Transpose

The arbitrary matrix A is transposed to matrix A^T.
The example is shown as

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

Trace

The trace of a square matrix A ($\mathrm{tr}A$) is defined to be the sum of elements on the main diagonal of A. The example is shown as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then $\mathrm{tr}A = \sum_i a_{ii} = 15$.

Determinant

For a 3×3 matrix A, its determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} \times & \times & \times \\ \times & a_{22} & a_{23} \\ \times & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} \times & \times & \times \\ a_{21} & \times & a_{23} \\ a_{31} & \times & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} \times & \times & \times \\ a_{21} & a_{22} & \times \\ a_{31} & a_{32} & \times \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} \times & \times \\ \times & a_{33} \end{vmatrix} - a_{11}a_{23} \begin{vmatrix} \times & \times \\ a_{32} & \times \end{vmatrix} - a_{12}a_{21} \begin{vmatrix} \times & \times \\ \times & a_{33} \end{vmatrix}$$

$$+ a_{12}a_{23} \begin{vmatrix} \times & \times \\ a_{31} & \times \end{vmatrix} + a_{13}a_{21} \begin{vmatrix} \times & \times \\ \times & a_{32} \end{vmatrix} - a_{13}a_{22} \begin{vmatrix} \times & \times \\ a_{31} & \times \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{22}a_{31}$$

Cramer's rule

In a system of n linear equations, represented in matrix multiplication form $A\mathbf{x} = \mathbf{b}$
where A is the $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are the n-th column vectors. $\mathbf{x} = (x_1, \cdots, x_n)^T$, $\mathbf{b} = (b_1, \cdots, b_n)^T$.

Then, if $|A| \neq 0$,

$$x_i = |A_i|/|A|, \quad A_i = \begin{pmatrix} a_{11} & \cdots & b_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{k1} & & b_{ki} & & a_{kn} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_{ni} & \cdots & a_{nn} \end{pmatrix}$$

This is Cramer's rule.

LU decomposition

$A = LU$ where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Direct method by LU decomposition

In linear equation $A\mathbf{x} = \mathbf{b}$, $LU\mathbf{x} = \mathbf{b}$ by using LU decomposition $A = LU$.

Here, we consider $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$.

In forward substitution,

$$\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21}y_1 \\ &\vdots \\ y_n &= b_n - \sum_{j=1}^{n-1} l_{nj}y_j \end{aligned}$$

In backforward substitution,

$$\begin{aligned} x_n &= y_n/u_{nn} \\ x_{n-1} &= (y_{n-1} - u_{n-1,n}x_n)/u_{n-1,n-1} \\ &\vdots \\ x_1 &= (y_1 - \sum_{j=2}^{n} u_{1,j}x_j)/u_{11} \end{aligned}$$

.

Constant multiple

$$c\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nn} \end{pmatrix}$$

where c is the scalar constant.

Inverse matrix

$AB = BA = I$
where A and B is the $n \times \ n$ matrices and I is the $n \times \ n$ unit matrix. In the case, the matrix B is uniquely determined by A and is called the inverse matrix of A. The inverse matrix of A is denoted by A^{-1}.

Product

The elements of the matrix product $C = AB$ is that $c_{ij} = [AB]_{ij} = \sum_k a_{ik} b_{kj}$ where A is an $n \times m$ matrix and B is an $m \times l$ matrix.

Addition and Subtraction

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the addition/subtraction is that

$$A{\pm}B = \begin{pmatrix} a_{11}{\pm}b_{11} & a_{12}{\pm}b_{12} & a_{13}{\pm}b_{13} \\ a_{21}{\pm}b_{21} & a_{22}{\pm}b_{22} & a_{23}{\pm}b_{23} \\ a_{31}{\pm}b_{31} & a_{32}{\pm}b_{32} & a_{33}{\pm}b_{33} \end{pmatrix}$$

Hadamard product

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the Hadamard product is that

$$A{\circ}B = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{pmatrix}$$

Hadamard division

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the Hadamard division is that

$$A/B = \begin{pmatrix} a_{11}/b_{11} & a_{12}/b_{12} & a_{13}/b_{13} \\ a_{21}/b_{21} & a_{22}/b_{22} & a_{23}/b_{23} \\ a_{31}/b_{31} & a_{32}/b_{32} & a_{33}/b_{33} \end{pmatrix}$$

Hadamard power

$$A^{(n)} = \begin{pmatrix} a_{11}^n & a_{12}^n & a_{13}^n \\ a_{21}^n & a_{22}^n & a_{23}^n \\ a_{31}^n & a_{32}^n & a_{33}^n \end{pmatrix}$$

where n is scalar.

Tensor product

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

then the tensor product is that

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & a_{13}B \\ a_{21}B & a_{22}B & a_{23}B \\ a_{31}B & a_{32}B & a_{33}B \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{12}b_{11} & a_{12}b_{12} & a_{12}b_{13} & a_{13}b_{11} & a_{13}b_{12} & a_{13}b_{13} \\ a_{11}b_{21} & a_{11}b_{22} & a_{11}b_{23} & a_{12}b_{21} & a_{12}b_{22} & a_{12}b_{23} & a_{13}b_{21} & a_{13}b_{22} & a_{13}b_{23} \\ a_{11}b_{31} & a_{11}b_{32} & a_{11}b_{33} & a_{12}b_{31} & a_{12}b_{32} & a_{12}b_{33} & a_{13}b_{31} & a_{13}b_{32} & a_{13}b_{33} \\ a_{21}b_{11} & a_{21}b_{12} & a_{21}b_{13} & a_{22}b_{11} & a_{22}b_{12} & a_{22}b_{13} & a_{23}b_{11} & a_{23}b_{12} & a_{23}b_{13} \\ a_{21}b_{21} & a_{21}b_{22} & a_{21}b_{23} & a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & a_{23}b_{21} & a_{23}b_{22} & a_{23}b_{23} \\ a_{21}b_{31} & a_{21}b_{32} & a_{21}b_{33} & a_{22}b_{31} & a_{22}b_{32} & a_{22}b_{33} & a_{23}b_{31} & a_{23}b_{32} & a_{23}b_{33} \\ a_{31}b_{11} & a_{31}b_{12} & a_{31}b_{13} & a_{32}b_{11} & a_{32}b_{12} & a_{32}b_{13} & a_{33}b_{11} & a_{33}b_{12} & a_{33}b_{13} \\ a_{31}b_{21} & a_{31}b_{22} & a_{31}b_{23} & a_{32}b_{21} & a_{32}b_{22} & a_{32}b_{23} & a_{33}b_{21} & a_{33}b_{22} & a_{33}b_{23} \\ a_{31}b_{31} & a_{31}b_{32} & a_{31}b_{33} & a_{32}b_{31} & a_{32}b_{32} & a_{32}b_{33} & a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} \end{pmatrix}$$

Eigenvalue (Algebraic method)

An eigen equation is written as $A\mathbf{u} = \lambda\mathbf{u}$ where λ is scalar and $\mathbf{u}$ is vector, known as the eigenvalue and eigenvector.

By rearranging above equation, we obtain: $A\mathbf{u} = \lambda\mathbf{u}$, $(A-\lambda I)\mathbf{u} = \mathbf{0}$. If this equation has a nontrivial solution ($\mathbf{u}\neq 0$), the determinant $|A-\lambda I| = 0$.

[2×2 matrix case]

When the matrix A is written as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the quadratic equation $\lambda^2 - (a_{11} - a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$ is obtained. By using quadratic formula,

$$\lambda = \frac{a_{11} - a_{22} \pm \sqrt{(a_{11} - a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}.$$

[3×3 matrix case]

When the matrix A is written as

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

we obtain the cubic equation $a\lambda^3 + b\lambda^2 + c\lambda + d = 0$ where
$a = -1$,
$b = a_{11} + a_{22} + a_{33}$,
$c = a_{21}a_{12} + a_{13}a_{31} + a_{32}a_{23} - a_{11}a_{22} - a_{11}a_{33} - a_{22}a_{33}$,
$d = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} - a_{11}a_{32}a_{23} - a_{22}a_{31}a_{13} - a_{33}a_{21}a_{12}$.

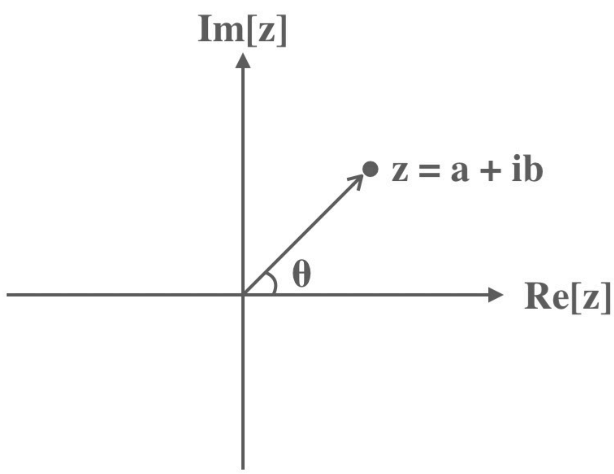
Therefore, we can solve the eigen equation in the case of the 3×3 matrix A by substituting above a, b, c and d for the cubic formula. The cubic formula is that

$$\begin{aligned}
\lambda_1 \quad &= -\frac{b}{3a} \\
&-\frac{1}{3a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d + \sqrt{(ab^3) - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3})} \\
&-\frac{1}{3a}\sqrt[3]{\frac{1}{2}(2b^3 - 9abc + 27a^2d - \sqrt{(ab^3) - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3})} \quad ,
\end{aligned}$$

$$\lambda_2 \quad = -\frac{b}{3a}$$

$$
\begin{aligned}
& -\frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d+\sqrt{(ab^3)-9abc+27a^2d)^2-4(b^2-3ac)^3})} \\
& -\frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d-\sqrt{(ab^3)-9abc+27a^2d)^2-4(b^2-3ac)^3})} \quad , \\
\lambda_3 \quad & = -\frac{b}{3a} \\
& -\frac{1-i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d+\sqrt{(ab^3)-9abc+27a^2d)^2-4(b^2-3ac)^3})} \\
& -\frac{1+i\sqrt{3}}{6a}\sqrt[3]{\frac{1}{2}(2b^3-9abc+27a^2d-\sqrt{(ab^3)-9abc+27a^2d)^2-4(b^2-3ac)^3})} \quad .
\end{aligned}
$$

In this Elixir library, the complex numbers in the above equations are calculated as Gaussian plane.



The real part and imaginary part are calculated by using arctangent's integral formula, written as

$$\arctan x = \int_0^x \frac{1}{z^2+1} dz.$$

This formula is treated as the numerical integration.

Singular value

Singular value decomposition (SVD) states:

$$A = U\Sigma V^T$$

where A and Σ is $n \times m$ matrix, U is $n \times n$ orthogonal matrix, U is $m \times m$ orthogonal matrix. In the case $m > n$,

$$\Sigma = \left(\begin{array}{ccc|c} \sigma_{11} & & O & \\ & \ddots & & O \\ O & & \sigma_{nn} & \end{array} \right)$$

where σ is singular value.

It can be replaced by an eigenvalue problem from the following relation.

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma^2 U^T,$$

$$\Sigma^2 = \begin{pmatrix} \sigma_{11}^2 & & O \\ & \ddots & \\ O & & \sigma_{nn}^2 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & & O \\ & \ddots & \\ O & & \lambda_{nn} \end{pmatrix}$$

where λ is eigenvalue of AA^T.

Diagonalization

An $n \times n$ matrix A is diagonalizable when A has n eigenvectors that are linear independent of each other. We consider the matrix P that is written as $P = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n]$ where $\mathbf{x}_i, i = 1, \cdots, n$ linear independent eigenvector of A.

$$AP = A[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] = [\lambda_1 \mathbf{x}_1, \lambda_2 \mathbf{x}_2, \cdots, \lambda_n \mathbf{x}_n]$$

where $\lambda_i, i = 1, \cdots, n$ eigenvalue of A. Since $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ are linear independent,

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix}$$

. This matrix is the diagonal matrix of A.

Jordan normal form

Since $(A - \lambda E)\mathbf{u} = \mathbf{x}$ and $(A - \lambda E)\mathbf{x} = \mathbf{0}$,

$$\left\{ \begin{array}{l} A\mathbf{u} = \mathbf{x} + \lambda \mathbf{u} \\ A\mathbf{x} = \lambda \mathbf{x} \end{array} \right. \tag{1}$$

.

Therefore,

$$A \left(\begin{array}{cc} \mathbf{x} & \mathbf{u} \end{array} \right) = \left(\begin{array}{cc} \mathbf{x} & \mathbf{u} \end{array} \right) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

.

$$P^{-1}AP = J$$

where

$$P = \begin{pmatrix} \mathbf{x} & \mathbf{u} \end{pmatrix}, J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

.

Eigenvalue and eigenvector (Power iteration method to solve maximum eigenvalue and eigenvector of n-th eigen equation)

An arbitrary (initial) vector $\mathbf{b}^0$ is written by the linear combination of eigenvectors $\sum_i c_i \mathbf{u}_i$ because eigenvectors are linearly independent.

$$\begin{aligned}\mathbf{b}^k \equiv A^k \mathbf{b}^0 &= A^k \sum_i c_i \mathbf{u}_i = \sum_{i=1} c_i \lambda_i^k \mathbf{u}_i \\ &= \lambda_1^k (c_1 \mathbf{u}_1 + \sum_{i=2} c_i \frac{\lambda_i^k}{\lambda_1^k} \mathbf{u}_i)\end{aligned}$$

where λ_1 is maximum value of the eigenvalue so that $|\frac{\lambda_i^k}{\lambda_1^k}| < 1$.

If k is a large enough number, we can write the eigenvector of the maximum eigenvalue, shown as

$$\mathbf{b}^k \simeq \lambda_1^k c_1 \mathbf{u}_1.$$

Moreover, we can write the maximum eigenvalue

$$\lambda_1 = \frac{(\mathbf{b}^k)^T A \mathbf{b}^k}{(\mathbf{b}^k)^T \mathbf{b}^k}.$$

Eigenvalue and eigenvector (Jacobi method to solve n-th eigen equation)

The Jacobi method is an iterative method for the numerical calculation of the eigenvalues and eigenvectors of a real symmetric matrix. (cf. Wikipedia)

Matrix norms

A is $n \times m$ matrix.

Frobenius norm:

$$||A||_F = \sqrt{\sum_i^n \sum_j^m |a_{ij}|^2} \quad .$$

L_1 norm:

$$||A||_1 = \max_j \sum_i^n |a_{ij}| \quad .$$

Max norm:

$$||A||_\infty = \max_i \sum_j^n |a_{ij}| \quad .$$

L_2 norm:

$$||A||_2 = \max_{ij} \sigma_{ij}$$

where σ is singular value of A.

Variance covariance matrix

A variance covariance matrix can be defined as

$$S = \begin{pmatrix} s_{xx} & s_{xy} \\ s_{yx} & s_{yy} \end{pmatrix}$$

where s_{xx} is variance value and s_{xy} is covariance value. $s_{xy} = \frac{1}{n}(\mathbf{x} - \bar{\mathbf{x}})(\mathbf{y} - \bar{\mathbf{y}})$, $\bar{\mathbf{x}} = \sum_{i=1}^n x_i/n$.

By the way, we can consider the Principal Component Analysis (PCA) by this variance covariance matrix with above power Iteration library.