Q-1

(a) 1) (i) $X|0\rangle$

(ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

2) (i) $|1\rangle$

(ii) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

3) $P(0) = 0\%$

(b) 1) (i) $H\,X|0\rangle$

(ii) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\underset{2\times2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

2) (i) $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

(ii) $\frac{1}{\sqrt{2}}\underset{2\times2}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}\underset{2\times1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3) $P(0) = 50\%$

(c) 1) (i) $XHX|0\rangle$

(ii) $\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

2) (i) $\frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$

(ii) $\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

3) $P(0) = 50\%$.

(d) 1) (i) $H \times H \times |0\rangle$

(ii) $\frac{1}{2}\cdot\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

2) (i) $-|1\rangle$

(ii) $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\-2\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

3) $P(0) = 0\%$.

Q-2

(1) $XHYH|1\rangle$

(i) $q \quad |0\rangle - \boxed{H} - \boxed{Y} - \boxed{H} - \boxed{X} -$

(ii) $P(1) = 100\%$.

(2) $HYHX|1\rangle$

(i) $q \quad |1\rangle - \boxed{X} - \boxed{H} - \boxed{Y} - \boxed{H} -$

(ii) $P(1) = 100\%$.

(3) $XYH|0\rangle$

(i) $q \quad |0\rangle - \boxed{H} - \boxed{Y} - \boxed{X} -$

(ii) $P(1) = 50\%$.

(4) $SHTH|0\rangle$

(i) $q \quad |0\rangle - \boxed{H} - \boxed{T} - \boxed{H} - \boxed{S} -$

(ii) $P(1) = 14.65\%$

Q-3

(1) $HXH = X$ False

$$\frac{1}{2}\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}_{2\times2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}_{2\times2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}_{2\times2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}_{2\times2} = \frac{1}{2}\begin{bmatrix}2 & 0\\0 & -2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \neq X$$

(2) $HZH = X$ True

$$\frac{1}{2}\underbrace{\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = X$$

(3) $(XH)^5 = I$ False

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -4 & -4 \end{bmatrix} \neq I$$

Q-4

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

a) $Y^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ $\quad Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$Y^\dagger Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b) For a matrix to be Hermitian

$Y^\dagger = Y^*$ which is not true

$\therefore$ Y is not Hermitian

c) $U = \alpha X + \beta Y + \gamma Z$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X^\dagger = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y^\dagger = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$U^{\dagger}U = (\alpha X^{\dagger} + \beta Y^{\dagger} + \gamma Z^{\dagger})(\alpha X + \beta Y + \gamma Z)$$

$$= \alpha^2 X^{\dagger}X + \beta^2 Y^{\dagger}Y + \gamma^2 Z^{\dagger}Z$$

$$X^{\dagger}X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Y^{\dagger}Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$Z^{\dagger}Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$U^{\dagger}U = \alpha^2 I + \beta^2 I + \gamma^2 I$$
$$= \alpha^2 + \beta^2 + \gamma^2$$
$$= 1 = I_{1\times 1}$$

$\therefore$ U is unitary