BME 384T – FALL 2021

3D Elastic (Hookean) material models

Learning Objectives

1. Vectors and tensors:

- (Review ***vector algebra***)
- Define the meaning of a ***tensor*** versus a ***matrix***
- Use ***index (Einstein) notation*** to perform tensor algebra

2. Perform kinematic analyses:

- Compare and contrast ***spatial*** versus ***material coordinates***
- Analyse ***large deformation kinematics*** with respect to material coordinates

3. Understand stress tensors and equilibrium equations:

- Define ***stress tensors*** with respect to ***spatial and material coordinates***
- Understand the stress ***equilibrium equations*** that govern soft tissue mechanics

now

4. Describe mechanical response and analyse constitutive relations:

- Describe the ***theoretical framework for constitutive relations*** for non-linear elasticity
- Derive ***isotropic*** constitutive equations
- Describe ***heart muscle microstructure*** and mechanical properties
- Formulate and analyse ***anisotropic*** constitutive models

Concept of a continuum

- Central to the formulation of basic formulation of mechanics is the concept of a material continuum.
- The classical definition of a material continuum is an isomorphism of the real number system in three-dimensional Euclidean space.
- Thus, the mass density of a continuum at a point P is defined by considering a sequence of volumes ΔV enclosing P.
- If the mass of particles in ΔV is denoted by ΔM, and if the ratio $\Delta M/\Delta V$ tends to a limit ΔP when $\Delta V \sim 0$, then ΔP is the mass density of the continuum at P.
- In the classical concept of a continuum, we ignore the discrete composition of matter, and assume that matter is uniformly distributed within the body in question

Continuum mechanics

Classical Physics → Classical Mechanics → Continuum mechanics

- The object of biomechanics is at the animal, organ, tissue, and cell level.
  - (1) At the cellular, tissue, organ and organism level, it is sufficient to take Newton's laws of motion as an axiom.
  - (2) The smallest volume of object we shall consider contains a very large number of atoms and molecules. It is convenient to consider the materials a continuum.

- The continuum assumption tends to be reasonable when $\delta/\lambda \ll 1$, where δ is a characteristic length scale of the microstructure and λ is a characteristic length scale of the physical problem of interest.
  - Forces felt by cells within the wall of a large artery due to the distending blood pressure
    - Microstructure: μm (size of the cell and ECM fibers)
    - Physical problem: mm (wall thickness)
    - $\delta/\lambda = \sim 0.001 \ll 1$ → Continuum assumption reasonable
  - Velocity of blood at the centerline of a large artery
    - Microstructure: μm (size of the red cell)
    - Physical problem: mm (wall diameter)
    - $\delta/\lambda = \sim 0.001 \ll 1$ → Continuum assumption reasonable
  - Velocity of blood in capillary
    - Microstructure: μm (size of the red cell, 5-8 μm)
    - Physical problem: μm (diameter of capillary, 5-8 μm)
    - $\delta/\lambda = \sim 1$ → Continuum assumption not work

- **The continuum approach greatly simplifies the analysis of mechanical problems.**

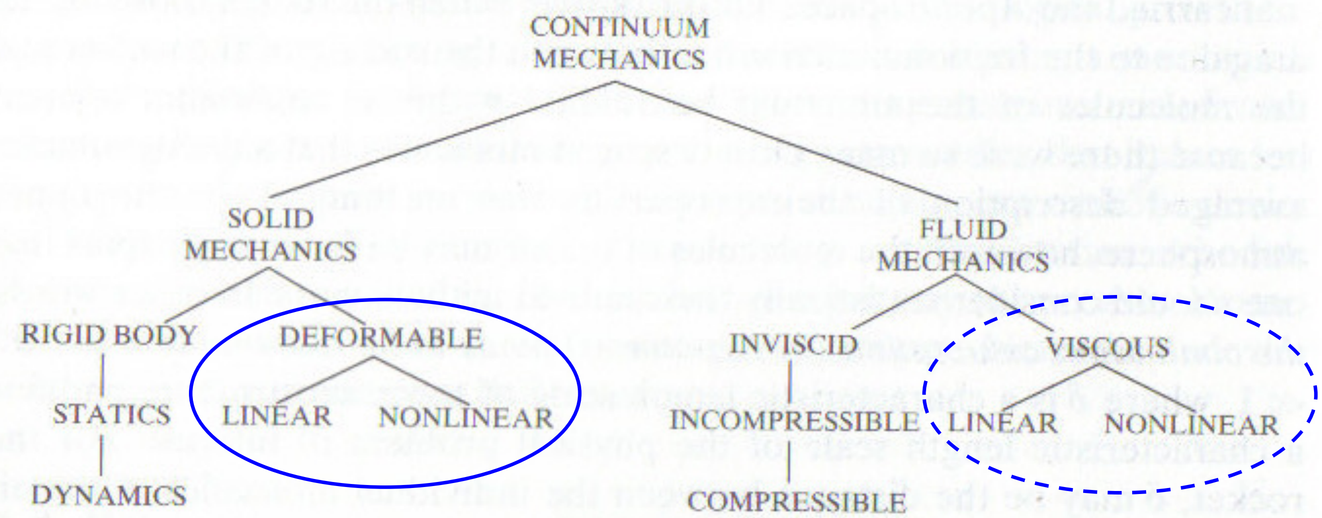


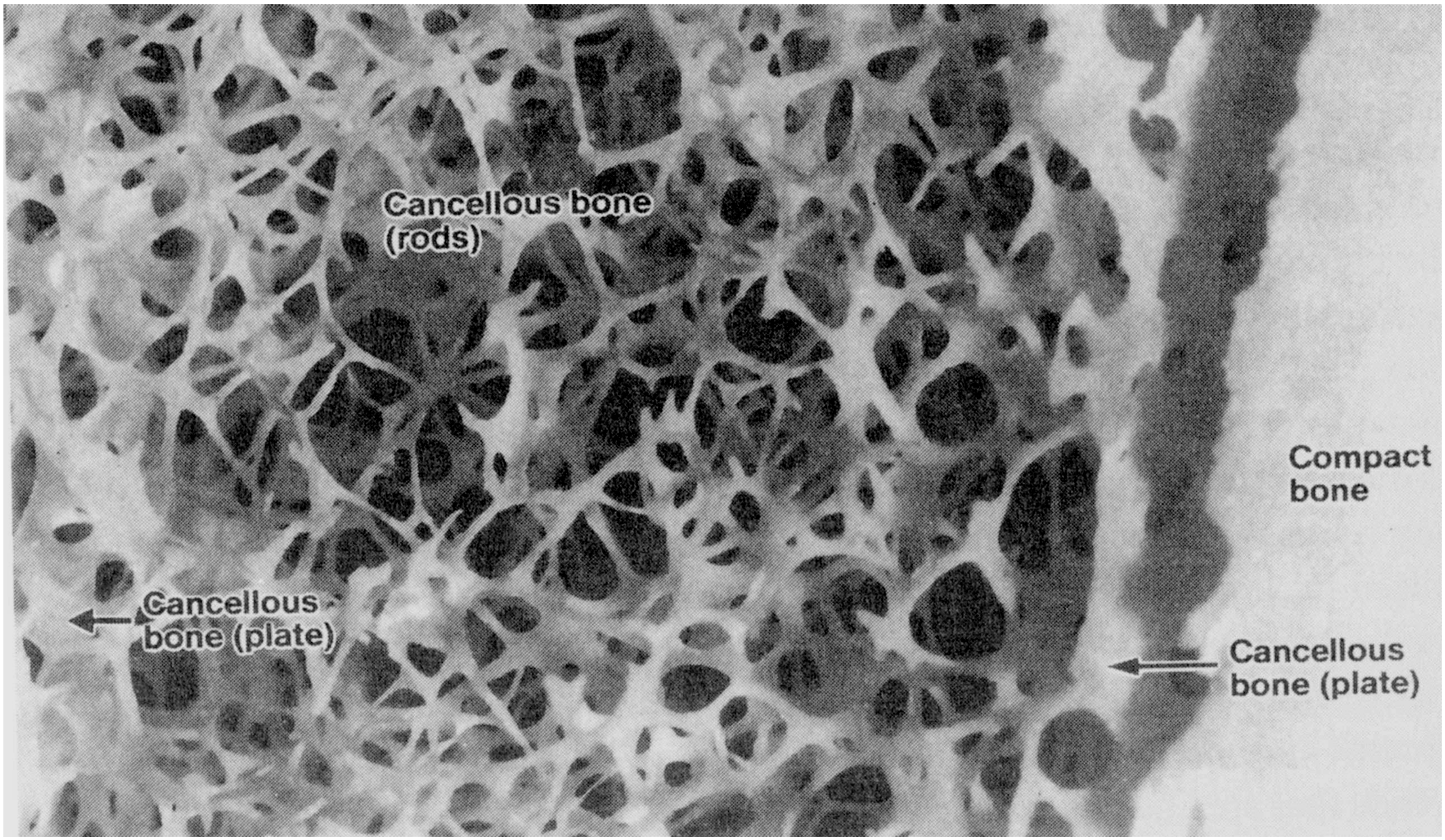
FIGURE 1.4 Flowchart of traditional divisions of study within continuum mechanics. Note that solid mechanics and fluid mechanics focus primarily on solidlike and fluidlike behaviors, not materials in their solid versus fluid/gaseous phases. Note, too, that linear and nonlinear refer to material behaviors, not the governing differential equations of motion. As we shall see in Chapter 11, many materials simultaneously exhibit solidlike (e.g., elastic) and fluidlike (e.g., viscous) behaviors, which gives rise to the study of viscoelasticity and the theory of mixtures, both of which are important areas within continuum biomechanics.

- Biomechanics focus on deformable solids (e.g. soft tissue) and viscous fluids (e.g. viscosity of blood)
- In this introductory course, we will focused only on mechanics of deformable biosolids.

Overall approach for solving boundary value Problems in solid mechanics

Displacements	**<->**	**Strain**	**<->**	**Constitutive model**	**<->**	**Stress**	**<->**	**Forces**
Measured or computed		*Computed*		*Assumed*		*Computed*		*Measured or computed*

Can we define a continuum for trabecular bone?



General method of approach

- Every continuum biomechanics problem can be addressed via the fundamental postulates of continuum mechanics problem by specifying three things:
  - (1) the geometry (i.e., the domain of interest),
  - (2) the constitutive relations (i.e., how the material responds to applied loads under conditions of interest),
  - (3) the applied loads (or associated boundary condtions)

- The key to success in this approach is often the identification of robust constitutive relations
  - A constitutive relation is but a mathematical descriptor of particular behaviors exhibited by a material under conditions of interest; it is not a descriptor of material itself.
  - Multiple theories will likely be needed to describe the myriad of behaviors exhibited by a given molecule, cell, tissue, or organ under different conditions.

- **Observation**
  - To observe the particular behaviors of interest and then, by induction, to delineate general characteristics of the material's response to applied load.
- **Hypotheses/Theory**
  - To formulate a general hypothesis and establish a theoretical frame work based on the axiomatic and deductive foundations of mathematics and mechanics.
- **Design Instrumentation**
  - To design and construct novel experimental system.
- **Experimentation or Simulation**
  - To perform experiments to test the hypothesis or theory, which includes identification of specific functional relationships between quantities of interest and calculation of the values of the associated material parameters.
- **Theory refinement and additional experiments**
  - Iterative procedure continues until the associated constitutive relation has predictive capability.
- **Final model**
  - Based on final model, one can answer applied questions of interest, often via numerical simulations and then animal and clinical trials.

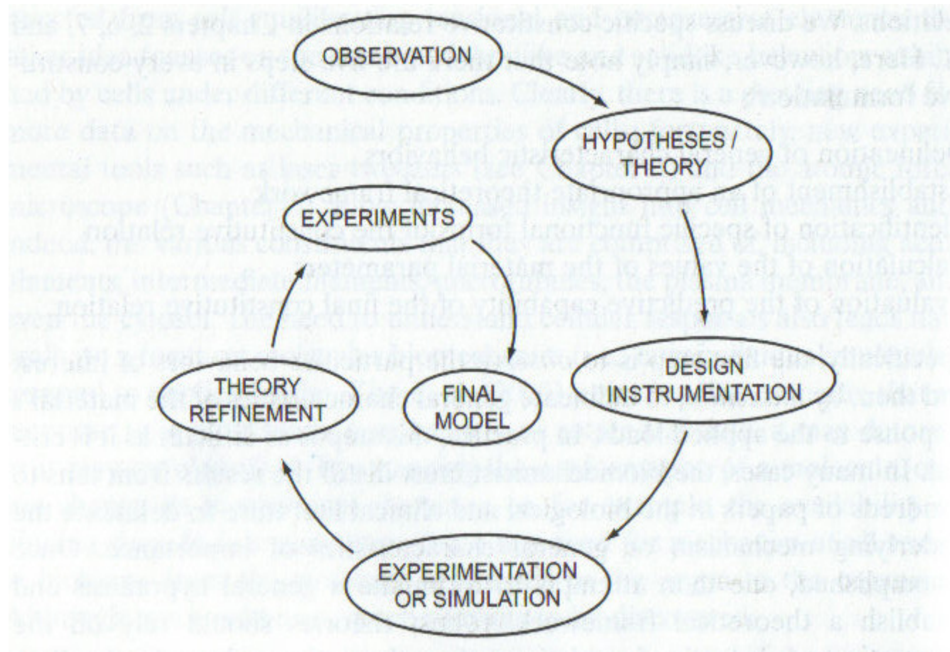


FIGURE 1.9 Illustration of the scientific approach employed by many in biomechanics. In particular, note that observations and experiments are equally important, but very different. The latter must be designed based on a hypothesis or theory for the purpose of testing an idea. Because science is "relative truth," we often need to iterate to improve our models of the physical and biological worlds.

Generalized Linear Elasticity

Overall approach for solving boundary value
Problems in solid mechanics

Displacements	**<->**	**Strain**	**<->**	**Constitutive model**	**<->**	**Stress**	**<->**	**Forces**
Measured or computed		*Computed*		*Assumed*		*Computed*		*Measured or computed*

§ 2-1: Concept of Stiffness

Hooke (1678): *As the force, so the extension (based on study of metallic springs).*

- F= k (x-x_0)=k Δx, f: force, k: spring constant or stiffness, x: deformed length, x_0: initial length
- Problem is that thick sample made of same material will appear "stiffer" – extend less.

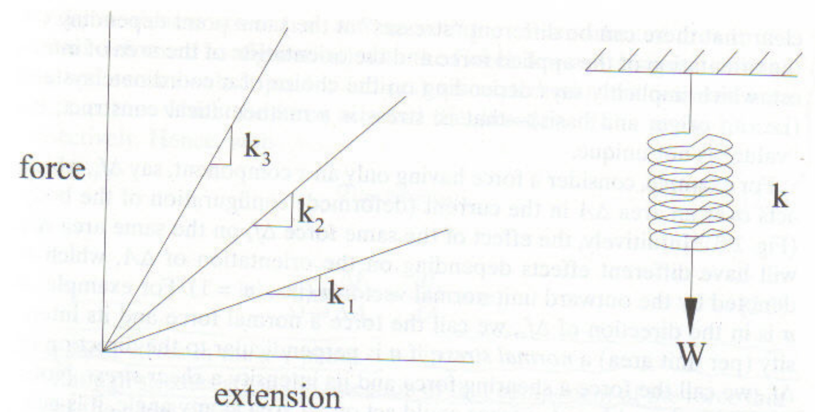
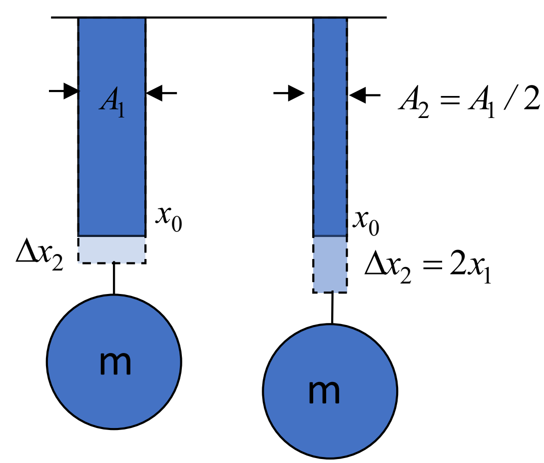


FIGURE 2.2 Force–extension behavior of three different metallic springs, which exhibit linear behaviors and thereby can be quantified by individual spring constants k (or stiffnesses). Although many springs exhibit a linear behavior, nonlinear springs exist as well.

Assume the material has a linear response to applied load

k_1 = F/ Δx_1 = mg/ Δx_1
k_2 = F/ Δx_2 = mg/ Δx_2 = mg/2Δx_1 = 2mg/Δx_1
➔ $k_2 > k_1$



§ 2-2: Concept of stress

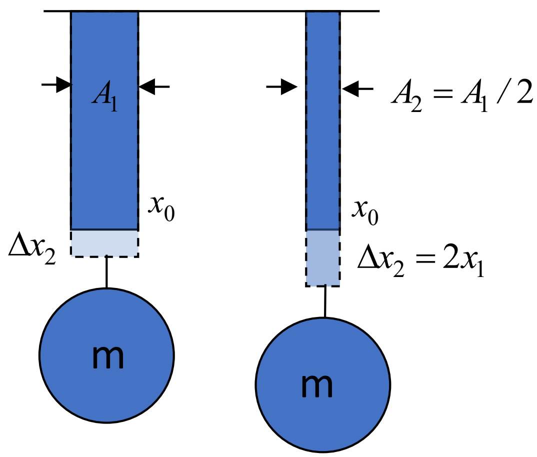
Euler (1757): *force intensity – force acting normal to an area divided by the valve of that area*

$$\sigma = \frac{F}{A}$$

Normalize the deformation $\varepsilon = \frac{x - x_0}{x_0} = \frac{\Delta x}{x_0}$

To material has a linear response to applied load: $\sigma = E\varepsilon$

- σ - stress
- ε - strain
- *E* – Young's modulus

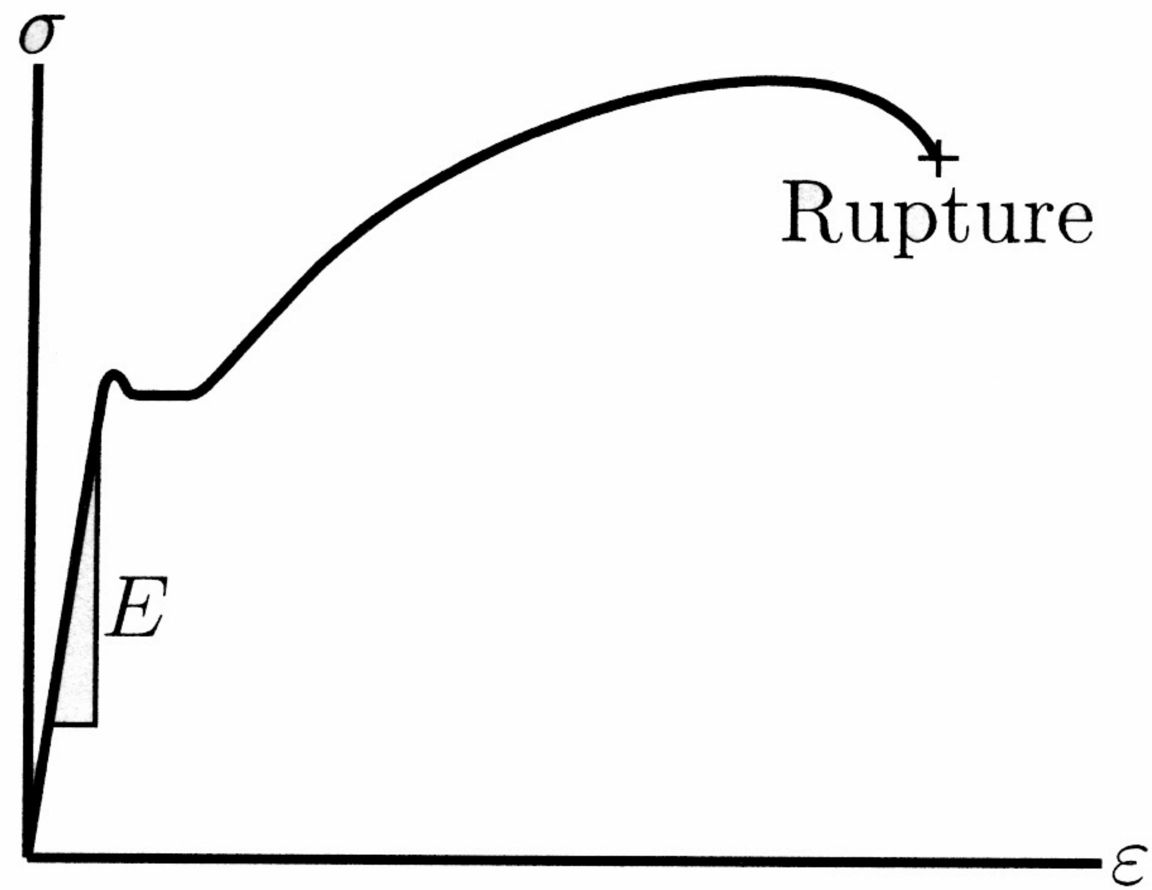


$$E_1 = \left(\frac{F}{A_1}\right) \Big/ \frac{\Delta x_1}{x_0} = \frac{F x_0}{A_1 \Delta x_1}$$

$$E_2 = \left(\frac{F}{A_1/2}\right) \Big/ \frac{2\Delta x_1}{x_0} = \frac{F x_0}{A_1 \Delta x_1}$$

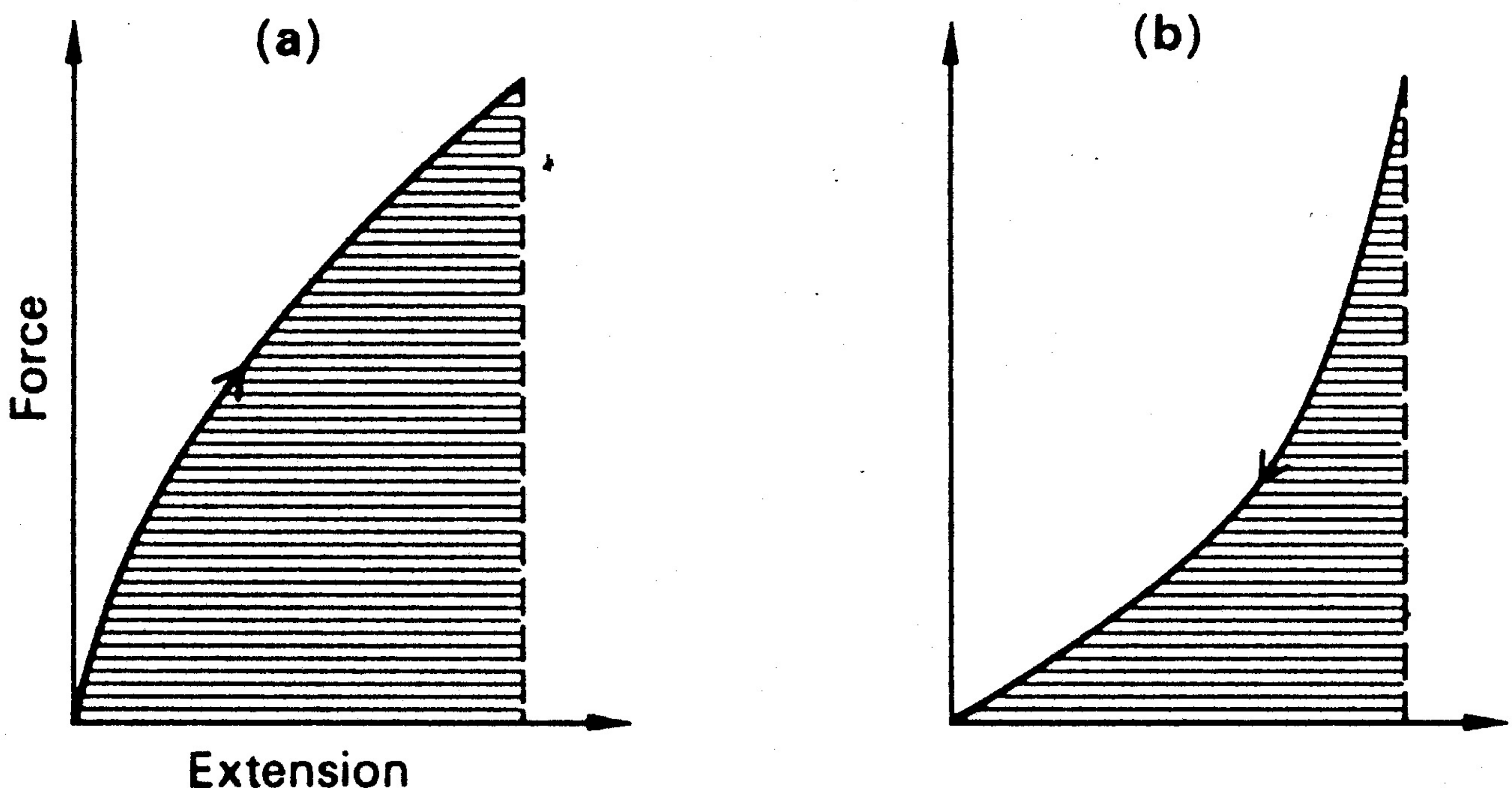
$$E_1 = E_2$$

➔ It is the stress (force per unit cross-sectional area) that is related to the strength of the materials



A representative stress-strain curve for a metallic material

The loading (a) and unloading (b) curve for an inelastic material.



•Focus on infinitesimal deformations (i.e. ones where strains are typically <0.001).

•Make no distinctions between the referenced and deformed states.

•Important for analysis of rigid biological materials, such as bone.

•Concepts covered will form the basis for more advanced studies presented in later chapters.

•Not utilized in the biomechanics of biological materials undergoing large deformations (e.g. soft tissues, cell membranes).

Elasticity

- The most common and easiest material response is an **elastic response**.

- An **elastic response** implies that:

  - The loading and unloading responses are identical

  - The material responds instantaneously to an applied load and its behavior is time-independent.

  - The material returns to its unloaded configuration when the external loads are removed (i.e. no residual deformation).

Generalized Linear Elasticity

- In the following we extend the concepts presented in the previous chapters to a generalized 3D state.

- As before, we focus on infinitesimal deformations (i.e. ones where $\varepsilon \leq 0.001$) and thus make no distinctions between the referenced and deformed states.

- Moreover, we extend the concept of **Hooke's law (σ=Eε)** to a generalized 3D state.

- While not utilized in the biomechanics of biological materials undergoing large deformations (e.g. soft tissues, cell membranes), the concepts covered will form the basis for more advanced studies presented in later chapters.

Generalized 3D Hooke's Law
81 material constants

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\
C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\
\cdots & \cdots & \cdots & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & & \\
 & & & & & & & &
\end{bmatrix}
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \\ \varepsilon_{32} \\ \varepsilon_{13} \\ \varepsilon_{21} \end{bmatrix}
$$

Voight Notation

$$\sigma_{11} = \sigma_1 \qquad \varepsilon_{11} = \varepsilon_1$$

$$\sigma_{22} = \sigma_2 \qquad \varepsilon_{22} = \varepsilon_2$$

$$\sigma_{33} = \sigma_3 \qquad \varepsilon_{33} = \varepsilon_3$$

$$\sigma_{23} = \sigma_{32} = \sigma_4 \qquad 2\varepsilon_{23} = 2\varepsilon_{32} = \varepsilon_4$$

$$\sigma_{13} = \sigma_{13} = \sigma_5 \qquad 2\varepsilon_{13} = 2\varepsilon_{13} = \varepsilon_5$$

$$\sigma_{12} = \sigma_{21} = \sigma_6 \qquad 2\varepsilon_{12} = 2\varepsilon_{21} = \varepsilon_6$$

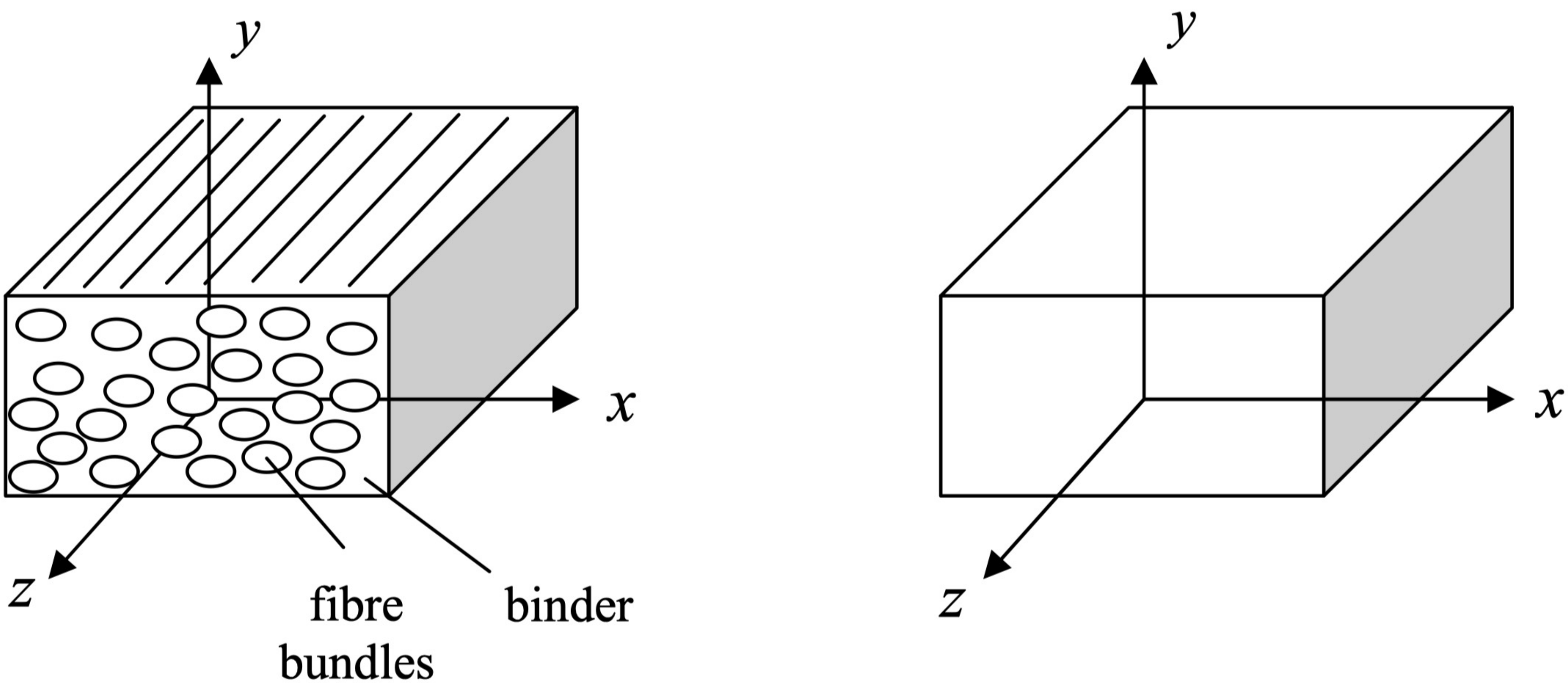
Voight Notation

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix}$$

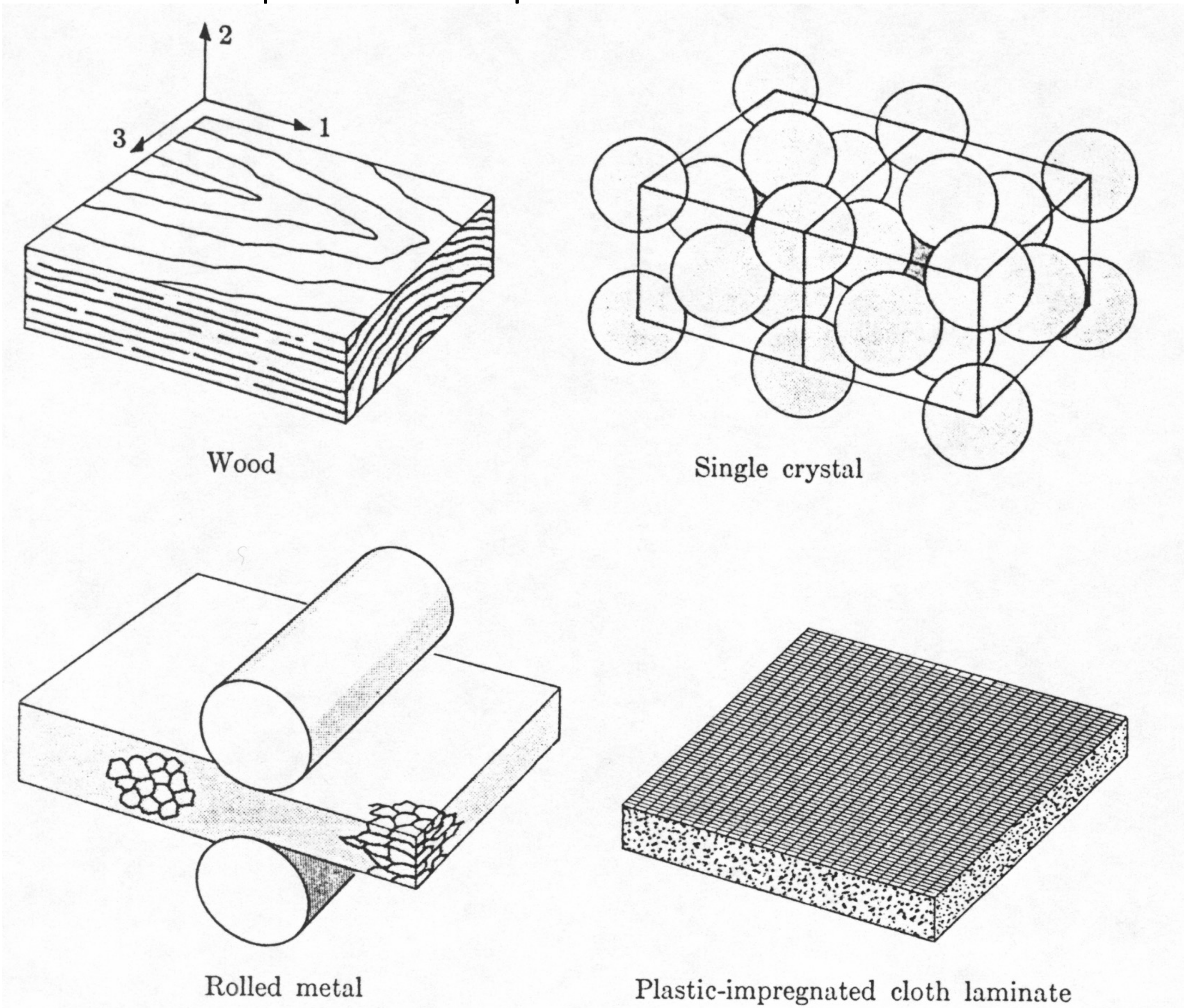
3D Hooke's Law
Enforcing symmetry conditions
=> 21 material constants

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

Examples of orthotropic materials.



Examples of orthotropic materials.



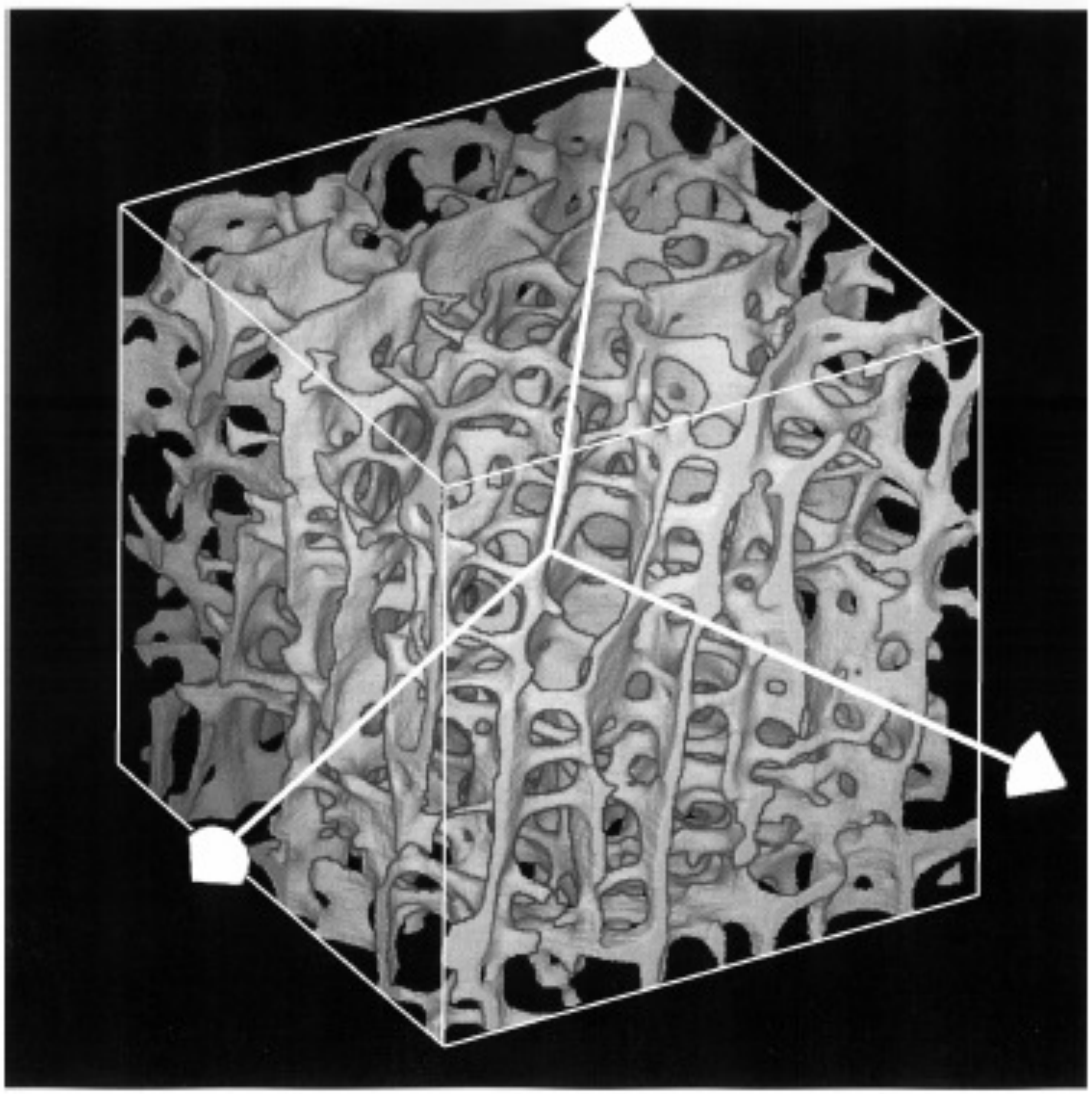


Figure 11.17. An illustration of the trabecular grain. By trabecular grain we mean a set of three ordered orthogonal directions, the first of which lies along the local predominant trabecular direction, which is locally the stiffest direction; the second and third directions are directions orthogonal to each other in the plane perpendicular to the first direction and represent directions of extrema in stiffness in the local region of the cancellous bone. The specimen is a 7-mm cube. Reprinted with permission from Yang et al. (1999).

3D Hooke's Law
Monoclinic with one plane of symmetry
⇒13 material constants

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

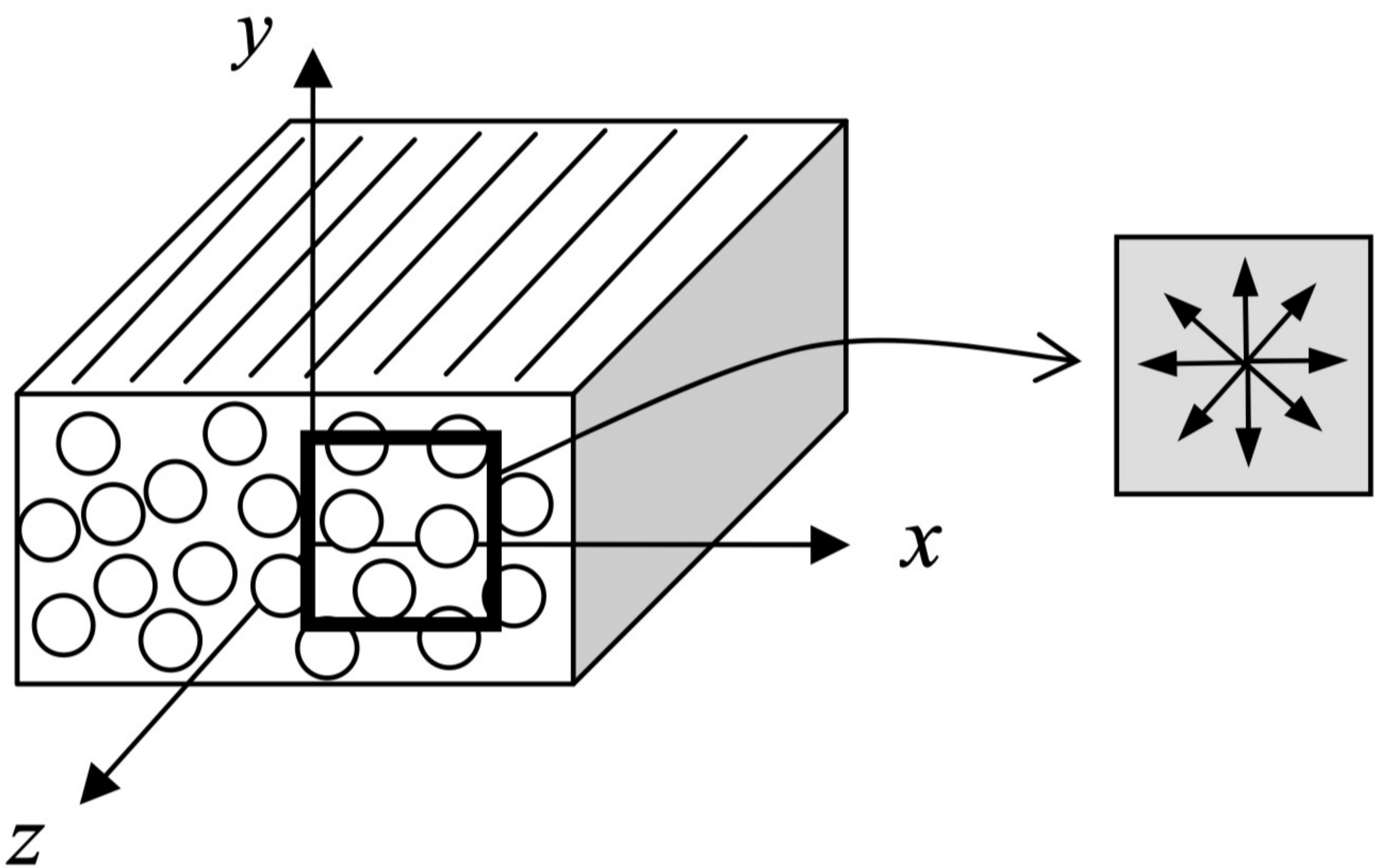
3D Hooke's Law
Orthotropy with three planes of symmetry
$\Rightarrow$9 material constants

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

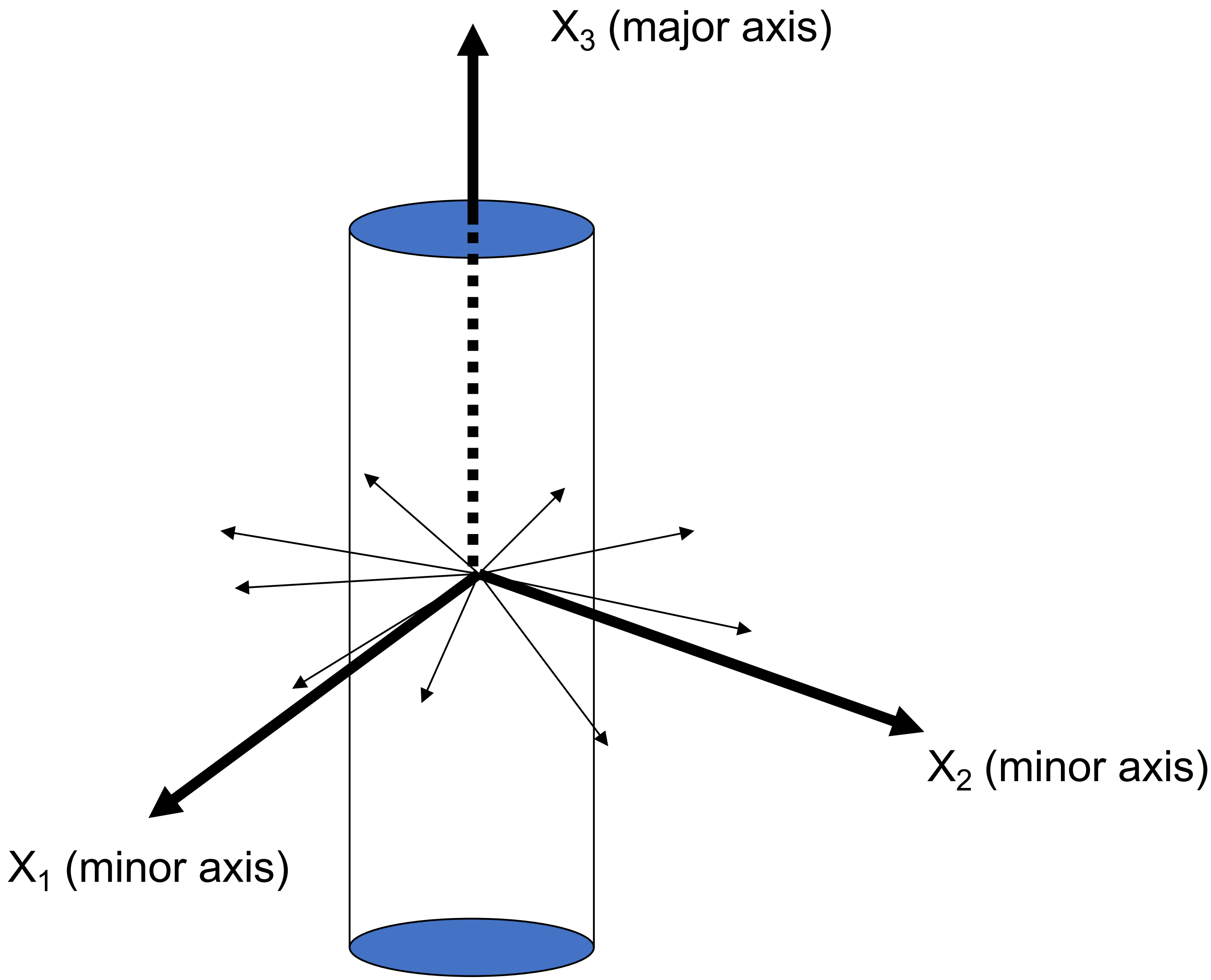
Orthotropic

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Example of a transverse isotropic materials.



Transverse Isotropy



3D Hooke's Law
Transverse Isotropy
⇒5 material constants

$$
C_{ij} = \begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
 & C_{22} & C_{23} & 0 & 0 & 0 \\
 & & C_{22} & 0 & 0 & 0 \\
 & & & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\
 & & & & C_{66} & 0 \\
 & & & & & C_{66}
\end{bmatrix}
$$

transverse isotropic

$$[S_{ij}] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_2 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E2 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_{12} \end{bmatrix}$$

Isotropic material

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ & & & & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}$$

Isotropic **C** rewritten in terms of the **Lamé coefficients** λ **and** μ.

$$C_{ij} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

isotropic

$$S_{ij} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ & & \frac{1}{E} & 0 & 0 & 0 \\ & & & \frac{1}{G} & 0 & 0 \\ & & & & \frac{1}{G} & 0 \\ & & & & & \frac{1}{G} \end{bmatrix}$$

$$G = \frac{E}{2(1+\nu)}$$

Isotropic

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{yy} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right]$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right]$$

$$\sigma_{xy} = \frac{E}{1+\nu}\varepsilon_{xy}$$

$$\sigma_{xz} = \frac{E}{1+\nu}\varepsilon_{xz}$$

$$\sigma_{yz} = \frac{E}{1+\nu}\varepsilon_{yz}$$

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\varepsilon_{xz} = \frac{1+\nu}{E}\sigma_{xz}$$

$$\varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{yz}$$

Interrelationships between elastic constants for a LEHI material

	λ, μ	E_Y, ν	μ, ν	E_Y, μ	k, ν
λ	λ	$\frac{\nu E_Y}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\mu(E_Y-2\mu)}{3\mu-E_Y}$	$\frac{3k\nu}{1+\nu}$
μ	μ	$\frac{E_Y}{2(1+\nu)}$	μ	μ	$\frac{3k(1-2\nu)}{2(1+\nu)}$
k	$\lambda+\frac{2}{3}\mu$	$\frac{E_Y}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E_Y}{3(3\mu-E_Y)}$	k
E_Y	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	E_Y	$2\mu(1+\nu)$	E_Y	$3k(1-2\nu)$
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	$\frac{E_Y}{2\mu}-1$	ν

Elastic coefficients in the stress-strain relationship for different material classes.

Material and coordinate system	Number of nonzero coefficients	Number of independent coefficients
Three-dimensional case		
Anisotropic	36	21
Generally Orthotropic (nonprincipal coordinates)	36	9
Specially Orthotropic (principal coordinates)	12	9
Specially Orthotropic, transversely isotropic	12	5
Isotropic	12	2
Two-dimensional case (*lamina*)		
Anisotropic	9	6
Generally Orthotropic (nonprincipal coordinates)	9	4
Specially Orthotropic (principal coordinates)	5	4
Balanced orthotropic, or square symmetric (principal coordinates)	5	3
Isotropic	5	2

SUMMARY

Voight Notation

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix}$$

3D Hooke's Law
Enforcing symmetry conditions ONLY
$\Rightarrow$ 21 material constants
$\Rightarrow$ This is our starting point

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

(Voigt notation)

3D Hooke's Law
Orthotropy with three planes of symmetry
⇒9 material constants

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

Orthotropic

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

COMPLIANCE MATRIX, *S*

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Orthotropic

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

3D Hooke's Law
Transverse Isotropy
$\Rightarrow$5 material constants

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{22} - C_{23}) & 0 & 0 \\ & & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

transverse isotropic

$$[S_{ij}] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_2 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E2 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_{12} \end{bmatrix}$$

Isotropic material

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & \frac{1}{2}(C_{11} - C_{12}) & 0 & 0 \\ & & & & \frac{1}{2}(C_{11} - C_{12}) & 0 \\ & & & & & \frac{1}{2}(C_{11} - C_{12}) \end{bmatrix}$$

Isotropic **C** rewritten in terms of the **Lamé coefficients λ and μ**.

$$C_{ij} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

Further restrictions on the form of C
Positive Definiteness

From Section 6.5:

Thus, C_{ij} has the properties that:

- All diagonal elements are positive (i.e. $C_{ii} > 0$, with no sum on i).
- $\det(C_{ij}) > 0$.
- $S_{ij} = (C_{ij})^{-1}$ exists and is also symmetric and positive definite.
- The following sub-matrices are also positive definite

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}, \begin{pmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{pmatrix}, \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

Further restrictions on the form of C

Positive Definiteness

From Section 6.5 – Isotropic material model:

As discussed above, the compliance matrix (and the various sub-matrices) is positive definite, so the diagonal elements are positive. As a result

$$E > 0, G > 0$$

$$\det \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} \\ -\frac{v}{E} & \frac{1}{E} \end{bmatrix} = \frac{1}{E}(1 - v^2) > 0 \tag{146}$$

$$\text{i.e. } v^2 < 1$$

and

$$\det \begin{bmatrix} \frac{1}{E} & -\frac{v}{E} & -\frac{v}{E} \\ & \frac{1}{E} & -\frac{v}{E} \\ & & \frac{1}{E} \end{bmatrix} = \frac{1}{E^3}(1 - 2v^2 - 3v^3) = \frac{1}{E^3}(1 - 2v)(1 + v)^2 > 0 \tag{147}$$

$$\text{i.e. } v < \tfrac{1}{2}$$

Thus

$$-1 < v < \frac{1}{2} \tag{148}$$

Eqn. (3-46) thus sets bounds on the possible values for v. It should be noted that for incompressible isotropic materials =0.5 (show it).

From Section 6.5 – Transverse Isotropic material model:

Again, since the compliance matrix is positive definite the diagonal elements are positive definete. That is,

$$E_1, E_3, G_{12}, G_{13} > 0 \tag{153}$$

Also,

$$\det\begin{bmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_1} \\ -\frac{v_{21}}{E_1} & \frac{1}{E_1} \end{bmatrix} = \frac{1}{E_1^2}(1 - v_{21}^2) > 0$$
$$\text{i.e. } -1 < v_{21}^2 < 1 \tag{154}$$

$$\det\begin{bmatrix} \frac{1}{E_1} & -\frac{v_{31}}{E_3} \\ -\frac{v_{13}}{E_1} & \frac{1}{E_3} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{v_{31}}{E_3} \\ -\frac{v_{31}}{E_3} & \frac{1}{E_3} \end{bmatrix} = \frac{1}{E_1 E_3}(1 - v_{31}^2 \tfrac{E_1}{E_3}) > 0$$
$$\text{i.e. } v_{31}^2 < \frac{E_3}{E_1} \text{ or } v_{13}v_{31} < 1 \tag{155}$$

$$\det\begin{bmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_1} & -\frac{v_{31}}{E_3} \\ -\frac{v_{21}}{E_1} & \frac{1}{E_1} & -\frac{v_{31}}{E_3} \\ -\frac{v_{13}}{E_1} & -\frac{v_{13}}{E_1} & \frac{1}{E_3} \end{bmatrix} = \frac{1}{E_1^2 E_3}\left[1 - 2\left(\frac{E_1}{E_3}\right)v_{21}v_{31}^2 - 2\left(\frac{E_1}{E_3}\right)v_{31}^2 - v_{21}^2\right]$$
$$= \left[1 - 2\left(\frac{E_1}{E_3}\right)v_{31}^2 - v_{21}^2\right](1 + v_{21}) > 0 \tag{156}$$

Since $1 + v_{21} > 0$,

$$1 - 2v_{31}^2\left(\frac{E_1}{E_3}\right) > 0 \quad \text{or} \quad 1 - 2v_{31}v_{13} > 0 \tag{157}$$

From Section 6.5 – Orthotropic material model:

Again the meanings and relationships of the constants in the compliance matrix can be obtained as outlined previously. The relations between C_{ij} and the engineering constants are given by:

$$\begin{aligned} C_{11} &= \frac{1 - v_{23}v_{32}}{E_2E_3\Delta}, \quad & C_{12} &= \frac{v_{21} + v_{31}v_{23}}{E_2E_3\Delta}, \quad & C_{23} &= \frac{v_{31} + v_{21}v_{32}}{E_2E_3\Delta}, \\ C_{22} &= \frac{1 - v_{13}v_{31}}{E_1E_3\Delta}, \quad & C_{23} &= \frac{v_{32} + v_{12}v_{31}}{E_1E_3\Delta}, \quad & C_{33} &= \frac{1 - v_{12}v_{21}}{E_1E_2\Delta}, \\ C_{44} &= G_{23}, & C_{55} &= G_{31},\ C_{66} = G_{12} \end{aligned} \tag{159}$$

where

$$\Delta = \frac{1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - 2v_{21}v_{32}v_{13}}{E_1E_2E_3} \tag{160}$$

To establish the restrictions for the engineering constants we follow the same procedures described previously.

$$\begin{aligned} &E_1 > 0,\ E_2 > 0,\ E_3 > 0, \quad G_{23} > 0,\ G_{31} > 0,\ G_{12} > 0 \\ &v_{21}^2 < \left(\frac{E_2}{E_1}\right), \qquad v_{12}^2 < \left(\frac{E_1}{E_2}\right) \\ &v_{32}^2 < \left(\frac{E_3}{E_2}\right), \qquad v_{23}^2 < \left(\frac{E_2}{E_3}\right) \\ &v_{13}^2 < \left(\frac{E_1}{E_3}\right), \qquad v_{31}^2 < \left(\frac{E_3}{E_1}\right) \end{aligned} \tag{161}$$

also,

$$1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - 2v_{21}v_{32}v_{13} > 0 \tag{162}$$