QUESTION ONE (COMPULSORY - 30 MARKS)

a) Let $A = \{2, 3, 5\}$, $B = \{1, 2, 4, 8, 7\}$, $C = (1, 7)$ and $D = \{2, 4\}$: Find: (5 marks)

(i) $A \cup B$

(ii) $B \cap C$

(iii) The power set of D, $\mathcal{P}(D)$

(iv) $A \times D$

(v) $B - C$

b) Give two examples of sets that are both open and closed and two that are neither closed nor open. No explanation required. (4 marks)

c) Use mathematical induction to show that

$$3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$$ (4 marks)

d) Show that $\sqrt{7}$ is not rational. (4 marks)

e) Let $a, b \in \mathbb{R}$. Show that if $a < b + \varepsilon$ for each $\varepsilon > 0$, then $a \leq b$. (2 marks)

f) Let $A = \left\{\frac{n+3}{2n-1} | n \in \mathbb{N}\right\}$ and $B = (-\infty, -1)$. Find (3 marks)

(i) $\sup A$

(ii) $\inf A$

(iii) $\sup B$

g) Let $A = (-\infty, 2) \cup (2, 4) \cup \{5\}$. Find: (3 marks)

(i) The set of boundary points of A, $\delta(S)$.

(ii) The exterior points of A.

(iii) The isolated points of A.

h) Determine whether or not the function below is continuous at $x = 2$

$$f(x) = \begin{cases} \frac{|2-x|}{x-2}, & \text{if } x \neq 2 \\ -1, & \text{if } x = 2 \end{cases}$$ (4 marks)

QUESTION 2 (20 MARKS)

a) (i) Show that the arbitrary intersection of closed sets is closed. (4 marks)

(ii) Using a counter example, show that the arbitrary union of closed sets is not necessarily closed. (2 marks)

b) Use mathematical induction to show that:

$$1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1.$$ (4 marks)

c) (i) Show that the relation R defined on the set of integers by

$$nRm \iff k|(n-m) \text{ for a fixed integer } k$$ (6 marks)

is an equivalence relation.

(ii) Write all the equivalence classes of the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ resulting from the relation R defined by

$$nRm \iff 4|(n-m).$$ (4 marks)

QUESTION 3 (20 MARKS)

a) Let $A = (-9, -2), B = [-2, 5], C = (0, 1) \cup (3, 6)$. Find: (3 marks)

(i) $A \cup B$

(ii) $B - C$

(iii) $B \cap C$

b) Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary collection of sets. Show that

$$\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right)^C = \bigcap_{\lambda \in \Lambda} A_\lambda^C.$$ (4 marks)

c) Show that a set is closed if and only if it's complement is open. (5 marks)

d) Let $f : \mathbb{Q} \to \mathbb{Z}$ be defined by $f\left(\frac{a}{b}\right) = a + b$. Determine whether or not f is well defined.

(3 marks)

e) Let $\mathbb{E}$ denote the set of all even numbers and $f : \mathbb{E} \to \mathbb{Z}$ defined by

$$f(x) = \frac{x+2}{2}.$$

Show that f is onto. (3 marks)

f) Let $A = (-\infty, -3) \cup \{-2\} \cup [-2, 0) \cup (3, 5)$. Find: (2 marks)

(i) Limit points of A that are not in A.

(ii) Closure of A ($\bar{A}$).

QUESTION 4 (20 MARKS)

a) Show that for $x \in \mathbb{R}$, if $|x| < \delta, \delta > 0$, then $-\delta < x < \delta$. (2 marks)

b) Let A be a set. Show that:

$$\inf(-A) = -\sup(A)$$ (4 marks)

c) Show that the set of all integers ($\mathbb{Z}$) is not a field. (2 marks)

d) Let $S \subseteq \mathbb{N}$. Show that if

  (i) $1 \in S$

  (ii) $n + 1 \in S$ whenever $n \in S$

then $S = \mathbb{N}$. (5 marks)

e) Show that $\lim_{x \to 2} 5x + 3 = 13$. (4 marks)

f) Sketch the graph of the function

$$f(x) = \begin{cases} 1, & x = 0, \\ x, & 0 < x < 1, \\ 2, & x = 1, \\ x, & 1 < x \leq 2, \\ -1, & 2 < x < 3, \\ 0, & x = 3. \end{cases}$$ (3 marks)

QUESTION 5 (20 MARKS)

a) Find the ranges of the following functions defined on $\mathbb{R}$,

$$f : x \mapsto \sin x, \qquad h : x \mapsto e^x, \qquad g : x \mapsto x^2.$$ (4 marks)

b) Suppose f and g are two functions defined on some neighborhood of c such that $\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$. Show that

$$\lim_{x \to c}(f(x) + g(x)) = l + m.$$ (5 marks)

c) Consider the following function:

$$f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ 2 + x^2 & \text{if } x > 1 \end{cases}$$

  (i) Find $\lim_{x \to 1^+} f(x)$ and $\lim_{x \to 1^-} f(x)$ (4 marks)

  (ii) Does $\lim_{x \to 1} f(x)$ exist. If so what is the limit? (2 marks)

d) Show that the intersection of an arbitrary family of closed sets is also closed. (5 marks)