QUESTION THREE [20 Marks]

(a) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factor decomposition of n. Prove that n is representable as a sum of two squares if and only if α_k is even whenever $p_k \equiv 3 \mod 4$. [7 Marks]

(b) Show that if a positive integer a is represented in decimal digits, i.e.,

$$a = (a_n a_{n-1} \cdots a_2 a_1 a_0) = a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_2 10^2 + a_1 10 + a_0,$$

then a is divisible by

(i) 7 if $(a_n a_{n-1} \cdots a_2 a_1) - 2(a_0)$ is divisible by 7. [4 Marks]

(ii) 11 if the alternating sum of its digits is divisible by 11. [2 Marks]

(iii) 13 if $(a_n a_{n-1} \cdots a_2 a_1) + 4(a_0)$ is divisible by 13. [4 Marks]

(c) Let n and m be integers. Prove that $\gcd(n, m) \cdot \operatorname{lcm}(n, m) = nm$. [3 Marks]

QUESTION FOUR [20 Marks]

(a) Determine all right angle triangles with integer side lengths whose hypotenuse has length 65. [7 Marks]

(b) Solve the following Diophantine equation: $X^2 + 2Y^2 = 1376$. [6 Marks]

(c) Compute $\left(\frac{163}{187}\right)$, where $\left(\frac{a}{b}\right)$ is a Legendre symbol. Give a reason for each step.

[7 Marks]

QUESTION FIVE [20 Marks]

(a) Show that there are no positive integers x, y and z such that $x^4 + y^4 = z^2$.

[8 Marks]

(b) Use continued fractions to solve the Diophantine equation: $5x + 3y = 4$.

[5 Marks]

(c) When Justine cashed her cheque, the absent minded teller gave her as many cents as she should have dollars (1 dollar is equivalent to 100 cents), and as many dollars as she should have cents. Equally absent minded Justine left with the cash without noticing the discrepancy. It was only after she spent 5 cents that she noticed now she had twice as much money as she should. What was the amount of her cheque? [7 Marks]

MMA 220: DISCRETE MATHEMATICS III

QUESTION ONE (Compulsory) [30 Marks]

(a) Show that the power function is completely multiplicative. [3 Marks]

(b) Show that if $a_1, a_2, \ldots, a_m$ is a complete residue system prime to m and $\gcd(k, m) = 1$, then $ka_1, ka_2, \ldots, ka_m$ is also such a system. [4 Marks]

(c) Kuria had x coins. When he sorted the coins into groups of 3, he remained with 2 coins as left overs. Again, when he sorted the coins into groups of 4, exactly one coin remained as a left over. Finally, when he sorted the coins into groups of 11 coins, he remained with 7 coins as left overs. Find the least value of x. [5 Marks]

(d) Calculate the following:

(i) Euler's totient function, $\phi(180)$. [2 Marks]

(ii) Number of divisors, $d(164)$. [2 Marks]

(iii) Möbius function, $\mu(84)$. [2 Marks]

(e) Show that if p is an odd prime then $2(p-3)! \equiv -1 \mod p$. [3 Marks]

(f) Find the solutions of the quadratic congruence $x^2 + 6x + 5 \equiv 0 \mod 7$. [3 Marks]

(g) Show that if $p \equiv 3 \mod 4$ then p cannot be written as a sum of two squares. [3 Marks]

(h) Obtain a continued fraction representation of $\sqrt{11}$. [3 Marks]

QUESTION TWO [20 Marks]

(a) Show that the quadratic congruence $x^2 \equiv 79 \mod 91$ has a solution and hence solve it. [12 Marks]

(b) State Wilson's theorem. [1 Mark]

(c) Prove that if $\gcd(a, p) = 1$ then $a^{\phi(p)} \equiv 1 \mod p$ where ϕ is Euler's totient function. [4 Marks]

(d) Use the result of part (c) above and Wilson's theorem to compute $(10886400 \times 6^{123456789}) \mod 11$. [3 Marks]