Uncovering Trend Rules

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Abstract

Trend rules are widely used to infer whether financial markets show an upward or downward trend. By taking suitable long or short positions, one can profit from a continuation of these trends. Conventionally, trend rules are based on moving averages (MAs) of prices rather than returns, which obscures how much weight is assigned to different historical time periods. In this paper, we show how to uncover the underlying historical weighting schemes of price MAs and combinations of price MAs. This leads to surprising and useful insights about popular trend rules, for example that some trend rules have inverted information decay (i.e., distant returns have more weight than recent ones) or hidden mean-reversion patterns. This opens the possibility for improving the trend rule by analyzing the added value of the mean reversion part. We advocate designing trend rules in terms of returns instead of prices, as they offer more flexibility and allow for adjusting trend rules to autocorrelation patterns in returns.

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JEL codes: C18, C53, C63, G11, G17

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Introduction

Trend rules are widely used to time financial markets. When historical price patterns persist to some extent into the future they can be exploited to predict the future direction of prices. Trend rules have their origin in technical analysis[1] and are based on technical indicators computed from historical prices. Undoubtedly the most popular technical indicators are moving averages (MAs) and combinations of MAs. The simplest trend rule uses one N-period MA and prescribes taking a long (short) position when the current price is above (below) this MA, thus capitalizing on the persistence of the trend. More complex trend rules use combinations of long term and short term MAs, or even multiple hierarchically stacked MAs designed to incorporate the acceleration or deceleration of a trend.

It is general practice to define MAs in terms of price levels.[2] This is surprising. Firstly, trend (or time series momentum) implies a degree of persistence in price *movements* and hence focuses on positive or negative *changes* in prices, rather than price *levels*. Indeed, a trend implies some dependence structure in the time series of *returns*.[3]

Secondly, using some combination of past price levels (as is done when using MAs of prices) obscures the weighting scheme assigned to separate historical periods because price levels cumulate returns over different historical periods. A return, in contrast, is unambiguously linked to a specific time period. Using trend rules defined in terms of returns therefore allows one to acknowledge the differences in importance or weight given to historical time periods. This is important as there will be some degree of *information decay* rendering the more distant history less relevant than the more recent past. In addition, it is important to know whether some implied historical weights are positive or negative, thus allowing for the distinction between trend persistence and mean-reversion.

Because of this greater transparency, we favor trend indicators defined in terms of returns. The contribution of our paper is that we show how to uncover the weighting schemes implied by conventional price MAs, both in a theoretical and an empirical fashion.

The analysis of weighting schemes in terms of returns reveals surprising and useful information about trend rules. As a first example, we analyze trend rules that combine a short and

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[1] In fact, trend rules are the only “objectifiable” technical analysis rules that can be subjected to statistical testing (as compared to “fuzzy” head-and-shoulder patterns, support lines and other graphical patterns attributed to historical price moves). See for example Brock, Lakonishok, and LeBaron (1992), Lo and McKinlay [1999], or more recently Zakamulin [2014].

[2] An exception is Okunev and White [2002] who analyze trend rules in foreign exchange markets and use MAs defined in terms of returns.

[3] In addition, price level series are generally integrated of order one and hence have undesirable statistical properties. It is therefore not without reason that financial econometrics focuses on relative first differences in prices or differences in log prices (viz. returns).

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a long price MA. Except for the special case where the short MA has length 1 (so the current price is compared to a MA of past prices), such rules have a hump-shaped weighting scheme: the weight of the most recent returns start low and increase up to a maximum, whereafter the weights decline again. Combinations of multiple trend rules may even lead to multiple humps.

Secondly, we analyze trend rules with a skip period. With such rules, the long MA is calculated over the period starting when the short MA ends. We show that such trend rules are simply rescaled versions of trend rules without a skip period, but with a longer window over which the long MA is computed.

Thirdly, the weighting scheme of the popular (but complex) MACD rule turns out to have a hump-shaped information decay as well. Moreover, it has as much negative weight as positive weight. As negative weights imply a mean reversion rule (positive past returns imply negative signals), the MACD rule is in fact just as much a trend rule as it is a mean reversion rule.

All these phenomena are hidden by definitions in terms of prices, but revealed by the analysis of return weights. The increased understanding of these trend rules allows one to improve these trend rules. For example, using the return weights it is possible to separately analyze the trend and mean reversion parts of the MACD rule, and it allows one to assess the impact of the implied mean reversion in combinations of multiple MA rules.

Apart from uncovering these phenomena, we extend the theoretical analyses by showing how the underlying weighting scheme of an implemented trend strategy can be uncovered from historical data. We illustrate this by revealing the weighting scheme implied by a trend rule applied to the S&P 500 index. The practical implication of this analysis is that, given only trend returns in a single market, we can determine the weighting scheme that was used to construct the trend signals.

Price Moving Averages (MAs)

Consider a stock market index, a currency or some other index series with historical log price series $\{p_t\}$.[4] We measure time in periods of equal unit length (this can be days, weeks, months). At time t, a simple price MA based on N time periods then takes the form:

$$MA(N)_t = \frac{1}{N}\sum_{i=0}^{N-1} p_{t-i} \tag{1}$$

[4] We use log prices since a change in log prices does not depend on the initial price level. Hence, a change in log prices is equivalent to a return. After all, a change of 10 index points when the index is 100 is quite different from a 10 point change when the index is 1,000. In practice trend rules are sometimes defined on price levels and later scaled by dividing by an average price level. This does not materially affect the weighting scheme or the conclusions of this paper, but unnecessarily complicates the theoretical derivation.

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Hence, the time t price can be written as $MA(1)_t$.

Typical trend rules combine a MA over a short period with a MA over a longer period. A (M,N)-trend indicator then is computed as:

$$trend(M,N)_t = MA(M)_t - MA(N)_t \tag{2}$$

with $M < N$. A special case is $M = 1$, in which case the current price is compared to the MA of log prices over the past N periods.

When the short moving average increases above or drops below the long moving average, there is a cross-over. When the trend indicator switches from positive to negative, this is interpreted as a sign that the momentum of the market may have become negative. Shorting the market will be profitable when the prediction indeed becomes true. As an alternative to these *directional bets*, also *proportional bets* can be taken, in which the size of the position depends not only on the sign of the trend indicator but also on its specific value (strength of trend).[5]

Analyzing MAs defined in prices

When MAs or trend indicators are defined in terms of (log) price levels, the question arises what the underlying weighting scheme of past periods is. In order to uncover this weighting scheme, we express each past log price level as the difference between the current log price level and the sum of intermediate log price changes:

$$p_{t-i} = p_t - \sum_{j=0}^{i-1} r_{t-j} \tag{3}$$

where $r_t = p_t - p_{t-1}$ is the log return. When applying eq.(3) to each price level comprised in the $MA(N)$ of eq.(1), we obtain the scheme as depicted in Exhibit 1.

[5] For the expected return, standard deviation and Information Ratio of directional versus proportional market timing strategies, we refer to Hallerbach [2014].

Exhibit 1: The diagram of price changes for a simple moving average price *MA(N)*.

term 1	term 2	term 3	term 4	…	term N
t	t-1	t-2	t-3	…	t-N-1
p_t	p_{t-1}	p_{t-2}			p_{t+1-N}
	$= p_t - r_t$	$= p_t - r_t$	$= p_t - r_t$		$= p_t - r_t$
		$-r_{t-1}$	$-r_{t-1}$		$-r_{t-1}$
			$-r_{t-2}$		$-r_{t-2}$
					$\cdots$
					$-r_{t-N-2}$

Combining eqs.(1) and (3) is equivalent to summing all the terms in Exhibit 1 and dividing by *N*. This yields the expression:[6]

$$MA(N)_t = \frac{1}{N}\sum_{i=0}^{N-1} p_{t-i} = p_t - \sum_{i=1}^{N-1}\left(\tfrac{N-i}{N}\right) r_{t+1-i} \tag{4}$$

Eq.(4) shows that a *MA(N)* represents the difference between the current log price level and a specific weighted sum of past returns.

When using eq.(4) for a *MA(M)* and substituting this result in the definition of the *(M,N)*-trend indicator, we obtain:

$$\begin{aligned} trend(M,N)_t &= MA(M)_t - MA(N)_t \\ &= \left[p_t - \sum_{i=1}^{M-1}\left(\tfrac{M-i}{M}\right) r_{t+1-i}\right] - \left[p_t - \sum_{i=1}^{N-1}\left(\tfrac{N-i}{N}\right) r_{t+1-i}\right] \\ &= \sum_{i=1}^{N-1}\left(\tfrac{N-i}{N}\right) r_{t+1-i} - \sum_{i=1}^{M-1}\left(\tfrac{M-i}{M}\right) r_{t+1-i} \end{aligned} \tag{5}$$

We conclude that the MA in log prices has a specific linear weighting scheme in terms of returns:

$$trend(M,N)_t = \sum_{i=1}^{N-1} w_i r_{t+1-i} \tag{6}$$

where the weights are given by the following expression:

[6] An entirely analytic and less intuitive derivation follows from inserting (3) into (1) and interchanging the double sum.

$$w_i = \begin{cases} \frac{i}{M} - \frac{i}{N} & \text{for } 1 \le i \le M-1 \\ 1 - \frac{i}{N} & \text{for } M \le i \le N-1 \end{cases} \tag{7}$$

In practice such trend rules are often scaled by dividing by a long-term (e.g. 10 years) standard deviation, in order to combine them with different time series rules, be it trend or otherwise. When viewing this trend rule as a weighted sum of returns directly, it makes sense to normalize these weights rather than dividing by a standard deviation. By doing so, the trend rule becomes a weighted average of past returns. In this case, the weights become (see Appendix):

$$w_i = \begin{cases} \frac{2}{N-M}\left(\frac{i}{M} - \frac{i}{N}\right) & \text{for } 1 \le i \le M-1, \\ \frac{2}{N-M}\left(1 - \frac{i}{N}\right) & \text{for } M \le i \le N-1. \end{cases}$$

A special case is the (1,*N*)-trend indicator:

$$trend(1,N)_t = \sum_{i=1}^{N-1} \left(\tfrac{N-i}{N}\right) r_{t+1-i} = ½(N-1) \sum_{i=1}^{N-1} \left(2\,\tfrac{1-i/N}{N-1}\right) r_{t+1-i} \tag{8}$$

where the last equality shows the normalized weights of the past returns.

Analyzing weighting schemes

In this section, we analyze the weights of $trend(M,N)$-rules in more detail. We show that $trend(M,N)$-rules carry an implied belief in short-term mean-reversion if they are compared with $trend(1,N)$-rules.

In Exhibit 2, we display the weighting schemes of $trend(M,N)$-rules, as given by eq.(7). The $trend(1,N)$-rules display a linear declining weight, which implies a linear discounting of past returns. The $trend(M,N)$-rule, with $M > 1$, however, shows a surprising hump in the weighting scheme: returns of M periods ago have a higher weight than returns over the last period. Furthermore, from eq.(7), we see that:

$$trend(M,N) = trend(1,N) - trend(1,M) \tag{9}$$

so the difference between using a $trend(M,N)$-rule or a $trend(1,N)$-rule is given by $-trend(1,M)$. Trend rules with negative weights are in fact mean reversion rules (i.e., positive past returns imply negative future returns and vice versa). As a result, if one faces the choice between a $trend(M,N)$ and a $trend(1,N)$, choosing the $trend(M,N)$-rule implies belief in short-term mean reversion.

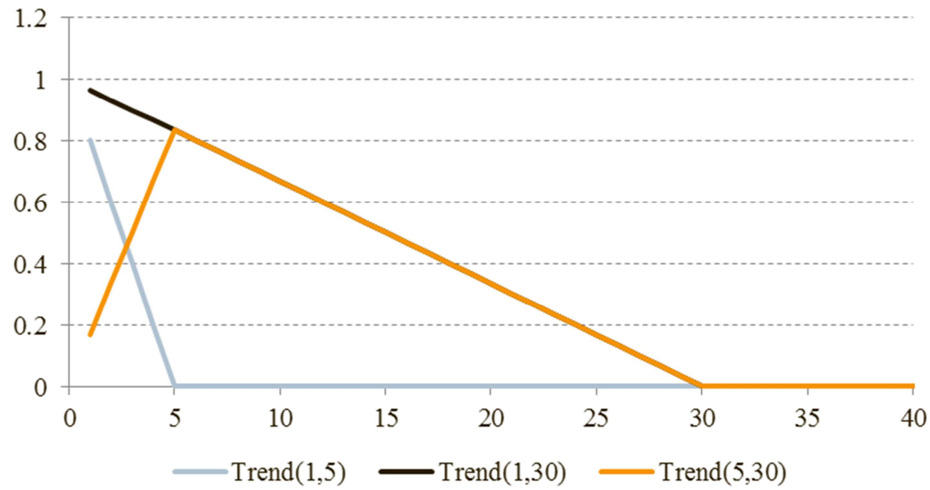

Exhibit 2: Weighting schemes of *trend*(*M*,*N*) rules.

Trends with a skip period

Because the short and long MA in a $trend(M, N)$-rule share an overlap of the first M periods, a skip period is sometimes used in the calculation of the trend rule. With a skip period, the MA of the previous M periods is compared with the MA of the N periods *before that*, i.e., from $N + M$ periods ago up to M periods ago. In this section, we show that adding a skip period is equivalent to prolonging the longest MA from N to $N + M$, apart from a scaling factor:

$$trend_{skip}(M, N)_t = \frac{M+N}{N} trend(M, N+M)_t \tag{9}$$

A trend rule with a skip period is defined as follows:

$$trend_{skip}(M, N)_t = \frac{1}{M}\sum_{i=0}^{M-1} p_{t-i} - \frac{1}{N}\sum_{i=0}^{N-1} p_{t-M-i} \tag{10}$$

For such rules, we can derive the weighting scheme similarly to how we derived eq.(7). We obtain:

$$trend_{skip}(M, N)_t = \sum_{i=1}^{M+N-1} w_i r_{t+1-i}\,, \tag{11}$$

with:

$$w_i = \begin{cases} \dfrac{i}{M} & \text{for } 1 \le i \le M-1 \\ \dfrac{N+M-i}{N} & \text{for } M \le i \le N+M-1 \end{cases} \tag{12}$$

For the first past M periods the weight increases linearly and after that it decreases linearly. Indeed, this weighting scheme shows remarkable resemblance to the hump-shaped weighting scheme of the $trend(M, N+M)$ -rule. We proceed by rewriting the weights of the $trend(M, N+M)$ -rule in eq.(7):

$$w_i = \begin{cases} \frac{i}{M} - \frac{i}{N+M} = \frac{N}{N+M}\frac{i}{M} & \text{for } 1 \le i \le M-1, \\ 1 - \frac{i}{N+M} = \frac{N}{N+M}\frac{N+M-i}{N} & \text{for } M \le i \le N+M-1, \end{cases} \tag{13}$$

which is indeed identical to (12), apart from the scaling factor.

Combining multiple trend rules

By combining multiple trend rules, an investor can benefit from diversification. Looking at the weighting schemes of the individual trend rules, however, reveals additional insights in such cases. After all, the weighting scheme of the average of multiple trend rules is simply the average of the weighting schemes. We show that this can lead to hump-shaped weighting schemes (cf. the $trend(M, N)$ -rule). If the different trend rules are suitably chosen, multiple humps might even occur. Rather than trying to obtain diversification through the combination of multiple trend rules which induce non-monotonic information decay, we propose to choose a weighting scheme *a priori*.

First, we combine the weighting scheme of two $trend(1, N)$ -rules. The resulting trend rule is the average of the two underlying trend rules. In practice, such trend rules are often combined after dividing by a long-term standard deviation (e.g. 10 years) to make their volatilities comparable. This, however, does not materially affect the form of the weighting scheme and hence our conclusions.

In Exhibit 3, we show the weighting scheme of the combination of a fast (1,40) and a slow (1,250) trend rule. The weighting scheme reveals a surprising inflection point at 40. Instead of such a discrete cut-off point, it might be more desirable to choose *a priori* a functional form that reflects a more gradual information decay, such as an exponential function.

Moreover, in Exhibit 4 we show the weighting scheme of a fast (5,40) and a slow (50,250) trend rule. As shown before, such trend rules have a hump-shaped weighting scheme and an implied belief in short-term mean reversion. Exhibit 4 clearly shows that a combination of such trend rules even has multiple humps. Although the combination of multiple trend rules seems perfectly reasonable for price series, an analysis of the weighting scheme reveals that the weight of returns increases between 1 and 5 periods ago, then decreases and increases again between 40 and 50 periods ago.

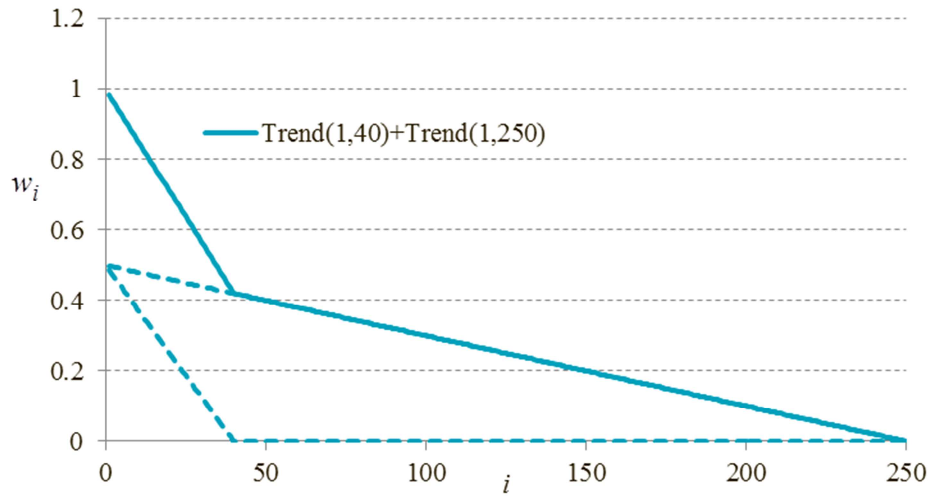

Exhibit 3: The weighting scheme of the average of a trend(1,40) and a trend(1,250) rule.

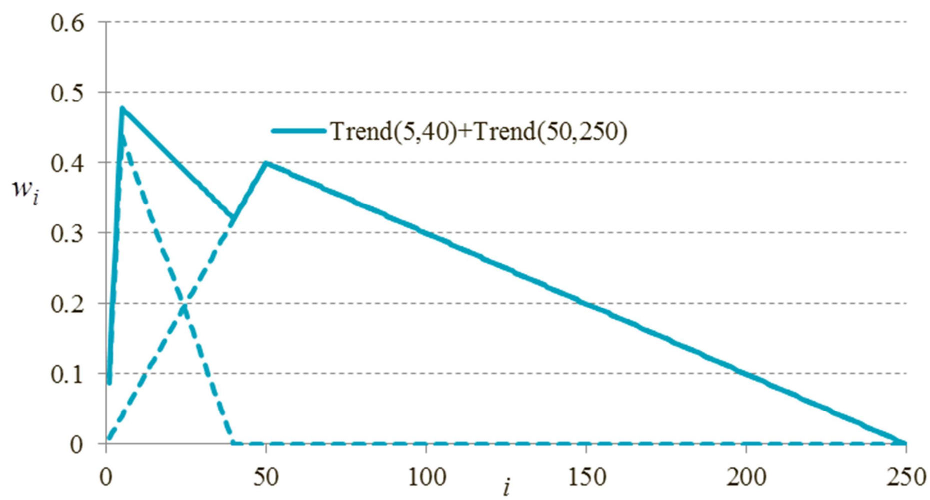

Exhibit 4: The weighting scheme of the average of a trend(5,40) and a trend(50,250) rule.

MACD

Another popular combination of trend rules is the MACD-rule (Moving Average Convergence Divergence), proposed by Appel [2005] in the 1970s. The MACD rule comprises a combination of three exponentially-weighted moving averages (EWMAs). In this section, we analyze the weighting scheme of the MACD rule in more detail. We show the functional form of the weighting scheme and we show that the MACD rule is just as much mean-reversion as it is trend: the sum of negative weights is equal to the sum of positive weights.

The MACD rule combines three EWMAs of a price series. First, a slow and a fast EWMA of a price series are computed, and next the MACD is defined as the difference between these EWMAs:

$$\mathrm{MACD}_t = \mathrm{EWMA}(p_t, \lambda_s) - \mathrm{EWMA}(p_t, \lambda_f) \tag{14}$$

where λ_s and λ_f are the persistence parameters of the slow and fast EWMA, respectively, satisfying $0 < \lambda_* \leq 1$. An N-period EWMA translates into a persistence of $\lambda = \frac{N-1}{N+1}$ (see Appel [2005]). Finally, the signal at time t, denoted by S_t, is an EWMA of the MACD (with a third persistence parameter λ), minus the MACD itself:

$$S_t = \mathrm{EWMA}(\mathrm{MACD}_t, \lambda) - \mathrm{MACD}_t \tag{15}$$

In the Appendix, we show that the signal of the MACD rule can also be written as a weighted sum of returns:

$$S_t = \sum_{i=1}^{t} r_{t+i-1} \left[(\lambda_s^i - \lambda_f^i) - (1-\lambda) \left(\frac{\lambda_s^i - \lambda^i}{1-\lambda/\lambda_s} - \frac{\lambda_f^i - \lambda^i}{1-\lambda/\lambda_f} \right) \right] =: \sum_{i=1}^{t} r_{t+i-1} w_i \tag{16}$$

These weights are plotted in Exhibit 5. Clearly, as in the case of multiple trend rules, more distant returns have higher weights than the most recent returns. While this may be perfectly desirable, this feature of the MACD rule is hidden by the definition in terms of EWMA rules, but revealed by its weighting scheme.

Furthermore, Exhibit 5 illustrates that at some point the weights become negative. In fact, in the Appendix we show that the sum of positive weights is equal to the sum of negative weights. As negative weights indicate mean reversion (positive past returns imply negative signals), the MACD rule – although typically presented as a trend rule – is just as much a mean reversion rule as it is a trend rule.

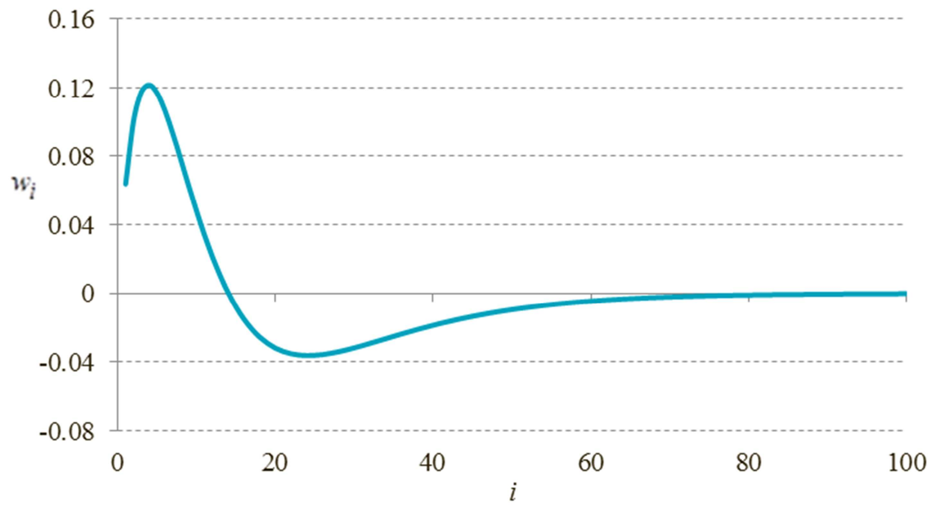

Exhibit 5: The weighting scheme of the MACD rule, with standard 26, 12, and 9-day parameters, i.e. $\lambda_s = 25/27 \approx 0.926$, $\lambda_f = 11/13 \approx 0.846$, and $\lambda = 8/10$ (see Appel [2005]). The sum of positive weights is equal to the sum of negative weights, so that the MACD rule is just as much a mean reversion rule as it is a trend rule (see Appendix).

Reverse Engineering

Besides revealing information about trend rules, expressing trend rules as weighted sums of returns also allows one to “reverse engineer” trend returns. That is, from the returns of a trend rule, we can – with large confidence – discover the trend rule that was used to generate the returns. The latter can even be done after a portfolio implementation rule is taken into account and after normalizing z-scores are taken, as is common in a multi-factor framework.[7] Our methodology is similar to the returns-based style analysis introduced by Sharpe [1988].

We illustrate this reverse engineering for an *MA*(5,45)-rule applied to daily S&P 500 data, obtained from Bloomberg, starting on 1-Jan-1950 and ending on 31-March-2014 (comprising 16,165 days). We take a z-score of the *MA*(5,45)-rule using the 1,300-day average and standard deviation. We further define a *signal* that is equal to 0 if the trend score is less than 0.3 in absolute value and equal to the sign of the score otherwise. The latter rule serves as a simple implementation rule that translates the scores into long, short, and neutral positions. Finally, we multiply the scores and the signals by the return of the next day to obtain the return series of the trend rule, $r_{score,t}$ and $r_{signal,t}$.

[7] The z-score of a score is computed as the initial score minus its average, divided by its standard deviation. The z-transformation makes scores from different sources mutually comparable and thus allows summing or averaging different scores into an overall score.

Given only these trend returns and the market returns, we uncover the underlying trend rule by running the following regression for scores as well as signals:

$$y_t = \alpha + \beta_1 r_t + \cdots + \beta_T r_{t+1-T} \tag{17}$$

Here r_t is the market return of day t and y_t is either the observed score $r_{score,t} / r_t$ or the observed signal $r_{signal,t} / r_t$ at time t.[8] The regression coefficients represent the empirical weights of the trend strategy.

We obtain estimates for the coefficients in eq.(17) through ordinary least squares regression with $T = 260$, for both the score and signal returns. The empirical weighting scheme is plotted in Exhibit 6, along with the theoretical weights of a *trend*(5,45)-rule. The empirical weighting scheme matches the theoretical weights quite closely, especially for the scores. For the signals we see some noise around the theoretical weights, but the empirical weights still match the theoretical weights quite closely.

The more non-linear the transformations applied to the trend scores, the less accurate the implied weights will be. The simple implementation rule with neutral positions used in this section is also a non-linear transformation and already reduces the R^2 of the regression from 94% for score returns to 67% for signal returns. More complex non-linear transformations will further reduce the R^2 and hence the confidence we have in the implied weights. Yet, even with the implementation rule used in this section, the underlying trend rule is convincingly identified. Applying *linear* transformations, in particular taking moving averages of moving averages, will not affect the accuracy of the implied weights since the regression itself is also linear.

[8] As this is a controlled experiment, we know that the returns of the trend series are a multiple of the returns of the market. In a non-controlled experiment, one would need to make sure that r_t is not close to 0, as the empirical weights would blow up.

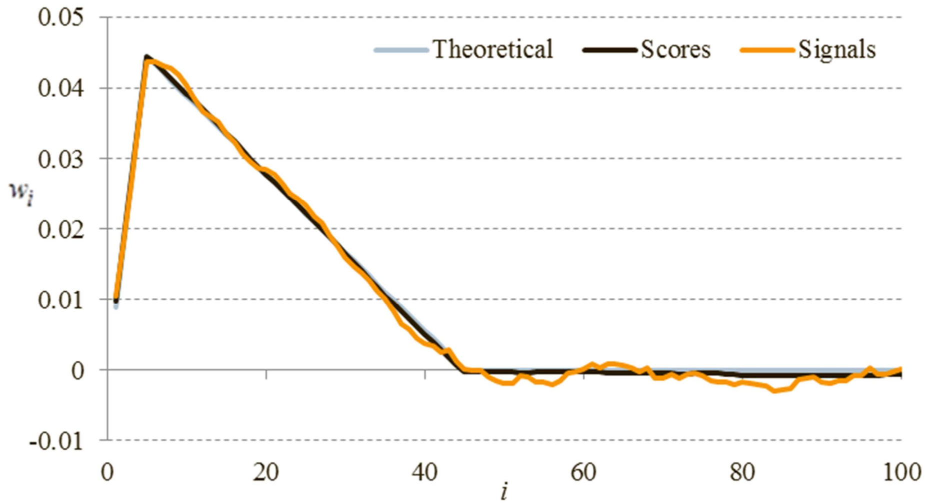

Exhibit 6: Implied weighting schemes of a *trend*(5,45) rule applied to daily S&P 500 data since Jan 1, 1950. The implied weighting schemes are obtained through eq. (17) with T = 260. "Score" refers to the implied weighting scheme of the trend score returns, "signal" to that of the trend signal returns, and "theoretical" to the theoretical weights of a *trend*(5,45) rule, cf. eq.(6). All weighting schemes are normalized so that the sum of all weights equals unity.

Summary and conclusions

Although it is general practice in trend analyses to use MAs in terms of price levels, we propose defining trend indicators in terms of returns. Since returns are unambiguously linked to separate time periods, trend rules formulated in terms of returns offer direct insight into the weight given to past time periods. This transparency allows for judging not only the degree of information decay implied by a trend rule, but also offers a better understanding of the implied patterns of trend persistence and mean-reversion. We showed how to uncover the weighting schemes implied by conventional price MAs (including the impact of a skip period) and we revealed the illusion of diversification resulting from combining different MAs.

We also analyzed the weighting scheme of the popular MACD trend rule, revealing that this rule is as much a trend rule as it is a mean reversion rule. Finally, we supplemented our theoretical analyses by showing how to empirically uncover the underlying weighting scheme of a trend strategy applied to a single market. Our results allow for evaluating the contribution of the implied trending and reverting patterns to the performance of a trend rule. In addition, the focus on return weights offers greater flexibility in specifying weighting schemes, opening the opportunity to improve a given trend rule by tuning its weights to autocorrelation patterns in the data. We leave this as a route for future research.

Appendix

In this Appendix, we express a *MA(N,M)* in terms of normalized weights and we derive the weighting scheme for the MACD trend rule.

Rewriting *MA(N,M)* in terms of normalized weights

Specifically, the sum of the return weights in eq.(7) is:

$$\begin{aligned}&\sum_{i=1}^{M-1}\left(\frac{i}{M}-\frac{i}{N}\right)+\sum_{i=M}^{N+M-1}\left(1-\frac{i}{N}\right)=\sum_{i=1}^{M-1}\frac{i}{M}+\sum_{i=M}^{N+M-1}1-\sum_{i=1}^{N-1}\frac{i}{N}\\&=\frac{1}{2}(M-1)+N-\frac{1}{2}(N-1)=\frac{1}{2}(M-N)\end{aligned} \tag{18}$$

Dividing the return weights in (7) by the expression above gives the normalized weights.

The return weighting scheme implied by the MACD trend rule

We now derive the return weighting scheme of the MACD rule. It comprises three exponentially weighted moving averages (EWMAs) of a price series and is computed in three steps.

In the first step, we calculate the difference between a fast and a slow price EWMA. We define the value of the slow EWMA at time t as $m_{s,t}$, and likewise the fast EWMA at time t as $m_{f,t}$. Their persistence parameters are λ_s and λ_f, respectively.[9] We assume the (log) price index is zero at time 0, so that:

$$p_t = r_1 + \ldots + r_t \tag{19}$$

For the slow EWMA, we have:

$$m_{s,t} = (1-\lambda_s)\sum_{k=0}^{t-1}\lambda_s^k p_{t-k} = (1-\lambda_s)\sum_{k=0}^{t-1}\lambda_s^k\sum_{u=1}^{t-k} r_u \tag{20}$$

Interchanging the double sum yields:

$$m_{s,t} = (1-\lambda_s)\sum_{u=1}^{t} r_u\sum_{k=0}^{t-u}\lambda_s^k = (1-\lambda_s)\sum_{u=1}^{t} r_u\frac{1-\lambda_s^{t-u+1}}{1-\lambda_s} = \sum_{u=1}^{t} r_u(1-\lambda_s^{t-u+1}) \tag{21}$$

Likewise, we find that $m_{f,t} = \sum_{u=1}^{t} r_u(1-\lambda_f^{t-u+1})$. The MACD at time t is defined as the difference between $m_{f,t}$ and $m_{s,t}$:

[9] Although the MACD algorithm does not prescribe whether to take prices or log prices, we use log prices for the ease of derivation. Using regular prices does not significantly affect the weights, but leads to slight inaccuracies as a result of multiplying returns rather than adding them.

$$\text{MACD}_t = \sum_{u=1}^{t} r_u (\lambda_s^{t-u+1} - \lambda_f^{t-u+1}) \tag{22}$$

In the second step, we calculate the EWMA of the MACD itself (with a third persistence parameter λ). The EWMA of the MACD is:

$$\text{EWMA}(m_{f,t} - m_{s,t}, \lambda) = \text{EWMA}(m_{f,t}, \lambda) - \text{EWMA}(m_{s,t}, \lambda) \tag{23}$$

so we proceed by rewriting $\text{EWMA}(m_{s,t}, \lambda)$:

$$\begin{aligned}
\text{EWMA}(\mathrm{m}_{s,t}, \lambda) &= (1-\lambda)\sum_{k=0}^{t-1} \lambda^k m_{s,t-k} = (1-\lambda)\sum_{k=0}^{t-1}\lambda^k \sum_{u=1}^{t-k} r_u (1-\lambda_s^{t-k-u+1}) \\
&= (1-\lambda)\sum_{u=1}^{t} r_u \left[\sum_{k=0}^{t-u} \lambda^k - \lambda_s^{t-u+1} \sum_{k=0}^{t-u} \frac{\lambda^k}{\lambda_s^k} \right] \\
&= (1-\lambda)\sum_{u=1}^{t} r_u \left[\frac{1-\lambda^{t-u+1}}{1-\lambda} - \lambda_s^{t-u+1} \frac{1-(\lambda/\lambda_s)^{t-u+1}}{1-\lambda/\lambda_s} \right] \\
&= (1-\lambda)\sum_{i=1}^{t} r_{t+i-1} \left[\frac{1-\lambda^i}{1-\lambda} - \frac{\lambda_s^i - \lambda^i}{1-\lambda/\lambda_s} \right]
\end{aligned} \tag{24}$$

Here, the last equality follows from substituting $i = t - u + 1$. Likewise, we have:

$$\text{EWMA}(\mathrm{m}_{f,t}, \lambda) = (1-\lambda)\sum_{i=1}^{t} r_{t+i-1} \left[\frac{1-\lambda^i}{1-\lambda} - \frac{\lambda_f^i - \lambda^i}{1-\lambda/\lambda_f} \right] \tag{25}$$

The EWMA of the MACD is thus given by:

$$\begin{aligned}
\text{EWMA}(\text{MACD}_t, \lambda) &= \text{EWMA}(m_{f,t}, \lambda) - \text{EWMA}(m_{s,t}, \lambda) \\
&= (1-\lambda)\sum_{i=1}^{t} r_{t+i-1} \left[\frac{\lambda_s^i - \lambda^i}{1-\lambda/\lambda_s} - \frac{\lambda_f^i - \lambda^i}{1-\lambda/\lambda_f} \right]
\end{aligned} \tag{26}$$

In the third and final step, we calculate the MACD signal at time t, denoted by S_t, as the difference between the MACD and the EWMA of the MACD:

$$\begin{aligned}
\mathrm{S}_t &= \text{MACD}_t - \text{EWMA}(\text{MACD}_t, \lambda) \\
&= \sum_{i=1}^{t} r_{t+i-1} \left[(\lambda_s^i - \lambda_f^i) - (1-\lambda)\left(\frac{\lambda_s^i - \lambda^i}{1-\lambda/\lambda_s} - \frac{\lambda_f^i - \lambda^i}{1-\lambda/\lambda_f} \right) \right] =: \sum_{i=1}^{t} r_{t+i-1} w_i
\end{aligned} \tag{26}$$

Using this weighting scheme, we can also show that the MACD rule incorporates as much mean reversion as it incorporates trend. In particular, the sum of positive weights is equal to the sum of negative weights. We prove this by showing that the total sum of weights is equal to 0. Evaluating all sums in eq.(26) and rewriting immediately yields:

$$
\begin{aligned}
\sum_{i=1}^{\infty} w_i &= \sum_{i=1}^{\infty}\left[(\lambda_s^i - \lambda_f^i) - (1-\lambda)\left(\frac{\lambda_s^i - \lambda^i}{1-\lambda/\lambda_s} - \frac{\lambda_f^i - \lambda^i}{1-\lambda/\lambda_f}\right)\right] \\
&= \frac{1}{1-\lambda_s} - \frac{1}{1-\lambda_f} - (1-\lambda)\left(\frac{\lambda_s}{\lambda_s-\lambda}\left(\frac{1}{1-\lambda_s} - \frac{1}{1-\lambda}\right) - \frac{\lambda_f}{\lambda_f-\lambda}\left(\frac{1}{1-\lambda_f} - \frac{1}{1-\lambda}\right)\right) \\
&= \frac{1}{1-\lambda_s} - \frac{1}{1-\lambda_f} - (1-\lambda)\left(\frac{\lambda_s}{(\lambda_s-\lambda)}\frac{\lambda_s-\lambda}{(1-\lambda_s)(1-\lambda)} - \frac{\lambda_f}{(\lambda_f-\lambda)}\frac{\lambda_f-\lambda}{(1-\lambda_f)(1-\lambda)}\right) \\
&= \frac{1}{1-\lambda_s} - \frac{1}{1-\lambda_f} - \frac{\lambda_s}{1-\lambda_s} + \frac{\lambda_f}{1-\lambda_f} = 0
\end{aligned}
\qquad (27)
$$

So the sums of positive and negative weights are indeed the same.

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