CoE 352 HW # 4 Kenneth Meyer klm5375 ①

Problem set 2.3 : # 4, 5, 7, 8, 12 (least squares)

* Should be use MATLAB for 4 & 5?

4. Gram-schmidt orthogonalization
* from columns a_1 & a_2 of [2 2 1; -1 2 2]', follow steps of Gram-schmidt to create orthonormal columns q_1 & q_2. What is R?

$$\Rightarrow A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \text{find } Q \text{ \& } R;\ A = QR.$$

(a_1) (a_2).

$\Rightarrow Q = [v_1 \ v_2] \Rightarrow v_1$ & v_2 form orthogonal basis for A.

$$v_1 = a_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$v_2 = a_2 - \frac{a_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{-2+4+2}{4+4+1}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - 4/9 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -17/9 \\ 10/9 \\ 14/9 \end{bmatrix} = v_2$$

$\Rightarrow$ columns of q!

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{4+4+1}} = \frac{1}{3} v_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{v_2}{\sqrt{(\frac{-17}{9})^2 + (\frac{10}{9})^2 + (\frac{14}{9})^2}} = \frac{1}{2.687} v_2 = \frac{1}{2.687}\begin{bmatrix} -17/9 \\ 10/9 \\ 14/9 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cc} q_1 & q_2 \end{array} \\ \boxed{Q =} & \begin{bmatrix} 2/3 & -.703 \\ 2/3 & .413 \\ 1/3 & .579 \end{bmatrix} \end{array}$$

$\Rightarrow A = QR \Rightarrow Q^T A = Q^T Q R$; $Q^T Q = I$ because Q is orthonormal,

$Q^T A = R$

$$\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -.703 & .413 & .579 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 7/3 & 4/3 \\ 0 & 2.687 \end{bmatrix} = R}$$

7. $b=(4,1,0,1)$, $x=(0,1,2,3)$, set up and solve the normal equation for the coefficients $\hat{u}=(C,D)$ in the nearest line $C+Dx$

line through 4 pts } try to solve $Au=b$; $\begin{bmatrix}1 & x\end{bmatrix}u=b \Rightarrow$
$$\begin{aligned}C+0D&=4\\C+1D&=1\\C+2D&=0\\C+3D&=1\end{aligned} \Rightarrow \overset{A}{\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}}\begin{bmatrix}C\\D\end{bmatrix}=\overset{b}{\begin{bmatrix}4\\1\\0\\1\end{bmatrix}}$$

$\Rightarrow$ minimize error; $\|b-Au\|^2=(b-Au)^T(b-Au)$

$\Rightarrow$ Solve $A^TA\hat{u}=A^Tb$ (normal eqn)

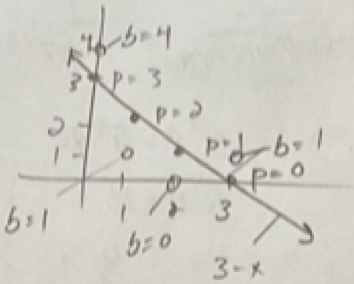
$$\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\hat{u}=\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}4\\1\\0\\1\end{bmatrix} \Rightarrow \boxed{\hat{u}=(\overset{C}{3},\overset{D}{-1}),\text{ nearest line is } 3-x}$$

$$\begin{bmatrix}4&6\\6&14\end{bmatrix}\underset{\begin{bmatrix}C\\D\end{bmatrix}}{\hat{u}}=\begin{bmatrix}6\\4\end{bmatrix} \Rightarrow \begin{aligned}4C+6D&=6\\6C+14D&=4\end{aligned} \Rightarrow \begin{aligned}-(2C+3D&=3)3\\(3C+7D&=2)2\\ \hline 0-5D&=5\end{aligned} \Rightarrow \boxed{D=-1}$$

$4C+6(-1)=6$
$4C=12 \rightarrow \boxed{C=3}$

8. Find projection $p=A\hat{u}$ from 7. Check these 4 values lie on $C+Dx$, compute error $e=b-p$, verify $A^Te=0$.

$$p=A\hat{u}=\begin{bmatrix}1&0\\1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} \Rightarrow p=\begin{bmatrix}3\\2\\1\\0\end{bmatrix}, \text{ check:}$$



$$e=b-p=\begin{bmatrix}4\\1\\0\\1\end{bmatrix}-\begin{bmatrix}3\\2\\1\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=e$$

$$A^Te=\begin{bmatrix}1&1&1&1\\0&1&2&3\end{bmatrix}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\checkmark,\ A^Te=0.$$ Not asked for, but total error $= e^Te = 1^2+(-1)^2+(-1)^2+1^2=4$.

12. closest parabola $C+Dx+Ex^2$ to some 4 points, write down unsolvable eqn for $u=(C,D,E)$, $Au=b$, set up normal eqn for $\hat{u}$. What is $\vec{e}$ when fitting $C+Dx+Ex^2+Fx^3$ to these 4 pts?

$$\begin{aligned}C+Dx_1+Ex_1^2&=b_1\\C+Dx_2+Ex_2^2&=b_2\\C+Dx_3+Ex_3^2&=b_3\\C+Dx_4+Ex_4^2&=b_4\end{aligned};\quad A^TA\hat{u}=A^Tb \Rightarrow \begin{bmatrix}1&1&1&1\\x_1&x_2&x_3&x_4\\x_1^2&x_2^2&x_3^2&x_4^2\end{bmatrix}\begin{bmatrix}1&x_1&x_1^2\\1&x_2&x_2^2\\1&x_3&x_3^2\\1&x_4&x_4^2\end{bmatrix}\overset{\hat{u}}{\begin{bmatrix}C\\D\\E\end{bmatrix}}=\begin{bmatrix}1&1&1&1\\x_1&x_2&x_3&x_4\\x_1^2&x_2^2&x_3^2&x_4^2\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

$\vec{x}_3=\begin{bmatrix}x_1^3\\ \vdots\\ x_4^3\end{bmatrix}$

linear means?

$e_2=b-p_2=b-(A_2\hat{u}_2)$ $\Rightarrow$ $*A_2\hat{u}_2+F\vec{x}_3=A_3\hat{u}_3$;
$e_3=b-p_3=b-(A_3\hat{u}_3)$ $\quad *e_3=b-(A_2\hat{u}_2+F\vec{x}_3) \Rightarrow e_3+F\vec{x}_3=b-A_2\hat{u}_2=e_2$!

The error term $\vec{e}_3$ when fitting $C+Dx+Ex^2+Fx^3$ to these 4 points will be equal to the error term of the quadratic fit, $\vec{e}_2$, minus the "cubic" term Fx^3 for each point; $\vec{e}_3=\vec{e}_2-F\vec{x}_3$, where $\vec{x}_3=\begin{bmatrix}x_1^3\\x_2^3\\x_3^3\\x_4^3\end{bmatrix}$.

2. Show that Newton's method for $u^2 - a = 0$ finds u^{k+1} as the average of u^k and a/u^k. Connect the new error $u^{k+1} - \sqrt{a}$ to the square of the old error $u^k - \sqrt{a}$: $u^{k+1} - \sqrt{a} = \frac{1}{2}\left(u^k + \frac{a}{u^k}\right) - \sqrt{a} = \left(u^k - \sqrt{a}\right)^2 / 2u^k$

Newton's method for $u^2 - a = 0$: $\longrightarrow$ $J\Delta u = -g$; $J(u^k)(u^{k+1} - u^k) = -g(u^k)$

$u^* = \sqrt{a}$, $J(u^k) = 2u^k = g'$

$\Rightarrow u^{k+1} = \dfrac{-g(u^k)}{J(u^k)} + u^k$

$\Rightarrow u^{k+1} = \dfrac{-\left((u^k)^2 - a\right)}{2u^k} + u^k$

$\Rightarrow u^{k+1} = \dfrac{-u^k + a/u^k}{2} + u^k = \boxed{\dfrac{u^k + a/u^k}{2}}$ $\Rightarrow$ this is the average of u^k and a/u^k.

* Newton squares the error at each step. * what are we supposed to "do"?

$u^k - \sqrt{a}$ is old error; new error $= (u^k - \sqrt{a})^2$.

$(u^k - \sqrt{a})^2 = (u^k)^2 - 2u^k\sqrt{a} + a = u^k\left(u^k + a/u^k - 2\sqrt{a}\right) = 2u^k\Bigg(\underbrace{\frac{u^k + \frac{a}{u^k}}{2}}_{u^{k+1}} - \sqrt{a}\Bigg)$

$= 2u^k\left(u^{k+1} - \sqrt{a}\right)$ $\Rightarrow$ the new error times $2u^k$, the jacobian, is equal to the square of the old error in our case.

$\Rightarrow$

5. "a)" Show Newton's method converges in one step, $u^1 = A^{-1}b$, with $g(u) = Au - b$. ④
"b)" given the fixed point iteration $u^{k+1} = H(u^k) = u^k - (Au^k - b)$, has $H' = I - A$.
Its convergence factor c is the maximum eigenvalue $|1 - \lambda(A)|$.
$\Rightarrow$ why is $c > 1$ for $A = K = (-1, 2, -1)$ matrix, but $c < 1$ for $A = K/2$?

Newton's method: $J(u^k)(u^{k+1} - u^k) = -g(u^k) = b - Au^k$

$J(u^k) = g' = A$ (vector differentiation)

$g(u) = Au - b = 0$

$\underline{u = A^{-1}b}$ $\Rightarrow$ exact solution

$\Rightarrow$ rearrange newton's method,

$u^{k+1} = -J(u^k)^{-1} g(u^k) + u$; sub in $J(u^k)$, u^k, and we see:

$u^{k+1} = -(A)^{-1}(Au^k - b) + u^k$

$u^{k+1} = -\cancel{u^k} + A^{-1}b + \cancel{u^k}$

exact solution

$\Rightarrow \boxed{u^{k+1} = A^{-1}b}$; $u^{k+1} - A^{-1}b = A^{-1}b - A^{-1}b = 0$ ✓ error = 0, it converges after one iteration.

$\Rightarrow u^{k+1}$ doesn't rely on u^k; it is always the same/doesn't change

$\boxed{u^1 = A^{-1}b}$, $u^0 = A^{-1}b \ldots$

"better" way to get answer they want: $u^1 = -(A^{-1})(Au^0 - b) + u^0$
$u^1 = -\cancel{u^0} + A^{-1}b + \cancel{u^0}$ $\Rightarrow$ $\underline{u^1 = A^{-1}b}$, same result.

"b)" The eigenvalues for K are equal to: $\lambda_K = 2 - 2\cos\left(\frac{k\pi}{n+1}\right)$, so $|\lambda| > 2$ because $|\cos\theta| < 1$.

The eigenvalues for $\frac{K}{2}$ are equal to $\frac{1}{2}$ those of K; $\lambda\vec{v} = A\vec{v}$
so $\frac{1}{2}\lambda\vec{v} = \frac{1}{2}A\vec{v}$. Hence, the eigenvalues of $\frac{1}{2}K$ are $\lambda_{K/2} = 1 - \cos\left(\frac{k\pi}{n+1}\right)$.

Given that the convergence factor c equals $|1 - \lambda_{max}|$, it can be shown that $c > 1$ because $0 \le \lambda_{K/2} \le 2$, so $|1 - \lambda_{K/2}| < 1$ for all λ, including λ_{max}.

5.) (Pset 2.3) *Not explained well in book

$a_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow$ find r_1, w_1, u_1, H_1

$r_1 = \begin{bmatrix} \|a_1\| \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = r_1}$, $w_1 = a - r = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = w_1}$

$\|a_1\| = \sqrt{4+4+1} = 3$

$u_1 = \frac{w_1}{\|w_1\|} = \frac{\langle -1, 2, 1 \rangle}{\sqrt{1+4+1}} = \boxed{\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = u_1}$

$H = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\left(\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \right)$

$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} = \boxed{\frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{bmatrix} = H_1}$

$\hookrightarrow \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\Rightarrow$ use $H_1 A$, below diagonal.

$H_1 A = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 9 & 4 \\ 0 & -8 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4/3 \\ 0 & -8/3 \\ 0 & -1/3 \end{bmatrix} \Rightarrow a_2 = \begin{bmatrix} -8/3 \\ -1/3 \end{bmatrix}$

$\Rightarrow r_2 = \begin{bmatrix} \|a_2\| \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{\sqrt{65}}{3} \\ 0 \end{bmatrix} = r_2}$, $w_2 = a_2 - r = \begin{bmatrix} -8/3 \\ -1/3 \end{bmatrix} - \begin{bmatrix} \sqrt{65}/3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{-8-\sqrt{65}}{3} \\ -1/3 \end{bmatrix} = w_2}$

$u_2 = \frac{w_2}{\|w_2\|} = \frac{w_2}{5.36} \Rightarrow \boxed{u_2 = \begin{bmatrix} -.998 \\ -.062 \end{bmatrix}}$

$H_2 = I - 2u_2u_2^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} -.998 \\ -.062 \end{bmatrix} \begin{bmatrix} -.998 & -.062 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -.992 & -.124 \\ 0 & -.124 & .992 \end{bmatrix}}$

only operate on this "square"

$H_2 H_1 A = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -.992 & -.124 \\ 0 & -.124 & .992 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 0 & -2 & 2 \end{bmatrix} \right) \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \approx \boxed{\begin{bmatrix} 3 & 4/3 \\ 0 & 2.687 \\ 0 & 0 \end{bmatrix} = R}$ H_2

* seems wrong as R is not square, but followed the steps where $H_1 = I - u_1u_1^T$ & $H_2 = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & I - u_2u_2^T & \\ 0 & & \end{bmatrix}$.

But, R is upper triangular.

Matlab coding Problem, Hand Calculations (HW 4 COE 352) ⑤

* there is a jacobian function in MATLAB, will compare answers.

$$\left.\begin{array}{ll} g_1 = u_1 + u_2^4 u_3 & = 0 \\ g_2 = -u_1 u_2 + 2u_2 + u_3^3 u_4 & = 0 \\ g_3 = -2u_1^2 + 4u_3 - u_4 & = 0 \\ g_4 = 2u_3^2 + 4u_4 & = 0 \end{array}\right\} \quad J = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_4} \\ \vdots & & & \\ \frac{\partial g_4}{\partial u_1} & & \cdots & \frac{\partial g_4}{\partial u_4} \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} 1 & 4u_2^3 u_3 & u_2^4 & 0 \\ -u_2 & -u_1 + 2 & 3u_3^2 u_4 & u_3^3 \\ -4u_1 & 0 & 4 & -1 \\ 0 & 0 & 4u_3 & 4 \end{bmatrix}$$

(I think) ✓

(matches w/ matlab code)

~~* might want to have this as a function...~~

~~(100% do)~~

~~* need to check error @ each step to see if it~~ is ~~below given tolerance.~~

<u>* Note</u>: a different number of iterations is reported for the second guess when using inv(A)*b vs. A\b, This is because the Jacobian J J(u) becomes ill-conditioned through the iterations,