QUESTION ONE (Compulsory) [30 Marks]

(a) Consider the following sets. $A=\{1,2,3,4\}$, $B=\{3,4,5,6\}$ and $C=\{1,3,5\}$. 3!

Define and determine the following sets;

(i) $B\setminus A$ [2 Marks]

(ii) Power set of C [3 Marks]

(b) Let $*$ be a binary operation on $\mathbb{Z}$ defined by $x * y = x + y - 1$ for all $x, y \in \mathbb{Z}$. Show that $*$ is both commutative and associative. [3 Marks]

(c) Consider the set $\mathbb{Z}_5 - \{\bar{0}\}$ of non zero residue classes modulo 5.

(i) Construct the Cayley table to verify that it is a group with respect to the usual multiplication. [3 Marks]

(ii) Determine the order of $\bar{3}$ in $\mathbb{Z}_5 - \{\bar{0}\}$. [1 Mark]

(d) (i) Define a bijective mapping. [2 Marks]

(ii) Let $g: \mathbb{R} \longmapsto \mathbb{R}$ be defined by $g(x) = 3x + 2$ for all $x \in \mathbb{R}$.

Show that g is a bijective mapping. [2 Marks]

(e) Let p, q, r and s be integers such that q and s are non zero

Define a relation $\sim$ such that $p/q \sim r/s$ if $ps = qr$. Show that $\sim$ is an equivalence relation. [3 Marks]

(f) Let G be a group;

(i) Define a left inverse of an element $a \in G$. [2Marks]

(ii) Show that $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$. [3 Marks]

(g) Consider the following permutations in their two line representation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 3 & 7 & 4 & 2 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 6 & 4 & 7 & 3 & 1 & 2 \end{pmatrix}$$

(i) Write their cycle representations. [2 Marks]

(ii) Compute the composition $g \circ h$. [2 Marks]

(h) Let R be a ring under the usual binary operation of addition and multiplication on R. State conditions required for R to be called a field. [2 Marks]

QUESTION TWO [20 Marks]

(a) Let G be a group and let H and K be subsets of G.

(i) Define a subgroup of G. [2 Marks]

(ii) Show that if $ab^{-1} \in$ H for $a, b \in$ H then H is a subgroup of G. [3Marks]

(iii) If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G. [8 Marks]

(b) Given that R is a non empty set and + and $*$ are the usual binary operations of addition and multiplication on R.

Explain clearly what is meant by these statements.

(i) R is a commutative ring [1Mark]

(ii) $u \in R$ is a unit. [1 Marks]

(iii) R is a division ring. [2 Marks]

(iv) R is an integral domain. [3 Marks]

QUESTION THREE [20 Marks.]

(a) Define a group homomorphism. [2 Marks]

(b) Let $\phi: G \to G^*$ be a group homomorphism from G to G^*.

(i) Show that if e is the identity element in G, then $\emptyset(e)$ is the identity e^* in G^* [5 Marks]

(iii) Let G be a group such that $a^{-1}b^{-1} = (ab)^{-1}$ for all $a, b \in G$

Show that G is abelian. [4 Marks]

(c) Consider the relation $R = \{(a, b) | a - b$ is a multiple of 3 $\}$

(i) Show that R is an equivalence relation on $\mathbb{Z}$. [3 Marks]

(ii) R partitions $\mathbb{Z}$ into residue classes denoted by $\mathbb{Z}_3$. Show that $\mathbb{Z}_3$ is a group under addition. [3 Marks]

(d) The set $G = \{1, -1, i, -i$ where $i = \sqrt{-1}\}$ is a group under the usual multiplication. Verify that G is cyclic. [3 Marks]

QUESTION FOUR [20 Marks]

(a) Let H be a subgroup of G.

(i) Define the left coset of H in G. [2 Marks]

(ii) Let $x, y \in G$. Show that $xH = yH$ if and only if $y^{-1}x \in H$.[6 Marks]

(b) The set $S_3 = \{(1), (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}$ of permutations is a group under mapping composition. Considering the subset

$H = \{(1), (12\ 3), (132)\}$ of S_3,

(i) use the Cayley table to show that H is a subgroup of S_3. [4Marks]

(ii) determine all the left cosets and right cosets of H in S_3. [6 Marks]

(iii) What conclusions can you make of your results in part (ii)? [2Marks]

QUESTION FIVE [20 Marks]

(a) (i) Define a zero divisor of a ring R. [2marks]

(ii) Prove that a ring R has no zero divisors if and only if the two cancellation laws hold for multiplication. [8Marks]

(b) (i) State the subring test. [2Marks]

(ii) Consider the subset S of $\mathbb{R}$ given by $S = \{a + b\sqrt{2} : a , b , \in \mathbb{Q} \}$.

Apply the subring test to show that S is a subring of $\mathbb{R}$. [3 Marks]

(c) Let $\emptyset: G \longrightarrow G^*$ be a homomorphism from the group G to the group G^*. If the set $ker\emptyset = \{x \in G | \emptyset(x) = e^* \in G^*\}$ is the kernel of $\emptyset$.

Show that $ker\emptyset$ is a subgroup of G. [5 Marks]