CHAPTER TWELVE

Forwards, Futures, Futures Options, Swaps

It does not matter if I speak; the future has already been determined.
Sophocles (496 B.C.–406 B.C.), *Oedipus Tyrannus*

This chapter continues the coverage of derivatives, financial contracts whose value depends on the value of some underlying assets or indices. Derivatives are essential to risk management, speculation, efficient portfolio adjustment, and arbitrage. Interest-rate-sensitive derivative securities require a separate chapter, Chap. 21.

12.1 Introduction

Many financial institutions take large positions in derivatives. For example, Chase Manhattan held U.S.$7.6 trillion (notional amount) in derivatives as of early 1998 [57]. Trading derivatives can be risky, however. The loss of U.S.$1.6 billion in the case of Orange County, California, led to its bankruptcy in December 1994 and the securities firms involved paying U.S.$739 million in subsequent settlements [527]; the value of an interest rate swap held by Sears in 1997 was a minus U.S.$382 million based on a notional principal of U.S.$996 million [884]; J.P. Morgan in 1997 had U.S.$659 million in nonperforming assets, 90% of which were defaults from Asian counterparties;[1] Union Bank of Switzerland (UBS) wrote off U.S.$699 million because of investment in the hedge fund, Long-Term Capital Management (LTCM).

Four types of derivatives stand out: futures contracts, forward contracts, options, and swaps. Futures contracts and forward contracts are contracts for future delivery of the underlying asset. The underlying asset can be a physical commodity (corn, oil, live cattle, pork bellies, precious metals, and so on) or financial instrument (bonds, currencies, stock indices, mortgage securities, other derivatives, and so on) [95, 470]. **Futures contracts** are essentially standardized forward contracts that trade on futures exchanges such as the Chicago Board of Trade (CBT) and the CME.[2] Futures and forward contracts can be used for speculation, hedging, or arbitrage between the spot and the deferred-delivery markets.

Futures and forward contracts are obligations on both the buyer and the seller. Options, we recall, are binding on only the seller. The buyer has the right, but not the obligation, to take a position in the underlying asset. Such a right naturally commands a premium. Options can be used to hedge downside risk, speculation, or arbitrage markets.

A **swap** is a contract between two parties to exchange cash flows in the future based on a formula. Typically, one party pays a fixed price to the other party in exchange for a market-determined floating price. Swaps can be used to reduce financing costs or to hedge. In reality, interest rate swaps and forex forward contracts make up banks' major derivatives holdings [60]. Like the forward contracts, swaps are traded by financial institutions and their corporate clients outside of organized exchanges.

The relevant riskless interest rate for many arbitragers operating in the futures market is the repo rate. A **repo** (**sale and repurchase agreement** or **RP**) is an agreement in which the owner of securities ("seller") agrees to sell them to a counterparty ("buyer") and buy them back at a slightly higher price later. The counterparty hence provides a loan. However, this loan has little risk as the lender keeps the securities if the seller fails to buy them back. (The lender essentially runs a pawnshop.) From the lender's perspective, this agreement is called a **reverse repo**. Overnight repo rates are lower than the federal funds rate. A loan of more than 1 day is called a **term repo**. The dollar interest is determined by

$$\text{dollar principal} \times \text{repo rate} \times \frac{\text{repo term}}{360}.$$

The Bank of England was the first central bank to introduce repos in 1830. The Federal Reserve uses the the repo market to influence short-term interest rates. When the Federal Reserve is doing repo, it is actually lending money, not borrowing it [827].

Throughout this chapter, r denotes the riskless interest rate. Other notations include, unless stated otherwise, the current time t, the maturity date of the derivative T, the remaining time to maturity $\tau \equiv T - t$ (all measured in years), the spot price S, the spot price at maturity S_T, the delivery price X, the forward or futures price F for a newly written contract, and the value of the contract f. A price with a subscript t usually refers to the price at time t. Continuous compounding will be assumed throughout this chapter.

12.2 Forward Contracts

Forward contracts are for delivery of the underlying asset for a certain **delivery price** on a specific time in the future. They are ideal for hedging purposes. Consider a corn farmer who enters into a forward contract with a food processor to deliver 100,000 bushels of corn for \$2.5 per bushel on September 27, 1995. Assume that the cost of growing corn is \$2.0 per bushel. Such a contract benefits both sides: the farmer, because he is assured of a buyer at an acceptable price, and the processor, because knowing the cost of corn in advance helps reduce uncertainty in planning. If the spot price of corn rises on the delivery date, the farmer will miss the opportunity of extra profits. On the other hand, if the price declines, the processor will be paying more than it would. A forward agreement hence limits both the risk and the potential rewards.

Problems may arise if one of the participants fails to perform. The food processor may go bankrupt, the farmer can go bust, the farmer might not be able to harvest 100,000 bushels of corn because of bad weather, or the cost of growing corn may skyrocket. More importantly, whichever way the corn price moves, either the food processor or the farmer has an incentive to default. Even corporate giants default on their forward contracts [767].

EXCHANGE RATES
Monday, March 20, 1995

	U.S.$ equiv.		Currency per U.S.$	
Country	*Mon.*	*Fri.*	*Mon.*	*Fri.*
Germany (Mark)	.7126	.7215	1.4033	1.3860
30-Day Forward	.7133	.7226	1.4019	1.3839
90-Day Forward	.7147	.7242	1.3991	1.3808
180-Day Forward	.7171	.7265	1.3945	1.3765

Figure 12.1: German mark exchange rate quotations. The forward German marks are at a premium to the spot mark: The forward exchange rates in terms of $/DEM exceed the spot exchange rate, perhaps because of lower inflation in Germany. Source: *Wall Street Journal*, March 21, 1995.

12.2.1 Forward Exchange Rate

Along with forex options, forward contracts provide an avenue to hedging currency risk. Figure 12.1 shows the spot exchange rate and forward exchange rates for the German mark. Consider a U.S. company that is expecting to receive DEM10 million in 3 months' time. By using a forward sale at the 3-month forward exchange rate of $.7147/DEM1, the firm will receive exactly U.S.$7,147,000 in 3 months' time. Compared with hedging by use of forex options, the forward hedge insulates the firm from any movements in exchange rates whether they are favorable or not.

➤ **Exercise 12.2.1** Selling forward DEM10 million as in the text denies the hedger the profits if the German mark appreciates. Consider the "60:40" strategy, whereby only 60% of the German marks are sold forward with the remaining unhedged. Derive the payoff function in terms of the $/DEM exchange rate 3 months from now.

Spot and Forward Exchange Rates

Let S denote the spot exchange rate and F the forward exchange rate 1 year from now (both in domestic/foreign terms). Use r_{f} and r_l to denote the annual interest rates of the foreign currency and the local currency, respectively. First formulated by Keynes[3] in 1923 [303], arbitrage opportunities will arise unless these four numbers satisfy a definite relation known as the **interest rate parity**:

$$\frac{F}{S} = \frac{e^{r_\ell}}{e^{r_{\mathrm{f}}}} = e^{r_\ell - r_{\mathrm{f}}}. \tag{12.1}$$

Here is the argument. A holder of the local currency can either (1) lend the money in the domestic market to receive e^{r_ℓ} one year from now or (2) convert the local currency in the spot market for the foreign currency, lend for 1 year in the foreign market, and convert the foreign currency into the local currency at the fixed forward exchange rate in the future, F, by selling forward the foreign currency now. As usual, no money changes hand in entering into a forward contract. One unit of local currency will hence become $Fe^{r_{\mathrm{f}}}/S$ 1 year from now in this case. If $Fe^{r_{\mathrm{f}}}/S > e^{r_\ell}$, an arbitrage profit can result from borrowing money in the domestic market and lending it in the foreign market. Conversely, if $Fe^{r_{\mathrm{f}}}/S < e^{r_\ell}$, an arbitrage profit can result from borrowing money in the foreign market and lending it in the domestic market. We conclude that $Fe^{r_f}/S = e^{r_\ell}$. It is straightforward to check that

$$\frac{F}{S} = \frac{1 + r_\ell}{1 + r_f} \tag{12.2}$$

under periodic compounding.

The interest rate parity says that if the domestic interest rate is higher than the foreign rate, the foreign country's currency will be selling at a premium in the forward market. Conversely, if the domestic interest rate is lower, the foreign currency will be selling at a discount in the forward market.

➤ **Exercise 12.2.2** (1) What does the table in Fig. 12.1 say about the relative interest rates between the United States and Germany? (2) Estimate German interest rates from Fig. 12.1 and Eq. (12.2) if the annualized U.S. rates for 30-day, 90-day, and 180-day T-bills are 5.66%, 5.9%, and 6.15%, respectively.

➤ **Exercise 12.2.3** Show that Eq. (11.6) can be simplified to

$$C = Fe^{-r\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}),$$
$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Fe^{-r\tau} N(-x)$$

without the explicit appearance of the exchange rate, where

$$x \equiv \frac{\ln(F/X) + (\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}.$$

Note that S, F, and X above are in domestic/foreign terms.

12.2.2 Forward Price

The payoff from holding a forward contract at maturity is $S_T - X$ (see Fig. 12.2). Contrast this with the European call's $\max(S_T - X, 0)$. Forward contracts do not involve any initial cash flow. The **forward price** is the delivery price that makes the forward contract zero valued; in other words, $f = 0$ when $F = X$. The delivery price cannot change because it is written in the contract, but the forward price may change after the contract comes into existence. In other words, the value of a forward contract, f, is zero at the outset, and it will fluctuate with the spot price thereafter. Apparently this value is enhanced when the spot price climbs and depressed when the spot price declines. The forward price also varies with the maturity of the contract, which can be verified by the data in Fig. 12.1.

For example, a repo is a forward contract on a Treasury security. It has a zero value initially because the Treasury security is exchanged for its fair value in cash and the repurchase price is set to the forward price [510].

The Underlying Asset Pays No Income

LEMMA 12.2.1 *For a forward contract on an underlying asset providing no income,*

$$F = Se^{r\tau}. \tag{12.3}$$

Proof: If $F > Se^{r\tau}$, an investor can borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F. At maturity, the asset

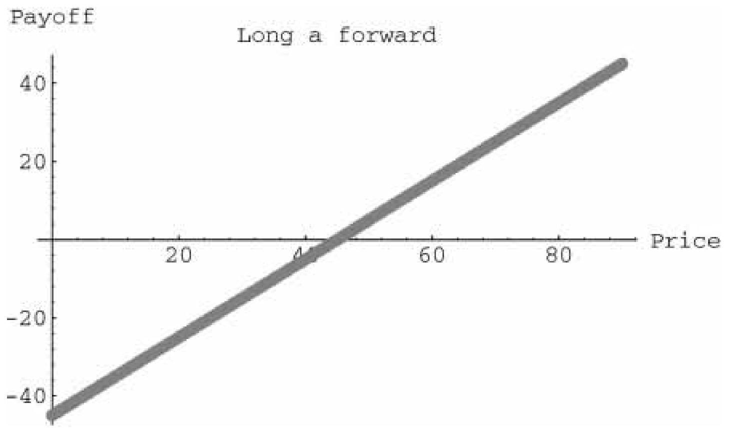

Figure 12.2: Payoff of forward contract. Shown is the payoff of a long forward contract with a delivery price of \$45 at maturity.

is sold for F and $Se^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - Se^{r\tau} > 0$.

If $F < Se^{r\tau}$ instead, an investor can short the underlying asset, invest the proceeds for τ years, and take a long position in the forward contract with delivery price F. At maturity, the asset is bought for F to close out the short position, leaving a profit of $Se^{r\tau} - F$.

EXAMPLE 12.2.1 A new 3-month forward contract on a 6-month zero-coupon bond should command a delivery price of $Se^{r/4}$, where r is the annualized 3-month riskless interest rate and S is the spot price of the bond. For instance, if $r = 6\%$ and $S = 970.87$, then the delivery price is $970.87 \times e^{0.06/4} = 985.54$.

The forward price, as previously mentioned, may not maintain equality to the delivery price as time passes. In fact, the value of a forward contract providing no income at any time before maturity should be

$$f = S - Xe^{-r\tau}.$$

We can verify this by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and one short position in the underlying asset. The cash will grow to X at maturity, which can be used to take delivery of the forward contract. The delivered asset will then close out the short position. Because the value of the portfolio is zero at maturity, its PV must be zero. Lemma 12.2.1 can be proved alternatively by the preceding identity because X must equal $Se^{r\tau}$ in order for $f = 0$.

➤ **Exercise 12.2.4** (1) Prove that a newly written forward contract is equivalent to a portfolio of one long European call and one short European put on the same underlying asset and expiration date with a common strike price equal to the forward price. (An option is said to be **at the money forward** if $X = F$.) (2) Prove the alternative put–call parity that says $C = P$ for the call and put in (1). (3) Derive Lemma 12.2.1 from the put–call parity. (4) Verify that, by substituting the forward price in Eq. (12.3) for the strike price X in the Black–Scholes formula of Theorem 9.3.4, we obtain $C = P$.

The Underlying Asset Pays Predictable Income

LEMMA 12.2.3 *For a forward contract on an underlying asset providing a predictable income with a present value of I,*

$$F = (S - I)\, e^{r\tau}. \tag{12.4}$$

Proof: If $F > (S - I)\, e^{r\tau}$, an investor can borrow S dollars for τ years, buy the underlying asset, and short the forward contract with delivery price F. At maturity, the asset is sold for F, and $(S - I)\, e^{r\tau}$ is used to repay the loan, leaving an arbitrage profit of $F - (S - I)\, e^{r\tau} > 0$. If $F < (S - I)\, e^{r\tau}$, an investor can short the underlying asset, invest the proceeds for τ years, and take a long position in the forward contract with delivery price F. At maturity, the asset is bought for F to close out the short position, and a profit of $(S - I)\, e^{r\tau} - F > 0$ is realized.

EXAMPLE 12.2.4 The above results can be extended to nonflat yield curves. A 10-month forward contract on a $50 stock that pays a dividend of $1 every 3 months has

$$\left(50 - e^{-r_3/4} - e^{-r_6/2} - e^{-3\times r_9/4}\right) e^{r_{10}\times(10/12)}$$

as the forward price, where r_i is the annualized i-month interest rate.

The value of a forward contract providing a predictable income with a PV of I is

$$f = (S - I) - Xe^{-r\tau}.$$

We can confirm this by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau} + I$, and one short position in the underlying asset. The cash will grow to X at maturity after paying the dividends to the original stockholder. There is a sufficient fund to take delivery of the forward contract, which then offsets the short position. Because the value of the portfolio is zero at maturity, its value must be zero at present, or $f - (S - I) + Xe^{-r\tau} = 0$.

The Underlying Asset Pays a Continuous Dividend Yield

A continuous dividend yield means that dividends are paid out continuously at an annual rate of q. The value of a forward contract at any time before maturity must equal

$$f = Se^{-q\tau} - Xe^{-r\tau}. \tag{12.5}$$

We can verify this by considering a portfolio of one long forward contract, cash amount $Xe^{-r\tau}$, and a short position in $e^{-q\tau}$ units of the underlying asset. All dividends are paid for by shorting additional units of the underlying asset. Hence the cash will grow to X at maturity, and the short position will grow to exactly one unit of the underlying asset. There is a sufficient fund to take delivery of the forward contract, which then offsets the short position. Because the value of the portfolio is zero at maturity, its PV must be zero. One consequence of Eq. (12.5) is that the forward price is

$$F = Se^{(r-q)\,\tau}. \tag{12.6}$$

➤ **Exercise 12.2.5** All the above cases satisfy the following relation:

$$f = (F - X)\, e^{-r\tau}. \tag{12.7}$$

Prove the preceding identity by an arbitrage argument.

12.3 Futures Contracts

Futures contracts are different from forward contracts in several ways. First, they are traded on a central exchange rather than on over-the-counter markets, leading to a more efficient and accurate price determination. Second, the establishment of a clearinghouse means that sellers and buyers do not face each other. As in the options market, the clearinghouse acts as a seller to all buyers and a buyer to all sellers. Credit risk inherent in forward contracts is hence minimized. Third, futures contracts are standardized instruments. They specify the delivery of a specific quantity of a specific commodity that meets quality standards at predetermined places and dates. This is in sharp contrast with forward contracts for which the only requirement is mutual agreement. Finally, gains and losses of futures contracts are **marked to market** daily. Hence the account is adjusted at the end of each trading day based on the **settlement price** to reflect the investor's gain or loss. The settlement price is the average of the prices at which the contract is traded immediately before the bell signaling the end of trading for the day.

EXAMPLE 12.3.1 The CBT July wheat contract specifies, among other things, that the wheat delivered be 5,000 bushels of no. 2 soft red wheat, no. 2 hard red winter wheat, no. 2 dark northern spring wheat, or no. 1 northern spring wheat on a date in the month of July chosen by the seller [95, 799].

The **contract size**, or simply the **size**, of a futures contract is the amount of the underlying asset to be delivered under the contract. For instance, it is 5,000 bushels for the corn futures contracts on the CBT and 1 million U.S. dollars for the Eurodollar futures contracts on the CME. A position can be **closed out** or **offset** by entering into a reversing trade to the original one. An investor who is long one November soybeans futures contract can close out the position by shorting one November contract, for example. A clearinghouse simply cancels offsetting positions from its book. Most futures contracts are closed out in this way rather than have the underlying asset delivered. In contrast, forward contracts are meant for delivery.

EXAMPLE 12.3.2 A farmer sold short corn futures, and the cost of growing corn rises now. He can offset his short futures position to reduce the losses. Consider another farmer who faces the problem that the crop is going to be different from the 100,000-bushel projection. Because corn futures trade in 5,000-bushel pieces, 20 contracts were sold to cover the anticipated 100,000-bushel crop. If the crop now appears to be only 80,000 bushels, the farmer can offset four of those contracts. Such flexibility is not available to forward contracts.

Because price changes in the futures contract are settled daily, the difference between them has been paid for in installments throughout the life of the contract. Hence the spot price rather than the initial futures price is paid on the delivery date. Marking to market nullifies any financial incentive for not making delivery. Suppose that a farmer enters into a forward contract to sell a food processor 100,000 bushels of corn at \$2.00 per bushel in November. If the price of corn rises to \$2.5 by November, the farmer has incentive to sell his harvest in the spot market at \$2.5 rather than to the processor at \$2.00. With marking to market, however, the farmer has transferred \$0.5 per bushel from his futures account to that of the food processor by November. When the farmer makes delivery, he is paid the spot price, \$2.5 per bushel. The farmer thus has little incentive to default. Note that the net price remains \$2.00 per bushel, the original delivery price.

The prospect of delivery ties the futures price to the spot price (or **cash price**), which makes hedging possible. On the delivery date, the settlement price of the futures contract is determined by the spot price. Before the delivery date, however, the futures price could be above or below the spot price.

Financial futures are futures contracts based on a financial instrument or index. Since the first financial futures were launched in 1972, the trading of financial futures has surpassed that of agricultural futures. The most popular equity financial futures today, the S&P 500 Index futures, were created in 1982 [865]. Like options on stock index, some financial futures are settled in cash rather than by delivery of the underlying asset. The S&P 500 Index futures contract, for instance, is settled in cash rather than by delivering 500 stocks. Each S&P 500 contract is on \$500 times the index (see Fig. 12.3). As another example, one futures contract on the Nikkei 225 Stock Average is on US\$5 times the index. This amounts to fixing the dollar–yen exchange rate. The index is price weighted and on a portfolio of 225 of the largest stocks traded on the Tokyo Stock Exchange.

The difference between the spot price and the futures price, $S - F$, is called the **basis**. For example, if the soybeans cost \$7.80 a bushel, and the November soybean futures contract is \$7.90 on the same day, the basis would be 10 cents under (-10 cents) the November contract. As another example, if the cash market price of a T-bond is 88-16 and the June adjusted futures price is 90-22, then the basis is 2-06 under the June contract. Note that T-bond futures are quoted in 32nds, and prices like 88 16/32 are written as 88-16. Basis can be positive or negative, but it should converge eventually to zero. If the basis moves toward zero, it is said to be **narrowing**, whereas it is said to be **widening** if it moves away from zero. Although basis cannot be predicted precisely, it is generally less volatile than either the futures price or the spot price.

EXAMPLE 12.3.3 Suppose that the spot price of wheat is \$4.225 per bushel and the July futures price of wheat is \$3.59 per bushel with a contract size of 5,000 bushels. The basis is $4.225 - 3.59 = 0.635$ per bushel. Imagine that the basis widens by \$.1 to \$.735 caused by, say, the futures price's falling to \$3.54 and the spot price's rising to \$4.275. A person with a short position in one futures contract and a long position in 5,000 bushels of wheat will make $5000 \times 0.1 = 500$ dollars in profit. If the basis narrows by \$.1 to \$0.535, the same investor will have a loss of an equal amount.

EXAMPLE 12.3.4 A firm to be paid £1 million in 60 days is worried that the pound will weaken. Suppose that a pound futures contract on the International Monetary Market (IMM) of the CME has a settlement date in 71 days. Because each contract controls 62,500 pounds, $1{,}000{,}000/62{,}500 = 16$ contracts are sold.

Monday, March 20, 1995
FUTURES PRICES

	Open	High	Low	Settle	Change	Lifetime High	Lifetime Low	Open Interest
GRAINS AND OILSEEDS								
CORN (CBT) 5,000 bu.; cents per bu.								
Mar	240 1/4	241	239 1/2	239 3/4	−1	282 1/2	220 1/2	2,432
May	246 3/4	248	246 1/4	246 1/2	−3/4	285	228	113,058
				. . .				
INTEREST RATE								
TREASURY BONDS (CBT)–$100,000; pts. 32nds of 100%								
Mar	104-31	105-05	104-18	104-20	−11	116-20	95-13	28,210
June	104-12	104-19	104-00	104-02	−11	113-15	94-27	328,566
				. . .				
LIBOR-1MO. (CME)–$3,000,000; points of 100%								
Apr	93.84	93.84	93.82	93.83	. . .	6.17	. . .	27,961
				. . .				
EURODOLLAR (CME)–$1 million; pts of 100%								
June	93.52	93.54	93.51	93.52	. . .	6.48	. . .	515,578
Sept	93.31	93.32	93.28	93.30	. . .	6.70	. . .	322,889
				. . .				
CURRENCY								
JAPAN YEN (CME)–12.5 million yen; $ per yen (.00)								
June	1.1363	1.1390	1.1240	1.1301	−.0030	1.1390	.9915	56,525
Sept	1.1490	1.1491	1.1385	1.1430	−.0029	1.1491	1.0175	2,282
				. . .				
DEUTSCHEMARK (CME)–125,000 marks; $ per mark								
June	.7249	.7255	.7124	.7147	−.0087	.7448	.5980	56,053
Sept	.7182	.7215	.7153	.7171	−.0087	.7415	.6290	1,776
				. . .				
BRITISH POUND (CME)–62,500 pds; $ per pound								
June	1.5910	1.5936	1.5680	1.5734	−.0102	1.6530	1.5330	21,050
Sept	1.5770	1.5830	1.5680	1.5704	−.0102	1.6480	1.5410	149
				. . .				
INDEX								
S&P 500 INDEX (CME)–$500 times index								
June	499.75	500.75	498.90	500.15	+.40	501.00	449.50	186,725
				. . .				

Figure 12.3: Futures quotations. Source: *Wall Street Journal*, March 21, 1995.

12.3.1 Daily Cash Flows

Consider a futures contract with n days to maturity. Let F_i denote the futures price at the end of day i for $0 \leq i \leq n$. The contract's cash flow on day i is $F_i - F_{i-1}$ because of daily settlement. Hence, the net cash flow over the life of the contract is

$$(F_1 - F_0) + (F_2 - F_1) + \cdots + (F_n - F_{n-1}) = F_n - F_0 = S_T - F_0. \tag{12.8}$$

Recall that F_n equals the spot price at maturity, S_T. Although a futures contract has the same accumulated payoff $S_T - F_0$ as a forward contract, the actual payoff may differ because of the reinvestment of daily cash flows and how $S_T - F_0$ is distributed over the n-day period. In contrast, no cash flows occur until settlement for forward contracts.

➤ **Exercise 12.3.1** Suppose that the interest rate is such that \$1 grows to \$$R$ over a 1-day period and that this rate applies to both borrowing and lending. Derive the payoff for a futures contract if the cash flow is reinvested through lending when it is positive and financed through borrowing when it is negative.

12.3.2 Forward and Futures Prices

Somewhat surprisingly, Cox et al. proved that the futures price equals the forward price if interest rates are nonstochastic [233]. This result often justifies treating a futures contract as if it were a forward contract, ignoring its marking-to-market feature.

Consider forward and futures contracts on the same underlying asset with n days to maturity. Suppose that the interest rate for day i is r_i and that \$1 at the beginning of day i grows to $R_i \equiv e^{r_i}$ by day's end. Let F_i be the futures price at the end of day i. Note that \$1 invested in the n-day discount bond at the end of day zero will be worth $R \equiv \prod_{j=1}^{n} R_j$ by the end of day n.

Starting from day one, we maintain $\prod_{j=1}^{i} R_j$ long futures positions at the end of day $i-1$ and invest the cash flow at the end of day i in riskless bonds maturing on day n, the delivery date. The cash flow from the position on day i is $(F_i - F_{i-1}) \prod_{j=1}^{i} R_j$ because day i starts with $\prod_{j=1}^{i} R_j$ contracts. This amount will be compounded until the end of day n to become

$$(F_i - F_{i-1}) \prod_{j=1}^{i} R_j \prod_{j=i+1}^{n} R_j = (F_i - F_{i-1}) \prod_{j=1}^{n} R_j = (F_i - F_{i-1})\, R.$$

The value at the end of day n is therefore

$$\sum_{i=1}^{n} (F_i - F_{i-1})\, R = (F_n - F_0)\, R = (S_T - F_0)\, R.$$

Observe that no investment is required for the strategy.

Suppose that the forward price f_0 exceeds the futures price F_0. We can short R forward contracts, borrow $f_0 - F_0$, and carry out the above strategy. The initial cash flow is $f_0 - F_0 > 0$. At the end of day n, the debt grows to $(f_0 - F_0)\, R$, and the net value is

$$f_0 R - S_T R - (f_0 - F_0)\, R + (S_T - F_0)\, R = 0.$$

Therefore $f_0 - F_0 > 0$ is a pure arbitrage profit. The case of $f_0 < F_0$ is symmetrical. This completes the proof.

With stochastic interest rates, forward and futures prices are no longer theoretically identical. In fact, this is the major reason for the price differences in the forex forward and futures markets [176, 275]. For short-term contracts, however, the differences tend to be small. In fact, the differences are significant only for longer-term contracts on interest-rate-sensitive assets. We shall assume that forward and futures prices are identical.

➤ **Exercise 12.3.2** Complete the proof by considering the $f_0 < F_0$ case.

➤ **Exercise 12.3.3** Suppose that interest rates are uncertain and that futures prices move in the same direction as interest rates. Argue that futures prices will exceed forward prices. Similarly, argue that if futures prices move in a direction opposite from that of interest rates, then futures prices will be exceeded by forward prices.

12.3.3 Stock Index Futures

Stock index futures originated in 1982 when the Kansas City Board of Trade introduced the Value Line Stock Index futures. There are now stock index futures based on the S&P 500 Index, the Nikkei 225 Stock Average futures, the NYSE Composite Index futures (traded on the New York Futures Exchange for $500 times the index), the Major Market Index (traded on the CBT for $500 times the index), the DJIA Index (traded on the CBT for $10 times the index; ticker symbol DJ), and so on.

Indices can be viewed as dividend-paying securities, the security being the basket of stocks comprising the index and the dividends being those paid by the stocks. If the index is broadly based, dividends can be assumed to be paid continuously. With q denoting the average annualized dividend yield during the life of the contract, the futures price is then

$$F = Se^{(r-q)\tau}. \tag{12.9}$$

EXAMPLE 12.3.5 The S&P 500 Index futures contract is based on the S&P 500 Index. The minimum fluctuation (**tick size**) is 0.05 point. Because the value of a contract is \$500 times the Index, a change of 0.05 represents a $\$500 \times 0.05 = \$25$ tick. Consider a 3-month futures contract on the S&P 500 Index. Suppose that the stocks underlying the index provide a dividend yield of 3% per annum, the current value of the index is 480, and the interest rate is 8%. The theoretical futures price is then $480 \times e^{0.05 \times 0.25} = 486.038$.

When Eq. (12.9) fails to hold, one can create arbitrage profits by trading the stocks underlying the index and the index futures. For example, when $F > Se^{(r-q)\tau}$, one can make profits by buying the stocks underlying the index and shorting futures contracts. One should do the reverse if $F < Se^{(r-q)\tau}$. These strategies are known as **index arbitrage** and are executed by computers, an activity known as **program trading**. Equation (12.9) is not applicable to the Nikkei 225 futures, however. Recall that one such contract is on the *dollar* amount equal to five times the index, which is measured in *yen*. But no securities whose value is $5 times the index exist; hence the arbitrage argument breaks down.

For indices that all stocks tend to pay dividends on the same date, we can estimate the dividends' dollar amount and timing. Then the index becomes a security providing known income, and Eq. (12.4) says the futures price is

$$F = (S - I)\, e^{r\tau}. \tag{12.10}$$

➤ **Exercise 12.3.4** Do Eqs. (12.9) and (12.10) assume that the stock index involved is not adjusted for cash dividends?

12.3.4 Forward and Futures Contracts on Currencies

Let S denote the domestic/foreign exchange rate and let X denote the delivery price of the forward contract. Use r_{f} to refer to the foreign riskless interest rate. A portfolio consisting of one long forward contract, cash amount $Xe^{-r\tau}$ in domestic currency, and one short position in the amount of $e^{-r_{\mathrm{f}}\tau}$ in foreign currency is clearly worth zero at time T. Hence its current value must be zero, that is, $f + Xe^{-r\tau} - Se^{-r_{\mathrm{f}}\tau} = 0$. The value X that makes $f = 0$ is the forward price (i.e., the forward exchange rate) $F = Se^{(r-r_{\mathrm{f}})\tau}$, which is exactly the interest rate parity.

12.3.5 Futures on Commodities and the Cost of Carry

Some commodities are held solely for investment (such as gold and silver), whereas others are held primarily for consumption. Arbitrage arguments can be used to obtain futures prices in the former case, but they give only upper bounds in the latter.

For a commodity held for investment purposes and with zero storage cost, the futures price is $F = Se^{r\tau}$ according to Eq. (12.3). In general, if U stands for the PV of the storage costs incurred during the life of a futures contract, then Eq. (12.4) implies that $F = (S + U)\, e^{r\tau}$ as storage costs are negative income. Alternatively, if u denotes the storage cost per annum as a proportion of the spot price, then Eq. (12.6) implies that $F = Se^{(r+u)\tau}$ as storage costs provide a negative dividend yield. For commodities held primarily for consumption, however, we can say only that

$$F \le (S + U)\, e^{r\tau}, \quad F \le Se^{(r+u)\tau},$$

respectively, because of the benefits of holding the physicals. These benefits are measured by the so-called **convenience yield** defined as the y such that

$$Fe^{y\tau} = (S + U)\, e^{r\tau}, \quad Fe^{y\tau} = Se^{(r+u)\tau}, \tag{12.11}$$

respectively.

We can frame the relation between the futures and spot prices in terms of the **cost of carry**, which is the storage cost plus the interest cost paid to carry the asset but less the income earned on the asset. For a stock paying no dividends, the cost of carry is r because it neither incurs storage costs nor generates any income; for a stock index, it is $r - q$ as income is earned at rate q; for a currency, it is $r - r_{\mathrm{f}}$; for a commodity with storage costs, it is $r + u$. Suppose the cost of carry is c and the convenience yield is y. For an investment asset $F = Se^{c\tau}$, and for a consumption asset $F = Se^{(c-y)\tau}$.

The cost of carry is often cast in monetary terms, called the **carrying charge** or the **carrying cost**. It measures the dollar cost of carrying the asset over a period and consists of interest expense I, storage costs U, minus cash flows generated by the asset D:

$$C \equiv I + U - D. \tag{12.12}$$

The cost of carry will be in dollar terms from now on unless stated otherwise. Similarly, the convenience yield can also be expressed in dollar terms:

$$\text{convenience yield} \equiv S + C - F.$$

As a consequence, the basis $S - F$ is simply the convenience yield minus the cost of carry:

$$\text{basis} = \text{convenience yield} - C. \tag{12.13}$$

Recall that the convenience yield is negligible for financial instruments and commodities held primarily for investment purposes. For such assets, changes in basis are due entirely to changes in the cost of carry.

Look up the futures prices of corn in Fig. 12.3. Because the prices for the near months are lower than the distant months, this market is said to be **normal**. The premium that the distant months command over near months is due to the greater carrying costs. In an **inverted** or **discount** market, the distant months sell at lower prices than the near months. A strong demand for cash grains or the willingness of elevator owners to store grains at less than the full storage costs can create an inverted market.

When the forward price equals the sum of the spot price and the carrying charge (in other words, zero convenience yield), the forward price is said to be at **full carry**. Forward and futures prices should be set at full carry for assets that have zero storage cost and can be sold short or in ample supply (see Exercises 12.3.7 and 12.3.8). As previously mentioned, commodities held for investment purposes should also reflect full carry. If the total cost of storing corn is, say, four cents per bushel a month and if futures prices reflect the full carrying cost, the prices for the different delivery months might look like the following table:

December	*March*	*May*	*July*	*September*
\$2.00	\$2.12	\$2.20	\$2.28	\$2.36

➤ **Exercise 12.3.5** For futures, the cost of carry may be called **cash and carry**, which is the strategy of buying the cash asset with borrowed funds. (1) Illustrate this point with futures contracts when the underlying asset pays no income. (2) Show that the cost of carry can be used to find the futures price in (1) set at full carry.

➤ **Exercise 12.3.6** A manufacturer needs to acquire gold in 3 months. The following options are open to her: (1) Buy the gold now or (2) go long one 3-month gold futures contract and take delivery in 3 months. If she buys the gold now, the money that has been tied up could be invested in money market instruments. This is the opportunity cost of buying physical gold. What is the cost of carry for owning 100 ounces of gold at \$350 per ounce for a year if the T-bills are yielding an annually compounded rate of 6%?

➤ **Exercise 12.3.7** Prove that $F \leq S + C$, where C is the net carrying cost per unit of the commodity to the delivery date.

➤ **Exercise 12.3.8** For a commodity that can be sold short, such as a financial asset, prove that $F \geq S + C - U$, where U is the net storage cost for carrying one unit of the commodity to the delivery date.

12.4 Futures Options and Forward Options

The underlying asset of a futures option is a futures contract. On exercise, the option holder takes a position in the futures contract with a futures price equal to the option's strike price. In particular, a futures call (put) option holder acquires a long (short, respectively) futures position. The option writer does the opposite: A futures call (put, respectively) option writer acquires a short (long, respectively) futures position. The futures contract is then marked to market immediately, and the futures position of the two parties will be at the prevailing futures price. The option holder can withdraw in cash the difference between the prevailing futures price and the strike price. Of course, the option's expiration date should precede the futures contract's delivery date.

The whole process works as if the option writer delivered a futures contract to the option holder and paid the holder the prevailing futures price minus the strike price in the case of calls. In the case of puts, it works as if the option writer took delivery a futures contract from the option holder and paid the holder the strike price minus the prevailing futures price. Note that the amount of money that changes hands on exercise is only the difference between the strike price and the prevailing futures price. See Fig. 12.4 for sample quotations.

EXAMPLE 12.4.1 An investor holds a July futures call on 5,000 bushels of soybeans with a strike price of 600 cents per bushel. Suppose that the current futures price

Monday, March 20, 1995

. . .

INTEREST RATE

T-BONDS (CBT)

$100,000; points and 64ths of 100%

Strike	*Calls – Settle*			*Puts – Settle*		
Price	*Apr*	*May*	*Jun*	*Apr*	*May*	*Jun*
102	2-06	2-26	2-47	0-03	0-23	0-43
103	1-12	1-44	. . .	0-08	0-40	. . .
104	0-30	1-05	1-29	0-26	1-01	1-25
105	0-07	0-39	. . .	1-03	1-34	. . .
106	0-01	0-21	0-40	1-61	. . .	2-35
107	0-01	0-10	. . .	. . .	. . .	. . .

. . .

T-NOTES (CBT)

$100,000; points and 64ths of 100%

Strike	*Calls – Settle*			*Puts – Settle*		
Price	*Apr*	*May*	*Jun*	*Apr*	*May*	*Jun*
102	2-26	. . .	2-45	0-01	. . .	0-20
103	1-28	. . .	1-60	0-02	0-19	0-35
104	0-36	. . .	1-19	0-11	. . .	0-57
105	0-07	0-32	0-51	0-45	1-06	1-25
106	0-01	. . .	0-28	1-39	. . .	2-01
107	0-01	0-05	0-14	. . .	. . .	2-51

. . .

LIBOR – 1 Mo. (CME)

$3 million; pts. of 100%

Strike	*Calls – Settle*			*Puts – Settle*		
Price	*Apr*	*May*	*Jun*	*Apr*	*May*	*Jun*
9325	0.58	0.51	0.45	0.00	0.01	0.03
9350	0.34	0.29	0.24	0.01	0.04	0.07
9375	0.11	0.10	0.09	0.03	. . .	0.17
9400	0.01	. . .	0.03	. . .	. . .	. . .
9425	. . .	. . .	. . .	. . .	. . .	. . .
9450	0.00	0.00	. . .	. . .	. . .	. . .

. . .

INDEX

S&P 500 STOCK INDEX (CME)

$500 times premium

Strike	*Calls – Settle*			*Puts – Settle*		
Price	*Apr*	*May*	*Jun*	*Apr*	*May*	*Jun*
490	12.55	14.80	16.75	2.45	4.75	6.75
495	8.75	11.25	13.35	3.65	6.15	8.25
500	5.70	8.10	10.20	5.55	7.95	10.05
505	3.45	5.55	7.50	8.25	10.35	12.30
510	1.85	3.55	5.30	11.65	. . .	15.00
515	0.85	2.15	3.55	. . .	. . .	. . .

. . .

Figure 12.4: Futures options quotations. Months refer to the expiration month of the underlying futures contract. Source: *Wall Street Journal*, March 21, 1995.

of soybeans for delivery in July is 610 cents. The investor can exercise the option to receive \$500 (5,000 $\times$ 10 cents) plus a long position in a futures contract to buy 5,000 bushels of soybeans in July. Similarly, consider an investor with a July futures put on 5,000 bushels of soybeans with a strike price of 620 cents per bushel. Suppose the current futures price of soybeans for delivery in July is 610 cents. The investor can exercise the option to receive \$500 (5,000 $\times$ 10 cents) plus a short position in a futures contract to buy 5,000 bushels of soybeans in July.

EXAMPLE 12.4.2 Suppose that the holder of a futures call with a strike price of \$35 exercises it when the futures price is \$45. The call holder is given a long position in the futures contract at \$35, and the call writer is assigned the matching short position at \$35. The futures positions of both are immediately marked to market by the exchange. Because the prevailing futures price is \$45, the long futures position (the position of the call holder) realizes a gain of \$10, and the short futures position (the position of the call writer) realizes a loss of \$10. The call writer pays the exchange \$10 and the call holder receives from the exchange \$10. The call holder, who now has a long futures position at \$45, can either liquidate the futures position at \$45 without costs or maintain it.

Futures options were created in 1982 when the CBT began trading options on T-bond futures. Futures options are preferred to options on the cash instrument in some markets on the following grounds. In contrast to the cash markets, which are often fragmented and over the counter, futures trading takes place in competitive, centralized markets. Futures options have fewer liquidity problems associated with shortages of the cash assets – selling a commodity short may be significantly more difficult than selling a futures contract. Futures options are also useful in implementing certain strategies. Finally, futures options are popular because of their limited capital requirements [746].

Forward options are similar to futures options except that what is delivered is a forward contract with a delivery price equal to the option's strike price. In particular, exercising a call (put) forward option results in a long (short, respectively) position in a forward contract. Note that exercising a forward option incurs no immediate cash flows. Unlike futures options, forward options are traded not on organized exchanges but in over-the-counter markets.

EXAMPLE 12.4.3 Consider a call with strike \$100 and an expiration date in September. The underlying asset is a forward contract with a delivery date in December. Suppose that the forward price in July is \$110. On exercise, the call holder receives a forward contract with a delivery price of \$100. If an offsetting position is then taken in the forward market, a \$10 profit in September will be ensured. Were the contract a call on the futures, the \$10 profit would be realized in July.

➤ **Exercise 12.4.1** With a conversion, the trader buys a put, sells a call, and buys a futures contract. The put and the call have the same strike price and expiration month. The futures contract has the same expiration month as the options, and its price is equal to the options' strike price. Argue that any initial positive cash flow of conversion is guaranteed profit.

12.4.1 Pricing Relations

Assume a constant, positive interest rate. This is acceptable for short-term contracts. Even under this assumption, which equates forward price with futures price, a forward option does not have the same value as a futures option. Let delivery take place at time T, the current time be zero, and the option on the futures or forward contract have expiration date t $(t \leq T)$. Note that the futures contract can be marked to market when the option is exercised. Example 12.4.3 established the following identities for the futures options and forward options when they are exercised at time t:

$$\text{value of futures option} = \max(F_t - X, 0) \tag{12.14}$$

$$\text{value of forward option} = \max(F_t - X, 0)\, e^{-r(T-t)} \tag{12.15}$$

Furthermore, a European futures option is worth the same as the corresponding European option on the underlying asset if the futures contract has the same maturity as the option. The reason is that the futures price equals the spot price at maturity. This conclusion is independent of the model for the spot price.

The put–call parity is slightly different from the one in Eq. (8.1). Whereas the undiscounted stock price was used in the case of stock options, it is the discounted futures/forward prices that should be used here (see also Exercise 12.2.4).

THEOREM 12.4.4 (Put–Call Parity). *For European options on futures contracts, $C = P - (X - F)\, e^{-rt}$. For European options on forward contracts, $C = P - (X - F)\, e^{-rT}$.*

Proof: Consider a portfolio of one short call, one long put, one long futures contract, and a loan of $(X - F)\, e^{-rt}$. We have the following cash flow at time t.

	$F_t \leq X$	$F_t > X$
A short call	0	$X - F_t$
A long put	$X - F_t$	0
A long futures	$F_t - F$	$F_t - F$
A loan of $(X - F)\, e^{-rt}$	$F - X$	$F - X$
Total	0	0

Because the net future cash flow is zero in both cases, the portfolio must have zero value today. This proves the theorem for futures option.

The proof for forward options is identical except that the loan amount is $(X - F)\, e^{-rT}$ instead. The reason is that the forward contract can be settled only at time T.

An American forward option should be worth the same as its European counterpart. In other words, the early exercise feature is not valuable.

THEOREM 12.4.5 *American forward options should not be exercised before expiration as long as the probability of their ending up out of the money is positive.*

Proof: Consider a portfolio of one long forward call, one short forward contract with delivery price F, and a loan of $(F - X)\, e^{-rT}$. If $F_t < X$ at t, the wealth at t is

$$0 + (F - F_t)\, e^{-r(T-t)} - (F - X)\, e^{-r(T-t)} = (X - F_t)\, e^{-r(T-t)} > 0.$$

If $F_t \geq X$ at t, the wealth at t is

$$(F_t - X)\, e^{-r(T-t)} + (F - F_t)\, e^{-r(T-t)} - (F - X)\, e^{-r(T-t)} = 0.$$

So the value of the forward call C satisfies $C-(F-X)\,e^{-rT}>0$. On the other hand, if the call is exercised immediately, the PV at time zero is only $\max(F-X,0)\,e^{-rT}$. The case of forward puts is proved in Exercise 12.4.3.

Early exercise may be optimal for American futures options. Hence an American futures option is worth more than the European counterpart even if the underlying asset generates no payouts [125].

THEOREM 12.4.6 *American futures options may be exercised optimally before expiration.*

➤ **Exercise 12.4.2** Prove Theorem 12.4.5 for American forward puts. (Hint: Show that $P-(X-F)\,e^{-rT}>0$ first.)

➤ **Exercise 12.4.3** Prove that $Fe^{-rt}-X\leq C-P\leq F-Xe^{-rt}$ for American futures options.

12.4.2 The Black Model

Black developed the following formulas for European futures options in 1976:

$$\begin{aligned} C &= Fe^{-rt}N(x)-Xe^{-rt}N(x-\sigma\sqrt{t}), \\ P &= Xe^{-rt}N(-x+\sigma\sqrt{t})-Fe^{-rt}N(-x), \end{aligned} \tag{12.16}$$

where [81]

$$x\equiv\frac{\ln(F/X)+(\sigma^2/2)\,t}{\sigma\sqrt{t}}.$$

Formulas (12.16) are related to those for options on a stock paying a continuous dividend yield. In fact, they are exactly Eq. (9.20) with the dividend yield q set to the interest rate r and the stock price S replaced with the futures price F. This observation also proves Theorem 12.4.6 based on the discussions in Subsection 9.6.2. For European forward options, just multiply the above formulas by $e^{-r(T-t)}$ as forward options differ from futures options by a factor of $e^{-r(T-t)}$ based on Eqs. (12.14) and (12.15).

Black's formulas can be expressed in terms of S instead of F by means of the substitution $F=Se^{(r-q)\,T}$. (The original formulas have the advantage of not containing q or T. The delta for the call is then $\partial C/\partial F=e^{-rt}N(x)$ and that for the put is $\partial P/\partial F=e^{-rt}[\,N(x)-1\,]$. The delta for the call can also be cast with respect to the spot price:

$$\frac{\partial C}{\partial S}=\frac{\partial C}{\partial F}\,\frac{\partial F}{\partial S}=e^{-rt}N(x)\,e^{(r-q)\,T}=e^{-r(t-T)-qT}N(x).$$

Other sensitivity measures can be easily derived [746, p. 345].

Besides index options and index futures, a third type of stock index derivative is the index futures option. European index futures options can be priced by Black's formulas. The S&P 500 Index and the DJIA span all three types of derivatives. The NYA has options and futures options. Although the SPX and the DJIA index options are European, their index futures options are American. The NYA index option and futures option are both American.

Binomial tree algorithm for pricing American futures calls:

```
input:   F, σ, t, X, r, n;
real     R, p, u, d, C[n+1];
integer i, j;
R := e^(r(t/n));
u := e^(σ√(t/n)); d := e^(−σ√(t/n));
p := (1 − d)/(u − d);    // Risk-neutral probability.
for (i = 0 to n) { C[i] := max(0, F u^(n−i) d^i − X); }
for (j = n − 1 down to 0)
        for (i = 0 to j)
                C[i] := max((p × C[ i ] + (1 − p) × C[ i + 1 ])/R, F u^(j−i) d^i − X);
return C[0];
```

Figure 12.5: Binomial tree algorithm for American futures calls.

➤ **Exercise 12.4.4** (1) Verify that, under the Black–Scholes model, a European futures option is worth the same as the corresponding European option on the cash asset if the options and the futures contract have the same maturity. The cash asset may pay a continuous dividend yield. (2) Then argue that, in fact, (1) must hold under any model.

12.4.3 Binomial Model for Forward and Futures Options

The futures price behaves like a stock paying a continuous dividend yield of r. Under the binomial model, the risk-neutral probability for the futures price is $p_{\mathrm{f}} \equiv (1-d)/(u-d)$ by formula (9.21). Here, the futures price moves from F to Fu with probability p_{f} and to Fd with probability $1-p_{\mathrm{f}}$. Figure 12.5 contains a binomial tree algorithm for pricing futures options. The binomial tree algorithm for forward options is identical except that (12.15) is used for the payoff when the option is exercised. So we replace $Fu^{n-i}d^i$ with $Fu^{n-i}d^ie^{-r(T-t)}$ and $Fu^{j-i}d^i$ with $Fu^{j-i}d^ie^{-r(T-t(j/n))}$ in Fig. 12.5.

Recall that the futures price is related to the spot price by $F = Se^{rT}$ if the underlying asset does not pay dividends. The preceding binomial model for futures prices implies that the stock price moves from $S = Fe^{-rT}$ to $S_u \equiv Fue^{-r(T-\Delta t)} = Sue^{r\Delta t}$ with probability p_{f} and to $S_d \equiv Sde^{r\Delta t}$ with probability $1-p_{\mathrm{f}}$ in a period of length Δt.

Options can be replicated by a portfolio of futures contracts and bonds. This avenue may be preferred to using stocks because the restrictions on shorting futures are looser than those on stocks. To set up an equivalent portfolio of h_{f} futures contracts and \$$B$ in riskless bonds to replicate a call that costs C_u if the stock price moves to S_u and C_d if the stock price moves to S_d, we set up

$$h_{\mathrm{f}}(Fu-F)+e^{r\Delta t}B = C_u, \quad h_{\mathrm{f}}(Fd-F)+e^{r\Delta t}B = C_d.$$

Solve the preceding equations to obtain

$$h_{\mathrm{f}} = \frac{C_u - C_d}{(u-d)\,F} \geq 0,$$

$$B = \frac{(u-1)\,C_d - (d-1)\,C_u}{(u-d)\,e^{r\Delta t}}.$$

Compared with the delta in Eq. (9.1), repeated below,

$$h = \frac{C_u - C_d}{S_u - S_d} = \frac{C_u - C_d}{(Su - Sd)\, e^{r\Delta t}},$$

we conclude that

$$h_{\mathrm{f}} = \frac{C_u - C_d}{(u - d)\, Se^{rT}} = he^{-r(T-\Delta t)} < h.$$

Hence the delta with futures never exceeds that with stocks.

As $0 < p_{\mathrm{f}} < 1$, we have $0 < 1 - p_{\mathrm{f}} < 1$ as well. This suggests the following method to solve the problem of negative risk-neutral probabilities mentioned in Subsection 9.3.1: Build the binomial tree for the futures price F of the futures contract expiring at the same time as the option; then calculate S from F at each node by means of $S = Fe^{-(r-q)(T-t)}$ if the stock pays a continuous dividend yield of q [470].

➤ **Exercise 12.4.6** Start with the standard tree for the underlying non-dividend-paying stock (i.e., a stock price S can move to Su or Sd with $(e^{r\Delta t} - d)/(u - d)$ as the probability for an up move). (1) Construct the binomial model for the futures prices based on that tree. (2) What if the stock pays a continuous dividend yield of q?

➢ **Programming Assignment 12.4.7** Write binomial tree programs to price futures options and forward options.

➢ **Programming Assignment 12.4.8** Write binomial tree programs to implement the idea of avoiding negative risk-neutral probabilities enunciated above.

12.5 Swaps

Swaps are agreements between two **counterparties** to exchange cash flows in the future according to a predetermined formula. There are two basic types of swaps: interest rate and currency. An **interest rate swap** occurs when two parties exchange interest payments periodically. **Currency swaps** are agreements to deliver one currency against another [767]. Currency swaps made their debut in 1979, and interest rate swaps followed suit in 1981. In the following decade the growth of their notional volume was so spectacular as to dwarf that of any other market. For instance, interest rate swaps alone stood at over U.S.$2 trillion in 1993. Swaps also spurred the growth of related instruments such as multiperiod options (interest rate caps and floors, etc.) and **forward-rate agreements**.

Currency and interest rate swaps are collectively called **rate swaps**. Swaps on commodities are also available. For example, a company that consumes 200,000 barrels of oil per annum may pay $2 million per year for the next 5 years and in return receive $200{,}000 \times S$, where S is the prevailing market price of oil per barrel. This transaction locks in the price for its oil at $10 per barrel. We concentrate on currency swaps here.

12.5.1 Currency Swaps

A currency swap involves two parties to exchange cash flows in different currencies. As an example, consider the following fixed rates available to party A and party B in U.S. dollars and Japanese yen:

	Dollars	*Yen*
A	D_A%	Y_A%
B	D_B%	Y_B%

Suppose A wants to take out a fixed-rate loan in yen, and B wants to take out a fixed-rate loan in dollars. A straightforward scenario is for A to borrow yen at Y_A% and for B to borrow dollars at D_B%.

Assume further that A is *relatively* more competitive in the dollar market than the yen market, and vice versa for B – in other words, $Y_B - Y_A < D_B - D_A$. Now consider this alternative arrangement: A borrows dollars, B borrows yen, and they enter into a currency swap, perhaps with a bank as the financial intermediary. With a swap, the counterparties exchange principal at the beginning and the end of the life of the swap. This act transforms A's loan into a yen loan and B's yen loan into a dollar loan. The total gain to all parties is $[(D_B - D_A) - (Y_B - Y_A)]$% because the total interest rate is originally $(Y_A + D_B)$% and the new arrangement has a smaller total rate of $(D_A + Y_B)$%. Of course, this arrangement will happen only if the total gain is distributed in such a way that the cost to each party is less than that of the original scenario.

EXAMPLE 12.5.1 Two parties, A and B, face the following borrowing rates:

	Dollars	*Yen*
A	9%	10%
B	12%	11%

Assume that A wants to borrow yen and B wants to borrow dollars. A can borrow yen directly at 10%, and B can borrow dollars directly at 12%. As the rate differential in dollars (3%) is different from that in yen (1%), a currency swap with a total saving of $3 - 1 = 2\%$ is possible. Note that A is relatively more competitive in the dollar market, and B in the yen market. Figure 12.6 shows an arrangement that is beneficial to all parties involved, in which A effectively borrows yen at 9.5% and B borrows dollars at 11.5%. The gain is 0.5% for A, 0.5% for B, and, if we treat dollars and yen identically, 1% for the bank.

With the arrangement in Fig. 12.6 and principal amounts of U.S.$1 million and 100 million yen, the bank makes an annual gain of $0.025 million and an annual loss of 1.5 million yen. The bank thus bears some currency risk, but neither A nor B bears any currency risk. Currency risk clearly can be redistributed but not eliminated.

➤ **Exercise 12.5.1** Use the numbers in Example 12.5.1 to construct the same effective borrowing rates without the bank as the financial intermediary.

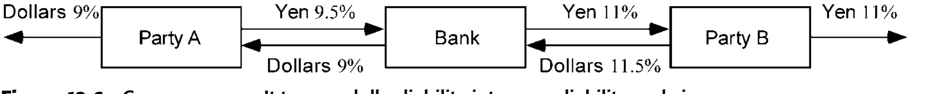

Figure 12.6: Currency swap: It turns a dollar liability into a yen liability and vice versa.

➤ **Exercise 12.5.2** Redesign the swap with the rates in Example 12.5.1 so that the gains are 1% for A, 0.5% for B, and 0.5% for the bank.

12.5.2 Valuation of Currency Swaps

As a Package of Cash Market Instruments

In the absence of default risk, the valuation of currency swap is rather straightforward. Take B in Fig. 12.6 as an example. The swap is equivalent to a long position in a yen bond paying 11% annual interest and a short position in a dollar bond paying 11.5% annual interest. The general pricing formula is thus $SP_{\mathrm{Y}} - P_{\mathrm{D}}$, where P_{D} is the dollar bond's value in dollars, P_{Y} is the yen bond's value in yen, and S is the \$/yen spot exchange rate. The value of a currency swap therefore depends on the term structures of interest rates in the currencies involved and the spot exchange rate. The swap has zero value when $SP_{\mathrm{Y}} = P_{\mathrm{D}}$.

EXAMPLE 12.5.2 Take a 2-year swap in Fig. 12.6 with principal amounts of U.S.\$1 million and 100 million yen. The payments are made once a year. Assume that the spot exchange rate is 90 yen/\$ and the term structures are flat in both nations – 8% in the U.S. and 9% in Japan. The value of the swap is

$$\frac{1}{90} \times \left(11 \times e^{-0.09} + 11 \times e^{-0.09\times 2} + 111 \times e^{-0.09\times 3}\right)$$
$$- \left(0.115 \times e^{-0.08} + 0.115 \times e^{-0.08\times 2} + 1.115 \times e^{-0.08\times 3}\right) = 0.074$$

million dollars for B.

As a Package of Forward Contracts

Swaps can also be viewed as a package of forward contracts. From Eq. (12.5), the forward contract maturing i years from now has a dollar value of

$$f_i \equiv (SY_i)\, e^{-qi} - D_i e^{-ri}, \tag{12.17}$$

where Y_i is the yen inflow at year i, S is the \$/yen spot exchange rate, q is the yen interest rate, D_i is the dollar outflow at year i, and r is the dollar interest rate. This formulation may be preferred to the cash market approach in cases involving costs of carry and convenience yields because forward prices already incorporate them [514]. For simplicity, flat term structures are assumed, but generalization is straightforward.

Take the swap in Example 12.5.2. Every year, B receives 11 million yen and pays \$0.115 million. In addition, at the end of the third year, B receives 100 million yen and pays \$1 million. Each of these transactions represents a forward contract. In particular, $Y_1 = Y_2 = 11$, $Y_3 = 111$, $S = 1/90$, $D_1 = D_2 = 0.115$, $D_3 = 1.115$, $q = 0.09$, and $r = 0.08$. Plug in these numbers to get $f_1 + f_2 + f_3 = 0.074$ million dollars, as in Example 12.5.2.

Equation (12.17) can be equivalently cast in terms of forward exchange rates as

$$f_i \equiv (F_i Y_i - D_i)\, e^{-ri},$$

where F_i is the i-year forward exchange rate. Even though the swap may have zero value (equivalently, $\sum_i f_i = 0$), it does not imply that each of the forward contracts, f_i, has zero value.

➤ **Exercise 12.5.3** Derive Eq. (12.17) with a forward exchange rate argument.

Additional Reading

Consult [88, 95, 346, 369, 470, 514, 646, 698, 746, 878, 879] for more information on derivative securities. The introduction of derivatives makes the price of the underlying asset more informative [151]. Black's model is very popular [94]. See [423, 894] for more "exotic" options and [521] for option pricing with default risk. Pointers to empirical studies on the relation between futures and forward prices can be found in [514].

NOTES

1. J.P. Morgan was acquired by Chase Manhattan in 2000, which became J.P. Morgan Chase.
2. The CBT developed futures contracts in 1865. Futures contracts were traded on the Amsterdam exchange in the seventeenth century.
3. Keynes (1883–1946) was one of the greatest economists in history [805, 806, 807].

https://doi.org/10.1017/CBO9780511546839.013 Published online by Cambridge University Press