2015
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Consider the differential equation,

$$\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - 2y = 0. \tag{1}$$

Using $p = \frac{dy}{dx}$, obtain the general and singular solutions of Eq. (1).

II. Consider the normal of the curve $y = y(x)$ at an arbitrary point $\mathrm{P}(x, y(x))$. Let the point N be the intersection of the normal with the x-axis (see Fig. 1.1). Assume that the curvature radius R of the curve at the point P is the double of the length PN. The curvature radius R at the point P is given by Eq. (2):

$$R = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} \Bigg/ \left|\frac{d^2y}{dx^2}\right|. \tag{2}$$

Answer the following questions.

1. Show that $y(x)$ satisfies Eq. (3):

$$1 + \left(\frac{dy}{dx}\right)^2 = 2\left|y\frac{d^2y}{dx^2}\right|. \tag{3}$$

2. Solve Eq. (3) for $y\frac{d^2y}{dx^2} > 0$. Give the name of the kind of curve that the obtained equation represents.

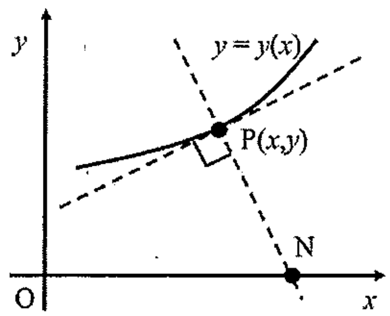

Fig.1.1

Problem 2

A quadratic form f of n real variables $x_i\ (i=1,2,\cdots,n)$ can be expressed as

$$f = x^{\mathrm{T}} A x \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{1}$$

where A is an $n \times n$ symmetric matrix, and x^{T} represents the transpose of vector x, i.e., a row vector consisting of $x_i\ (i=1,2,\cdots,n)$. The norm $\|x\|$ of vector x is defined as $\|x\| \equiv \sqrt{\sum_{i=1}^{n} x_i^2}$. Answer the following questions.

I. Obtain the 3×3 symmetric matrix A when the quadratic form of the following Eq. (2) is expressed with Eq. (1):

$$f = 6x_1^2 + 6x_2^2 + 5x_3^2 - 4x_2x_3 + 4x_3x_1 - 2x_1x_2. \tag{2}$$

II. Consider the general $n \times n$ symmetric matrix A. Let $\lambda_i\ (i=1,2,\cdots,n)$ be eigenvalues of matrix A, and $u_i\ (i=1,2,\cdots,n)$ be the corresponding eigenvectors of matrix A. When $\lambda_j \neq \lambda_k$, prove that u_j and u_k are orthogonal.

III. Consider an $n \times n$ square matrix $U = (u_1 u_2 \cdots u_n)$ that consists of n column eigenvectors $u_i\ (i=1,2,\cdots,n)$ of an $n \times n$ symmetric matrix A. Here, we assume that all u_i are mutually orthogonal and $\|u_i\| = 1$ for all u_i. Prove that $\|x\| = \|Ux\|$ is satisfied for any vector x.

IV. We define $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ as $y = Ux$ by using U in III. Express the quadratic form f of Eq. (1) using $y_i\ (i=1,2,\cdots,n)$ and $\lambda_i\ (i=1,2,\cdots,n)$ in II. Show also the deriving process.

V. Under the condition of $\|x\| = 1$, obtain the maximal value of the quadratic form f of Eq. (2) and find the corresponding $\boldsymbol{x}$.

Problem 3

For an arbitrary function $g(z)$, let us define the functions $g^+(x)$ and $g^-(x)$, respectively, as $g^+(x) \equiv \lim_{y\to+0} g(x+iy)$ and $g^-(x) \equiv \lim_{y\to-0} g(x+iy)$. Here, i is the imaginary unit, x and y are real numbers, and z is a complex number. For a continuous function $\varphi(x)$ in the interval $-1 \le x \le 1$, $f(z)$ is defined as:

$$f(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\varphi(x)}{x-z}\,dx. \tag{1}$$

Answer the following questions.

I. For the real interval $-1 < x < 1$, show that Eq. (2) is satisfied:

$$f^+(x) - f^-(x) = \varphi(x). \tag{2}$$

II. Prove that the function $X(z) = \sqrt{z^2-1}$ satisfies the following relation on the real axis:

$$\frac{X^+(x)}{X^-(x)} = \begin{cases} -1 & (|x| < 1) \\ 1 & (|x| > 1) \end{cases}. \tag{3}$$

Here, the branch cut of $X(z)$ is chosen along the real interval $-1 \le x \le 1$.

III. Derive $f(z)$ of Eq. (1) in the case of $\varphi(x) = X^+(x)$ with $X(z)$ defined in II.

Problem 4

Consider the three points, $\mathrm{P}(\cos\theta, \sin\theta, 1)$, $\mathrm{Q}(-\cos\theta, -\sin\theta, -1)$ and $\mathrm{R}(\cos 2\theta, \sin 2\theta, -1)$, expressed in terms of a real number θ in the *xyz* orthogonal coordinate system with the origin O. Answer the following questions.

I. Calculate the length of the line segment $\overline{\mathrm{PQ}}$.

II. Express the area S of the triangle PQR using θ.

III. Let the point M be the intersection point of the line segment $\overline{\mathrm{PR}}$ with the xy plane. Express the coordinates of the point M using θ.

IV. Answer the following questions, supposing that θ varies continuously from 0 to π.

1. Sketch the locus of the point M on the xy plane.
2. Calculate the area of the region swept by the line segment $\overline{\mathrm{OM}}$.

Problem 5

The Fourier transform $F(\omega)$ of a function $f(t)$ is defined as

$$F(\omega) \equiv \int_{-\infty}^{\infty} f(t)\exp(i\omega t)dt, \tag{1}$$

where i is the imaginary unit, and t and ω are real numbers. Answer the following questions.

I. Prove that $F(-\omega) = \overline{F(\omega)}$ for any real function $f(t)$. Here, $\overline{F(\omega)}$ denotes the complex conjugate of $F(\omega)$.

II. The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$(f * g)(t) \equiv \int_{-\infty}^{\infty} f(s)g(t-s)ds. \tag{2}$$

Here, s is a real number. Let $G(\omega)$ be the Fourier transform of $g(t)$. Express the Fourier transform of $(f * g)(t)$ using $F(\omega)$ and $G(\omega)$. Show also the deriving process.

III. Let us consider a real function $h(t)$ satisfying $h(t) = 0$ for $t < 0$ and its Fourier transform $H(\omega)$. Now we define an even function $r(t)$ and an odd function $x(t)$ as follows:

$$r(t) = \frac{h(t) + h(-t)}{2}, \tag{3}$$

$$x(t) = \frac{h(t) - h(-t)}{2}. \tag{4}$$

Using these two, $h(t)$ is expressed as $h(t) = r(t) + x(t)$. We use $R(\omega)$ and $X(\omega)$ to denote the Fourier transforms of $r(t)$ and $x(t)$, respectively. Answer the following questions.

1. Show that $R(\omega)$ is real for any ω.
2. Show that $X(\omega)$ is purely imaginary for any ω.
3. Let $h(t) = \begin{cases} \exp(-t) & (t \geq 0) \\ 0 & (t < 0) \end{cases}$ and calculate $R(\omega)$ and $X(\omega)$. Then, schematically draw each graph of the real part of $R(\omega)$ and the imaginary part of $X(\omega)$ as a function of ω.
4. Let $r(t) = \begin{cases} 1/4 & (|t| \leq 2) \\ 0 & (|t| > 2) \end{cases}$ and calculate $R(\omega)$ and $X(\omega)$. Then, schematically draw each graph of the real part of $R(\omega)$ and the imaginary part of $X(\omega)$ as a function of ω.

Problem 6

Consider the arrival time distributions of people coming to a restaurant. Each person comes one by one. Suppose that the n_0-th person arrives at the restaurant at the time t_0. Let $f_n(t)$ be the probability density of (n_0+n)-th person $(n \geq 1)$ arriving at the restaurant at the time (t_0+t). Here, $t > 0$. Assume that $f_1(t)$ is given by

$$f_1(t) = \lambda e^{-\lambda t}, \tag{1}$$

regardless of n_0 and t_0. Here, e is the base of the natural logarithm and λ is a positive constant. Answer the following questions.

I. Find the expectation value of the arrival time of the (n_0+1)-th person.

II. Show that $f_2(t) = \lambda^2 t e^{-\lambda t}$, $f_3(t) = \frac{1}{2}\lambda^3 t^2 e^{-\lambda t}$, and $f_4(t) = \frac{1}{6}\lambda^4 t^3 e^{-\lambda t}$.

III. Speculate the functional form of $f_n(t)$ from the results of II and prove it by mathematical induction.

IV. Let m be the number of the new persons coming to the restaurant from t_0 to t_0+T $(T>0)$, i.e., in the time interval (t_0, t_0+T). The probability distribution of m, $h(m,T)$, obeys the Poisson distribution:

$$h(m,T) = \frac{1}{m!}(\lambda T)^m e^{-\lambda T}. \tag{2}$$

Calculate the expectation value of m. Here, the formula $e^{\alpha} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}$ may be used.

2016
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.

2. Notify your proctor if you find any printing or production errors.

3. Answer three problems out of the six problems in the problem booklet.

4. You are given three answer sheets. Use one answer sheet for each problem. You may use the reverse side if necessary.

5. Print your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with a pair of scissors.

6. You may use the blank sheets of the problem booklet as working space and for draft solutions, but you must not detach them.

7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.

8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the general solution of the following differential equation:

$$\frac{d^4y}{dx^4}-2\frac{d^3y}{dx^3}+2\frac{dy}{dx}-y=9e^{-2x}. \tag{1}$$

Here, e denotes the base of the natural logarithm.

II. Find the value of the following integral:

$$\int_0^1 x^m (\log x)^n \, dx. \tag{2}$$

Here, m and n are non-negative integers.

III. We define $I(m)$ as

$$I(m) \equiv \int_0^1 x^m \arccos x \, dx. \tag{3}$$

Here, m is a non-negative integer. Use the principal values of inverse trigonometric functions.

1. Find the value of $I(0)$.
2. Find the value of $I(1)$.
3. Express $I(m)$ in terms of m and $I(m-2)$ when $m \geq 2$.
4. Find the value of $I(m)$.

Problem 2

Consider the column vectors $\mathbf{a}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{a}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

I. When $\mathbf{A} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$, obtain the three-dimensional column vector $\mathbf{x}$ which meets

$$\mathbf{Ax} - \mathbf{b} = \mathbf{0}. \tag{1}$$

II. Any $m \times n$ real matrix $\mathbf{B}$ is expressed using orthonormal matrices $\mathbf{U}(m \times m)$ and $\mathbf{V}(n \times n)$ as

$$\mathbf{B} = \mathbf{U\Sigma V}^T,\quad \Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & \sigma_r & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},\quad r = \mathrm{rank}(\mathbf{B}). \tag{2}$$

$\sigma_1, \sigma_2, \cdots, \sigma_r$ are positive real numbers, and they are called singular values of $\mathbf{B}$. $\mathbf{P}^T$ means the transposed matrix of a matrix $\mathbf{P}$. Then, express $\mathbf{BB}^T$ and $\mathbf{B}^T\mathbf{B}$ using matrices $\mathbf{U}$, $\mathbf{V}$, Σ and their transposed matrices, respectively.

Let $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2)$ in the following questions.

III. Find the eigenvalues and corresponding eigenvectors for $\mathbf{BB}^T$.

IV. Find singular values of $\mathbf{B}$ and orthonormal matrices $\mathbf{U}$ and $\mathbf{V}$ used in Equation (2).

V. Find the two-dimensional column vector $\mathbf{x}$ which minimizes the norm

$$\|\mathbf{Bx} - \mathbf{b}\|^2 = (\mathbf{Bx} - \mathbf{b})^T(\mathbf{Bx} - \mathbf{b}). \tag{3}$$

Problem 3

Consider a mapping $w = f(z)$ of a domain D on the complex z plane to a domain Δ on the complex w plane. Points on the complex z and w planes correspond to complex numbers $z = x + iy$ and $w = u + iv$, respectively. Here, x, y, u and v are real numbers, and i is the imaginary unit.

I. Let $w = \sin z$.

1. Express u and v as functions of x and y, respectively.
2. Suppose the domain $D_1 = \{(x, y) \mid 0 \le x \le \frac{\pi}{2}, y \ge 0\}$ on the z plane is transformed to a domain on the w plane. Show the transformed domain on the w plane by drawing the transformed images corresponding to the three half-lines: $x = 0$, $x = \frac{\pi}{2}$ and $x = c$ at $y \ge 0$ on the z plane. Here, c is a real constant on $0 < c < \frac{\pi}{2}$.

II. If a real function $g(x, y)$ has continuous first and second partial derivatives and satisfies Laplace's equation $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0$ in a domain Ω on a plane, $g(x, y)$ is said to be harmonic in Ω.

Suppose that a function $f(z) = u(x, y) + iv(x, y)$ is holomorphic in D on the z plane:

1. Show both $u(x, y)$ and $v(x, y)$ are harmonic in D on the z plane.
2. Suppose a function $h(u, v)$ is harmonic in Δ on the w plane, show a function $H(x, y) = h(u(x, y), v(x, y))$ is harmonic in D on the z plane.

III. Suppose a function $h(u, v)$ is harmonic in the domain $\Delta_1 = \{(u, v) \mid u \ge 0, v \ge 0\}$ on the w plane and satisfies the following boundary conditions:

$$h(0, v) = 0 \quad (v \ge 0), \tag{1}$$

$$h(u, 0) = 1 \quad (u \ge 1), \tag{2}$$

$$\frac{\partial h}{\partial v}(u, 0) = 0 \quad (0 \le u \le 1). \tag{3}$$

1. Let $z = \arcsin w$ and $H(x, y) = h(u, v)$. Find the boundary conditions for $H(x, y)$ corresponding to Equations (1), (2) and (3). Use the principal values of inverse trigonometric functions.
2. Find the function $H(x, y)$ which satisfies the boundary conditions obtained in Question III.1.
3. Find $h(u, 0)$ on the interval $0 \le u \le 1$.

Problem 4

In a three-dimensional Cartesian coordinate system xyz, consider the positional relationship among three planes defined by Equations (1)–(3), and the positional relationship among the three planes and a sphere defined by Equation (4).

$$a_{11}x + a_{12}y + a_{13}z = b_1, \tag{1}$$
$$a_{21}x + a_{22}y + a_{23}z = b_2, \tag{2}$$
$$a_{31}x + a_{32}y + a_{33}z = b_3, \tag{3}$$
$$x^2 + y^2 + z^2 = 3, \tag{4}$$

where a_{ij} and b_i $(i, j = 1, 2, 3)$ are constants.

For the three planes, let $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be the coefficient matrix and $\mathbf{B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$ be the augmented coefficient matrix.

I. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -c \end{pmatrix}$ where c is a positive constant.

1. Find $\text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{B})$.
2. Among the three planes, the plane that is tangential to the sphere defined by Equation (4) at a point P (1, 1, 1) is called Plane 1. Between the other two planes, the plane with the shorter distance to P is called Plane 2. Find the distance between P and Plane 2. Then, find the volume of the part of the sphere existing between Planes 1 and 2.

II. When the three planes intersect in a line, find $\text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{B})$.

III. Suppose that the three planes are tangential to the sphere at three different points. Illustrate all possible positional relationships among the three planes and the sphere. In addition, for each relationship, find $\text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{B})$.

Problem 5

I. A function $f(x)$ is continuous and defined on the interval $0 \leq x \leq \pi$. If $f(x)$ is extended to the interval $-\pi \leq x \leq \pi$ as an odd function, it can be expanded in the following Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} (b_n \sin nx), \tag{1}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, 3, \cdots). \tag{2}$$

Here, $f(0) = f(\pi) = 0$.

1. Find the Fourier sine series for the following function $f(x)$:

$$f(x) = x(\pi - x) \quad (0 \leq x \leq \pi). \tag{3}$$

2. Derive the following equation using the result obtained in Question I.1,

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}. \tag{4}$$

II. A two-variable function $f(x, y)$ is continuous and defined in the region $0 \leq x \leq \pi$ and $0 \leq y \leq \pi$. Using a similar method to Question I, $f(x, y)$ can be expanded in the following double Fourier sine series:

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn} \sin mx \sin ny), \tag{5}$$

$$B_{mn} = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x, y) \sin mx \sin ny \, dx dy \quad (m, n = 1, 2, 3, \cdots). \tag{6}$$

Here, $f(0, y) = f(\pi, y) = f(x, 0) = f(x, \pi) = 0$.

1. Find the double Fourier sine series for the following function $f(x, y)$:

$$f(x, y) = x(\pi - x) \sin y \quad (0 \leq x \leq \pi, 0 \leq y \leq \pi). \tag{7}$$

2. Function $u(x,y,t)$ is defined in the region $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$. Obtain the solution for the following partial differential equation of $u(x,y,t)$ by the method of separation of variables:

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{8}$$

where c is a positive constant and the following boundary and initial conditions apply:

$$u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \tag{9}$$
$$u(x,y,0) = x(\pi - x)\sin y. \tag{10}$$

Problem 6

Company A owns multiple factories $i\ (i=1,2,\cdots)$. Suppose that the probability of producing defective goods in a factory i is P_i, and that N_i goods are randomly sampled and shipped from the factory. Here, P_i is sufficiently small, and each factory does not affect any other.

I. Show the probability $f(i,k)$, which is the probability of k defective goods existing within N_i goods shipped from a factory i. Here, k is a non-negative integer.

II. Show that $f(i,k) \to \dfrac{e^{-\lambda_i}\lambda_i^k}{k!}$ when $N_i \to \infty$. Here, when calculating the limit of $f(i,k)$, λ_i is a constant, where $\lambda_i = N_i P_i$.

In the following questions, assume that $f(i,k) = \dfrac{e^{-\lambda_i}\lambda_i^k}{k!}$.

III. Suppose that goods are shipped from two factories as shown in Table 1. Find the probability of two defective goods being contained within all shipped goods.

Table 1

Factory number (i)	Probability of defectiveness (P_i)	Number of shipped goods (N_i)
1	0.01	500
2	0.02	300

IV. Find the probability of k defective goods being contained within all shipped goods under the same conditions as in Question III.

V. Suppose that $P_i = 0.001\,i$ in five factories $i\ (i=1,2,3,4,5)$ and the same number (N_c) of goods are shipped from all these factories.

Find the maximum value of N_c for which the expected number of defective goods out of all shipped goods is equal to or less than 3.

2017
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of the answer sheet represent the problem number you answer (P 1, P 2,..., P 6) on that sheet and also the class of the master's course (M) or doctoral course (D) applicants. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks on each sheet with scissors.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the value of the following definite integral:

$$I = \int_2^4 \frac{dx}{\sqrt{(x-2)(4-x)}}. \tag{1}$$

II. Find the general solution and the singular solution of the following differential equation:

$$y = x\frac{dy}{dx} + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2. \tag{2}$$

III. Find the general solution of the following differential equation:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 8y = x^2. \tag{3}$$

Problem 2

Answer the following questions about the square matrix A of order 3:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & -1 \\ -2 & -2 & 1 \end{pmatrix}. \tag{1}$$

I. Find all eigenvalues of A.

II. Find the matrix A^n, where n is a natural number.

III. The square matrix B of order 3 is diagonalizable and meets $AB = BA$. Prove that any eigenvector p of A is also an eigenvector of B.

IV. Find the square matrix B of order 3 that meets $B^2 = A$, where B is diagonalizable and all eigenvalues of B are positive.

V. The square matrix X of order 3 is diagonalizable and meets $AX = XA$. When $\mathrm{tr}(AX) = d$, find the maximum of $\det(AX)$ as a function of d.

Here, d is positive real and all eigenvalues of X are positive. In addition, $\mathrm{tr}(M)$ is the trace (the sum of the main diagonal elements) of the square matrix M, and $\det(M)$ is the determinant of the matrix M.

Problem 3

Answer the following questions. Here, i, e, and $\log$ denote the imaginary unit, the base of the natural logarithm, and the natural logarithm, respectively.

I. Consider the definite integral I expressed as

$$I = \int_0^{2\pi} \frac{\cos\theta \, d\theta}{(2+\cos\theta)^2}. \tag{1}$$

1. Find a complex function $G(z)$ of a complex variable z when we rewrite I as an integral of a complex function as

$$\oint_{|z|=1} G(z) dz, \tag{2}$$

where the integration path is a unit circle in the counter clockwise direction.

2. Find all poles and the respective orders and residues.
3. Evaluate the integral I.

II. Let a function of a real variable θ with real parameters α and β be

$$f(\theta;\alpha,\beta) = 1 + e^{2i\beta} + \alpha e^{i(\theta+\beta)}. \tag{3}$$

Consider the definite integral

$$F(\alpha,\beta) = \int_0^{2\pi} d\theta \frac{d}{d\theta}[\log f(\theta;\alpha,\beta)]. \tag{4}$$

1. Find a complex function $G(z)$ of a complex variable z when we rewrite $F(\alpha,\beta)$ as an integral of a complex function as

$$\oint_{|z|=1} G(z) dz, \tag{5}$$

where the integration path is a unit circle in the counter clockwise direction.

2. Find all poles and the respective orders and residues.
3. Evaluate $F(\alpha,\beta)$ by classifying cases with respect to α and β. Ignore the case in which the integration path passes through any poles.

Problem 4

For the real numbers θ and α within the regions $0 \leq \theta < 2\pi$ and $0 \leq \alpha \leq \pi$, consider the line L that passes through two points: point P($\cos\theta$, $\sin\theta$, 1) and point Q($\cos(\theta+\alpha)$, $\sin(\theta+\alpha)$, -1) in a three-dimensional Cartesian coordinate system xyz.

I. Represent the line L as a linear function of a parameter t. Here, the point on the line L at $t=0$ should represent the point Q and the point at $t=1$ should represent the point P.

II. Find the surface S swept by the line L as an equation of x, y and z when θ varies in the region $0 \leq \theta < 2\pi$. Let C be the intersection lines of the surface S with the plane $y=0$. Find the equation of C in terms of x and z, and sketch the shape of C.

Next, examine the Gaussian curvature of the surface S. Generally, when the position vector $\boldsymbol{r}$ of a point R on a curved surface is represented using parameters u and v by

$$\boldsymbol{r}(u,v) = (x(u,v), y(u,v), z(u,v)), \tag{1}$$

the Gaussian curvature K is represented as the following equation:

$$K = \frac{(\boldsymbol{r}_{uu}\cdot\boldsymbol{e})(\boldsymbol{r}_{vv}\cdot\boldsymbol{e}) - (\boldsymbol{r}_{uv}\cdot\boldsymbol{e})^2}{(\boldsymbol{r}_u\cdot\boldsymbol{r}_u)(\boldsymbol{r}_v\cdot\boldsymbol{r}_v) - (\boldsymbol{r}_u\cdot\boldsymbol{r}_v)^2}, \tag{2}$$

where $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are first-order partial differentials of $\boldsymbol{r}(u,v)$ with respect to the parameters u and v, and $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$ and $\boldsymbol{r}_{vv}$ are second-order partial differentials of $\boldsymbol{r}(u,v)$ with respect to the parameters u and v. $(\boldsymbol{a}\cdot\boldsymbol{b})$ represents the inner product of two three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and $\boldsymbol{e}$ is the unit vector of the normal direction at the point R.

III. Let the point W be the intersection of the surface S and the x axis in the region $x > 0$. Calculate the Gaussian curvature of S at the point W for α within the region $0 \leq \alpha < \pi$.

IV. For α within the region $0 \leq \alpha < \pi$, prove that the Gaussian curvature is less than or equal to 0 at arbitrary points on the surface S.

Problem 5

The Laplace transform $F(s) = L[f(t)]$ of a function $f(t)$, where $t \geq 0$, is defined as

$$F(s) = \int_0^\infty f(t) e^{-st} dt . \tag{1}$$

Here, s is a complex number, and e is the base of the natural logarithm. Answer the following questions. Show the derivation process with your answer.

I. Prove the following relations:

1. $L\left[t^n\right] = \dfrac{n!}{s^{n+1}}$, where n is a natural number.
2. $L\left[\dfrac{df(t)}{dt}\right] = sF(s) - f(0)$, where $f(t)$ is a differentiable function.
3. $L[e^{at} f(t)] = F(s-a)$, where a is a real number.

II. Solve the following differential equation using a Laplace transformation for $t \geq 0$:

$$t\frac{d^2 f(t)}{dt^2} + (1+3t)\frac{df(t)}{dt} + 3f(t) = 0, \quad f(0) = 1, \quad \left.\frac{df}{dt}\right|_{t=0} = -3 . \tag{2}$$

You can use the relation $L[tf(t)] = -\dfrac{d}{ds}F(s)$, if necessary.

III. The point $\mathrm{P}(x(t), y(t))$, which satisfies the following simultaneous differential equations, passes through the point (a, b) when $t = 0$. a and b are real numbers.

$$\begin{cases} \dfrac{dx(t)}{dt} = -x(t) \\ \dfrac{dy(t)}{dt} = x(t) - 2y(t) . \end{cases} \tag{3}$$

1. Solve Equation (3) using a Laplace transformation for $t \geq 0$.

2. Express the relation between x and y by eliminating t from the solution of III. 1.
3. For both $(a,b)=(1,1)$ and $(-1,1)$, draw the trajectories of point P when t varies continuously from 0 to infinity.

Problem 6

A product factory manufactures 2 types of products: *product-I* and *product-II*. *Part-A* is necessary for *product-I*, and both *part-A* and *part-B* are necessary for *product-II*. There are parts that have standard quality and parts that do not have standard quality among *part-A* and *part-B*. All parts are delivered from the part factory to the product factory, but there is no quality check of any part. The qualities of *part-A* and *part-B* are independent, and they will not affect each other. The probabilities that *part-A* and *part-B* have standard quality are a and b, respectively.

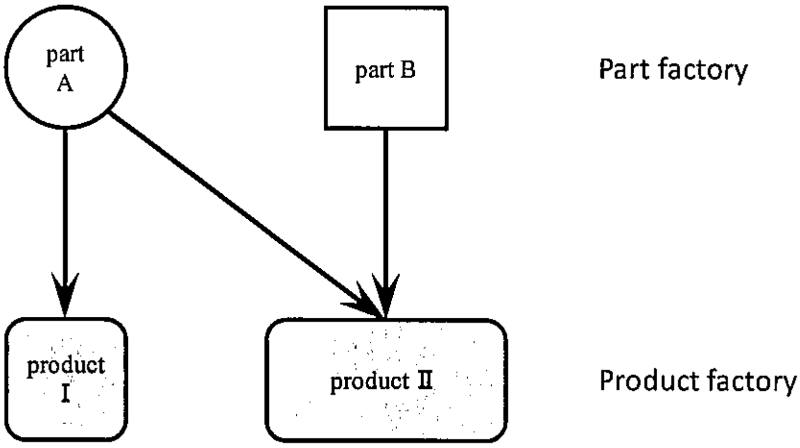

Figure 6.1

A final quality inspection is made in the product factory for *product-I* and for *product-II* before shipment. The inspection judges whether the quality of each product meets the standard or not. The inspections will not affect each other. The product inspection is not perfect: namely, products that have standard quality pass the product inspection as acceptable with the probability x. The products that do not have standard quality pass the product inspection as acceptable with the probability y.

Answer the following questions:

I. A *product-I* is randomly sampled and inspected once. Here, the probability that *product-I* can be manufactured with standard quality is defined as follows:

- The probability that *product-I* has standard quality is c if *part-A* has standard quality.
- *Product-I* will never have standard quality if *part-A* does not have standard quality.

1. Show the probability that the selected *product-I* passes the product inspection as acceptable.

2. Show the probability that the selected *product-I* actually has standard quality after it has passed the product inspection as acceptable.

II. A *product-II* is randomly sampled and inspected n times. Here, the probability that *product-II* can be manufactured with standard quality is defined as follows:

- The probability that *product-II* has standard quality is c if both *part-A* and *part-B* have standard quality.
- The probability that *product-II* has standard quality is d if only either *part-A* or *part-B* has standard quality.
- *Product-II* will never have standard quality if both *part-A* and *part-B* do not have standard quality.

1. Show the probability that the selected *product-II* has standard quality.
2. Show the probability that the selected *product-II* actually has standard quality after it has passed all product inspections (i. e., n times) as acceptable.

2018
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Find the general solutions of the following differential equations.

1. $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}-3y=e^x\cos x \tag{1}$$

2. $$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{4}{x^2}y=\left(\frac{2\log x}{x}\right)^2 \tag{2}$$

II. Answer the following questions for the partial differential equation represented in Equation (3) and the boundary conditions represented in Equations (4)-(7):

$$\frac{\partial^2u(x,y)}{\partial x^2}+\frac{\partial^2u(x,y)}{\partial y^2}=0 \qquad (0\le x,\ 0\le y\le 1), \tag{3}$$

$$\begin{cases} \lim\limits_{x\to+\infty} u(x,y)=0 & (4)\\ \left.\dfrac{\partial u(x,y)}{\partial y}\right|_{y=0}=0 & (5)\\ u(x,1)=0 & (6)\\ \left.\dfrac{\partial u(x,y)}{\partial x}\right|_{x=0}=1+\cos\pi y. & (7) \end{cases}$$

1. Find the solution which satisfies Equations (3) and (4) in the form of $u(x,y)=X(x)\cdot Y(y)$.
2. Find the solution satisfying Equations (5) and (6) for the solution of Question II.1.
3. Find the solution of the partial differential equation (3) satisfying all the boundary conditions given in Equations (4)-(7), using the solution of Question II.2.

Problem 2

I. Suppose that λ is an eigenvalue of a regular matrix $\boldsymbol{P}$, prove that:

1. λ is not zero.
2. λ^{-1} is an eigenvalue of $\boldsymbol{P}^{-1}$ and λ^{n} is an eigenvalue of $\boldsymbol{P}^{n}$, where n is a positive integer.

II. Suppose $\boldsymbol{P}$ is an orthogonal matrix. When the following symmetric matrix $\boldsymbol{A}$ can be diagonalized by $\boldsymbol{P}$, find the matrix $\boldsymbol{P}$ and obtain the diagonalized matrix.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

III. When a matrix $\boldsymbol{P}$, and vectors $\boldsymbol{r}$ and $\boldsymbol{x}$ are given as

$$\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ p & p^2 & p^3 \\ q & q^2 & q^3 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} r \\ r^2 \\ r^3 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where p, q, and r are non-zero real numbers that differ from each other.

1. Find the condition that p and q must satisfy in order for $\boldsymbol{P}$ to be a regular matrix.
2. When $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{r}$ has a single solution, obtain $\boldsymbol{x}$. Here, $\boldsymbol{P}^{\mathrm{T}}$ is the transposed matrix of $\boldsymbol{P}$.

IV. The matrix $\boldsymbol{P}_n$ is an n-th order square matrix ($n \geq 2$), as shown below, where p and q are real numbers that differ from each other.

$$
\boldsymbol{P}_n = \begin{pmatrix} p+q & q & 0 & \cdots & 0 & 0 \\ p & p+q & \ddots & \ddots & \vdots & \vdots \\ 0 & p & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & q & 0 \\ \vdots & \vdots & \ddots & \ddots & p+q & q \\ 0 & 0 & \cdots & 0 & p & p+q \end{pmatrix}
$$

1. Obtain the recurrence formula satisfied by the determinant of $\boldsymbol{P}_n$, $|\boldsymbol{P}_n|$.
2. Express the determinant $|\boldsymbol{P}_n|$ in terms of p, q, and n, using the recurrence formula in Question IV.1.

Problem 3

Answer the following questions concerning complex functions defined over the z-plane ($z = x + iy$), where i denotes the imaginary unit.

I. For the function $f(z) = \dfrac{z}{(z^2+1)(z-1-ia)}$, where a is a positive real number:

1. Find all the poles and respective residues of $f(z)$.
2. Using the residue theorem, calculate the definite integral

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x-1-ia)}\,dx. \tag{1}$$

II. Consider the function $g(z) = \dfrac{z}{(z^2+1)(z-1)}$ and the closed counter-clockwise path of integration C, which consists of the upper half circle C_1 with radius R ($z = R\,e^{i\theta}$, $0 \le \theta \le \pi$), the line segment C_2 on the real axis ($z = x$, $-R \le x \le 1-r$), the lower half circle C_3 with its center at $z = 1$ ($z = 1 - r\,e^{i\theta}$, $0 \le \theta \le \pi$), and the line segment C_4 on the real axis ($z = x$, $1 + r \le x \le R$). Here, e denotes the base of the natural logarithm, and let $r > 0$, $r \ne \sqrt{2}$ and $R > 1 + r$. An example for the case $0 < r < 1$ is illustrated in Figure 3.1.

Answer the following questions.

1. Calculate the integral $\int_C g(z)\,dz$.
2. Using the result from Question II.1, calculate the following value

$$\lim_{\varepsilon \to +0}\left[\int_{-\infty}^{1-\varepsilon} g(x)\,dx + \int_{1+\varepsilon}^{\infty} g(x)\,dx\right]. \tag{2}$$

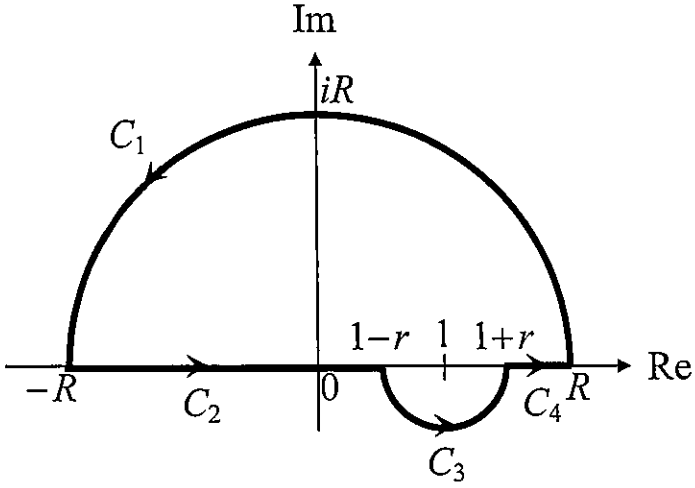

Figure 3.1

Problem 4

I. Consider surfaces presented by the following sets of equations, with parameters u and v in a three-dimensional orthogonal coordinate system xyz. Show the equations for the surfaces without the parameters and sketch them. Here, a, b, and c are non-zero real constants.

1. $x = au\cosh v, \quad y = bu\sinh v, \quad z = u^2.$
2. $x = a\dfrac{u-v}{u+v}, \quad y = b\dfrac{uv+1}{u+v}, \quad z = c\dfrac{uv-1}{u+v}.$

II. In a three-dimensional orthogonal coordinate system xyz, consider the surface S represented by the following equation, where a and b are real constants.

$$z = x^2 - 2y^2 + ax + by \tag{1}$$

1. Determine the normal vector at a point (x, y, z) on the surface S.
2. Determine the equation for the surface T which is obtained by rotating the surface S around the z-axis by $\pi/4$. Here, the positive direction of rotation is counter-clockwise when looking at the origin from the positive side of the z-axis.
3. Consider the surface S', which is the portion of the surface S in $-1 \le x \le 1$ and $-1 \le y \le 1$. Determine the area of the projection of the surface S' onto the yz plane.
4. Calculate the length of the perimeter for the surface S' when $a = b = 0$.
5. Calculate the Gaussian curvature of the surface S at the point $\left(0, \dfrac{1}{4}, -\dfrac{1}{8}\right)$ when $a = b = 0$.

Problem 5

Let $f(t)$ be a periodic function of period T, $f(t+T)=f(t)$ ($T>0$), and be expanded in the complex Fourier series as follows:

$$f(t)=\sum_{n=-\infty}^{\infty} F_n \exp(-i\,\omega_n t). \tag{1}$$

Here, i is the imaginary unit, and t is a real number. Answer the following questions.

I. Express ω_n using T and n.

II. Let M be a positive-integer constant and $\delta(t)$ be the delta function, and define

$$\hat{f}(t)=\sum_{m=0}^{M-1} f(t)\,\delta(t-m\Delta t), \qquad \Delta t=\frac{T}{M}. \tag{2}$$

Express

$$\lim_{\varepsilon\to+0}\int_{-\varepsilon}^{T-\varepsilon} \hat{f}(t)\exp(i\,\omega_k t)\,dt \tag{3}$$

using F_n ($n=-\infty,\cdots,-1,0,1,\cdots,\infty$). Here k is an arbitrary integer.

III. Δt is given in Question II. Express F_j ($j=0,1,2,\cdots,M-1$) using

$$f(0),\ f(\Delta t),\ f(2\Delta t),\cdots,\ f((M-1)\Delta t),$$

when $F_n=0$ ($n<0$ or $n\geq M$).

IV. Calculate F_j in Question III when $f(l\Delta t)=(-1)^l$ ($l=0,1,2,\cdots,M-1$).

Problem 6

There are n children queuing in a line. You have m candies and will begin handing out 1 or 2 candies to each child, starting from the first child in the line. You hand out the candies until reaching the end of the line or until there are no candies left. Answer the following questions. Note that n and m are positive integers.

I. Show the number of distribution patterns of candies if $n = m = 4$.

II. Show the number of distribution patterns of candies if $m \geq 2n$.

III. Define X_m as the number of distribution patterns of candies if $n \geq m$. Show the recurrence formula satisfied by X_m.

IV. Obtain X_m using the recurrence formula in Question III.

V. Consider the situation where the number of children is larger than the number of candies. Define $P(m)$ as the ratio of the number of distribution patterns (where the distribution finishes by giving 2 candies) to the total number of distribution patterns. $P(m)$ converges as m increases. Compute the convergence value.

VI. Consider the situation where $m \geq 2n$. The following rules are added to the way of handing out the candies: For the first child in the line, the probability of receiving 1 candy is $1/2$ and the probability of receiving 2 candies is $1/2$. If a child receives 1 candy, the probability of the next child receiving 1 candy is $1/2$ and the probability of receiving 2 candies is $1/2$. If a child receives 2 candies, the probability of the next child receiving 1 candy is $3/4$ and the probability of receiving 2 candies is $1/4$. Compute the probability that the n-th child in the line receives 2 candies.

2019
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in Japanese or English.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklet for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Obtain the general solution of the following differential equation:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^3. \tag{1}$$

II. Obtain the general solution of the following differential equation:

$$x^2\frac{dy}{dx} - x^2y^2 + xy + 1 = 0. \tag{2}$$

Note that $y = \frac{1}{x}$ is a particular solution.

III. Let I_n be defined by:

$$I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx, \tag{3}$$

where n is a non-negative integer.

1. Calculate I_0, I_1, and I_2.
2. Calculate I_n for $n \geq 2$.

Problem 2

I. Answer the following questions about the matrix $\boldsymbol{P}$:

$$\boldsymbol{P} = \begin{pmatrix} 0 & 0 & \frac{3}{2} \\ 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}. \tag{1}$$

1. Obtain all eigenvalues of the matrix $\boldsymbol{P}$ and the corresponding eigenvectors with unit norms.

2. Obtain $\boldsymbol{P}^2$ and $\boldsymbol{P}^3$.

II. Let $\boldsymbol{A}$ be the real matrix given by the block diagonal matrix:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & c & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & d & 0 \end{pmatrix}. \tag{2}$$

Express succinctly the necessary and sufficient condition on a, b, c, d, and e, such that there exists a positive integer m for which $\boldsymbol{A}^m$ is the identity matrix (proof is not required).

III. The matrix $\boldsymbol{M}$ is a square matrix of order 12 with all elements taking either 0 or 1, such that each row and column has exactly one element being 1. Let k_0 be the minimum value of the positive integer k such that $\boldsymbol{M}^k$ is the identity matrix. For all possible matrices $\boldsymbol{M}$, give the maximum value of k_0 (proof is not required).

Problem 3

In the following, z denotes a complex number and i is the imaginary unit. The real part and the imaginary part of z are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively.

I. Answer the following questions.

1. Give the solutions of $z^5 = 1$ in polar form. Plot the solutions on the complex plane.
2. The mapping f is defined by $f\colon z \mapsto f(z) = \exp(iz)$. Plot the image of the region $D = \{z\colon\ \mathrm{Re}(z) \geq 0,\ 1 \geq \mathrm{Im}(z) \geq 0\}$ under f on the complex plane.
3. Find the residue of the function $z^2 \exp\left(\frac{1}{z}\right)$ at $z = 0$.

II. Consider the complex function: $f(z) = \frac{(\log z)^2}{(z+a)^2}$, where a is a positive real number. The closed path C shown in Figure 3.1 is defined by $C = C_+ + C_R + C_- + C_r$, where $R > a > r > 0$. Here, $\log z$ takes the principal value on the path C_+. Answer the following questions.

1. Apply the residue theorem to calculate the contour integral $\oint_C f(z)\, dz$.
2. Use the result of Question II.1 to calculate the integral: $\int_0^{\infty} \frac{\log x}{(x+a)^2}\, dx$.

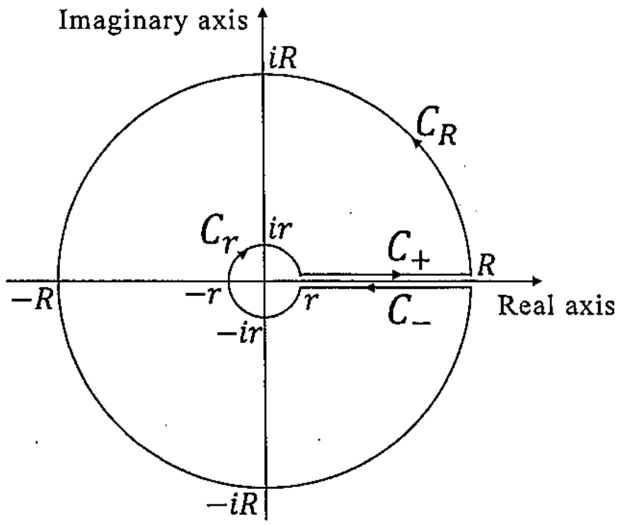

Figure 3.1

Problem 4

Answer the following questions on shapes in the three-dimensional orthogonal coordinate system xyz.

I. Consider the surface S_1 represented by the equation $x^2 + 2y^2 - z^2 = 0$. Find the equations expressed in x, y, and z of the normal line and the tangent plane T to the surface S_1 at the point $\mathrm{A}(2, 0, 2)$.

II. Consider the surface S_2 represented by the following set of equations with the parameters u and v:

$$\begin{cases} x = \dfrac{1}{\sqrt{2}} \cosh u \cos v & (1) \\ y = \dfrac{1}{2} \cosh u \sin v - \dfrac{1}{\sqrt{2}} \sinh u & (2) \\ z = \dfrac{1}{2} \cosh u \sin v + \dfrac{1}{\sqrt{2}} \sinh u \ , & (3) \end{cases}$$

where u and v are real numbers, and $0 \leq v < 2\pi$.
Let S_3 be the surface obtained by rotating the surface S_2 around the x-axis by $-\pi/4$. Here, the positive direction of rotation is the direction of the semi-circular arrow on the yz-plane shown in Figure 4.1.

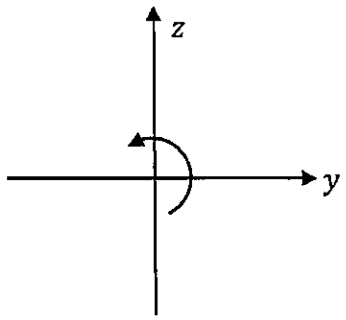

Figure 4.1

Answer the following questions.

1. Find the matrix $\boldsymbol{R}$ that represents the linear transformation rotating a shape around the x-axis by $-\pi/4$.
2. Find an equation expressed in x, y, and z for the surface S_3.
3. Find an equation expressed in x, y, and z for the surface S_2.

III. Consider the solid V that is enclosed by the surface S_3 obtained in Question II.2 and by the two planes $z = 1$ and $z = -1$. Answer the following questions.

1. Calculate the area of the cross section obtained by cutting the solid V with the xz-plane.
2. Calculate the area of the cross section obtained by cutting the solid V with the plane T obtained in Question I.

Problem 5

Consider the continuously differentiable function $f(x)$ of the real variable x. Let $f(x) \to 0$ as $|x| \to \infty$. $f(x)$, its derivative $f'(x)$, and $xf(x)$ are absolutely integrable. The Fourier transform of the function $f(x)$ is denoted by $\mathcal{F}\{f(x)\}(u)$ or equivalently by $\hat{f}(u)$, and defined by

$$\mathcal{F}\{f(x)\}(u) = \hat{f}(u) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-iux) dx, \tag{1}$$

where u is a real variable and i is the imaginary unit. The Fourier transform is defined in the same way for other functions.

I. Express $\mathcal{F}\{f'(x)\}(u)$ in terms of $\hat{f}(u)$ and u.

II. Express $\dfrac{d\hat{f}(u)}{du}$ in terms of $\mathcal{F}\{xf(x)\}(u)$.

III. Let the function $f(x) = \exp(-ax^2)$, where a is a positive real constant $(a > 0)$. The following relation holds for $f(x)$:

$$f'(x) = -2axf(x). \tag{2}$$

Apply the Fourier transform on both sides of Eq. (2) to obtain a first–order ordinary differential equation in $\hat{f}(u)$. Solve this ordinary differential equation to obtain $\hat{f}(u)$. Note that the integration constant in the solution of this ordinary differential equation can be obtained by calculating $\hat{f}(0)$ with the help of Eq. (1) and the value of the following improper integral:

$$\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}. \tag{3}$$

IV. Consider the function $h(x,t)$ of the real variables x and t. Let $h(x,t)$ be defined for $-\infty < x < \infty$ and $t \geq 0$, and satisfy the following partial differential equation:

$$\frac{\partial h(x,t)}{\partial t} = \frac{\partial^2 h(x,t)}{\partial x^2} \qquad (t > 0), \tag{4}$$

given the initial condition

$$h(x,0) = \exp(-ax^2) \qquad (a > 0). \tag{5}$$

1. Apply the Fourier transform with respect to the variable x on both sides of the partial differential equation (4) to obtain an ordinary differential equation with $\hat{h}(u,t) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,t)\exp(-iux)dx$ and the independent variable t.
2. By solving the ordinary differential equation found in Question IV.1, obtain $\hat{h}(u,t)$.
3. Use the inverse Fourier transform with respect to the variable u to obtain a solution $h(x,t)$ satisfying Eq. (4) and Eq. (5).

V. Consider the continuous function $g(x)$ and its Fourier transform $\hat{g}(u)$. Let $g(x) \to 0$ as $|x| \to \infty$ and $g(x)$ be absolutely integrable. The convolution of the functions $f(x)$ and $g(x)$ is defined by

$$(f * g)(x) \equiv \int_{-\infty}^{\infty} f(y)g(x-y)dy. \tag{6}$$

1. Express $\mathcal{F}\{(f * g)(x)\}(u)$ in terms of $\hat{f}(u)$ and $\hat{g}(u)$.
2. Here, the function $h(x,t)$ satisfies Eq. (4), given the initial condition $h(x,0) = g(x)$. Use the result of Question V.1 to find an integral representation of a solution $h(x,t)$, where $t > 0$.

Problem 6

Consider n random variables $X_1, X_2, \cdots, X_n$ that can take the values 0 and 1. Here, n is an integer greater than or equal to 4. The probability of an event A is denoted by $P(A)$, and the conditional probability of the event A given an event B is denoted by $P(A|B)$. The intersection between the event A and the event B is denoted by $A \wedge B$. Answer the following questions.

I. Let us assume that the $X_1, X_2, \cdots, X_n$ are independent. In addition, assume that each X_k $(k = 1, 2, \cdots, n)$ takes the value 1 with the probability p and the value 0 with the probability $1-p$, i.e., $P(X_k = 1) = p$ and $P(X_k = 0) = 1 - p$.

1. Find the expected value and the variance of the sum of the $X_1, X_2, \cdots, X_n$.
2. The random variables $X_1, X_2, \cdots, X_n$ are arranged in the row $X_n \cdots X_2 X_1$. Let Y be the integer value obtained by regarding that row as an n-digit binary number. For example, in the case that $n = 4$, $Y = 5$ when the row $X_4 X_3 X_2 X_1$ is 0101, and $Y = 13$ when the row $X_4 X_3 X_2 X_1$ is 1101. Y is a random variable that takes integer values from 0 to $2^n - 1$. Obtain the expected value and variance of Y.

II. The values of the random variables $X_1, X_2, \cdots, X_n$ are obtained sequentially according to the following steps. First, X_1 takes the value 1 with the probability p and the value 0 with the probability $1-p$. Then, X_k $(k = 2, 3, \cdots, n)$ takes the same value as X_{k-1} with the probability q and the value different from X_{k-1} with the probability $1-q$, i.e.,
$P(X_k = 1 | X_{k-1} = 1) = P(X_k = 0 | X_{k-1} = 0) = q$
and $P(X_k = 1 | X_{k-1} = 0) = P(X_k = 0 | X_{k-1} = 1) = 1 - q$.

1. Let $P(X_k = 1)$ be represented by r_k, where k is an integer varying from 1 to n. Derive a recurrence equation for r_k. Solve this recurrence equation to express r_k with p, q, and k.
2. Obtain the probability $P(X_1 = 1 \ \wedge \ X_2 = 0 \ \wedge \ X_3 = 1 \ \wedge \ X_4 = 0)$.
3. Obtain the probability $P(X_3 = 1 \mid X_1 = 0 \ \wedge \ X_2 = 1 \ \wedge \ X_4 = 1)$.

2020
The Graduate School Entrance Examination
Mathematics
1:00 pm－3:30 pm

GENERAL INSTRUCTIONS

Answers should be written in English or Japanese.

1. Do not open the problem booklets, whether in English or Japanese, until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answer three problems (two problems for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation) out of the six problems in the problem booklet.
4. You are given three answer sheets (two answer sheets for examinees of Department of Civil Engineering, Department of Systems Innovation, Department of Nuclear Engineering and Management, and Department of Technology Management for Innovation). Use one answer sheet for each problem. You may use the reverse side if necessary.
5. Fill in your examinee number and the problem number in the designated places at the top of each answer sheet. The wedge-shaped marks on the top edge of each answer sheet represent the problem number that you answer (P 1, P 2, …, P 6) and also the class of the course (master M, doctor D) that you are applying. At the end of the examination, follow your proctor's instructions and cut out carefully the two corresponding wedge marks per sheet.
6. You may use the blank sheets of the problem booklets for rough papers without detaching them.
7. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. You may not take the booklets or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

Problem 1

I. Answer the following questions about the differential equation:

$$\cos x \frac{d^2y}{dx^2} - \sin x \frac{dy}{dx} - \frac{y}{\cos x} = 0 \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right). \tag{1}$$

1. A particular solution of Eq. (1) is of the form of $y = (\cos x)^m$ (m is a constant). Find the constant m.
2. Find the general solution of Eq. (1), using the solution of Question I.1.

II. Find the value of the following integral:

$$I = \int_1^\infty x^5 e^{-x^4+2x^2-1} dx. \tag{2}$$

Note that, for a positive constant α, the relation $\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$ holds.

III. Express the general solution of the following differential equation in the form of $f(x,y) = C$ (C is a constant) using an appropriate function $f(x,y)$:

$$(x^3 y^n + x)\frac{dy}{dx} + 2y = 0 \qquad (x > 0, y > 0), \tag{3}$$

where n is an arbitrary real constant.

Problem 2

Consider the following matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & -1 \\ -2 & 1 & 1 \\ -1 & 1 & \alpha \end{pmatrix}, \tag{1}$$

where α is a real number. In the following, the transpose of a vector $\boldsymbol{v}$ is denoted by $\boldsymbol{v}^{\mathrm{T}}$.

I. Obtain α when the sum of the three eigenvalues of the matrix $\boldsymbol{A}$ is 7.

II. Obtain α when the product of the three eigenvalues of the matrix $\boldsymbol{A}$ is -16.

III. Let $\|\boldsymbol{A}\|$ be the maximum of $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$ for the set of real vectors $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ that satisfy $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = 1$. Obtain α when $\|\boldsymbol{A}\| = 4$.

IV. In the following questions, $\alpha = 4$.

1. Obtain all eigenvalues of the matrix $\boldsymbol{A}$ and their corresponding normalized eigenvectors.
2. Find the range of $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{y}$ for the real vectors $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ that satisfy $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y} = 1$ and $y_1 - y_2 - 2y_3 = 0$.
3. Find the range of $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{z}$ for the real vectors $\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$ that satisfy $\boldsymbol{z}^{\mathrm{T}}\boldsymbol{z} = 1$ and $z_1 + z_2 + z_3 = 0$.

Problem 3

In the following, z denotes a complex number, and x and ε denote real numbers. The imaginary unit is denoted by i.

I. Answer the following questions about the function $f_n(z) = 1/(z^n - 1)$. Here, n is an integer greater than or equal to 2.

1. For the case that $n = 3$, find all singularities of $f_n(z)$.

2. Calculate the residue value at a singularity p_0 of $f_n(z)$ and give a simple expression of the residue in terms of n and p_0.

3. For a contour C given by the closed curve $|z| = 2$ and oriented in the counter-clockwise direction, evaluate the contour integral $\oint_C f_n(z)dz$.

II. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{-\infty}^{1-\varepsilon}\frac{1}{x^3-1}dx+\int_{1+\varepsilon}^{\infty}\frac{1}{x^3-1}dx\right]. \quad (1)$$

III. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{0}^{1-\varepsilon}\frac{\cos x}{x^4-1}dx+\int_{1+\varepsilon}^{\infty}\frac{\cos x}{x^4-1}dx\right]. \quad (2)$$

IV. Obtain the following limit value:

$$\lim_{\varepsilon\to+0}\left[\int_{0}^{1-\varepsilon}\frac{\sin\left(x^2-\frac{\pi}{4}\right)}{x^4-1}dx+\int_{1+\varepsilon}^{\infty}\frac{\sin\left(x^2-\frac{\pi}{4}\right)}{x^4-1}dx\right]. \quad (3)$$

Problem 4

In the three-dimensional orthogonal coordinate system xyz, the unit vectors along the x, y, and z directions are $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively. Using the parameter θ $(0 \leq \theta \leq \pi)$, we define two curves by their vector functions $\mathbf{P}(\theta)$ and $\mathbf{Q}(\theta)$:

$$\mathbf{P}(\theta) = x(\theta)\mathbf{i} + y(\theta)\mathbf{j}, \tag{1}$$

$$\mathbf{Q}(\theta) = \mathbf{P}(\theta) + z(\theta)\mathbf{k}, \tag{2}$$

where

$$x(\theta) = \frac{3}{2}\cos(\theta) - \frac{1}{2}\cos(3\theta), \tag{3}$$

$$y(\theta) = \frac{3}{2}\sin(\theta) - \frac{1}{2}\sin(3\theta). \tag{4}$$

Here, $z(\theta)$ is a continuous function satisfying $z(0) > 0$ and $z(\pi) < 0$, and the curve parametrized by $\mathbf{Q}(\theta)$ lies on the sphere of radius 2, centered at the origin $(0, 0, 0)$ of the coordinate system. The positive direction of a curve corresponds to increasing values of the parameter θ. Note that the curvature is the reciprocal of the radius of curvature. Answer the following questions.

I. As θ is varied from 0 to π, calculate the arc length of the curve parametrized by $\mathbf{P}(\theta)$.

II. Obtain $z(\theta)$.

III. Let α be the angle between the tangent of the curve parametrized by $\mathbf{Q}(\theta)$ and the unit vector $\mathbf{k}$. Calculate $\cos(\alpha)$.

IV. Find the curvature $\kappa_P(\theta)$ of the curve parametrized by $\mathbf{P}(\theta)$. Here, $\theta = 0$ and $\theta = \pi$ are excluded.

V. Let $\kappa_Q(\theta)$ be the curvature of the curve parametrized by $\mathbf{Q}(\theta)$. Express $\kappa_Q(\theta)$ in terms of $\kappa_P(\theta)$ and α. Here, $\theta = 0$ and $\theta = \pi$ are excluded.

Problem 5

The Laplace transform of the function $f(t)$, defined for $t \geq 0$, is denoted by $F(s) = \mathcal{L}[f(t)]$ and its definition is given by

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) \exp(-st)\, dt, \tag{1}$$

where s is a complex number. In the following, the set of all complex numbers is denoted by $\mathbb{C}$, and the set of the complex numbers with positive real parts is denoted by $\mathbb{C}^+$.

I. Consider the following function $g(t)$ defined for $t \geq 0$:

$$g(t) = \int_0^\infty \frac{\sin^2(tx)}{x^2}\, dx. \tag{2}$$

1. Find the Laplace transform $G(s) = \mathcal{L}[g(t)]$ $(s \in \mathbb{C}^+)$ of the function $g(t)$.
2. Obtain the value of the following integral using the result of Question I.1:

$$\int_{-\infty}^\infty \frac{\sin^2(x)}{x^2}\, dx. \tag{3}$$

II. Consider the function $u(x,t)$ that satisfies the following partial differential equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad (0 < x < 1, t > 0), \tag{4}$$

under the boundary conditions:

$$\begin{cases} \left.\dfrac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0 \quad (t \geq 0), & (5) \\ u(1,t) = 1 \quad (t \geq 0), & (6) \\ u(x,0) = \dfrac{\cosh(x)}{\cosh(1)} \quad (0 < x < 1). & (7) \end{cases}$$

1. The Laplace transform of $u(x,t)$ is denoted by $U(x,s) = \mathcal{L}[u(x,t)]$ $(s \in \mathbb{C}^+)$. Derive the ordinary differential equation and boundary conditions for $U(x,s)$ with respect to the independent variable x. Here, the function $u(x,t)$ can be assumed to be bounded. The following relations can also be used:

$$\mathcal{L}\left[\frac{\partial u(x,t)}{\partial x}\right] = \frac{\partial U(x,s)}{\partial x}, \tag{8}$$

$$\mathcal{L}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \frac{\partial^2 U(x,s)}{\partial x^2}. \tag{9}$$

2. Using an analytic function $Q(s)$ $(s \in \mathbb{C})$, the function $U_{\mathrm{c}}(x,s)$ is defined as follows:

$$U_{\mathrm{c}}(x,s) = \frac{\cosh(x)}{(s-1)\cosh(1)} - \frac{\cosh(x\sqrt{s})}{Q(s)} \quad (0 \le x \le 1). \tag{10}$$

When the function $U(x,s) = U_{\mathrm{c}}(x,s)$ satisfies the differential equation and the boundary conditions derived in Question II.1 for $s \in \mathbb{C}^+$, find the function $Q(s)$.

3. Using the function $Q(s)$ derived in Question II.2, the sequence of complex numbers $\{a_r\}$ $(r = 1, 2, \cdots)$ is defined by arranging all of the roots of $Q(s) = 0$ $(s \in \mathbb{C})$ in ascending order of their absolute values. In this case, the following limits $R_r(x,t)$ are finite for $t \ge 0, 0 \le x \le 1$, and $r \ge 1$:

$$R_r(x,t) = \lim_{s \to a_r} (s - a_r) U_{\mathrm{c}}(x,s) \exp(st), \tag{11}$$

and the solution of the partial differential equation (4) is given by

$$u(x,t) = \sum_{r=1}^{\infty} R_r(x,t). \tag{12}$$

Determine $R_1(x,t)$, $R_2(x,t)$, and $R_r(x,t)$ for $r \ge 3$.

Problem 6

Consider a game where points are awarded in n independent trials. In each trial, either $+1$ or -1 is awarded and both outcomes have the same probability of $1/2$. Let X_k be the point awarded in the k^{th} trial $(1 \le k \le n)$, and $S_k = \sum_{i=1}^{k} X_i$. In the following questions, n is an even integer such that $n \ge 4$, and t is an even integer such that $2 \le t \le n$.

I. Obtain the probability for $S_4 = 0$.

II. Let $P_n(t)$ be the probability for $S_n = t$. Find $P_n(t)$.

III. Let $P_n^+(t)$ be the probability for $S_1 = 1$ and $S_n = t$. Find $P_n^+(t)$.

IV. Let $P_n^-(t)$ be the probability for $S_1 = -1$ and $S_n = t$. Find $P_n^-(t)$.

V. Let $Q_n(t)$ be the probability that all of the variables $\{S_j\}$ $(j = 1, 2, \cdots, n-1)$ are greater than zero and $S_n = t$. Express $Q_n(t)$ with $P_n^+(t)$ and $P_n^-(t)$. Then, express $Q_n(t)$ with $P_n(t)$.

VI. Obtain the probability that all of the variables $\{S_j\}$ $(j = 1, 2, \cdots, n)$ are greater than zero.

2021 年 度
大 学 院 入 学 試 験 問 題
数 学 1 （主に微分積分・微分方程式）
問題番号 M1
解答時間 40 分

注 意 事 項

1. 試験開始の合図があるまで，問題文を見ないこと。
2. 解答用紙 6 枚および下書用紙 2 枚を使用すること。
3. 解答用紙および下書用紙の裏面の使用は禁止する。
4. すべての解答用紙および下書用紙の上方の指定された箇所に，受験番号を忘れずに記入すること。
5. 日本語または英語で解答すること。
6. 解答は解答用紙の実線の枠内に記入すること。
7. 解答に関係のない記号，符号などを記入した答案は無効とする。
8. 日本語の問題文は 3-4 ページ，英語の問題文は 5-6 ページに書かれている。
9. 問題文のスクロール，拡大および縮小はしてよい。キーボード操作は禁止する。

・解答には結果だけでなく導出過程も含めること。

・ネットワークトラブルが生じた場合でも解答を続けること。

2021
The Graduate School Entrance Examination
Mathematics 1 (Primarily from the fields of Differential and Integral Calculus, Differential Equations)
Problem Number M1
Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not look at the Problems until the start of the examination has been announced.
2. Use 6 Answer Sheets and 2 Draft Sheets.
3. Do not use the back faces of the Answer Sheets or the Draft Sheets.
4. Fill in your examinee number in the designated places at the top of all the Answer Sheets and the Draft Sheets.
5. Answers must be written in Japanese or English.
6. Answers must be marked within the solid frame on the Answer Sheets.
7. Any Answer Sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. The Problems are described in Japanese on pages 3-4 and in English on pages 5-6.
9. Scrolling, expansion and reduction of the Problems are permitted. Keyboard operation is prohibited.

• Show the derivation processes as well as the results.

• Continue the answer even if network trouble occurs.

Mathematics 1 (Primarily from the fields of Differential and Integral Calculus, Differential Equations)

Answer both Questions I and II, where x is a real variable.

I. Answer the following questions.

1. Find the derivative $\frac{\mathrm{d}y(x)}{\mathrm{d}x}$ of the following real function $y(x)$ defined for $0 < x < 1$:

$$y(x) = (\arccos x)^{\log x}\,, \tag{1}$$

where $0 < \arccos x < \pi/2$.

2. Calculate the following indefinite integral:

$$\int \frac{x^2 + x + 2}{x^3 - px^2}\,\mathrm{d}x\,, \tag{2}$$

where p is a real constant.

3. Calculate the following definite integral:

$$I = \int_0^{\sin\theta} \frac{\arctan(\arcsin x)}{\sqrt{1 - x^2}}\,\mathrm{d}x\,, \tag{3}$$

where $0 < \theta < \pi/2$.

Continued on the next page.

II. Consider that complex-valued functions $p(x)$ and $q(x)$ satisfy the simultaneous ordinary differential equations below:

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = -ibq(x)\exp(-2iax)\,, \tag{4}$$

$$\frac{\mathrm{d}q(x)}{\mathrm{d}x} = -ibp(x)\exp(2iax)\,. \tag{5}$$

Here, i is the imaginary unit, and a and b are real constants. Answer the following questions.

1. Derive the simultaneous ordinary differential equations for complex-valued functions $f(x)$ and $g(x)$, based on the change of variables $f(x) = p(x)\exp(iax)$ and $g(x) = q(x)\exp(-iax)$.

2. Show that the value of $|f(x)|^2 + |g(x)|^2$ is independent of x, where $|A|$ denotes the absolute value of a complex number A.

3. Let $a = 0.8$ and $b = 0.6$. Solve the simultaneous ordinary differential equations derived in Question II.1 using the initial values $f(0) = 1$ and $g(0) = 0$, and obtain $f(x)$ and $g(x)$.

2021 年 度
大 学 院 入 学 試 験 問 題
数 学 2 （主に線形代数）
問題番号 M2
解答時間 40 分

注　意　事　項

1. 試験開始の合図があるまで，問題文を見ないこと。
2. 解答用紙 6 枚および下書用紙 2 枚を使用すること。
3. 解答用紙および下書用紙の裏面の使用は禁止する。
4. すべての解答用紙および下書用紙の上方の指定された箇所に，受験番号を忘れずに記入すること。
5. 日本語または英語で解答すること。
6. 解答は解答用紙の実線の枠内に記入すること。
7. 解答に関係のない記号，符号などを記入した答案は無効とする。
8. 日本語の問題文は 3-4 ページ，英語の問題文は 5-6 ページに書かれている。
9. 問題文のスクロール，拡大および縮小はしてよい。キーボード操作は禁止する。

・解答には結果だけでなく導出過程も含めること。

・ネットワークトラブルが生じた場合でも解答を続けること。

2021
The Graduate School Entrance Examination
Mathematics 2 (Primarily from the field of Linear Algebra)
Problem Number M2
Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not look at the Problems until the start of the examination has been announced.
2. Use 6 Answer Sheets and 2 Draft Sheets.
3. Do not use the back faces of the Answer Sheets or the Draft Sheets.
4. Fill in your examinee number in the designated places at the top of all the Answer Sheets and the Draft Sheets.
5. Answers must be written in Japanese or English.
6. Answers must be marked within the solid frame on the Answer Sheets.
7. Any Answer Sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. The Problems are described in Japanese on pages 3-4 and in English on pages 5-6.
9. Scrolling, expansion and reduction of the Problems are permitted. Keyboard operation is prohibited.

• Show the derivation processes as well as the results.

• Continue the answer even if network trouble occurs.

Mathematics 2 (Primarily from the field of Linear Algebra)

Answer all Questions I, II and III.

I. Answer the following questions concerning the matrix $\boldsymbol{A}$ given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{pmatrix}. \tag{1}$$

In the following, $\boldsymbol{I}$ is the 3×3 identity matrix, $\boldsymbol{O}$ is the 3×3 zero matrix, n is an integer greater than or equal to 0 and t is a real number.

1. Obtain all eigenvalues of the matrix $\boldsymbol{A}$.

2. Find coefficients a, b and c of the following equation satisfied by $\boldsymbol{A}$:

$$\boldsymbol{A}^3 + a\boldsymbol{A}^2 + b\boldsymbol{A} + c\boldsymbol{I} = \boldsymbol{O}\,. \tag{2}$$

3. Obtain $\boldsymbol{A}^{2n+1}$.

4. Since Equation (2) is satisfied, the following equation holds:

$$\exp(t\boldsymbol{A}) = p\boldsymbol{A}^2 + q\boldsymbol{A} + r\boldsymbol{I}\,. \tag{3}$$

Express coefficients p, q and r in terms of t without using the imaginary unit.

Continued on the next page.

II. Consider a discrete-time system where stochastic transitions between the two states (A and B) occur as shown in Figure 2.1. The transition probability in unit time from the state A to B is α and from the state B to A is β. Note that $0 < \alpha < 1$ and $0 < \beta < 1$. Variables n and k represent discrete time and are integers greater than or equal to 0 .

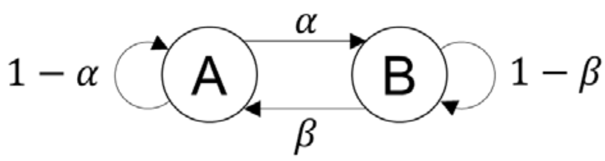

Figure 2.1

Answer the following questions.

1. Let $P_{\mathrm{A}}(n)$ be the probability that the state is A at time n and $P_{\mathrm{B}}(n)$ be the probability that the state is B at time n. Let $\boldsymbol{P}(n) = \begin{pmatrix} P_{\mathrm{A}}(n) \\ P_{\mathrm{B}}(n) \end{pmatrix}$. Express matrix $\boldsymbol{M}$ using α and β, assuming $\boldsymbol{P}(n+1) = \boldsymbol{M}\boldsymbol{P}(n)$.

2. Obtain all eigenvalues and the corresponding eigenvectors of matrix $\boldsymbol{M}$.

3. As time tends towards infinity, the probability that the state is A and the probability that the state is B converge towards constant values. Obtain each value.

4. Assume $R_{\mathrm{A}}(n) = P_{\mathrm{A}}(n) - \lim_{k \to \infty} P_{\mathrm{A}}(k)$. Express $R_{\mathrm{A}}(n+1)$ by using $R_{\mathrm{A}}(n)$.

III. Assume vectors $\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \dots,\ \boldsymbol{a}_m$ are linearly independent in a vector space V, where m is an integer greater than or equal to 3. Obtain the condition that m must satisfy in order for $\boldsymbol{a}_1 + \boldsymbol{a}_2,\ \boldsymbol{a}_2 + \boldsymbol{a}_3, \dots,\ \boldsymbol{a}_{m-1} + \boldsymbol{a}_m$ and $\boldsymbol{a}_m + \boldsymbol{a}_1$ to be linearly independent.

2021 年 度

大 学 院 入 学 試 験 問 題

数 学 3 （主に複素関数論）

問題番号 M3

解答時間 40 分

注 意 事 項

1. 試験開始の合図があるまで，問題文を見ないこと。
2. 解答用紙 6 枚および下書用紙 2 枚を使用すること。
3. 解答用紙および下書用紙の裏面の使用は禁止する。
4. すべての解答用紙および下書用紙の上方の指定された箇所に，受験番号を忘れずに記入すること。
5. 日本語または英語で解答すること。
6. 解答は解答用紙の実線の枠内に記入すること。
7. 解答に関係のない記号，符号などを記入した答案は無効とする。
8. 日本語の問題文は 3-4 ページ，英語の問題文は 5-6 ページに書かれている。
9. 問題文のスクロール，拡大および縮小はしてよい。キーボード操作は禁止する。

・解答には結果だけでなく導出過程も含めること。

・ネットワークトラブルが生じた場合でも解答を続けること。

2021

The Graduate School Entrance Examination

Mathematics 3 (Primarily from the field of Complex Function Theory)

Problem Number M3

Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not look at the Problems until the start of the examination has been announced.
2. Use 6 Answer Sheets and 2 Draft Sheets.
3. Do not use the back faces of the Answer Sheets or the Draft Sheets.
4. Fill in your examinee number in the designated places at the top of all the Answer Sheets and the Draft Sheets.
5. Answers must be written in Japanese or English.
6. Answers must be marked within the solid frame on the Answer Sheets.
7. Any Answer Sheet with marks or symbols irrelevant to your answers is considered to be invalid.
8. The Problems are described in Japanese on pages 3-4 and in English on pages 5-6.
9. Scrolling, expansion and reduction of the Problems are permitted. Keyboard operation is prohibited.

• Show the derivation processes as well as the results.

• Continue the answer even if network trouble occurs.

Mathematics 3 (Primarily from the field of Complex Function Theory)

Answer all Questions I, II and III. In the following, z denotes a complex number, i the imaginary unit, e the base of the natural logarithm, and $|z|$ the absolute value of z.

I. Answer the following questions on the complex function $M(z) = \dfrac{mz}{mz - z + 1}$. Here, m is a complex number such that $|m| = 1$ and $m \neq 1$.

1. Find all fixed points of $M(z)$ which satisfy $M(z) = z$.
2. Express the derivative of $M(z)$ at $z = 0$ by using m.
3. Find m for which the circle $\left|z - \dfrac{1-i}{2}\right| = \dfrac{1}{\sqrt{2}}$ on the complex z plane is mapped onto the real axis through $M(z)$.

II. Deduce the conditions for z and, on the complex z plane, draw the area of z in which the imaginary part of the complex function $J(z) = e^{-i\alpha}z + e^{i\alpha}z^{-1}$ is positive. Here, α is a real number and $0 < \alpha < \pi/2$.

Continued on the next page.

III. To calculate the definite integral $I = \int_0^\infty \frac{x^\beta}{(x^2+1)^2} \mathrm{d}x$, consider the line integral of the complex function $f(z) = \frac{z^\beta}{(z^2+1)^2}$ on the complex plane. Here, β is a real number and $0 < \beta < 1$. The closed integration path $C = C_1 + C_R + C_2 + C_r$ $(0 < r < 1 < R)$ is defined with semicircles and line segments as shown in Figure 3.1. Answer the following questions.

1. Using the residue theorem, calculate the line integral $\oint_C f(z)\mathrm{d}z$.
2. Express $\int_{C_1} f(z)\mathrm{d}z + \int_{C_2} f(z)\mathrm{d}z$ with the definite integral $\int_r^R \frac{x^\beta}{(x^2+1)^2} \mathrm{d}x$.
3. Obtain $\lim_{R\to\infty} \int_{C_R} f(z)\mathrm{d}z$.
4. Obtain $\lim_{r\to 0} \int_{C_r} f(z)\mathrm{d}z$.
5. Using the previous results, calculate the definite integral I .

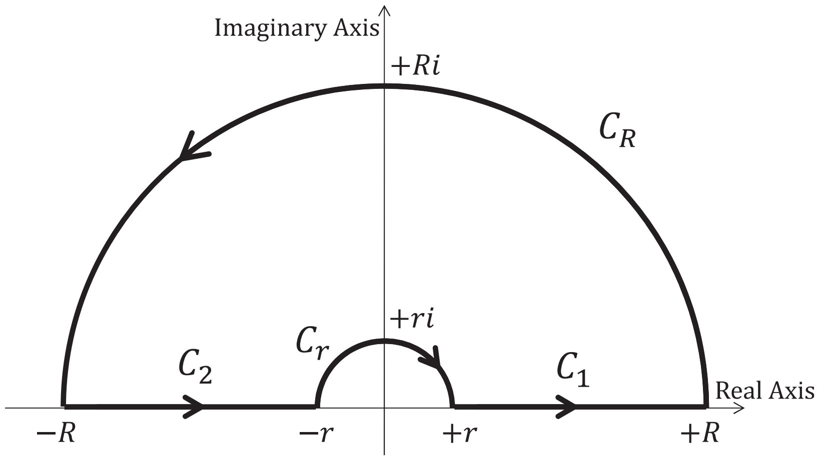

Figure 3.1

2022

The Graduate School Entrance Examination

Mathematics 1

Primarily from the fields of
"Differential and Integral Calculus, Differential Equations"
and "Series, Fourier Analysis, Integral Transform"

Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not open the problem booklet until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. The problems are described in Japanese on pages 2-4 and in English on pages 8-10.
4. Answer all questions.
5. 2 answer sheets are given. Use one answer sheet for each Question (I and II). You may use the reverse side if necessary.
6. Write the question number (I or II) that you answer on the answer sheet in the upper left box.
7. Fill in your examinee number in the designated place at the top of each answer sheet.
8. Answers must be written in Japanese or English.
9. You may use the blank pages of the problem booklet for drafts without detaching them.
10. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
11. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

日本語の注意事項はおもて表紙にある。

Mathematics 1 (Primarily from the fields of "Differential and Integral Calculus, Differential Equations" and "Series, Fourier Analysis, Integral Transform")

Answer both Questions I and II.

I. Consider an ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ in the xy-plane. Here, a and b are constants satisfying $a > b > 0$. Answer the following questions.

1. Find the equation of the tangent line at a point (X,Y) on the ellipse in the first quadrant.

2. The tangent line obtained in Question I.1 intersects the x- and y-axes. Find the coordinates (X,Y) at the tangent point that minimizes length of the segment connecting the two intersects and obtain the minimum length of the segment.

3. Consider a region bounded by the segment obtained in Question I.2 and the x- and y-axes, and let C_1 be a cone formed by rotating the region around the x-axis. Next, let C_2 be a cone with the maximum volume while having the same surface area (including a base area) as the cone C_1 . Find $\dfrac{S_2}{S_1}$, where S_1 and S_2 are the base areas of the cones C_1 and C_2 , respectively.

Continued on a later page.

II. Consider a real-valued function $f(t)$ for a real variable t defined for $0 \leq t < \infty$. The Laplace transform is defined as

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t) \mathrm{d}t \ , \tag{1}$$

where s is a complex variable whose real part is positive. Under the condition that the improper integral in the right-hand side does not diverge, answer the following questions.

1. When the conditions

$$\lim_{t \to \infty} e^{-st} f(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-st} f'(t) = 0 \tag{2}$$

are satisfied, show the following equation holds:

$$\mathcal{L}[f''(t)] = -f'(0) - sf(0) + s^2 \mathcal{L}[f(t)] \ . \tag{3}$$

Note that $f'(t)$ and $f''(t)$ are defined as

$$f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} \quad \text{and} \quad f''(t) = \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} \ . \tag{4}$$

2. Calculate the Laplace transform of $g(t) = e^{-at} \cos(\omega t)$ and $h(t) = e^{-at} \sin(\omega t)$ defined for $0 \leq t < \infty$ by showing derivation processes using Equation (1). Note that a and ω are positive real numbers.

3. Solve the differential equation

$$f''(t) + 6f'(t) + 13f(t) = 0 \ , \tag{5}$$

where the initial values are $f(0) = 5$ and $f'(0) = -11$.

2022

The Graduate School Entrance Examination

Mathematics 2

Primarily from the fields of

"Vector, Matrix, Eigenvalue (Linear Algebra)"

and "Curve and Surface"

Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not open the problem booklet until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. The problems are described in Japanese on pages 2-4 and in English on pages 8-10.
4. Answer all questions.
5. 2 answer sheets are given. Use one answer sheet for each Question (I and II). You may use the reverse side if necessary.
6. Write the question number (I or II) that you answer on the answer sheet in the upper left box.
7. Fill in your examinee number in the designated place at the top of each answer sheet.
8. Answers must be written in Japanese or English.
9. You may use the blank pages of the problem booklet for drafts without detaching them.
10. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
11. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

日本語の注意事項はおもて表紙にある。

Mathematics 2 (Primarily from the fields of "Vector, Matrix, Eigenvalue (Linear Algebra)" and "Curve and Surface")

Answer both Questions I and II.

I. Real symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined as follows:

$$\boldsymbol{A} = \begin{pmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{pmatrix}, \tag{1}$$

$$\boldsymbol{B} = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix}. \tag{2}$$

1. Obtain $\boldsymbol{AB}$.

Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined as Equations (1) and (2) satisfy $\boldsymbol{AB} = \boldsymbol{BA}$.

2. In general, two real symmetric matrices that are commutative for multiplication are simultaneously diagonalizable. Prove this for the case where all the eigenvalues are mutually different.

3. Suppose a three-dimensional real vector $\boldsymbol{v}$ whose norm is 1 is an eigenvector of $\boldsymbol{A}$ in Equation (1) corresponding to an eigenvalue a as well as an eigenvector of $\boldsymbol{B}$ in Equation (2) corresponding to an eigenvalue b . That is, $\boldsymbol{Av} = a\boldsymbol{v}$, $\boldsymbol{Bv} = b\boldsymbol{v}$, and $\|\boldsymbol{v}\| = 1$. Obtain all the sets of $(\boldsymbol{v}, a, b)$.

Continued on a later page.

II. Answer the following questions concerning the curved surface given by Equation (3) in the Cartesian coordinate system xyz . Note that $\boldsymbol{m}^{\mathrm{T}}$ indicates transpose of $\boldsymbol{m}$.

$$f(x,y,z) = 2(x^2 + y^2 + z^2) + 4yz + \frac{z-y}{\sqrt{2}} = 0 \tag{3}$$

1. When the function $f(x,y,z)$ is expressed in the following form, derive the real symmetric matrix $\boldsymbol{A}$ of order 3 and the vector $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

$$f(x,y,z) = (x \quad y \quad z)\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + 2\boldsymbol{b}^{\mathrm{T}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \tag{4}$$

2. Suppose that the matrix $\boldsymbol{A}$ derived in Question II.1 is diagonalized as $\boldsymbol{A} = \boldsymbol{P}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{P}$ using an orthogonal matrix $\boldsymbol{P}$ of order 3 and a diagonal matrix $\boldsymbol{D}$, which is given by Equation (5):

$$\boldsymbol{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} . \tag{5}$$

Obtain a set of $\boldsymbol{P}$ and $\boldsymbol{D}$ satisfying $d_1 \geq d_2 \geq d_3$.

3. Express the function f using X, Y, and Z, obtained by applying the coordinate transformation defined by $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boldsymbol{P}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, using $\boldsymbol{P}$ derived in Question II.2.

4. Consider a region surrounded by the curved surface given by Equation (3) and a plane defined by $y - z - \sqrt{2} = 0$. Obtain the volume of this region.

2022

The Graduate School Entrance Examination

Mathematics 3

Primarily from the fields of

"Function Theory, Complex Number"

and "Probability and Statistics, Information Mathematics, etc."

Answer Time 40 minutes

GENERAL INSTRUCTIONS

1. Do not open the problem booklet until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. The problems are described in Japanese on pages 2-5 and in English on pages 8-11.
4. Answer all questions.
5. 2 answer sheets are given. Use one answer sheet for each Question (I and II). You may use the reverse side if necessary.
6. Write the question number (I or II) that you answer on the answer sheet in the upper left box.
7. Fill in your examinee number in the designated place at the top of each answer sheet.
8. Answers must be written in Japanese or English.
9. You may use the blank pages of the problem booklet for drafts without detaching them.
10. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
11. You may not take the booklet or answer sheets with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

日本語の注意事項はおもて表紙にある。

Mathematics 3 (Primarily from the fields of "Function Theory, Complex Number" and "Probability and Statistics, Information Mathematics, etc.")

Answer both Questions I and II.

I. In Questions I.1 and I.2, z denotes a complex number, i the imaginary unit, and $|z|$ the absolute value of z.

1. Calculate the following integral, where C is the closed path on the complex plane as shown in Figure 3.1.

$$I_1 = \oint_C \frac{z}{(z-i)(z-1)} \mathrm{d}z \tag{1}$$

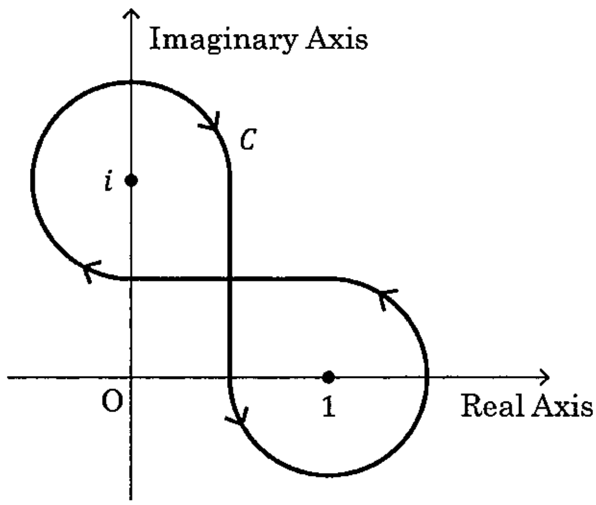

Figure 3.1

Continued on a later page.

2. Consider the definite integral I_2 expressed as

$$I_2 = \int_0^{2\pi} \frac{\mathrm{d}\theta}{10 + 8\cos\theta} \quad . \tag{2}$$

2.1. Find a complex function $G(z)$ when I_2 is rewritten as an integral of a complex function as

$$I_2 = \oint_{|z|=1} G(z)\mathrm{d}z \quad . \tag{3}$$

Note that the integration path is a unit circle centered at the origin on the complex plane oriented counterclockwise. Show the derivation process with your answer.

2.2. Find all singularities of $G(z)$.

2.3. Using the residue theorem, obtain I_2.

Continued on a later page.

II. Consider a situation where products are produced sequentially. The events producing defective products are independent and identically distributed, and a defective product is produced with a probability of ϕ $(0 \leq \phi \leq 1)$. We consider the changes of the probability distributions before and after observing production results. In the following questions, N (≥ 1) denotes the number of products observed.

1. By defining $v_i = 1$ if the i-th product is a defective product, and $v_i = 0$ if it is not defective, we get a series $\boldsymbol{v} = (v_1, \cdots, v_N)$, where the values can be 0 or 1. Let $N_d(\boldsymbol{v})$ be the number of observations with value of 1 in $\boldsymbol{v}$, obtain the occurrence probability of $\boldsymbol{v}$.

Suppose that the probability of producing a defective product follows the Beta distribution

$$\mathrm{Beta}_{\mathrm{a,b}}(x) = \frac{1}{\mathrm{B(a,b)}}\ x^{\mathrm{a}-1}(1-x)^{\mathrm{b}-1} \qquad (0 \leq x \leq 1), \tag{4}$$

for real numbers a (> 1) and b (> 1). Note that the Beta function $\mathrm{B(a,b)}$ is defined as

$$\mathrm{B(a,b)} = \int_0^1 t^{\mathrm{a}-1}(1-t)^{\mathrm{b}-1}\mathrm{d}t\ . \tag{5}$$

We now consider the changes of a and b before and after observing production results in the Beta distribution, which gives the probability of producing a defective product, using the Bayesian estimation. In the Bayesian estimation, the parameter θ (in this case, ϕ) that determines the probability is treated as the random variable and we assume that its distribution is described by $\pi(\theta)$. We calculate $\pi(\theta|A)$, that is the probability distribution of θ when an event A is observed with the occurrence probability $P(A)$, by Equation (6). Here, $\pi(\theta|A)$ is the posterior probability, $P(A|\theta)$ is the conditional occurrence probability that the event A is observed under θ, and $\pi(\theta)$ is the prior probability.

$$\pi(\theta|A) = \frac{\pi(\theta)P(A|\theta)}{P(A)} \tag{6}$$

Continued on a later page.

2. We assume that ϕ, the probability of producing a defective product, follows the prior probability $\mathrm{Beta}_{\mathrm{a,b}}(\phi)$. Let $Q(\boldsymbol{v}|\phi)$ be the conditional occurrence probability of $\boldsymbol{v}$ under ϕ and $Q_{\mathrm{a,b}}(\boldsymbol{v})$ be the occurrence probability of $\boldsymbol{v}$. Obtain the posterior probability after $\boldsymbol{v}$ occurs.

3. Suppose that $Q(\boldsymbol{v}|\phi)$ in Question II.2 is the occurrence probability obtained in Question II.1 and let $\mathrm{a} = 2,\ \mathrm{b} = 50,$ obtain $Q_{2,50}(\boldsymbol{v})$.

4. In Question II.3, show that the posterior probability becomes the Beta distribution $\mathrm{Beta}_{\mathrm{a}',\mathrm{b}'}(\phi)$, and obtain a' and b'.

5. Obtain ϕ that gives the maximum likelihood estimate (that maximizes the posterior probability) in Question II.4.

2023

The Graduate School Entrance Examination

Mathematics

13:00 – 15:00

GENERAL INSTRUCTIONS

1. Do not open the problem booklet until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. The problems are described in Japanese on pages 2-12 and in English on pages 16-26.
4. Answer all problems.
5. Six answer sheets are given. Use one answer sheet for each Problem (from 1 to 6). You may use the reverse side if necessary.
6. Write the problem number (1 to 6) that you answer on the answer sheet in the upper left box.
7. Fill in your examinee number in the designated place at the top of each answer sheet.
8. Answers must be written in Japanese or English.
9. You may use the blank pages of the problem booklet for drafts without detaching them.
10. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
11. Do not take the answer sheets or the booklet with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

日本語の注意事項はおもて表紙にある。

Problem 1

Answer all the following questions.

I. Find the following limit value:

$$\lim_{x \to 0} \frac{b^x - c^x}{ax} \qquad (a, b, c > 0) . \tag{1}$$

II. Find the general solutions of the following differential equations.

1. $$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \log x \qquad (x > 0) \tag{2}$$

2. $$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 2x^2 + 2x \tag{3}$$

III. Let a_n be defined by

$$a_n = \frac{n!}{n^{n+\frac{1}{2}} e^{-n}} \quad , \tag{4}$$

where n is a positive integer and e is the base of natural logarithm.
Find $\lim_{n \to \infty} \frac{a_n}{a_{n+1}}$.
Note that the function $y = x^{-1}$ $(x > 0)$ is convex downward.

Problem 2

Consider expressing the following matrix $\boldsymbol{A}$ in a form of $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, using a diagonal matrix $\boldsymbol{D}$ and a regular matrix $\boldsymbol{P}$. Here, a is a real number.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & a \\ 0 & a & 2 \end{pmatrix} \tag{1}$$

I. When $a = 1$, find a diagonal matrix $\boldsymbol{D}$.

II. When $a = 1$, prove $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} > 0$ for any three-dimensional non-zero real vector $\boldsymbol{x}$. $\boldsymbol{x}^{\mathrm{T}}$ represents the transpose of $\boldsymbol{x}$.

III. Find the condition of a which satisfies $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} > 0$ for any three-dimensional non-zero real vector $\boldsymbol{x}$.

IV. Assume that a satisfies the condition obtained in Question III.
For a real vector $\boldsymbol{b} = \begin{pmatrix} a \\ 0 \\ -1 \end{pmatrix}$, express the minimum value of the function $f(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$ by using a.

Problem 3

In the following, $z = x + iy$ and $w = u + iv$ represent complex numbers, where i is the imaginary unit, and x, y, u and v are real numbers.

I. In order to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} \mathrm{d}x \quad , \tag{1}$$

consider the complex function $f(z) = \dfrac{1}{z^6 + 1}$.

1. Find all singularities of $f(z)$.
2. By applying the residue theorem, determine the value of I.

II. Two domains, which are banded and semi-infinite on the complex z-plane, are defined as:

$$D_1 = \left\{x + iy \middle| 0 \le x \le \frac{\pi}{2},\ y \ge 0\right\} \text{ and } D_2 = \left\{x + iy \middle| x \ge 0,\ -\frac{\pi}{2} \le y \le 0\right\}.$$

Consider the mapping $w = g(z)$ from the complex z-plane to the complex w-plane with an analytic function $g(z)$. Let D_1^* and D_2^* be the images of D_1 and D_2, respectively, through this mapping.

1. When $g(z) = \cos z$, sketch the domain D_1^*.
2. When $g(z) = (\cosh z)^3$, sketch the domain D_2^*.

Problem 4

In the three-dimensional orthogonal xyz coordinate system, consider the region V that satisfies Equations (1) and (2).

$$x^2 + y^2 - z^2 \geq 0 \tag{1}$$

$$x^2 + y^2 + 2x \leq 0 \tag{2}$$

I. Sketch the cross-sectional shape of the region V at $z = 1$.

II. Obtain the surface area of the region V.

Problem 5

Answer all the following questions.

I. Let a periodic function $f(x)$ satisfy the condition $f(x+\pi) = f(x-\pi)$. Find the Fourier series expansion of $f(x)$ for each case, where $f(x)$ is expressed as follows for the interval $-\pi \le x \le \pi$.

1. $f(x) = x \quad (-\pi < x < \pi), \quad f(-\pi) = f(\pi) = 0$
2. $f(x) = x^2$

For the Fourier series expansion, the following equations should be used.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\mathrm{d}x \tag{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \tag{3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,\mathrm{d}x \tag{4}$$

II. Consider the function shown in Figure 5.1 as

$$V(t) = A\left|\sin\left(\frac{\omega t}{2}\right)\right| \qquad \left(\omega = \frac{2\pi}{T}\right) . \tag{5}$$

Note that A and T are positive real numbers. The complex Fourier series expansion of $V(t)$ is given as

$$V(t) = -\frac{2A}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{4n^2-1}e^{in\omega t} . \tag{6}$$

Let $I(t)$ be the periodic solution that satisfies the ordinary differential equation

$$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = V(t) . \tag{7}$$

Note that L and R are positive real numbers.

Find the coefficient C_n, when the complex Fourier series expansion of $I(t)$ is expressed as

$$I(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\omega t} . \tag{8}$$

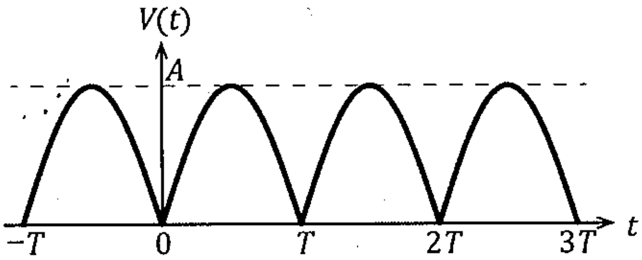

Figure 5.1

Problem 6

Consider a light whose state alternately and repeatedly switches between the OFF (no light) state and the ON (light) state. For each OFF and ON state, the duration, represented by T_0 and T_1 respectively, changes at each transition and is independent.

By using t, which represents elapsed time from the initiation of each state, T_0 and T_1 follow the exponential distribution whose probability density functions are described respectively as

$$f_0(t) = \lambda_0 e^{-\lambda_0 t} \qquad (\lambda_0 > 0)\,, \tag{1}$$

and

$$f_1(t) = \lambda_1 e^{-\lambda_1 t} \qquad (\lambda_1 > 0)\ . \tag{2}$$

Here, for example, $P_0(a,b)$, which is the probability that the condition $a \leq T_0 \leq b$ $(0 \leq a \leq b)$ is satisfied, can be calculated as

$$P_0(a,b) = \int_a^b f_0(t)\mathrm{d}t\ . \tag{3}$$

Assume that the light switches from the ON state to the OFF state at time $\tau = 0$. Answer the following questions.

I. Calculate the expected value and the standard deviation of T_0.

II. Calculate the expected value and the standard deviation of $T_0 + T_1$.

III. Consider a situation where time tends towards infinity and the condition $(\lambda_0 + \lambda_1)\tau \to \infty$ approximately holds.

1. Calculate the probability that the light is in the OFF state.
2. Calculate the expected value of the remaining time from the current state to the next state transition of the light.

IV. At the time $\tau = \tau_x$ $(\tau_x > 0)$, calculate the probability that the light is in the ON state for the first occasion after $\tau = 0$.

2024

The Graduate School Entrance Examination

Mathematics

13:00 - 15:30

GENERAL INSTRUCTIONS

1. Do not open the problem booklet until the start of the examination is announced.
2. Notify your proctor if you find any printing or production errors.
3. Answers must be written in Japanese or English. The problems are described in Japanese on pages 2–12 and in English on pages 16–26.
4. Answer three problems out of the six problems in the problem booklet.
5. Three answer sheets are given. Use one answer sheet for each Problem (from 1 to 6). You may use the reverse side if necessary.
6. Write the problem number (1 to 6) that you answer in the upper left box of the answer sheet.
7. Fill in your examinee number in the designated place at the top of each answer sheet.
8. You may use the blank pages of the problem booklet for drafts without detaching them.
9. Any answer sheet with marks or symbols irrelevant to your answers is considered to be invalid.
10. Do not take the answer sheets or the booklet with you after the examination.

Examinee Number	No.

Write your examinee number in the space provided above.

日本語の注意事項はおもて表紙にある。

Problem 1

I. Find the general solution $y(x)$ of the following differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-y), \tag{1}$$

where $0 < y < 1$.

II. Find the value of the following definite integral, I :

$$I = \int_{-1}^{1} \frac{\arccos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{\pi}{3}x\right)}\,\mathrm{d}x, \tag{2}$$

where $0 \leq \arccos\left(\frac{x}{2}\right) \leq \pi$.

III. For any positive variable x, we define $f(x)$ and $g(x)$ respectively as

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{(2m)!} x^{2m} \tag{3}$$

and

$$g(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x). \tag{4}$$

For any non-negative integer n, $I_n(x)$ is defined as

$$I_n(x) = \int_0^x \left\{\frac{g(X)}{f(X)}\right\}^n \mathrm{d}X. \tag{5}$$

Here, you may use

$$\exp(x) = \sum_{m=0}^{\infty} \frac{1}{m!} x^m. \tag{6}$$

1. Calculate $f(x)^2 - g(x)^2$.
2. Express $I_{n+2}(x)$ using $I_n(x)$.

Problem 2

Answer the following questions about a real symmetric matrix, $\boldsymbol{A}$:

$$\boldsymbol{A}=\begin{pmatrix}0 & 1 & 2\\ 1 & 0 & 2\\ 2 & 2 & 3\end{pmatrix}. \tag{1}$$

I. Find all the different eigenvalues of matrix $\boldsymbol{A}$, $\lambda_1, \cdots, \lambda_r$ $(\lambda_1 < \cdots < \lambda_r)$.

II. Find all the eigenspaces $W(\lambda_1), \cdots, W(\lambda_r)$ corresponding to $\lambda_1, \cdots, \lambda_r$, respectively.

III. Find an orthonormal basis, $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$, which belongs to either of $W(\lambda_1), \cdots, W(\lambda_r)$, obtained in Question II.

IV. Find the spectral decomposition of $\boldsymbol{A}$:

$$\boldsymbol{A}=\sum_{i=1}^{r}\lambda_i \boldsymbol{P}_i, \tag{2}$$

where $\boldsymbol{P}_i$ is the projection matrix onto $W(\lambda_i)$.

V. Find $\boldsymbol{A}^n$, where n is any positive integer.

Problem 3

Answer the following questions. Here, for any complex value z, $\bar{z}$ is the complex conjugate of z, $\arg z$ is the argument of z, $|z|$ is the absolute value of z, and i is the imaginary unit.

I. Sketch the region of z on the complex plane that satisfies the following:

$$z\bar{z} + \sqrt{2}(z + \bar{z}) + 3i(z - \bar{z}) + 2 \leq 0. \tag{1}$$

II. Answer the following questions on the complex valued function $f(z)$ below.

$$f(z) = \frac{z^2 - 2}{(z^2 + 2i)z^2} \tag{2}$$

1. Find all the poles of $f(z)$ as well as the orders and residues at the poles.
2. By applying the residue theorem, find the value of the following integral I_1. Here, the integration path C is the circle on the complex plane in the counterclockwise direction which satisfies $|z + 1| = 2$.

$$I_1 = \oint_C f(z)\,\mathrm{d}z \tag{3}$$

III. Answer the following questions.

1. Let $g(z)$ be a complex valued function, which satisfies

$$\lim_{|z|\to\infty} g(z) = 0 \tag{4}$$

for $0 \leq \arg z \leq \pi$.

Let C_R be the semicircle, with radius R, in the upper half of the complex plane with the center at the origin.

Show

$$\lim_{R\to\infty} \int_{C_R} e^{iaz} g(z)\,\mathrm{d}z = 0, \tag{5}$$

where a is a positive real number.

2. Find the value of the following integral, I_2:

$$I_2 = \int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x. \tag{6}$$

Problem 4

I. In the two-dimensional orthogonal xy coordinate system, consider the curve L represented by the following equations with the parameter t $(0 \leq t \leq 2\pi)$. Here, a is a positive real constant.

$$x(t) = a(t - \sin t) \tag{1}$$

$$y(t) = a(1 - \cos t) \tag{2}$$

1. Obtain the length of the curve L when t varies in the range of $0 \leq t \leq 2\pi$.
2. Obtain the curvature at an arbitrary point of the curve L.
   Here, $t = 0$ and $t = 2\pi$ are excluded.

II. In the three-dimensional orthogonal xyz coordinate system, consider the curved surface represented by the following equations with the parameters u and v (u and v are real numbers).

$$x(u, v) = \sinh u \cos v \tag{3}$$

$$y(u, v) = 2 \sinh u \sin v \tag{4}$$

$$z(u, v) = 3 \cosh u \tag{5}$$

1. Express the curved surface by an equation without the parameters.
2. Sketch the xy-plane view at $z = 5$ and the xz-plane view at $y = 0$, respectively, of the curved surface. In the sketches, indicate the values at the intersection with each of the axes.
3. Express the unit normal vector $\boldsymbol{n}$ of the curved surface by u and v.
   Here, the z-component of $\boldsymbol{n}$ should be positive.
4. Let κ be the Gaussian curvature at the point $u = v = 0$.
   Calculate the absolute value of κ.

Problem 5

I. We consider a continuous and absolutely integrable function $f(t)$ of a real variable t and denote the Fourier transform of the function $f(t)$ as $\mathcal{F}\{f(t)\}$. We define a function $F(\omega)$ by the following formula:

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt\,, \tag{1}$$

where ω is a real variable and i is the imaginary unit.

1. We define $g(t) = f(at)$ for a constant a satisfying $a > 0$ and

$$G(\omega) = \mathcal{F}\{g(t)\} = \mathcal{F}\{f(at)\}. \tag{2}$$

   Express $G(\omega)$ using the function F.
2. When $f(t) = \exp(-t^2)$ and $a = 2$, sketch the graph of the Fourier transformed functions $F(\omega)$ and $G(\omega)$ defined by Equation (2) as a function of ω to show the difference between them.
3. We define $h(t) = f(t) \exp(-ibt)$ for a constant b satisfying $b > 0$ and

$$H(\omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\{f(t) \exp(-ibt)\}. \tag{3}$$

   Express $H(\omega)$ using the function F.
4. When $f(t) = \exp(-t^2)$ and $b = 2$, sketch the graph of the Fourier transformed functions $F(\omega)$ and $H(\omega)$ defined by Equation (3) as a function of ω to show the difference between them.

II. Let N be a positive integer. We define a discrete Fourier transform $D_1, \cdots, D_N$ by the following formula:

$$D_m = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} c_n \exp\left(-i\frac{2\pi}{N}nm\right), \tag{4}$$

for a complex sequence $c_1, \cdots, c_N$. Here, m is an integer satisfying $1 \leq m \leq N$.

1. Calculate $S(n, n')$:

$$S(n, n') = \frac{1}{N} \sum_{m=1}^{N} \exp\left\{i\frac{2\pi}{N}(n - n')m\right\}. \tag{5}$$

   Here, n is an integer satisfying $1 \leq n \leq N$, and n' is an integer satisfying $1 \leq n' \leq N$.

2. Let U_{mn} be a complex number satisfying

$$D_m = \sum_{n=1}^{N} U_{mn} c_n. \tag{6}$$

   Show that the matrix $\mathbf{U} = [U_{mn}]_{1 \leq m \leq N, 1 \leq n \leq N}$ is a unitary matrix.

3. Derive an equation for the inverse discrete Fourier transform c_n from $D_1, \cdots, D_N$.
   Here, n is an integer satisfying $1 \leq n \leq N$.

4. For any complex value z, $\bar{z}$ is the complex conjugate of z. We define Q by

$$Q = \sum_{n=1}^{N} (\overline{c_n} c_{n+1} + \overline{c_{n+1}}\, c_n). \tag{7}$$

   Express Q in terms of D_m and $\overline{D_m}$. Here, we impose the condition $c_{N+1} = c_1$.

Problem 6

Consider an electric vehicle charging station with a single charger installed and let us observe the number of vehicles at the station at regular time intervals.

Arriving vehicles at the station are lined up in the queue in the order of arrival, and only the first vehicle in the queue can be charged. In the interval between one observation and the next observation, assume that one new vehicle arrives with probability $p\ (0 < p < 1)$, and that the vehicle charging at the head of the queue completes charging with probability $q\ (0 < q < 1)$. Here, assume that p and q are constants and $p + q < 1$.

The queue can accommodate $N\ (N \geq 2)$ vehicles, including the vehicle being charged at the head of the queue, and the $(N + 1)$ -th vehicle shall give up and leave the station without queuing up. The vehicle which completes charging leaves the station immediately.

In the interval between one observation and the next observation, either only one or no vehicles arrive at the station and either only one or no vehicles complete charging. Moreover, assume that both arrival of new vehicle and completion of charging for the first vehicle do not occur together in any one interval.

I. When there are $i\ (0 < i < N)$ vehicles in the queue, find the probability for the following condition: no new vehicle arrives and the first vehicle does not leave in the interval between one observation and the next observation.

Let $\pi_i\ (0 \leq i \leq N)$ be the probability that i vehicles are in the queue in the steady state.

II. Express the relationship between π_i and π_{i+1}. Here, $i \leq N - 1$.

III. Express π_i using p, q and N.

IV. Find the expected value of the number of vehicles at the station in the steady state using p, q and N. Here, $p < q$.