 1 & 6\\ 2 & 4 & 1 & 1\\ 1 & 1 & 1 & 0\end{array}\right]$$

Again, by Gaussian elimination, we have the row echelon form:

$$\left[\begin{array}{ccc|c}3 & 9 & 1 & 6\\ 2 & 4 & 1 & 1\\ 1 & 1 & 1 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}3 & 9 & 1 & 6\\ 0 & -2 & 1/3 & -3\\ 0 & 0 & 1/3 & 1\end{array}\right] = \begin{cases}3a_1+9c_1+b_1=6\\ -2c_1+\frac{1}{3}b_1=-3\\ \frac{1}{3}b_1=1\end{cases} = \begin{cases}a_1=-5\\ c_1=2\\ b_1=3\end{cases}$$

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$\therefore$ The replicating portfolio is: $(a_1,c_1,b_1)^T=(-5,2,3)^T$

$\therefore C_0=-5(2)+2(4.5)+3(1)=2$ ✓ 2