Question 1

(30 points)

a) Let V be a vector space over the field $\mathbb{R}$. When is the set $B \subset V$ a basis for V? (2 pts)

b) Let $W = \{w_1, w_2, w_3, w_4\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$ be a set of vectors in $\mathbb{R}^3$.

i) Is the set W linearly independent? If yes, prove it and if not, show how it is linearly dependent by expressing one vector in the set as a linear combination of the other vectors. (5 pts)

ii) Is W a basis for a subspace of $\mathbb{R}^3$? If NOT construct a basis from the given set by deleting some vectors. (3 pts)

iii) What is the dimension of the subspace spanned by W? (1 pts)

b) Let $P_2 = \{a + bx + cx^2 | \, a, b, c \in \mathbb{R}\}$ be the vector space of polynomials of degree 2 over $\mathbb{R}$. Show that $Q = \{a + cx^2 | \, a, c \in \mathbb{R}\}$ is a vector subspace of V and give its dimension. (3 pts)

c) Let U, V be vector subspaces of $\mathbb{R}^3$ and $T : U \to V$ be a linear map defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 - 2x_3 \\ x_1 + 3x_2 - x_3 \\ -x_1 + x_2 + x_3 \end{bmatrix}.$$

i) Define the image and kernel of a linear map T. (2 pts)

ii) Give the matrix of the Linear map T above. (2 pts)

iii) Find the range and kernel of T. (6 pts)

d) Given the matrix $A = \begin{bmatrix} 3 & 4 \\ 2 & -4 \end{bmatrix}$, calculate the eigenvalues of A and their corresponding eigenvectors, hence diagonalize A. (7 pts)

Question 2

(20 points)

a) Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

i) Calculate the eigenvalues of A. (4 pts)

ii) For each eigenvalue in (i) above, calculate its eigenvector. (9 pts)

iii) Is the matrix A diagonalizable? If Yes diagonalize A and if NOT explain why! (4 pts)

b) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Show that the spectrum of T cannot be empty. (3 pts)

426173

a) Given a square matrix A, define its characteristic polynomial and its minimal polynomial. (2 pts)

b) State the Cayley Hamilton's Theorem. (2 pts)

b) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.

i) Evaluate the minimal polynomial of A. (3 pts)

ii) Use the Cayley Hamilton's theorem to show that for the polynomial $f(x) = x^3 - 5x^2 + 6x$, $f(A) = I_3$ where I_3 is the identity matrix in $\mathbb{R}^3$. (5 pts)

iii) Use the Cayley Hamilton Theorem to evaluate the inverse of A.(4 pts)

iv) Evaluate the eigenvalues and eigenvectors of A^7. (4 pts)

Question 4 **(20 points)**

a) Evaluate the transition matrix T from the basis $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ to the basis $B_2 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. (8 pts)

b) Show that the transition matrix T above is bijective and hence evaluate its inverse. (6 pts)

c) A vector $v = \begin{bmatrix} 4 \\ 2 \\ 8 \end{bmatrix}$ is written in the coordinates of the standard basis B for $\mathbb{R}^3$. Write this vector in the coordinates of basis B_1 and in the coordinates of basis B_2. (6 pts)

Question 5 **(20 points)**

a) Suppose A has eigenvalues $0,\ 3,\ 5$ with independent eigenvectors $u,\ v,\ w$ respectively.

i) Give a basis for the null space and a basis for the column space of A. (4 pts)

ii) Find a particular solution to $Ax = v + w$, and hence evaluate all solutions. (6 pts)

ii) Does the system of linear equations $Ax = u$, have any solutions? Explain your answer. (3 pts)

b) Suppose A and B have the same eigenvalues $\lambda_1, ..., \lambda_n$ with the same independent eigenvectors $v_1, ..., v_n$. Show that $A = B$. **Hint:** Any vector v is a linear combination $c_1v_1 + +c_nv_n$. What is Av? And what is Bv? (7 pts)