Strong 2.2 #1,5,6,11 $ coding problem. (leapfrog or trapezoidal)

1. Leapfrog matrix for $u''+u=0$ is $G_L = \begin{bmatrix} 1 & h \\ -h & 1-h^2 \end{bmatrix}$, λ_1, λ_2.

a) $\lambda_1 + \lambda_2 = e^{i\theta} + e^{-i\theta}$, find $\cos\theta = 1 - \frac{1}{2}h^2$ for $h \le 2$.

$\Rightarrow \lambda_1 + \lambda_2 = (\cos\theta + i\cancel{\sin\theta}) + (\cos(-\theta) + i\cancel{\sin(-\theta)}) = 2\cos\theta \Rightarrow \boxed{\cos\theta = \frac{\lambda_1+\lambda_2}{2}}$

* also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ * what does "find $\cos\theta$" mean?

b) $h=2$, find eigenvalues $ eigenvectors * use $\cos\theta = 1 - \frac{1}{2}h^2$?....

$G_L = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \Rightarrow \det\begin{vmatrix} 1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 + 4 = 0 \Rightarrow \lambda^2 = -3 \Rightarrow \lambda_1 = i\sqrt{3}$, $\lambda_2 = -i\sqrt{3}$

$V_{\lambda_1} = \begin{bmatrix} 1-i\sqrt{3} & 2 \\ -2 & -1-i\sqrt{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1+i\sqrt{3} \\ 1-i\sqrt{3} & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1+i\sqrt{3} \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = -\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x_2$, $x_2 = x_2$

$\rightarrow \frac{1}{2}(-1+i\sqrt{3})(1+i\sqrt{3})$

$\rightarrow \frac{1}{2}(-1+\cancel{i^2}(3)) = -2$ $\quad V_{\lambda_1} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + i\begin{bmatrix} -\sqrt{3}/2 \\ 0 \end{bmatrix}$

$V_{\lambda_2} = \begin{bmatrix} 1+i\sqrt{3} & 2 \\ -2 & -1+i\sqrt{3} \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1-i\sqrt{3} \\ 0 & 0 \end{bmatrix} \Rightarrow \left. \begin{matrix} x_1 = -\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)x_2 \\ x_2 = x_2 \end{matrix} \right\} V_{\lambda_2} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + i\begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{\begin{matrix} \lambda_1 = i\sqrt{3}, & \vec{V}_{\lambda_1} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - i\begin{bmatrix} \sqrt{3}/2 \\ 0 \end{bmatrix} \\ \lambda_2 = -i\sqrt{3}, & \vec{V}_{\lambda_2} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + i\begin{bmatrix} \sqrt{3}/2 \\ 0 \end{bmatrix} \end{matrix}}$

c) $h=3$; find λ_1 $ λ_2 $ verify $\lambda_1\lambda_2 = 1$, but $|\lambda_{max}| > 1$

$G_L = \begin{bmatrix} 1 & 3 \\ -3 & -8 \end{bmatrix} \Rightarrow \det\begin{vmatrix} 1-\lambda & 3 \\ -3 & -8-\lambda \end{vmatrix} = -(1-\lambda)(8+\lambda) + 9 = 0$

$\Rightarrow \lambda^2 + 7\lambda - 8 + 9 = 0 \Rightarrow \lambda^2 + 7\lambda + 1 = 0 \Rightarrow \lambda = \frac{-7 \pm \sqrt{49-4}}{2} \Rightarrow \lambda = -\frac{7}{2} \pm \frac{\sqrt{45}}{2}$

$\lambda_1\lambda_2 = \left(-\frac{7}{2} + \frac{\sqrt{45}}{2}\right)\left(-\frac{7}{2} - \frac{\sqrt{45}}{2}\right) = \frac{49}{4} - \frac{45}{4} = 1$ ✓ $(\lambda_1\lambda_2 = 1)$

$\lambda_2 = -\frac{7}{2} - \frac{\sqrt{45}}{2}$, $|\lambda_2| > 1$ by inspection ✓ $(|\lambda_{max}| > 1)$

$\Rightarrow$

②

5. $A^T = -A$ (skew-symmetric)

+ what are we supposed to "show"?

$$\frac{du}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} u \quad \text{or} \quad \begin{aligned} u_1' &= cu_2 - bu_3 \\ u_2' &= au_3 - cu_1 \\ u_3' &= bu_1 - au_2 \end{aligned}$$

a) $||u(t)||^2 = u_1^2 + u_2^2 + u_3^2$, derivative $= 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$. Substitute.

$$\Rightarrow 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) = \frac{du}{dt}||u(t)||^2$$

Expo $\Rightarrow \cancel{2u_1u_2c} - \cancel{2bu_1u_3} + \cancel{2au_2u_3} - \cancel{2cu_1u_2} + \cancel{2bu_1u_3} - \cancel{2au_2u_3} = \frac{du}{dt}$

$$\Rightarrow 0 = \frac{du}{dt} \Rightarrow \boxed{||u(t)||^2 = \text{constant} = ||u(0)||^2}$$

✓

Also: $\int 0 = \int \frac{du}{dt} \Rightarrow c = u(t) = u(0) = u(1) = \ldots\ldots$

$u(t) = u(0)$

$$\boxed{||u(t)||^2 = ||u(0)||^2}$$

b) $Q = e^{At}$ is orthogonal; prove $Q^T = e^{-At}$ from the series $Q = e^{At} = I + At + (At)^2/2! + \ldots$

If $e^{At} = I + At + (At)^2/2!$, ~~then~~ $e^{-At} = I - At + (At)^2/2! - \frac{(At)^3}{3!} + \ldots$.

If Q is orthogonal (as given), $QQ^T = I$. Thus, if $e^{At}e^{-At} = I$, we can prove that $Q^T = e^{-At}$.

3 terms $\Rightarrow e^{At}e^{-At} = (I + At + (At)^2/2! + \ldots)(I - At + (At)^2/2! + \ldots)$

$$= \left(I - \cancel{At} + \cancel{(At)^2/2!} + \cancel{At} - \cancel{(At)^2} + \frac{(At)^3}{\cancel{2!}} + \frac{\cancel{(At)^2}}{\cancel{2!}} - \frac{\cancel{(At)^3}}{\cancel{2!}} + \frac{(At)^4}{4} + \ldots\right)$$

$$= \left(I + \frac{(At)^4}{4} + \ldots\right)$$

2 terms $\Rightarrow e^{At}e^{-At} = (I + At + \ldots)(I - At + \ldots)$

$$= (I - (At)^2 + \ldots)$$

When more terms are used in the series e^{At} to multiply out $e^{At}e^{-At}$, (when 2 terms were used to approximate the series) residuals (e.g. $-(At)^2$ when 2 terms were used to approximate the series) vanish. Therefore, as the number of terms $\to \infty$, any residual terms will either be cancelled out or will be approximately 0, as $\lim_{x\to\infty} \frac{(At)^x}{x!} = 0$. Therefore, $Q^T = e^{-At}$, which means $QQ^T = e^{At}e^{-At} = I$.

$\hookrightarrow \lim_{x\to\infty} \frac{3^x}{x!} = 0$

6. Trapezoidal Rule Conserves Energy, $\|u\|^2$ when $\underline{u'=Au}$ & $A^T=-A$

$(I-\frac{\Delta t}{2}A)U_{n+1} = (I+\frac{\Delta t}{2}A)U_n$ (2nd)

<u>Given</u>: G_T is orthogonal because $A^T=-A$

$\Rightarrow U_{n+1} = G_T U_n$; $G_T = (I-\frac{\Delta t}{2}A)^{-1}(I+\frac{\Delta t}{2}A)$

$\Rightarrow G_T^T U_{n+1} = U_n$ ① (because $G_T^T G_T = I$; G_T is orthogonal)

$$G_T^T U_{n+1}(U_{n+1}+U_n) = U_n(U_{n+1}+U_n)$$

$$G_T^T U_{n+1}U_{n+1} + G_T^T U_{n+1}U_n = U_nU_{n+1} + U_nU_n$$

$$G_T^T U_{n+1}U_{n+1} + \cancel{G_T^T U_{n+1}U_n} = \underset{U_n G_T U_n}{\cancel{G_T^T U_{n+1}}} + \cancel{G_T^T U_{n+1}U_n}$$

$\cancel{+① \ (G_T^T U_{n+1})^T = (U_n)^T}$

$\cancel{\Rightarrow U_{n+1}^T (G_T^T)^T = U_n^T}$

$\cancel{\Rightarrow U_{n+1}^T G_T = U_n^T}$

$$G_T(G_T^T U_{n+1}U_{n+1}) = G_T(U_n G_T U_n)$$

$\Rightarrow U_{n+1}U_{n+1} = (G_TU_n)(G_TU_n)$

$\cancel{\Rightarrow U_{n+1}U_{n+1} = G_TU_n}$

$\Rightarrow U_{n+1}^2 = (G_TU_n)^2$

Given that G_T is orthogonal, so $\|G_TU_n\|^2 = \|U_n\|^2$. Thus,

$\Rightarrow \|U_{n+1}\|^2 = \|G_TU_n\|^2 = \|U_n\|^2$ ✓

$\Rightarrow \|U_{n+1}\|^2 = \|U_n\|^2$

11. $H = \frac{1}{2}p^T M^{-1} p + \frac{1}{2}u^T K u$ = energy for oscillating linear springs & masses. (4)

u = position, p = momentum.

Given $p' = \frac{\partial H}{\partial u}$ & $u' = \frac{\partial H}{\partial p}$, derive Newton's law $Mu'' + Ku = 0$.

$* \ \frac{d}{dx}(x^T B) = B$!!

$* \ \frac{d}{dx}(x^T B x) = 2Bx$

$$-\frac{\partial H}{\partial u} = p' = 0 + \frac{1}{2}(2Ku) = Ku \Rightarrow p' = -Ku \quad ①$$

$$\frac{\partial H}{\partial p} = u' = \frac{1}{2}(2M^{-1}p) + 0 = M^{-1}p \Rightarrow u' = M^{-1}p \quad ②$$

② $\Rightarrow$ $Mu' = p$ (right multiply by M, $MM^{-1} = I$)

$$\frac{d}{dt}(Mu') = \frac{d}{dt}(p)$$

$$Mu'' = p'$$

↓ plug into ①

$$Mu'' = -Ku$$

$$\boxed{Mu'' + Ku = 0}$$