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Optimal design and defense of networks under link attacks

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Keywords:
Deferred acceptance mechanism, matching, experiment, experience transmission.

JEL codes:
C78, C92, D47, D82

Working Paper

Optimal design and defense of networks under link attacks*

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Abstract

Networks facilitate the exchange of goods and information and create benefits. We consider a network with complementary nodes, i.e. nodes need to be connected to generate a positive payoff. This network may face intelligent attacks on links. To study how the network should be designed and protected, we develop a strategic model inspired by Dziubiński and Goyal (2013) with two players: a Designer and an Adversary. First, the Designer forms costly protected and non-protected links. Then, the Adversary attacks k links given that protected links cannot be removed by her attacks. The Designer designs a network that minimizes its costs given that it has to resist the attacks of the Adversary. We establish that in equilibrium the Designer forms a minimal 1-link-connected network which contains only protected links, or a minimal $(k+1)$-link-connected network which contains only non-protected links, or a network which contains one protected link and $(n-1)(k+1)/2$ non-protected links. We also examine situations where the Designer has a limited number of protected links and situations where protected links are imperfect, i.e. protected links are removed by attacks with some probabilities. We show that if the available number of protected links is limited, then there exists an equilibrium network which contains several protected and non-protected links. In the imperfect defense framework, we provide conditions under which the results of the benchmark model are preserved.

JEL Classification: D74, D85.

Key Words: Attacks on links, Network defense, Network design.

1 Introduction

Networks can be seen as communication structures. They are composed of nodes and links, where links represent the flow of information. Networks represent a crucial feature in our society, and are of

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particular interest in different fields such as military defense, telecommunication or computer networks. Some networks can be damaged by natural disasters or intelligent attacks. Attacks can affect nodes (agents, computers, telecommunication antennas, ...) or links (roads, communications flows, ...), and may disconnect a network.[1] In this paper, we examine situations where attacks target links. To illustrate this type of situations, suppose a firm which consists of several production units (nodes of the network). Each production unit produces a part of the product and the parts are assembled by a given production unit. The links of the network allow the parts of the good to be transferred among the units. If one unit is not connected to the rest of the units, its part cannot be transferred and the product has no value. Recall that during the Second World War, the production units for the weapons (nodes) were buried, so they were impossible to target, and attacks had to target the roads (links) in order to destroy the production process of the enemy. Therefore, the issue was to design a network of communication between the production units that the enemy could not disconnect.

Our goal is to examine how to design and protect the network in an optimal way, such that the network remains connected after an intelligent link attack.[2] We say that a network is designed and protected in an optimal way when the costs associated to the design and the protection of the network are minimized. We consider a two-stage game with two players: a Designer (D) and an Adversary (A).

- Stage 1. The Designer moves first and chooses both a set of protected, and a set of non-protected links. Protected links cannot be removed by the attacks of the Adversary.
- Stage 2. After observing the network (strategy) formed by the Designer, the Adversary attacks the network by allocating attacks to specific links. The number of attacks, k, available for the Adversary is given.

Creating protected and non-protected links is costly for the Designer. The benefits obtained by the Designer at the end of the game depends on the connectivity of the residual network, that is the network obtained after the attack of the Adversary. If the residual network is connected, then the Designer wins the game: her benefits are equal to 1 and the benefits of the Adversary are 0. If the residual network is not connected, then the Adversary wins the game: her benefits are equal to 1 and the benefits of the Designer are 0. The payoff obtained by the Designer is equal to the difference between the benefits and the costs associated to her strategy. The payoff obtained by the Adversary is equal to its benefits. If we take again our military example, and assume that node $i-1$ is the supplier of node i, then the Designer has to maintain a path between each pair of nodes $i-1$ and i to obtain some end goods. In other words, the residual network has to be connected to allow some production.

We are interested in the Sub-game Perfect Equilibrium (SPE) of the two-stage game. We assume that the cost of protected links and non-protected links are sufficiently low so that the Designer has some profitable strategies which allow the residual network to be connected. First, given the number of nodes and the available number of attacks of the Adversary, we provide the minimal cost for each number of protected links chosen by the Designer when she designs a network that resists an optimal attack

[1] A network is connected if no set of nodes is isolated from the rest of nodes.

[2] Note that an intelligent attack can also be seen as the worst case of a random or natural attack.

of the Adversary. Second, we establish that three non-empty networks may arise in equilibrium in the benchmark model.

1. A minimal $(k+1, n)$-link-connected network which contains no protected links.[3]
2. A minimal $(1, n)$-link-connected network which contains only protected links.
3. A network g which contains one protected link and $(n-1)(k+1)/2$ non-protected links.

The first network is the unique SPE when the cost of forming non-protected links is sufficiently low relative to the cost of forming protected links. The second network is the unique SPE when the cost of forming non-protected links is sufficiently high relative to the cost of forming protected links. The third network is the unique SPE for intermediate relative costs (cost of a protected link / cost of a non-protected link) when the number of nodes is odd and the number of attacks is even.

Additionally to the benchmark model described above, we study some variations of the game to develop a larger understanding of optimal design of networks. We take into account two types of limitations concerning protections. First, we consider that D has a number of protected links that is smaller than in the benchmark model.[4] Then, we consider situations where each protected link has a probability π to be removed when it is attacked by A.[5]
In the framework where the number of protected links available for D is low, we show that for intermediate relative costs (cost of a protected link / cost of a non-protected link), strategy where D designs a network which contains both protected links and non-protected links is a SPE. In the framework where protected links are removed by attacks with some probabilities, we provide conditions under which the results obtained in our benchmark model are preserved.

We now relate our paper to the existing literature on networks. This literature has become broader in the recent years (*Jackson* [18], *Goyal* [10] and *Vega-Redondo* [24]). The two seminal papers on the formation of social and economic networks are the paper of *Jackson and Wolinsky* [19] and the paper of *Bala and Goyal* [3]. *Bala and Goyal* [4] and *Haller and Sarangi* [14] introduce imperfectly reliable links in the *Bala and Goyal* [3] model. *Bala and Goyal* [4] show that for certain ranges of linking cost and probability of failure, the equilibrium network is at least $(2, n)$-link-connected, i.e. two nodes are connected by at least two paths. *Haller and Sarangi* [14] extend the model of *Bala and Goyal* [4] by allowing heterogeneity in probabilities of link failure. These authors model random link failure but not an intelligent attack that seeks to interrupt the communication flow. In this paper, we study the robustness of a network that must be designed and protected to resist an intelligent attack on links.
A growing literature on attacked networks studies situations where the Adversary attacks the nodes. Dziubinski and Goyal (DG, [9]) study the optimal design and defense of network under an intelligent attack on nodes. In DG's framework, there are two players: the Designer and the Adversary; the Designer can

[3] A network g, which contains n nodes, is a minimal $(k+1, n)$-link-connected network, if it is not possible to disconnect it by removing k links, and such that there is no network which cannot be disconnected by removing k links and contains a smaller number of links.

[4] If we take again our military example, the ennemy may not have enough resources to protect the whole network.

[5] Despite the effort of the Designer (of the army) to protect the communication flow, the Adversary (the enemy) may still be able to succeed in destroying protected links with some probabilities.

form links between n nodes, and/or protect these nodes to ensure their survival. The model we propose is close to the model of DG. The major differences between the DG's framework and our framework are the following.

- The Adversary attacks nodes in the DG's framework while she attacks links in our framework;
- In our framework, the Designer wins the game if every node of the population is able to communicate with each other node in the residual network. In the DG's framework, the Designer wins the game if the residual network is connected whatever the number of nodes removed by the Adversary. Thus, our setting is based on the complementarity of nodes while DG assume that nodes are substitutable.

DG show that in a SPE, the Designer protects 0 or 1 node. If the Designer protects 0 node, then she designs a minimal $(k+1, n)$-node-connected network.[6] We obtain the same type of networks when the Designer uses no protection in the DG's framework and in our framework. At first sight, this result seems intriguing since the Adversary attacks nodes in DG's paper and links in our paper. However, a minimal $(k+1, n)$-node-connected network defined as in DG is also a network that contains the minimal number of links and resists the Adversary who attacks links. In DG, if the Designer uses protections, she designs a star network[7] and protects 1 node, the central node. In our framework, when D uses protections, she designs either a network which contains 1 protected link and $(n-1)(k+1)/2$ non-protected links, or a network which contains $n-1$ protected links. The results differ because in our framework every node needs to be connected with each other node in the residual network. Moreover, we establish that if we limit the number of available protections, then there exist situations where D designs networks which contain several protected and non-protected links. This result follows the discontinuity in the number of non-protected links that each protection allows the Designer to save.

DG examine imperfect defense through an example. They assume that the protections used by the Designer can fail when they are attacked by the Adversary. More precisely, an attack on an unprotected target always destroys the target, and an attack on a protected target destroys the target with a positive probability. A recent independent work of *Landwehr* [21] extends the analysis of imperfect defense. He shows that for a certain range of protection cost and cost of forming links, strategies that use both protections and several links are equilibria.

Hoyer and De Jaegher [17] consider a framework where the Designer has to shape the network and form enough links in order to resist the attacks. In this framework, the Designer cannot protect specific parts of the network. The authors study the optimal way to design a network under link or node deletion with various cost levels. They show that if the costs of forming links are low, a regular network[8] with a sufficient number of links is the optimal network for the Designer. If costs are high and links are attacked, then a star network is optimal for the Designer. The difference with our paper (except for the fact that they do not use protected links) is that in our framework, nodes are complementary and the Designer cannot sacrifice any node to minimize her costs. *Haller* [13] extends the model of *Hoyer and De Jaegher*

[6] A minimal $(k+1, n)$-node-connected network is a network, which contains n nodes, that cannot be disconnected by removing k nodes, and such that there is no network which cannot be disconnected by removing k nodes and contains a smaller number of links.

[7] A star network is a network where one node, the central one, is linked with all other nodes, and other nodes are only linked with the central node.

[8] A network where all nodes have the same number of links.

by adding the possibility for two nodes to be connected by more than one link. In that case, it is more difficult for the Adversary to disconnect the network. The possibility for multiple links between nodes can be seen as a different way to protect a connection between specific nodes than ours.

A part of the literature on attacked networks examines the role played by the contagion of attacks in networks. *Goyal and Vigier* [11] extend the work of DG by allowing the contagion of attacks (or threats). They find that the star network with a protected central node remains an equilibrium network. *Cabrales, Gottardi and Vega-Redondo* [6] and *Baccara and Bar-Isaac* [2] study the contagion of attacks in networks respectively in the field of financial firms where a financial risk can spread between connected firms and in the field of criminal networks where connectivity increases vulnerability because of external threats.[9] *Cerdeiro, Dziubinski and Goyal* [7] and *Acemoglu, Malekian and Ozdaglar* [1] identify nodes to players. *Cerdeiro, Dziubinski and Goyal* [7] propose a three-stage game. First, the Designer chooses the network. Second, each player observes the network and chooses independently and simultaneously if they invest in protection or not. Third, the Adversary observes the protected network and chooses the players to infect. In *Acemoglu, Malekian and Ozdaglar* [1] nodes/players are connected in a random network. Players have to invest in protection to be immune. Their investment depends on their links and the probability of being infected in the random network. This model allows to examine for instance the impact of a contagious disease on the individual behavior. These papers are different from the present one for two reasons. First, we take into account situations where an attack on a link can remove only this specific link. Indeed, literature on contagious attacks reflects situations such as epidemics or virus spreading while our paper is focused on the study of specific link removal (for military strategies for instance). Second, in our model nodes cannot influence the architecture of the network by their decision.[10]

The rest of the paper is organized as follows. In section 2, we present the model setup. In section 3, we present our main results. In section 4, we extend our model by examining situations where the number of protected links available for the Designer is limited, and situations where protected links have some probabilities to be removed by an attack. In section 5, we conclude.

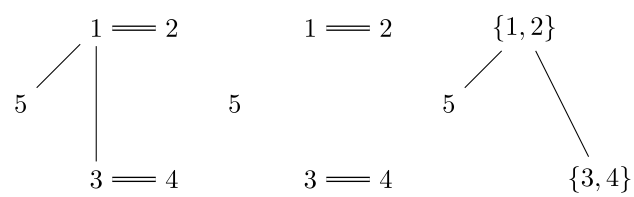
[9] *McBride and Hewitt* [22] study the best way to dismantle a criminal network with imperfect information on its architecture. There also exists a literature which examines the particular cases of terrorist attacks, transportation network security, and homeland security (see *Brown, Carlyle, Salmeron and Wood* [5], *Tambe* [23], and *Hong* [16].)

[10] Additionally to economic, several fields study problems close to the one we deal with. In an early graph theoretic work, *Harary* [15] exhibits a family of (k, n)-node-connected networks with a total number of links that is minimal. This family of networks is crucial to establish our results. *Groetschel, Monma and Stoer* [12] study a situation where a firm has to prevent a communication network to be disconnected given that there exist possibilities of communication failure. As some connections may be interrupted, the firm has to design the least costly network that guarantees the best service for the consumers. Moreover, there also exists a literature on the design of survivable networks (see the survey of *Kerivin and Mahjoun* [20]) in Computer Science. *Cunningham* [8] studies network security and considers a model where the Designer allocates a different number of defense units to each link. A defended link has a level of resistance that depends on the number of defense units the Designer has allocated to it. The Adversary allocates attack units to remove a link. A link is removed if more attack units than defense units have been allocated to this link. The author proposes an algorithm which exhibits how some links have to be reinforced in order to protect the network.

2 Model setup

To simplify the notation, we set $\llbracket a, b \rrbracket = \{a, \dots, b\}$. Let $\mathcal{E}$ be the set of even natural numbers and $\mathcal{O}$ be the set of odd natural numbers. Moreover, $\lfloor x \rfloor$ and $\lceil x \rceil$ are the smallest integer smaller or equal to x and the smallest integer higher or equal to x respectively.

Network. An *undirected network* g is an ordered pair of disjoint sets (N, E) such that E is a subset of the set $N \times N$ of unordered pairs of N. The set $N = \llbracket 1, n \rrbracket$, with $n \geq 4$, is the set of nodes and E is the set of links. If g is a network, $E(g)$ is the link set of g. A link $\{i, j\}$ is said to join the nodes i and j and is denoted ij. If there exists a link between i and j in g, then i and j are *adjacent*, and nodes i and j are *incident* to the link ij. Let $\mathcal{G}$ be the set of undirected networks with n nodes. We define $\mathbf{E} = \{ij : i \in N, j \in N \setminus \{i\}\}$ as the set of links of the network where all links have been formed. We consider two types of links: the *protected* ones and the *non-protected* ones. Let $E_P(g)$ be the set of protected links in g and $E_{NP}(g)$ be the set of non-protected links in g, with $E(g) = E_P(g) \cup E_{NP}(g)$, and $E_P(g) \cap E_{NP}(g) = \emptyset$. To simplify notation, we let $|E_P| = p$. Protected links are not removed by the attacks considered in the benchmark model and non-protected links are removed by these attacks. Let $d_i(g)$ be the number of links incident to the node i in g, that is the *degree* of node i in g. We say that $g' = (N', E')$ is a subnetwork of $g = (N, E)$ if $N' \subseteq N$ and $E' \subseteq E$. A *path* between two nodes i_0 and i_L of a network g is a finite alternating sequence of *distinct* nodes and links $i_0, i_0i_1, i_1, \dots, i_{L-1}i_L, i_L$ where $i_\ell \in N$ for all $\ell \in \llbracket 0, L \rrbracket$ and $i_\ell i_{\ell+1} \in E(g)$ for all $\ell \in \llbracket 0, L-1 \rrbracket$. A *cycle* is a path where $i_0 = i_L$. A network g is *connected* if for each pair of nodes $(i, j) \in N \times N \setminus \{i\}$, there exists a path between them. Subnetwork $g' = (N', E')$ is a component of network g if g' is connected and there is no connected subnetwork of g, $g'' = (N'', E'')$, such that $N' \subseteq N''$. By convention, a node $i \in N$ such that $d_i(g) = 0$ is a component. We define $g_P = (N, E_P)$ as the subnetwork of g which contains only the protected links of g. Given a link $ij \in E_P(g)$, the network $g \setminus ij$ is obtained by contracting the link ij; that is, to get $g \setminus ij$ we identify the nodes i and j and remove all resulting links joining a node to itself. The E_P-contraction of network g, $\hat{g}^{E_P}$, is obtained from g by sequences of links contraction for all links in $E_P(g)$.[11] We illustrate the E_P-contraction of network g in Figure 1.



(a) Network g (b) Network g_P (c) Network $\hat{g}^{E_P}$

Figure 1: Illustration of E_P-contraction

[11]In the E_P-contraction, two nodes may be linked by several links; in this case it is a *multigraph*. Formal definition of a multigraph is given in the appendix.

Two player game. The players are the Designer (D) and the Adversary (A). We consider a two-stage game where D moves first and A moves at the second stage. A strategy for D consists in a pair $\boldsymbol{g} = (E(g), E_P(g))$, with $g \in \mathcal{G}$ and $E_P(g) \subseteq E(g)$. In other words, D designs a network g and she protects a subset of links of g. Let G be the set of strategies of D. Given a maximal number of attacks, $k \in [\![0, n-3]\!]$, a strategy for A is a function $E_a : G \to 2^{\mathbf{E}}$, $E_a : \boldsymbol{g} = (E(g), E_P(g)) \mapsto E_a(\boldsymbol{g})$, with $E_a(\boldsymbol{g}) \subseteq E(g)$ and $|E_a(\boldsymbol{g})| \leq k$. In other words, A attacks a subset of links formed by D given that she can attack at most k links. To sum up, at the first stage D chooses a strategy $\boldsymbol{g} = (E(g), E_P(g))$, and at the second stage, A attacks a subset of links formed by D, given that A attacks at most k links.

Residual network and benefits. Given the strategy $\boldsymbol{g} = (E(g), E_P(g))$ played at the first stage by D and the removal of the set of links $E_a(\boldsymbol{g}) \subseteq E(g)$ at the second stage by A, we obtain a *residual network* g_R such that $E(g_R) = E_P(g) \cup (E_{NP}(g) \setminus E_a(\boldsymbol{g}))$. By construction, $g_R = (N, E(g_R))$ is a subnetwork of g. The benefits of D are given by:

$$\phi(g_R) = \begin{cases} 1, \text{ if } g_R \text{ is connected,} \\ 0, \text{ otherwise.} \end{cases} \tag{1}$$

Network and costs. Both protected and non-protected links are costly. We assume linear costs: each protected link has a strictly positive cost $c_P > 0$ and each non-protected link has a strictly positive cost $c_L > 0$. We assume that $c_P > c_L$. The cost of a defended network is

$$c(\boldsymbol{g}) = c_L |E_{NP}(g)| + c_P |E_P(g)|. \tag{2}$$

If the cost of protected or non-protected links is too high, then D cannot use strategy where she forms protected links or non-protected links. Therefore, to obtain non trivial results, we assume that the cost of protected and non-protected links are sufficiently low to allow D to form the number of protected and non-protected links necessary to protect the network (given k). More specifically, in the rest of the paper we assume that $c_P < 1/(n-1)$ and $c_L < 1/(\lceil n(k+1)/2 \rceil)$.

Payoffs. The payoff of the Designer from choosing $\boldsymbol{g} = (E(g), E_P(g))$ when the Adversary chooses $E_a(\boldsymbol{g}) \subseteq E(g)$ is

$$\Pi^D(\boldsymbol{g}, E_a(\boldsymbol{g})) = \phi(g_R) - c(\boldsymbol{g}) \tag{3}$$

The payoff associated with g_R obtained by A is $1 - \phi(g_R)$.

To sum up, the objective of the Designer is to obtain a connected residual network at a minimal cost. The objective of the Adversary is to obtain a residual network that is disconnected. Hence, her goal is to isolate a part of the network.

We now provide some illustrations where the payoff function given in equation 3 captures the payoff of D. Suppose that D has n production units identified to nodes. Let y_i be the output of production unit i, and δ_i be a Kronecker index, such that $\delta_i = 1$ if there is a path between $i \in [\![2, n]\!]$ and production unit $i-1$, and $\delta_i = 0$ otherwise. Here, production unit $i-1$ can be interpreted as the *unique* supplier of production unit i. We assume that $y_1 = \gamma$, $\gamma > 0$, and $y_i = \delta_{i-1}\gamma$ for $i \in N \setminus \{1\}$. If the total output D obtained from the production units is $Y = y_n$, then the total output function is in line with our payoff function. Similarly, if $Y = \min_{i\in N}\{y_i\}$ or $Y = \prod_{i\in N}(y_i)^{\rho_i}$ with $\rho_i > 0$, then the total output function is in line with our payoff function.

Moreover, let nodes be identified to cities and links be identified to communication flows between the cities. Public authorities may have an incentive to maintain communication between all the cities when some communication flows are broken because of a natural disaster or a strategic attack. Indeed, if some cities are isolated from the others, then it is difficult for the public authorities to rescue inhabitants of these cities.

Sub-game Perfect Equilibrium (SPE). At equilibrium, for every $\boldsymbol{g}$, A chooses a set of links $E_a(\boldsymbol{g}) \subseteq E_{NP}(g)$ resulting in g_R such that $\phi(g_R)$ is minimized. Given the best response outcome g_R, D achieves payoff $\phi(g_R) - c(\boldsymbol{g})$ when choosing $\boldsymbol{g}$. At the specific equilibrium, D chooses strategy $\boldsymbol{g}$ that maximizes her payoff. More formally, a SPE is a list $((E^\star(g^\star), E_P^\star(g^\star)), E_a^\star(\boldsymbol{g}^\star))$, with $E_P^\star(g^\star) \subseteq E^\star(g^\star)$, $E_a^\star(\boldsymbol{g}^\star) \subseteq E_{NP}^\star(g^\star)$ that prescribes the following strategic choices. At Stage 2, A plays a best response $E_a^\star(\boldsymbol{g})$ to $\boldsymbol{g} = (E(g), E_P(g))$, we have

$$E_a^\star(\boldsymbol{g}) \in \underset{E_a(\boldsymbol{g})\subseteq E(g)}{\operatorname{argmin}} \{\phi(g_R)\}.$$

Note that $E_a^\star(\boldsymbol{g}) \subseteq E_{NP}(g)$ since attacks cannot remove protected links. Let $g_R^\star$ be the residual network obtained when D plays strategy $\boldsymbol{g}$ and A plays a best response to $\boldsymbol{g}$. Given the best response outcome $g_R^\star$, D achieves payoff $\phi(g_R^\star) - c(\boldsymbol{g})$ when choosing $\boldsymbol{g}$. At Stage 1, D plays $\boldsymbol{g}^\star = (E^\star(g^\star), E_P^\star(g^\star))$ such that

$$\boldsymbol{g}^\star \in \underset{\boldsymbol{g}\in G}{\operatorname{argmax}}\{\phi(g_R^\star) - c(\boldsymbol{g})\}.$$

With a slight abuse of notation, we say that $\boldsymbol{g}^\star$ is a sub-game perfect equilibrium.

Specific architectures. The *empty network* is the network which contains no links. A network g which contains n nodes is a (κ, n)-link-connected network if any subnetwork g' obtained from g by removing $\kappa - 1$ links is connected, and there exists a subnetwork g' obtained from g by removing κ links that is not connected. Let $\mathcal{G}(\kappa, n)$ be the set of minimal networks with n nodes which are (κ, n)-link-connected, i.e. if $g \in \mathcal{G}(\kappa, n)$, then there does not exist a (κ, n)-link-connected network, g', such that $|E(g')| < |E(g)|$. It is easy to see that every node i of a minimal (κ, n)-link-connected network g satisfies $d_i(g) \geq \kappa$, as otherwise it could be separated by removing all its neighbors. Consequently, if n or κ are even, then the number of links in a minimal (κ, n)-link-connected network is at least $n\kappa/2$. Moreover, due to the *handshake* lemma, if n and κ are odd, then the number of links in a minimal (κ, n)-link-connected network is at least $(n\kappa + 1)/2$. As was shown by Harary [15], these conditions are also sufficient. The proof of this result is constructive − Harary describes how to obtain the desired family of graphs. The minimal (κ, n)-link-connected networks described by Harary are called (κ, n)-*Harary-networks.* To give the reader some idea of how (κ, n)-Harary-networks look like, we provide some examples in Figure 2 with

5 nodes. For full description of the construction the interested reader is referred to Harary [15].

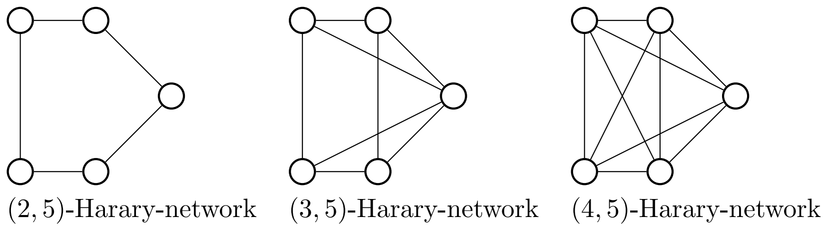


Figure 2: Example of (κ, n)-Harary-networks

Specific strategies. Let us consider that the number of links that A may attack is k. Strategy $(\emptyset, \emptyset)$ is the strategy where D forms the *empty* network, and so she does not protect any link. Strategy $\boldsymbol{g}^F$ is the strategy where D uses $n-1$ protected links and does not form any non-protected links. We define strategy $\boldsymbol{g}^{k+1}$ as the strategy of D where she uses no protected links and designs a minimal $(k+1, n)$-link-connected network. Strategy $\boldsymbol{g}_1^{k+1}$ is the strategy where D uses one protected link, and designs a network g such that the E_P-contraction of network g is a minimal $(k+1, n-1)$-link-connected network.

3 Model Analysis

First, we provide the optimal cost function for each pair (p, k) given that D builds a network that A cannot disconnect with k attacks. First, we define two useful functions.

$$C_1(p,k) = \begin{cases} \frac{c_L}{2}[(n-p)(k+1)] + pc_P, & \text{if } (n-p)(k+1) \in \mathcal{E}, \\ \\ \frac{c_L}{2}[(n-p)(k+1)+1] + pc_P, & \text{if } (n-p)(k+1) \in \mathcal{O}, \end{cases}$$

and

$$C_2(p,k) = (n-2p)\left(k+1-\frac{n-2p-1}{2}\right)c_L + pc_P.$$

Second, we set $p_1(k, n)$ and $p_2(k, n)$ as follows:

$$p_1(k,n) = \frac{4\ n - 3\ k - 5 - \sqrt{9\ k^2 - 8\ k\ n + 30\ k - 8\ n + 25}}{8},$$

$$p_2(k,n) = \frac{4\ n - 3\ k - 5 + \sqrt{9\ k^2 - 8\ k\ n + 30\ k - 8\ n + 25}}{8}.$$

Proposition 1 *The optimal cost function associated with the pair (p, k) given that D builds a network* *that A cannot disconnect with k attacks, is*

$$C^{\star}(p,k) = \begin{cases} C_1(p,k), & \text{for } p \in [\![0, n-2]\!] \setminus [\![\lfloor p_1(k,n) \rfloor + 1, \lceil p_2(k,n) \rceil - 1]\!], \\ \\ C_2(p,k), & \text{for } p \in [\![\lfloor p_1(k,n) \rfloor + 1, \lceil p_2(k,n) \rceil - 1]\!], \\ \\ (n-1)c_P, & \text{for } p = n-1. \end{cases} \tag{4}$$

Proof The proof is given in Appendix. □

Let us provide the intuition of Proposition 1. First, observe that each additional protected link formed by D allows to merge two components of $g_P = (N, E_P)$. Therefore, the number of components of $g_P = (N, E_P)$ decreases by one for each additional protected link formed by D. Consequently, the number of components in $g_P = (N, E_P)$ is $n - p$. Second, observe that if $\boldsymbol{g}$ is a SPE, then each component of $g_P = (N, E_P)$ is incident to at least $k+1$ non-protected links, otherwise A can disconnect network g with k attacks. When $n-p$ or $k+1$ is even, there exist situations such that if each component of $g_P = (N, E_P)$ is incident to $k+1$ non-protected links, then A cannot disconnect network g with k attacks. Similarly, when $n-p$ and $k+1$ are odd, there exist situations such that if each component of $g_P = (N, E_P)$ is incident to $k+1$ non-protected links except one which is incident to $k+2$ non-protected links, then A cannot disconnect network g with k attacks. For these cases we have $C^{\star}(p,k) = C_1(p,k)$. We illustrate this type of situations in the following example.

Example 1 Suppose $N = [\![1, 10]\!]$ and $k = 6$. Consider the case where $p = 5$. We describe a strategy $\boldsymbol{g}$ that allows D to incur a cost of forming links equal to $C_1(5,6)$. First, D forms protected links between node $a \in [\![1, 10]\!]$ and node $b \in [\![1, 10]\!]$ in g if a and b are linked in g^3 given in Figure 3, that is $g_P = g^3$. Moreover, D forms a non-protected link between node $a \in [\![1, 10]\!]$ and node $b \in [\![1, 10]\!]$ in g if a and b are linked in g^1 or g^2 given in Figure 3. Note that g^1 is a complete network and g^2 is a $(3,5)$-Harary-network. Finally, we observe that each component in g^3 is incident to at least 7 non-protected links and there is no possibility for A to disconnect g with 6 attacks.

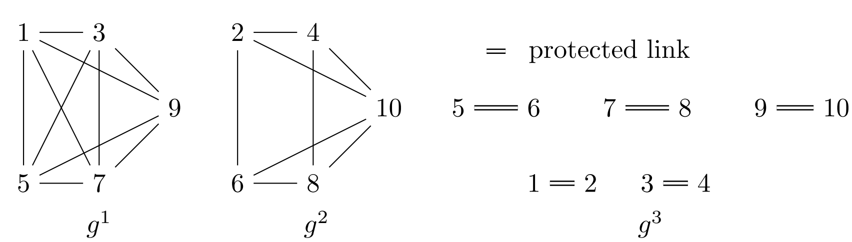


Figure 3: Networks associated with Example 1

Recall that each component of $g_P = (N, E_P)$ is incident to at least $k+1$ non-protected links. This fact implies that there exist some situations where D has to incur cost of forming links given by C_2. We illustrate these situations through the following example.

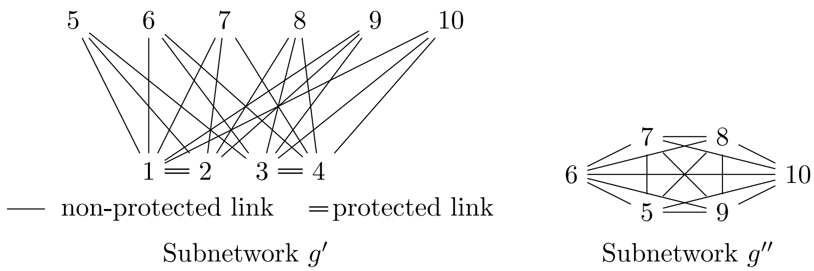


Figure 4: Subnetworks associated with Example 2

Example 2 Suppose $N = [\![1, 10]\!]$ and $k = 7$. Consider the case where $p = 2$. We assume that D forms a protected link between nodes 1 and 2 and between nodes 3 and 4. Network $g_P = (N, \{12, 34\})$ contains 8 components; we denote by $\mathcal{C}$ the component which contains nodes 1 and 2, and we denote by $\mathcal{C}'$ the component which contains nodes 3 and 4. Each of these components has to be incident to 8 non-protected links, otherwise there exists a strategy that allows A to disconnect the network formed by D. Note that each node $a \in [\![5, 10]\!]$ can form at most 5 non-protected links with other nodes in $[\![5, 10]\!]$. We illustrate this remark in subnetwork g'' in Figure 4. Consequently, each node $a \in [\![5, 10]\!]$ has to form at least 3 non-protected links with nodes in $[\![1, 4]\!]$. We illustrate this remark in subnetwork g' in Figure 4. It is worth noting that nodes 5, 6, 7, 8, 9 and 10 have to form a total of 18 non-protected links with nodes 1, 2, 3 and 4. As for components $\mathcal{C}$ and $\mathcal{C}'$, they should be incident to a total of 16 non-protected links. By the *pigeon hole principle*, the function C_1 cannot be satisfied in this situation. More precisely, a network that cannot be disconnected by A with 7 attacks has to contain 33 non-protected links instead of 32. Function C_2 captures this type of situations.

In Example 2, we have assumed that D uses protected links to form two components which contain two nodes in $g_P = (N, E_P)$. Assume that D forms a unique component in $g_P = (N, E_P)$. For instance, D forms a protected link between nodes 1 and 2 and between nodes 2 and 3. Then, each node $a \in [\![4, 10]\!]$ can form at most 6 non-protected links with other nodes in $[\![4, 10]\!]$. Consequently, since $k = 7$ each node $a \in [\![4, 10]\!]$ has to form at least 2 non-protected links with nodes in $[\![1, 3]\!]$. The component which contains nodes 1, 2 and 3 is incident to 14 non-protected links. It follows that D forms $21+14 = 35$ non-protected links instead of 33 links in Example 2. This example illustrates that D designs $g_P = (N, E_P)$ in order to maximize its number of components which contain strictly more than one node.

We now generalize Example 2 to provide some intuition for $p_1(k, n)$ and $p_2(k, n)$. Let p be the number of components which contain two nodes. We observe that components which contain one node need to be incident to at least $k + 1$ non-protected links. The minimal total number of non-protected links between these components and components which contain two nodes is equal to $(n-2p)((k+1)-(n-2p-1))$. Moreover, to minimize the number of links, the total number of non-protected links incident to components which contain two nodes should be equal to $(k+1)p$. Equation $(n-2p)((k+1)-(n-2p-1)) = (k+1)p$ is satisfied for $p = p_1(k, n)$ and $p = p_2(k, n)$. For $p \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$, the number of non-protected links required to resist the attacks of A is given by the function cost C_2.

Finally, we observe that if D forms $n-1$ protected links, then she has no incentive to form any non-protected links. Consequently, $C^\star(n-1,k) = (n-1)c_P$.

We now characterize the SPE according to the costs of forming protected and non-protected links. Since the costs of links formation are sufficiently low by assumption, strategy $(\emptyset, \emptyset)$ is not a SPE.

Proposition 2 *We assume that $k < n-2$.*

1. *Suppose that n or $k+1$ is even.*
   (a) *If $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, then $\boldsymbol{g}^{k+1}$ is the unique SPE.*
   (b) *If $c_P/c_L < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, then $\boldsymbol{g}^{F}$ is the unique SPE.*
2. *Suppose that n and $k+1$ are odd, $k < n-3$.*
   (a) *If $c_P/c_L < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, then $\boldsymbol{g}^{F}$ is the unique SPE.*
   (b) *If $(k+2)/2 > c_P/c_L > \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, then $\boldsymbol{g}_1^{k+1}$ is the unique SPE.*
   (c) *If $(k+2)/2 < c_P/c_L$, then $\boldsymbol{g}^{k+1}$ is the unique SPE.*

Proof We prove successively the two parts of the proposition.

1. Suppose that n or $k+1$ is even and $k < n-2$. By Proposition 1, we know that for $p \in [\![1, n-2]\!]$, we have $C^\star(p,k) - C^\star(0,k) > pc_P - c_L(k+1)/2$. Therefore, for $p \in [\![1, n-2]\!]$, $C^\star(p,k) - C^\star(0,k) > 0$ if $c_P/c_L > (k+1)/2$. Similarly, for $p \in [\![1, n-2]\!]$, we have $C^\star(n-1,k) - C^\star(p,k) > 0$ if $c_P/c_L > \left(\frac{n-p}{n-p-1}\right)\left(\frac{k+1}{2}\right)$. Moreover, $C^\star(n-1,k) - C^\star(0,k) > 0$ if $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$.
   First, assume that $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. Then, $c_P/c_L > (k+1)/2$ and $\boldsymbol{g}^{k+1}$ is the unique SPE. Second, assume that $c_P/c_L < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. Then, $c_P/c_L < \left(\frac{n-p}{n-p-1}\right)\left(\frac{k+1}{2}\right)$ for all $p \in [\![1, n-2]\!]$. It follows that if $c_P/c_L < \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, then $\boldsymbol{g}^F$ is the unique SPE.
2. Suppose that $k+1$ and n are odd, and $k < n-2$. By Proposition 1, we know that $C(0,k) - C(1,k) > 0$ if $c_P/c_L < (k+2)/2$, $C(1,k) - C(n-1,k) > 0$ if $c_P/c_L < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, and $C(0,k) - C(n-1,k) > 0$ if $c_P/c_L < \left(\frac{k+1}{2}\right) + \left(\frac{k+2}{2(n-1)}\right)$. Note that $(k+2)/2 > \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$ for $k < n-3$. By using the same argument as in the previous point, we establish that there is no SPE associated with $p \in [\![2, n-2]\!]$. Assume that $c_P/c_L < \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$, then $c_P/c_L < \left(\frac{k+1}{2}\right) + \left(\frac{k+2}{2(n-1)}\right)$ and $\boldsymbol{g}^F$ is the unique SPE. Assume that $c_P/c_L > \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$. There are two possibilities. If $c_P/c_L < (k+2)/2$, then $C(1,k) - C(n-1,k) < 0$ and $C(1,k) - C(0,k) < 0$. It follows that $\boldsymbol{g}_1^{k+1}$ is the unique SPE. If $c_P/c_L > (k+2)/2$, then $C(0,k) - C(n-1,k) < 0$ and $C(0,k) - C(1,k) < 0$. It follows that $\boldsymbol{g}^{k+1}$ is the unique SPE.

□

Note that in Proposition 2 inequality $(k+2)/2 > c_P/c_L > \left(\frac{n-1}{n-2}\right)\left(\frac{k+1}{2}\right)$ is not satisfied for $k = n-3$. Consequently, for $k = n-3$ there does not exist situation where $\boldsymbol{g}_1^{k+1}$ is a SPE.

Let us provide the intuition of Proposition 2. To simplify notation, we set $p_1 = p_1(k,n)$ and $p_2 = p_2(k,n)$. First, we consider the first part of Proposition 2: n or $k+1$ is even. In Figure 5, we use Proposition 1 to draw the *average* number of non-protected links that each protected link allows to save given that A cannot disconnect the network designed by D with k attacks. The slope of line $(d_{\tilde{p}})$, $s_{\tilde{p}}$, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D does not form any protected links and the situation where D forms $\tilde{p}$ protected links, with $\tilde{p} \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$. The slope of line (d), s_d, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D does not form any protected links and the situation where D forms $p \in [\![1, n-2]\!] \setminus [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$ protected links. The slope of line (d'), $s_{d'}$, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D does not form any protected links and the situation where D forms $n-1$ protected links. Finally, the slope of line (e), s_e, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D forms p' protected links and D forms $n-1$ protected links.

A line $(d_{\tilde{p}})$ that is built with any $\tilde{p} \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$ satisfies $|s_{\tilde{p}}| < |s_d|$, and $|s_d| < |s_{d'}|$. Similarly, a line (e) that is built with any $p' \in [\![1, n-2]\!]$ satisfies $1/|s_e| < 1/|s_{d'}|$.

Suppose $c_P/c_L > |s_{d'}|$. Then, costs of forming non-protected links in $\boldsymbol{g}^{k+1}$ are lower than the costs of forming protected links in $\boldsymbol{g}^F$. Moreover, if $c_P/c_L > |s_{d'}|$, then $c_P/c_L > |s_d| > |s_{\tilde{p}}|$. Recall that these inequalities are satisfied when $\tilde{p}$ is replaced by any $p \in [\![1, n-2]\!]$. It follows that the cost of forming links is minimized for strategy $\boldsymbol{g}^{k+1}$. Suppose $c_P/c_L < |s_{d'}|$. We have $c_L/c_P > 1/|s_{d'}|$, and if $c_L/c_P > 1/|s_{d'}|$, then $c_L/c_P > 1/|s_e|$. Reall that the inequality is satisfied when p' is replaced by any $p \in [\![1, n-2]\!]$. It follows that the cost of forming links is minimized for strategy $\boldsymbol{g}^F$. Finally, note that by construction, $|s_{d'}| = \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, and we obtain the result given in part 1. of Proposition 2.

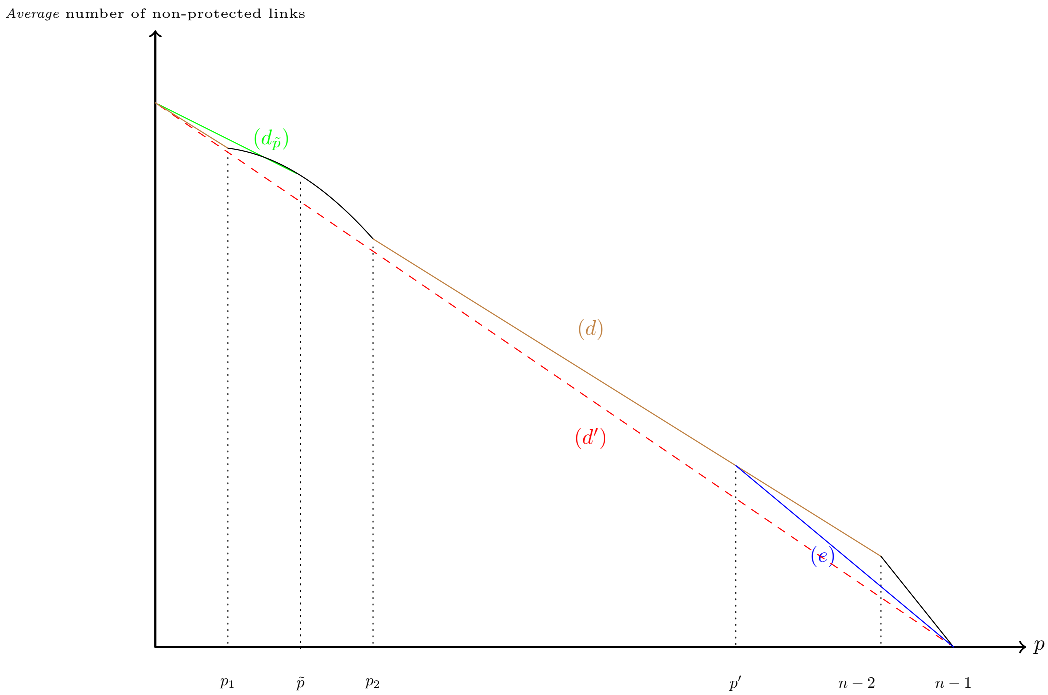


Figure 5: Intuition of Proposition 2 when $n(k+1) \in \mathcal{E}$

We now consider second part of Proposition 2: n and $k+1$ are odd, and $k < n-3$. The intuition is similar to the situation where n or $k+1$ is even except for $p = 0$ and $p = 1$. Consequently, we focus only on three cases: $p = 0$, $p = 1$ and $p = n-1$. In Figure 6, we use Proposition 1 to draw the average number of non-protected links that each protected link allows D to save when n and $k+1$ are odd. The slope of (e_1), s_1, can be interpreted as the *average* number of non-protected links that the first protected link allows to save. The slope of (e_2), s_2, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D forms one protected link and the situation where D forms $n-1$ protected links. The slope of (e_3), s_3, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D forms no protected links and the situation where D forms $n-1$ protected links.

Suppose that $c_P/c_L < |s_2|$. Then, $c_P/c_L < |s_3|$. It follows that $\boldsymbol{g}^F$ is the unique SPE. Suppose that $c_P/c_L > |s_2|$. Then, there are two possibilities. If $c_P/c_L > |s_1|$, then $\boldsymbol{g}^{k+1}$ is the unique SPE. If $c_P/c_L < |s_1|$, then $\boldsymbol{g}_1^{k+1}$ is the unique SPE.

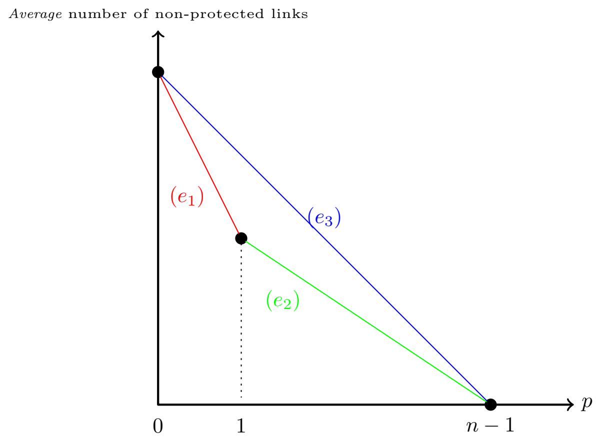


Figure 6: Intuition of Proposition 2 when $n(k+1) \in \mathcal{O}$ and $k < n-3$

We now compare the results obtained in our framework where A attacks links and the results obtained in DG's framework where A attacks nodes (Proposition 1, [9]). Recall that in DG's paper, the non-empty networks formed by D are either a star network with a protected central node, or a minimal $(k+1, n)$-node-connected network without any protection. Observe that the strategy $\boldsymbol{g}^{k+1}$ is a SPE when the cost of links (non-protected links in our case) is sufficiently low relative to the cost of protection in both frameworks.

However, when A attacks nodes, D uses at most one protection in equilibrium. The role played by protection is different since in DG, if D builds a star network, one protection is sufficient to protect the network and resist the attack of A. In our framework, D may use more than one protected link in a SPE: when the cost of protected links is sufficiently low relative to the cost of non-protected links, D designs a $(1, n)$-link-connected network which contains only protected links. Protecting a network under link-attack is more costly than protecting a network under node-attack. This is due to the fact that we require the survival of every node in our framework, while this requirement is not true in the DG's framework.

4 Limited number of protected links and imperfect protected links

In this section, we assume that protected links that D can use are limited. We take into account two types of limitations. First, we consider that D has an available number of protected links that is smaller than $n-1$. In particular, we are interested in situations where the number of protected links is smaller than $\lfloor p_2 \rfloor$. Then we consider situations where each protected link has a probability π to be removed when it is attacked by A.

4.1 Limited number of protected links

Proposition 2 establishes that in a SPE, it is not possible to obtain a situation where D uses both non-protected links and protected links in order to protect the network against A when n or $k+1$ is even. We now examine a situation where the maximal number of protected links, $\bar{p}$, that D can form is strictly smaller than $n-1$. More precisely, we are interested in the situation where $\bar{p} \in [\![\lfloor p_1(k,n)\rfloor+1, \lceil p_2(k,n)\rceil - 1]\!]$.

Let $\boldsymbol{g}_{p_1}^{k+1}$ be the strategy where D uses $\lfloor p_1(k,n)\rfloor$ protected links, and designs a network g such that the E_P-contraction of network g cannot be disconnected by removing k non-protected links and g contains $(1/2)\lceil (n-\lfloor p_1(k,n)\rfloor)(k+1)\rceil$ non-protected links.[12] In the following, we establish that there exist some situations where the strategy $\boldsymbol{g}_{p_1}^{k+1}$ is the unique SPE. To simplify the analysis, we restrict our attention to situations where $k+1$ is even.

Proposition 3 *Assume that* $k+1$ *is even. Moreover, assume that* $\lfloor p_1(k,n)\rfloor \geq 1$*,* $\lceil p_2(k,n)\rceil - \lfloor p_1(k,n)\rfloor \geq 4$*, and* $\bar{p} \in [\![\lfloor p_1(k,n)\rfloor+1, \lceil p_2(k,n)\rceil - 1]\!]$*.*[13] *There exists* $\epsilon > 0$ *such that if* $c_P/c_L < (k+1)/2$ *and* $c_P/c_L > (k+1)/2 - \epsilon$*, then* $\boldsymbol{g}_{p_1}^{k+1}$ *is the unique SPE.*

Proof By inspecting the proof of Lemma 2, for each $p \in [\![\lfloor p_1(k,n)\rfloor+1, \bar{p}]\!]$, we observe that the total number of non-protected links formed in a SPE g, K, is strictly greater than $(n-p)(k+1)/2$. We set $\sigma = K - (n-p)(k+1)/2 > 0$. For $p \in [\![\lfloor p_1(k,n)\rfloor+1, \bar{p}]\!]$, we have $C^\star(\lfloor p_1(k,n)\rfloor, k) - C^\star(p,k) = 0$, if $c_P/c_L = (k+1)/2 - \sigma/(p - \lfloor p_1(k,n)\rfloor) = \mathcal{B}_p$. We have $\mathcal{B}_p < (k+1)/2$. We consider $p^\star \in [\![\lfloor p_1(k,n)\rfloor+1, \bar{p}]\!]$ such that $\mathcal{B}_{p^\star} \leq \mathcal{B}_p$ for all $p \in [\![\lfloor p_1(k,n)\rfloor+1, \bar{p}]\!]$, and we set $\epsilon = ((k+1)/2 - \mathcal{B}_{p^\star})/2$.

Assume that $c_P/c_L < (k+1)/2$ and $c_P/c_L > (k+1)/2 - \epsilon$. Since $c_P/c_L < (k+1)/2$, $C_1(\lfloor p_1(k,n)\rfloor, k) < C_1(p,k)$ for all $p \in [\![0, \lfloor p_1(k,n)\rfloor - 1]\!]$ and there is no SPE associated with $p \in [\![0, \lfloor p_1(k,n)\rfloor - 1]\!]$. Moreover, since $c_P/c_L > (k+1)/2 - \epsilon > \mathcal{B}_{p^\star}$ there is no SPE associated with $p \in [\![\lfloor p_1(k,n)\rfloor+1, \bar{p}]\!]$. Consequently, the unique SPE is $\boldsymbol{g}_{p_1}^{k+1}$. □

We use Figure 5 to provide intuition of Proposition 3. To simplify notation, we set $p_1 = p_1(k,n)$ and $p_2 = p_2(k,n)$. Recall that the slope of line (d), s_d, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D does not form any protected links and the situation where D forms $p \in [\![1, \lfloor p_1\rfloor]\!]$ protected links; and the slope of line $(d_{\tilde{p}})$, $s_{\tilde{p}}$, can be interpreted as the *average* number of non-protected links that each protected link allows to save between the situation where D does not form any protected links and the situation where D forms $\tilde{p}$ protected links, with $\tilde{p} \in [\![\lfloor p_1\rfloor+1, \lceil p_2\rceil - 1]\!]$. Note that for all $\tilde{p} \in [\![\lfloor p_1\rfloor+1, \lceil p_2\rceil - 1]\!]$, we have $|s_d| > |s_{\tilde{p}}|$. If $|s_d| > c_P/c_L > |s_{\tilde{p}}|$, then the minimal costs incurred by D, when she forms no protected links, is higher than the minimal costs incurred by D when she forms $\lfloor p_1\rfloor$ protected links. Similarly, the minimal costs incurred by D when she forms $\tilde{p} \in [\![\lfloor p_1\rfloor+1, \bar{p}]\!]$ are higher than the minimal costs incurred by D when she forms $\lfloor p_1\rfloor$ protected links. It follows that the unique SPE is $\boldsymbol{g}_{p_1}^{k+1}$.

DG [9] show that when A attacks nodes, there exist situations where the SPE consists in a star network

[12] A process which allows to obtain such E_P-contraction is given in the appendix (Lemma 2, part 3.B).

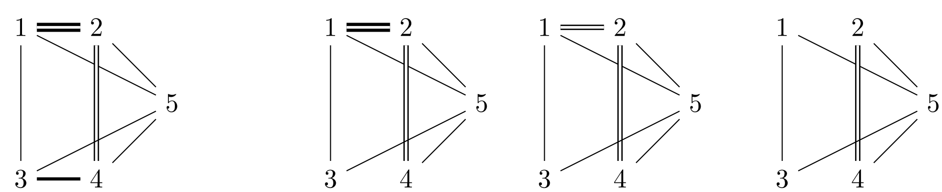
[13] To establish the existence of such intervals, consider $N = [\![1, 80]\!]$ and $k = 70$. We have $\lfloor p_1(k,n)\rfloor = 9$, and $\lceil p_2(k,n)\rceil = 17$.

with a protected central node. In this case D uses both protections and links to protect her network. In our framework, D also uses both protected and non-protected links to protect her networkif the number of protected links available to D belongs to $[\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$. We find this result because of the existence of a discontinuity in the number of non-protected links that each protected link allows the Designer to save.

4.2 Imperfect protected links

We now assume that each protected link has a probability $\pi \in (0,1)$ to be removed when it is attacked by A. To simplify notation, the set of links of network g is denoted by E instead of $E(g)$. Consider for each E_a, network $\dot{g} = (N, \dot{E})$ with $\dot{E} = E_P \cup \dot{E}_{NP}$, with $\dot{E}_{NP} = E_{NP} \setminus E_a$. We define a realization of network $\dot{g}$ as a subnetwork $\dot{g}_r = (N, E_r)$ of $\dot{g}$ with $E_r \subseteq \dot{E}$, and $|\dot{E}| - |E_r| \leq k - |E_{NP} \cap E_a|$. We illustrate these networks in the following example.

Example 3 Suppose $N = [\![1, 5]\!]$, $E_{NP}(g) = \{13, 15, 25, 34, 35, 45\}$, $E_P(g) = \{12, 24\}$ and $E_a(\boldsymbol{g}) = \{12, 34\}$. Network $\dot{g}$ is drawn in Figure 7 (b). Two realizations of network $\dot{g}$ are drawn in Figure 7 (c) and (d). Note that $\dot{g}_r^1$ occurs with probability $1 - \pi$, and $\dot{g}_r^2$ occurs with probability π.



(a) g and attacked links (b) Network $\dot{g}$ (c) Realization $\dot{g}_r^1$ (d) Realization $\dot{g}_r^2$

Figure 7: Networks of Example 3

Let $\lambda(\dot{g}_r | \dot{g}, E_P, E_a)$ be the probability that $\dot{g}_r$ is realized given $\dot{g}$, E_P and E_a. We have

$$\lambda(\dot{g}_r | \dot{g}, E_P, E_a) = \prod_{ij \in E_r, ij \in E_P \cap E_a} (1 - \pi) \prod_{ij \in \dot{E} \setminus E_r, ij \in E_P \cap E_a} \pi.$$

The expected benefits obtained by D when she chooses strategy $\boldsymbol{g}$ and A chooses E_a is

$$\sum_{\dot{g}_r, E_r \subseteq \dot{E}} \lambda(\dot{g}_r | \dot{g}, E_P, E_a) \phi(\dot{g}_r).$$

We assume that the costs incurred by D when she chooses strategy $\boldsymbol{g}$ are given by equation 2. The expected payoffs obtained by D is the difference between the expected benefits and the costs of forming protected and non-protected links.

We first examine a situation where probability π is sufficiently low to preserve our results.

Proposition 4 *We assume that n or $k+1$ is even, and $c_L > 1-(1-\pi)^k$.*

1. *If $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$, then strategy $\boldsymbol{g}^{k+1}$ is the unique SPE.*
2. *Assume that $1-(1-\pi)^k < c_L(n-p)(k+1)/2 - c_P(n-1-p)$ for all $p \in [\![0, k-1]\!]$. Then, $\boldsymbol{g}^F$ is the unique SPE.*

Proof Let N_P be the set of nodes incident to protected links in g. First, we establish that D chooses a strategy such that $(N_P, E(g))$ is acyclic. Observe that the expected benefits for D, associated with a network where optimal strategies for A consists in targeting no non-protected links, is at least $(1-\pi)^k$. This expected payoff is obtained when $|E_P| \geq k$ and optimal strategies for A consists in targeting no non-protected links. The maximal expected benefits that D can obtain in a network is 1; it arises when D chooses strategy $\boldsymbol{g}^{k+1}$. It follows that an additional link allows to obtain an additional benefit which is at most equal to $1-(1-\pi)^k$. This additional link induces an additional cost at least equal to c_L. Since $c_L > 1-(1-\pi)^k$ and $c_P > c_L$, D has no incentive to form a link that increases her expected benefits without allowing to reduce the number of non-protected links she forms. It follows that D uses a strategy where network $(N_P, E(g))$ is acyclic.
Second, we establish successively the two parts of the proposition.

1. Suppose that $c_P/c_L > \left(\frac{n}{n-1}\right)\left(\frac{k+1}{2}\right)$. By Proposition 2 part 1., we know that $C^\star(0,k) < C^\star(p,k)$ for all $p \in [\![1, n-1]\!]$. Moreover, given that D chooses a strategy such that $g_P = (N, E_P)$ is acyclic, the expected benefits obtained by D when she forms p protected links is $(1-\pi)^p$ for $p \in [\![1, k-1]\!]$ and $(1-\pi)^k$ for $p \in [\![k, n-1]\!]$. The expected benefits obtained by D when she forms 0 protected link is 1. Since $1 > (1-\pi)^p$, for $p > 0$ and $C^\star(0,k) < C^\star(p,k)$ for all $p \in [\![1, n-1]\!]$, the expected payoff obtained by D is maximized when she plays strategy $\boldsymbol{g}^{k+1}$.
2. Suppose that $1-(1-\pi)^k < c_L(n-p)(k+1)/2 - c_P(n-p-1)$, for all $p \in [\![0, k-1]\!]$. Since $(1-\pi)^p < 1$, we have $(1-\pi)^p - (1-\pi)^k < c_L(n-p)(k+1)/2 - c_P(n-1-p)$ for all $p \in [\![0, k-1]\!]$. Therefore, $(1-\pi)^p - c_L(n-p)(k+1)/2 - pc_P < (1-\pi)^k - c_P(n-1)$, for all $p \in [\![0, k-1]\!]$. Moreover, since $0 < 1-(1-\pi)^k < c_L n(k+1)/2 - c_P(n-1)$, we have $c_P/c_L < \frac{n(k+1)}{2(n-1)}$. We have $(n-p)c_L(k+1)/2 - (n-p-1)c_P > 0$, for $p \in [\![1, n-2]\!]$, when $c_P/c_L < \frac{n(k+1)}{2(n-1)}$. Consequently, $(1-\pi)^k - c_P(n-1) > (1-\pi)^k - c_L(n-p)(k+1)/2 - pc_P$ for all $p \in [\![k, n-1]\!]$. It follows that $\boldsymbol{g}^F$ is the unique SPE.

□

In Proposition 4, we consider situations where probability π is low relative to the cost of forming non-protected links. We now examine other situations through an example.

Example 4 Suppose that $N = [\![1, 5]\!]$ and $k = 2$. We draw networks that maximize the expected payoff of D given that A plays an optimal strategy when $p = 1, \ldots, 5$. Thick lines identify attacks on links that maximize the expected payoff of player A in these networks. Note that D has no incentive to form networks which contains at least 6 protected links since $c_P > c_L$. Observe that the expected payoff of D associated with network g^1 is not modified when π changes, while her expected payoff associated with all other networks drawn in Figure 4 decreases with π. Consequently, given c_P and c_L there exists a

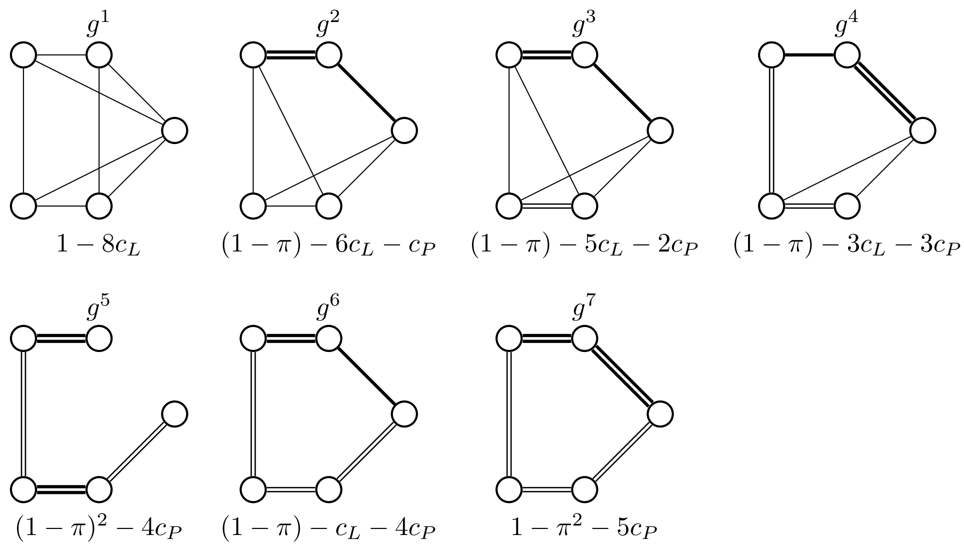


Figure 8: Networks of Example 4

probability $\bar{\pi}$ such that for $\pi > \bar{\pi}$ network g^1 is a SPE. Moreover, for $\pi = 0.2$, $c_P = 0.02$, and $c_L = 0.018$ network g^7 is a SPE.[14] Note that in g^7, each node is incident to 2 protected links and $k = 2$.

Example 4 establishes two main insights. First, networks without protected links are the unique SPE in situation where π is sufficiently high. Second, there exist situations where the Designer designs a network where each node is incident to m protected links, with $m = k$. Note that since $c_P > c_L$, D has no incentive to form a network where each node is incident to $k+1$ protected links.

Moreover, by Proposition 2, if π is sufficiently low, $n(k+1)$ is odd and $k < n-3$, then there exist values for c_L, c_P such that $\boldsymbol{g}_1^{k+1} = \boldsymbol{g}_1^3$ is a SPE. In Example 4, since $n = 5$ and $k = 2$ we have $k = n - 3$, and network g^2 cannot be a SPE in our benchmark model.

DG [9] examine the impact of imperfect defense in a framework where D protects nodes instead of links. They use an example and provide two insights. First, there exist parameters such that the SPE obtained in the perfect defense model remain equilibria in the imperfect defense model, i.e. the empty network, the center protected star, and the minimal $(k+1, n)$-node-connected networks remain equilibria. Second, they establish that richer strategies than those played by D in the perfect defense model may appear in equilibrium. In particular, there exist situations where an optimal strategy for D is to protect multiple nodes and create a network which generalizes the center protected star network, or to design a $(2, n)$-Harary-network and to protect all the nodes.

It is worth noting that imperfect defense has the same type of impact in the framework of DG and in our framework. First, if the probability of success of attacks π is sufficiently high and the cost of forming non-protected links is sufficiently low, then the strategy $\boldsymbol{g}^{k+1}$ is the unique optimal strategy for D. Sec-

[14]Note that DG [9] and Landwehr [21] establish that in models with imperfect defense, there exist situations where D designs a $(2, n)$-Harary-networks in equilibrium.

ond, the set of strategies candidate to be an equilibrium is larger in the imperfect defense framework than in the perfect defense framework. In particular, for sufficiently high π, D has an incentive to increase the number of protections relative to a situation where $\pi = 0$: there exist situations where D protects all the nodes in DG's framework, and there exist situations where D designs a network where each node is incident to k protected links in our framework. Third, in both frameworks it is difficult to obtain general results when imperfect defense is introduced. However, Landwehr [21] provides equilibrium strategies for D when the number of attacks is very small. In particular, he establishes that if $k = 2$, then there exist 6 types of strategies that D may play in equilibrium according to the value of π, c_P, and c_L.

5 Conclusion

In this paper, we have studied the optimal way to design and protect a network under link attack. In our benchmark model, the number of protected links available for the Designer is not bounded, and protected links cannot be removed by the Adversary. Our main findings in this model are the following. There exist three types of network that are SPE according to the value of the parameters of the model. First, if the relative cost (cost of a protected link / cost of a non-protected link) is low relative to the number of attacks, then D forms a $(1, n)$-link-connected network which contains only protected links. Second, if the relative cost is high relative to the number of attacks, then the Designer forms a minimal $(k+1, n)$-connected network which contains only non-protected links. Third, for intermediate relative costs, there exist situations where the Designer forms a network which contains one protected link and $(n-1)(k+1)/2$ non-protected links. To sum up, in this paper we provide the minimal costs that D incurs to protect her network against an intelligent attack (i.e. the worst attack).

We also examine situations where the number of protected links available for the Designer is limited. In that case, we establish that for intermediate relative costs, the Designer forms a network which contains several protected and non-protected links. Finally, we discuss the case of imperfect protected links. We cannot provide a full characterization of SPE in the imperfect defense model, but we provide conditions under which results obtained in the framework with perfect defense are preserved. Moreover, we establish through an example that the set of equilibria is larger in the framework with imperfect defense links than in the framework with perfect defense.

In this paper, we have assumed that the Designer incurs the same costs if she forms protected links that are adjacent and if she forms protected links that are not adjacent. It would be interesting to examine a situation where it is more costly for the Designer to form protected links that are not adjacent.

Appendix

First, to simplify the presentation of the proofs, we provide some useful definitions. In a *multigraph* multiple links and loops are allowed.[15] Formally, a multigraph g is an ordered triple $(V(g), E(g), \psi_g)$ consisting of a non-empty set of nodes, $V(g)$, a set of links, $E(g)$, disjoint of $V(g)$, and an incidence

[15]By definition a network does not contain a loop, that is a link joining a node to itself; neither does it contain multiple links, that is, several links joining the same two nodes.

function ψ_g that associates with each link an *unordered* pair of nodes of g. If e is a link and i and j are nodes such that $\psi_g(e) = ij$, then e is said to join i and j. We observe that if ψ_g is injective and there is no $e \in E(g)$ such that $\psi(e) = ii$ with $i \in V(g)$, then g is a network. A *bipartite* multigraph is one whose nodes set can be partitioned into two subsets X and Y so that each link is incident to a node in X and a node in Y.

Second, we observe that in a SPE $(E(g), E_P(g))$, subnetwork $(N, E_P(g))$ of g contains no cycle, otherwise D has formed a costly protected link that is not useful. It follows that if the number of protections used by D is equal to p, then the number of nodes of $\hat{g}^{E_P}$ is always equal to $n-p$. Indeed, let $\mathcal{C}_1(g_P), \ldots, \mathcal{C}_\ell(g_P), \ldots, \mathcal{C}_m(g_P)$ be the components of g_P with $\mathfrak{c}_\ell(g_P)$ the number of nodes of the component $\mathcal{C}_\ell(g_P)$. We have simultaneously $\sum_{\ell=1}^m \mathfrak{c}_\ell(g_P) = n$ and $\sum_{\ell=1}^m (\mathfrak{c}_\ell(g_P) - 1) = p$. It follows that $m = n - p$. By construction, the number of nodes in $\hat{g}^{E_P}$ is equal to the number of components in the network $g_P = (N, E_P)$.

Third, we build the multigraph $\hat{g} = (\hat{N}_g, \hat{E}_g, \psi_g)$ from g as follows. Let $\{\mathcal{C}_1(g_P), \ldots, \mathcal{C}_{n-p}(g_P)\}$ be the set of components of $g_P = (N, E_P)$, we set $\hat{N}_g = [\![1, n-p]\!]$.[16] Moreover, $\hat{E}_g = \{e_{ij} : ij \in E_{NP}(g)\}$ and $\psi_g(e_{ij}) = ab$ with $i \in \mathcal{C}_a(g_P)$, $j \in \mathcal{C}_b(g_P)$ and $a, b \in [\![1, n-p]\!]$. For $a, b \in [\![1, n-p]\!]$, we define $\mathfrak{m}_{ab}(\hat{g}) = |\{e_{i,j} \in \hat{E}_g : \psi_g(e_{ij}) = ab\}|$ as the number of links between nodes a and b in $\hat{g}$.

Note that if $\hat{g}$ cannot be disconnected by removing k non-protected links, then g cannot be disconnected by removing k non-protected links.

To prove Proposition 1, we need to distinguish two cases: $p \geq \lceil n/2 \rceil$ and $p < \lceil n/2 \rceil$.

Lemma 1 *Suppose $p \geq \lceil n/2 \rceil$, $p \neq n-1$. For (p, k) the minimal cost associated with a network that A cannot disconnect with k attacks is $C_1(p, k)$.*

Proof We establish that for $p \geq \lceil n/2 \rceil$, $p \neq n-1$, there exists a strategy $\boldsymbol{g} \in G$ for D such that (1) $\hat{g}$ is a minimal $(k+1, n-p)$-link-connected network or multigraph, and (2) $\hat{g}$ contains $(n-p)(k+1)/2$ non-protected links if $n-p$ or $k+1$ are even, and contains $((n-p)(k+1)+1)/2$ non-protected links if $n-p$ and $k+1$ are odd. Thus, we are interested in building $\hat{g}$ that induces a network g whose costs of link formation satisfy $C_1(p, k)$.

1. First, we establish that if $p \geq \lceil n/2 \rceil$, then the maximal number of links incident to a node is at least $n-2$ in $\hat{g}$. To achieve this goal, we examine a strategy where the components in g have a size equal to $\lfloor n/(n-p) \rfloor$ or $\lfloor n/(n-p) \rfloor + 1$. Let $x, r \in \mathbb{N}$, $r < n-p$ be such that $(n-p)x + r = n$. By the *euclidian algorithm*, x and r exist. We assume that D chooses E_P such that $g_P = (N, E_P)$ contains $n-p-r$ components with x nodes and r components with $x+1$ nodes. By construction of $\hat{g}$, the maximal number of links between $a \in [\![1, n-p]\!]$ and $b \in [\![1, n-p]\!]$ is $\mathfrak{c}_a(g_P)\mathfrak{c}_b(g_P)$ in $\hat{g}$. We provide the maximal number of links incident to any node $a \in [\![1, n-p]\!]$ given that $g_P = (N, E_P)$ contains $n-p-r$ components with x nodes and r components with $x+1$ nodes. The maximal number of links incident to any node $a \in [\![1, n-p]\!]$ is at least equal to $x^2(n-p-1)$. By definition, $x \geq (p+1)/(n-p)$ since $r \leq n-p-1$. It follows that the maximal number of links incident to node $a \in [\![1, n-p]\!]$ is at least equal to $((p+1)/(n-p))^2(n-p-1)$. Let

[16] Observe that the set of nodes of the E_P-contraction of g is equal to $\hat{N}_g$.

$f : p \mapsto ((p+1)/(n-p))^2(n-p-1)$. Then $f'(p) = \frac{p+1}{(n-p)^3}(p^2 - p(3n+1) + 2n^2 - n - 2) > 0$ since $p^2 - p(3n+1) + 2n^2 - n - 2 \geq (n-2)^2 - (n-2)(3n+1) + 2n^2 - n - 2 = 4 > 0$. Moreover, $f((2n-1)/3) = 4\left(\frac{n-2}{3}\right) > n-2$. It follows that for $x = 2$ and $r = n-p-1$ the maximal number of links incident to any node $a \in [\![1, n-p]\!]$ is at least equal to $n-2$. Since $x \mapsto x^2(n-p-1)$ is increasing with x, and $x \mapsto (n-r)/(n-p)$ is decreasing with r, for $x = 2$ and $r = 0$ the maximal number of links incident to any node $a \in [\![1, n-p]\!]$ is at least equal to $n-2$. Consequently, for $p \geq \lceil n/2 \rceil$ the maximal number of links incident to any node $a \in [\![1, n-p]\!]$ is at least equal to $n-2$.

2. We now deal with the non-protected links D forms between the nodes in $\hat{g}$. Let $y, \rho \in I\!N$, $\rho < n-p-1$ be such that $(n-p-1)y + \rho = k+1$. To build $\hat{g}$ we need to define a network $g_0 = ([\![1, n-p]\!], E)$. Suppose $\rho = 1$. If $n-p$ is even, then we have for $a, b \in [\![1, n-p]\!]$, $[a+b = n-p+1 \implies ab \in E(g_0)]$ and there is no other link in g_0; if $n-p$ is odd, then we have for $a, b \in [\![1, n-p-1]\!]$, $[a+b = n-p \implies ab \in E(g_0)]$ and there is a link between node 1 and $\lceil n-p \rceil$ in g_0. We have $\mathfrak{m}_{ab}(\hat{g}) = y$ if $ab \notin E(g_0)$ and $\mathfrak{m}_{ab}(\hat{g}) = y+1$ if $ab \in E(g_0)$. Suppose $\rho > 1$. Then, we build a $(\rho, n-p)$-Harary network g_0. We have $\mathfrak{m}_{ab}(\hat{g}) = y$ if $ab \notin E(g_0)$ and $\mathfrak{m}_{ab}(\hat{g}) = y+1$ if $ab \in E(g_0)$. Note that due to point 1. and the construction chosen for $\hat{g}$, it is possible for D to form $y+1$ links between two nodes in $\hat{g}$ since $y+1 \leq n-2$ for $p < n-2$, and $\rho = 0$ for $p = n-2$.

3. We now establish that $\hat{g}$ is $(k+1, n-p)$-link-connected. Suppose $\hat{g}$ is a network, then by construction, it is a $(k+1, n-p)$-Harary-network, and it is $(k+1, n-p)$-link-connected. Suppose $\hat{g}$ is not a network, so $y \geq 1$. It is sufficient to show that there is no subset $X \subseteq [\![1, n-p]\!]$ that can be disconnected from the subset $[\![1, n-p]\!] \setminus X$ in $\hat{g}$ by removing k links. Suppose $\rho > 1$. It is equivalent to say that $\hat{g}$ is $(k+1, n-p)$-link-connected and it is not possible to simultaneously disconnect y complete networks and one $(\rho, n-p)$-link-connected network by removing k links. Since the number of links between nodes in X and nodes in $N \setminus X$ are equal to $|X|(n-|X|)$ in the complete network, it is clear that in the complete network the subsets X that require the smallest number of attacks to be disconnected satisfies $|X| = 1$. Hence, the most optimal attacks of A in $\hat{g}$ consists in removing all the links incident to one node, and it is not possible to disconnect $\hat{g}$ by removing k links. Finally, by construction of $\hat{g}$, the cost of links associated with $\hat{g}$ is given by $C_1(p, k)$. We use similar arguments for $\rho = 1$.

□

To simplify the presentation of the following lemma, we let $p_1 = p_1(k, n)$ and $p_2 = p_2(k, n)$.

Lemma 2 *Suppose $p < \lceil n/2 \rceil$. For (p, k) the minimal cost associated with a network that A cannot disconnect with k attacks is $C^\star(p, k)$.*

Proof Suppose $p < \lceil n/2 \rceil$. We assume that numbers of nodes in components $\mathcal{C}_1(g_P), \mathcal{C}_2(g_P), \ldots, \mathcal{C}_{n-p}(g_P)$ satisfy $\mathfrak{c}_1(g_P) \leq \mathfrak{c}_2(g_P) \leq \ldots \leq \mathfrak{c}_{n-p}(g_P)$. If g cannot be disconnected by k attacks, then there are at least $k+1$ non-protected links in g between each component of g_P and other components of g_P.

1. Suppose that $\mathfrak{c}_{n-p}(g_P) \geq 3$. Then, since $p < \lceil n/2 \rceil$, we have $\mathfrak{c}_1(g_P) = \mathfrak{c}_2(g_P) = 1$. Suppose that

there exists $\ell^\star \in [\![2, n-p]\!]$ such that

$$\Xi_1 = (k+1)\ell^\star - \sum_{\substack{\ell,\ell' \in [\![1,\ell^\star]\!] \\ \ell \neq \ell'}} \mathfrak{c}_\ell(g_P)\mathfrak{c}_{\ell'}(g_P) > (k+1)(n-p-\ell^\star) = \Psi_1.$$

Ξ_1 is the minimal number of links between components in $[\![1, \ell^\star]\!]$ and components in $[\![\ell^\star+1, n-p]\!]$ that allows each component in $[\![1, \ell^\star]\!]$ to be incident to at least $k+1$ non-protected links. Ψ_1 is the number of links such that each component in $[\![\ell^\star+1, n-p]\!]$ is incident to $k+1$ non-protected links. It follows that the minimal number of non-protected links required to obtain a network which is not disconnected by k attacks contains $\lceil (n-p)(k+1)/2 \rceil + \Xi_1 - \Psi_1$ links. We note that $\Xi_1 - \Psi_1$ decreases if g is replaced by a network similar to g except that a node i and a protected link which belong to $\mathcal{C}_{n-p}(g_P)$ are removed and put in $\mathcal{C}_1(g_P)$. It follows that the costs of forming links are minimized when there are $n-2p$ components which contain 1 node and p components which contain 2 nodes.

2. Given k, we examine p such that the number of non-protected links formed in g is higher than $\lceil (n-p)(k+1)/2 \rceil$. We know that there are $n-p$ components, p components contain 2 nodes and $n-2p$ components contain 1 node. By construction $\mathcal{C}_1(g_P), \ldots, \mathcal{C}_{n-2p}(g_P)$ are the components of g_P which contain 1 node and $\mathcal{C}_{n-2p+1}(g_P), \ldots, \mathcal{C}_{n-p}(g_P)$ of g_P are the components which contain 2 nodes. The number of non-protected links in g required to protect components $\mathcal{C}_1(g_P), \ldots, \mathcal{C}_{n-2p}(g_P)$ is equal to the minimal number of non-protected links required to protect components $\mathcal{C}_{n-2p+1}(g_P), \ldots, \mathcal{C}_{n-p}(g_P)$ if p satisfies

$$(n-2p)((k+1)-(n-2p-1)) = (k+1)p.$$

This equation is satisfied for p_1 and p_2. Let $\Gamma = (n-2p)((k+1)-(n-2p-1))$ and $\Psi_2 = (k+1)p$. For $p \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$, we have $\Psi_2 < \Gamma$. In that case the minimal number of non-protected links formed by D given that she builds a network that A cannot disconnect with k attacks is

$$(n-2p)((k+1)-(n-2p-1)) + \frac{(n-2p)(n-2p-1)}{2} = (n-2p)\left(k+1-\frac{n-2p-1}{2}\right).$$

Note that, by construction, $(n-2p)\left(k+1-\frac{n-2p-1}{2}\right) > \lceil (n-p)(k+1)/2 \rceil$.

3. We now provide networks that A cannot disconnect with k attacks, which minimize the costs of forming links given (p,k). Let $x, r \in \mathbb{N}$, $r < x$ and $xp + r = \Gamma$ and $x' \in \mathbb{N}$, $x'(n-2p) = \Gamma$, with $x' = k+1-(n-2p-1)$. There are two possibilities.

(A) Assume that $p \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$. We now build $\hat{E}_g$ as follows. First, all distinct nodes $a, b \in [\![1, n-2p]\!]$ are linked in $\hat{g}$. Second, there is no link between nodes $a, b \in [\![n-2p+1, n-p]\!]$. We use the following process, called (P1), to form links between nodes in $[\![1, n-2p]\!]$ and nodes in $[\![n-2p+1, n-p]\!]$. The process consists in Γ steps. We let $\ell_t = t \mod n-2p$, with $\ell_t \in [\![1, n-2p]\!]$. We define E_{t-1} the set of non-protected links formed during steps $1, \ldots, t-1$ of the process and $d_\ell(E_{t-1})$ the number of links between node $\ell \in [\![n-2p+1, n-p]\!]$ and nodes in $[\![1, n-2p]\!]$ at Step

$t-1$. We let $\mathcal{D}^{\min}(t-1) = \underset{\ell' \in [\![n-2p+1,n-p]\!]}{\arg\min} \{d_{\ell'}(E_{t-1})\}$ and $\ell^{\min}_{t-1} = \min \mathcal{D}^{\min}(t-1)$. At Step 1, we add a link between nodes 1 and $n-2p+1$; we obtain E_1. At Step t, we add a link between node ℓ_t and node $\ell^{\min}_{t-1}$ to the set of links E_{t-1}; we obtain E_t. The process stops at step Γ. $\hat{E}_g$ all links in E_Γ and all links between nodes in $[\![1,n-2p]\!]$. By construction, subnetwork $([\![1,n-2p]\!], \hat{E}_g)$ is complete and the bipartite multigraph $g' = ([\![1,n-p]\!], E_\Gamma)$ is such that each node in $[\![1,n-2p]\!]$ is incident with $x' = k+1-(n-2p-1)$ links and each node in $[\![n-2p+1,n-p]\!]$ is incident with at least $k+1$ links, and at least one node in $[\![1,n-2p]\!]$ is incident to $x > k+1$ links. It follows that $\hat{g}$ is $(k+1,n-p)$-link-connected. By construction and by point 2, D incurs $C_2(p,k)$ to form links in g.

(B) Assume that $p \notin [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$. Suppose $k+1 \leq n-p-1$. Then, we build $\hat{g}$ as a $(k+1,n-p)$-Harary-network and we obtain the result. Suppose $k+1 > n-p-1$. Then there exists $a \in [\![1,n-2p]\!]$ and $b \in [\![n-2p+1,n-p]\!]$ such that $\mathfrak{m}_{ab}(\hat{g}) = 2$. Observe that if $\mathfrak{m}_{ab}(\hat{g}) = 1$ for all $a \in [\![1,n-2p]\!]$ and $b \in [\![n-2p+1,n-p]\!]$, then the maximal number of links incident to each node $b \in [\![n-2p+1,n-p]\!]$ is at least $k+1$ in $\hat{g}$ since $n-2p+4(p-1) = n+2p-4 \geq k+1$, for $p \geq 1$. We now build $\hat{E}_g$ as follows. First, all distinct nodes $a,b \in [\![1,n-2p]\!]$ are linked in $\hat{g}$. Second, we deal with links between nodes in $[\![1,n-2p]\!]$ and nodes in $[\![n-2p+1,n-p]\!]$.
(B.0) Let $\xi_1, \rho_1 \in I\!N$, $\rho_1 < \xi_1$ be such that $\xi_1 p + \rho_1 = (n-2p)(k+1-(n-2p-1))$. We use a process similar to (P1) except that it stops when each node in $[\![1,n-2p]\!]$ have formed $k+1-(n-2p-1)$ links with nodes in $[\![n-2p+1,n-p]\!]$. Due to the process, ρ_1 nodes in $[\![n-2p+1,n-p]\!]$ are adjacent to ξ_1+1 nodes and $p-\rho_1$ nodes in $[\![n-2p+1,n-p]\!]$ are adjacent to ξ_1 nodes. Moreover, each node in $[\![1,n-2p]\!]$ is incident to $k+1$ links.
Third, we deal with links between nodes in $[\![n-2p+1,n-p]\!]$. There are two possibilities.
(B.1) Suppose $p-\rho_1$ is even. Let $x_2, r_2 \in I\!N$, $r_2 < x_2$ such that $x_2(p-1)+r_2 = k+1-(\xi_1+1)$, with $\xi_1 < k$ since $p \notin [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$. We successively build additional links. Recall that at the end of the process similar to (P1), there is a set of nodes $X = \{a_1, \ldots, a_{p-\rho_1}\}$, $X \subseteq [\![n-2p+1,n-p]\!]$, which are involved in ξ_1 links. We use three steps to add links. At Step 1, we add a link between two distinct nodes a_ℓ and $a_{\ell'}$ if $a_\ell, a_{\ell'} \in X$ and $\ell + \ell' = p - \rho_1 + 1$. Note that each node in $[\![n-2p+1,n-p]\!]$ is involved in $\xi+1$ links at the end of Step 1. At Step 2, there are three possibilities. If $r_2 > 1$, then we build a (r_2,p)-Harary network on the set of nodes $[\![n-2p+1,n-p]\!]$. If $r_2 = 1$ and p is even we add a link between nodes $a \in [\![n-2p+1,n-p]\!]$ and $b \in [\![n-2p+1,n-p]\!]$ if $a+b = 2n-p+1$. If $r_2 = 1$ and p is odd we add a link between nodes $a \in [\![n-2p+1,n-p]\!]$ and $b \in [\![n-2p+1,n-p]\!]$ if $a+b = 2n-p+2$ and we add a link between node $n-2p+1$ and a node in $[\![1,n-2p]\!]$. At Step 3, we add x_2 links between each pair of vertices in $[\![n-2p+1,n-p]\!]$.
(B.2) Suppose $p-\rho_1$ is odd. Let $x_3, r_3 \in I\!N$, $r_3 < x_3$ such that $x_3(p-1)+r_3 = k+1-(\xi_1+1)$. We successively build additional links. Recall that at the end of process (P1), there is a set of nodes $X = \{a_0, a_1, \ldots, a_{p-\rho_1-1}\}$, $X \subseteq [\![n-2p+1,n-p]\!]$, which are involved in ξ_1 links. At Step 1, we add a link between two distinct nodes a_ℓ and $a_{\ell'}$ if $a_\ell, a_{\ell'} \in X$ and $\ell + \ell' = p - \rho_1$. Note that a_0 is involved in 0 link with other nodes in $[\![n-2p+1,n-p]\!]$ at the end of Step 1. There are two possibilities for Step 2.
(B.2.a.) Suppose r_3 and p are odd. Step 2 consists in building a (r_3,p)-Harary network on the set of

nodes $[\![n-2p+1, n-p]\!]$ where a_0 is the node with the highest degree of the (r_3, p)-Harary network. We use the same type of arguments as in (B.1) to deal with the case where $r_3 = 1$, except that a_0 is incident to two links. At Step 3, we add x_3 links between each pair of vertices in $[\![n-2p+1, n-p]\!]$.
(B.2.b.) Suppose r_3 or p are even. Step $2'$ consists in building a (r_3, p)-Harary network on the set of nodes $[\![n-2p+1, n-p]\!]$ and to add a link between node a_0 and a node in $[\![1, n-2p]\!]$ which is involved in less than two links with a_0. We use the same type of arguments as in (B.1) to deal with the case where $r_3 = 1$. At Step $3'$, we add x_3 links between each pair of vertices in $[\![n-2p+1, n-p]\!]$.

We now show that the cost of forming links is given by C_1. We restrict our attention to the situation where $n-p$ is even and $k+1$ is odd since all other possibilities are solved with the same type of arguments. Since $n-p$ is even, there are two possibilities, either $n-2p$ and p are odd, or $n-2p$ and p are even. We show that due to our process, the total number of non-protected links in g is $n(k+1)/2$.
− Suppose $n-2p$ and p are odd. At the end of the process (B.0) the total number of links formed, $(n-2p)(k+1-(n-2p-1))$, is odd since $(k+1)$ and $(n-2p)$ are odd. There are two possibilities at the end of (B.0), either ξ_1 is even, or ξ_1 is odd.
First, suppose that ξ_1 is even. Since $(n-2p)(k+1-(n-2p-1))$ and p are odd and ξ_1 is even, ρ_1 is odd. It follows that $p-\rho_1$ is even. We use process (B.1). Since ξ_1 is even and $(k+1)$ is odd, $(k+1)-(\xi_1+1)$ is even. Moreover, $(p-1)$ is even. Consequently, r_2 is even and all nodes in $[\![n-2p+1, n-p]\!]$ are incident to r_2 links in the (r_2, p)-Harary-network built in (B.1). Therefore, each node is incident to $k+1$ links in g in the end of (B.1).
Second, suppose that ξ_1 is odd. Since $(n-2p)(k+1-(n-2p-1))$, p and ξ_1 are odd, ρ_1 is even. It follows that $p-\rho_1$ is odd and we use process (B.2). Moreover, since $p-1$ is even and $k+1-(\xi_1+1)$ is odd, r_3 is odd. We use process (B.2.a.) and each node is incident to $k+1$ links in g at the end of process (B.2.a.).
− Suppose $n-2p$ and p are even. At the end of the process (B.0) the total number of non-protected links, $(n-2p)(k+1-(n-2p-1))$, is even since $(n-2p)$ is even. Since p and $(n-2p)(k+1-(n-2p-1))$ are even, ρ_1 is even. It follows that $p-\rho_1$ is even. We use process (B.1) and since p is even, all nodes in $[\![n-2p+1, n-p]\!]$ are incident to r_2 links in the (r_2, p)-Harary-network. At the end of process (B.1), each node is incident to $k+1$ non-protected links in g.

By using similar arguments as in the situation where $p \in [\![\lfloor p_1 \rfloor + 1, \lceil p_2 \rceil - 1]\!]$, we obtain the results: $\hat{g}$ is minimally $(k+1, n-p)$-link-connected, and D incurs $C_1(p, k)$ to form links of g.

□

Proof of Proposition 1 Lemmas 1 and 2 provide the proof for $p \neq n-1$. For $p = n-1$, we observe that non-protected links are not useful for D. □

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