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RANDOM ECONOMIES WITH MANY INTERACTING AGENTS

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1. Introduction

Standard microeconomic theory considers the characteristics of an individual economic agent, and in particular his preferences, as fixed initial data. To borrow a phrase from Koopmans (1957, p. 161), it does not allow him 'to indulge in a certain randomness in his responses to given circumstances'. In this paper we will allow such indulgence: (i) the preferences of an economic agent may be random, and (ii) the probability law which governs that randomness may depend on the agent's environment. In a pure exchange economy, the environment of an agent will be specified by the preferences and endowments of the other agents.

Hildenbrand (1971) developed the equilibrium analysis of large exchange economies under (i) but excluding (ii). Thus, the state $\omega(a)$ describing preferences and endowments of the agent a was allowed to be random, but these random variables were assumed to be stochastically independent. The results of Hildenbrand (1971) imply in particular the existence of a price system such that the resulting per capita excess demand gets small if the number of economic agents gets large. More precisely, and granting some regularity conditions, we may choose a price system p such that

$$\lim_{|A| \uparrow \infty} \frac{1}{|A|} \sum_{a \in A} \zeta(\omega(a), p) = 0, \text{ in probability,} \tag{1}$$

where A is the set of agents, $|A|$ the number of agents, and $\zeta(\omega(a), p)$ the excess demand of the agent $a \in A$ induced by p if his state is $\omega(a)$.

It is not altogether surprising that randomness alone, without interaction, does not seriously affect the existence of price equilibria. In essence, this was anticipated in Koopmans' (1957, p. 165) remark that 'changes in consumers'

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preferences would be a much less important source of uncertainty if in fact such changes occurred for different consumers independently of each other. The law of large numbers would in that case cut down the variability in the distribution of aggregate demand at constant prices over the various commodities. It is through waves of imitation . . . that interacting preferences become an important source of uncertainty'. The remark also suggests that, in view of the existence of equilibrium prices, independence may be dropped as long as one imposes conditions on the underlying probability space (e.g., strong mixing) which guarantee a suitable law of large numbers. This approach has been used by Malinvaud (1972) and by Bhattacharya and Majumdar (1973). Our purpose, however, is to limit all probabilistic assumptions to the individual agents' characteristics. We will then analyse, for some specific models, to which extent equilibrium prices can be derived from these local data. It turns out that even short range interaction may propagate through the economy and may indeed 'become an important source of uncertainty'. First of all, the microeconomic characteristics may no longer determine the macroeconomic phase, that is, the global probability law which governs the joint behavior of all economic agents. And in that case a given global phase typically will *not* satisfy a law of large numbers like eq. (1).

As in Hildenbrand (1971), we will limit the discussion to pure exchange economies. To formalize (i) and (ii) above, we will use as our initial data on an agent $a \in A$ a conditional probability law $\pi_a(\,.\,|\eta)$ on his possible states, given his environment η. Our basic problem will be the existence of price equilibria in a sense similar to eq. (1). It is clear from the outset that no limit law like eq. (1) can be expected in all generality, without restrictions on the structure and the range of interaction. We will thus concentrate on a special class of models, the homogeneous Markov economies of sect. 3. In view of the existence of price equilibria these economies look quite promising. The Markov property corresponds to short range interaction so that, in a sense, we are not too far away from the independent case. The homogeneity assumptions are such that ergodic theory applies, and so one may hope to obtain limit laws like eq. (1) by means of an ergodic theorem. It turns out that a homogeneous Markov economy does indeed admit a price equilibrium as long as our microeconomic characteristics $\pi_a(\,.\,|\,.\,)$ $(a \in A)$ determine uniquely the macroeconomic phase. But if the interaction is both strong and complex enough then the local data may not suffice to characterize the global situation. This is exemplified by the Ising economies of sect. 4, and here we encounter situations where stabilization of the economy by means of a price system will 'almost never' work. Ising economies also provide insight into the different effects of cyclic and anticyclic interaction.

Mathematically, this paper is a survey on recent research in Physics and Probability on interacting particle systems, combined with results of Hildenbrand (1971) on random demand and Debreu (1970) on the existence of price equilibria. The Ising economies of sect. 4 are of course just an economic reinterpretation of the famous Ising model in Statistical Mechanics where it serves to throw some light on critical phenomena like spontaneous magnetization and the coexistence of water and ice. In our context, these critical phenomena reappear as the breakdown of price equilibria.

2. A microeconomic approach to random economies

In this section we develop a microeconomic concept of a pure exchange economy with l commodities and with agents whose behavior is dependent on their environment and subject to random fluctuation.

Let $A \neq \phi$ be a countable set, the set of *economic agents*. Any finite subset of A will be called a *coalition*. An economic agent $a \in A$ is characterized by his *preferences*, an element $\precsim(a)$ in the set $\mathscr{P}$ of continuous complete preorderings on the commodity space R^l_+, and by his *initial endowment*, a vector $e(a) \in R^l_+$. Let us say that these two parameters constitute the *state of the agent*. Thus, the state of the agent $a \in A$ is given by a point $\omega(a) \in S$, where S is a suitable subset of $\mathscr{P} \times R^l_+$. In order to avoid measure theoretic technicalities in the sequel, let us assume right away that S, the set of possible states, is the same for all agents and is *finite*.[1] Moreover we impose, as in Hildenbrand (1971), the regularity condition that all preferences covered by S are monotonic and strongly convex.

In a deterministic model the economy is known if we know the state of each agent. In other words, an economy is usually specified by a map

$$\omega: A \to S$$

which, for each agent $a \in A$, indicates his state $\omega(a)$. In our present context we call such a map a *state of the economy*. Let us denote by Ω the set S^A of all possible states of the economy, and by $\mathscr{F}$ the σ-field on Ω which is generated by the individual states $\omega \to \omega(a)$ $(a \in A)$.

If we want to admit randomness, the most straightforward approach is to view the state of the economy as a random variable over some underlying probability space; cf. Hildenbrand (1971), Malinvaud (1972), Bhattacharya and Majumdar (1973). Equivalently, one is inclined to say that an economy should be specified by a probability measure μ on $(\Omega, \mathscr{F})$. Note however that, in a sense, this is no longer a purely microeconomic concept. In order to specify μ it is not enough to observe the individual agents and to determine, separately for each agent $a \in A$, his probability distribution

$$\mu_a(s) \equiv \mu[\omega(a) = s] \quad (s \in S).$$

In addition, one has to know the probabilities which govern the joint behavior of each coalition. Thus, the task of actually constructing μ involves collective as well as individual data. Only in the case where the different agents behave

[1] A general version of what follows would involve some compactness assumption, e.g. (γ) in Hildenbrand (1971, p. 421).

independently from each other, the individual distributions μ_a determine the global measure μ; cf. (2.4) below.

Let us now present a different approach to randomness in a pure exchange economy. Technically, it consists in using conditional rather than absolute probabilities as the initial data. Economically, these conditional probabilities will describe how the state of an economic agent – here we have mainly his preferences in mind – is affected by his environment, that is, by the states of the others – and here it makes sense to think both of preferences and endowments.

(2.1) *Definition.* An *environment* of the economic agent $a \in A$ is a map $\eta: A - \{a\} \to S$ which specifies the states of the other agents. The *characteristics* of the agent are given by a probability kernel π_a which to each environment η associates a probability measure $\pi_a(.\,|\eta)$ on S. Thus, $\pi_a(s|\eta)$ is the probability that the agent $a \in A$ assumes the state $s \in S$, given the environment η.

Note that knowledge of π_a does not presuppose information on the probabilities which govern the environment of the agent $a \in A$; in principle, π_a is obtained by letting the agent answer a number of questionnaires. Thus the following concept of a random economy is indeed a microeconomic one, in the sense that no a priori information on aggregates is required.

(2.2) *Definition.* A *random pure exchange economy* is a triple $\mathscr{E} = (A, S, \Pi)$ where Π is a collection of characteristics $\pi_a (a \in A)$. Let us call Π the (*microeconomic, local*) *characteristics* of the economy. Any probability measure μ on $(\Omega, \mathscr{F})$ which is compatible with Π in the sense that

$$(2.3)^2 \qquad \mu[\omega(a) = s|\eta] = \pi_a(s|\eta) \quad \mu\text{ – almost surely} \quad (a \in A, s \in S)$$

will be called a (*macroeconomic, global*) *phase* of the economy. We say that the local characteristics are *consistent* if they admit at least one global phase.

(2.4) *Remarks.* 1) If A is finite then the macroeconomic phase is uniquely determined by the microeconomic characteristics of the economy; cf. Spitzer (1971).

2) Hildenbrand (1971) studied pure exchange economies where the state of an economic agent is random, but independent of the states of the other agents. This means that the characteristics $\pi_a(.\,|\eta)$ actually do not depend on the environment η. Thus (2.3) implies that μ is the product measure on $\Omega = S^A$ with marginals $\mu_a(.) = \pi_a(.\,|\eta)$. In particular the economy has only one phase.

3) Malinvaud (1972) and Bhattacharya–Majumdar (1973) studied random economies over some fixed probability space $(\Omega, \mathscr{F}, \mu)$, replacing independence by conditions like strong mixing or symmetric dependence. In our terminology, they fixed some phase of the economy and discussed equilibrium prices under

[2]The left side denotes a version of the conditional probability of $\omega(a) = s$ with respect to the σ-field generated by the environments of a.

the assumption that a suitable law of large numbers prevails in that phase. Our approach is different in that we only assume information on the microeconomic characteristics. In particular we will formalize in sect. 4 Malinvaud's "cases in which the dependence may strongly affect 'neighbors' " in terms of a Markov property of the local characteristics, and then investigate to which extent such short range interaction is inherited by the global phases of the economy.

Let us now recall the role of prices. A price is a vector $p = (p_1, \ldots, p_l) \in R^l_+$ with $\Sigma_i p_i = 1$. Under a price p, an economic agent $a \in A$ in the state $\omega(a) = (\precsim(a), e(a)) \in S$ is supposed to demand a commodity vector $\varphi(\omega(a), p)$ which is $\precsim(a)$ – maximal in his budget set $\{e \in R^l_+ | p.e \leqq p.e(a)\}$. This vector is uniquely determined due to our assumptions on S. Thus p induces a well defined individual excess demand $\zeta(\omega(a), p) \equiv \varphi(\omega(a), p) - e(a)$, and a global per capita excess demand,

$$\frac{1}{|A|} \sum_{a \in A} \zeta(\omega(a), p)$$

– which makes sense as long as A is finite. From now on we will concentrate on the following question: Can we choose a price p such that for large economies the excess demand per capita is approximately zero? As in Hildenbrand (1971) we will make this statement precise by turning it into a limit law.

Let us fix, as the basic framework of our analysis, an economy $\mathscr{E} = (A, S, \Pi)$ with a countably infinite number of agents:

$$(2.5) \qquad |A| = \infty.$$

We shall now formulate our question above for large but finite subeconomies, or coalitions, of $\mathscr{E}$.

The following definition is preliminary in that the condition '(A_n) exhausts A' will be specified later. It does not only mean $\cup_{n=1}^{\infty} A_n = A$ but also that the coalitions A_n are 'good representatives' of A; cf. the definition preceding (3.6).

(2.6) *Definition.* We say that a price p *equilibrates*, or *stabilizes, the phase* μ of the economy $\mathscr{E}$ if

$$(2.7) \qquad \lim \frac{1}{|A_n|} \sum_{a \in A_n} \zeta(\omega(a), p) = 0 \quad \mu\text{ – almost surely}$$

whenever (A_n) is an increasing sequence of coalitions which exhausts A. Let us say that p *equilibrates*, or *stabilizes, the economy* $\mathscr{E}$ if p equilibrates each phase of $\mathscr{E}$.

Thus the question is twofold. First,

(2.8) can we stabilize a given phase?

and, secondly,

(2.9) can we stabilize just on the basis of our microeconomic data, irrespective of the specific phase which may prevail?

We will now investigate both parts for a special class of economies where the structure of interaction is particularly transparent.

3. Markovian interaction

Assume that the state of an agent $a \in A$, and in particular his choice of preferences, is influenced by just a finite number of other agents, his *reference group*. Let us call these reference persons the *neighbors* of the agent. Denoting the set of neighbors of a by $N(a)$, we may formalize our assumption as follows:

(3.1) $\pi_a(.|\eta) = \pi_a(.|\eta')$ as soon as the environments η and η' coincide on $N(a)$ $(a \in A)$.

To introduce the notion of neighbors means to endow the countable set A with the structure of a graph. A mathematical theory of Markovian interaction on general graphs is now developing, and we refer to Preston (forthcoming) for its present state. In this paper we limit ourselves to the case, best understood so far, where A carries a lattice structure. That is, we assume from now on

(3.2) $A = Z^v \equiv \{a = (a_1, \ldots, a_v) | a_i \text{ integer}\}$

for some $v \geqq 1$ and define the reference group of an agent $a \in A$ as

(3.3) $N(a) \equiv \{b \in A | \, \|b-a\| = 1\}$,

where $\|a\| = (\Sigma_i a_i^2)^{\frac{1}{2}}$. Thus, any agent has $2v$ neighbors which may influence his individual state.

(3.4) *Definition.* An economy $\mathscr{E} = (A, S, \Pi)$ with (3.2) and (3.3) whose local characteristics are consistent and satisfy (3.1) is called a *Markov economy.* We say that $\mathscr{E}$ is *homogeneous* if Π is translation invariant, that is, if all agents react in the same manner to their environment. A phase μ is called homogeneous if μ is a translation invariant measure.[3]

Clearly, an extremely homogeneous structure of dependence like (3.4) can't claim much economic realism, although some of its specific features are noteworthy from an economic point of view. For example, it seems quite remarkable

[3]For $a \in A$ consider the shift $\theta_a: \Omega \to \Omega$ defined through $[\theta_a\omega](b) = \omega(a+b)$. Translation invariance of Π means $\pi_{a+b}(.|\eta) = \pi_a(.|\eta_0\theta_b)$ where $\eta_0\theta_b(c) = \eta(b+c)$ $(a, b, c \in A)$. Translation invariance of μ means $\mu_0\theta_a = \mu$ $(a \in A)$.

that inhomogeneities and disparities may arise in the macroeconomic picture even if the microeconomic characteristics are entirely homogeneous (cf. the symmetry breakdown in Ising economies discussed below). However, in view of a systematic approach to dependence among economic agents, our discussion of the lattice case can only serve as an introduction to some of the techniques involved, and to the kind of critical phenomena one should expect in general.

Let us now fix a homogeneous Markov economy $\mathscr{E} = (A, S, \Pi)$ and write

$$\Phi(\mathscr{E}) = \text{the set of all phases of } \mathscr{E},$$

$$\Phi_0(\mathscr{E}) = \text{the set of all homogeneous phases of } \mathscr{E}.$$

Consistency of the microeconomic characteristics is known to imply

$$|\Phi(\mathscr{E})| \geqq |\Phi_0(\mathscr{E})| \geqq 1, \tag{3.5}$$

and both inequalities can be strict; cf. Spitzer (1971). The case $|\Phi(\mathscr{E})| > |\Phi_0(\mathscr{E})|$ is often called a *symmetry breakdown.* It means that, although the individual agents are all governed by the same conditional probability law, the global phase may be inhomogeneous, and in particular the individual distributions μ_a may vary from agent to agent. The case $|\Phi(\mathscr{E})| > 1$ is often called a *phase transition*, suggesting that fixed local characteristics do not exclude a spontaneous switch from one global phase to another.

Both $\Phi(\mathscr{E})$ and $\Phi_0(\mathscr{E})$ are metrizable simplices with respect to the weak topology on the space of measure over the compact space Ω; cf. Choquet (1969), Georgii (1972). Thus, by Choquet's integral representation theorem, each phase (resp. each homogeneous phase) can be written as a mixture of extreme points in $\Phi(\mathscr{E})$ [resp. $\Phi_0(\mathscr{E})$]. Let us call each extreme point of $\Phi_0(\mathscr{E})$ a *pure phase.*

Returning to our questions (2.8) and (2.9) we say that an increasing sequence (A_n) of coalitions *exhausts* A if it expands to A in approximately the same manner as the coalitions $B_n = \{a \in A \mid \|a\| \leqq n\}$. To be precise, we require $A_n \subseteq B_n$ and the existence of some integer N and some $\delta > 0$ such that A_n is the disjoint union of at most N boxes parallel to the axes of the lattice A and satisfies $|A_n|\,|B_n|^{-1} \geqq \delta$.

(3.6) *Theorem. Any pure phase can be stabilized. In particular we can equilibrate the economy as soon as it admits only one phase.*

Proof. The key fact is that any pure phase is ergodic; cf. Georgii (1972). Thus Pitt's extension of Wiener's ν-dimensional ergodic theorem applies and yields

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sum_{a\in A_n} \zeta(\omega(a), p) = E_\mu[\zeta(\omega(0), p)] \quad \mu-\text{almost surely} \tag{3.7}$$

whenever (A_n) exhausts A and μ is pure; cf. Georgii (1972). E_μ denotes expectation with respect to the pure phase μ. We are thus left with the problem of choosing the price p such that the right side vanishes, that is, such that a given agent is satisfied at least on the average. But this is solved combining a result of Hildenbrand on random demand [Hildenbrand (1971, prop. 1)] with a result of Debreu (1970, p. 390) on price equilibria as indicated in Hildenbrand (1971, p. 421). As to the second statement, note that a phase is pure as soon as it is unique due to (3.5).

The theorem answers our question (2.9) in the affirmative if there is no phase transition. Let us now quote two conditions on the microeconomic characteristics which guarantee that this is the case.

For the rest of the paper we assume that the conditional probabilities in (3.1) are all strictly positive. Then a theorem of Averintzev (1970) asserts that the microeconomic characteristics are consistent if and only if they can be represented in the form

$$\pi_a(s|\eta) = Z(a,\eta)^{-1} \exp\,[\gamma(a,s) + \sum_{b\in N(a)} U(a,b,s,\eta(b))]. \tag{3.8}$$

For a quick proof via the Möbius inversion formula as well as for the extension to general graphs we refer to Preston (forthcoming). $Z(a, \eta)$ is just a normalization factor to guarantee $\Sigma_s\, \pi_a(s|\eta) = 1$. The function U satisfies

$$U(a, a', ., .) = 0 \quad \text{if} \quad \|a-a'\| \neq 1, \tag{3.9}$$

which corresponds to the Markov property,[4] and homogeneity of the local characteristics is equivalent to the relations

$$U(a+b, a'+b, ., .) = U(a, a', ., .), \quad \gamma(a+b, .) = \gamma(a, .). \tag{3.10}$$

This representation makes explicit to which extent the individual agent's state is *inner directed*, and to which extent it is *outer directed*, i.e. dependent on his environment. The endogeneous component is governed by the function γ, the exogeneous component by the coupling factors $U(a, a', s, s')$ which describe the intensity of interaction between the agents a and a' if their respective states are s and s'.

(3.11) *Remark.* (3.8) may be used to characterize homogeneous phases by a variational principle: $\Phi_0(\mathscr{E})$ coincides with the set of all those translation invariant probability measures μ on Ω which minimize $e_{U,\gamma}(\mu) - h(\mu)$, the *free energy per capita* under μ. Here $h(\mu)$ denotes the *entropy per capita*, and

$$e_{U,\gamma}(\mu) = E_\mu[\gamma(a, \omega(a)) + \tfrac{1}{2} \sum_{b\in N(a)} U(a, b, \omega(a), \omega(b))],$$

the *energy per capita* under μ. In particular we see that among all homogeneous random economies μ on Ω which realize a fixed intensity level $e_{U,\gamma}(\mu) = e_0$,

[4] If we replace (3.9) by the condition $\Sigma_{a'} \max_{s,s'} U(a, a', s, s') < \infty$ then we obtain a model for infinite range interaction 'decaying at infinity', and the results of this section remain valid in that case; cf. Georgii (1972).

our homogeneous phases are exactly those with maximal entropy $h(\mu)$. In this sense, the phases may be viewed as equilibrium states of an economy whose structure of interaction is described by the functions γ and U. The variational characterization of homogeneous phases is due to Lanford and Ruelle; a probabilistic proof can be found in Föllmer (1973).

The representation (3.8) is unique if we require $\gamma(.,s_0) = U(.,.,s_0,.) = U(.,.,.,s_0) = 0$ for some reference state $s_0 \in S$. With this normalization we have $U = 0$ if and only if there is no interaction at all, and we know that in this case there is no phase transition; cf. (2.4). Condition (3.13) extends this fact to the case of *weak interaction.*

(3.12) *Theorem.* [*Spitzer* (*1971*), *Dobrushin* (*1968*)]. *There is no phase transition if either*

(3.13) $\max |U(.,.,.,.)|$ *is small enough,*

i.e. if the economic agents are sufficiently inner directed, or

(3.14) $\nu = 1$,

i.e. if the structure of interaction is one-dimensional.

The theorem suggests that we should be ready for the case $|\Phi(\mathscr{E})| > 1$ as soon as the interaction is both strong and complex enough. Note that in the presence of phase transition our result (3.6) is of very limited scope: if there are two different phases then there is also an infinity of non-pure phases due to the convexity of $\Phi(\mathscr{E})$. In order to see more clearly how question (2.8) is affected by phase transition, let us now look at a famous example which, despite its seeming simplicity, exhibits already the kind of critical phenomena one has to expect.

4. Ising economies

Let $\mathscr{E} = (A, S, \Pi)$ be a homogeneous Markov economy. We say that $\mathscr{E}$ is *egalitarian* if it is egalitarian both in initial endowments and in attitudes:

(4.1) $e(a) = e \in {}^{\circ l}_{+} \quad (a \in A)$

i.e., the initial endowments are well determined (not random) and all equal to the same commodity vector e in the interior of R^l_+, and

(4.2) π_a is rotation invariant $\quad (a \in A)$

i.e., each agent reacts in the same manner to neighbors above, below etc.

(4.3) *Definition.* An *Ising economy* is an egalitarian economy with two goods and two exclusive preferences.

From now on we assume that $\mathscr{E}$ is an Ising economy. Since the endowments are all equal to the same vector $e = (e_1, e_2) \in R^2_+$, we may now use as the set of possible states

$$S = \{+1, -1\}.$$

The interpretation of $\omega(a) = +1$ is that the agent $a \in A$ wants as much as possible of the first good without caring about the second. In the same way $\omega(a) = -1$ means exclusive preference of the second good. Due to rotation invariance, the representation (3.8) can now be written in the form

$$\pi_a(\pm 1|\eta) = Z(\eta)^{-1} \exp\,[\pm(\gamma + J \sum_{b \in N(a)} \eta(b))] \tag{4.4}$$

for certain constants γ and J. If $J > 0$ then an economic agent has a propensity to go with the trend as it is reflected in the states of his neighbors. If $J < 0$ then he rather tends to go against the trend.

(4.5) *Definition.* The Ising economy is called *cyclic* if $J > 0$ and *anticyclic* if $J < 0$. It is called *outer directed* if $\gamma = 0$.

Consider a phase $\mu \in \Phi_0(\mathscr{E})$ and let us first see for which price $p = (p_1, p_2)$ the expected excess demand of an agent $a \in A$ vanishes in that phase. By homogeneity of μ, such a price p will induce an expected excess demand

$$E_\mu[\sum_{a \in A_0} \zeta(\omega(a), p)] = (0, 0) \tag{4.6}$$

for any coalition $A_0 \subseteq A$. Under any price p the excess demand of an agent is

$$\zeta(+1, p) = \left(\frac{p_2}{p_1} e_2, -e_1\right) \text{ resp. } \zeta(-1, p) = \left(-e_1, \frac{p_1}{p_2} e_2\right)$$

if his state is $+1$ resp. -1. Writing $\mu_1 = \mu[\omega(a) = +1]$ and $\mu_2 = \mu[\omega(a) = -1]$ – which does not depend on a by homogeneity – we see that (4.6) requires

$$\mu_1\left(\frac{p_2}{p_1} e_2, -e_1\right) + \mu_2\left(-e_1, \frac{p_1}{p_2} e_1\right) = (0, 0),$$

which is equivalent to the condition

$$\frac{p_2}{p_1} = \frac{e_1}{e_2}\,\frac{\mu_2}{\mu_1}. \tag{4.7}$$

Now let us turn to the problem of stabilizing the economy resp. of stabilizing a given phase. We know by (3.6) and (3.12) that we can equilibrate $\mathscr{E}$ if $\nu = 1$ or if the interaction is weak enough, that is, if J is sufficiently small in absolute value. Let us therefore assume $\nu \geqq 2$, and let us consider the *cyclic case* with arbitrary $J > 0$. Here it is known that there is no phase transition if $\gamma \neq 0$; cf. Spitzer (1971). Thus (3.6) yields the following:

(4.8) *Proposition. A cyclic Ising economy can be stabilized as soon as there is a residuum of autonomy in the agents' choice of preferences.*

Now we add the assumption that $\mathscr{E}$ is outer directed: $\gamma = 0$. Then there is a critical value J_0 depending on the dimension $\nu \geqq 2$ such that for $J > J_0$ there are exactly two pure phases; cf. Georgii (1972). These two pure phases, call them μ^1 and μ^2, satisfy the relation

$$(4.9) \qquad \mu_1^1/\mu_2^1 = \mu_2^2/\mu_1^2 > 1,$$

where we use the notation of (4.7). Let us now look at the limit behavior of *per capita* excess demand in this case, i.e. for

$$(4.10) \qquad \nu \geqq 2, \quad \gamma = 0, \quad J > J_0.$$

Denoting expectation with respect to μ^i by E^i $(i = 1, 2)$, we have, by the ergodicity of pure phases and by Wiener's ν-dimensional ergodic theorem [recall the proof of (3.6)],

$$(4.11) \qquad \frac{1}{|A_n|} \sum_{a \in A_n} \zeta(\omega(a), p \to E^i[\zeta(\omega(0), p)]$$

μ^i – almost surely for $i = 1, 2$ whenever (A_n) exhausts A. (4.7) and (4.9) show that there is no price p which makes the right side of (4.11) vanish simultaneously for $i = 1$ and $i = 2$. In particular we see that we cannot stabilize the economy. But the situation is worse than that. Take any homogeneous phase $\mu \in \Phi_0(\mathscr{E})$. By the remarks preceding (3.6) we can write

$$(4.12) \qquad \mu = \lambda\mu^1 + (1-\lambda)\mu^2$$

for some $\lambda \in [0, 1]$. But (4.12) implies that (4.11) holds μ – almost surely on $\Omega^i (i = 1, 2)$, where $\Omega = \Omega^1 + \Omega^2$ is a disjoint partition such that Ω^i carries μ^i [recall that different ergodic measures are orthogonal; Jacobs (1963)]. We just saw that there is no way of annihilating the right side in (4.11) for $i = 1, 2$ with one and the same p. This means that there is no way of stabilizing the phase μ as soon as μ puts positive weight both on Ω^1 and Ω^2, i.e. if $\lambda \in (0, 1)$ – which happens 'almost always'. In this sense we may say:

(4.13) *Proposition. An outer directed cyclic Ising economy with strong and complex interaction can 'almost never' be stabilized.*

This discussion of cyclic Ising economies is complete in the case $\nu = 2$ where it is known that all phases are homogeneous. For $\nu \geqq 3$ and strong enough interaction there is, in addition to the 'interval' $\Phi_0(\mathscr{E})$, an infinity of non-homogeneous phases, and in this sense matters get even worse; cf. Spitzer (1971).

A final remark on the *anticyclic case*. Here the agents tend to act against the trend, and this type of behavior is often recommended in view of maintaining an equilibrium situation. In an Ising economy, one might thus be tempted to expect that equilibration becomes easier for $J < 0$. In a sense, the contrary is true. Even for $\gamma \neq 0$, in the presence of a certain autonomy in the individual's decisions, there may occur a phase transition; cf. Spitzer (1971). And even for $\nu = 2$ there are non-homogeneous phases as soon as $J < -J_0$; cf. Spitzer (1971). In another sense, however, the situation does indeed look better than in the cyclic case. Di Liberto (1973) just showed $|\Phi_0(\mathscr{E})| = 1$ for $\nu = 2$ and sufficiently strong interaction. This means that the economy can indeed be stabilized if one has some a priori reason to disregard the infinity of non-homogeneous phases.

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