Q1)

Shifting an array x_n by an amount m just amounts to taking the inverse Fourier transform (F^{-1}) of $e^{-2\pi ikm/N}$ (which is just the Fourier transform of a delta function), for N the total number of points and $k \in [0, N-1]$, multiplied by the Fourier transform (F) of the original array (x_n), which we can write as X_k. This relation can written as,

$$\{x_{n-m}\} = F^{-1}(\{\exp(-\frac{i2\pi km}{N}) \cdot X_k\})_n \tag{1}$$

and is implemented in the function `shift`. The shifted Gaussian and the original Gaussian are displayed in Figure 1.

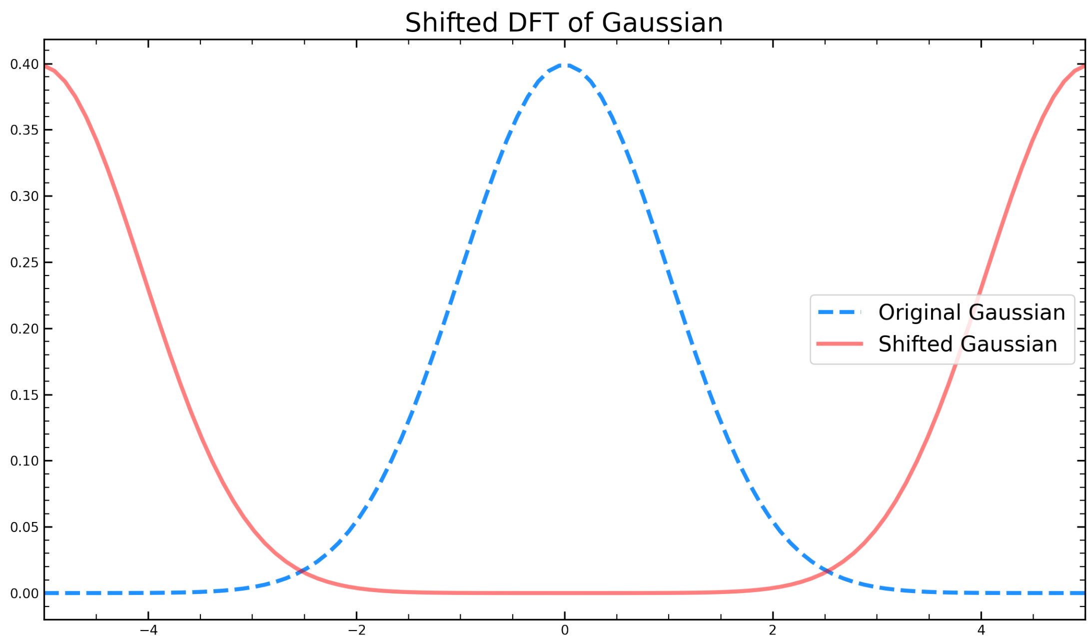

Figure 1: Shifted Gaussian using convolution.

Q2)

We can obtain the correlation function following the inverse DFT of the DFT of f multiplied by the complex conjugate of the DFT of g (the conjugate DFT just amounts to taking the DFT of $g(-x)$, which for an array simply implies taking the DFT of the flipped array, having assumed $g(x)$ (equivalently the array) to be real). The correlation function of a Gaussian of mean $= 0$ and $\sigma = 1$ with itself is presented in Figure 2, and has been normalized by its maximum such that the functions are easily comparable (to visualize).

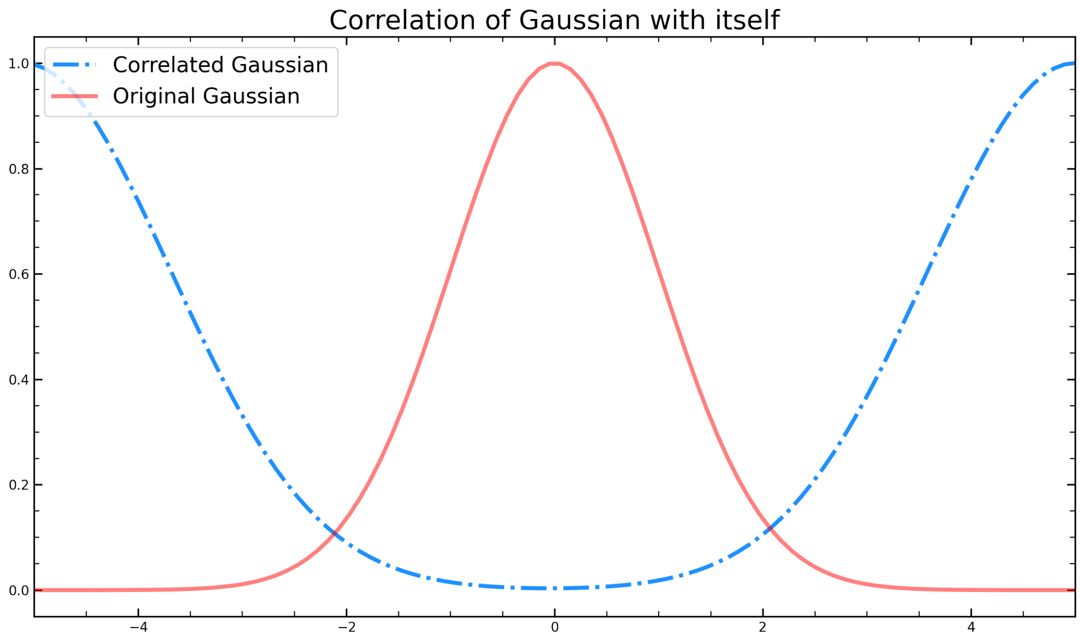

Figure 2: Correlation function of Gaussian with itself.

Q3)

We can use the two functions defined in Q1 and Q2 to obtain the correlation function of a shifted Gaussian with an unshifted Gaussian. After plotting a series of correlation functions for a variety of different shift amounts (by different fractions of N), the plots in Figure 3, 4, 5, and 6, were obtained.

The correlation function is a minimum when the maxima of the original and shifted Gaussian completely overlap (which occurs for a shift of an integer multiple of N), and a maximum when the minima overlap.

We also see that correlation output shifts in the same direction as the shifted Gaussian (we see this as the Gaussian is shifted rightward, and the correlation function follows). **Note that if we take the correlation of a unshifted Gaussian with a shifted Gaussian, the reverse pattern is observed - that is, the correlation output shifts in the opposite direction of the shifted Gaussian.**

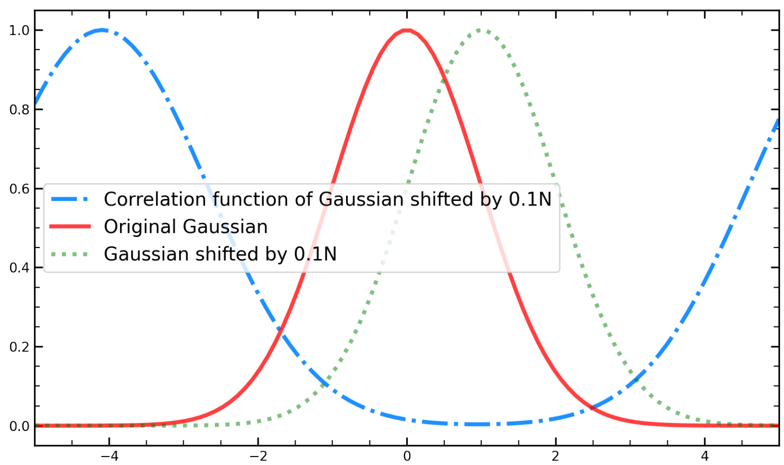

Figure 3: Correlation function for Gaussian shifted by 0.1N.

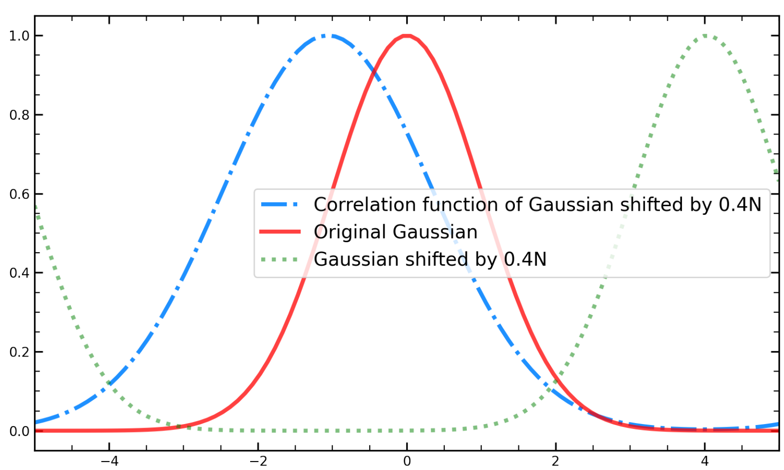

Figure 4: Correlation function for Gaussian shifted by 0.4N.

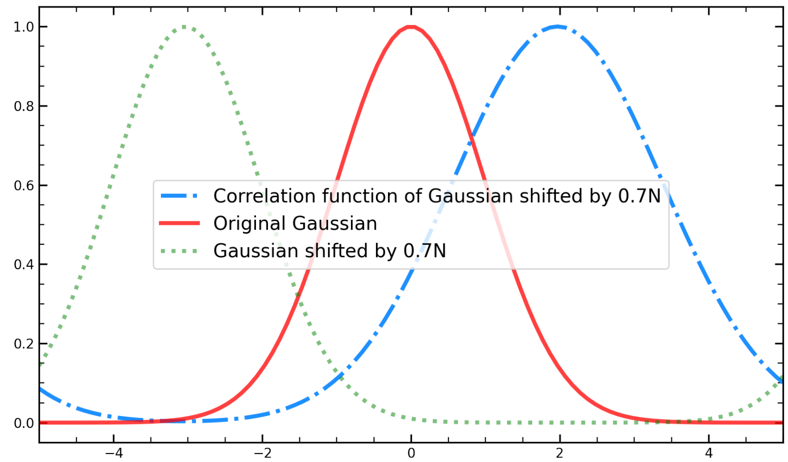

Figure 5: Correlation function for Gaussian shifted by 0.7N.

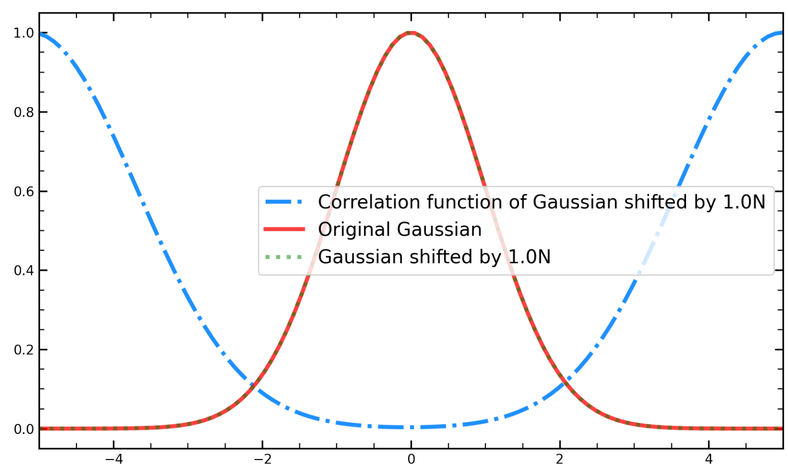

Figure 6: Correlation function for Gaussian shifted by 1.0N.

Q4)

We can add zeros to the end of input arrays to avoid circular convolution/correlation, and obtain the desired linear convolution. This is due to the fact that 'padding' the signal/array with zeros effectively returns us to the assumption that the signal/array is zero outside of the known range (where the standard DFT assumption is periodicity outside of the known range). Therefore, the new signal/array is periodic with period $N + N_0$, for N_0 the number of zeros we have appended to the signal/array.

To do this, we can modify the two arrays (which may be of different sizes), such that the final output arrays are of length 2 times the smaller array plus 5, given the function appends 5 zeros to each array in the final step before taking the convolution. An example of this 'safe convolution' for a Cosine function and Gaussian is presented below in Figure 7.

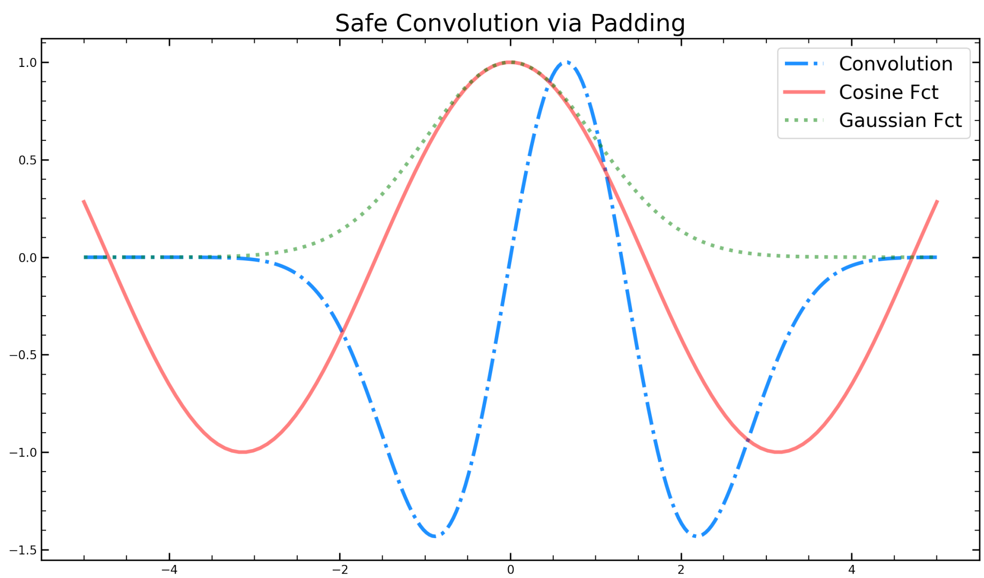

Figure 7: Safe convolution of two functions after padding arrays with zeros to circumvent the periodicity of the DFT. The convolution function has been normed by its largest value to allow for easy comparison.

Q5a)

This can be done by recognizing that the sum can be written as the sum of a geometric series. Expanding the series out we get,

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = 1 + e^{-2\pi ik/N} + e^{-4\pi ik/N} + e^{-6\pi ik/N} + ... = \sum_{x=0}^{N-1} \exp(-2\pi ik/N)^x \tag{2}$$

which is very similar to the geometric series closed-form formula,

$$\sum_{x=0}^{N} ar^x = a\left(\frac{1-r^{N+1}}{1-r}\right) \implies a = 1 \quad \text{and} \quad r = \exp(-2\pi ik/N) \tag{3}$$

and noting our sum runs from $[0, N-1]$, we can rewrite our series in similar form as,

$$\sum_{x=0}^{N-1} \exp(-2\pi ik/N)^x = \left(\frac{1-\exp(-2\pi ik/N)^{N+1-1}}{1-\exp(-2\pi ik/N)}\right) = \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)}. \tag{4}$$

Q5b)

Taking the limit as $k \to 0$ we get,

$$\lim_{k\to 0} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} = \lim_{k\to 0} \frac{(2\pi i)\exp(-2\pi ik)}{(\frac{2\pi i}{N})\exp(-2\pi ik/N)} = \lim_{k\to 0} \frac{N\exp(-2\pi ik)}{\exp(-2\pi ik/N)} = \frac{N\exp(0)}{\exp(0)} = N \tag{5}$$

after applying L'hopital's rule. Now we can show that the limit as $k \to n$ for $\{n : n \in \mathbb{Z}, n \neq mN, m \in \mathbb{Z}\}$. This limit is,

$$\lim_{k\to n} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} = \frac{1-1}{1-\exp(-2\pi in/N)} = \frac{0}{1-\exp(-2\pi in/N)} = 0 \tag{6}$$

given the denominator is only 0 for n being an integer multiple of N (where we recover the result from the limit above). However, for all n not an integer multiple of N we have a non-zero denominator given,

$$\exp(-2\pi in/N) \neq 1 \ \forall \ \frac{n}{N} \notin \mathbb{Z} \tag{7}$$

and hence the limit evaluates to 0 given the 0 in the numerator.

Q5c)

Considering a non-integer value of k in our sine function, $f(x) = \sin(2\pi kx/N)$, we can write down the Fourier transform, $F(k')$, of $f(x)$ as,

$$F(k') = \sum_{x=0}^{N-1} f(x)\exp(-2\pi ik'x/N) = \sum_{x=0}^{N-1} \sin(2\pi kx/N)\exp(-2\pi ik'x/N) \tag{8}$$

where we can use the trigonometric identity for sine to simplify. This gives us,

$$F(k') = \sum_{x=0}^{N-1} \left(\frac{e^{2\pi ikx/N} - e^{-2\pi ikx/N}}{2i}\right)\exp(-2\pi ik'x/N) \tag{9}$$

which implies we can write $F(k')$ as,

$$F(k') = \sum_{x=0}^{N-1} \left(\frac{e^{2\pi i(k-k')x/N} - e^{-2\pi i(k+k')x/N}}{2i}\right) = \frac{1}{2i}\left(\sum_{x=0}^{N-1} e^{-2\pi i(k'-k)x/N} - \sum_{x=0}^{N-1} e^{-2\pi i(k+k')x/N}\right). \tag{10}$$

Writing these sums as we evaluated them in Q5b) we get,

$$F(k') = \frac{1}{2i}\left(\frac{1-\exp(-2\pi i(k'-k))}{1-\exp(-2\pi i(k'-k)/N)} - \frac{1-\exp(-2\pi i(k+k'))}{1-\exp(-2\pi i(k+k')/N)}\right). \tag{11}$$

Therefore, we can compare our analytic expression of the DFT of a non-integer sine function with the FFT, and show that the two are within $\approx$ machine precision of each other (see Figure 8).

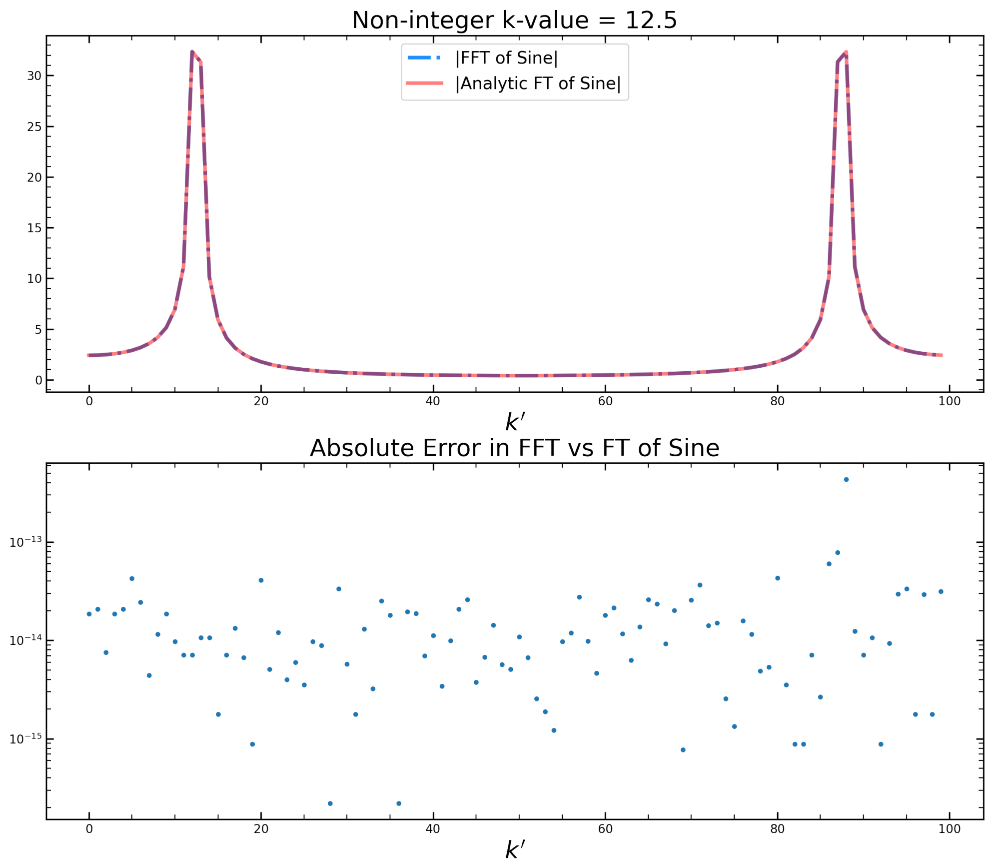

Figure 8: Comparison of our derived analytic DFT of a non-integer sine function (for non-integer $k = 12.5$) and the FFT of the same function. Note that the plot is of the absolute value of the FT, which nicely displays the expected δ functions. A plot of the absolute error between the two real values of the two transforms is presented in the second plot, where the average error is $\approx 10^-14$, which is close to machine precision.

Clearly, the FFT agrees very well with our analytic expression for the FT of a non-integer sine wave. Given the error is on the order of $\approx 10^{-14}$, this is around machine precision. In the first plot for the non-integer k-value we have something similar to a delta function, although it appears to be somewhat spread out relative to the standard spike. It may be noted that the two spikes are occurring where expected, at $k' = k$ and $k' = -k$ indexed from the last element in the array (due to circular DFT). Increasing the number of points in x and hence k' we would likely see smoother spikes that more closely resemble delta functions.

Q5d)

Multiplying our non-integer sine function by the suggested window function, and taking the FFT of the product, the following plot was obtained (Figure 9).

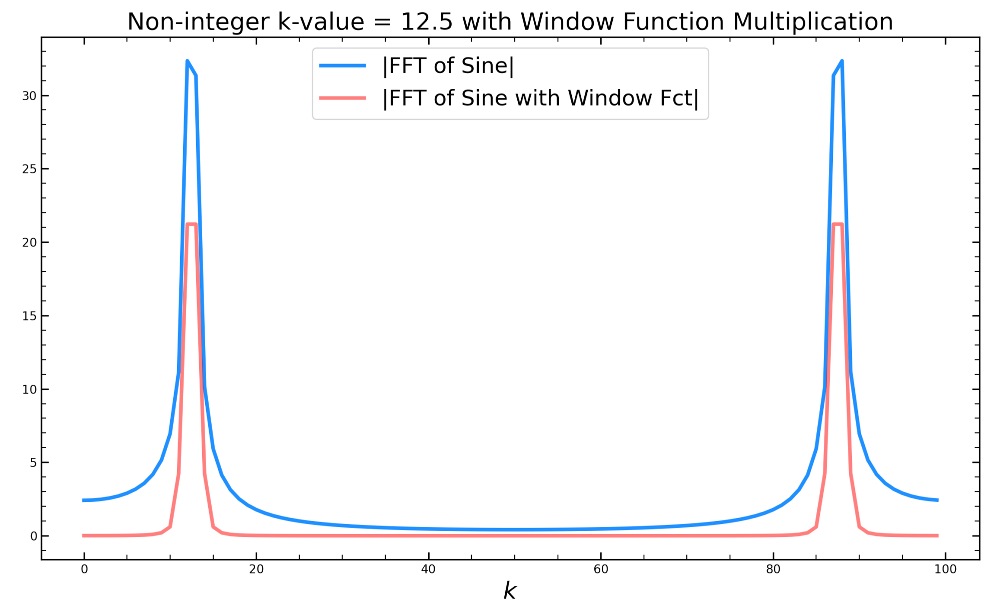

Figure 9: Absolute values of the real-valued FFT of the non-integer sine function and FFT of the product of the non-integer sine function and the window function, $f(x) = 0.5 - 0.5\cos(2\pi x/N)$. As we see above, the multiplication of sine by the suggested window function drastically reduces the spectral leakage we observed in Q5c). We can see this directly on the above plot, where the two 'delta' functions for the FFT of sine multiplied by the window function exhibit the characteristic 'spike' shape far more than the spikes resulting from the direct FFT of sine (they are less spread out at their base).

Q5e)

The FFT of the window function is displayed below in Figure 10. The expected values of the FT are plotted as horizontal lines, and it is clear that the FFT numerically agrees (points intersect with the horizontal lines where expected).

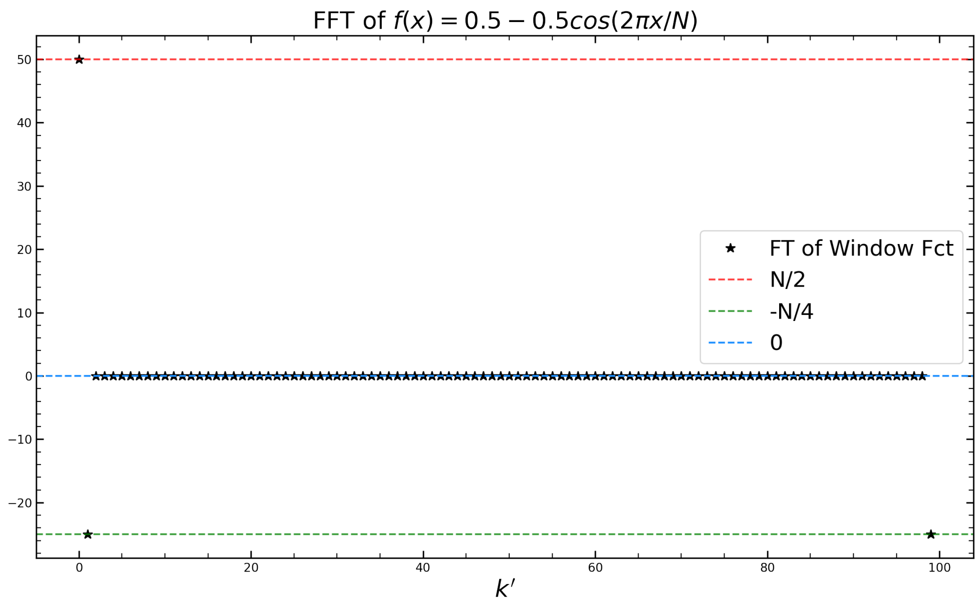

Figure 10: Real-valued part of the FFT of the window function.

Our window function above takes on only 3 possible values, given it is zero (or within machine precision of zero) for all points outside of the first 2 points and the last point. Therefore, we can write our function as,

$$F(\text{Window}(x))_{k'} = F_{\text{Wind}}(k') = \begin{cases} \frac{N}{2} & k' = 0 \\ \frac{-N}{4} & k' = 1, -1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where the Fourier transform is circulant and hence, $k' = -1$ is just the last point in the array above, which we showed had a value of $-N/4$. Rewriting this as a single equation of unit sample functions (delta function with value 1 for an argument of 0), we get,

$$F_{\text{Window}(x)}(k') = \frac{N}{2}\delta(k') - \frac{N}{4}\delta([k'-1]\text{mod}N) - \frac{N}{4}\delta([k'+1]\text{mod}N) \tag{13}$$

where we have included the modulo operations to account for the circular nature of the DFT, which is periodic with period N.

Given we have the Fourier transform of the window function, we can compute the Fourier transform of the windowed non-integer sine function we dealt with in the previous parts by exploiting the convolution identity. This implies we can write the Fourier transform of the product of two functions in real space as the convolution of the Fourier transforms of two the functions,

$$F(f(x) \cdot g(x))_k = F(f(x))_k * F(g(x))_k. \tag{14}$$

For the functions we are dealing with, where $f(x) = \sin(2\pi kx/N)$ and $g(x) = \text{Window}(x)$ this becomes,

$$F(f(x) \cdot g(x))_{k'} = \frac{1}{N} \sum_{n=0}^{N-1} F(sin(2\pi kx/N))_{k'-n} \left(\frac{N}{2}\delta(n) - \frac{N}{4}\delta([n-1]\text{mod}N) - \frac{N}{4}\delta([n+1]\text{mod}N) \right) \quad (15)$$

where we can exploit the properties of the δ function and discrete convolution to obtain the final expression,

$$F(f(x) \cdot g(x))_{k'} = \frac{1}{2}F(f(x))[k'] - \frac{1}{4}F(f(x))[(k'-1)\text{mod}N] - \frac{1}{4}F(f(x))[(k'+1)\text{mod}N] \quad (16)$$

Therefore, the windowed Fourier transform is just the un-windowed Fourier transform (DFT of non-integer sine function we computed previously) evaluated at 3 points per k'-value, where the 3 points are $k', k'-1$, and $k'+1$ (modulo N) - and hence, just its nearest neighbours. The plot of the analytic DFT and FFT of the windowed-function is presented in Figure 11.

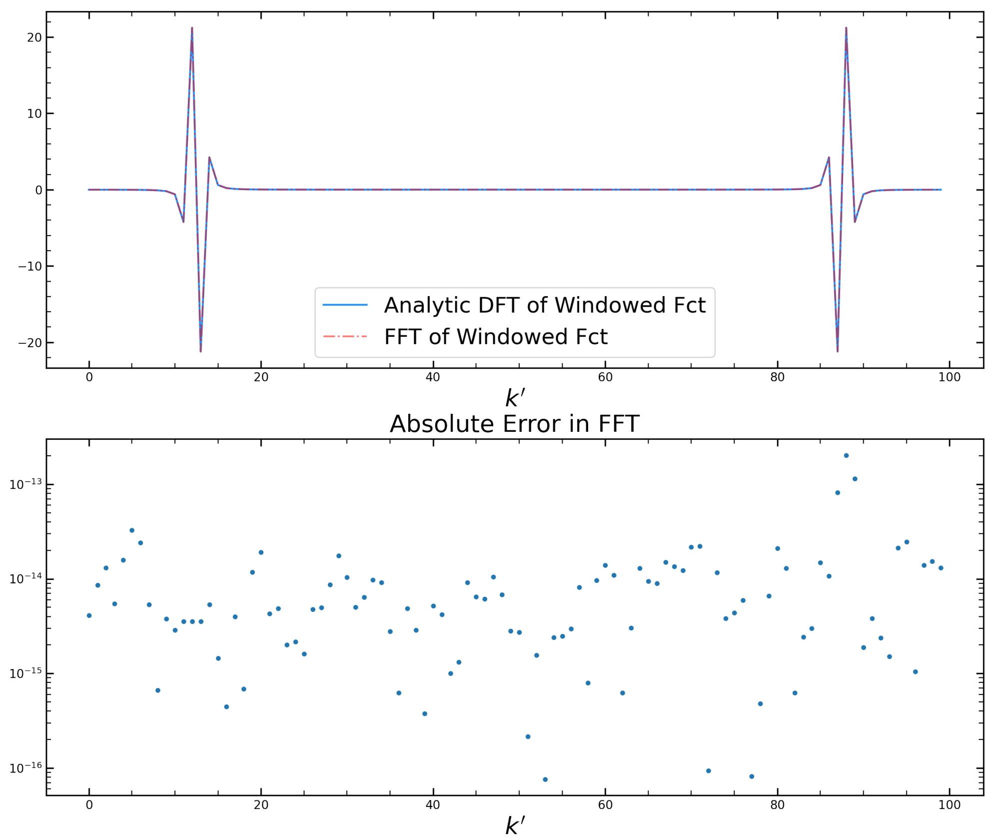

Figure 11: Comparison of the analytic DFT of the windowed-sine function, and the FFT of the windowed-function. Clearly, they agree very well, to within near machine precision.

Q6a)

We can write the power spectrum of a random walk as the FT of the auto-correlation function of the random walk. Therefore, given the autocorrelation function we derived in class was $N - ndx/2$, we can just the take the Fourier transform of this and show that it scales as $1/k^2$.

In class, we found the autocorrelation function of the random walk to be,

$$\langle f(x)f(x+dx)\rangle = N - \frac{ndx}{2} \tag{17}$$

for a given scale factor n, which implies we can write the power spectrum as,

$$P(k) = \sum_{dx=0}^{N-1}\left(N - \frac{ndx}{2}\right)e^{-2\pi ikdx/N}. \tag{18}$$

Evaluating the DFT of the autocorrelation function, we will obtain a function that scales as $1/k^2$.

Q6b)

To do this, we can generate a random walk using the function suggested, and pass it to our correlation function defined in Q2 to obtain the autocorrelation function of the random walk. By definition the Fourier transform of the autocorrelation is the power spectrum, so taking the FFT of the autocorrelation function, we can plot the power spectrum and show that indeed goes as $\frac{1}{k^2}$. The power spectrum of the random walk is displayed in Figure 12 alongside a scaled $1/k^2$ relation, and it is obvious given the general agreement that it also scales with k in the manner.

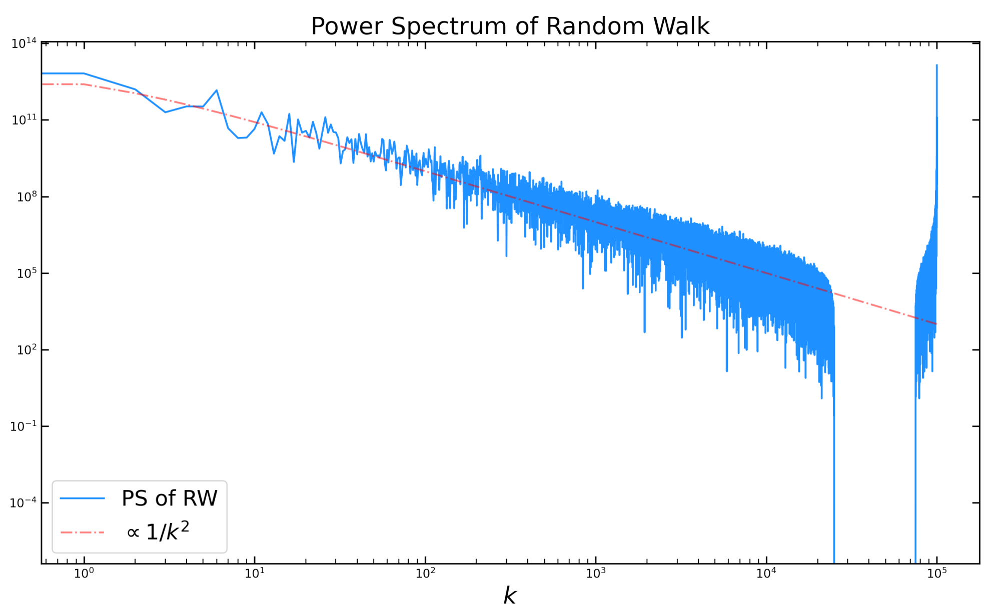

Figure 12: Power spectrum of a random walk for $N = 10^5$ steps, alongside a scaled $1/k^2$ relation of $10^{13}/k^2$. The $k = 0$ has been omitted. Clearly, the two curves are in general agreement.