CHAPTER FIFTEEN

Continuous-Time Derivatives Pricing

This problem of time in the art of music is of capital importance.
Igor Stravinsky (1882–1971), *Poetics of Music*

After a short introduction to partial differential equations, this chapter presents the partial differential equation that the option value should satisfy in continuous time. The general methodology is then applied to derivatives, including options on a stock that pays continuous dividends, futures, futures options, correlation options, exchange options, path-dependent options, currency-related options, barrier options, convertible bonds, and options under stochastic volatility. This chapter also discusses the correspondence between the partial differential equation and the martingale approach to pricing.

15.1 Partial Differential Equations

A two-dimensional second-order partial differential equation has the following form:

$$p\,\frac{\partial^2\theta}{\partial x^2}+q\,\frac{\partial^2\theta}{\partial x\,\partial y}+r\,\frac{\partial^2\theta}{\partial y^2}+s\,\frac{\partial\theta}{\partial x}+t\,\frac{\partial\theta}{\partial y}+u\theta+v=0,$$

where p, q, r, s, t, u, and v may be functions of the two independent variables x and y as well as the dependent variable θ and its derivatives. It is called **elliptic**, **parabolic**, or **hyperbolic** according to whether $q^2<4pr$, $q^2=4pr$, or $q^2>4pr$, respectively, over the domain of interest. For this reason, q^2-4pr is called the **discriminant**. Note that the solution to a partial differential equation is a function.

Partial differential equations can also be classified into **initial-value** and **boundary-value problems**. An initial-value problem propagates the solution forward in time from the values given at the initial time. In contrast, a boundary-value problem has known values that must be satisfied at both ends of the relevant intervals [391]. If the conditions for some independent variables are given in the form of initial values and those for others as boundary conditions, we have an **initial-value boundary problem**.

A standard elliptic equation is the two-dimensional **Poisson equation**:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=-\rho(x,y).$$

The **wave equation**,

$$\frac{\partial^2 \theta}{\partial t^2} - \frac{1}{v^2}\frac{\partial^2 \theta}{\partial x^2} = 0,$$

is hyperbolic. The most important parabolic equation is the **diffusion (heat) equation**,

$$\frac{1}{2} D \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial t} = 0,$$

which is a special case of the Fokker–Planck equation. Given the initial condition $\theta(x, 0) = f(x)$ for $-\infty < x < \infty$, its unique bounded solution for any $t > 0$ is

$$\frac{1}{\sqrt{2\pi Dt}} \int_{-\infty}^{\infty} f(z)\, e^{-(x-z)^2/(2Dt)}\, dz \tag{15.1}$$

when $f(x)$ is bounded and piecewise continuous for all real x. The solution clearly depends on the entire initial condition. Any two-dimensional second-order partial differential equation can be reduced to generalized forms of the Poisson equation, the diffusion equation, or the wave equation according to whether it is elliptic, parabolic, or hyperbolic.

➤ **Exercise 15.1.1** Verify that the diffusion equation is indeed satisfied by integral (15.1).

15.2 The Black–Scholes Differential Equation

The price of any derivative on a non-dividend-paying stock must satisfy a partial differential equation. The key step is recognizing that the same random process drives both securities; it is **systematic**, in other words. Given that their prices are perfectly correlated, we can figure out the amount of stock such that the gain from it offsets exactly the loss from the derivative, and vice versa. This removes the uncertainty from the value of the portfolio of the stock and the derivative at the end of a short period of time and forces its return to be the riskless rate in order to avoid arbitrage opportunities.

Several assumptions are made: (1) the stock price follows the geometric Brownian motion $dS = \mu S\, dt + \sigma S\, dW$ with constant μ and σ, (2) there are no dividends during the life of the derivative, (3) trading is continuous , (4) short selling is allowed, (5) there are no transactions costs or taxes, (6) all securities are infinitely divisible, (7) there are no riskless arbitrage opportunities, (8) the term structure of riskless rates is flat at r, and (9) there is unlimited riskless borrowing and lending. Some of these assumptions can be relaxed. For instance, μ, σ, and r can be deterministic functions of time instead of constants. In what follows, t denotes the current time (in years), T denotes the expiration time, and $\tau \equiv T - t$.

15.2.1 Merton's Derivation

Let C be the price of a derivative on S. From Ito's lemma (Theorem 14.2.2),

$$dC = \left(\mu S \frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S}\, dW.$$

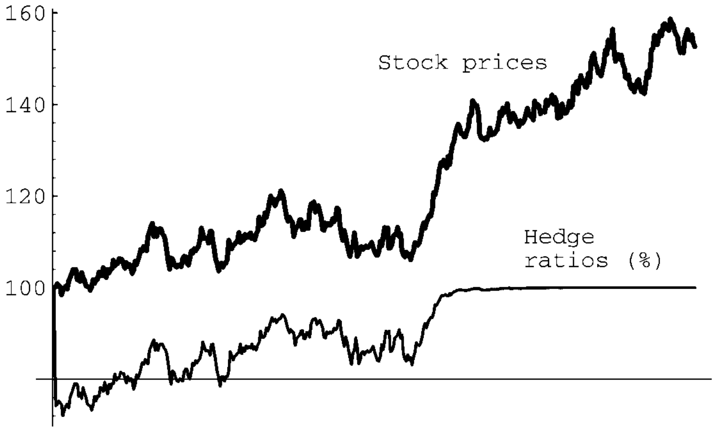

Figure 15.1: Stock price and delta (hedge ratio). Recall that delta is defined as $\partial C/\partial S$. The strike price here is \$95.

Note that the same W drives both C and S. We now show that this random source can be eliminated by being short one derivative and long $\partial C/\partial S$ shares of stock (see Fig. 15.1). Define Π as the value of the portfolio. By construction, $\Pi = -C + S(\partial C/\partial S)$, and the change in the value of the portfolio at time dt is

$$d\Pi = -dC + \frac{\partial C}{\partial S}\, dS.$$

Substitute the formulas for dC and dS into the preceding equation to yield

$$d\Pi = \left(-\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt.$$

Because this equation does not involve dW, the portfolio is riskless during dt time and hence earns the instantaneous return rate r; that is, $d\Pi = r\,\Pi\, dt$, so

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r \left(C - S\frac{\partial C}{\partial S} \right) dt.$$

Equate the terms to obtain finally

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC. \tag{15.2}$$

This is the celebrated **Black–Scholes differential equation** [87].

The Black–Scholes differential equation can be expressed in terms of sensitivity numbers:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC. \tag{15.3}$$

(Review Section 10.1 for the definitions of sensitivity measures.) Identity (15.3) leads to an alternative way of computing Θ numerically from Δ and Γ. In particular, if a portfolio is delta-neutral, then the above equation becomes

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rC.$$

A definite relation thus exists between Γ and Θ.

➤ **Exercise 15.2.1** (1) Verify that the Black–Scholes formula for European calls in Theorem 9.3.4 indeed satisfies Black–Scholes differential equation (15.2). (2) Verify that the value of a forward contract on a non-dividend-paying stock satisfies Black–Scholes differential equation (15.2).

➤ **Exercise 15.2.2** It seems reasonable to expect the predictability of stock returns, as manifested in the drift of the Ito process, to have an impact on option prices. (One possibility is for the log price $R \equiv \ln S$ to follow a trendy Ornstein–Uhlenbeck process instead of Brownian motion.) The analysis in the text, however, implies that the drift of the process is irrelevant; the same Black–Scholes formula stands. Try to resolve the issue.

➤ **Exercise 15.2.3** Outline an argument for the claim that the Black–Scholes differential equation results from the BOPM by taking limits.

Continuous Adjustments

The portfolio Π is riskless for only an infinitesimally short period of time. If the delta $\partial C/\partial S$ changes with S and t, the portfolio must be continuously adjusted to ensure that it remains riskless.

Number of Random Sources

There is no stopping at the single-factor random source. In the presence of two random sources, three securities suffice to eliminate the uncertainty: Use two to eliminate the first source and the third to eliminate the second source. To make this work, the factors must be traded. A **traded security** is an asset that is held solely for investment by a significant number of individuals. Generally speaking, a market is complete only if the number of traded securities exceeds the number of random sources [76].

Risk-Neutral Valuation

Like the BOPM, the Black–Scholes differential equation does not depend directly on the risk preferences of investors. All the variables in the equation are independent of risk preferences, and the one that does depend on them, the expected return of the stock, does not appear in the equation. As a consequence, any risk preference can be used in pricing, including the risk-neutral one.

In a risk-neutral economy, the expected rate of return on all securities is the riskless rate r. Prices are then obtained by discounting the expected value at r. Lemma 9.2.1 says the same thing of the BOPM. The risk-neutral assumption greatly simplifies the analysis of derivatives. It is emphasized that it is the *instantaneous* return rate of the stock that is equal to r (see Comment 14.4.1 for the subtleties).

➤ **Exercise 15.2.4** Explain why the formula

$$e^{-r\tau}\int_X^\infty (y-X)\,\frac{1}{\sigma y\sqrt{2\pi\tau}}\exp\left[-\frac{\{\ln(y/S)-(r-\sigma^2/2)\tau\}^2}{2\sigma^2\tau}\right]dy,$$

is equivalent to the Black–Scholes formula for European calls. (Hint: Review Eq. (6.10).)

15.2.2 Solving the Black–Scholes Equation for European Calls

The Black–Scholes differential equation can be solved directly for European options. After Eq. (15.2) is transformed with the change of variable $C(S,\tau) \equiv B(S,\tau)\, e^{-r\tau} X$, the partial differential equation becomes

$$-\frac{\partial B}{\partial \tau} + rS\,\frac{\partial B}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 B}{\partial S^2} = 0,$$

where $B(0,\tau) = 0$ for $\tau > 0$ and $B(S,0) = \max(S/X - 1, 0)$ for $S > 0$. With transformations $D(x,\tau) \equiv B(S,\tau)$ and $x \equiv (S/X)\,e^{r\tau}$, we end up with the diffusion equation:

$$-\frac{\partial D}{\partial \tau} + \frac{1}{2}\,(\sigma x)^2\,\frac{\partial^2 D}{\partial x^2} = 0,$$

where $D(0,\tau) = 0$ for $\tau > 0$ and $D(x,0) = \max(x-1, 0)$ for $x > 0$. After one more transformation $u \equiv \sigma^2 \tau$, the function $H(x,u) \equiv D(x,\tau)$ satisfies

$$-\frac{\partial H}{\partial u} + \frac{1}{2}\,x^2\,\frac{\partial^2 H}{\partial x^2} = 0,$$

where $H(0,u) = 0$ for $u > 0$ and $H(x,0) = \max(x-1,0)$ for $x > 0$. The final transformation $\Theta(z,u)\,x \equiv H(x,u)$ where $z \equiv (u/2) + \ln x$, lands us at

$$-\frac{\partial \Theta}{\partial u} + \frac{1}{2}\,\frac{\partial^2 \Theta}{\partial z^2} = 0.$$

The boundary conditions are $|\Theta(z,u)| \leq 1$ for $u > 0$ and $\Theta(z,0) = \max(1 - e^{-z}, 0)$. The above diffusion equation has the solution

$$\begin{aligned}
\Theta(z,u) &= \frac{1}{\sqrt{2\pi u}} \int_0^\infty (1 - e^{-y})\, e^{-(z-y)^2/(2u)}\, dy \\
&= \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-(z-y)^2/(2u)}\, dy - \frac{1}{\sqrt{2\pi u}} \int_0^\infty e^{-y} e^{-(z-y)^2/(2u)}\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-z/\sqrt{u}}^\infty e^{-\omega_1^2/2}\, d\omega_1 - \frac{1}{\sqrt{2\pi}\, x} \int_{-(z-u)/\sqrt{u}}^\infty e^{-\omega_2^2/2}\, d\omega_2 \\
&= N\left(\frac{z}{\sqrt{u}}\right) - \frac{1}{x}\, N\left(\frac{z-u}{\sqrt{u}}\right)
\end{aligned}$$

by formula (15.1) with the change of variables $\omega_1 \equiv (y-z)/\sqrt{u}$ and $\omega_2 \equiv (y - z + u)/\sqrt{u}$. Hence

$$H(x,u) = \Theta(\ln x + (u/2), u)\, x = x N\left(\frac{\ln x + (u/2)}{\sqrt{u}}\right) - N\left(\frac{\ln x - (u/2)}{\sqrt{u}}\right).$$

Retrace the steps to obtain

$$\begin{aligned}
C(S,\tau) &= H\left(\frac{S}{X} e^{r\tau}, \sigma^2\tau\right) e^{-r\tau} X \\
&= SN\left(\frac{\ln(S/X) + r\tau + \sigma^2\tau/2}{\sqrt{\sigma^2\tau}}\right) - e^{-r\tau} XN\left(\frac{\ln(S/X) + r\tau - \sigma^2\tau/2}{\sqrt{\sigma^2\tau}}\right),
\end{aligned}$$

which is precisely the Black–Scholes formula for the European call.

➤ **Exercise 15.2.5** Solve the Black–Scholes differential equation for European puts.

15.2.3 Initial and Boundary Conditions

Solving the Black–Scholes differential equation depends on the initial and the boundary conditions defining the particular derivative. These conditions spell out the values of the derivative at various values of S and t. For European calls (puts), the key terminal condition is $\max(S(T) - X, 0)$ ($\max(X - S(T), 0)$, respectively) at time T. There are also useful boundary conditions. The call value is zero when $S(t) = 0$, and the put is $Xe^{-r(T-t)}$ when $S(t) = 0$ (see Exercise 8.4.5). Furthermore, as S goes to infinity, the call value is S and the put value zero. The accuracy is even better if $S - Xe^{-r(T-t)}$ is used in place of S for the European call as $S \to \infty$. Although these boundary conditions are not mathematically necessary, they improve the accuracy of numerical methods [879].

The American put is more complicated because of early exercise whose boundary $\overline{S}(t)$ is unknown a priori. Recall that the exercise boundary specifies the stock price at each instant of time when it becomes optimal to exercise the option. The formulation that guarantees a unique solution is

$$\begin{aligned}
\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = rP \quad &\text{and} \quad P > X - S \quad \text{for } \overline{S} < S < \infty\\
&\qquad\quad P = X - S \quad \text{for } 0 \leq S < \overline{S}\\
\frac{\partial P}{\partial S} = -1 \quad &\text{and} \quad P = X - S \quad \text{for } S = \overline{S}\\
&\qquad\qquad\, P = 0 \quad \text{for } S \to \infty
\end{aligned}$$

plus the obvious terminal condition [154]. The region $0 \leq S < \overline{S}$ is where early exercise is optimal. The exercise boundary is a continuous decreasing function of τ for American puts and a continuous increasing function of τ for American calls [575].

➤ **Exercise 15.2.6** Verify that the Black–Scholes differential equation is violated where it is optimal to exercise the American put early; i.e., $X - S$ does not satisfy the equation.

15.3 Applications

15.3.1 Continuous Dividend Yields

The price for a stock that continuously pays out dividends at an annualized rate of q follows

$$\frac{dS}{S} = (\mu - q)\, dt + \sigma\, dW,$$

where μ is the stock's rate of return. This process was postulated in Subsection 9.6.4 for the stock index and the exchange rate. In a risk-neutral economy, $\mu = r$.

Consider a derivative security whose value f depends on a stock that pays a continuous dividend yield. From Ito's lemma (Theorem 14.2.2),

$$df = \left((\mu - q)\, S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S}\, dW.$$

Set up a portfolio that is short one derivative security and long $\partial f/\partial S$ shares. Its value is $\Pi = -f + (\partial f/\partial S)\, S$, and the change in the value of the portfolio at time dt is given by $d\Pi = -df + (\partial f/\partial S)\, dS$. Substitute the formulas for df and dS to yield

$$d\Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt.$$

The total wealth change is simply the above amount plus the dividends, $d\Pi + qS(\partial f/\partial S)\, dt$. As this value is not stochastic, the portfolio must be instantaneously riskless:

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right) dt + qS \frac{\partial f}{\partial S}\, dt = r\,\Pi\, dt.$$

Simplify to obtain

$$\frac{\partial f}{\partial t} + (r-q)\, S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

For European calls, the boundary conditions are identical to those of the standard option except that its value should be $Se^{-q(T-t)}$ as S goes to infinity. The solution appeared in Eq. (9.20). For American calls, the formulation that guarantees a unique solution is

$$\begin{aligned}
\frac{\partial C}{\partial t} + (r-q)\, S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC \quad \text{and} \quad & C > S - X && \text{for } 0 \le S < \overline{S},\\
& C = S - X && \text{for } \overline{S} < S < \infty,\\
\frac{\partial C}{\partial S} = 1 \quad \text{and} \quad & C = S - X && \text{for } S = \overline{S},\\
& C = 0 && \text{for } S = 0,
\end{aligned}$$

plus the terminal condition $C = \max(S - X, 0)$, of course [879].

15.3.2 Futures and Futures Options

The futures price is related to the spot price by $F = Se^{(r-q)(T-t)}$. By Example 14.3.6, $dF/F = \sigma\, dW$. The futures price can therefore be treated as a stock paying a continuous dividend yield equal to r. This is the rationale behind the Black model.

➤ **Exercise 15.3.1** Derive the partial differential equation for futures options.

15.3.3 Average-Rate and Average-Strike Options

To simplify the notation, assume that the option is initiated at time zero. The arithmetic average-rate call and put have terminal values given by

$$\max\left(\frac{1}{T}\int_0^T S(u)\, du - X, 0\right), \quad \max\left(X - \frac{1}{T}\int_0^T S(u)\, du, 0\right),$$

respectively. Arithmetic average-rate options are notoriously hard to price. In practice, the prices are usually sampled at discrete points in time [598].

If the averaging is done geometrically, the payoffs become

$$\max\left(\exp\left[\frac{\int_0^T \ln S(u)\,du}{T}\right] - X, 0\right), \quad \max\left(X - \exp\left[\frac{\int_0^T \ln S(u)\,du}{T}\right], 0\right),$$

respectively. The geometric average $\exp[\,\frac{1}{T}\int_0^T \ln S(u)\,du\,]$ is lognormally distributed when the underlying asset's price is lognormally distributed (see Example 14.3.5). Lookback calls and puts on the average have terminal payoffs $\max(S(T) - \frac{1}{T}\int_0^T S(u)\,du, 0)$ and $\max(\frac{1}{T}\int_0^T S(u)\,du - S(T), 0)$, respectively.

The partial differential equation satisfied by the value V of a European arithmetic average-rate option can be derived as follows. Introduce the new variable $A(t) \equiv \int_0^t S(u)\,du$. It is not hard to verify that $dA = S\,dt$. Ito's lemma (Theorem 14.2.2) applied to V yields

$$dV = \left(\mu S\,\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A}\right) dt + \sigma S\,\frac{\partial V}{\partial S}\,dW.$$

Consider the portfolio of short one derivative and long $\partial V/\partial S$ shares of stock. This portfolio must earn riskless returns because of lack of randomness. Therefore

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + S\,\frac{\partial V}{\partial A} = rV.$$

➤ **Exercise 15.3.2** Show that geometric average-rate options satisfy

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} + (\ln S)\,\frac{\partial V}{\partial A} = rV,$$

where $A(t) \equiv \int_0^t \ln S(u)\,du$.

15.3.4 Options on More than One Asset: Correlation Options

For a correlation option whose value depends on the prices of two assets S_1 and S_2, both of which follow geometric Brownian motion, the partial differential equation is

$$\frac{\partial C}{\partial t} + \sum_{i=1}^{2} rS_i\,\frac{\partial C}{\partial S_i} + \sum_{i=1}^{2} \frac{\sigma_i^2 S_i^2}{2}\,\frac{\partial^2 C}{\partial S_i^2} + \rho\sigma_1\sigma_2 S_1 S_2\,\frac{\partial^2 C}{\partial S_1 \partial S_2} = rC. \tag{15.4}$$

➤ **Exercise 15.3.3** (1) Justify Eq. (15.4). (2) Generalize it to n assets.

15.3.5 Exchange Options

An **exchange option** is a correlation option that gives the holder the right to exchange one asset for another. Its value at expiration is thus

$$\max(S_2(T) - S_1(T), 0),$$

where $S_1(T)$ and $S_2(T)$ denote the prices of the two assets at expiration. The payoff implies two ways of looking at the option: as a call on asset 2 with a strike price equal to the future price of asset 1 or as a put on asset 1 with a strike price equal to the future value of asset 2.

Assume that the two underlying assets do not pay dividends and that their prices follow

$$\frac{dS_1}{S_1} = \mu_1\, dt + \sigma_1\, dW_1, \qquad \frac{dS_2}{S_2} = \mu_2\, dt + \sigma_2\, dW_2,$$

where ρ is the correlation between dW_1 and dW_2. The option value at time t is

$$V(S_1, S_2, t) = S_2 N(x) - S_1 N(x - \sigma\sqrt{T-t}),$$

where

$$x \equiv \frac{\ln(S_2/S_1) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$\sigma^2 \equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2. \tag{15.5}$$

This is called **Margrabe's formula** [640].

Margrabe's formula can be derived as follows. Observe first that $V(x, y, t)$ is **homogeneous of degree one** in x and y, meaning that $V(\lambda S_1, \lambda S_2, t) = \lambda V(S_1, S_2, t)$. An exchange option based on λ times the prices of the two assets is thus equal in value to λ original exchange options. Intuitively, this is true because of

$$\max(\lambda S_2(T) - \lambda S_1(T), 0) = \lambda \times \max(S_2(T) - S_1(T), 0)$$

and the perfect market assumption [660]. The price of asset 2 relative to asset 1 is $S \equiv S_2/S_1$. Hence the option sells for $V(S_1, S_2, t)/S_1 = V(1, S_2/S_1, t)$ with asset 1 as numeraire. The interest rate on a riskless loan denominated in asset 1 is zero in a perfect market because a lender of one unit of asset 1 demands one unit of asset 1 back as repayment of principal. Because the option to exchange asset 1 for asset 2 is a call on asset 2 with a strike price equal to unity and the interest rate equal to zero, the Black–Scholes formula applies:

$$\frac{V(S_1, S_2, t)}{S_1} = V(1, S, t) = SN(x) - 1 \times e^{-0\times(T-t)} N(x - \sigma\sqrt{T-t}),$$

where

$$x \equiv \frac{\ln(S/1) + (0 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(S_2/S_1) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Suppose the option holder sells $V_1 \equiv \partial V/\partial S_1$ units of asset 1 short and buys $-V_2 \equiv -\partial V/\partial S_2$ units of asset 2. Because $V(\cdot)$ is homogeneous of degree one in S_1 and S_2, the position has zero value because $V - V_1 S_1 - V_2 S_2 = 0$ by Euler's theorem (see Exercise 15.3.6). Hence $dV - V_1\, dS_1 - V_2\, dS_2 = 0$. From Ito's lemma (Theorem 14.2.2),

$$dV = V_1\, dS_1 + V_2\, dS_2 + \frac{\partial V}{\partial t}\, dt + \frac{V_{11}\sigma_1^2 S_1^2 + 2V_{12}\sigma_1\sigma_2\rho S_1 S_2 + V_{22}\sigma_2^2 S_2^2}{2}\, dt,$$

where $V_{ij} \equiv \partial^2 V/(\partial S_i \partial S_j)$. Hence

$$\frac{\partial V}{\partial t} + \frac{V_{11}\sigma_1^2 S_1^2 + 2V_{12}\sigma_1\sigma_2\rho S_1 S_2 + V_{22}\sigma_2^2 S_2^2}{2} = 0 \tag{15.6}$$

with the following initial and boundary conditions:

$$V(S_1, S_2, T) = \max(0, S_2 - S_1),$$
$$0 \le V(S_1, S_2, t) \le S_2 \quad \text{if } S_1, S_2 \ge 0\,.$$

Margrabe's formula is not much more complicated if S_i pays out a continuous dividend yield of q_i, $i = 1, 2$. We simply replace each occurrence of S_i with $S_i e^{-q_i(T-t)}$ to obtain

$$\begin{aligned} V(S_1, S_2, t) &= S_2 e^{-q_2(T-t)} N(x) - S_1 e^{-q_1(T-t)} N(x - \sigma\sqrt{T-t}), \\ x &\equiv \frac{\ln(S_2/S_1) + (q_1 - q_2 + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ \sigma^2 &\equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2. \end{aligned} \tag{15.7}$$

➤ **Exercise 15.3.4** A call on the **maximum of two assets** pays $\max(S_1(T), S_2(T))$ at expiration. Replicate it by a position in one of the assets plus an exchange option.

➤ **Exercise 15.3.5** Consider a call on the **minimum of two assets with strike price** X. Its terminal value is $\max(\min(S_1(T), S_2(T)) - X, 0)$. Show that this option can be replicated by a long position in two ordinary calls and a short position in one call on the maximum of two assets at the same strike price X, which has a terminal payoff of $\max(\max(S_1(T), S_2(T)) - X, 0)$.

➤ **Exercise 15.3.6 (Euler's Theorem)**. Prove that

$$\sum_{i=1}^{n} x_i \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i} = f(x_1, x_2, \dots, x_n)$$

if $f(x_1, x_2, \dots, x_n)$ is homogeneous of degree one in $x_1, x_2, \dots, x_n$.

➤ **Exercise 15.3.7** (1) Derive Margrabe's formula from the alternative view that a European exchange option is a put on asset 1 with a strike price equal to the future value of asset 2. (2) Derive the Black–Scholes formula from Margrabe's formula.

➤ **Exercise 15.3.8** Verify variance (15.5) for Margrabe's formula.

➤ **Exercise 15.3.9 (Put–Call Parity)** Prove that $V(S_2, S_1, t) - V(S_1, S_2, t) + S_2 = S_1$.

➤ **Exercise 15.3.10** Derive Eq. (15.6) from Eq. (15.4).

15.3.6 Options on Foreign Currencies and Assets

Correlation options involving foreign currencies and assets were first covered in Section 11.5. Analysis of such options can take place in either the domestic market or the foreign market before being converted back into the domestic currency [734].

In what follows, $S(t)$ denotes the spot exchange rate in terms of the domestic value of one unit of foreign currency. We know from Subsection 11.5.1 that foreign currency is analogous to a stock paying a continuous dividend yield equal to the foreign riskless interest rate r_{f} in foreign currency. Therefore $S(t)$ follows the geometric Brownian motion process,

$$\frac{dS}{S} = (r - r_{\mathrm{f}})\, dt + \sigma_{\mathrm{s}}\, dW_{\mathrm{s}}(t),$$

in a risk-neutral economy. The foreign asset is assumed to pay a continuous dividend yield of q_{f}, and its price follows

$$\frac{dG_{\mathrm{f}}}{G_{\mathrm{f}}} = (\mu_{\mathrm{f}} - q_{\mathrm{f}})\, dt + \sigma_{\mathrm{f}}\, dW_{\mathrm{f}}(t)$$

in foreign currency. The correlation between the rate of return of the exchange rate and that of the foreign asset is ρ; in other words, ρ is the correlation between dW_s and dW_f.

Foreign Equity Options

From Eq. (9.20), European options on the foreign asset G_f with the terminal payoffs $S(T) \times \max(G_f(T) - X_f, 0)$ and $S(T) \times \max(X_f - G_f(T), 0)$ are worth

$$C_f = G_f e^{-q_f \tau} N(x) - X_f e^{-r_f \tau} N(x - \sigma_f \sqrt{\tau}),$$
$$P_f = X_f e^{-r_f \tau} N(-x + \sigma_f \sqrt{\tau}) - G_f e^{-q_f \tau} N(-x),$$

in foreign currency, where

$$x \equiv \frac{\ln(G_f/X_f) + \left(r_f - q_f + \sigma_f^2/2\right)\tau}{\sigma_f \sqrt{\tau}}$$

and X_f is the strike price in foreign currency. They will fetch SC_f and SP_f, respectively, in domestic currency. These options are called **foreign equity options** struck in foreign currency.

➤ **Exercise 15.3.11** The formulas of C_f and P_f suggest that a foreign equity option is equivalent to S domestic options on a stock paying a continuous dividend yield of q_f and a strike price of $X_f e^{(r-r_f)\tau}$. Verify that this observation is indeed valid.

➤ **Exercise 15.3.12** The dynamics of the foreign asset value in domestic currency, SG_f, depends on the correlation between the asset price and the exchange rate (see Example 14.3.5). (1) Why is ρ missing from the option formulas? (2) Justify the equivalence in Exercise 15.3.11 with (1).

Foreign Domestic Options

Foreign equity options fundamentally involve values in the foreign currency. However, consider this: Although a foreign equity call may allow the holder to participate in a foreign market rally, the profits can be wiped out if the foreign currency depreciates against the domestic currency. What is really needed is a call in *domestic* currency with a payoff of $\max(S(T)\, G_f(T) - X, 0)$. This is called a **foreign domestic option**.

To foreign investors, this call is an option to exchange X units of domestic currency (foreign currency to them) for one share of foreign asset (domestic asset to them) – an exchange option, in short. By formula (15.7), its price in foreign currency equals

$$G_f e^{-q_f \tau} N(x) - \frac{X}{S} e^{-r\tau} N(x - \sigma\sqrt{\tau}),$$

where

$$x \equiv \frac{\ln(G_f S/X) + (r - q_f + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$$

and $\sigma^2 \equiv \sigma_s^2 + 2\rho\sigma_s\sigma_f + \sigma_f^2$. The domestic price is therefore

$$C = SG_f e^{-q_f \tau} N(x) - X e^{-r\tau} N(x - \sigma\sqrt{\tau}).$$

Similarly, a put has a price of

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - SG_f e^{-q_f\tau} N(-x).$$

➤ **Exercise 15.3.13** Suppose that the domestic and the foreign bond prices in their respective currencies with a par value of one and expiring at T also follow geometric Brownian motion processes. Their current prices are B and B_f, respectively. Derive the price of a forex option to buy one unit of foreign currency with X units of domestic currency at time T. (This result generalizes Eq. (11.6), which assumes deterministic interest rates.)

Cross-Currency Options

A cross-currency option, we recall, is an option in which the currency of the strike price is different from the currency in which the underlying asset is denominated [775]. An option to buy 100 yen at a strike price of 1.18 Canadian dollars provides one example. Usually, a third currency, the U.S. dollar, is involved because of the lack of relevant exchange-traded options for the two currencies in question (yen and Canadian dollars in the above example) in order to calculate the needed volatility. For this reason, the notations below will be slightly different.

Let S_A denote the price of the foreign asset and S_C the price of currency C that the strike price X is based on. Both S_A and S_C are in U.S. dollars, say. If S is the price of the foreign asset as measured in currency C, then we have the **triangular arbitrage** $S = S_A/S_C$.[1] Assume that S_A and S_C follow the geometric Brownian motion processes $dS_A/S_A = \mu_A\, dt + \sigma_A\, dW_A$ and $dS_C/S_C = \mu_C\, dt + \sigma_C\, dW_C$, respectively. Parameters σ_A, σ_C, and ρ can be inferred from exchange-traded options. By Exercise 14.3.6,

$$\frac{dS}{S} = (\mu_A - \mu_C - \rho\sigma_A\sigma_C)\, dt + \sigma_A\, dW_A - \sigma_C\, dW_C,$$

where ρ is the correlation between dW_A and dW_C. The volatility of dS/S is hence $(\sigma_A^2 - 2\rho\sigma_A\sigma_C + \sigma_C^2)^{1/2}$.

➤ **Exercise 15.3.14** Verify that the triangular arbitrage must hold to prevent arbitrage opportunities among three currencies.

➤ **Exercise 15.3.15** Show that both forex options and foreign domestic options are special cases of cross-currency options.

➤ **Exercise 15.3.16** Consider a portfolio consisting of a long call on the foreign asset and X long puts on currency C. The strike prices in U.S. dollars of the call (X_A) and put (X_C) are such that $X = X_A/X_C$. Prove the portfolio is worth more than the cross-currency call when all options concerned are European. (A cross-currency call has a terminal payoff of $S_C \times \max(S - X, 0)$ in U.S. dollars.)

Quanto Options

Consider a call with a terminal payoff $\widehat{S} \times \max(G_f(T) - X_f, 0)$ in domestic currency, where $\widehat{S}$ is a constant. This amounts to fixing the exchange rate to $\widehat{S}$. For instance, a call on the Nikkei 225 futures, if it existed, fits this framework with $\widehat{S} = 5$ and G_f denoting the futures price. A guaranteed exchange rate option is called a **quanto option** or simply a **quanto**. The process $U \equiv \widehat{S}\, G_f$ in a risk-neutral economy follows

$$\frac{dU}{U} = (r_{\rm f} - q_{\rm f} - \rho\sigma_{\rm s}\sigma_{\rm f})\, dt + \sigma_{\rm f}\, dW \tag{15.8}$$

in domestic currency [470, 878]. Hence it can be treated as a stock paying a continuous dividend yield of $q \equiv r - r_{\rm f} + q_{\rm f} + \rho\sigma_{\rm s}\sigma_{\rm f}$. Apply Eq. (9.20) to obtain

$$C = \widehat{S}\,[\, G_{\rm f}\, e^{-q\tau} N(x) - X_{\rm f} e^{-r\tau} N(x - \sigma_{\rm f}\sqrt{\tau})\,],$$
$$P = \widehat{S}\,[\, X_{\rm f}\, e^{-r\tau} N(-x + \sigma_{\rm f}\sqrt{\tau}) - G_{\rm f} e^{-q\tau} N(-x)\,],$$

where

$$x \equiv \frac{\ln(G_{\rm f}/X_{\rm f}) + \left(r - q + \sigma_{\rm f}^2/2\right)\tau}{\sigma_{\rm f}\sqrt{\tau}}.$$

Note that the values do not depend on the exchange rate.

In general, a **quanto derivative** has nominal payments in the foreign currency that are converted into the domestic currency at a fixed exchange rate. A **cross-rate swap**, for example, is like a currency swap except that the foreign currency payments are converted into the domestic currency at a fixed exchange rate. Quanto derivatives form a rapidly growing segment of international financial markets [17].

➤ **Exercise 15.3.17** Justify Eq. (15.8).

15.3.7 Convertible Bonds with Call Provisions

When a CB with call provisions is called, its holder has the right either to convert the bond (**forced conversion**) or to redeem it at the call price. Assume that the firm and the investor pursue an optimal strategy whereby (1) the investor maximizes the value of the CB at each instant in time through conversion and (2) the firm minimizes the value of the CB at each instant in time through call.

Let the market value $V(t)$ of the firm's securities be determined exogenously and independent of the call and conversion strategies, which can be justified by the **Modigliani–Miller irrelevance theorem**. Minimizing the value of the CB therefore maximizes the stockholder value. The market value follows $dV/V = \mu\, dt + \sigma\, dW$. Assume that the firm in question has only two classes of obligations: n shares of common stock and m CBs with a conversion ratio of k. The stock may pay dividends, and the bond may pay coupon interests. The **conversion value** per bond is

$$C(V, t) = zV(t),$$

where $z \equiv k/(n + mk)$. Each bond has \$1,000 par value, and T stands for the maturity date.

Let $W(V, t)$ denote the market value at time t of one convertible bond. From assumption (1), the bond never sells below the conversion value as

$$W(V, t) \geq C(V, t). \tag{15.9}$$

In fact, the bond can never sell at the conversion value except immediately before a dividend date. This is because otherwise its rate of return up to the next dividend date would not fall below the stock's; actually, it would be higher because of the higher priority of bondholders. Therefore the bond sells above the conversion value, and the investor does not convert it. As a result, relation (15.9) holds with strict inequality between dividend dates, and conversion needs to be considered only at dividend or call dates.

We now consider the implications of the call strategy. When the bond is called, the investor has the option either to redeem at the call price $P(t)$ or convert it for $C(V, t)$. The value of the bond if called is hence given by

$$V_c(V, t) \equiv \max(P(t), C(V, t)).$$

There are two cases to consider.

1. $C(V, t) > P(t)$ **when the bond is callable:** The bond will be called immediately because, by a previous argument, the bond sells for at least the conversion value $C(V, t)$, which is the value if called. Hence,

$$W(V, t) = C(V, t). \tag{15.10}$$

2. $C(V, t) \leq P(t)$ **when the bond is callable:** Note that the call price equals the value if called, V_c. The bond should be called when its value if not called equals its value if called. This holds because, in accordance with assumption (2), the firm will call the bond when the value if not called exceeds $V_c(V, t)$ and will not call it otherwise. Hence

$$W(V, t) \leq V_c(V, t) = P(t), \tag{15.11}$$

and the bond will be called when its value if not called equals the call price.

Finally, the Black–Scholes differential equation implies that

$$\frac{\partial W}{\partial t} + rV\frac{\partial W}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 W}{\partial V^2} = rW.$$

The boundary conditions for the above differential equation are summarized below.

- They include relation (15.9), Eq. (15.10), and relation (15.11) (the latter two when the bond is callable and under their respective conditions), and the maturity value

$$W(V, T) = \begin{cases} zV(T), & \text{if } zV(T) \geq 1000 \\ 1000, & 1000 \times m \leq V(T) \leq 1000/z. \\ V(T)/m, & V(T) \leq 1000 \times m \end{cases}$$

  These three conditions above correspond to the cases when the firm's total value (1) is greater than the total conversion value, (2) is greater than the total par value but less than the total conversion value, and (3) is less than the total par value.
- $0 \leq mW(V, t) \leq V(t)$ because the bond value cannot exceed the firm value.
- $W(0, t) = 0$.
- $W(V, t) \leq B(V, t) + zV(t)$ because a CB is dominated by a portfolio of an otherwise identical fixed-rate bond $B(V, t)$ and stock with a total value equal to the conversion value. $B(V, t)$ is easy to calculate under constant interest rates.
- When the bond is not callable and $V(t)$ is high enough to make negligible the possibility of default, it behaves like an option to buy a fraction z of the firm. Hence $\lim_{V\to\infty} \partial W(V, t)/\partial V = z$.

- On a dividend date, $W(V, t^-) = \max(W(V - D, t^+), zV(t))$, where t^- denotes the instant before the event and t^+ the instant after. This condition takes into account conversion just before the dividend date.
- $W(V, t^-) = W(V - mc, t^+) + c$ on a coupon date and when the bond is not callable, where c is the amount of the coupon.
- $W(V, t^-) = \min(W(V - mc, t^+) + c, V_c(V, t))$ on a coupon date and when the bond is callable.

The partial differential equation has to be solved numerically by the techniques in Section 18.1.

➤ **Exercise 15.3.18** Suppose that the CB is continuously callable once it becomes callable, meaning that it is callable at any instant after a certain time t^*. Argue that the $C(V, t) > P(t)$ case needs to be considered only at $t = t^*$ and not thenceforth for the call provision. (Hence $W(V, t^-) = \min(W(V - mc, t^+) + c, P(t))$ on any coupon date $t > t^*$.)

15.4 General Derivatives Pricing

In general, the underlying asset S may not be traded. Interest rate, for instance, is not a traded security, whereas stocks and bonds are. Let S follow the Ito process $dS/S = \boldsymbol{\mu}\, dt + \boldsymbol{\sigma}\, dW$, where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ may depend only on S and t. Let $f_1(S, t)$ and $f_2(S, t)$ be the prices of two derivatives with dynamics $df_i/f_i = \mu_i\, dt + \sigma_i\, dW$, $i = 1, 2$. Note that they share the same Wiener process as S.

A portfolio consisting of $\sigma_2 f_2$ units of the first derivative and $-\sigma_1 f_1$ units of the second derivative is instantaneously riskless because

$$\begin{aligned}\sigma_2 f_2\, df_1 - \sigma_1 f_1\, df_2 &= \sigma_2 f_2 f_1(\mu_1\, dt + \sigma_1\, dW) - \sigma_1 f_1 f_2(\mu_2\, dt + \sigma_2\, dW)\\ &= (\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt,\end{aligned}$$

which is devoid of volatility. Therefore

$$(\sigma_2 f_2 f_1 \mu_1 - \sigma_1 f_1 f_2 \mu_2)\, dt = r(\sigma_2 f_2 f_1 - \sigma_1 f_1 f_2)\, dt,$$

or $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1)$. After rearranging the terms, we conclude that

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \quad \text{for some } \lambda.$$

Any derivative whose value depends on only S and t and that follows the Ito process $df/f = \mu\, dt + \sigma\, dW$ must thus satisfy

$$\frac{\mu - r}{\sigma} = \lambda \quad \text{or,} \quad \text{alternatively, } \mu = r + \lambda\sigma. \tag{15.12}$$

We call λ the **market price of risk**, which is independent of the specifics of the derivative. Equation (15.12) links the excess expected return and risk. The term $\lambda\sigma$ measures the extent to which the required return is affected by the dependence on S.

Ito's lemma can be used to derive the formulas for μ and σ:

$$\mu = \frac{1}{f}\left(\frac{\partial f}{\partial t} + \boldsymbol{\mu} S \frac{\partial f}{\partial S} + \frac{1}{2}\boldsymbol{\sigma}^2 S^2 \frac{\partial^2 f}{\partial S^2}\right), \qquad \sigma = \frac{\boldsymbol{\sigma} S}{f}\frac{\partial f}{\partial S}.$$

Substitute the preceding equations into Eq. (15.12) to obtain

$$\frac{\partial f}{\partial t} + (\boldsymbol{\mu} - \lambda \boldsymbol{\sigma})\, S\, \frac{\partial f}{\partial S} + \frac{1}{2}\, \boldsymbol{\sigma}^2 S^2\, \frac{\partial^2 f}{\partial S^2} = r f. \tag{15.13}$$

The presence of $\boldsymbol{\mu}$ shows that the investor's risk preference is relevant, and the derivative may be dependent on the underlying asset's growth rate and the market price of risk. Only when the underlying variable is the price of a traded security can we assume that $\boldsymbol{\mu} = r$ in pricing.

Provided certain conditions are met, such as the underlying processes' being Markovian, the approach to derivatives pricing by solving partial differential equations is equivalent to the martingale approach of taking expectation under a risk-neutral probability measure [290, 692]. The fundamental theorem of asset pricing, Theorem 13.2.3, as well as other results in Subsection 13.2.1, also continues to hold in continuous time. This suggests the following risk-neutral valuation scheme for Eq. (15.13): Discount the expected payoff of f at the riskless interest rate under the revised process $dS/S = (\boldsymbol{\mu} - \lambda \boldsymbol{\sigma})\, dt + \boldsymbol{\sigma}\, dW$. Although the same symbol W is used, a convention adopted throughout the book for convenience, it is important to point out that W is no longer the original Wiener process. In fact, a change of probability measure has taken place, and W is a Wiener process with respect to the risk-neutral probability measure.

Assume a constant interest rate r. Then any European-style derivative security with payoff f_T at time T has value $e^{-r(T-t)} E_t^{\pi}[\, f_T\,]$, where E_t^{π} takes the expected value under a risk-neutral probability measure given the information up to time t. As a specific application, consider the futures price F. With a delivery price of X, a futures contract has value $f = e^{-r(T-t)} E_t^{\pi}[\, S_T - X\,]$. Because F is the X that makes f zero, it holds that $0 = E_t^{\pi}[\, S_T - F\,] = E_t^{\pi}[\, S_T\,] - F$, i.e., $F = E_t^{\pi}[\, S_T\,]$. This extends the result for the binomial model in Exercise 13.2.10 to the continuous-time case.

➤ **Exercise 15.4.1** Suppose that $S_1, S_2, \ldots, S_n$ pay no dividends and follow $dS_i / S_i = \boldsymbol{\mu}_i\, dt + \boldsymbol{\sigma}_i\, dW_i$. Let $\boldsymbol{\rho}_{jk}$ denote the correlation between dW_j and dW_k. Show that

$$\frac{\partial f}{\partial t} + \sum_i (\boldsymbol{\mu}_i - \lambda_i \boldsymbol{\sigma}_i)\, S_i\, \frac{\partial f}{\partial S_i} + \frac{1}{2} \sum_i \sum_k \boldsymbol{\rho}_{ik} \boldsymbol{\sigma}_i \boldsymbol{\sigma}_k S_i S_k\, \frac{\partial^2 f}{\partial S_i\, \partial S_k} = r f \tag{15.14}$$

when the derivative f depends on more than one state variable $S_1, S_2, \ldots, S_n$.

➤ **Exercise 15.4.2** A **forward-start option** is like a standard option except that it becomes effective only at time τ^* from now and with the strike price set at the stock price then (the option thus starts at the money). Let $C(S)$ denote the value of an at-the-money European forward-start call, given the stock price S. (1) Show that $C(S)$ is a linear function in S under the Black–Scholes model. (2) Argue that the value of a forward-start option is $e^{-r\tau^*} C(E^{\pi}[\, S(\tau^*)\,]) = e^{-r\tau^*} C(S e^{(r-q)\,\tau^*})$, where q is the dividend yield.

15.5 Stochastic Volatility

The Black–Scholes formula displays bias in practice. Besides the smile pattern mentioned in Subsection 9.4.3, (1) volatility changes from month to month, (2) it is mean reverting in that extreme volatilities tend to return to the average over time, (3) it seems to fall as the price of the underlying asset rises [346, 823], and (4) out-of-the-money options and options on low-volatility assets are underpriced. These findings led to the study of stochastic volatility.

Stochastic volatility injects an extra source of randomness if this uncertainty is not perfectly correlated with the one driving the stock price. In this case, another traded security besides stock and bond is needed in the replicating portfolio. In fact, if volatility were the price of a traded security, there would exist a self-financing strategy that replicates the option by using stocks, bonds, and the volatility security.

Hull and White considered the following model:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW_1, \qquad \frac{dV}{V} = \mu_{\mathrm{v}}\, dt + \sigma_{\mathrm{v}}\, dW_2,$$

where $V \equiv \sigma^2$ is the instantaneous variance [471]. Assume that μ depends on S, σ, and t, that μ_{v} depends on σ and t (but not S), that dW_1 and dW_2 have correlation ρ, and that the riskless rate r is constant. From Eq. (15.14),

$$\frac{\partial f}{\partial t} + (\mu - \lambda\sigma)\, S\, \frac{\partial f}{\partial S} + (\mu_{\mathrm{v}} - \lambda_{\mathrm{v}}\sigma_{\mathrm{v}})\, V\, \frac{\partial f}{\partial V}$$
$$+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\sigma\sigma_{\mathrm{v}} SV \frac{\partial^2 f}{\partial S \partial V} + \sigma_{\mathrm{v}}^2 V^2 \frac{\partial^2 f}{\partial V^2}\right) = rf.$$

Because stock is a traded security (but volatility is not), the preceding equation becomes

$$\frac{\partial f}{\partial t} + r\, S\, \frac{\partial f}{\partial S} + (\mu_{\mathrm{v}} - \lambda_{\mathrm{v}}\sigma_{\mathrm{v}})\, V\, \frac{\partial f}{\partial V}$$
$$+ \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\sigma\sigma_{\mathrm{v}} SV \frac{\partial^2 f}{\partial S \partial V} + \sigma_{\mathrm{v}}^2 V^2 \frac{\partial^2 f}{\partial V^2}\right) = rf.$$

After two additional assumptions, $\rho = 0$ (volatility is uncorrelated with the stock price) and $\lambda_{\mathrm{v}}\sigma_{\mathrm{v}} = 0$ (volatility has zero systematic risk), the equation becomes

$$\frac{\partial f}{\partial t} + r\, S\, \frac{\partial f}{\partial S} + \mu_{\mathrm{v}} V\, \frac{\partial f}{\partial V} + \frac{1}{2}\left(\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \sigma_{\mathrm{v}}^2 V^2 \frac{\partial^2 f}{\partial V^2}\right) = rf.$$

A series solution is available for the model.

The volatility risk was assumed not to be priced [579]. To assume otherwise, we need to model risk premium on the variance process [440]. When the volatility follows an uncorrelated Ornstein–Uhlenbeck process, closed-form solutions exist [823].

Additional Reading

A rigorous derivation of the Black–Scholes differential equation can be found in [681]. See [212, 408, 446, 861, 883] for partial differential equations, [531] for the approximation of the early exercise boundary, [769] for the derivation of Margrabe's formula based on the binomial model, [575, 744, 746, 894] for currency-related options, [122, 221, 491, 697] for pricing CBs, and [424, 470, 615] for the bias of the Black–Scholes option pricing model. We followed [120] in Subsection 15.3.7. Martingale pricing in continuous time relies on changing the probability measure with the **Girsanov theorem** [289]. That using stochastic volatility models can result in some pricing improvement has been empirically documented [44]. In cases in which the

Black–Scholes model has been reasonably supported by empirical research, gains from complicated models may be limited, however [526]. Intriguingly, the Black–Scholes formula continues to hold as long as all traders believe that the stock prices are lognormally distributed, even if that belief is objectively wrong [194].

NOTE

1. Triangular arbitrage had been known for centuries. See Montesquieu's *The Spirit of Laws* [676, p. 179].