

## Transpose

The arbitrary matrix  $A$  is transposed to matrix  $A^T$ .

The example is shown as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

## Trace

The trace of a square matrix  $A$  ( $\text{tr}A$ ) is defined to be the sum of elements on the main diagonal of  $A$ . The example is shown as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then  $\text{tr}A = \sum_i a_{ii} = 15$ .

## Determinant

For a  $3 \times 3$  matrix  $A$ , its determinant is

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} \times & \times & \times \\ \times & a_{22} & a_{23} \\ \times & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} \times & \times & \times \\ a_{21} & \times & a_{23} \\ a_{31} & \times & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} \times & \times & \times \\ a_{21} & a_{22} & \times \\ a_{31} & a_{32} & \times \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}a_{22} \begin{vmatrix} \times & \times \\ \times & a_{33} \end{vmatrix} - a_{11}a_{23} \begin{vmatrix} \times & \times \\ a_{32} & \times \end{vmatrix} - a_{12}a_{21} \begin{vmatrix} \times & \times \\ \times & a_{33} \end{vmatrix} \\ &\quad + a_{12}a_{23} \begin{vmatrix} \times & \times \\ a_{31} & \times \end{vmatrix} + a_{13}a_{21} \begin{vmatrix} \times & \times \\ \times & a_{32} \end{vmatrix} - a_{13}a_{22} \begin{vmatrix} \times & \times \\ a_{31} & \times \end{vmatrix} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31} \end{aligned}$$

## Cramer's rule

In a system of  $n$  linear equations, represented in matrix multiplication form  $A\mathbf{x} = \mathbf{b}$

where  $A$  is the  $n \times n$  matrix and  $\mathbf{x}$  and  $\mathbf{b}$  are the  $n$ -th column vectors.  $\mathbf{x} = (x_1, \dots, x_n)^T$ ,  $\mathbf{b} = (b_1, \dots, b_n)^T$ .

Then, if  $|A| \neq 0$ ,

$$x_i = |A_i|/|A|, \quad A_i = \begin{pmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{k1} & & b_{ki} & & a_{kn} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{pmatrix}$$

This is Cramer's rule.

## LU decomposition

$A = LU$  where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

## Direct method by LU decomposition

In linear equation  $A\mathbf{x} = \mathbf{b}$ ,  $LU\mathbf{x} = \mathbf{b}$  by using LU decomposition  $A = LU$ .

Here, we consider  $L\mathbf{y} = \mathbf{b}$  and  $U\mathbf{x} = \mathbf{y}$ .

In forward substitution,

$$\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21}y_1 \\ &\vdots \\ y_n &= b_n - \sum_{j=1}^{n-1} l_{nj}y_j \end{aligned}$$

In backforward substitution,

$$\begin{aligned} x_n &= y_n/u_{nn} \\ x_{n-1} &= (y_{n-1} - u_{n-1,n}x_n)/u_{n-1,n-1} \\ &\vdots \\ x_1 &= (y_1 - \sum_{j=2}^n u_{1,j}x_j)/u_{11} \end{aligned}$$

## Constant multiple

$$c \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nn} \end{pmatrix}$$

where  $c$  is the scalar constant.

## Inverse matrix

$$AB = BA = I$$

where  $A$  and  $B$  is the  $n \times n$  matrices and  $I$  is the  $n \times n$  unit matrix. In the case, the matrix  $B$  is uniquely determined by  $A$  and is called the inverse matrix of  $A$ . The inverse matrix of  $A$  is denoted by  $A^{-1}$ .

## Product

The elements of the matrix product  $C = AB$  is that  $c_{ij} = [AB]_{ij} = \sum_k a_{ik}b_{kj}$  where  $A$  is an  $n \times m$  matrix and  $B$  is an  $m \times l$  matrix.

## Addition and Subtraction

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the addition/subtraction is that

$$A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{pmatrix}$$

## Hadamard product

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the Hadamard product is that

$$A \circ B = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{pmatrix}$$

## Hadamard division

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the Hadamard division is that

$$A/B = \begin{pmatrix} a_{11}/b_{11} & a_{12}/b_{12} & a_{13}/b_{13} \\ a_{21}/b_{21} & a_{22}/b_{22} & a_{23}/b_{23} \\ a_{31}/b_{31} & a_{32}/b_{32} & a_{33}/b_{33} \end{pmatrix}$$

## Hadamard power

$$A^{(n)} = \begin{pmatrix} a_{11}^n & a_{12}^n & a_{13}^n \\ a_{21}^n & a_{22}^n & a_{23}^n \\ a_{31}^n & a_{32}^n & a_{33}^n \end{pmatrix}$$

where  $n$  is scalar.

## Tensor product

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the tensor product is that

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} & a_{13}b_{11} & a_{13}b_{12} & a_{13}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} & a_{13}b_{21} & a_{13}b_{22} & a_{13}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} & a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} & a_{23}b_{11} & a_{23}b_{12} & a_{23}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & a_{23}b_{21} & a_{23}b_{22} & a_{23}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} & a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} & a_{32}b_{11} & a_{32}b_{12} & a_{32}b_{13} & a_{33}b_{11} & a_{33}b_{12} & a_{33}b_{13} \\ a_{31}b_{21} & a_{31}b_{22} & a_{31}b_{23} & a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} & a_{33}b_{21} & a_{33}b_{22} & a_{33}b_{23} \\ a_{31}b_{31} & a_{31}b_{32} & a_{31}b_{33} & a_{32}b_{31} & a_{32}b_{32} & a_{32}b_{33} & a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{pmatrix}$$

## Eigenvalue (Algebraic method)

An eigen equation is written as  $A\mathbf{u} = \lambda\mathbf{u}$  where  $\lambda$  is scalar and  $\mathbf{u}$  is vector, known as the eigenvalue and eigenvector.

By rearranging above equation, we obtain:  $A\mathbf{u} = \lambda\mathbf{u}$ ,  $(A - \lambda I)\mathbf{u} = \mathbf{0}$ . If this equation has a nontrivial solution ( $\mathbf{u} \neq 0$ ), the determinant  $|A - \lambda I| = 0$ .

[2×2 matrix case]

When the matrix  $A$  is written as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the quadratic equation  $\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$  is obtained. By using quadratic formula,

$$\lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}.$$

[3×3 matrix case]

When the matrix  $A$  is written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we obtain the cubic equation  $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$  where

$$a = -1,$$

$$b = a_{11} + a_{22} + a_{33},$$

$$c = a_{21}a_{12} + a_{13}a_{31} + a_{32}a_{23} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33},$$

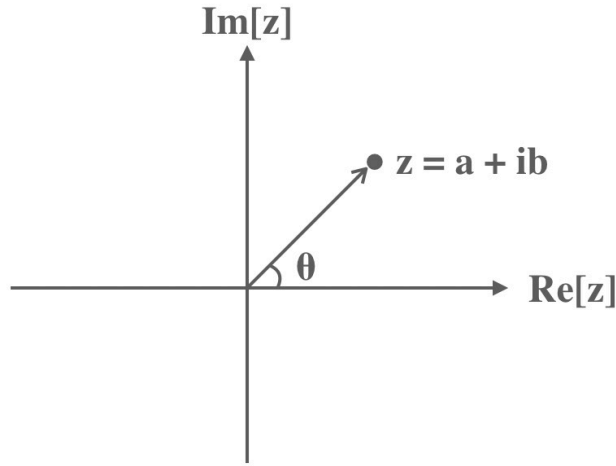
$$d = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{11}a_{32}a_{23} - a_{22}a_{31}a_{13} - a_{33}a_{21}a_{12}.$$

Therefore, we can solve the eigen equation in the case of the 3×3 matrix  $A$  by substituting above  $a$ ,  $b$ ,  $c$  and  $d$  for the cubic formula. The cubic formula is that

$$\begin{aligned} \lambda_1 &= -\frac{b}{3a} \\ &\quad - \frac{1}{3a} \sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d + \sqrt{(ab^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3})} \\ &\quad - \frac{1}{3a} \sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d - \sqrt{(ab^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3})}, \\ \lambda_2 &= -\frac{b}{3a} \end{aligned}$$

$$\begin{aligned}
& -\frac{1+i\sqrt{3}}{6a} \sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d+\sqrt{(ab^3-9abc+27a^2d)^2-4(b^2-3ac)^3})} \\
& -\frac{1-i\sqrt{3}}{6a} \sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d-\sqrt{(ab^3-9abc+27a^2d)^2-4(b^2-3ac)^3})} \quad , \\
\lambda_3 &= -\frac{b}{3a} \\
& -\frac{1-i\sqrt{3}}{6a} \sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d+\sqrt{(ab^3-9abc+27a^2d)^2-4(b^2-3ac)^3})} \\
& -\frac{1+i\sqrt{3}}{6a} \sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d-\sqrt{(ab^3-9abc+27a^2d)^2-4(b^2-3ac)^3})} \quad .
\end{aligned}$$

In this Elixir library, the complex numbers in the above equations are calculated as Gaussian plane.



The real part and imaginary part are calculated by using arctangent's integral formula, written as

$$\arctan x = \int_0^x \frac{1}{z^2 + 1} dz.$$

This formula is treated as the numerical integration.

## Singular value

Singular value decomposition (SVD) states:

$$A = U\Sigma V^T$$

where  $A$  and  $\Sigma$  is  $n \times m$  matrix,  $U$  is  $n \times n$  orthogonal matrix,  $V$  is  $m \times m$  orthogonal matrix. In the case  $m > n$ ,

$$\Sigma = \left( \begin{array}{ccc|c} \sigma_{11} & & & O \\ & \ddots & & \\ & & \sigma_{nn} & \\ O & & & O \end{array} \right)$$

where  $\sigma$  is singular value.

It can be replaced by an eigenvalue problem from the following relation.

$$AA^T = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma^2 U^T,$$

$$\Sigma^2 = \left( \begin{array}{ccc} \sigma_{11}^2 & & O \\ & \ddots & \\ & & \sigma_{nn}^2 \\ O & & & \lambda_{nn} \end{array} \right) = \left( \begin{array}{ccc} \lambda_{11} & & O \\ & \ddots & \\ & & \lambda_{nn} \end{array} \right)$$

where  $\lambda$  is eigenvalue of  $AA^T$ .

## Diagonalization

An  $n \times n$  matrix  $A$  is diagonalizable when  $A$  has  $n$  eigenvectors that are linear independent of each other. We consider the matrix  $P$  that is written as  $P = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$  where  $\mathbf{x}_i, i = 1, \dots, n$  linear independent eigenvector of  $A$ .

$$AP = A[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] = [\lambda_1 \mathbf{x}_1, \lambda_2 \mathbf{x}_2, \dots, \lambda_n \mathbf{x}_n]$$

where  $\lambda_i, i = 1, \dots, n$  eigenvalue of  $A$ . Since  $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$  are linear independent,

$$P^{-1}AP = \left( \begin{array}{ccc} \lambda_1 & & O \\ & \ddots & \\ & & \lambda_n \end{array} \right)$$

. This matrix is the diagonal matrix of  $A$ .

## Jordan normal form

Since  $(A - \lambda E)\mathbf{u} = \mathbf{x}$  and  $(A - \lambda E)\mathbf{x} = \mathbf{0}$ ,

$$\begin{cases} A\mathbf{u} = \mathbf{x} + \lambda\mathbf{u} \\ A\mathbf{x} = \lambda\mathbf{x} \end{cases} \quad (1)$$

Therefore,

$$A \begin{pmatrix} \mathbf{x} & \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{x} & \mathbf{u} \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$P^{-1}AP = J$$

where

$$P = \begin{pmatrix} \mathbf{x} & \mathbf{u} \end{pmatrix}, J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

## Eigenvalue and eigenvector (Power Iteration method)

An arbitrary (initial) vector  $\mathbf{b}^0$  is written by the linear combination of eigenvectors  $\sum_i c_i \mathbf{u}_i$  because eigenvectors are linearly independent.

$$\begin{aligned} \mathbf{b}^k &\equiv A^k \mathbf{b}^0 = A^k \sum_i c_i \mathbf{u}_i = \sum_{i=1} c_i \lambda_i^k \mathbf{u}_i \\ &= \lambda_1^k (c_1 \mathbf{u}_1 + \sum_{i=2} c_i \frac{\lambda_i^k}{\lambda_1^k} \mathbf{u}_i) \end{aligned}$$

where  $\lambda_1$  is maximum value of the eigenvalue so that  $|\frac{\lambda_i^k}{\lambda_1^k}| < 1$ .

If  $k$  is a large enough number, we can write the eigenvector of the maximum eigenvalue, shown as

$$\mathbf{b}^k \simeq \lambda_1^k c_1 \mathbf{u}_1.$$

Moreover, we can write the maximum eigenvalue

$$\lambda_1 = \frac{(\mathbf{b}^k)^T A \mathbf{b}^k}{(\mathbf{b}^k)^T \mathbf{b}^k}.$$

## Matrix norms

$A$  is  $n \times m$  matrix.

Frobenius norm:

$$\|A\|_F = \sqrt{\sum_i^n \sum_j^m |a_{ij}|^2}.$$

$L_1$  norm:

$$\|A\|_1 = \max_j \sum_i^n |a_{ij}|.$$

Max norm:

$$\|A\|_\infty = \max_i \sum_j^n |a_{ij}|.$$

$L_2$  norm:

$$\|A\|_2 = \max_{ij} \sigma_{ij}$$

where *sigma* is singular value of  $A$ .



## Variance covariance matrix

A variance covariance matrix can be defined as

$$S = \begin{pmatrix} s_{xx} & s_{xy} \\ s_{yx} & s_{yy} \end{pmatrix}$$

where  $s_{xx}$  is variance value and  $s_{xy}$  is covariance value.  $s_{xy} = \frac{1}{n}(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{y} - \bar{\mathbf{y}})$ ,  $\bar{\mathbf{x}} = \sum_{i=1}^n x_i/n$ .

By the way, we can consider the Principal Component Analysis (PCA) by this variance covariance matrix with above power Iteration library.