CHAPTER TEN

Sensitivity Analysis of Options

Cleopatra's nose, had it been shorter, the whole face of the world would have been changed.

Blaise Pascal (1623–1662)

Understanding how the value of a security changes relative to changes in a given parameter is key to hedging. Duration, for instance, measures the rate of change of bond value with respect to interest rate changes. This chapter asks similar questions of options.

10.1 Sensitivity Measures ("The Greeks")

In the following, $x \equiv [\, \ln(S/X) + (r + \sigma^2/2)\,\tau\,]/(\sigma\sqrt{\tau})$, as in the Black–Scholes formula of Theorem 9.3.4, and $N'(y) = (1/\sqrt{2\pi})\, e^{-y^2/2} > 0$ is the density function of the standard normal distribution.

10.1.1 Delta

For a derivative such as option, **delta** is defined as $\Delta \equiv \partial f/\partial S$, where f is the price of the derivative and S is the price of the underlying asset. The delta of a portfolio of derivatives on the same underlying asset is the sum of the deltas of individual derivatives. The delta used in the BOPM to replicate options is the discrete analog of the delta here. The delta of a European call on a non-dividend-paying stock equals

$$\frac{\partial C}{\partial S} = N(x) > 0, \tag{10.1}$$

and the delta of a European put equals $\partial P/\partial S = N(x) - 1 < 0$. See Fig. 10.1 for an illustration. The delta of a long stock is of course one.

A position with a total delta equal to zero is said to be **delta-neutral**. Because a delta-neutral portfolio is immune to small price changes in the underlying asset, creating it can serve for hedging purposes. For example, a portfolio consisting of a call and $-\Delta$ shares of stock is delta-neutral. So one can short Δ shares of stock to hedge a long call. In general, one can hedge a long position in a derivative with a delta of Δ_1 by shorting Δ_1/Δ_2 units of another derivative with a delta of Δ_2.

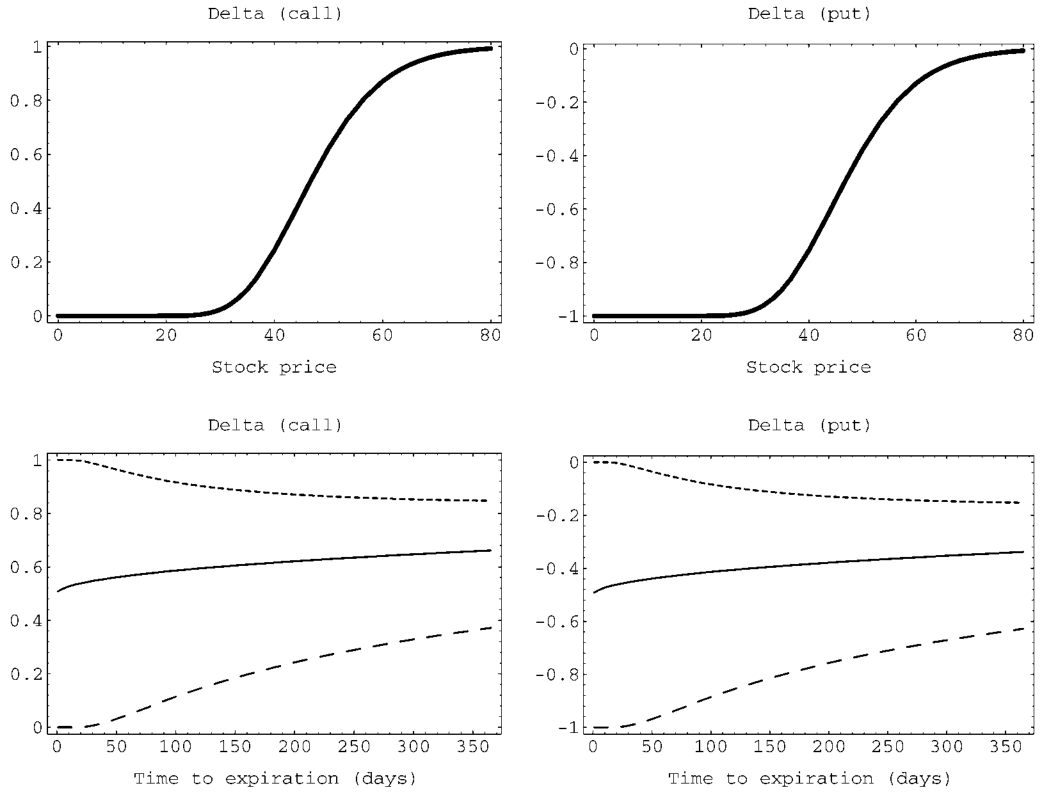

Figure 10.1: Option delta. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted curves use $S = 60$ (in-the-money call or out-of-the-money put), the solid curves use $S = 50$ (at-the-money option), and the dashed curves use $S = 40$ (out-of-the-money call or in-the-money put).

➤ **Exercise 10.1.1** Verify Eq. (10.1) and that the delta of a call on a stock paying a continuous dividend yield of q is $e^{-q\tau} N(x)$.

➤ **Exercise 10.1.2** Prove that $\partial P / \partial X = e^{-r\tau} N(-x + \sigma\sqrt{\tau})$.

➤ **Exercise 10.1.3** Show that at-the-money options have the maximum time value.

➤ **Exercise 10.1.4** What is the **charm**, defined as $\partial \Delta / \partial \tau$, of a European option?

10.1.2 Theta

Theta, or **time decay**, is defined as the rate of change of a security's value with respect to time, or $\Theta \equiv -\partial \Pi / \partial \tau$, where Π is the value of the security. For a European call on a non-dividend-paying stock,

$$\Theta = -\frac{S N'(x)\, \sigma}{2\sqrt{\tau}} - r X e^{-r\tau} N(x - \sigma\sqrt{\tau}) < 0.$$

The call hence loses value with the passage of time. For a European put,

$$\Theta = -\frac{S N'(x)\, \sigma}{2\sqrt{\tau}} + r X e^{-r\tau} N(-x + \sigma\sqrt{\tau}),$$

which may be negative or positive. See Fig. 10.2 for an illustration.

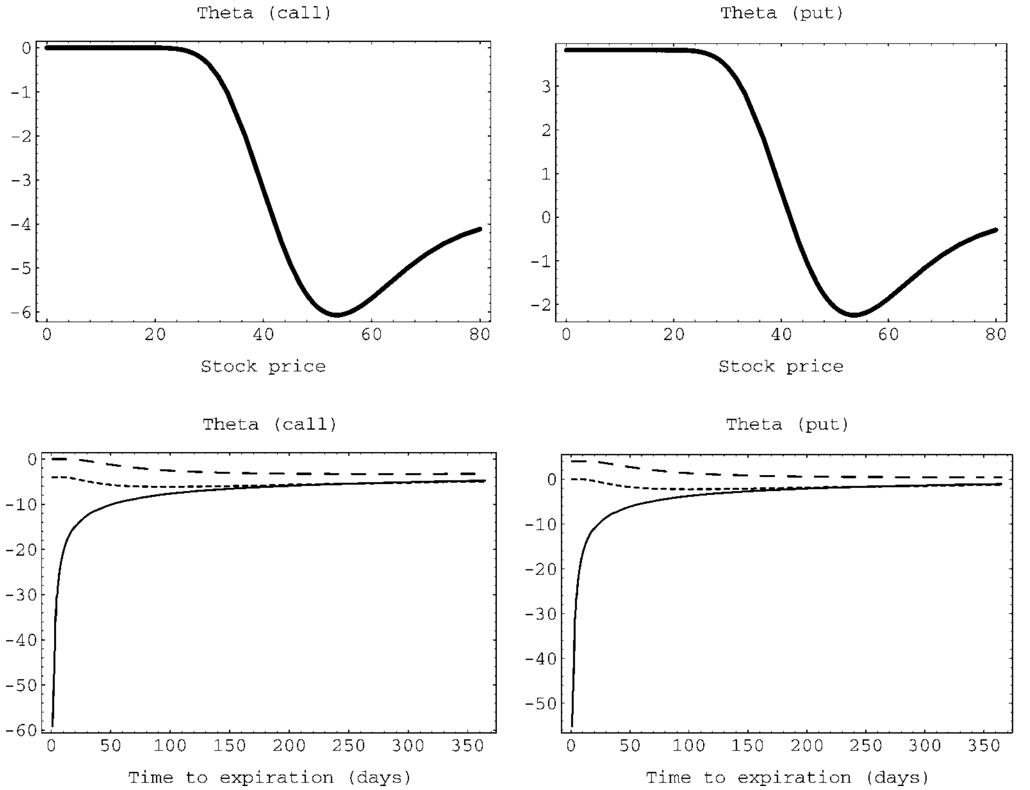

Figure 10.2: Option theta. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted curves uses $S = 60$ (in-the-money call or out-of-the-money put), the solid curves use $S = 50$ (at-the-money option), and the dashed curves use $S = 40$ (out-of-the-money call or in-the-money put).

➤ **Exercise 10.1.5** (1) At what stock price is the theta of a European call smallest? (2) Show that the theta of an American put is always negative.

10.1.3 Gamma

The **gamma** of a security is the rate of change of its delta with respect to the price of the underlying asset, or $\Gamma \equiv \partial^2 \Pi / \partial S^2$. The gamma measures how sensitive the delta is to changes in the price of the underlying asset. A portfolio with a high gamma needs in practice be rebalanced more often to maintain delta neutrality. The delta and the gamma have obvious counterparts in bonds: duration and convexity. The gamma of a European call or put on a non-dividend-paying stock is $N'(x)/(S\sigma\sqrt{\tau}) > 0$. See Fig. 10.3 for an illustration.

10.1.4 Vega

Volatility often changes over time. The **vega**[1] (sometimes called **lambda**, **kappa**, or **sigma**) of a derivative is the rate of change of its value with respect to the volatility of the underlying asset, or $\Lambda \equiv \partial \Pi / \partial \sigma$. A security with a high vega is very sensitive to small changes in volatility. The vega of a European call or put on a non-dividend-paying stock is $S\sqrt{\tau}\, N'(x) > 0$, which incidentally solves Exercise 9.4.1. A positive

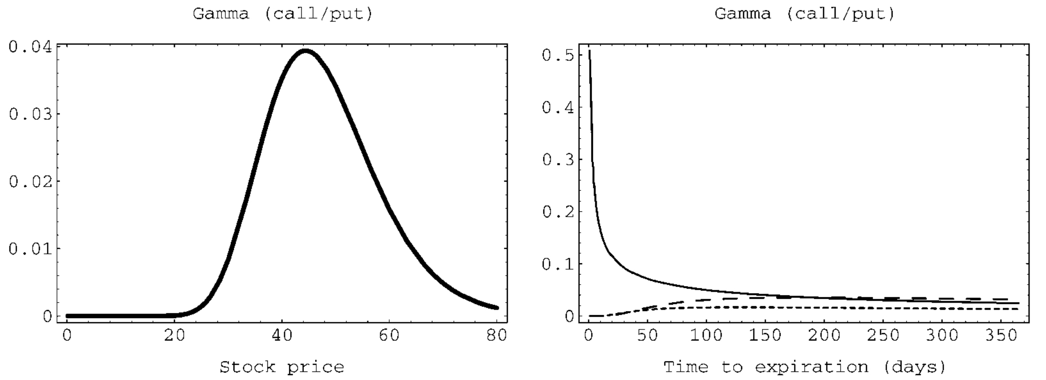

Figure 10.3: Option gamma. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted curve uses $S = 60$ (in-the-money call or out-of-the-money put), the solid curves use $S = 50$ (at-the-money option), and the dashed curve uses $S = 40$ (out-of-the-money call or in-the-money put).

vega is consistent with the intuition that higher volatility increases option value. See Fig. 10.4 for an illustration.

➤ **Exercise 10.1.6** Prove that the vega as a function of σ is unimodal for $\sigma > 0$. A function is **unimodal** if it is first increasing and then decreasing, thus having a single peak.

10.1.5 Rho

The **rho** of a derivative is the rate of change in its value with respect to interest rates, or $\rho \equiv \partial \Pi / \partial r$. The rhos of a European call and a European put on a non-dividend-paying stock are $X\tau e^{-r\tau} N(x - \sigma\sqrt{\tau}) > 0$ and $-X\tau e^{-r\tau} N(-x + \sigma\sqrt{\tau}) < 0$, respectively. See Fig. 10.5 for an illustration.

➤ **Exercise 10.1.7** (1) What is the **speed**, defined as $\partial \Gamma / \partial S$, of a European option? (2) What is the **color**, defined as $\partial \Gamma / \partial \tau$, of a European option?

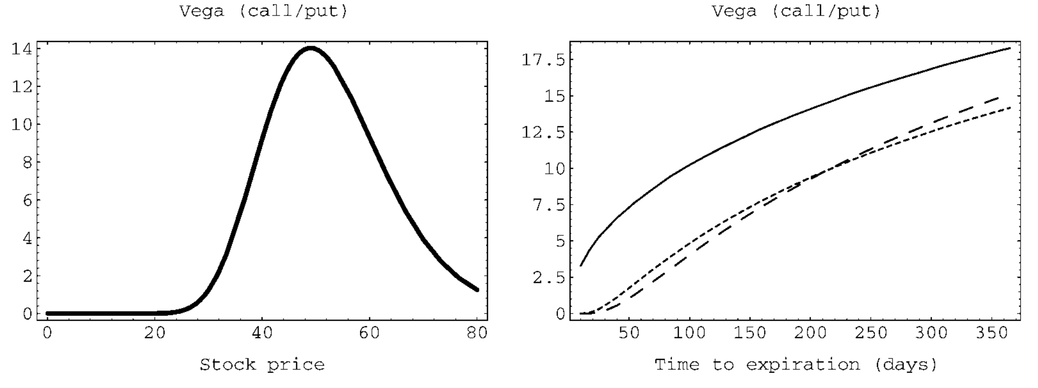

Figure 10.4: Option vega. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted curve uses $S = 60$ (in-the-money call or out-of-the-money put), the solid curves use $S = 50$ (at-the-money option), and the dashed curve uses $S = 40$ (out-of-the-money call or in-the-money put).

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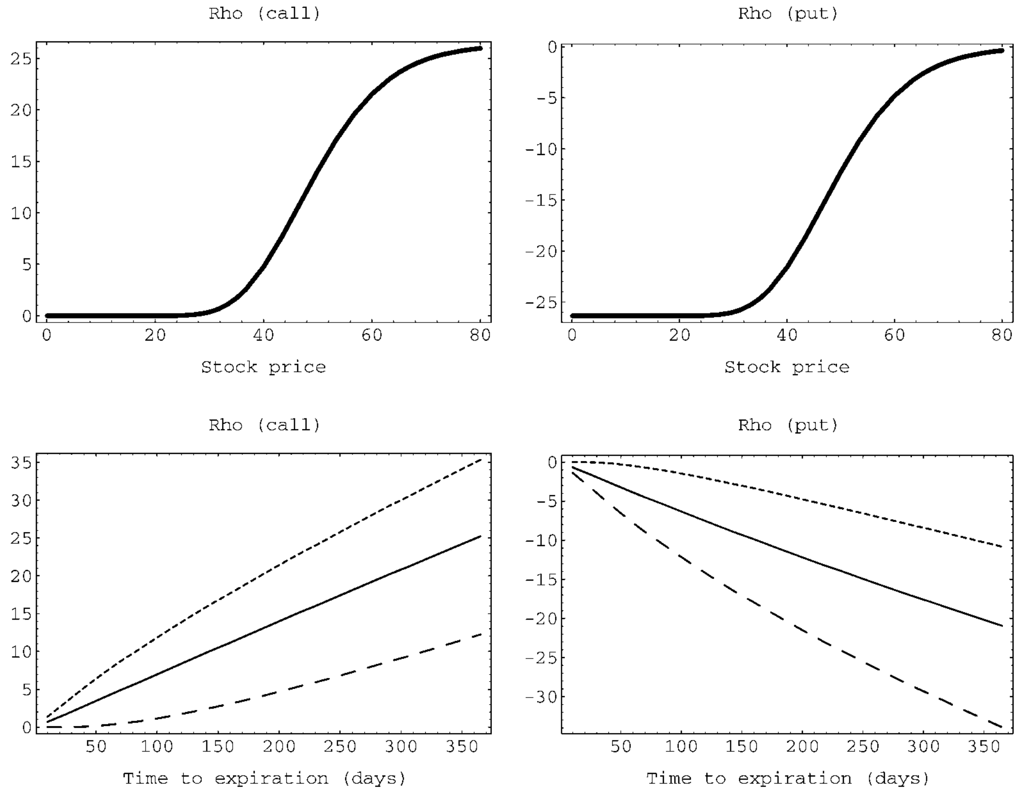

Figure 10.5: Option rho. The default parameters are $S = 50$, $X = 50$, $\tau = 201$ (days), $\sigma = 0.3$, and $r = 8\%$. The dotted curves use $S = 60$ (in-the-money call or out-of-the-money put), the solid curves use $S = 50$ (at-the-money option), and the dashed curves use $S = 40$ (out-of-the-money call or in-the-money put).

10.2 Numerical Techniques

Sensitivity measures of derivatives for which closed-form formulas do not exist have to be computed numerically. Take delta as an example. It is defined as $\Delta f/\Delta S$, where ΔS is a small change in the stock price and Δf is the resulting change in the derivative's price. A standard method computes $f(S-\Delta S)$ and $f(S+\Delta S)$ and settles for

$$\frac{f(S+\Delta S)-f(S-\Delta S)}{2\Delta S}.$$

The computation time for this numerical differentiation scheme roughly doubles that for evaluating the derivative security itself.

A preferred approach is to take advantage of the intermediate results of the binomial tree algorithm. When the algorithm reaches the end of the first period, f_u and f_d are computed. Recall that these values correspond to derivative values at stock prices Su and Sd, respectively. Delta is then approximated by

$$\frac{f_u - f_d}{Su - Sd}.$$

Binomial tree algorithm for the delta of American puts on a non-dividend-paying stock:

input: $S, u, d, X, n, \widehat{r}\,(u > e^{\widehat{r}} > d$ and $\widehat{r} > 0)$;
real $R, p, P\,[\,n+1\,]$;
integer i, j;
$R := e^{\widehat{r}}$;
$p := (R - d)/(u - d)$;
for $(i = 0$ to $n)$ $\{P\,[\,i\,] := \max\left(0,\, X - Su^{n-i} d^{i}\right); \}$
for $(j = n - 1$ down to $1)$
    for $(i = 0$ to $j)$
        $P[\,i\,] := \max((p \times P[\,i\,] + (1 - p) \times P[\,i + 1\,])/R,\, X - Su^{j-i} d^{i})$;
return $(P[\,0\,] - P[\,1\,])/(Su - Sd)$;

Figure 10.6: Binomial tree algorithm for the delta of American puts on a non-dividend-paying stock. Adapted from Fig. 9.13.

The extra computational effort beyond the original binomial tree algorithm is essentially nil. See Fig. 10.6 for an algorithm.

Other sensitivity measures can be similarly derived. Take gamma. At the stock price $(Suu + Sud)/2$, delta is approximately $(f_{uu} - f_{ud})/(Suu - Sud)$, and at the stock price $(Sud + Sdd)/2$, delta is approximately $(f_{ud} - f_{dd})/(Sud - Sdd)$. Gamma is the rate of change in deltas between $(Suu + Sud)/2$ and $(Sud + Sdd)/2$, that is,

$$\frac{\frac{f_{uu} - f_{ud}}{Suu - Sud} - \frac{f_{ud} - f_{dd}}{Sud - Sdd}}{(Suu - Sdd)/2}. \tag{10.2}$$

In contrast, numerical differentiation gives

$$\frac{f(S + \Delta S) - 2f(S) + f(S - \Delta S)}{(\Delta S)^2}.$$

As we shall see shortly, numerical differentiation may give inaccurate results.

Strictly speaking, the delta and the gamma thus computed are the delta at the end of the first period and the gamma at the end of the second period. In other words, they are not the sensitivity measures at the present time but at times τ/n and $2(\tau/n)$ from now, respectively, where n denotes the number of periods into which the time to expiration τ is partitioned. However, as n increases, such values should approximate delta and gamma well. The theta, similarly, can be computed as

$$\frac{f_{ud} - f}{2(\tau/n)}.$$

As for vega and rho, there is no alternative but to run the binomial tree algorithm twice. In Eq. (15.3), theta will be shown to be computable from delta and gamma.

10.2.1 Why Numerical Differentiation Fails

A careful inspection of Eq. (9.8) reveals why numerical differentiation fails for European options. First, the option value is a continuous piecewise linear function of the current stock price S. Kinks develop at prices $Xu^{-j} d^{-(n-j)}$, $j = 0, 1, \ldots, n$. As a result, if ΔS is suitably small, the delta computed by numerical differentiation will be a ladderlike function of S, hence not differentiable at the kinks. This bodes ill for

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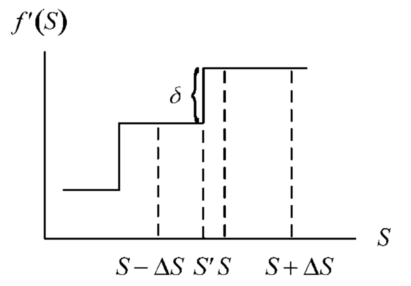

Figure 10.7: Numerical differentiation for delta and gamma.

numerical gamma. In fact, if ΔS is suitably small, gamma computed through numerical differentiation will be zero most of the time because $f'(S-\Delta S) = f'(S) = f'(S+\Delta S)$ unless S is near a kink. However, another problem arises when S is near a kink. Assume that S is to the right of the kink at S' and $S - \Delta S < S' < S$. Hence $f'(S) = f'(S+\Delta S)$ and $f'(S) - f'(S-\Delta S) = \delta$ for some constant $\delta > 0$ (see Fig. 10.2.1). Numerical gamma now equals

$$\frac{f(S+\Delta S) - 2f(S) + f(S-\Delta S)}{(\Delta S)^2} = \frac{\delta(S' - S + \Delta S)}{(\Delta S)^2}.$$

This number can become huge as ΔS decreases, and the common practice of reducing the step size ΔS will not help.

➤ **Exercise 10.2.1** Why does the numerical gamma in definition (10.2) not fail for the same reason?

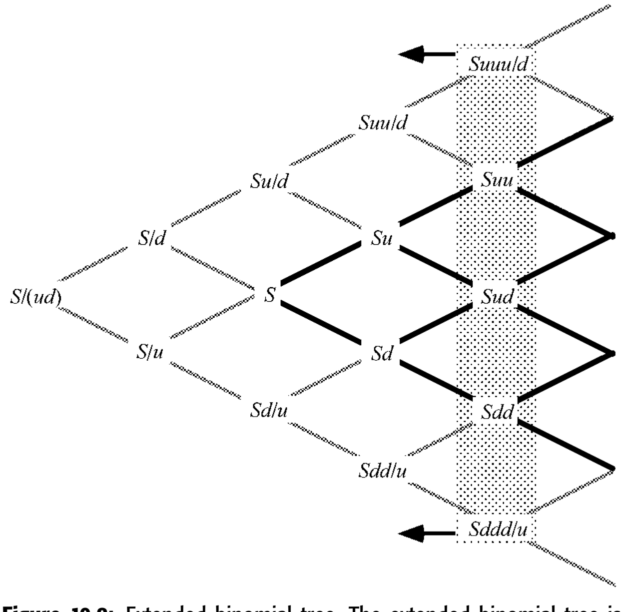

Figure 10.8: Extended binomial tree. The extended binomial tree is constructed from the original binomial tree (bold lines) but with time extended beyond the present by two periods.

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10.2.2 Extended Binomial Tree Algorithms

It is recommended that delta and gamma be computed on the binomial tree, not at the current time, but one and two periods from now, respectively [718]. An improved method starts the binomial tree two periods *before* now, as in Fig. 10.8. Delta is then computed as

$$\frac{f_{u/d} - f_{d/u}}{(Su/d) - (Sd/u)},$$

and gamma is computed as

$$\frac{\frac{f_{u/d} - f}{(Su/d) - S} - \frac{f - f_{d/u}}{S - (Sd/u)}}{[\,(Su/d) - (Sd/u)\,]/2}.$$

➢ **Programming Assignment 10.2.2** Implement the extended binomial tree algorithm for numerical delta and gamma. Compare the results against numerical differentiation and closed-form solutions.

NOTE

1. Vega is not Greek.