CHAPTER TWENTY-SEVEN

Fixed-Income Securities

Neither a borrower nor a lender be.
Shakespeare (1564–1616), *Hamlet*

Bonds are issued for the purpose of raising funds. This chapter concentrates on bonds, particularly those with embedded options. It ends with a discussion of key rate durations.

27.1 Introduction

A bond can be secured or unsecured. A **secured** issue is one for which the issuer pledges specific assets that may be used to pay bondholders if the firm defaults on its payments. Many bond issues are **unsecured**, however, with no specific assets acting as collateral. Long-term unsecured issues are called **debentures**, whereas short-term unsecured issues such as **commercial paper** are referred to as **notes**.

It is common for a bond issue to include in the indenture provisions that give either the bondholder and/or the issuer an option to take certain actions against the other party. The bond **indenture** is the master loan agreement between the issuer and the investor. A common type of embedded option in a bond is a call feature, which grants the issuer the right to retire the debt, fully or partially, before the maturity date. An issue with a put provision, as another example, grants the bondholder the right to sell the issue back to the issuer. Here the advantage to the investor is that if interest rates rise after the issue date, reducing the bond's price, the investor can force the issuer to redeem the bond at, say, par value. A convertible bond (CB) is an issue giving the bondholder the right to exchange the bond for a specified number of shares of common stock. Such a feature allows the bondholder to take advantage of favorable movements in the price of the issuer's stock. An **exchangeable bond** allows the bondholder to exchange the issue for a specified number of common stock shares of a corporation different from the issuer of the bond. Some bonds are issued with warrants attached as part of the offer. A warrant grants the holder the right to purchase a designated security at a specified price.

27.2 Treasury, Agency, and Municipal Bonds

Strictly speaking, only default-free bonds without options deserve the name "fixed-income security." In the U.S. market, almost all fixed-income securities in this narrow

Outstanding U.S. Treasury securities (U.S. $ billions)

1980	616.4	**1985**	1,360.2	**1990**	2,195.8	**1995**	3,307.2
1981	683.2	**1986**	1,564.3	**1991**	2,471.6	**1996**	3,459.7
1982	824.4	**1987**	1,675.0	**1992**	2,754.1	**1997**	3,456.8
1983	1,024.4	**1988**	1,821.3	**1993**	2,989.5	**1998**	3,355.5
1984	1,176.6	**1989**	1,945.4	**1994**	3,126.0	**1999**	3,281.0

Figure 27.1: Outstanding U.S. Treasury securities 1980–1999. Prices are quoted in 1/32 of a percent. Source: *U.S. Treasury*.

sense are issued by the Treasury. Nevertheless, in reality the term fixed-income security is used rather loosely and has come to describe even bonds with uncertain payments. Figure 27.1 tabulates the U.S. Treasury securities in terms of outstanding volume, and Figure 27.2 provides a view of the immense U.S. Treasuries market.

In early 1996 the Treasury announced plans to issue bonds whose nominal payments are indexed to inflation so that their payments are fixed in real terms [147]. The index for measuring the inflation rate is the nonseasonally adjusted U.S. City Average All Items Consumer Price Index for All Urban Consumers (CPI-U) published monthly by the Bureau of Labor Statistics. When the bond matures, the principal will be adjusted to reflect all the inflation there is during the life of the bond. (The British government issued index-linked securities in 1981 [135].) On January 29, 1997, U.S.$7 billion of 3 3/8% 10-year inflation-indexed Treasury notes were auctioned (see Fig. 27.3) [759].[1] The interest rate set at auction will remain fixed throughout the term of the security. Semiannual interest payments will be based on the inflation-adjusted principal at the time the interest is paid. At maturity, the securities will be redeemed at the greater of their inflation-adjusted principal or their par amount at original issue.

Federal agency debt can be issued by Federal agencies, which are direct arms of the U.S. government, or various **government-sponsored enterprises** (**GSE**s), which were created by Congress to fund loans to such borrowers as homeowners, farmers, and students [395]. GSEs are privately owned, publicly chartered entities that raise funds in the marketplace. Examples include Federal Home Loan Banks (FHLBanks), the Federal National Mortgage Association (FNMA or "Fannie Mae"), the Federal Home Loan Mortgage Corporation (FHLMC or "Freddie Mac"), and the Student Loan Marketing Association (SLMA or "Sallie Mae"). Although there are no Federal guarantees on the securities issued by the GSEs, the perception that the government would ultimately cover any defaults causes the yields on these securities to be below those on most corporate securities. This may change in the future, however [404].

Municipal bonds are fixed-income securities issuedby state, state authorities, or local governments to finance capital improvements or support a government's general financing needs. There are two major categories of municipal bonds: revenue bonds and general obligation bonds. **Revenue bonds** are issued to raise funds for a particular project such as a toll road or a hospital that is projected to generate enough income to pay principal and interest to bondholders. **General obligation bonds** are backed by the taxing power of the issuer such as city, county, or state. They are often considered less risky than revenue bonds. Investors are attracted to municipal bonds because the interest is exempt from federal income taxes and, in some cases, state

TREASURY BONDS, NOTES & BILLS
Monday, March 20, 1995

GOVT. BONDS & NOTES

Rate	*Maturity Mo/Yr*	*Bid*	*Asked*	*Chg.*	*Ask Yld.*
3 7/8	Mar 95n	99:29	99:31	−1	5.05
8 3/8	Apr 95n	100:03	100:05	−1	5.79
3 7/8	Apr 95n	99:24	99:26	.....	5.54
5 7/8	May 95n	99:31	100:01	.....	5.55
8 1/2	May 95n	100:12	100:14	.....	5.38
10 3/8	May 95	100:20	100:22	.....	5.53
		...			
7 1/8	Feb 23	95:26	95:28	−11	7.48
6 1/4	Aug 23	85:21	85:23	−12	7.47
7 1/2	Nov 24	100:14	100:16	−15	7.46
7 5/8	Feb 25	102:23	102:25	−10	7.39

U.S. TREASURY STRIPS

Mat.	*Type*	*Bid*	*Asked*	*Chg.*	*Ask Yld.*
May 95	ci	99:04	99:04	+1	5.84
May 95	np	99:04	99:04	+1	5.95
Aug 95	ci	97:22	97:23	.....	5.82
Aug 95	np	97:19	97:19	.....	6.10
		...			
Nov 24	ci	11:17	11:20	−4	7.39
Nov 24	bp	11:21	11:24	−4	7.35
Feb 25	ci	11:25	11:29	−3	7.25
Feb 25	bp	12:04	12:08	−3	7.15

TREASURY BILLS

Maturity	*Days to Mat.*	*Bid*	*Asked*	*Chg.*	*Ask Yld.*
Mar 23 '95	1	5.43	5.33	+0.42	5.40
Mar 30 '95	8	4.96	4.86	−0.04	4.93
Apr 06 '95	15	5.61	5.51	+0.04	5.60
Apr 13 '95	22	5.50	5.40	+0.01	5.49
		...			
Dec 14 '95	267	5.93	5.91	−0.01	6.21
Jan 11 '96	295	5.95	5.93	−0.01	6.25
Feb 08 '96	323	5.96	5.94	−0.01	6.28
Mar 07 '96	351	5.98	5.96	.....	6.34

Figure 27.2: Treasuries quotations. Colons represent 32nds. The final column for T-bills shows the annualized BEYs as computed by Eq. (3.10). T-bill quotes are in hundredths and are on a discount basis. All yields are based on the asked quote. n, Treasury note; ci, stripped coupon interest; bp, T-bond, stripped principal; tt, T-note, stripped principal. Source: *Wall Street Journal*, March 21, 1995.

and local taxes as well. Municipal bonds are quoted in percent of par and 1/32 of a percent like T-bonds.

27.3 Corporate Bonds

Both stock and bond markets offer an efficient way for corporations to raise capital. Bonds have the advantage of not diluting the stockholder's equity. Compared with

TREASURY BONDS, NOTES & BILLS
Thursday, January 7, 1999

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INFLATION-INDEXED TREASURY SECURITIES

Rate	*Mat.*	*Bid/Asked*	*Chg.*	**Yld.*	*Accr. Prin.*
3.625	07/02	99-15/16	−01	3.768	1024
3.375	01/07	96-23/24	−02	3.850	1035
3.625	01/08	98-11/12	−05	3.831	1015
3.625	04/28	98-03/04	+17	3.734	1014

*-Yld. to maturity on accrued principal.

Figure 27.3: Inflation-indexed Treasuries quotations. Source: *Wall Street Journal*, January 8, 1999.

bank loans, bonds often allow corporations to borrow at a lower interest rate than the rates available from their banks. With bonds, a corporation also borrows money at a fixed rate for a longer term than it could at a bank because most banks do not make long-term fixed-rate loans.

Corporate bonds are quoted in points and eighths of a point. A bond with $10,000 par value quoted at 95 6/8, for example, has a price of $9575. Because each bond must be customized to reflect the concerns of both the issuer and the investors, corporate bonds are not standardized. Bondholders, by making loans to the issuer, are legally the issuer's creditors, not owners like stockholders.

27.3.1 Callable and Putable Bonds

The holder of a callable bond sells the issuer an option to purchase the bond from the time it is first callable until the maturity date. The position of the bondholder is therefore

$$\text{long a callable bond} = \text{long a noncallable bond} + \text{sold a call option.}$$

In terms of price, we have

$$\text{callable bond price} = \text{noncallable bond price} - \text{call option price.} \tag{27.1}$$

The issuer may be entitled to call the bond at the first call date and any time thereafter (**continuously callable**) or at the first call date and all subsequent coupon payment dates (**discretely callable**). The call price may also vary over time. Typically, there is an initial call protection period, after which the bond is callable with a call price that declines to par over its remaining life [329]. Take for example a bond with 20 years to maturity and callable in 5 years at 106. The bondholder is essentially long a hypothetical 20-year noncallable bond and short a call option granting the issuer the right to call the bond 5 years from now for a price of 106.

The issuer will call a bond when the bond yield in the market for a new issue net of the underwriting fees and tax is lower than the current issue's coupon rate. Whether to exercise the option hinges on the future bond payments if the bond is not called. The price of the callable bond when it is callable will remain near its call

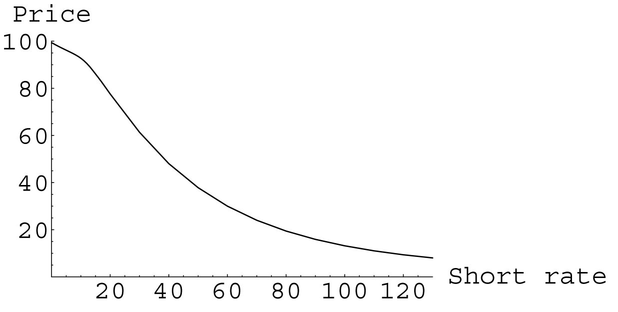

Figure 27.4: Price compression of callable bonds. The five-year 10% bond callable at par is priced by the CIR model.

price when interest rates are low, a phenomenon known as **price compression**. This is because the bond is likely to be called (see Fig. 27.4).

The holder of a **putable bond** has the right to sell the bond to the issuer at a designated price and time [238, 536]. The position of a putable bondholder can be described as

$$\text{long a putable bond} = \text{long a nonputable bond} + \text{long a put option}.$$

The price of a putable bond thus is

$$\text{putable bond price} = \text{nonputable bond price} + \text{put option price}.$$

A bond may carry both call and put options. If these options are exercisable at par on the same date, the bond is usually called an **extendible bond** [371].

27.3.2 Bonds with Sinking-Fund Provisions

The scheduled principal payments of a bond with sinking-fund provisions are spread out over many years. Two methods can be used to retire the required principal amount. Either the issuer can purchase the required amount in the open market and deliver the bonds to the trustee or it can call the required amount at par by random selection. Which method to execute depends on the prevailing interest rates. If rates are high, the issuer will choose to satisfy its sinking-fund requirement by market purchases. If rates are low, however, par calls will be chosen. When a sinking fund and a call option are both included in a bond issue, there are situations in which it is optimal to call only part of an issue. Almost all sinking-fund bond issues contain call options [848].

➤ **Exercise 27.3.1** Sketch a method to price a callable bond with sinking-fund provisions.

27.3.3 Convertible Bonds

CBs grant the bondholder the right to acquire the stock of the issuing corporation under specific conditions. The CB contract will state either a conversion ratio or a **conversion price**. A conversion ratio, we recall, specifies the number of shares to be obtained through conversion. The ratio is always adjusted proportionately for stock splits and stock dividends. Alternatively, the conversion ratio may be expressed in terms of a conversion price defined as

$$\text{conversion price} \equiv \frac{\text{par value of CB}}{\text{conversion ratio}}.$$

This price represents the cost per share through conversion. The conversion privilege may extend for all or only some portion of the bond's life. There are typically other embedded options in a CB, the most common being the right of the issuer to call or put the issue [328]. Conditions are usually imposed on the exercise of the options; for example, the stock price must be trading at a certain premium to the conversion price for the CB to be called. The contract may contain a **refix clause** in that the conversion price is set to the stock price if it is lower than the conversion price on the refix day [221].

The conversion value, we recall, is the value of the CB if it is converted immediately,

$$\text{conversion value} = \text{market price of stock} \times \text{conversion ratio}. \tag{27.2}$$

It is also called the **parity** in the market (but see Exercise 27.3.3). The market price of a CB must be at least its conversion value and **straight value** – the bond value without the conversion option. The price that an investor effectively pays for the stock if the CB is purchased and then converted is called the **market conversion price**:

$$\text{market conversion price} \equiv \frac{\text{market price of CB}}{\text{conversion ratio}}.$$

As the market conversion price cannot be lower than the market price of the stock, the bondholder pays a premium per share in the amount of

$$\text{market conversion price} - \text{market price of stock}.$$

The premium is usually expressed as a percentage of the market price of stock.

EXAMPLE 27.3.1 A CB with a par value of \$10,000 and a conversion price of \$80 would imply a conversion ratio of $10000/80 = 125$. Given the CB's quoted price, 103, the purchase price is $1.03 \times 10000 = 10300$ dollars. At the current stock price of \$78, the premium per share is $10300/125 - 78 = 4.4$ dollars.

CBs exhibit the characteristics of both bond and stock. If the stock price is so low that the straight value is much higher than the conversion value, the CB behaves as a fixed-rate bond. On the other hand, if the stock price is so high that the conversion value is much higher than the straight value, the CB trades as an equity instrument. Between these two extremes, the CB trades as a hybrid security. Both the straight value and the parity act as a floor for the CB price, giving it a call-like behavior (see Fig. 27.5 for illustration).

It is simpler to price a CB on a *per-share basis*, that is, the hypothetical CB that can be converted for *one share*. The actual price, of course, equals the per-share price times the conversion ratio. As a reasonable first approximation, assume constant interest rates [847]. Then binomial tree algorithms for American options such as the one in Fig. 9.17 can be used to price CBs after the payoff function is modified. At maturity, the choice is between the conversion price plus coupon and the stock, whereas, before maturity, the choice is between the CB and the stock. See Fig. 27.6 for

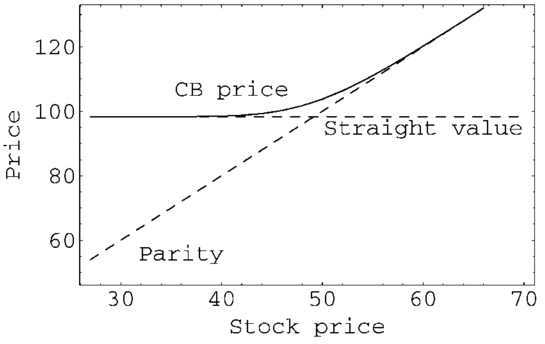

Figure 27.5: CB price vs. stock price. Both the straight value and the parity provide a floor. This particular CB has a conversion ratio of two and a conversion price of \$50. The plot is similar to the one in Fig. 7.3 for calls on a stock.

an algorithm. Sensitivity measures can also be computed (review Chap. 10). Many of equity options' properties continue to hold for CBs. For example, higher stock price volatilities increase the CB's value.

➤ **Exercise 27.3.2** Use an arbitrage argument to show that a CB must trade for at least its conversion value.

➤ **Exercise 27.3.3** Can you find one fault with formula (27.2)?

➤ **Exercise 27.3.4** Prove that, much as with American calls, it never pays to convert the bond when the stock does not pay cash dividends and the interest rate remains constant.

Binomial tree algorithm for pricing convertible bonds on a stock that pays a known dividend yield:

input: $S, P, \sigma, t, n, \delta\, (1 > \delta > 0), m, r, c$;
real $R, p, u, d, C[\,n+1\,], v$;
integer i, j;
$R := e^{r(t/n)}$;
$u := e^{\sigma\sqrt{t/n}}$; $d := e^{-\sigma\sqrt{t/n}}$;
$p := (R-d)/(u-d) =$;
for $(i = 0$ to $n)$ { $C[\,i\,] := \max\left(Su^{n-i}d^{i}(1-\delta)^{m},\ P+c\right)$; }
for $(j = n-1$ down to $0)$
    for $(i = 0$ to $j)$ {
        if [the period $(j, j+1\,]$ contains an ex-dividend date] $m := m-1$;
        $v := (p \times C[\,i\,] + (1-p) \times C[\,i+1\,])/R$;   // Backward induction.
        if [the period $[\,j, j+1\,)$ contains a coupon payment date] $v := v + c$;
        $C[\,i\,] := \max\left(Su^{j-i}d^{i}(1-\delta)^{m},\ v\right)$;
    }
return $C[\,0\,]$;   // On a per-share basis.

Figure 27.6: Binomial tree algorithm for CBs on a stock paying a dividend yield. S is the current stock price, m stores the total number of ex-dividend dates at or before expiration that occurs t years from now, δ is the dividend yield for each cash dividend, P is the conversion price, and c is the coupon payment per share. Note that the conversion price is already on a per-share basis. The partition should be fine enough that coupon payment dates and ex-dividend dates are separated by at least one period.

➤ **Exercise 27.3.5** Argue that under the binomial CB pricing model of Fig. 27.6, the CB price converges to the stock price as the stock price increases.

➤ **Exercise 27.3.6** Like warrants, CBs can be converted into newly issued shares; they are in fact equivalent under certain assumptions (see Exercise 11.1.10, part (2)). On a per-share basis, the conversion price plus the final coupon payment acts very much like the strike price. But unlike with warrants, exercising the conversion option does not require paying the "strike price." Derive the pricing relation between European warrants and CBs with a European-style conversion option. Assume that the issuer pays no dividends and ignore the dilution issue.

➢ **Programming Assignment 27.3.7** Implement the algorithm in Fig. 27.6.

27.3.4 Notes

A floating-rate note has coupon payments pegged to the yield of a particular interest rate such as LIBOR or the yield on a Treasury security. The interest rate that the borrower pays is reset periodically. For example, the rate might be reset every 6 months to the current T-bill rate plus 100 basis points. Variations on floating-rate notes include call features, issued by the firm, and conversion features that allow the investor to transfer to a fixed-rate note. In addition, the coupon rate may have a floor or a cap. Put features, whereby the holder can redeem the investment at par at particular coupon payment dates or after some predetermined date, may also be present. Of course, if there is no spread, a floating-rate note will sell at par on reset dates.

Structured notes are securities in which the issuer sells a note and simultaneously enters into a swap or derivative transaction to eliminate its exposure to the customized terms of the note structure [237]. Each structured note is customized with unique features that match the preferences of the investor. Consider an investor who believes that the yield curve will flatten. A security designed to reflect that view could have its coupon payments linked to the shape of the yield curve. Thus the coupon on the note might be reset semiannually to a rate that depends on the yield spread between the 30-year and the 2-year Treasury yields.

27.4 Valuation Methodologies

Several valuation methodologies were mentioned for corporate bonds before: yield to worst, yield to call, yield to par call, yield spread, and static spread. This section covers additional methodologies.

27.4.1 The Static Cash Flow Yield Methodology

In the **static cash flow yield methodology**, the yield to maturity of a bond is compared with that of the on-the-run Treasury security with a similar maturity. The difference is the yield spread. Because the yield must make certain assumptions about the future cash flow, it is also called the (**static**) **cash flow yield**. There are two obvious problems with this methodology. First, the yields fail to account for the term structure of interest rates. Second, interest rate volatility may alter the cash flow of bonds with embedded options.

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Exercise 27.4.1 Assume any stochastic discrete-time short rate model. Consider a risky corporate zero that is not currently in default. When the firm defaults, it stays in the default state until the maturity when the investor receives zero dollar. Let p_i denote the risk-neutral probability that the bond defaults at time i given that it has not defaulted earlier; the p_is depend on only the time, not the short rate. Prove that

$$\frac{\text{price of } n\text{-period corporate zero}}{\text{price of } n\text{-period Treasury zero}} = \prod_{i=1}^{n} p_i .$$

(This result generalizes Eq. (5.6) in the static world. The algorithm in Fig. 5.9 can be used to retrieve the p_is.)

27.4.2 The Option Pricing Methodology

The value of an embedded option is the price difference between the bond with the option feature and an otherwise identical bond without the option. This insight leads the option pricing methodology to decompose fixed-income securities with option features into an option and an option-free component. The callable bond is a quintessential example, as demonstrated in Eq. (27.1). For instance, consider a coupon bond with a maturity date of June 2015 and callable in June 2012 at 102. Its bondholder as of April 1, 2000, effectively owns a noncallable bond with 15 years and 2 months to maturity and is short a call option, granting the issuer the right to call away 3 years of cash flows beginning June 1, 2012, for a strike price of 102.

The **option pricing methodology** applies the option pricing framework such as the Black–Scholes model in Section 24.7 to estimate the option price. The binomial tree algorithm for American options in Fig. 9.13 can be modified to price the embedded call option. After the option is priced, the **implied noncallable bond price** is then calculated as the sum of the callable bond price and the call option price. Finally, the **option-adjusted yield** is calculated as the yield that makes the PV of the cash flow of the hypothetical noncallable bond equal the implied price. The callable bond is said to be priced fairly if its option-adjusted yield equals the yield for an "equivalent" noncallable bond. It is said to be rich (overvalued) if the option-adjusted yield is lower, and cheap (undervalued) if the option-adjusted yield is higher [325, 330].

This methodology suffers from several difficulties. The Black–Scholes model is not satisfactory for pricing fixed-income securities as argued in Section 24.7. And there may not exist a benchmark with which to compare the option-adjusted yield to get the yield spread. Finally, this methodology does not incorporate the shape of the yield curve, which affects the value of all interest-rate-sensitive securities [329].

Exercise 27.4.2 Argue that when interest rates rise, the price of a callable bond will not fall as much as the price of its noncallable component.

Exercise 27.4.3 For bonds with embedded options, traditional duration measures such as modified duration lose relevance because of cash flow uncertainties. The duration of a callable bond after the call option is adjusted for is commonly referred to as the **option-adjusted duration** (**OAD**) and is defined as $\text{OAD}_{\mathsf{c}} \equiv -\frac{\partial \text{price}_{\mathsf{c}}}{\partial y} / \text{price}_{\mathsf{c}}$. (The subscript "c" refers to callable measures.) The convexity of a callable bond after the call option is adjusted for is commonly referred to as the **option-adjusted convexity** (**OAC**) and is defined as $\text{OAC}_{\mathsf{c}} \equiv -\frac{\partial^2 \text{price}_{\mathsf{c}}}{\partial y^2} / \text{price}_{\mathsf{c}}$. (1) Prove that

$\text{OAD}_c = (\text{price}_{nc}/\text{price}_c) \times \text{duration}_{nc} \times (1-\Delta)$, where $\Delta \equiv \partial(\text{call price})/\partial\text{price}_{nc}$ is the delta of the embedded call option. (The subscript "nc" refers to noncallable measures.) (2) Show that

$$\text{OAC}_c = \frac{\text{price}_{nc}}{\text{price}_c} \times [\,\text{convexity}_{nc} \times (1-\Delta) - \text{price}_{nc} \times \Gamma \times (\text{duration}_{nc})^2\,],$$

where $\Gamma \equiv \partial^2(\text{call price})/(\partial\text{price}_{nc})^2$ is the gamma of the embedded call option.

27.4.3 The Option-Adjusted Spread Methodology

The final methodology is the **option-adjusted spread** (**OAS**) [34, 323]. This popular approach takes into account the embedded option features of fixed-income securities and tackles the difficulties faced by the previous methods. Unlike the previous methods, OAS analysis does not attempt to predict a bond's redemption date. The binomial interest rate tree of Chap. 23 is used to illustrate the main ideas. Generalization to other models is straightforward. Specifically, we use the calibrated binomial interest rate tree in Fig. 23.8 and callable bonds as the basis for our numerical calculations. Recall that a spread is defined as the incremental return applied to every short rate on the tree. It measures the extent to which the bond's rate of return exceeds riskless returns. The OAS generalizes the spread concept to bonds with uncertain cash flows.

Given an OAS, the procedure to calculate the model price is essentially the same as the one for option-free bonds except that, at each node, the exercise of the call must be considered. If the PV of *future* cash flows as determined by backward induction at any node exceeds the call price, the call will be exercised and the lower call price becomes the bond value. This usually happens when interest rates are low.

Consider a 3-year callable bond with a market price of 99.696 and an annual coupon rate of 5%. The call provision is a discrete par call that may be exercised on any coupon payment date. When the call provision is exercised, it eliminates all coupon payments scheduled to take place after the call date, and the par value plus the currently due coupon is paid at that time. We want to derive the OAS associated with the 99.696 observed price of the bond. In Fig. 27.7, the OAS is verified to be 50 basis points over the short rates. The same bond without the call option would have fetched 100.569 as shown in Fig. 23.14 under the same OAS. The call option depresses the bond value, as expected.

An algorithm for finding the OAS of callable bonds is given in Fig. 27.8. Besides callable bonds, OAS analysis can be extended to other corporate bond structures. The general procedure is summarized below.

1. Estimate the spot rate curve.
2. Calibrate the interest rate model.
3. Develop rules for exercising the embedded options.
4. Add the OAS to the short rates on the tree.
5. Compute the model price.
6. Iterate 4 and 5 by varying the OAS until the model price matches the market price.

There are many choices for the root-finding algorithms. The simple bisection method cannot fail. The Newton–Raphson method, albeit faster, may not apply at certain OASs because of nondifferentiability. See [727] and Programming Assignments 27.4.9 and 27.4.10 for additional choices.

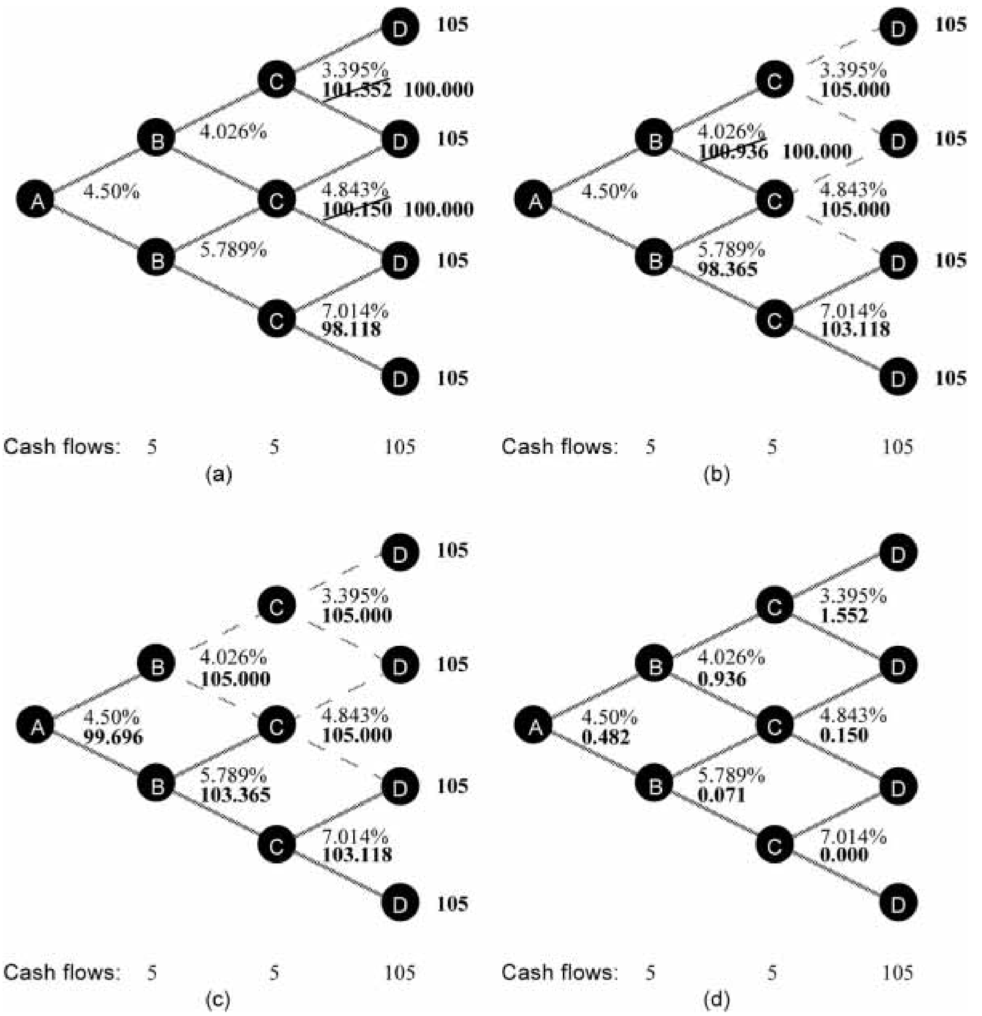

Figure 27.7: Calculation of the OAS of callable bonds. The price trees are based on a 3-year callable bond paying an annual coupon rate of 5%. The call provision is a discrete par (100) call that may be exercised on any coupon payment date. The underlying binomial interest rate tree is produced from Fig. 23.8 by the addition of a constant spread of 50 basis points to each short rate. The dotted lines signal early exercise. (a) PVs at C nodes are computed. Prices over 100 are replaced with 100 because of the exercise of the option. (b) PVs at B nodes are computed after the addition of 5 to prices at C nodes. Again, prices over 100 are replaced with 100. (c) Repeat the steps in (b) for node A. The price at each node has the PV of the *remaining* cash flows. Because the model price matches the market price, 99.696, 0.5% is the OAS. (d) Calculate the option value (this part is incorrect; see text).

➤ **Exercise 27.4.4** Argue that the OAS does not assume parallel shifts in the term structure.

➤ **Exercise 27.4.5** Explain why the OAS of a callable bond decreases as the interest rate volatility increases, other things being equal.

➤ **Exercise 27.4.6** For a putable bond, how does its OAS behave (1) when the market price decreases, other things being equal, and (2) when the coupon rate decreases, other things being equal and with the market price at par?

Algorithm for computing the OAS of callable bonds:

```
input:   P, n, cp[n], r[1..n], C[0..n], v[1..n], ε;
real     P[1..n+1], s;
integer  i, j;
s := 0; // Initial guess.
P[1] := ∞;
while [ | P[1] − P | > ε ] {
        for (i = 1 to n+1) { P[i] := C[n]; }   // Initialization.
        for (i = n down to 1)   // Sweep the column backward in time.
            for (j = 1 to i)   // Backward induction.
                 P[j] := C[i−1] + min( cp[i−1], (P[j]+P[j+1]) / (2×(1+r[i]×v[i]^(j−1)+s)) );
        Update s;
}
return s;
```

$$P[\,j\,] := C[\,i-1\,] + \min\left(\mathrm{cp}[\,i-1\,],\ \frac{P[\,j\,]+P[\,j+1\,]}{2\times\left(1+r[\,i\,]\times v[\,i\,]^{j-1}+s\right)}\right);$$

Figure 27.8: Algorithm for computing the OAS of callable bonds. P is the market price, $\mathrm{cp}[\,i\,]$ is the call price i periods from now ($\mathrm{cp}[\,i\,] = \infty$ if the bond is not callable then), $r[\,i\,]$ is the baseline rate for period i, $C[\,i\,]$ contains the cash flow occurring exactly i periods from now (equivalently, end of the $(i-1)$th period), $v[\,i\,]$ is the multiplicative ratio for the rates in period j, and n is number of periods. All numbers are measured by the period. Note that the coupon payment on a call date will be paid whether or not the bond is called. For putable bonds, (1) replace min() with max(), (2) replace the call prices in cp[ ] with the put prices, and (3) make $\mathrm{cp}[i] = -\infty$ if the bond is not putable then.

➤ **Exercise 27.4.7** Argue that using Monte Carlo simulation to price callable (putable) bonds tends to underestimate (overestimate, respectively) their values.

➢ **Programming Assignment 27.4.8** Implement the algorithm in Fig. 27.8 with the differential tree method, which is based on the Newton–Raphson method.

➢ **Programming Assignment 27.4.9** Implement the algorithm in Fig. 27.8 with the secant method.

➢ **Programming Assignment 27.4.10** Implement the algorithm in Fig. 27.8 with the Ridders method.

Valuing the Embedded Option and Option-Adjusted Yield

In contrast with the explicit valuation of the embedded call option in the option pricing methodology, option price is a by-product of OAS analysis. It is calculated in the following way. First, the implied noncallable bond price is the PV of the bond afte the OAS is added to the short rates on the tree. Then the value of the embedded call option is determined by subtraction of the callable bond's market price from the implied noncallable bond price.

Refer to Fig. 27.7(d) for the calculation of the embedded option's value below. At C nodes the intrinsic value of the call option is the greater of zero and the price of the underlying bond minus the call price. For instance, the top C node's intrinsic value is

$$\max(0, 101.552 - 100) = 1.552.$$

Similarly, the third C node's intrinsic value is zero because the option is not exercised. At B nodes, we find the PV of the option values at C nodes and take as the option value the greater of the intrinsic value at B and the PV of the option value at the successor C nodes. At the top B node, for instance, the intrinsic value is 0.936 as it is exercised. The PV of the option values at the two successor C nodes is

$$\frac{1.552 + 0.150}{2 \times 1.04026} = 0.818.$$

The option value is therefore 0.936. Similarly, the option value at the bottom B node is

$$\frac{0.150}{2 \times 1.05789} = 0.071.$$

The option value is finally

$$\frac{0.936 + 0.071}{2 \times 1.045} = 0.482.$$

The implied price of the underlying fixed-rate bond is hence $99.696 + 0.482 = 100.178$. However, this cannot be right. Should not the price be 100.569, the price of the same bond without the call option as calculated in Fig. 23.14 under the same OAS of 50 basis points? And by Eq. (27.1), should not the embedded call's value be $100.569 - 99.696 = 0.873$ instead of 0.482? Indeed, the way 0.482 was calculated assumed an underlying bond that is callable. Hence 100.178 is the price of a bond that is *not* option free as desired.

Recall that the option-adjusted yield is the interest rate that makes the PV of the cash flows for the bond equal the implied price of the noncallable bond. Because the hypothetical noncallable bond with an implied price of 100.569 has 3 years to maturity and pays a 5% annual coupon, the option-adjusted yield is 4.792% compounded annually.

➤ **Exercise 27.4.11** Correct Fig. 27.7(d).

Option-Adjusted Spread Duration and Convexity

OAS analysis can be used to assess how prices move as interest rates change. This is done by first changing the short rate by a small amount Δr, say, plus 10 basis points. Next the interest rate tree is revised to reflect the new short rate (for example, Δr is added to every short rate on the Ho–Lee model's interest rate tree). With the OAS held constant, the new model price P_+ is computed. Finally, the market price P_0 and the new model price are used to estimate the effective duration $(P_0 - P_+)/(P_0 \, \Delta r)$ by Eqs. (4.6) [325, 848]. This duration is called the **OAS duration** [55, 330]. The effective convexity is more computation intensive as it requires the model price P_- after the short rate is decremented by a small amount Δr. It equals $(P_+ + P_- - 2 \times P_0)/(P_0(\Delta r)^2)$ by Eq. (4.16). For multifactor interest rate models, the above procedures must be repeated for each factor.

A popular alternative is to add Δr to the whole spot rate curve and then calibrate the interest rate model with the same volatility. The rest of the computation is identical [36, 329, 429]. Specifying a term structure movement outside the interest rate model is certainly not theoretically sound. However, this method is easy to apply; besides, to do otherwise requires a lot of confidence in the model [731].

Holding Period Returns

The HPR assesses the bond over a holding period. The FV at the horizon consists of the projected principal and interest cash flows, the interest on the reinvestment thereof, and the projected horizon price. The monthly total return is then

$$\left(\frac{\text{total future amount}}{\text{price of the bond}}\right)^{1/\text{number of months}} - 1.$$

To calculate the above return, reinvestment rates and interest rate dynamics are all needed. The OAS can be easily combined with the HPR analysis. First we create a few static interest rate scenarios for the holding period. We then calculate the HPR for each scenario by assuming that the OAS remains unchanged at the horizon. This methodology is better than simply comparing the OASs of two securities.

➤ **Exercise 27.4.12** Assume a flat prevailing spot rate curve and continuous compounding. Prove that for an option-free nonbenchmark bond, any calibrated interest rate tree will compute a spread that equals the yield spread. (The yield spread is the difference between the yields to maturity of benchmark and nonbenchmark bonds. The spread is the incremental return over the short rate on the tree in the sense of Subsection 23.3.1.)

27.5 Key Rate Durations

Although duration is essential for identifying interest rate risk exposures, it has many limitations. For example, the assumed yield curve shift is usually not realistic. Also, when it becomes desirable to isolate a security's sensitivity to rate changes in various maturities, a vector of durations is needed instead of just a single number. Key rate durations were proposed by Ho to address these concerns [453] and have proved to be an effective and intuitive tool for risk management [260, 390, 455].

The idea is to break the effective duration into a vector of durations called key rate durations. This decomposition measures sensitivities to different segments of the yield curve so that when the key rate durations are added up, it gives approximately the original effective duration. Securities with identical effective duration can have very different key rate durations. By revealing what segments of the yield curve affect the security value most, key rate durations isolate the risks.

To define key rate durations, we start with a set of 11 key rates: 3 months, and 1, 2, 3, 5, 7, 10, 15, 20, 25, and 30 years. Other choices are also possible. Let $d(i)$ denote the amount of change at the ith key rate. A key rate's effect on other rates decline *linearly*, reaching zero at the adjacent key rates and beyond. The first and the last key rates need to be handled separately, of course. See Fig. 27.9, in which $t(i)$ denotes the ith key rate's term. Each key rate change induces a custom spot rate curve shift called the **basic key rate shift**. The corresponding key rate duration is defined as $(P_0 - P_+)/[\,P_0 d(i)\,]$. The price P_+ is calculated as follows. Use the custom shift defined by the basic key rate shift to derive the new spot rate curve, which is used to calibrate the interest rate model. The security is then priced, with the same OAS if necessary. The resulting 11 durations are the key rate durations.

EXAMPLE 27.5.1 Increase the 10-year key rate by 10 basis points. Because 5 years lie between the 10-year key rate and the next key rate to the right, the 15-year key rate, the effect of the change in the 10-year key rate falls off at a rate of $10/5 = 2$

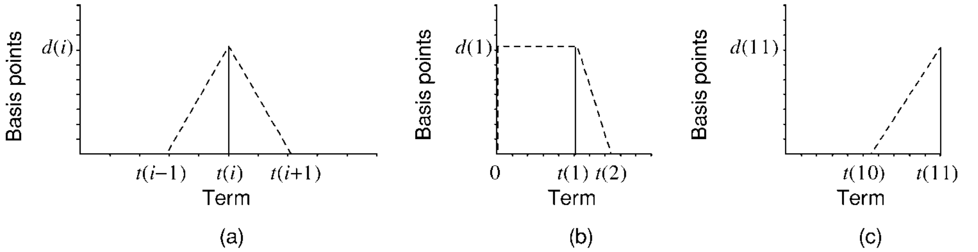

Figure 27.9: Term structure movements used by key rate durations. An increase of $d(i)$ basis points at (a) the ith key rate, $2 \leq i \leq 10$, (b) the first key rate, and (c) the last, eleventh key rate. Note that $t(1) = 0.25$ and $t(11) = 30$. Variations are possible. For example, we can break (b) into two segments such that the one between 0 and $t(1)$ is a mirror image of (c) and the one centered at $t(1)$ is a triangle like (a) [848].

basis points per year. Hence the 11-year rate increases by 8 basis points, the 12-year rate increases by 6 basis points, and so on. The 15-year rate and all the rates beyond 15 years are unchanged. The 10-year key rate move also affects rates of term less than 10 years. Because 3 years lie between the 7-year key rate and the 10-year key rate, the 10-basis-point increase in the 10-year rate falls off by $10/3 = 0.333$ basis points per year. Hence the 9-year rate increases by $10 - 0.333 = 9.667$ basis points, the 8.5-year rate increases by $10 - 1.5 \times 0.333 = 9.5$ basis points, and so on. The 7-year rate and all the rates below it are unchanged.

The sum of key rate durations can approximate the duration with respect to any custom shift to the spot rate curve as follows. Describe a custom shift by the function $s(t), 0 \leq t \leq 30$. Then $s(t)$ is linearly approximated by the sum of the basic key rate shifts defined by $d(i) \equiv s(t(i)), i = 1, 2, \ldots, 11$, at the 11 key rates (see Fig. 27.10 for illustration). Hence the duration with respect to $s(t)$ is expected to be roughly the sum of key rate durations.

One problem with key rate durations is that the basic key rate shifts are independent of the pricing model. Another is that the basic key rate shifts can introduce negative forward rates.

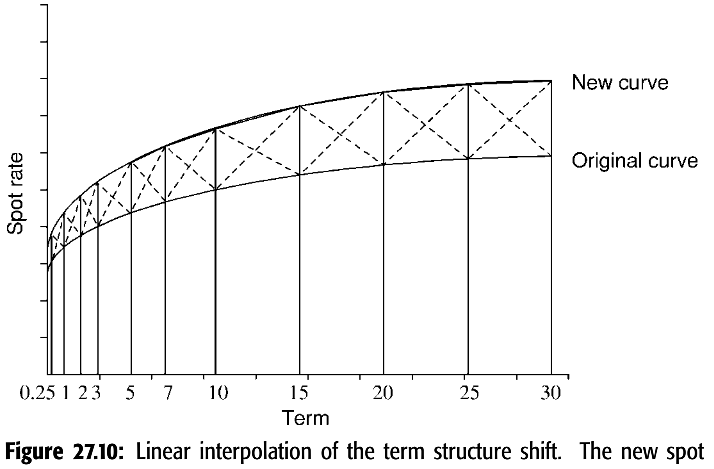

Figure 27.10: Linear interpolation of the term structure shift. The new spot rate curve is approximated when the 11 basic key rate shifts are added to the original spot rate curve. Although the new curve and the approximated curve are guaranteed to agree at only the key rates, they are quite close.

Additional Reading

See [152] for the mispricing of U.S. Treasury callable bonds and [777] for more information on inflation-indexed Treasury securities. U.S. Treasury security prices largely react to the arrival of public information on the economy, particularly the employment, Producer Price Index, and federal funds target rate announcements [358]. Real interest rate changes do not seem important in moving either bond or stock prices [146]. See [328, 712, 837, 889] for more information on bonds with embedded options and [372] for criticisms of the analytical frameworks for embedded options. Consult [117] for hedging with interest rate options and [317, 328, 799] for **bond swaps**. An empirical study of the call policy finds many inconsistencies with the theory [553]. Corporations issue CBs for various reasons [559]. CBs can serve as an alternative to venture capital [599]. See [21, 562, 651, 762, 880] for more information on the OAS, [218] for suggestions beyond the OAS, and [198] for various versions of OAS. The OAS is criticized in [349]. Consult [481, 513, 535, 583] for credit risk and [253, 254, 398] for **credit derivatives**, which protect investors from defaults or rating downgrades.

Reasons for the accumulation of public debts extend far beyond "wars and rebellions" [414]. Misgivings about national debts are common throughout history. Adam Smith (1723–1790) believed such debts would "in the long run probably ruin all the great nations of Europe." He was also concerned about the "pretended payment" that disguised public bankruptcy [808]. Montesquieu endorsed the use of a sinking fund to "procure the public confidence" [676]. J.S. Mill (1806–1873) asserted that the transfer of interest payment is a "serious evil" [665].

NOTE

1. Its CUSIP number is 9128272M3. All U.S. securities issued in book-entry or certificate form after 1970 are identified by a nine-digit **CUSIP number**. CUSIP stands for "Committee on Uniform Securities Identification Procedures." The CUSIP numbering system, developed by the American Bankers Association, was expanded in 1989 to include foreign securities, to be identified by the nine-digit CUSIP International Numbering System [355].