HW7, COE 352 Kenneth Meyer klm5375

6.2 # 1, 2, 3, 4 4* - MATLAB

1. $e^{a\Delta t} - \left[\frac{1+(a\Delta t/2)}{1-(a\Delta t/2)}\right] = e^{a\Delta t} - \left[\left(1+\frac{a\Delta t}{2}\right)\left(1+\frac{a\Delta t}{2}+\left(\frac{a\Delta t}{2}\right)^2+\left(\frac{a\Delta t}{2}\right)^3 \dots\right)\right]$

$= e^{a\Delta t} - \left[1+\frac{a\Delta t}{2}+\left(\frac{a\Delta t}{2}\right)^2+\frac{a\Delta t}{2}+\left(\frac{a\Delta t}{2}\right)^2+\left(\frac{a\Delta t}{2}\right)^3+\left(\frac{a\Delta t}{2}\right)^3+\left(\frac{a\Delta t}{2}\right)^4+\dots\right]$

squared terms: $\left(\frac{a\Delta t}{2}\right)^2, \left(\frac{a\Delta t}{2}\right)^2 = \frac{1}{2}(a\Delta t)^2$ ✓, $\boxed{c = 1/2}$

cubed terms: $\left(\frac{a\Delta t}{2}\right)^3 + \left(\frac{a\Delta t}{2}\right)^3 = \frac{1}{4}(a\Delta t)^3$

=> this term should be $\frac{-(a\Delta t)^3}{3!}$ via $e^{a\Delta t}$ series expansion. $\frac{1}{4}(a\Delta t)^3 - \frac{1}{6}(a\Delta t)^3$

therefore the error is $DE \approx \frac{1}{12}(\Delta t)^3 u'''$. $\longleftrightarrow$ $= \underline{\frac{1}{12}(a\Delta t)^3}$

2. $\frac{1}{2}\left(3e^{a\Delta t} - 4 + e^{-a\Delta t}\right) - a\Delta t e^{a\Delta t}$ (1) simplified (1) to $-2 + \frac{1}{2}e^{-a\Delta t} + \frac{3}{2}e^{a\Delta t} - a\Delta t e^{a\Delta t}$

$\left(e^{a\Delta t} = 1 + a\Delta t + \frac{(a\Delta t)^2}{2!} + \frac{(a\Delta t)^3}{3!} + \dots\right)$

$= -\cancel{2} + \frac{1}{2}\left(\cancel{1} + (-a\Delta t) + \frac{(-a\Delta t)^2}{2!} + \frac{(-a\Delta t)^3}{3!} + \dots\right) + \frac{3}{2}\left(\cancel{1} + a\Delta t + \frac{(a\Delta t)^2}{2!} + \frac{(a\Delta t)^3}{3!} + \dots\right)$

$- a\Delta t\left(1 + a\Delta t + \frac{(a\Delta t)^2}{2!} + \frac{(a\Delta t)^3}{3!} + \dots\right)$

* cancel terms w/o $a\Delta t$; $= -\frac{1}{\cancel{2}}\cancel{a\Delta t} + \frac{1}{4}(a\Delta t)^2 - \frac{(a\Delta t)^3}{12} + \frac{3}{\cancel{2}}\cancel{a\Delta t} + \frac{3}{4}(a\Delta t)^2 + \frac{1}{4}(a\Delta t)^3$

$-\cancel{a\Delta t} - \cancel{a\Delta t^2} - \frac{(a\Delta t)^3}{2}$ (omit higher order terms)

$= -\frac{(a\Delta t)^3}{12} + \frac{1}{4}(a\Delta t)^3 - \frac{(a\Delta t)^3}{2} = \left(-\frac{1}{12} - \frac{1}{4}\right)(a\Delta t)^3 = \boxed{-\frac{1}{3}(a\Delta t)^3}$

$\boxed{c = -\frac{1}{3}}$ ✓ leading error

$\left(-\frac{1}{3}(a\Delta t)^3 = c(a\Delta t)^3;\ c = -\frac{1}{3}\right)$

=>

3. $(u_{n+1} - u_n) - \Delta t\, u'_{n+1} \approx \left(\Delta t\, u_n' + \frac{(\Delta t)^2}{2} u_n''\right) - \Delta t\left(u_n' + \Delta t\, u_n''\right) \approx -\frac{(\Delta t)^2}{2} u_n''$.

* for $u' = u$ & $u_0 = 1$, find the exact error in u_1.

* if $u_0 = 1$, and $u' = u$; $u_0' = 1$. $\rightarrow$ does $u'' = u'$?

additionally, if $u' = u$, $u'' = u'$. Hence, *unsure abt this

$u_n' = u_n'' = u_0' = 1$. Therefore,

$$\Delta t\, u_n' + \frac{(\Delta t)^2}{2} u_n'' - \Delta t\left(u_n' + \Delta t\, u_n''\right) = \cancel{\Delta t}(1) + \frac{(\Delta t)^2}{2}(1) - \cancel{\Delta t}(1 + \Delta t(1)) = \boxed{-\frac{(\Delta t)^2}{2}}$$

could have pluged in directly

(Exact error in u_1)

4. Runge-Kutta (see .m file).

$$\frac{u_{n+1} - u_n}{\Delta t} = \frac{1}{3}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$k_1 = f(u_n, t_n)/2$

$k_2 = f(u_n + \Delta t\, k_1, t_{n+1/2})/2$

$k_3 = f(u_n + \Delta t\, k_2, t_{n+1/2})/2$

$k_4 = f(u_n + 2\Delta t\, k_3, t_{n+1})/2$

Trying on: $u' = -100u + 100 \sin t$,

$\Delta t = -.0275$ & $-.028$.

(these are close to stability limit of $-.0278$)

$\Rightarrow -a\Delta t < 2.78$; in $u' = -100u + 100 \sin t$, $a = -100$!

$t_{n+1/2} = t_n + \frac{\Delta t}{2}$?

* I'm not achieving instability, not 100% sure why.

Assignment 7 (cont) Kenneth Meyer ③

6.3, # 3, 5, 9

3. $U_{j,n+1} = \sum a_m U_{j+m,n}$, $G = \sum a_m e^{imk\Delta x}$ .. $G_{exact} = e^{ick\Delta t}$,
show consistency when $\sum a_m = 1$, $\sum m a_m = c\Delta t/\Delta x = r$.
↳ first order accuracy!

- if $\sum m a_m = \frac{c\Delta t}{\Delta x}$ and $\sum a_m = 1$, $m = \frac{c\Delta t}{\Delta x}$; $\underline{m\Delta x = c\Delta t}$.
- Expanding $\sum a_m e^{imk\Delta x}$: ($\sum a_m$ vanishes because $a_m = 1$)

$$\sum a_m e^{imk\Delta x} = 1 + i\underline{mk\Delta x} + \tfrac{1}{2}(imk\Delta x)^2 + \ldots$$

* didn't really need to expand, could just sub in.

$$= 1 + ikc\Delta t + \tfrac{1}{2}(ikc\Delta t)^2 + \ldots$$

$$\boxed{G_{exact} = e^{ick\Delta t}}$$ (basically subbed in $m\Delta x = c\Delta t$).

$\sin^2 + \cos^2 = 1$; $\sin^2 = 1 - \cos^2$

5. $|G|^2 = (1 - r^2 + r^2\cos k\Delta x)^2 + (ir\sin k\Delta x)^2$

$= (1 - r^2 + r^2\cos k\Delta x) + (-r^2 + r^4 - r^4\cos k\Delta x) + (r^2\cos k\Delta x - r^4\cos k\Delta x + r^4\cos^2 k\Delta x)$

$+ i^2 r^2 \sin^2 k\Delta x = -r^2\sin^2 k\Delta x = -r^2(1-\cos^2 k\Delta x)$

? $\longrightarrow + i^2 r^2 \sin^2 k\Delta x$ + $-r^4\cos k\Delta x + r^4\cos^2 k\Delta x + r^2 - r^2\cos^2 k\Delta x$ (sign?)

$= 1 - 2r^2 + 2r^2\cos k\Delta x + r^4 - 2r^4\cos k\Delta x + r^4\cos^2 k\Delta x$

$= 1 - r^2 + r^4 + 2r^2\cos k\Delta x - 2r^4\cos k\Delta x - r^2\cos^2 k\Delta x + r^4\cos^2 k\Delta x$

$= 1 - (r^2 - r^4)(1 - \cos k\Delta x)^2$ ✓

~~proof~~

shorter proof: $\boxed{r^2 \le 1, \text{ so } 0 \le r^2 - r^4 \le 1/4. \ |G|^2 = 1 - \text{positive \#, so } |G|^2 \le 1.}$

(thought I needed to prove $0 \le |G^2| \le 1$, which is done below)

*so, minimum value of $G = 0$.

$\boxed{\begin{aligned} &\bullet\ r' = 2r - 4r^3 = 0 \rightarrow 2r(1 - 2r^2) \Rightarrow 1 = 2r^2,\ r = \sqrt{\tfrac{1}{2}} \\ &r(\sqrt{\tfrac{1}{2}}) = (\tfrac{1}{2} - \tfrac{1}{4})^2 = \tfrac{1}{4}\ !! \end{aligned}}$

$\underline{|G|^2 = 1 - (r^2 - r^4)(1 - \cos k\Delta x)^2}$. Because $|\cos k\Delta x| \le 1$, the maximum value $(1-\cos k\Delta x)^2$
can hold is $(1-(-1))^2 = 4$. We check the maximum value of $r^2 - r^4$,
if $r^2 \le 1$, $r^2 - r^4$,
which is $\frac{1}{4}$, as seen boxed above ($r^2 - r^4 = 0$ @ endpoints $r=-1, r=1$).
Hence, for all possible $k\Delta x$ on $-1 \le r \le 1$, the maximum value of
$|G|^2$ occurs when $(1 - \cos k\Delta x) = 0$, and is $|G|^2 = 1$. Proving $r^2 - r^4 > 0$
likely would also suffice; $0 \le r^2 - r^4 \le \frac{1}{4}$, so G will be less than 1.
(or equal to)

9. $\frac{1}{2}(U_{j+1,n} + U_{j-1,n})$ replaces U_{jn}. Subtracting u_{jn};

$\frac{1}{2}U_{j+1,n} - U_{j,n} + \frac{1}{2}U_{j-1,n}$ => this is essentially a second difference formula; looks similar to $\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$. Thus, the substitution/replacement used comes from a second difference, which in turn improves the stability of Lax-Friedrichs as it was a simple centered difference originally.

(4)