Numerical Analysis HW 2 — TODO: 1. check I have been using V & V^T correctly in SVD 2. finish #1. ①

1. 3.3 & 3.4

3.3 x: m-vector, A is $m \times n$ matrix

a) $\|x\|_\infty \le \|x\|_2$ b) $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$ c) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$ d) $\|A\|_2 \le \sqrt{m}\,\|A\|_\infty$

a) $\|x\|_\infty \le \|x\|_2$

$$\max_{1\le i\le m}|x_i| \overset{?}{\le} \left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}$$

$$\left(\max_{1\le i\le m}|x_i|\right)^2 \le \sum_{i=1}^{m}|x_i|^2$$

$$\cancel{\left(\max_{1\le i\le m}|x_i|\right)^2} \le \cancel{\left(\max_{1\le i\le m}x_i\right)^2} + \sum_{j=1, j\ne i}^{m}|x_i|^2$$

$$0 \le \sum_{j=1, j\ne i}^{m}|x_i|^2 \quad \checkmark \text{ true } \forall x$$

Example (equality)

$x = (2,0,0)$

$$\|x\|_\infty \overset{?}{=} \|x\|_2$$

$$\max_{1\le i\le m}|x_i| \overset{?}{=} \left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}$$

$$|x_1| \overset{?}{=} \sqrt{x_1^2+x_2^2+x_3^2}$$

$$|x_1| = \sqrt{x_1^2+\cancel{0^2}+\cancel{0^2}}$$

$$|x_1| = \sqrt{|x_1|^2}$$

$$2=2 \quad \checkmark$$

b) $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$

$$\left(\left(\sum_{i=1}^{m}|x_i|^2\right)^{1/2}\right)^2 \le \left(\sqrt{m}\left(\max_{1\le i\le m}|x_i|\right)\right)^2$$

$$\sum_{i=1}^{m}|x_i|^2 \le m\left(\max_{1\le i\le m}|x_i|\right)^2$$

$$\underbrace{|x_1|^2+\dots+|x_m|^2}_{m \text{ terms}} \le m\left(\max_{1\le i\le m}|x_i|\right)^2 = \sum_{i}^{m}\max_{1\le i\le m}|x_i|^2$$

$\Updownarrow$ LHS, RHS of inequality

- true, because each side has the same number (m) terms, and $\forall\, x_i \in x$, $x_i \le \max_{1\le i\le m}|x_i|$, so each term on the LHS is less than or equal to those on the right.

Ex: $x = (1,1,1)$

$\|x\|_2 = \sqrt{3}$

$\|x\|_\infty = 1$

$\sqrt{m} = \sqrt{3}$

$\Rightarrow \sqrt{3} = \sqrt{3}(1) \quad \checkmark$

$\Longrightarrow$

3.3, HW2, Numerical Analysis (cont)

c) $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$ * A is m×n.

$$\|A\|_\infty = \sup_{x\in\mathbb{C}^n} \frac{\|Ax\|_\infty}{\|x\|_\infty} \le \sup \frac{\|Ax\|_2}{\|x\|_\infty} \le \sup \frac{\|Ax\|_2}{\frac{\|x\|_2}{\sqrt{n}}} = \sqrt{n}\sup \frac{\|Ax\|_2}{\|x\|_2}$$

$$\Rightarrow \underline{\|A\|_\infty \le \sqrt{n}\,\|A_2\|}$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\|A\|_\infty = 2$

$\|A\|_2 = \sqrt{2}$, $\sqrt{a}\|A\|_2 = \sqrt{2}(\sqrt{2}) = 2$ ✓

d) $\|A\|_2 \le \sqrt{m}\,\|A\|_\infty$

$$\|A\|_2 = \sup_{\substack{x\in\mathbb{C}^n \\ \|x\|_2=1}} \|Ax\|_2 = \underbrace{\sqrt{\rho(A^*A)} \le \sqrt{\sum_i^m \max(A_i)^2}}_{\text{inequality occurs here}} = \sqrt{m\,(\max(A_i))^2} = \sqrt{m}\,\|A\|_\infty$$

secondary proof, more formal

$$\Rightarrow \boxed{\|A\|_2 \le \sqrt{m}\,\|A\|_\infty}$$

$$\sup_{x\in\mathbb{C}^n} \frac{\|Ax\|_2}{\|x\|_2} \le \sup_{x\in\mathbb{C}^n} \frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_2} \le \sup_{x\in\mathbb{C}^n} \frac{\sqrt{m}\,\|Ax\|_\infty}{\|x\|_\infty} = \sqrt{m}\,\|A\|_\infty$$ ✓

consequence of $\|x\|_2 \le \sqrt{m}\,\|x\|_\infty$ $\|x\|_\infty \le \|x\|_2$

ex: $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$. $\rightarrow$ $\|A\|_\infty = 1$, $\|A\|_2 = \sqrt{3}$; $\sqrt{3}\,\|A\|_\infty = \|A\|_2$ ✓ ($\sqrt{m}$!)

3.4) $A \in \mathbb{R}_{m \times n}$, $B \in \mathbb{R}_{\mu \times \nu}$, $\mu \le m$, $\nu \le n$, is a submatrix of A

a) B can be obtained by multiplying A by some matrices, D_L and D_R, on the left and righthand side, respectively. $IA = AI = A$, and to remove the i^{th} row from A when trying to form submatrix B, set the i^{th} diagonal entry of D_L to zero. To delete the j^{th} column from A, set the j^{th} diagonal entry of the modified identity matrix D_R to zero.

To summarize,

$$B = D_L A D_R, \quad D_L \in \mathbb{R}_{m\times m},\ D_{L_{ij}} = \begin{cases} 0 & \text{if } i \ne j \text{ or } i^{th} \text{ row to be deleted} \\ 1 & \text{else} \end{cases}$$

$$D_R \in \mathbb{R}_{n\times n},\ D_{R_{ij}} = \begin{cases} 0 & \text{if } i \ne j \text{ or } j^{th} \text{ column of } A \text{ to be deleted} \\ 1 & \text{else} \end{cases}$$

b) Show $\|B\|_p \le \|A\|_p \ \forall p,\ 1 \le p \le \infty$

$\|B\|_p$

$\Updownarrow$

$\|D_L A D_r\|_p$

$\Updownarrow$

$\|D_L A D_r\|_p \le \|D_L\|_p \|A D_r\|_p \le \|D_L\|_p \|A\|_p \|D_R\|_p$

$\Updownarrow$ — $\|D\|_p = \max_{1\le i\le m} |d_i| = 1$ for D_L and D_r

$\|D_L\|_p \|A\|_p \|D_R\|_p = (1)\|A\|_p(1)$ *assumed $\mu \ge 1, \nu \ge 1$ for $B \in \mathbb{R}^{\mu\times\nu}$

$\Rightarrow \boxed{\|B\|_p \le \|A\|_p}$ ✓

2. SVD

4.1) Determine SVDs of following matrices

a) $\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} = \overset{M\times N}{A} \Rightarrow A = \underset{}{\overset{m\times k}{U}}\,\overset{k\times k}{S}\,\overset{n\times k}{V^T}$, eig($AA^T$) $\rightarrow U$, eig(A^TA) $\rightarrow V$.

$A = A^T \Rightarrow AA^T = A^TA = \begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix} = \begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix}$ * check SVD for diag. mat.

$\Rightarrow \det(A - \lambda I) = 0$ gives eigenvalues,

$\Rightarrow \det\begin{bmatrix}9-\lambda & 0\\ 0 & 4-\lambda\end{bmatrix} = (9-\lambda)(4-\lambda) = 0 \Rightarrow \lambda_1 = 9,\ \lambda_2 = 4.$

$\Rightarrow S = \begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}\sqrt{\lambda_1} & 0\\ 0 & \sqrt{\lambda_2}\end{bmatrix}$

$\Rightarrow$ eigenvects: find null($AA^T - \lambda I$).

λ_1: $\begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix} - \begin{bmatrix}9 & 0\\ 0 & 9\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & -5\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0 \Rightarrow v_2 = 0,\ v_1$ free $\Rightarrow \begin{bmatrix}1\\ 0\end{bmatrix}$

λ_2: $\begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix} - \begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 0 \Rightarrow v_1 = 0,\ v_2$ free. $\Rightarrow \begin{bmatrix}0\\ 1\end{bmatrix}$

$\Rightarrow V = \begin{bmatrix}| & |\\ v_1 & v_2\\ | & |\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow A = USV^T \Rightarrow U = AVS^{-1}$ b/c orthonormal

$U = \frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

inverse of 2×2: $\frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$S^{-1} = \frac{1}{6}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$

$U = \frac{1}{6}\begin{bmatrix}3 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix} = \frac{1}{6}\begin{bmatrix}6 & 0\\ 0 & -6\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$$\Rightarrow \boxed{A = USV^T = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$$

4.1 (cont)

b) $\begin{bmatrix}2&0\\0&3\end{bmatrix} \to A^TA = AA^T = \begin{bmatrix}4&0\\0&9\end{bmatrix} \to \lambda_1 = 9,\ \lambda_2 = 4 \to \Sigma = \begin{bmatrix}3&0\\0&2\end{bmatrix}$

V: $\lambda_1 : AA^T - \lambda_1 I = \begin{bmatrix}-5&0\\0&0\end{bmatrix} \Rightarrow v_1 = 0,\ v_2 \text{ free} \to v_1 = \begin{bmatrix}0\\1\end{bmatrix}$

$\lambda_2 : AA^T - \lambda_2 I = \begin{bmatrix}0&0\\0&5\end{bmatrix} \to v_2 = 0,\ v_1 \text{ free} \to v_2 = \begin{bmatrix}1\\0\end{bmatrix}$ $\Bigg\}\ V = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$U = AV\Sigma^{-1} = \begin{bmatrix}2&0\\0&3\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\left(\frac{1}{6}\begin{bmatrix}2&0\\0&3\end{bmatrix}\right)$

$= \frac{1}{6}\begin{bmatrix}0&2\\3&0\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix} = \frac{1}{6}\begin{bmatrix}0&6\\6&0\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\Rightarrow \boxed{A = U\Sigma V^T = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3&0\\0&2\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}}$

c) $\begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix} : A^TA \to V.\quad A^TA = \begin{bmatrix}0&0&0\\2&0&0\end{bmatrix}\begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix} = \begin{bmatrix}0&0\\0&4\end{bmatrix}$

$\det(A^TA - \lambda I) = \begin{bmatrix}-\lambda&0\\0&4-\lambda\end{bmatrix} = -\lambda(4-\lambda) = 0 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 0$

$\Rightarrow \sigma_1 = \sqrt{\lambda_1} = 2;\quad S = \begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix}$

V: $\text{null}(A^TA - \lambda_1 I) \Rightarrow \begin{bmatrix}-4&0\\0&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} \to v_1 = 0,\ v_2 \text{ free} \to v_1 = \begin{bmatrix}0\\1\end{bmatrix}$

$\text{null}(A^TA - \lambda_2 I) \Rightarrow \begin{bmatrix}0&0\\0&4\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} \to v_1 \text{ free},\ v_2 = 0 \to v_2 = \begin{bmatrix}1\\0\end{bmatrix}$ $\Bigg\}\ V^T = \begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\to \cancel{U = AV\Sigma^{-1} = \begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}}$ * Σ^{-1} not well-defined;

$AV = (U\Sigma V^T)V \implies \begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix} = U\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\begin{bmatrix}0&2\\0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = U\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix}$ * U is 3×3

$\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix} = U\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix} \Rightarrow$ any U s.t. $UU^{-1} = I$ and $U\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix} = \begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix}$ will work.

$\Rightarrow$ let $U = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$\Rightarrow \boxed{A = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&0\\0&0\\0&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}}$

2) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A \Rightarrow A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \cancel{1} - 2\lambda + \lambda^2 \cancel{-1}$

$\Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda_1 = 2,\ \lambda_2 = 0 \Rightarrow S = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

$\sigma_1 = \sqrt{2} \quad \sigma_2 = 0$

$\lambda_1 = 2$

$(A^TA - \lambda_1 I) = \begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} \longrightarrow$ find eigenvectors, which are columns of V.

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Rightarrow v_1 = v_2 \longrightarrow V_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow v_1 = -v_2 \longrightarrow V_{\lambda_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \longrightarrow$

$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $V^T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ * but must be unitary $\rightarrow$ cols are unit vectors

$\Rightarrow AV = U\Sigma \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = U\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow U =$ any 2x2 matrix s.t. $UU^{-1} = I$ and

$U\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ * $2/\sqrt{2} = \sqrt{2}$

$\Rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & ? \end{bmatrix} \rightarrow$ let ? = 1 to make U unitary!

$\rightarrow U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$

4.1 (cont)

e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\det(A^TA - \lambda I) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 4 = (\cancel{2}-\lambda-\cancel{2})(2-\lambda+2) = 0$

$\Rightarrow \lambda(\lambda-4) = 0 \Rightarrow \lambda_1 = 4, \ \lambda_2 = 0$

$\sigma_1 = 2, \ \sigma_2 = 0$

$S = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$

v_1: $\begin{bmatrix} 2-4 & 2 \\ 2 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to -2v_1 + 2v_2 = 0 \longrightarrow v_1 = v_2$

$v_{\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

v_2: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \longrightarrow 2v_1 + 2v_2 = 0 \to v_1 = -v_2 \to v_{\lambda_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ * want to be unitary $\to V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$AV = U\Sigma \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 2/\sqrt{2} & 0 \\ 2/\sqrt{2} & 0 \end{bmatrix} = U\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $\longrightarrow \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$

$\longrightarrow U = 2 \times 2$ s.t. $\left.\begin{aligned} 2u_{11} + 0u_{12} &= 2/\sqrt{2} \\ 2u_{21} + 0u_{22} &= 2/\sqrt{2} \end{aligned}\right\}$ $u_{11} = 1/\sqrt{2}$, $u_{21} = 1/\sqrt{2}$, u_{12} & u_{22} free, make unitary

$\Rightarrow U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

↳ arbitrary col.

$\to \boxed{A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$

Numerical Analysis HW # 2 (cont)

$$\begin{bmatrix} - v_1 - \\ - v_2 - \end{bmatrix}$$

5.3 a) (optional) $A = \begin{bmatrix} -2 & 11 \\ -10 & 5 \end{bmatrix}$

in python: $A = U \Sigma V^T = \begin{bmatrix} -0.707 & -0.707 \\ -0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 14.142 & 0 \\ 0 & 7.071 \end{bmatrix} \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}$

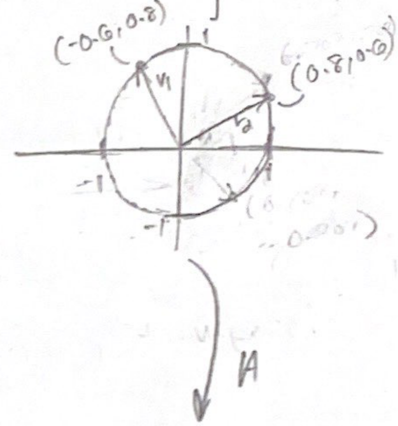
with least # negatives:

$$A = U \Sigma V^T = \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix} \begin{bmatrix} 14.142 & 0 \\ 0 & 7.071 \end{bmatrix} \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

b) Singular values. $\sigma_1 = 14.142$, $\sigma_2 = 7.071$

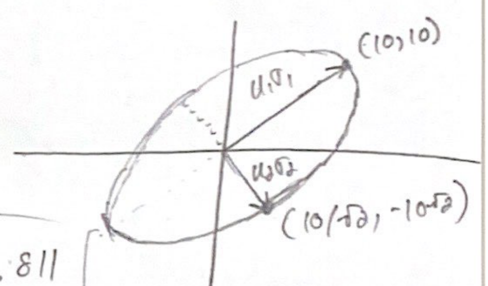
Left singular vectors: $u_1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$, $u_2 \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix}$

Right singular vectors: $v_1 = \begin{bmatrix} -0.6 \\ 0.8 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$



c) Norms of A

$$\|A\|_1 = \max_{1 \le j \le n} \|a_j\|_1 = \|a_2\| = |11| + |5| = \boxed{16}$$



$$\|A\|_2 = \max\{\sigma\} = \sigma_1 = \boxed{14.142}$$

$$\|A\|_F = \sqrt{\sigma_1^2 + \dots + \sigma_n^2} = \sqrt{(14.142)^2 + (7.071)^2} = \boxed{15.811}$$

$$\|A\|_\infty = \max_{1 \le i \le n} \|a_i^*\|_1 = \|a_2\| = |-10| + |5| = \boxed{15}$$

d) Find A^{-1}, indirectly via the SVD

$\underbrace{A^{-1}A}_{I} = A^{-1} U \Sigma V^T$

$V = A^{-1} U \Sigma$

$V \Sigma^{-1} = A^{-1} U$

$V \Sigma^{-1} U^T = A^{-1}$

$$A^{-1} = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 0.0707 & 0 \\ 0 & 0.1414 \end{bmatrix} \begin{bmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{bmatrix}$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} 0.05 & -0.11 \\ 0.1 & -0.02 \end{bmatrix}}$$

$\Rightarrow$

e) ~~freq~~ find λ_1, λ_2 of A (done in python)

$\lambda_1 = 1.5 + 9.887i$

$\lambda_2 = 1.5 - 9.887i$

f) $\det A = \lambda_1 \lambda_2$?

$\det(A) = -2(5) + (11)(+10) = 100$ ✓

$\lambda_1\lambda_2 = (1.5 + 9.887i)(1.5 - 9.887i) = 1.5^2 - (9.887)^2 i^2 = 2.25 + 97.7500..$

$= 100$ ✓

$|\det A| = \sigma_1\sigma_2$?

$|\det A| = |100| = 100$ ✓

$\sigma_1\sigma_2 = (14.142)(7.071) = 100$ ✓ ← used more decimals in the calculation

g) The area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$ is equal to πab, where a and b are the axes of the ellipsoid, so $\pi ab = \pi\sigma_1\sigma_2 = \boxed{100\pi}$.

3. Projections

6.2 $E \in \mathbb{R}^{m \times m}$, $Ex = (x + Fx)/2$, $F \in \mathbb{R}^{m \times m}$ s.t. $F(x) \to (x_m, \ldots, x_1)^*$

$(x_1, \ldots, x_m)$

Q: what type of projector is E and what are its entries?

· is E a projector? $(E^2 = E?)$

$Ex = (I + F)x/2 \quad \Rightarrow E = I + F$

$E^2 = (I+F)(I+F) = I^2 + IF + FI + F^2 = I + 2F + F^2$

$\Rightarrow E^2 x = (x + 2Fx + \underbrace{FFx}_{x})/2$ $\quad$ * $FFx = Fx^* = x$!

$= (2x + 2Fx)/2$

$\Rightarrow E^2 x = (x + Fx)/2$ ✓ $\quad E^2 = E$!

E is a projector. Now, what type of projector is it?

$Ex \rightarrow \left(\frac{x_1+x_m}{2}, \frac{x_2+x_{m-1}}{2}, \ldots, \frac{x_m+x_1}{2} \right)$

$E = \frac{(I+F)}{2}$, F flips entries $\longrightarrow$ $F = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}_m$

$\Rightarrow$ E = An "x" of $\frac{1}{2}$s along both diagonals,

$\begin{bmatrix} \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \vdots & \frac{1}{2} & \cdots & \frac{1}{2} & \\ & \frac{1}{2} & \cdots & \frac{1}{2} & \\ \frac{1}{2} & & & & \frac{1}{2} \end{bmatrix}$. If m is odd, the middle entry, $m+\left(\frac{m}{2}\right)+1$, will be equal to 1.

$\hookrightarrow$ entries of E

Does $E = E^*$? Yes.

$E = I + F$

$E^* = I^* + F^* = I + ? \longrightarrow F = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix} \Rightarrow F^* = F!$

$= I + F \Rightarrow E^* = E$

Check $\Rightarrow$ $\boxed{E \text{ is an orthogonal projector}}$

~~right, think it's orthogonal.~~

4. Oblique Projection

- $P \in \mathbb{R}^{m \times m}$ is an oblique projection to some subspace of $\mathbb{R}^m$
- A given, full rank, $Range(P) = Range(A)$.
- B full rank, ~~the~~ $range(B) \perp Null(P)$

a) P projects to $\mathbb{R}^n$; $Range(P) = Range(A)$

$$\Updownarrow$$

$$\dim(Range(P)) = \dim(Range(A)) = m$$

and because A is full rank,

$\Rightarrow \boxed{A \in \mathbb{R}^{m \times n}}$ because of ① so $rank(A) = col(A) = n$.

$range(B) \perp null(P)$

$$\Downarrow$$

$$\dim(range(B)) + \dim(null(B)) = m$$

$$\dim(range(B)) + (m-n) = m$$

$$\dim(range(B)) = n \longrightarrow \boxed{B \in \mathbb{R}^{m \times n}}$$

b) Show $P = A(B^TA)^{-1}B^T$ ... A and B are projection operators...

$\rightarrow$ going to assume

- A spans the subspace P projects onto

① $I - P$ projects onto $null(P)$, ② $range(B) \perp null(P)$.

① $(I-P)v = 0$

② $B^T(I-P)v = 0$ (inner product $= 0$ b/c $\perp$).

$B^Tv = B^TPv = B^TAy$ $\quad Range(P) = Range(A) \longrightarrow Pv = Ay$ for some $y \in \mathbb{R}^n$, b/c $A^{m \times n}$, $v \in \mathbb{R}^m$

$$(B^TA)^{-1}B^Tv = y$$

$$A(B^TA)^{-1}B^Tv = Ay$$

$$A(B^TA)^{-1}B^Tv = Pv \implies \boxed{P = A(B^TA)^{-1}B^T}$$