CS 101 Discrete Structures

Logical Thinking

Nutchakorn Ngamsaowaros and Wanida Putthividhya
Dept. of Computer Science, Thammasat University

Logical Thinking: Formal Logic

- Inquiry Problems
- Logical connectives and Propositions
- Truth Tables
- Logical Equivalence
- Exercises

Inquiry Problems 1

- Westley, standing with his hands behind his back, claims that he is holding a quarter in his left hand and a $20 bill in his right hand. You believe he is lying.
- What would you have to show to demonstrate that he is lying?
  - Invent a diagram, chart, or symbols to illustrate all the possible scenarios.

Inquiry Problems 2

- Buttercup knows whether or not Westley is lying. She promises that if Westley is lying, she will give you a cookie. Buttercup always keeps her promises.
- Suppose she does not give you a cookie; what can you conclude?
- Suppose she gives you a cookie; what can you conclude? Illustrate your thinking in some organized way.

Inquiry Problems 3

- Camp Halcyon and Camp Placid are two summer camps with the following daily policies on pool use and cleanup duties.
- Camp Halcyon's Policy: If you used the pool in the afternoon and you didn't clean up after lunch, then you must clean up after dinner.
- Camp Placid's Policy: You must do at least one of the following: (a) Stay out of the pool in the afternoon, (b) clean up after lunch, or (c) clean up after dinner.
- How do these policies differ? Explain your reasoning.

Propositions

- **Definition 1.1** A ***statement*** (also known as a ***proposition***) is a declarative sentence that is either true or false, but not both.
- Example:
  - 7 is odd
  - 1 + 1 = 4
  - If it is raining, then the ground is wet.
  - Our professor is from Venus.

Note that we don't need to be able to decide whether a sentence is true or false in order for it to be a 'statement'.

How can a declarative sentence fail to be a statement?

There are two main ways.

- A declarative sentence may contain an unspecified term:
  - Ex. "x is even."
  - In this case, x is called a ***free variable***. The truth of the sentence depends on the value of x, so if that value is not specified, we can't regard this sentence as a statement.
- A sentence is **self-referential**:
  - Ex. "This sentence is false."
  - If we say the sentence is true, then it claims to be false;
  - if we say the sentence is false, then it appears to be true.

Connectives

- conjunction **AND** — $\wedge$ หรือ **&&**
- **inclusive** disjunction **OR** (**inclusive-OR**) — $\vee$ หรือ $+$ หรือ ||
- **exclusive** disjunction **OR** (**exclusive-OR**) — $\underline{\vee}$ หรือ $\oplus$
- **negation** — ~ หรือ $\neg$ หรือ **!**
- **implication** — $\rightarrow$
- **double implication** — $\leftrightarrow$

Connectives : conjunction

- Let **p**: **It is raining. q**: **It is cold.** be two propositions
  - the **conjunction** of **p** and **q** is

    p ∧ **q** : It is raining **AND** it is cold.

- The ***truth value*** of the ***compound proposition*** **p** ∧ **q** (the **conjunction** of **p** and **q**) can be described as the following truth table (ตารางค่าความจริง)

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Connectives : conjunction (2)

- Ex1.2. Let p : **A decade is 10 years.**

  q : **A millennium is 100 years.**

  be two **propositions**.

  Show the **compound proposition** p ∧ q ______________________________________

  and its **truth value**.

  Ans.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Connectives : conjunction (3)

- <u>Ex1.3</u>. Let **p** : **X < 10** where **X** is set to 7 นั่นคือ **p** : 7 **< 10**

  q : **Y > 5** where **Y** is set to 5 นั่นคือ **q** : 5 **> 5**

Show the **compound proposition p ∧ q** ______________________

and fill in its **truth value** in the table.

<u>Ans</u>. **p ∧ q** :

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

X	Y	p	q	p ∧ q
7	5	T	F	F
-10	10			
10	10			
10	5			

Connectives : inclusive disjunction

- Let **p: Mana is a programmer.,**
  q: Manee is a programmer.

  be two **propositions.**
  - the **(inclusive) disjunction** of **p** and **q** is
    p ∨ q : Mana is a programmer **OR** Manee is a programmer.

- the **truth value** of the **compound proposition p ∨ q** (the **inclusive disjunction** of **p** and **q**) can be described as the following **truth table** (ตารางค่าความจริง)

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Connectives : inclusive disjunction (2)

- Ex1.4. Let p : **A millennium is 100 years.,**

  q : **A millennium is 1000 years.**

  be two **propositions**.

  Show the **compound proposition** p ∨ q and its **truth value**.

  Ans.

Connectives : inclusive disjunction (3)

- <u>Ex1.5</u>. Let **p** : **X < 10** where **X** is set to 7 นั่นคือ **p** : 7 **< 10**

  q : **Y > 5** where **Y** is set to 5 นั่นคือ **q** : 5 **> 5**

  be two **propositions**.

  Show the **compound proposition p ∨ q**

  and fill in its **truth value** in the table.

  <u>Ans</u>. **p ∨ q** :

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

x	y	p	q	p ∨ q
7	5	T	F	T
-10	10			
10	10			
10	5			

Connectives : inclusive disjunction (4)

- <u>Ex1.6.</u> Think about the **propositions p, q** such that the **compound proposition** (**inclusive-OR**) **p ∨ q** will be **false** only when x ∈ I is within the range [0,100]

p : ....

q : ....

p ∨ q : ....

....

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Connectives : inclusive disjunction (5)

Q. Consider the statement

"Students who have taken calculus **OR** statistics can take this class."

- Anne has taken both calculus and statistics.

  Can Anne take this class? Ans.
- Bob has taken calculus but not statistics.

  Can Bob take this class? Ans.
- Cathy has taken statistics but not calculus.

  Can Cathy take this class? Ans.
- David has not taken both statistics and calculus.

  Can David take this class? Ans.

Connectives : exclusive disjunction

- Let **p**: **It is raining.**, **q**: **It is warm.** be two **propositions**.
  - the **(exclusive) disjunction** of **p** and **q** is

    p ⊕ q : **Either** it is raining **OR** it is warm.

    (but not both)
- Let **p**: **John is studying in a high school.**,

  q: **John is studying in a university.**

  be two **propositions**.
  - the **(exclusive) disjunction** of **p** and **q** is

    p ⊕ q : **Either** John is studying in a high school **OR** John is studying in a university.

    (but not both)

Connectives : exclusive disjunction (2)

- the **truth value** of the **compound proposition p ⊕ q** (the **exclusive disjunction** of **p** and **q**) can be described as the following **truth table** (ตารางค่าความจริง)

p	q	p ⊕ q
T	T	F
T	F	T
F	T	T
F	F	F

Connectives : exclusive disjunction (3)

- Ex1.7. The statement

  "Students who have taken **Either** calculus **OR** statistics can take this class."

  means that

  "Students who have taken calculus **OR** statistics, **but not both**, can take this class."

Connectives : exclusive disjunction (4)

- In the situation that there is **no dedicate XOR operator** for programmers, we could use our logical thinking background to create a *logically-equivalent statement*

p	q	$p \oplus q$	$\sim p$	$\sim q$	$p \vee q$	$\sim p \vee \sim q$	$(p \vee q) \wedge (\sim p \vee \sim q)$
T	**T**	**F**	F	F	T	F	**F**
T	**F**	**T**	F	T	T	T	**T**
F	**T**	**T**	T	F	T	T	**T**
F	**F**	**F**	T	T	F	T	**F**

Connectives : negation

- Let **p**: **Paris is the capital of England.** be a **proposition.**
  - the **negation** of **p** is **¬ p**

    ¬ p : **It is not the case that**
    Paris is the capital of England.

    ¬ p : **It is false that**
    Paris is the capital of England.

    ¬ p : Paris is **NOT** the capital of England.

- the **truth value** of the **compound proposition ¬ p** หรือ **! p** (the **negation** of **p**) can be described as the following **truth table** (ตารางค่าความจริง)

p	¬ p
T	F
F	T

Connectives : negation (2)

- Ex1.8. Negation of inequalities

$\neg\ (x > 2) \equiv x \leq 2$	$\neg\ (x \geq 2) \equiv x < 2$
$\neg\ (x \leq 2) \equiv x > 2$	$\neg\ (x < 2) \equiv x \geq 2$

Connectives : negation (3)

- Ex1.9: Let **p : x < 10** be a **proposition**.
  Show the **proposition ¬ p**
  and fill in its **truth value** in the table.

Ans. ¬ **p** :

p	¬ p
T	F
F	T

x	นั่นคือ	p	¬ p
7	**p : 7 < 10**	**T**	**F**
-10	**p : -10 < 10**		
10	**p : 10 < 10**		
50	**p : 50 < 10**		

Connectives : implications

- Let p : **I am elected.**,
  q : **I will lower taxes.**
  be two **propositions**.

- The **implication** of p and q is $p \rightarrow q$

$p \rightarrow q$: **If** p, **then** q

If I am elected, **then** I will lower taxes.

$p \rightarrow q$: p **only if** q

I am elected **only if** I will lower taxes.

$p \rightarrow q$: p **implies** q

I am elected **implies** that I will lower taxes.

$p \rightarrow q$: p **is sufficient for** q

Being elected **is sufficient for** me to lower taxes.

Connectives : implications (2)

- Let **p** : **I am elected.**,

  q : **I will lower taxes.** be two **propositions**.
  - The **implication** of p and q is $p \rightarrow q$

$p \rightarrow q$: **q whenever/when/if p**

I will lower taxes **whenever** I am elected.

$p \rightarrow q$: **q follows from p**

Lowering taxes **follows from** being elected.

$p \rightarrow q$: **q is necessary for p**

Lowering taxes **is necessary for** being elected.

Connectives : implications (3)

- An **implication p → q** is sometimes called a **conditional proposition** or **conditional statement**
  - **p** is called the **hypothesis** (antecedent)
  - **q** is called the **conclusion** (consequence)

Connectives : implications (4)

- The **truth value** of the **conditional proposition $p \rightarrow q$** is defined by the following truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note! a **conditional proposition** that is **true** because the **hypothesis** is **false** is said to be **true by default** or **vacuously true**.

Connectives : implications (5)

- Ex 1.10. Let **p** : $x > 0$, **q** : $x^2 > 0$ be two propositions.
  Determine the truth value of the conditional proposition
  $p \to q$ in the following cases

x	x^2	p	q	$p \to q$
10				
1				
-10				
0				

Connectives : implications (7)

- Ex1.11.

  Let **p** be the statement "Dan studies discrete structures."

  Let **q** be the statement "Dan will find a good job."

  Write the **conditional proposition** **p** → **q** as a statement in English.

  Ans.

Connectives : implications :
Converse, Inverse, and Contrapositive Propositions

- the **converse** (ประพจน์บทกลับ) of the **conditional proposition p → q** is the proposition **q → p**.

- the **inverse** (ประพจน์ผกผัน) of the **conditional proposition p → q** is the proposition **¬ p → ¬ q**.

- the **contrapositive** (ประพจน์แย้งสลับที่) of the **conditional proposition p → q** is the proposition **¬ q → ¬ p**.

Connectives : implications (11)

- Ex1.13. Write down the **converse**, **inverse**, and the **contrapositive** of the **implication**

  "The home team wins **whenever** it is raining."

  Ans. From the implication given,

  p : It is raining.

  q : The home team wins.

original statement : **If** it is raining, **then** the home team wins.

converse : **If** the home team wins, **then** it is raining.

inverse : **If** it **is not** raining, **then** the home team **does not** win.

contrapositive : **If** the home team **does not** win,
then it **is not** raining.

Connectives : implications (12)

- **Q.** Consider the **truth values** of the following **propositions** :

  Let **p** : **1 > 2**, **q** : **4 < 8** be two **propositions**.

  $\mathbf{p \rightarrow q}$ is ....

  $\mathbf{q \rightarrow p}$ is ....

$\mathbf{\neg p \rightarrow \neg q}$ is ....

$\mathbf{\neg q \rightarrow \neg p}$ is ....

Connectives : double implications

- A **double implication p ↔ q** is sometimes called a **bi-conditional proposition** or **bi-conditional statement**

- the truth value of the **bi-conditional proposition p ↔ q** is defined by the following truth table

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Note! the **bi-conditional proposition p ↔ q** is **true** precisely when both the **conditional proposition p → q** and **q → p** are **true**.

Connectives : double implications (2)

- Let **p** : **You can take the flight.**,
  q : **You buy a ticket.**
  be two **propositions**.

- The **double implication** of **p** and **q** is **$p \leftrightarrow q$**
  $p \leftrightarrow q$: **p if and only if q** หรือ **p iff q**
  You can take the flight **iff** you buy a ticket.
  $p \leftrightarrow q$: **p only if q, and conversely** หรือ
  if p then q, and conversely

<u>Ex</u>. You can take the flight **only if** you buy a ticket, **and conversely**.

Connectives : double implications (3)

- Ex1.14. Write the **proposition** in an appropriate symbolic form and then, determine the **truth value** of the **proposition**

"1 < 5 if and only if 2 < 8"

define **p** : ……………………………………………

define **q** : ……………………………………………

symbolic form ……………………………………

truth value of the **proposition** …………

Exercises: Connectives

- <u>Ex 1.15</u>. พิจารณาคำพูดต่อไปนี้และแปลงเป็นประโยคสัญลักษณ์ทางตรรกศาสตร์ที่สอดคล้องกับความหมายของอาจารย์ผู้พูด

    นักศึกษาต้องได้เกรดวิชา CS101 อย่างน้อย C

    นักศึกษาถึงจะจบการศึกษาได้

    Let **p** : students graduate from CSTU.

    q : students pass CS101 with C or higher.

Exercises: Connectives (2)

- **<u>Ex 1.16</u>.** พิจารณากฎเหล็กของคุณป้าที่ต้องการให้หลานสาวจอมแก่น น้องอัณณา ทานข้าวเย็นให้หมดก่อนที่จะได้ทานขนมหวานของโปรด และแปลงเป็นประโยคสัญลักษณ์ทางตรรกศาสตร์ที่สอดคล้องกับความหมายของคุณป้า

  อัณณา หนูต้องทานข้าวเย็นให้หมด หนูถึงจะได้ทานขนมนะคะ

  Let **p** : Ana finishes her dinner.

  q : Ana gets her sweet treat.

  <u>Ans</u>.

Exercises: Connectives (3)

- Ex 1.16. (cont.) Should it be the proposition **p** -> **q** ?

  อัณณา หนูต้องทานข้าวเย็นให้หมด หนูถึงจะได้ทานขนมนะคะ

  p : Ana finishes her dinner. **q** : Ana gets her sweet treat.

> "**If** Ana finishes her dinner, **then** Ana gets her sweet treat."

Ans.

Exercises: Connectives (4)

- Ex 1.16. (cont.) Should it be the proposition $p \wedge q$?

  อัณณา หนูต้องทานข้าวเย็นให้หมด หนูถึงจะได้ทานขนมนะคะ

  p : Ana finishes her dinner. q : Ana gets her sweet treat.

> "Ana finishes her dinner **AND** Ana gets her sweet treat."

Ans.

Exercises: Connectives (5)

- Ex 1.16. (cont.) Should it be the proposition **p** **<->** **q** **?**

  อัณณา หนูต้องทานข้าวเย็นให้หมด หนูถึงจะได้ทานขนมนะคะ

  p : Ana finishes her dinner. **q** : Ana gets her sweet treat.

"**If** Ana finishes her dinner **then** Ana gets her sweet treat, **and conversely**"

Ans.

Exercises: Connectives (6)

- Ex1.17. พิจารณาคำพูดต่อไปนี้ของอาจารย์ผู้สอนและแปลงเป็นประโยคสัญลักษณ์ทางตรรกศาสตร์ที่สอดคล้องกับความหมายของอาจารย์ผู้สอน

  "นักศึกษาที่จะสอบผ่านวิชาCS101 ก็เฉพาะนักศึกษาที่ตั้งใจเรียนเท่านั้นหล่ะ"

  Let **p** : students study hard enough.

  q : students pass CS101.

Ans.

Exercises: Connectives (7)

- Ex1.17. (cont.) Should it be the proposition **p -> q** ?

  นักศึกษาที่จะสอบผ่านวิชาCS101 ก็เฉพาะนักศึกษาที่ตั้งใจเรียนเท่านั้นหล่ะ

  p : students study hard enough. **q** : students pass CS101.

"**If** students study hard enough, **then** they will pass CS101."

Ans.

Exercises: Connectives (8)

- Ex1.17. (cont.) Should it be the proposition **q -> p ?**

  นักศึกษาที่จะสอบผ่านวิชาCS101 ก็เฉพาะนักศึกษาที่ตั้งใจเรียนเท่านั้นหล่ะ

  p : students study hard enough. **q** : students pass CS101.

> “**If** students pass CS101, **then** they study hard enough.”

Ans.

Exercises: Connectives (9)

- Ex1.17. (cont.) Should it be the proposition **p** <-> **q** ?

  นักศึกษาที่จะสอบผ่านวิชาCS101 ก็เฉพาะนักศึกษาที่ตั้งใจเรียนเท่านั้นหล่ะ

  p : students study hard enough. **q** : students pass CS101.

> "Students pass CS101 **if and only if** they study hard enough."

Ans.

Exercises: Connective (10)

- <u>Ex1.18.</u> พิจารณา คำพูดในการขอเลิกกับแฟนสาวของผู้ชายคนหนึ่ง

“เพราะเธอดี(เกินไป) ฉันจึงทิ้งเธอ”

<u>กำหนดให้</u> **p**: เธอดี **q**: ฉันจึงทิ้งเธอ

- คำพูดของผู้ชายคนนี้ เขียนได้เป็น p -> q

จากคำพูดของผู้ชาย ตอบคำถามต่อไปนี้

1. ตอนนี้ เค้าเลิกกับผู้หญิงแล้ว แปลว่า ผู้ชายบอกว่าผู้หญิงคนนี้เป็นคนดี ใช่หรือไม่?
2. ผู้หญิงถามนศ.ว่า แสดงว่า ถ้าฉันเป็นคนเลว ผู้ชายเค้าจะไม่ทิ้งฉันไป ใช่มั้ย? นศ.จะอธิบายอย่างไร?
3. ดังนั้น ผู้หญิงจะสรุปได้ว่าอะไร?

Logical Equivalences : definition

- Two **propositions** having the same **truth values** no matter what truth values their constituent **propositions** are said to be **logically equivalent**
  - that is, the **truth tables** of the two **propositions** are **identical**

- Let **p** and **q** be **propositions**.

  We say that **p** and **q** are **logically equivalent**,

  denoted as $\mathbf{p} \equiv \mathbf{q}$ or $\mathbf{p} \Leftrightarrow \mathbf{q}$, if $\mathbf{p} \leftrightarrow \mathbf{q}$ is a **tautology**.

Logical Equivalences (2)

- To determine whether **propositions p** and **q** are **logically equivalent,**
  - write the **truth tables** for **p** and **q**
    - if all of the entries for **p** and **q** are always the same, then **p** and **q** are **logically equivalent**
    - if some entry is **true** for one of **p** or **q** and **false** for the other, then **p** and **q** are not equivalent
  - apply appropriate **logical equivalence laws** to transform compound propositions or reduce constituted terms

Derivation Rules: Equivalence Rules

Equivalence	Name
$p \Leftrightarrow \neg\neg p$	double negation
$p \to q \Leftrightarrow \neg p \vee q$	implication
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	commutativity
$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$ $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	associativity

- all **equivalence rules** could be checked using truth tables
- A $\Leftrightarrow$ B or A $\equiv$ B says that A and B are **logically equivalent**.