CHAPTER
TWENTY-ONE

Interest Rate Derivative Securities

I never gamble.
J.P. Morgan, Sr. (1837–1913)

Interest-rate-sensitive securities are securities whose payoff depends on the levels and/or evolution of interest rates. The interest rate derivatives market is enormous. The global notional principal of over-the-counter derivative contracts was an estimated U.S.$72 trillion as of the end of June 1998, of which 67% were interest rate instruments and 31% were forex instruments [51]. The use of such derivatives in portfolio risk management has made possible economical and efficient alteration of interest rate sensitivities [325]. Throughout this book, **interest rate derivative securities** exclude fixed-income securities with embedded options.

21.1 Interest Rate Futures and Forwards

An interest rate futures contract is a futures contract whose underlying asset depends solely on the level of interest rates. Figure 21.1 gives an idea of the diversity of interest rate futures.

21.1.1 Treasury Bill Futures

The first financial futures contract was based on a fixed-income instrument, the Government National Mortgage Association (GNMA or "Ginnie Mac") mortgage-backed certificates whose trading began in 1972 at the CBT. The IMM of the CME followed 3 months later with futures contracts based on the 13-week T-bill [95].

The T-bill futures contract traded on the IMM is based on the 13-week (3-month) T-bill with a face value of $1 million. The seller of a T-bill futures contract agrees to deliver to the buyer at the delivery date a T-bill with 13 weeks remaining to maturity and a face value of $1 million. The T-bill delivered can be newly issued or seasoned. The futures price is the price at which the T-bill will be sold by the short and purchased by the buyer. The contract allows for the delivery of 89-, 90-, or 91-day T-bills after price adjustments.

T-bills are quoted in the spot market in terms of the annualized discount rate of formula (3.9). In contrast, the futures contract is quoted, not directly in terms of

Monday, March 20, 1995
FUTURES PRICES

. . .

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
				. . .				
				INTEREST RATE				
TREASURY BONDS (CBT) – $100,000; pts. 32nds of 100%								
Mar	104-31	105-05	104-18	104-20	−11	116-20	95-13	28,210
June	104-12	104-19	104-00	104-02	−11	113-15	94-27	328,566
Sept	104-06	104-06	103-19	103-21	−11	112-15	94-10	14,833
Dec	103-23	103-23	103-04	103-07	−12	111-23	93-27	1,372
Mr96	102-26	103-00	102-26	102-26	−13	103-17	93-13	232
June	102-15	102-15	102-13	102-13	−13	104-28	93-06	46
				. . .				
TREASURY NOTES (CBT) – $100,000; pts. 32nds of 100%								
Mar	105-00	105-02	104-27	104-28	−3	111-07	98-11	24,870
June	104-16	104-20	104-11	104-13	−3	105-22	97-27	225,265
				. . .				
5 YR TREAS NOTES (CBT) – $100,000; pts. 32nds of 100%								
Mar	103-31	03-315	103-27	03-275	−2.5	104-11	99-15	19,230
June	103-18	103-21	03-155	03-165	−2.0	104-01	99-06	179,928
				. . .				
2 YR TREAS NOTES (CBT) – $200,000; pts. 32nds of 100%								
Mar	02-085	02-085	102-07	102-07	−1/4	02-105	99-252	5,776
June	103-30	01-302	01-282	101-29	+1/4	02-015	99-24	26,904
				. . .				
30-DAY FEDERAL FUNDS (CBT) – $5 million; pts. of 100%								
Mar	94.05	94.05	94.05	94.05	−.01	94.44	93.28	2,905
Apr	93.97	93.97	93.96	93.97	. . .	93.98	93.05	4,331
				. . .				
TREASURY BILLS (CME) – $1 mil.; pts. of 100%								
	Open	High	Low	Settle	Chg	Discount Settle	Discount Chg	Open Interest
June	94.04	94.06	94.04	94.05	+.02	5.95	−.02	16,964
Sept	93.83	93.83	93.80	93.82	+.01	6.18	−.01	10,146
Dec	93.64	93.66	93.64	93.66	−.01	6.34	+.01	9,082
				. . .				
LIBOR-1MO. (CME) – $3,000,000; points of 100%								
Apr	93.84	93.84	93.82	93.83	. . .	6.17	. . .	27,961
				. . .				
EURODOLLAR (CME) – $1 million; pts of 100%								
	Open	High	Low	Settle	Chg	Yield Settle	Yield Chg	Open Interest
June	93.52	93.54	93.51	93.52	. . .	6.48	. . .	515,578
Sept	93.31	93.32	93.28	93.30	. . .	6.70	. . .	322,889
				. . .				
Mr04	91.65	91.65	91.65	91.65	−.01	8.35	+.01	1,910
				. . .				

Figure 21.1: Interest rate futures quotations. Expanded from Fig. 12.3.

yield, but instead on an index basis that is related to the discount rate as

$$\text{index price} = 100 - (\text{annualized discount yield} \times 100).$$

The price is therefore merely a different way of quoting the interest rate. For example, if the yield is 8%, the index price is $100 - (0.08 \times 100) = 92$. Alternatively, the discount yield for the futures contract can be derived from the price of the futures contract:

$$\text{annualized discount yield} = \frac{100 - \text{index price}}{100}.$$

The invoice price that a buyer of \$1 million face-value of 13-week T-bills must pay at the delivery date is found by first computing the dollar discount:

$$\text{dollar discount} = \text{annualized discount yield} \times \$1{,}000{,}000 \times (T/360),$$

where T is the number of days to maturity. The invoice price is

$$\text{invoice price} = \$1{,}000{,}000 - \text{dollar discount}.$$

Combining the two preceding equations, we arrive at

$$\text{invoice price} = \$1{,}000{,}000 \times [\, 1 - \text{annualized discount yield} \times (T/360)\,],$$

where $T = 89$, 90, or 91. For example, suppose that the index price for a T-bill futures contract is 92.52. The discount yield for this T-bill futures contract is $(100 - 92.52)/100 = 7.48\%$, and the dollar discount for the T-bill to be delivered with 91 days to maturity is

$$0.0748 \times \$1{,}000{,}000 \times \frac{91}{360} = \$18{,}907.78.$$

The invoice price is thus $1{,}000{,}000 - 18{,}907.78 = 981{,}092.22$ dollars.

The "tick" for the T-bill futures contract is 0.01. A change of 0.01 translates into a one-basis-point change in the discount yield. The dollar price change of a tick is therefore $0.0001 \times \$1{,}000{,}000 \times (90/360) = \$25$ for a 90-day contract. The contract is quoted and traded in half-tick increments.

Suppose that the futures contract matures in t years and its underlying T-bill matures in $t^* > t$ years, the difference between them being 90 days. Let $S(t)$ and $S(t^*)$ denote the continuously compounded riskless spot rates for terms t and t^*, respectively. Because no income is paid on the bill, the futures price equals $F = e^{S(t)\,t - S(t^*)\,t^*}$ by Lemma 12.2.1. This incidentally shows the duration of the T-bill futures to be $t^* - t$ if the yield curve is flat. Let $\overline{r} \equiv (t^* - t)^{-1} \ln(1/F) = [\, S(t^*)\,t^* - S(t)\,t\,]/(t^* - t)$ be the continuously compounded forward rate for the time period between t and t^*. The futures price is therefore the price the bill will have if the 90-day interest rate at the delivery date proves to be $\overline{r}$. Rearrange the terms to obtain

$$S(t) = \frac{S(t^*)\,t^* - \overline{r}(t^* - t)}{t}.$$

The rate derived by the right-hand-side formula above is called the **implied repo rate**. As argued above, arbitrage opportunities exist if the implied repo rate differs from the actual T-bill rate $S(t)$.

EXAMPLE 21.1.1 Suppose that the price of a T-bill maturing in 138 days (0.3781 year) is 95 and the futures price for a 90-day (0.2466-year) T-bill futures contract maturing in 48 days is 96.50. The implied repo rate is

$$\frac{138 \times (1/0.3781) \ln(1/0.95) - 90 \times (1/0.2466) \ln(1/0.965)}{48} = 0.1191,$$

or 11.91%.

➤ **Exercise 21.1.1** Derive the formula for the change in the T-bill futures' invoice price per tick with 91 days to maturity.

Duration-Based Hedging

Under continuous compounding, the duration of a T-bill is its term to maturity. Suppose that one anticipates a cash inflow of L dollars at time t and plans to invest it in 6-month T-bills. To address the concern that the interest rate may drop at time t by using Treasury bill futures, one should buy the following number of contracts:

$$\frac{L \times 0.5}{\text{T-bill futures contract price} \times 0.25}$$

by Eq. (4.13). The general formula is

$$\frac{L \times \text{duration of the liability}}{\text{interest rate futures contract price} \times \text{duration of the futures}}$$

under the assumption of parallel shifts.

EXAMPLE 21.1.2 A firm holds 6-month T-bills with a total par value of $10 million. The term structure is flat at 8%. The current T-bill value is hence $\$10{,}000{,}000 \times e^{-0.08/2} = \$9{,}607{,}894$, and the current T-bill futures price is $\$1{,}000{,}000 \times e^{-0.08/4} = \$980{,}199$. The firm hedges by selling $(9{,}607{,}894 \times 0.5)/(980{,}199 \times 0.25) = 19.6$ futures contracts.

21.1.2 The Eurodollar Market

Money deposited outside its nation of origin is called **Eurocurrency**. Eurocurrency trading involves the borrowing and the lending of time deposits. The most important Eurocurrency is the Eurodollar, and the interest rate banks pay on Eurodollar time deposits is LIBOR. The 1-month LIBOR is the rate offered on 1-month deposits, the 3-month LIBOR is the rate offered on 3-month deposits, and so on. LIBOR plays the role in international financial markets that prime rates do in domestic ones.

LIBOR is quoted actual over 360; therefore the interest rate is stated as if the year had 360 days even though interest is paid daily. Consider a $1 million loan with an annualized interest rate of the 6-month LIBOR plus 0.5%. The life of this loan is divided into 6-month periods. For each period, the rate of interest is set 0.5% above the 6-month LIBOR at the beginning of the period. For example, if the current LIBOR is 8% and the period has 182 days, the interest due at the end of the period is

$$\$1{,}000{,}000 \times 0.085 \times \frac{182}{360} = \$42{,}972.22.$$

The LIBOR rate underestimates the effective rate. For example, the effective annual rate corresponding to a 6-month LIBOR quote of 7% is the slightly higher 7.2231%

because

$$\left(1+0.07\times\frac{182}{360}\right)\left(1+0.07\times\frac{183}{360}\right)-1\approx 0.072231.$$

Because the LIBOR yields are quoted in terms of an add-on interest rate, they are directly comparable with those on domestic CDs.

21.1.3 Eurodollar Futures

Eurodollar futures started trading in 1981 and are now traded on both the IMM and the London International Financial Futures and Options Exchange (LIFFE). This contract is one of the most heavily traded futures contracts in the world. The 3-month Eurodollar is the underlying instrument for the Eurodollar futures contract, which has a delivery date ranging from 3 months to 10 years.

As with the T-bill futures, this contract is for \$1 million of face value and is traded on an index price basis with a tick of 0.01. The quoted futures price is equal to 100 minus the annualized yield, which is also called the **implied** LIBOR **rate**. The contract month specifies the month and year in which the futures contract expires. For example, the June contract in Fig. 21.1 is quoted as 93.52. This implies a Eurodollar interest rate quote of 6.48% for the 3-month period beginning June and a contract price of

$$\$1{,}000{,}000\times[\,1-(0.0648/4)\,]=\$983{,}800.$$

Unlike the T-bill futures, the Eurodollar futures contract is a cash-settlement contract. The parties settle in cash for the value of a Eurodollar time deposit based on the LIBOR at the delivery date. The final marking to market sets the contract price to

$$\$1{,}000{,}000\times[\,1-(\text{LIBOR}/4)\,].$$

The LIBOR above is the actual 90-day rate on Eurodollar deposits with quarterly compounding. So the futures price converges to $100\times(1-\text{LIBOR})$ by design. Equivalently, the implied LIBOR rate converges to the spot LIBOR rate.

The T-bill futures is a contract on a price unlike that of the Eurodollar futures, which is a contract on an interest rate. Eurodollar futures prices move linearly with the bank discount yield; a 1% change in yield always causes a

$$\$1{,}000{,}000\times\frac{1}{100}\times\frac{90}{360}=\$2{,}500$$

change in price, which implies a tick value of \$25. This is in sharp contrast to T-bill futures, whose prices move linearly with T-bill prices.

EXAMPLE 21.1.3 Consider a floating-rate liability of \$10 million with an interest rate at 1% above the prevailing 3-month LIBOR on the interest payment date in September. There happen to be Eurodollar futures expiring on the payment date with a futures price of 93.33. Selling 10 such contracts locks in a LIBOR rate of 6.67% for the 4-month period beginning September. Because the borrowing cost is 1% over LIBOR, the locked-in borrowing rate is 7.67%. The interest payment is $10\times 0.0767/4=0.19175$ million dollars at this rate (see Fig. 21.2).

LIBOR	*5%*	*6%*	*7%*	*8%*	*9%*
Futures price	95	94	93	92	91
Interest expense	150,000	175,000	200,000	225,000	250,000
Loss on futures	41,750	16,750	−8,250	−33,250	−58,250
Net borrowing cost	191,750	191,750	191,750	191,750	191,750

Figure 21.2: Locking in the borrowing cost with Eurodollar futures. The interest expense is $\$10{,}000{,}000 \times (1\% + \text{LIBOR})/4$, and the loss on futures is $10 \times \$1{,}000{,}000 \times (\text{futures price} - 93.33)/(4 \times 100)$. See Example 21.1.3 for explanations.

The preceding single-period example readily extends to multiperiod situations. The way to transform a floating-rate liability into a fixed-rate liability is by selling a strip of futures whose expiration dates coincide with interest payment dates. The locked-in borrowing rates are the implied LIBOR rates plus the applicable spread. Here the interest generated by the marking-to-market feature is ignored.

A LIBOR term structure can be established from the implied 3-month LIBOR rates. Suppose now that June 1995 is the expiration date of the June 1995 Eurodollar futures contract. The Eurodollar futures prices are in Fig. 21.3. The actual 3-month LIBOR from June to September is 6.48%. Because the maturity is 92 days away, the return is $6.48\% \times (92/360) = 1.656\%$. The 6-month rate could be established by the June and the September Eurodollar futures contracts as follows. The implied rate from September to December is $6.70\% \times (91/360) = 1.694\%$. The implied rate over the 6-month period is thus $(1.01656 \times 1.01694) - 1 = 3.378\%$, which gives an annualized rate of $3.378\% \times (360/183) = 6.645\%$. This is also the borrowing rate that can be

	Futures Price	*Implied LIBOR*	*Days Settle*		*Futures Price*	*Implied LIBOR*	*Days Settle*
June	93.52	6.48	92	Dec	92.37	7.63	91
Sept	93.30	6.70	91	Mar00	92.38	7.62	92
Dec	93.11	6.89	91	June	92.31	7.69	92
Mar96	93.10	6.90	92	Sept	92.25	7.75	91
June	93.02	6.98	92	Dec	92.17	7.83	90
Sept	92.97	7.03	91	Mar01	92.18	7.82	92
Dec	92.87	7.13	90	June	92.11	7.89	92
Mar97	92.88	7.12	92	Sept	92.05	7.95	91
June	92.82	7.18	92	Dec	91.97	8.03	90
Sept	92.78	7.22	91	Mar02	92.01	7.99	92
Dec	92.70	7.30	90	June	91.96	8.04	92
Mar98	92.71	7.29	92	Sept	91.90	8.10	91
June	92.65	7.35	92	Dec	91.82	8.18	90
Sept	92.61	7.39	91	Mar03	91.83	8.17	92
Dec	92.53	7.47	90	June	91.76	8.24	92
Mar99	92.54	7.46	92	Sept	91.70	8.30	91
June	92.49	7.51	92	Dec	91.61	8.39	91
Sept	92.45	7.55	91	Mr04	91.65	8.35	92

Figure 21.3: Eurodollar futures prices. The yields are the hypothetical implied 3-month LIBOR rates as of June 1995.

locked in for the 6-month period. The procedure can be repeated to yield a LIBOR term structure.

EXAMPLE 21.1.4 What about the 12-month implied LIBOR rate beginning June 1996? It is determined by the June, September, December, and March Eurodollar futures contracts and is given by the value f satisfying

$$1+f\times\frac{365}{360}=\left(1+6.98\%\times\frac{92}{360}\right)\left(1+7.03\%\times\frac{91}{360}\right)$$
$$\times\left(1+7.13\%\times\frac{90}{360}\right)\left(1+7.12\%\times\frac{92}{360}\right).$$

Solving this expression yields 7.257%.

➤ **Exercise 21.1.2** Calculate the annualized implied 9-month LIBOR rate between June 1995 and March 1996 from the data in Fig. 21.3.

21.1.4 Treasury Bond Futures

The CBT trades 2-year, 5-year, and 10-year T-note futures and T-bond futures. All require delivery of the underlying securities, but a number of eligible securities can be delivered against the contract.

Initiated in 1977, the T-bond futures contract calls for $100,000 face value in deliverable-grade U.S. T-bonds. The bond delivered must have at least 15 years to maturity or to the first call date if callable. The short position can choose any business day in the delivery month to deliver, although contracts are rarely settled by actual delivery. The underlying instrument for the T-bond futures contract is $100,000 par value of a hypothetical 20-year 8% coupon bond. For example, if the March contract settles at 104-20, the buyer is entitled to receive this coupon bond for a price of $104\frac{20}{32}$% of $100,000, or $104,625. The minimum price fluctuation for the T-bond futures contract is 1/32 of 1%. The dollar value of that is $\$100,000\times(1/32)\%=\$31.25$; hence the minimum price fluctuation is thus $31.25.

Although prices and yields of the T-bond futures are quoted in terms of this hypothetical bond, the seller of the futures contract has the choice of several actual Treasury bonds that are acceptable for delivery. A well-defined mechanism computes from the quoted price the effective futures price for all the bonds that satisfy delivery requirements. The invoice amount is determined by

$$\begin{aligned}\text{invoice price} = {} & (\text{settlement price}\times\text{contract size}\times\text{conversion factor})\\ & +\text{accrued interest}.\end{aligned}\tag{21.1}$$

The **conversion factor** is the price of a $1 (face value) coupon bond with a maturity – measured from the first day in the delivery month – equal to the delivered bond if it were priced to yield 8%, compounded semiannually. Maturities are rounded down to the nearest quarter. For example, 24-years-and-5-months becomes 24-years-and-3-months. Let ω be the percentage of the coupon that due the holder as defined in Eq. (3.19). The bond is worth

$$\frac{c}{(1.04)^{\omega}}+\frac{c}{(1.04)^{\omega+1}}+\frac{c}{(1.04)^{\omega+2}}+\cdots+\frac{1+c}{(1.04)^{\omega+n-1}}-c(1-\omega),$$

where c is the semiannual coupon rate and n is the number of remaining coupon payments. The last term is the accrued interest. As coupon payments are made at 6-month intervals, ω is either $1/2$ or 1. For example, if the first coupon payment occurs 3 months hence, then $\omega = 1/2$.[1]

Consider a 13% coupon bond with 19 years and 2 months to maturity. In calculating the conversion factor, the bond is assumed to have exactly 19 years to maturity, and the nearest coupon payment will be made 6 months from now. On the assumption that the discount rate is 8% per annum with semiannual compounding, the bond has the value

$$\sum_{i=1}^{38} \frac{6.5}{(1.04)^i} + \frac{100}{(1.04)^{38}} = 148.42.$$

The conversion factor is therefore 1.4842. Consider an otherwise identical bond with 19 years and 4 months to maturity. For the purpose of calculating the conversion factor, the bond is assumed to have exactly 19 years and 3 months to maturity. There are 39 coupon payments, starting 3 months from now. The value of the bond is

$$\sum_{i=0}^{38} \frac{6.5}{(1.04)^{i+0.5}} + \frac{100}{(1.04)^{38.5}} - 3.25 = 148.66.$$

The conversion factor is therefore 1.4866.

EXAMPLE 21.1.5 Suppose the T-bond futures contract settles at 96 and the short elects to deliver a T-bond with a conversion factor of 1.15. The price is $\$100{,}000 \times 0.96 \times 1.15 = \$110{,}400$. The buyer of the contract must also pay the seller accrued interest on the bond delivered as dictated by Eq. (21.1).

The party with the short position can choose from among the deliverable bonds the "cheapest" one to deliver. Because the party with the short position receives the invoice price equal to

$$\text{(quoted futures price} \times \text{conversion factor)} + \text{accrued interest}$$

and the cost of purchasing a bond is

$$\text{quoted cash price} + \text{accrued interest},$$

the **cheapest-to-deliver bond** is the one for which

$$\text{quoted cash price} - \text{(quoted futures price} \times \text{conversion factor)}$$

is least. One can find this by examining all deliverable bonds. The cheapest-to-deliver bond may change from day to day.

In addition to the option to deliver any acceptable Treasury issue (sometimes referred to as the **quality** or **swap option**), the short position has two more options. First, it decides when in the delivery month delivery actually takes place. This is called the **timing option**. (The futures contract stops trading 7 business days before the end of the delivery month.) The other option is the right to give notice of intent to deliver up to 8 P.M. Chicago time on the date when the futures settlement price has been fixed. This option is referred to as the **wild card option**. These three options, in sum referred to as the **delivery option**, mean that the long position can never be sure which T-bond will be delivered and when [325, 470]. Such complexity has helped provide liquidity.

➤ **Exercise 21.1.3** A deliverable 8% coupon bond must have a conversion factor of one regardless of its maturity as long as the accrued interest is zero. Why?

➤ **Exercise 21.1.4** Calculate the conversion factor for a 13% coupon bond with 19 years and 11 months to maturity at the delivery date.

Valuation

The delivery option creates value for the short position. It also makes pricing difficult. If both the cheapest-to-deliver bond and the delivery date are known, however, the T-bond futures contract becomes a futures contract on a security with known income. Lemma 12.2.3 says the futures price F is related to the cash bond price S by $F = (S - I)\, e^{r\tau}$, where I is the PV of the coupons during the life of the futures contract and r is the riskless interest rate for the period.[2]

EXAMPLE 21.1.6 Consider a cheapest-to-deliver bond with 9% coupon and a conversion factor of 1.0982. Its next coupon date is 120 days from now, and delivery will take place in 270 days' time followed by the next coupon date after 33 days. The term structure is flat at 8%, continuously compounded. Assume that the current quoted bond price is 108. We first figure out the accrued interest, say 1.533. The cash bond price is therefore 109.533. The futures price if the contract were written on the 9% bond would be

$$\left[\, 109.533 - 4.5 \times e^{0.08 \times (120/365)} \,\right] \times e^{0.08 \times (270/365)} = 111.309.$$

At delivery, there are 150 days of accrued interest $(270 - 120 = 150)$. The quoted futures price if the contract were written on the 9% bond would therefore be

$$111.309 - \left(4.5 \times \frac{150}{150 + 33} \right) = 107.620.$$

Because the contract is in fact written on a standard 8% bond and 1.0982 standard bonds are considered equivalent to each 9% bond, $107.620/1.0982 = 97.997$ should be the quoted futures price.

Hedging

Recall that a basis-point value (BPV) is the price change of a debt instrument given a one-basis-point (0.01%) change in its yield. For example, if the yield on a bond changes from 8% to 8.01% and the resulting price changes by \$70, its BPV is \$70. The greater the BPV, the greater the interest rate exposure. The BPV of a futures contract is the BPV of the cheapest-to-deliver instrument divided by its conversion factor. A conversion factor of α means that the price sensitivity of the bond is approximately α times that of the futures, and α futures contracts need to be sold to immunize the price change of every \$100,000 face value of the underlying bond. The BPV of the T-bond futures contract increases as interest rates decline. This is in sharp contrast to Eurodollar futures for which each 0.01% change in rates changes the price by precisely \$25.

In order to use futures to alter a portfolio's duration, it is necessary to calculate both the sensitivity of the bond portfolio to yield changes and the sensitivity of the futures prices to yield changes. For the futures contract to succeed as a hedge, its price movements should track those of the underlying bond closely. This is indeed the case [95, 746].

EXAMPLE 21.1.7 An investor holds a bond portfolio with a Macaulay duration (MD) of 5.0 and a market value of $100 million. Let the bond equivalent yield (BEY) be 10%. The number of futures contracts to buy in order to increase the MD to 9.0 can be determined as follows. Assume that the BPV of the futures is $85. The BPV of the portfolio equals

$$\frac{5.0}{1+(0.10/2)} \times \$100{,}000{,}000 \times 0.0001 = \$47{,}619.$$

The desired BPV can be derived from the targeted MD by

$$\frac{9.0}{1+(0.10/2)} \times \$100{,}000{,}000 \times 0.0001 = \$85{,}714.$$

As the BPV needs to be increased by $38,095, the number of futures contracts to buy is $38095/85 \approx 448$.

Bond futures and stock index futures can be combined to synthetically change the allocation of assets. A manager would like to achieve the equivalent of selling $100 million in bonds and buying $100 million in stock by using futures. First, follow the steps in Example 21.1.7 to figure out the BPV of $100 million worth of bonds and then the number of bond futures to sell with the same BPV. The number of stock index futures to buy is determined by $100,000,000 divided by the value of the contract, which equals $500 times the value of the S&P 500 Index for the S&P 500 Index futures contract, for example.

21.1.5 Treasury Note Futures

The T-note futures contract is modeled after the T-bond futures contract. For instance, the underlying instrument for the 10-year T-note futures contract is $100,000 par value of a hypothetical 10-year 8% T-note. Again, there are several acceptable Treasury issues that may be delivered. For the 10-year futures, for example, an issue is acceptable if the maturity is not less than 6.5 years and not more than 10 years from the first day of the delivery month. The delivery options granted to the short position and the minimum price fluctuation are the same as those of the T-bond futures [325]. An exception is the 2-year T-note futures, whose face value is $200,000 and whose prices are quoted in terms of one quarter of 1/32 of a dollar.

➤ **Exercise 21.1.5** A pension fund manager wants to take advantage of the high yield offered on T-notes, but unusually high policy payouts prohibit such investments. It is, however, expected that cash flow will return to normal in September. What can be done?

21.1.6 Forward Rate Agreements

First used in 1982, forward rate agreements (FRAs) are cash-settled forward contracts between two parties with a payoff linked to the future level of a reference rate [647]. The interest is based on a hypothetical deposit and paid at a predetermined future date. The buyer of an FRA pays the difference between interest on this hypothetical loan at a fixed rate and interest on the same loan at the prevailing rate. The market is primarily an interbank market and represents the over-the-counter equivalent of the exchange-traded futures contracts on short-term rates.

Formally, suppose that X represents the annualized fixed rate and y represents the actual annualized reference rate prevailing at the settlement date T. The net cash payment to the buyer of an FRA at the end of the contract period, $T + m$, is

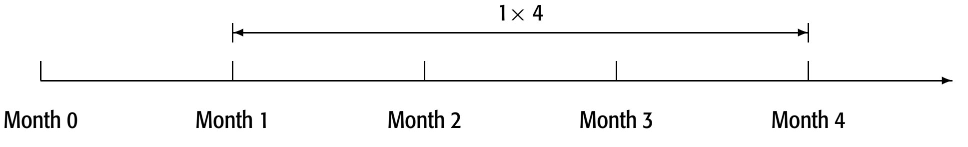
$$(y - X) \times \mathcal{N} \times \frac{m}{360}. \tag{21.2}$$

Here m represents the "deposit" period in days, and $\mathcal{N}$ is the hypothetical loan amount (the **notional principal**). Generally, the net payment due is discounted back to the settlement date with the reference rate as the discount rate. The cash payment is thus

$$(y - X) \times \mathcal{N} \times \frac{m}{360} \times \frac{1}{1 + y(m/360)}$$

at the settlement date T.

The quote convention identifies the points in time when the contract begins and ends. Thus FRAs covering the period starting in 1 month and ending in 4 months are referred to as 1×4 (one-by-four) contracts. They are on the 3-month LIBOR (see the following diagram):



Similarly, FRAs on the 6-month LIBOR for settlement 1 month forward are 1×7 contracts. On any given day, forward rates are available for both 3- and 6-month LIBOR 1 month, 2 months, 3 months, 4 months, 5 months, and 6 months forward. On each subsequent day, new contracts are offered again.

EXAMPLE 21.1.8 A bank will, in 3 months, lend \$1 million to a client for 6 months. To hedge the rate commitment the client demands, it uses FRAs to lock in the funding cost. The bank asks for a quote on 3×9 LIBOR and gets 5.5%. It then offers a fixed rate of 6% to the client. Suppose that the 6-month LIBOR becomes 6.2% 3 months from now. The loss 9 months from now on the actual lending is $(0.062 - 0.06) \times \$1{,}000{,}000 \times (182/360) = \$1{,}011.11$, whereas the gain from the FRA is $(0.062 - 0.055) \times \$1{,}000{,}000 \times (182/360) = \$3{,}538.89$.

Pricing FRAs amounts to deriving the fair fixed forward rate assuming there is no default risk on the part of the FRA writer. The forward rate for the time period $[T, T + m]$ equals

$$f_{\mathrm{L}}(T, T + m) \equiv \left[\frac{d_{\mathrm{L}}(T)}{d_{\mathrm{L}}(T + m)} - 1 \right] \Big/ \Delta t.$$

Here $\Delta t \equiv m/360$, and $d_{\mathrm{L}}(t)$ denotes the PV of a Eurodollar deposit that pays \$1 t days from now (see Exercise 5.6.3). Hence $f_{\mathrm{L}}(T, T + m)$ is the desired fixed contract rate X that makes the FRA zero valued now. In general, the PV of the FRA in

Eq. (21.2) equals

$$[\, f_{\rm L}(T, T+m) - X\,] \times \Delta t \times d_{\rm L}(T+m) = d_{\rm L}(T) - (1 + X\Delta t)\, d_{\rm L}(T+m) \tag{21.3}$$

per Eurodollar of notional principal.

➤ **Exercise 21.1.6** (1) Prove Exercise 5.6.3 with an arbitrage argument. (2) Verify Eq. (21.3).

21.2 Fixed-Income Options and Interest Rate Options

This section covers options on fixed-income securities and interest rates. We use "fixed-income options" for the former and "interest rate options" for the latter. The over-the-counter market for fixed-income options began in the mid-1970s with essentially put options on mortgages [724]. Almost all exchange-traded interest rate options are European.

With fixed-income options, one buys puts to hedge against rate rises and calls to hedge against rate falls. With yield-based interest rate options, the situation is reversed: A call buyer anticipates that interest rates will go up, whereas a put buyer anticipates that the rates will go down. Many of the trading strategies in Chap. 7 remain applicable here.

21.2.1 Options on Treasuries

Consider a European option that expires at date T and is written on a T-bill with a maturity of m days when the option expires. The strike price of the option, x (in percentage), is a discount rate whose corresponding number in percentage of par is

$$X = \left(1 - \frac{x}{100} \times \frac{m}{360}\right) \times 100$$

under the 360-day year. The payoff to a call at expiration is $\max(d(m) - X, 0)$, where $d(m)$ is the date-T value of a \$1 face-value zero-coupon bond maturing at date $T + m$. The payoff to a put at expiration is $\max(X - d(m), 0)$.

Treasury options are not very liquid. A thin market exists for exchange-traded options on specific T-bonds. Prices are quoted in points and 1/32 of a point, with each point representing 1% of the principal value, or \$1,000. The amount paid on exercise is equal to the strike price times the underlying principal plus accrued interest. For example, the settlement price of an option with strike 90 is $\$100{,}000 \times (90/100) = \$90{,}000$ plus accrued interest.

21.2.2 Interest Rate Options on Treasury Yields

The CBOE trades European-style, cash-settled interest rate options on the following Treasury yields: (1) the short-term rate based on the annualized discount rate on the most recently auctioned 13-week T-bill (ticker symbol IRX), (2) the 5-year rate based on the yield to maturity of the most recently auctioned 5-year T-note, (3) the 10-year rate based on the yield to maturity of the most recently auctioned 10-year T-note, and (4) the 30-year rate based on the yield to maturity of the most recently auctioned 30-year Treasury bond (ticker symbol TYX). See Fig. 21.4.

Friday, August 28, 1998

OPTIONS ON SHORT-TERM INTEREST RATES (IRX)

Strike		Calls-Last			Puts-Last	
Price	Sep	Oct	Dec	Sep	Oct	Nov
50	...	...	...	1 7/16	...	...
55	...	...	5/16	...	...	...

...

5 YEAR TREASURY YIELD OPTION (FVX)

Strike		Calls-Last			Puts-Last	
Price	Sep	Oct	Nov	Sep	Oct	Nov
52 1/2	...	...	7/8	...	...	...

...

10 YEAR TREASURY YIELD OPTION (TNX)

Strike		Calls-Last			Puts-Last	
Price	Sep	Oct	Mar	Sep	Oct	Nov
55	...	...	7/8	...	...	...

...

30 YEAR TREASURY YIELD OPTION (TYX)

Strike		Calls-Last			Puts-Last	
Price	Sep	Oct	Dec	Sep	Oct	Nov
50	...	...	4 1/8	3/16	...	...
52 1/2	...	...	...	5/8	...	...
55	7/16	...	...	1 15/16	...	...
57 1/2	...	5/16	...	...	...	...

...

Figure 21.4: Treasury yield quotations. Source: *Wall Street Journal*, August 31, 1998.

The underlying values for these options are 10 times the underlying rates. As a result, an annualized discount rate of 3.25% on the 13-week T-bill would place the underlying value of the IRX at 32.50, and a yield to maturity of 6.5% on the 30-year T-bond would place the underlying value of the TYX at 65.00. Clearly, every one-percentage-point change in interest rates makes the underlying value change by 10 points in the same direction. Like equity options, these options use the $100 multiplier for the contract size. The final settlement value is determined from quotes on the last trading day as reported by the Federal Reserve Bank of New York at 2:30 P.M. Central Time. The payoff, if exercised, is equal to $100 times the difference between the settlement value and the strike price. For example, an investor holding an expiring TYX July 66 call with a settlement value of 69 at expiration will exercise the call and receive $300.

➤ **Exercise 21.2.1** An investor owns T-bills and expects rising short-term rates and falling intermediate-term rates. To profit from this reshaping of the yield curve, he sells T-bills, deposits the cash in the bank, and purchases puts on the 10-year Treasury yield. Analyze this strategy.

21.2.3 Interest Rate Caps and Floors

A cap or a floor is a series of interest rate options. The underlying asset could be an interest rate or the price of a fixed-income instrument. Options on a cap are called **captions**, and options on a floor are called **flotions** [325, 346]. Captions and flotions are compound options.

Simple interest rates are often used in the specification of interest rate derivatives. The payoff to the cap at expiration is

$$\max\left(\frac{r-x}{1+r(m/360)}, 0\right) \times \frac{m}{360} \times \mathcal{N},$$

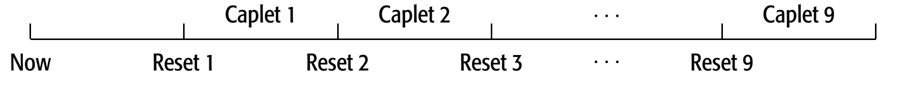
where x is the cap rate expressed in terms of simple interest rate and r is the m-day interest rate (both are annualized). Similarly, the payoff to a floor with a floor rate x written on the m-day interest rate is

$$\max\left(\frac{x-r}{1+r(m/360)}, 0\right) \times \frac{m}{360} \times \mathcal{N}.$$

The payoffs $\max(r-x, 0) \times (m/360) \times \mathcal{N}$ and $\max(x-r, 0) \times (m/360) \times \mathcal{N}$ are discounted because they are received at the maturity of the underlying, which is m days from the cap's or the floor's expiration date.

EXAMPLE 21.2.1 Take a 6-month European call on the 6-month LIBOR with a 7% strike rate. The face value is \$10 million, and the expiration date is 6 months (183 days) from now. This call gives the buyer the right to receive $\max(r-7\%, 0) \times (182/360) \times$ \$10,000,000, where r is the 6-month LIBOR rate prevailing in 6 months' time. The payoff is received at the maturity date of the underlying interest rate, which is $183 + 182 = 365$ days from now. The PV of the above payoff at the expiration of the cap is therefore the above equation divided by $1 + r(182/360)$.

EXAMPLE 21.2.2 Consider the cap in Fig. 21.5. Although a total of 10 6-month periods are involved, the cap contains only 9 options whose payoffs are determined on reset dates. The underlying interest rate for the first period is the interest rate today, which is known; hence no option is involved here. See the following time line:



The cash flow is depicted in Fig. 21.6. The mechanics of floors are similar. Besides the 6-month LIBOR, other candidates for the underlying interest rate are the prime rate, the T-bill rate, the CD rate, and the commercial paper rate.

➤ **Exercise 21.2.2** Verify that an FRA to borrow at a rate of r can be replicated as a portfolio of one long caplet and one short floorlet with identical strike rate x and expiration date equal to the settlement date of the FRA.

Reference rate	six-month LIBOR
Strike rate	8%
Length of agreement	five years
Frequency of settlement	every six months
Notional principal amount	\$100 million

Figure 21.5: An interest rate cap.

Year	*LIBOR*	*Loan cash flow*	*Cap payoff*	*Net cash flow*
0.0	8%	+100.0000		+100.0000
0.5	9%	−4.0556	0.0000	−4.0556
1.0	9%	−4.5625	0.5069	−4.0556
1.5	9%	−4.5625	0.5069	−4.0556
2.0	9%	−4.5625	0.5069	−4.0556
2.5	9%	−4.5625	0.5069	−4.0556
3.0	9%	−4.5625	0.5069	−4.0556
3.5	9%	−4.5625	0.5069	−4.0556
4.0	9%	−4.5625	0.5069	−4.0556
4.5	9%	−4.5625	0.5069	−4.0556
5.0		−104.5625	0.5069	−104.0556

Figure 21.6: Cash flow of a capped loan. Assume that the LIBOR starts at 8% and then moves to 9% half a year from now and stays there. The loan cash flow and the interest rate option's cash flow are calculated based on the prevailing LIBOR rate at the beginning of each half-year period. For instance, at year one the loan's interest is $100 \times 9\% \times (182.5/360) = 4.5625$ million dollars. A half-year is assumed to have exactly 182.5 days for simplicity.

21.2.4 Caps/Floors and Fixed-Income Options

Consider a caplet on the m-day LIBOR with strike rate x. Let the notional principal amount be \$1 for simplicity. At expiration, if the actual interest rate is r, the caplet pays

$$\max\left(\frac{r(m/360) - x(m/360)}{1 + r(m/360)}, 0\right).$$

Interestingly, this caplet is equivalent to $\alpha \equiv 1 + x(m/360)$ puts on the m-day zero-coupon bond with a strike price of $1/\alpha$ and the identical expiration date as the payoff of the puts is

$$\alpha \times \max\left(\frac{1}{1 + x(m/360)} - \frac{1}{1 + r(m/360)}, 0\right)$$
$$= \max\left(\frac{r(m/360) - x(m/360)}{1 + r(m/360)}, 0\right).$$

Similarly, a floorlet is equivalent to α calls with the same strike price. A cap is therefore a package of puts on zeros, and a floor is a package of calls on zeros. An interest rate collar, then, is equivalent to buying a package of puts and selling a package of calls.

The Black model is widely used in practice to price caps and floors. For caps, it applies Eqs. (12.16) with F denoting the implied forward rate for the period between the cap's expiration date T and date $T + m$. This amounts to assuming that the forward rate is lognormally distributed during the period $[\,T, T + m\,]$. The resulting formula is then multiplied by the notional principal and finally multiplied by $(m/360)/[\,1 + F(m/360)\,]$.

➤ **Exercise 21.2.3** Prove the equivalency for the floorlet.

21.2.5 Yield Curve Options

A **yield curve option** is a European option written on the difference between two reference rates. A call based on the difference between the yields on the 20-year T-bond and the 12-month T-bill, for example, has a payoff at expiration given by $\max((y_{20y} - y_{12m}) - X, 0)$, where X is the strike spread between the two reference yields [149, 616].

21.3 Options on Interest Rate Futures

The most popular exchange-traded interest rate options are those on T-bond futures, T-note futures (traded on the CBT), and Eurodollar futures (traded on the IMM). These futures options are all American style. The prices are quoted as a percentage of the principal amount of the underlying debt security. For options on Eurodollar futures, the price is quoted to two decimal places, and one contract is for the delivery of futures contracts with a face value of \$1 million. For options on T-bond and T-note futures (except the 2-year note), the price is quoted to the nearest $1/64$ of 1%, and one contract is for the delivery of futures contracts with a face value of \$100,000. Figure 21.7 shows sample quotations. For example, an investor holding a December call with a strike price of 98, having paid 1-08 for it, will make a net profit of

$$\$100{,}000 \times \frac{101.00 - 98.00 - 1.125}{100} = \$1875$$

if the T-bond futures price rises to 101-00. Futures options on fixed-income securities have largely replaced options on the same securities as the vehicle of choice for institutional investors [325].

The option on T-bond futures is an option on the futures price itself, that is, the futures price of the fictitious 20-year 8% Treasury bond. The size of the contract is \$100,000. For example, with futures prices at 95, a call struck at 94 has an intrinsic value of \$1,000 and a put strike at 100 has an intrinsic value of \$5,000. Prices are quoted in multiples of 1/64 of 1% of a \$100,000 T-bond futures contract. Each 1/64 point (tick size) is worth \$15.625 [375]. Options cease trading in the month before the delivery month of the underlying futures contract.

A 10-year T-note futures contract has a face value at maturity of \$100,000. The tick size is 1/64 of a point (\$15.625/contract). Options cease trading in the month before the delivery month of the underlying futures contract. The 5-year T-note futures option is identical. The 2-year T-note futures option is identical except that the face value is \$200,000 and the tick size is 1/128 of a point (\$15.625/contract).

Options on Eurodollar futures are based on the quoted Eurodollar futures price. Like the underlying futures, the size of the contract is \$1 million, and each 0.01 change in price carries a value of \$25. The option premium is quoted in terms of basis points. For example, a premium quoted as 20 implies an option price of $20 \times \$25 = \$500$. Take a 3-month put on the June Eurodollar futures contract at a strike price of 93. The expiration date of the put is in June, and the underlying asset is June Eurodollar futures. The terminal payoff is $[\max(93 - F, 0)/100\,] \times (90/360) \times \$1{,}000{,}000$, where F is the futures price in June.

Monday, March 20, 1995

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INTEREST RATE

T-BONDS (CBT)

$100,000; points and 64ths of 100%

Strike	Calls – Settle			Puts – Settle		
Price	Apr	May	Jun	Apr	May	Jun
102	2-06	2-26	2-47	0-03	0-23	0-43
103	1-12	1-44	. . .	0-08	0-40	. . .
104	0-30	1-05	1-29	0-26	1-01	1-25
105	0-07	0-39	. . .	1-03	1-34	. . .
106	0-01	0-21	0-40	1-61	. . .	2-35
107	0-01	0-10	. . .	. . .	. . .	. . .

. . .

T-NOTES (CBT)

$100,000; points and 64ths of 100%

Strike	Calls – Settle			Puts – Settle		
Price	Apr	May	Jun	Apr	May	Jun
102	2-26	. . .	2-45	0-01	. . .	0-20
103	1-28	. . .	1-60	0-02	0-19	0-35
104	0-36	. . .	1-19	0-11	. . .	0-57
105	0-07	0-32	0-51	0-45	1-06	1-25
106	0-01	. . .	0-28	1-39	. . .	2-01
107	0-01	0-05	0-14	. . .	. . .	2-51

. . .

5 YR TREAS NOTES (CBT)

$100,000; points and 64ths of 100%

Strike	Calls – Settle			Puts – Settle		
Price	Apr	May	Jun	Apr	May	Jun
10200	1-33	. . .	1-47	0-01	0-07	0-15
10250	1-02	. . .	1-23	0-01	0-12	0-22
10300	0-36	. . .	1-01	0-03	0-21	0-32
10350	0-13	. . .	0-46	0-12	0-33	0-45
10400	0-03	0-20	0-31	0-34	. . .	0-62
10450	0-01	0-11	0-21	. . .	. . .	. . .

. . .

EURODOLLAR (CME)

$ million; pts. of 100%

Strike	Calls – Settle			Puts – Settle		
Price	Jun	Sep	Dec	Jun	Sep	Dec
9300	0.56	0.49	0.52	0.04	0.20	0.41
9325	0.34	0.34	0.39	0.07	0.29	0.52
9350	0.17	0.22	0.29	0.15	0.41	0.67
9375	0.05	0.11	0.19	0.28	0.55	. . .
9400	0.01	0.06	0.12	0.49	0.75	0.98
9425	0.00	0.03	0.07	0.73	0.96	1.18

. . .

LIBOR – 1 Mo. (CME)

$3 million; pts. of 100%

Strike	Calls – Settle			Puts – Settle		
Price	Apr	May	Jun	Apr	May	Jun
9325	0.58	0.51	0.45	0.00	0.01	0.03
9350	0.34	0.29	0.24	0.01	0.04	0.07
9375	0.11	0.10	0.09	0.03	. . .	0.17
9400	0.01	. . .	0.03	. . .	. . .	. . .
9425	. . .	. . .	. . .	. . .	. . .	. . .
9450	0.00	0.00	. . .	. . .	. . .	. . .

. . .

Figure 21.7: Interest rate futures option quotations. The months refer to the expiration month of the underlying futures contract. Source: *Wall Street Journal*, March 21, 1995.

21.3.1 Hedging Floating-Rate Liabilities

Exposure to fluctuations in short-term rates can be hedged with Eurodollar futures options. Let us redo the calculations behind the table in Fig. 21.2 (see Example 21.1.3), this time using Eurodollar futures options instead. The liability, we recall, is a floating-rate debt of $10 million with an interest rate at 1% above the prevailing 3-month LIBOR on the payment date. There happen to be Eurodollar futures options expiring on the payment date. Purchasing 10 futures put options at the strike price of 93 caps the borrowing cost at $200,000 when the liability is due (consult Fig. 21.8). Of course, there is a cost in the cap represented by the option premium.

LIBOR	*5%*	*6%*	*7%*	*8%*	*9%*
Futures price	95	94	93	92	91
Interest expense	150,000	175,000	200,000	225,000	250,000
Put payout	0	0	0	25,000	50,000
Net borrowing cost	150,000	175,000	200,000	200,000	200,000

Figure 21.8: Capping the borrowing cost with Eurodollar futures options. The interest expense is $10{,}000{,}000 \times (1\% + \text{LIBOR})/4$, and the put payout is $10 \times 1{,}000{,}000 \times (93 - \text{futures price})/(4 \times 100)$.

21.4 Interest Rate Swaps

Two parties enter into an interest rate swap to exchange periodic interest payments. The dollar amount each counterparty pays to the other is the agreed periodic interest rate times the notional principal. The benchmarks popular for the floating rate are those on various money market instruments [226].

21.4.1 "Plain Vanilla" Interest Rate Swaps

In a "plain vanilla" interest rate swap, one party periodically pays a cash flow determined by a fixed interest rate (the **fixed leg**) and receives a cash flow determined by a floating interest rate (the **floating leg**). The other party does the opposite. In other words, two parties swap floating-rate debt and fixed-rate debt. Unlike currency swaps, no principal is exchanged. The fixed rate that makes a swap's value zero is called the **swap rate**.

A swap has four major components: notional principal amount, interest rates for the parties, frequency of cash exchanges, and duration of the swap. A "$40 million, 2-year, pay fixed, receive variable, semi" swap, for example, means that the notional principal is $40 million, one party makes a fixed-rate payment every 6 months based on $40 million, and the counterparty makes a floating-rate payment every 6 months based on $40 million, for a period of 2 years.

The floating-rate payment is linked to some short-term interest rate such as the 6-month LIBOR. The fixed rate for a plain vanilla swap is usually quoted as some spread over benchmark U.S. Treasuries. For example, a quote of "30 over" for a 5-year swap says that the fixed rate will be set at the 5-year Treasury yield plus 30 basis points. Although the net cash flow is established at the beginning of the period, it is usually paid out at the end of the period (**in arrears**) instead of at the beginning of the period (**in advance**).

Consider a 2-year swap with a notional principal of $10 million and semiannual payments. The fixed rate is 20 basis points above the 2-year Treasury rate, and the floating rate is the 6-month LIBOR. Suppose the Treasury rate is currently 5%. The fixed rate is therefore 5.2%. The fixed-rate payer will be paid according to the 6-month LIBOR rates determined at dates 0, 0.5 year, 1 year, and 1.5 years. If the LIBOR rates at the above four dates are 5.5%, 6%, 5.9%, and 5%, the fixed-rate payer has the following cash flow:

Payment date	*0.5 year*	*1 year*	*1.5 years*	*2 years*
Received amount	275,000	300,000	295,000	250,000
Paid amount	260,000	260,000	260,000	260,000

Note that the applicable rates are divided by two because only one-half-year's interest is being paid. In reality, only the losing party pays the difference.

Treasury rates are quoted differently from LIBOR rates. Consider a fixed rate quoted as a BEY under the actual/365 day count convention. To compare this yield with a LIBOR rate, which differs in the number of days on which they are quoted, either the 6-month LIBOR rate must be multiplied by 365/360 or the BEY must be multiplied by 360/365. Hence, if a 0.4% spread over the LIBOR is on the basis of a 365-day year with semiannual compounding, the rate becomes

$$\text{LIBOR} + 0.4 \times (360/365) = \text{LIBOR} + 0.3945\%$$

under the 360-day year. There are other complications. For instance, the timing of the cash flows for the fixed-rate payer and that of the floating-rate payer are rarely identical; the fixed-rate payer may make payments annually, whereas the floating-rate payer may make payments semiannually, say. The way in which interest accrues on each leg of the transaction may also differ because of different day count conventions.

Suppose A wants to take out a floating-rate loan linked to the 6-month LIBOR and B wants to take out a fixed-rate loan. They face the following borrowing rates:

	Fixed	*Floating*
A	F_A%	6-month LIBOR + S_A%
B	F_B%	6-month LIBOR + S_B%

Clearly A can borrow directly at LIBOR plus S_A% and B can borrow at F_B%. The total interest rate is the LIBOR plus $(S_A + F_B)$%. Suppose that $S_B - S_A < F_B - F_A$. In other words, A is *relatively* more competitive in the fixed-rate market than in the floating-rate market, and vice versa for B. Consider the alternative whereby A borrows in the fixed-rate market at F_A%, B borrows in the floating-rate market at LIBOR plus S_B%, and they enter into a swap, perhaps with a bank as the financial intermediary. These transactions transform A's loan into a floating-rate loan and B's loan into a fixed-rate loan, as desired. The new arrangement pays a total of the LIBOR plus $(S_B + F_A)$%, a saving of $(S_A + F_B - S_B - F_A)$%. Naturally these transactions will be executed only if the total gain is distributed in such a way that every party benefits; that is, A pays less than the LIBOR plus S_A%, B pays less than F_B%, and the bank enjoys a positive spread.

EXAMPLE 21.4.1 Consider the following borrowing rates that A and B face:

	Fixed	*Floating*
A	8%	6-month LIBOR + 1%
B	11%	6-month LIBOR + 2%

Party A desires a floating-rate loan, and B wants a fixed-rate loan. Clearly A can borrow directly at the LIBOR plus 1%, and B can borrow at 11%. As the rate differential in fixed-rate loans (3%) is different from that in floating-rate loans (1%), a swap with a total saving of $3 - 1 = 2\%$ is possible. Party A is relatively more competitive in the fixed-rate market, whereas B is relatively more competitive in the floating-rate market. So they borrow in the respective markets in which they are competitive. Then each enters into a swap with the bank. There are hence two separate swap agreements, as shown in Fig. 21.9. The outcome: A effectively borrows at the LIBOR

Figure 21.9: Plain vanilla interest rate swaps.

plus 0.5% and B borrows at 10%. The distribution of the gain is 0.5% for A, 1% for B, and 0.5% for the bank. From the bank's point of view, the swap with A is like paying 7.5% and receiving LIBOR "**flat**" (i.e., no spread to the LIBOR), and the swap with B is like receiving 8% and paying LIBOR flat. Suppose the swap's duration is 10 years and the 10-year Treasury yield is 7%. The bank would quote such a swap as 50–100, meaning it is willing to enter into a swap (1) to receive the LIBOR and pay a fixed rate equal to the 10-year Treasury rate plus 50 basis points and (2) to pay the LIBOR and receive a fixed rate equal to the 10-year Treasury rate plus 100 basis points. The difference between the Treasury rate paid and received (50 basis points here) is the bid–ask spread.

As in the preceding example, counterparties are seldom involved directly in swaps. Instead, swaps are usually executed between counterparties and market makers or between market makers. The market maker faces the risk of having to locate another counterparty or holding an unmatched position if one leg of the swap defaults. It does not, however, risk the loss of principal as there is no exchange of principal to begin with.

A **par swap curve** can be constructed from zero-valued swaps of various maturities. The bootstrapping algorithm in Fig. 5.4 may be applied to give the theoretical **zero-coupon swap curve**, which can be used to price any swap.

➤ **Exercise 21.4.1** Party A wants to take out a floating-rate loan, and B wants to take out a fixed-rate loan. They face the borrowing rates below:

	Fixed	*Floating*
A	F_A%	LIBOR $+S_A$%
B	F_B%	LIBOR $+S_B$%

Party A agrees to pay the bank a floating rate of (LIBOR $-S'_A$)% in exchange for a fixed rate of $(F_A + F'_A)$%, and B agrees to pay the bank a fixed rate of $(F_B + F'_B)$% in exchange for a floating rate of (LIBOR$-S'_B$)%. Prove that

$$0 < S_A + F'_A + S'_A < S_A + F_B - F_A + F'_B + S'_B < S_A + F_B - S_B - F_A$$

must hold for both A and B to enter into a swap with the bank in which A effectively takes out a floating-rate loan and B a fixed-rate loan.

Applications to Asset/Liability Management

By changing the cash flow characteristics of assets, a swap can provide a better match between assets and liabilities. Consider a commercial bank with short-term deposits that are repriced every 6 months at the 6-month LIBOR minus 20 basis points. It faces a portfolio mismatch problem because its customers borrow long term. To tackle this maturity mismatch, the bank enters into a swap agreement whereby it pays a fixed rate of 10% semiannually and receives the 6-month LIBOR. This arrangement

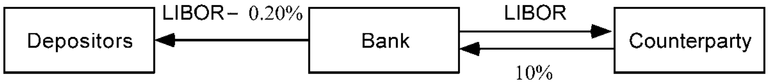

Figure 21.10: Interest rate swaps for asset/liability management.

transforms the floating-rate liability into a fixed-rate liability of 9.8%, as shown in Fig. 21.10.

As another application, suppose that a bank has a \$100 million 10-year-term commercial loan paying a fixed rate of 10%. Interest is paid semiannually, and the principal is paid at the end of the 10-year period. The bank issues 6-month CDs to fund the loan. The interest rate that the bank plans to pay on such CDs is a 6-month LIBOR plus 50 basis points. Clearly, if the 6-month LIBOR rises above 9.5%, the bank loses money. A life insurance company faces a different problem. It has committed itself to paying 9% on a **guaranteed investment contract** (**GIC**) it issued for the next 10 years. The amount of the GIC is also \$100 million. The insurance company invests this money on a floating-rate instrument that earns a rate equal to the 6-month LIBOR plus 150 basis points. The rate is reset semiannually. Clearly, if the 6-month LIBOR falls below 7.5%, the insurance company loses money.

Interest rate swaps may allow both parties to lock in a spread. Suppose there exists a 10-year interest rate swap with a notional principal of \$100 million. The terms are for the bank to pay 8.5% annual rate and receive LIBOR every 6 months and for the insurance company to pay LIBOR and receive 8.4% every 6 months. The bank's and the insurance company's cash flows every 6 months now appear in Fig. 21.11. Hence, the bank locks in a spread of 100 basis points, however the 6-month LIBOR turns out. Similarly, the insurance company locks in a spread of 90 basis points.

Like currency swaps, an interest rate swap can be interpreted as either a package of cash flows from buying and selling cash market instruments or as a package of forward contracts. We conduct the following analysis in the absence of default risk.

Valuation of Swaps as a Package of Cash Market Instruments

Assume for the purpose of analysis that the counterparties exchange the notional principal of $\mathcal{N}$ dollars at the end of the swap's life. It is then easy to see that a fixed-rate payer is long a floating-rate bond and short a fixed-rate bond. The value of the swap is therefore $P_2 - P_1$ from the fixed-rate payer's perspective, where P_1 (P_2, respectively) is the value of the fixed-rate (floating-rate, respectively) bond underlying the swap. The value of the swap is $P_1 - P_2$ for the floating-rate payer. As shown in Subsection 4.2.3, the floating leg should be priced at par immediately after a payment date, i.e., $P_2 = \mathcal{N}$, if the rate used for discounting the future cash flow is

	The bank			*The insurance company*		
	Loan	*Swap*	*Total*	*Investment/GIC*	*Swap*	*Total*
Inflow	10%	LIBOR	10% + LIBOR	LIBOR + 1.5%	8.4%	9.9% + LIBOR
Outflow	LIBOR + 0.5%	8.5%	9% + LIBOR	9%	LIBOR	9% + LIBOR
Spread			1%			0.9%

Figure 21.11: Locking in the spread with interest rate swaps.

the floating rate underlying the swap. For example, it was the LIBOR plus 1% for the swap in Fig. 21.9. Because the swap when first entered into has zero value, $P_1 = \mathcal{N}$.

Let the fixed-rate payments (C dollars each) and the floating-rate payments be made at times $t_1, t_2, \dots, t_n$ from now. By the above analysis, $P_1 = \sum_{i=1}^{n} Ce^{-r_i t_i} + \mathcal{N} e^{-r_n t_n}$, where r_i is the spot rate for time t_i. As for the floating-rate bond, $P_2 = (\mathcal{N} + C^*)\, e^{-r_1 t_1}$, where C^* is the known floating-rate payment to be made at the next payment time t_1.

EXAMPLE 21.4.2 A party agrees to pay the 6-month LIBOR plus 1% every 6 months and receive 9% annual interest rate paid every 6 months on a notional principal of \$10 million. All rates are compounded semiannually. Assume that the 6-month LIBOR rate at the last payment date was 9%. There are two more payment dates, at 0.3 and 0.8 years from now, and the relevant continuously compounded rates for discounting them are 10.1% and 10.3%, respectively. From these data,

$$P_1 = 0.45 \times e^{-0.101 \times 0.3} + 10.45 \times e^{-0.103 \times 0.8} = 10.0600 \text{ (million)},$$

$$P_2 = (10 + 0.5)\, e^{-0.101 \times 0.3} = 10.1866 \text{ (million)}.$$

The swap's value is hence $P_2 - P_1 = 0.1266$ (million) for the fixed-rate payer.

The duration of an interest rate swap from the perspective of the fixed-rate payer is

duration of floating-rate bond − duration of fixed-rate bond.

Most of the interest rate sensitivity of a swap results from the duration of the fixed-rate bond because the duration of the floating-rate bond is less than the time to the next reset date (see Subsection 4.2.3).

A party who is long a floating-rate bond and short a fixed-rate bond loses the principal and must continue servicing its fixed-rate debt if the floating-rate note issuer defaults. In contrast, the fixed-rate payer in a swap does not need to continue payment if the counterparty defaults. Hence the observation that a swap is equivalent to a portfolio of floating-rate and fixed-rate bonds holds only in the absence of a default risk.

➤ **Exercise 21.4.2** A firm buys a \$100 million par of a 3-year floating-rate bond that pays the 6-month LIBOR plus 0.5% every 6 months. It is financed by borrowing \$100 million for 3 years on terms requiring 10% annual interest rate paid every 6 months. Show that these transactions create a synthetic interest rate swap.

Valuation of Swaps as a Package of Forward Rate Agreements

Consider a swap with a notional principal of $\mathcal{N}$ dollars. Let the fixed-rate payments (C dollars each) and the floating-rate payments (based on future annual rates $f_1, f_2, \dots, f_n$) be made at times $t_1, t_2, \dots, t_n$ from now. The rate f_i is determined at t_{i-1}. At time t_i, C dollars is exchanged for $(f_i/k) \times \mathcal{N}$, where $k \equiv 1/(t_i - t_{i-1})$ is the payment frequency per annum and f_i is compounded k times annually (e.g., $k = 2$ for semiannual payments). For the fixed-rate payer, this swap is essentially a forward contract on the floating rate, say the 6-month LIBOR, whereby it agrees to pay C dollars in exchange for delivery of the 6-month LIBOR. Similarly, the floating-rate payer is essentially short a forward contract on the 6-month LIBOR. Hence an interest rate swap is equivalent to a package of FRAs.

From Eq. (12.7), the value of the forward contract to take delivery of the floating rate equals $[(f_i/k)\mathcal{N} - C]\, e^{-r_i t_i}$, with r_i denoting the time-t_i spot rate and f_i the forward rate. The first exchange at time t_1 has PV of $(C^* - C)\, e^{-r_1 t_1}$, where C^* is based on a floating rate currently known. The value of the swap is hence

$$(C^* - C)\, e^{-r_1 t_1} + \sum_{i=2}^{n} \left(\frac{f_i}{k} \mathcal{N} - C \right) e^{-r_i t_i} \tag{21.4}$$

for the fixed-rate payer. For the floating-rate payer, simply reverse the sign.

EXAMPLE 21.4.3 Consider the swap between B and a bank in Fig. 21.9. Party B receives the 6-month LIBOR plus 1% and pays 9%. All rates are compounded semiannually. Payments occur every 6 months on a notional principal of \$10 million. Assume the 6-month LIBOR rate at the last payment date was 9%. There are two more payment dates, at 0.3 and 0.8 years from now, and the relevant continuously compounded rates for discounting them – 6-month LIBOR plus 1% – are 10.1% and 10.3%. The annualized continuously compounded 6-month forward rate 0.3 year from now is $(0.8 \times 0.103 - 0.3 \times 0.101)/(0.8 - 0.3) = 0.1042$. It becomes $2 \times (e^{0.1042/2} - 1) = 0.106962$ under semiannual compounding. The swap value,

$$(0.5 - 0.45)\, e^{-0.101 \times 0.3} + \left(\frac{0.106962}{2} \times 10 - 0.45 \right) e^{-0.103 \times 0.8} = 0.1266 \text{ (million)}$$

by formula (21.4), is in complete agreement with Example 21.4.2.

Because formula (21.4) defines the value of a swap, it can also serve as its **replacement value** for the fixed-rate payer. In other words, if the floating-rate payer terminates the swap, this is the amount the fixed-rate payer would request for compensation so that the floating side could be replaced without increasing the fixed rate. Let V stand for the replacement value. Clearly if $V < 0$, then the counterparty would choose not to default on this agreement because it can be sold at a profit. The risk exposure is hence $\max(0, V)$.

As with other synthetic securities, interest rate swaps are not redundant even though it can be replicated by forward contracts. Several reasons have been cited [325]. First, they offer longer maturities than forward contracts. Second, interest rate swaps incur less transactions costs than a package of forward contracts that involve multiple transactions. Third, interest rate swaps are more liquid than forward contracts, particularly long-term forward contracts.

➤ **Exercise 21.4.3** Use Example 21.4.3's data to calculate the fixed rate that makes the swap value zero.

➤ **Exercise 21.4.4** Verify the equivalence of the two views on interest rate swaps.

➤ **Exercise 21.4.5** Consider a swap with zero value. How much up-front premium should the fixed-rate payer pay in order to lower the fixed-rate payment from C to $\widehat{C}$?

➤ **Exercise 21.4.6** Replicate swaps with interest rate caps and floors.

21.4.2 More Interest Rate Swaps

The number of different types of swaps is almost limitless. A **callable swap** allows the fixed-rate payer to terminate the swap at no penalty and prevents losses during declining rates. The premium may be amortized over the term of the swap. A **putable swap** allows the floating-rate payer to terminate the swap early. An investor can sell swaps short (**reverse swap**) in order to pay floating rates and receive fixed rates. This strategy gains under declining-rate environments and loses under rising-rate environments. A **basis swap** is a swap in which the two legs of the swap are tied to two different floating rates. A basis swap becomes a **yield-curve swap** if the two legs are based on short- and long-term interest rates, say the 6-month LIBOR and the 10-year Treasury yield. Such a swap can be used to control the exposure to changes in the yield-curve shape. In **deferred swaps** (or **forward swaps**), parties do not begin to exchange interest payments until some future date. In an **extendible swap**, one party has the option to extend the life of the swap beyond the specified period. A swap can be an agreement to exchange a fixed interest rate in one currency for a floating interest rate in another currency, in other words, a combination of a plain vanilla interest rate swap and a currency swap. In an **amortizing swap**, the principal is reduced in a way that corresponds to, say, the amortization schedule on a loan. In an **accreting swap**, the principal increases according to a schedule.

➤ **Exercise 21.4.7** A firm holds long-duration corporate bonds. It uses swaps to create synthetic floating-rate assets at attractive spreads to LIBOR and to shorten the duration much as A does in Fig. 21.9. Why should the fact that most corporate bonds are callable trouble the firm? How may callable swaps help?

Swaptions

A **swaption** is an option to enter into an interest rate swap. It is almost always European. The swap rate and the duration of the interest rate swap as measured from the option expiration date are specified in the contract. The market generally quotes on the fixed-rate part of the swap. So swaptions can be either **receiver swaptions** (the right to receive fixed and pay floating rates, or **floating-for-fixed**) or **payer swaptions** (the right to pay fixed and receive floating rates, or **fixed-for-floating**). The buyer of a receiver swaption benefits as interest rates fall, and the buyer of a payer swaption benefits as interest rates rise.

EXAMPLE 21.4.4 A firm plans to issue a 5-year floating-rate loan in 1 year and then convert it into a fixed-rate loan using interest rate swaps. To establish a floor for the swap rate, it purchases a 1-year swaption for the right to swap a fixed rate, say 8% per year, for a floating rate for a period of 5 years starting 1 year from now. If the fixed rate on a 5-year swap in 1 year's time turns out to be less than 8%, the company will enter into a swap agreement in the usual way. However, if it turns out to be greater than 8%, the company will exercise the swaption.

A fixed-for-floating interest rate swap can be regarded as an agreement to exchange a fixed-rate bond for a floating-rate bond. At the start of a swap, the value of the floating-rate bond equals the principal of the swap. A payer swaption can therefore be regarded as an option to exchange a fixed-rate bond for the principal of the swap, in other words, a put on the fixed-rate bond with the principal as the strike price. Similarly, a receiver swaption is a call on the fixed-rate bond with the principal as the strike price. When the Black model is used in pricing swaptions, the underlying asset is the forward rate for the interest rate swap [844].

➤ **Exercise 21.4.8** Prove that a cap is more valuable than an otherwise identical swaption.

Index-Amortizing Swaps

Amortizing swaps whose principal declines (amortizes) when interest rates decline are called **index-amortizing swaps** (**IAS**s). The principal is reduced by an amortizing schedule based on the spot interest rate. The amortizing schedule may not apply until after a lockout period. As mortgage prepayments usually pick up with declining rates, these instruments can partially hedge the prepayment risk of MBSs [848].

Formally, let T be the maturity of the swap with initial principal $\mathcal{N}$. The contract receives a fixed rate c and pays a floating rate $r(t)$. Let the lockout period be T^* years during which the principal is fixed at $\mathcal{N}$. For $t > T^*$, the remaining principal at time t changes according to

$$\mathcal{N}_t = \mathcal{N}_{t-1}(1 - a_t),$$

where $\mathcal{N}_{T^*} = \mathcal{N}$ and a_t is the amortizing amount. Hence $\mathcal{N}_t$ is $\mathcal{N}_{t-1}$ reduced by the amortizing schedule amount a_t. An amortizing schedule may look like

$$a_t = \begin{cases} 0 & \text{if } r(t) > k_0 \\ b_0 & \text{if } k_0 \geq r(t) > k_1 \\ b_1 & \text{if } k_1 \geq r(t) > k_2 \\ b_2 & \text{if } k_2 \geq r(t) > k_3 \\ b_3 & \text{if } k_3 \geq r(t) > k_4 \\ b_4 & \text{if } k_4 \geq r(t) > k_5 \\ 1 & \text{if } k_5 \geq r(t) \end{cases},$$

where $k_0 > k_1 > \cdots > k_5$ and $b_0 < b_1 < \cdots < b_4 < 1$ are positive constants. The preceding amortizing schedule depends on the interest rate $r(t)$. If the rate is larger than k_0, no reduction in principal occurs and $a_t = 0$; if it lies between k_0 and k_1, a reduction of b_0 occurs; if it lies between k_1 and k_2, a reduction of b_1 occurs, and so on.

The time-t cash flow to the IAS can be written as $[\, c - r(t-1)\,]\mathcal{N}_{t-1}$, which is determined at time $t-1$. Clearly the principal $\mathcal{N}_j$ depends on the whole interest rate path before time j, making the IAS a path-dependent derivative. An efficient algorithm will be presented in Programming Assignment 29.1.3.

Differential Swaps

A **differential swap** is an interest rate swap in which the interest rates for the two legs are linked to different currencies and the actual interest payments are denominated in the same currency by fixed exchange rates. For example, consider a swap with a dollar-based interest rate x and a DEM-based interest rate y. Both payments are to be denominated in U.S. dollars. Let the \$/DEM exchange rate be fixed at $\widehat{s}$. Then the settlement amount is $(y\widehat{s} - x)\mathcal{N}$ dollars. Clearly, differential swaps are a type of quanto derivative, thus the alternative name **quanto swaps**.

Additional Reading

For more information on interest rate derivatives, consult [95, 155, 325, 470, 746, 827, 837] for interest rate futures, [397, 538] for interest rate options, [54, 369, 474, 514, 608, 746, 821] for interest rate swaps, [449, 510, 792] for IASs, and [873] for differential swaps. The duration of T-bond futures is discussed in [554, 737]. See [95, p. 267] for the origin of Eurodollars. Finally, see [175] for the pricing of **equity swaps**.

NOTES

1. The conversion factor is independent of the prevailing interest rates. The theoretically sounder conversion ratio should be the price of the delivered bond divided by the price of the 20-year 8% coupon bond, both discounted at the prevailing interest rates. View www.cbot.com/ourproducts/financial/convbond.html for the CBT's regularly updated table of conversion factors.
2. Bond futures prices are in general lower than bond forward prices (see Exercise 12.3.3).