CHAPTER NINE

Option Pricing Models

Life can only be understood backwards; but it must be lived forwards.
Søren Kierkegaard (1813–1855)

Although it is rather easy to price an option at expiration, pricing it at any prior moment is anything but. The no-arbitrage principle, albeit valuable in deriving various bounds, is insufficient to pin down the exact option value without further assumptions on the probabilistic behavior of stock prices. The major task of this chapter is to develop option pricing formulas and algorithms under reasonable models of stock prices. The powerful binomial option pricing model is the focus of this chapter, and the celebrated Black–Scholes formula is derived.

9.1 Introduction

The major obstacle toward an option pricing model is that it seems to depend on the probability distribution of the underlying asset's price and the risk-adjusted interest rate used to discount the option's payoff. Neither factor can be observed directly. After many attempts, some of which were very close to solving the problem, the breakthrough came in 1973 when Black (1938–1995) and Scholes, with help from Merton, published their celebrated option pricing model now universally known as the **Black–Scholes option pricing model** [87].[1] One of the crown jewels of finance theory, this research has far-reaching implications. It also contributed to the success of the CBOE [660]. In 1997 the Nobel Prize in Economic Sciences was awarded to Merton and Scholes for their work on "the valuation of stock options."

The mathematics of the Black–Scholes model is formidable because the price can move to any one of an infinite number of prices in any finite amount of time. The alternative **binomial option pricing model** (**BOPM**) limits the price movement to two choices in a period, simplifying the mathematics tremendously at some expense of realism. All is not lost, however, because the binomial model converges to the Black–Scholes model as the period length goes to zero. More importantly, the binomial model leads to efficient numerical algorithms for option pricing. The BOPM is the main focus of this chapter.

Throughout this chapter, C denotes the call value, P the put value, X the strike price, S the stock price, and D the dividend amount. Subscripts are used to emphasize or differentiate different times to expiration, stock prices, or strike prices. The

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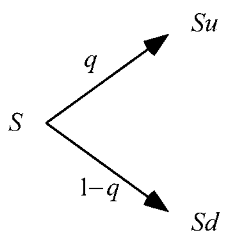
Figure 9.1: Binomial model for stock prices.



symbol PV(x) stands for the PV of x at expiration unless stated otherwise. Let $\hat{r} > 0$ denote the continuously compounded riskless rate per period and $R \equiv e^{\hat{r}}$ its **gross return**.

9.2 The Binomial Option Pricing Model

In the BOPM, time is discrete and measured in periods. The model assumes that if the current stock price is S, it can go to Su with probability q and Sd with probability $1 - q$, where $0 < q < 1$ and $d < u$ (see Fig. 9.1). In fact, $d < R < u$ must hold to rule out arbitrage profits (see Exercise 9.2.1). It turns out that six pieces of information suffice to determine the option value based on arbitrage considerations: S, u, d, X, $\hat{r}$, and the number of periods to expiration.

➤ **Exercise 9.2.1** Prove that $d < R < u$ must hold to rule out arbitrage profits.

9.2.1 Options on a Non-Dividend-Paying Stock: Single Period

Suppose that the expiration date is only one period from now. Let C_u be the price at time one if the stock price moves to Su and C_d be the price at time one if the stock price moves to Sd. Clearly,

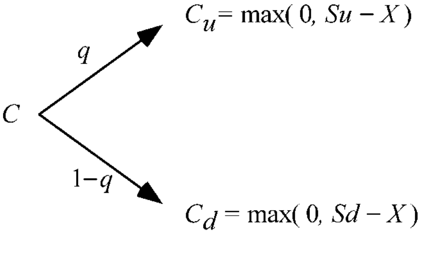
$$C_u = \max(0, Su - X), \qquad C_d = \max(0, Sd - X).$$

See Fig. 9.2 for illustration.

Now set up a portfolio of h shares of stock and B dollars in riskless bonds. This costs $hS + B$. We call h the **hedge ratio** or **delta**. The value of this portfolio at time one is either $hSu + RB$ or $hSd + RB$. The key step is to choose h and B such that the portfolio replicates the payoff of the call:

$$hSu + RB = C_u,$$
$$hSd + RB = C_d.$$

Figure 9.2: Value of one-period call in BOPM.



Solve the above equations to obtain

$$h = \frac{C_u - C_d}{Su - Sd} \geq 0, \tag{9.1}$$

$$B = \frac{uC_d - dC_u}{(u-d)\, R}. \tag{9.2}$$

An **equivalent portfolio** that replicates the call *synthetically* has been created. An equivalent portfolio is also called a **replicating portfolio** or a **hedging portfolio**. By the no-arbitrage principle, the European call should cost the same as the equivalent portfolio, or $C = hS + B$. As it is easy to verify that

$$uC_d - dC_u = \max(0, Sud - Xu) - \max(0, Sud - Xd) < 0,$$

the equivalent portfolio is a levered long position in stocks.

For American calls, we have to consider immediate exercise. When $hS + B \geq S - X$, the call should not be exercised immediately; so $C = hS + B$. When $hS + B < S - X$, on the other hand, the option should be exercised immediately for we can take the proceeds $S - X$ to buy the equivalent portfolio plus some more bonds; so $C = S - X$. We conclude that $C = \max(hS + B, S - X)$. For non-dividend-paying stocks, early exercise is not optimal by Theorem 8.4.1 (see also Exercise 9.2.6). Again, $C = hS + B$.

Puts can be similarly priced. The delta for the put is $(P_u - P_d)/(Su - Sd) \leq 0$, where $P_u = \max(0, X - Su)$ and $P_d = \max(0, X - Sd)$. The European put is worth $hS + B$, and the American put is worth $\max(hS + B, X - S)$, where $B = \{(uP_d - dP_u)/[(u-d)\, R]\}$.

➤ **Exercise 9.2.2** Consider two securities, A and B. In a period, security A's price can go from \$100 to either (a) \$160 or (b) \$80, whereas security B's price can move to \$50 in case (a) or \$60 in case (b). Price security B when the interest rate per period is 10%.

9.2.2 Risk-Neutral Valuation

Surprisingly, the option value is independent of q, the probability of an upward movement in price, and hence the expected gross return of the stock, $qSu + (1-q)\, Sd$, as well. It therefore does not directly depend on investors' **risk preferences** and will be priced the same regardless of how risk-averse an investor is. The arbitrage argument assumes only that more deterministic wealth is preferred to less. The option value does depend on the sizes of price changes, u and d, the magnitudes of which the investors must agree on.

After substitution and rearrangement,

$$hS + B = \frac{\left(\frac{R-d}{u-d}\right) C_u + \left(\frac{u-R}{u-d}\right) C_d}{R} > 0. \tag{9.3}$$

Rewrite Eq. (9.3) as

$$hS + B = \frac{pC_u + (1-p)\, C_d}{R}, \tag{9.4}$$

where

$$p \equiv \frac{R-d}{u-d}. \tag{9.5}$$

As $0 < p < 1$, it may be interpreted as a probability. Under the binomial model, the expected rate of return for the stock is equal to the riskless rate $\widehat{r}$ under $q = p$ because $pSu + (1-p)\, Sd = RS$.

An investor is said to be **risk-neutral** if that person is indifferent between a certain return and an uncertain return with the same expected value. Risk-neutral investors care about only expected returns. The expected rates of return of all securities must be the riskless rate when investors are risk-neutral. For this reason, p is called the **risk-neutral probability**. Because risk preferences and q are not directly involved in pricing options, any risk attitude, including risk neutrality, should give the same result. The value of an option can therefore be interpreted as the expectation of its discounted future payoff in a **risk-neutral economy**. So it turns out that the rate used for discounting the FV is the riskless rate in a risk-neutral economy. Risk-neutral valuation is perhaps the most important tool for the analysis of derivative securities.

We will need the following definitions shortly. Denote the **binomial distribution** with parameters n and p by

$$b(j; n, p) \equiv \binom{n}{j} p^j (1-p)^{n-j} = \frac{n!}{j!\,(n-j)!}\, p^j (1-p)^{n-j}.$$

Recall that $n! = n \times (n-1) \cdots 2 \times 1$ with the convention $0! = 1$. Hence $b(j; n, p)$ is the probability of getting j heads when tossing a coin n times, where p is the probability of getting heads. The probability of getting at least k heads when tossing a coin n times is this **complementary binomial distribution function** with parameters n and p:

$$\Phi(k; n, p) \equiv \sum_{j=k}^{n} b(j; n, p).$$

Because getting fewer than k heads is equivalent to getting at least $n-k+1$ tails,

$$1 - \Phi(k; n, p) = \Phi(n-k+1; n, 1-p). \tag{9.6}$$

➤ **Exercise 9.2.3** Prove that the call's expected gross return in a risk-neutral economy is R.

➤ **Exercise 9.2.4** Suppose that a call costs $hS + B + k$ for some $k \neq 0$ instead of $hS + B$. How does one make an arbitrage profit of M dollars?

➤ **Exercise 9.2.5** The standard arbitrage argument was used in deriving the call value. Use the risk-neutral argument to reach the same value.

9.2.3 Options on a Non-Dividend-Paying Stock: Multiperiod

Consider a call with two periods remaining before expiration. Under the binomial model, the stock can take on three possible prices at time two: Suu, Sud, and Sdd (see Fig. 9.2.3). Note that, at any node, the next two stock prices depend on only the

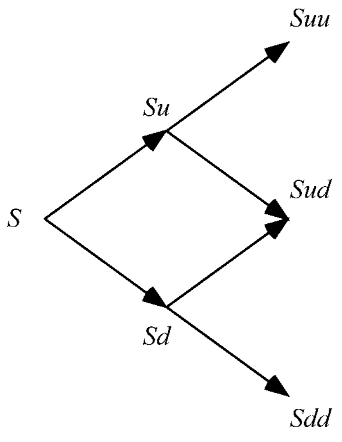


Figure 9.3: Stock prices in two periods. This graph is called a **binomial tree**, although **binomial lattice** is a better term as real tree branches do not merge.

current price, not on the prices of earlier times. This memoryless property is a key feature of an efficient market.[2]

Let C_{uu} be the call's value at time two if the stock price is Suu. Thus

$$C_{uu} = \max(0, Suu - X).$$

C_{ud} and C_{dd} can be calculated analogously:

$$C_{ud} = \max(0, Sud - X), \qquad C_{dd} = \max(0, Sdd - X).$$

See Fig. 9.4 for illustration. We can obtain the call values at time one by applying the same logic as that in Subsection 9.2.2 as follows:

$$C_u = \frac{pC_{uu} + (1-p)\,C_{ud}}{R}, \qquad C_d = \frac{pC_{ud} + (1-p)\,C_{dd}}{R}. \tag{9.7}$$

Deltas can be derived from Eq. (9.1). For example, the delta at C_u is $(C_{uu} - C_{ud})/(Suu - Sud)$.

We now reach the current period. An equivalent portfolio of h shares of stock and $\$B$ riskless bonds can be set up for the call that costs C_u (C_d) if the stock price goes to Su (Sd, respectively). The values of h and B can be derived from Eqs. (9.1) and (9.2).

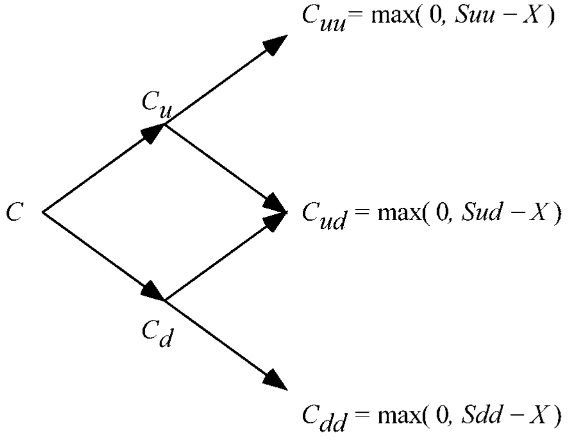


Figure 9.4: Value of a two-period call before expiration.

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Because the call will not be exercised at time one even if it is American, $C_u > Su - X$ and $C_d > Sd - X$. Therefore,

$$hS + B = \frac{pC_u + (1-p)\,C_d}{R} > \frac{(pu + (1-p)\,d)\,S - X}{R} = S - \frac{X}{R} > S - X.$$

So the call again will not be exercised at present, and

$$C = hS + B = \frac{pC_u + (1-p)\,C_d}{R}.$$

The above expression calculates C from the two successor nodes C_u and C_d and none beyond. The same computation happens at C_u and C_d, too, as demonstrated in Eqs. (9.7). This recursive procedure is called **backward induction** because it works backward in time [27, 66]. Now,

$$\begin{aligned} C &= \frac{p^2 C_{uu} + 2p(1-p)\,C_{ud} + (1-p)^2 C_{dd}}{R^2} \\ &= \frac{p^2 \times \max\,(0, Su^2 - X) + 2p(1-p) \times \max\,(0, Sud - X) + (1-p)^2 \times \max\,(0, Sd^2 - X)}{R^2}. \end{aligned}$$

The general case is straightforward: Simply carry out the same calculation at every node while moving backward in time. In the n-period case,

$$C = \frac{\sum_{j=0}^{n} \binom{n}{j}\, p^j (1-p)^{n-j} \times \max(0,\, Su^j d^{n-j} - X)}{R^n}. \tag{9.8}$$

It says that the value of a call on a non-dividend-paying stock is the expected discounted value of the payoff at expiration in a risk-neutral economy. As this C is the only option value consistent with no arbitrage opportunities, it is called an **arbitrage value**. Note that the option value depends on S, X, $\widehat{r}$, u, d, and n. Similarly, the value of a European put is

$$P = \frac{\sum_{j=0}^{n} \binom{n}{j}\, p^j (1-p)^{n-j} \times \max(0,\, X - Su^j d^{n-j})}{R^n}.$$

The findings are summarized below.

LEMMA 9.2.1 *The value of a European option equals the expected discounted payoff at expiration in a risk-neutral economy.*

In fact, every derivative can be priced as if the economy were risk-neutral [420]. For a European-style derivative with the terminal payoff function $\mathcal{D}$, its value is

$$e^{-\widehat{r}n} E^{\pi}[\,\mathcal{D}\,],$$

where E^{π} means that the expectation is taken under the risk-neutral probability.

Because the value of delta changes over time, the maintenance of an equivalent portfolio is a dynamic process. The dynamic maintaining of an equivalent portfolio does not depend on our correctly predicting future stock prices. By construction, the portfolio's value at the end of the current period, which can be either C_u or C_d, is precisely the amount needed to set up the next portfolio. The trading strategy is hence **self-financing** because there is neither injection nor withdrawal of funds throughout and changes in value are due entirely to capital gains.

Let a be the minimum number of upward price moves for the call to finish in the money. Obviously a is the smallest nonnegative integer such that $Su^a d^{n-a} \geq X$, or

$$a = \left\lceil \frac{\ln(X/Sd^n)}{\ln(u/d)} \right\rceil. \tag{9.9}$$

Hence,

$$\begin{aligned} C &= \frac{\sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} (Su^j d^{n-j} - X)}{R^n} \\ &= S \sum_{j=a}^{n} \binom{n}{j} \frac{(pu)^j [\,(1-p)\,d\,]^{n-j}}{R^n} - \frac{X}{R^n} \sum_{j=a}^{n} \binom{n}{j} p^j (1-p)^{n-j} \\ &= S \sum_{j=a}^{n} b(j; n, pue^{-\widehat{r}}) - Xe^{-\widehat{r}n} \sum_{j=a}^{n} b(j; n, p) \end{aligned} \tag{9.10}$$

The findings are summarized below.

THEOREM 9.2.2 *The value of a European call and the value of a European put are*

$$C = S\Phi(a; n, pue^{-\widehat{r}}) - Xe^{-\widehat{r}n}\Phi(a; n, p),$$
$$P = Xe^{-\widehat{r}n}\Phi(n-a+1; n, 1-p) - S\Phi(n-a+1; n, 1-pue^{-\widehat{r}}),$$

respectively, where $p \equiv (e^{\widehat{r}} - d)/(u-d)$ *and* a *is the minimum number of upward price moves for the option to finish in the money.*

The option value for the put above can be obtained with the help of the put–call parity and Eq. (9.6). It can also be derived from the same logic as underlies the steps for the call but with $\max(0, S-X)$ replaced with $\max(0, X-S)$ at expiration. It is noteworthy that with the random variable $\boldsymbol{S}$ denoting the stock price at expiration, the options' values are

$$C = S \times \text{Prob}_1[\,\boldsymbol{S} \geq X\,] - Xe^{-\widehat{r}n} \times \text{Prob}_2[\,\boldsymbol{S} \geq X\,], \tag{9.11}$$

$$P = Xe^{-\widehat{r}n} \times \text{Prob}_2[\,\boldsymbol{S} \leq X\,] - S \times \text{Prob}_1[\,\boldsymbol{S} \leq X\,], \tag{9.11$'$}$$

where Prob_1 uses pu/R and Prob_2 uses p for the probability that the stock price moves from S to Su. Prob_2 expresses the probability that the option will be exercised in a risk-neutral world. Exercise 13.2.12 will offer an interpretation for Prob_1.

A market is **complete** if every derivative security is attainable [420]. There are $n+1$ possible states of the world at expiration corresponding to the $n+1$ stock prices $Su^i d^{n-i}$, $0 \leq i \leq n$. Consider $n+1$ state contingent claims, the ith of which pays \$1 at expiration if the stock price is $Su^i d^{n-i}$ and zero otherwise. These claims make the market complete for European-style derivatives that expire at time n. The reason is that a European-style derivative that pays p_i dollars when the stock price finishes at $Su^i d^{n-i}$ can be replicated by a portfolio consisting of p_i units of the ith state contingent claim for $0 \leq i \leq n$. In the case of **continuous trading** in which trading is allowed for each period, two securities suffice to replicate every possible derivative and make the market complete (see Exercise 9.2.10) [289, 434].

The existence of risk-neutral valuation is usually taken to *define* arbitrage freedom in a model in that no self-financing trading strategies can earn arbitrage profits. In fact, the existence of risk-neutral valuation does imply arbitrage freedom for discrete-time models such as the BOPM. The converse proposition, that arbitrage freedom implies the existence of a risk-neutral probability, can be rigorously proved; besides, this probability measure is unique for complete markets. The "equivalence" between arbitrage freedom in a model and the existence of a risk-neutral probability is called the (first) **fundamental theorem of asset pricing** .

➤ **Exercise 9.2.6** Prove that early exercise is not optimal for American calls.

➤ **Exercise 9.2.7** Show that the call's delta is always nonnegative.

➤ **Exercise 9.2.8** Inspect Eq. (9.10) under $u \to d$, that is, zero volatility in stock prices.

➤ **Exercise 9.2.9** Prove the put–call parity for European options under the BOPM.

➤ **Exercise 9.2.10** Assume the BOPM. (1) Show that a state contingent claim that pays \$1 when the stock price reaches $Su^i d^{n-i}$ and \$0 otherwise at time n can be replicated by a portfolio of calls. (2) Argue that continuous trading with bonds and stocks can replicate any state contingent claim.

➤ **Exercise 9.2.11** Consider a single-period binomial model with two risky assets S_1 and S_2 and a riskless bond. In the next step, there are only two states for the risky assets, $(S_1 u_1, S_2 u_2)$ and $(S_1 d_1, S_2 d_2)$. Show that this model does not admit a risk-neutral probability for certain u_1, u_2, d_1, d_2, and R. (Hence it is not arbitrage free.)

A Numerical Example

A non-dividend-paying stock is selling for \$160 per share. From every price S, the stock price can go to either $S \times 1.5$ or $S \times 0.5$. There also exists a riskless bond with a continuously compounded interest rate of 18.232% per period. Consider a European call on this stock with a strike price of \$150 and three periods to expiration. The price movements for the stock price and the call value are shown in Fig. 9.5. The call value is found to be \$85.069 by backward induction. The same value can also be found as the PV of the expected payoff at expiration:

$$\frac{(390 \times 0.343) + (30 \times 0.441)}{(1.2)^3} = 85.069.$$

Observe that the delta value changes with the stock price and time.

Any mispricing leads to arbitrage profits. Suppose that the option is selling for \$90 instead. We sell the call for \$90 and invest \$85.069 in the replicating portfolio with 0.82031 shares of stock as required by delta. To set it up, we need to borrow $(0.82031 \times 160) - 85.069 = 46.1806$ dollars. The fund that remains, $90 - 85.069 = 4.931$ dollars, is the arbitrage profit, as we will see shortly.

Time 1. Suppose that the stock price moves to \$240. The new delta is 0.90625. Buy $0.90625 - 0.82031 = 0.08594$ more shares at the cost of $0.08594 \times 240 = 20.6256$ dollars financed by borrowing. Our debt now totals $20.6256 + (46.1806 \times 1.2) = 76.04232$ dollars.

Time 2. Suppose the stock price plunges to \$120. The new delta is 0.25. Sell $0.90625 - 0.25 = 0.65625$ shares for an income of $0.65625 \times 120 = 78.75$ dollars. Use this income to reduce the debt to $(76.04232 \times 1.2) - 78.75 = 12.5$ dollars.

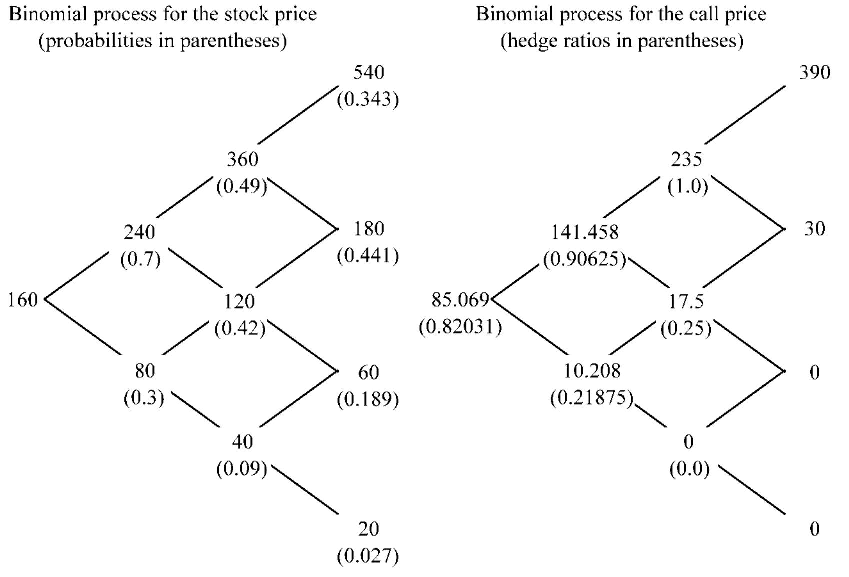

Figure 9.5: Stock prices and European call prices. The parameters are $S = 160$, $X = 150$, $n = 3, u = 1.5, d = 0.5, R = e^{0.18232} = 1.2, p = (R - d)/(u - d) = 0.7, h = (C_u - C_d)/(Su - Sd) = (C_u - C_d)/S$, and $C = [\, pC_u + (1 - p)\, C_d\,]/R = (0.7 \times C_u + 0.3 \times C_d)/1.2$.

Time 3 (The case of rising price). The stock price moves to \$180, and the call we wrote finishes in the money. For a loss of $180 - 150 = 30$ dollars, we close out the position by either buying back the call or buying a share of stock for delivery. Financing this loss with borrowing brings the total debt to $(12.5 \times 1.2) + 30 = 45$ dollars, which we repay by selling the 0.25 shares of stock for $0.25 \times 180 = 45$ dollars.

Time 4 (The case of declining price). The stock price moves to \$60. The call we wrote is worthless. Sell the 0.25 shares of stock for a total of $0.25 \times 60 = 15$ dollars to repay the debt of $12.5 \times 12 = 15$ dollars.

9.2.4 Numerical Algorithms for European Options

Binomial Tree Algorithms

An immediate consequence of the BOPM is the **binomial tree algorithm** that applies backward induction. The algorithm in Fig. 9.6 prices calls on a non-dividend-paying stock with the idea illustrated in Fig. 9.7. This algorithm is easy to analyze. The first loop can be made to take $O(n)$ steps, and the ensuing double loop takes $O(n^2)$ steps. The total running time is therefore quadratic. The memory requirement is also quadratic. To adapt the algorithm in Fig. 9.6 to price European puts, simply replace $\max(0, Su^{n-i}d^i - X)$ in Step 1 with $\max(0, X - Su^{n-i}d^i)$.

The binomial tree algorithm starts from the last period and works its way toward the current period. This suggests that the memory requirement can be reduced if the space is reused. Specifically, replace $C[\,n+1\,][\,n+1\,]$ in Fig. 9.6 with a one-dimensional array of size $n+1$, $C[\,n+1\,]$. Then replace step 1 with

$$C[\,i\,] := \max(0, Su^{n-i}d^i - X);$$

Binomial tree algorithm for pricing calls on a non-dividend-paying stock:

```
input:   S, u, d, X, n, r̂ (u > e^r̂ > d, r̂ > 0);
real     R, p, C[n+1][n+1];
integer i, j;
R := e^r̂;
p := (R − d)/(u − d);
for (i = 0 to n)
        1. C[n][i] := max(0, S u^(n−i) d^i − X);
for (j = n − 1 down to 0)
        for (i = 0 to j)
              2.1. C[j][i] := (p × C[j+1][i]+
                        (1 − p) × C[j+1][i+1])/R;
return C[0][0];
```

Figure 9.6: Binomial tree algorithm for calls on a non-dividend-paying stock. $C[j][i]$ represents the call value at time j if the stock price makes i downward movements out of a total of j movements.

Step 2.1 should now be modified as follows:

$$C[i] := \{p \times C[i] + (1-p) \times C[i+1]\}/R;$$

Finally, $C[0]$ is returned instead of $C[0][0]$. The memory size is now linear. The one-dimensional array captures the strip in Fig. 9.7 and will be used throughout the book.

We can make further improvements by observing that if $C[j+1][i]$ and $C[j+1][i+1]$ are both zeros, then $C[j][i]$ is zero, too. We need to let the i loop within the double loop run only from zero to $\min(n-a, j)$ instead of j, where a is defined in Eq. (9.9). This makes the algorithm run in $O(n(n-a))$ steps, which may be substantially smaller than $O(n^2)$ when a is large. The space requirement can be similarly reduced to $O(n-a)$ with a smaller one-dimensional array $C[n-a+1]$. See Fig. 9.8, in which the one-dimensional array implements the strip in that figure.

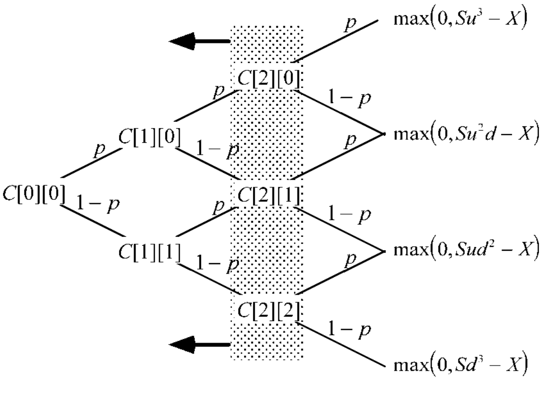

Figure 9.7: Backward induction on binomial trees. Binomial tree algorithms start with terminal values computed in step 1 of the algorithm in Fig. 9.6. They then sweep a strip backward in time to compute values at intermediate nodes until the root is reached.

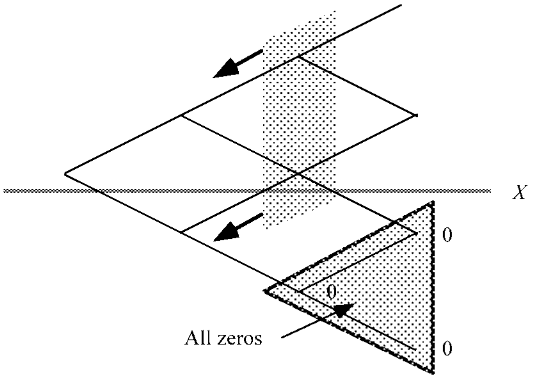

Figure 9.8: Skipping zero-valued nodes to improve efficiency. The stock expires worthless if it finishes below the horizontal line. Zeros at the terminal nodes propagate through the tree depicted here for a call. Such nodes can be skipped by binomial tree algorithms. Note that nodes at the same horizontal level have an identical stock price if $ud = 1$.

➢ **Programming Assignment 9.2.12** Implement the binomial tree algorithms for calls and puts.

An Optimal Algorithm

To reduce the running time to $O(n-a)$ and the memory requirement to $O(1)$, note that

$$b(j;n,p) = \frac{p(n-j+1)}{(1-p)j}\, b(j-1;n,p).$$

The following program computes $b(j;n,p)$ in $b[\,j\,]$ for $a \le j \le n$:

$$b[\,a\,] := \binom{n}{a} p^a (1-p)^{n-a};$$

$$\texttt{for}\quad (j = a+1\ \texttt{to}\ n)$$

$$b[\,j\,] := b[\,j-1\,] \times p \times (n-j+1)/[\,(1-p) \times j\,];$$

It clearly runs in $O(n-a)$ steps. With the $b(j;n,p)$ available, risk-neutral valuation formula (9.10) is trivial to compute. The case of puts is similar. As for the memory requirement, we need only a single variable instead of a whole array to store the $b(j;n,p)$s as they are being sequentially computed. The algorithm appears in Fig. 9.9. This linear-time algorithm computes the discounted expected value of $\max(S-X,0)$. It can be adapted to price any European option. For example, if the payoff function is $\max(\sqrt{S-X},0)$, we simply replace $D-X$ with $\sqrt{D-X}$ in the algorithm. The above technique cannot be applied to American options because of the possibility of early exercise. As a result, algorithms for American options usually run in quadratic time instead of in linear time. The performance gap between pricing American and European options seems inherent in general.

➤ **Exercise 9.2.13** Modify the linear-time algorithm in Fig. 9.9 to price puts.

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Linear-time, constant-space algorithm for pricing calls on a non-dividend-paying stock:

```
input:     S, u, d, X, n, r̂ (u > e^r̂ > d and r̂ > 0);
real       R, p, b, D, C;
integer    j, a;
a := ⌈ln(X/Sd^n)/ln(u/d)⌉;
p := (e^r̂ − d)/(u − d);
R := e^(n r̂);
b := p^a (1 − p)^(n−a); // b(a; n, p) is computed.
D := S × u^a d^(n−a);
C := b × (D − X)/R;
for (j = a + 1 to n) {
        b := b × p × (n − j + 1)/((1 − p) × j);
        D := D × u/d;
        C := C + b × (D − X)/R;
}
return C;
```

Figure 9.9: Optimal algorithm for European calls on a stock that does not pay dividends. Variable b stores $b(j; n, p)$ for $j = a, a+1, \ldots, n$, in that order, and variable C accumulates the summands in Eq. (9.10) by adding up $b(j; n, p) \times (Su^j d^{n-j} - X)/e^{n\widehat{r}}$, $j = a, a+1, \ldots, n$.

➢ **Programming Assignment 9.2.14** Implement the algorithm in Fig. 9.9 and benchmark its speed. Because variables such as b and D can take on extreme values, they should be represented in logarithms to maintain precision.

The Monte Carlo Method

Now is a good time to introduce the **Monte Carlo method**. Equation (9.8) can be interpreted as the expected value of the random variable $\boldsymbol{Z}$ defined by

$$\boldsymbol{Z} = \max(0, Su^j d^{n-j} - X)/R^n \quad \text{with probability } b(j; n, p), \quad 0 \le j \le n.$$

To approximate the expectation, throw n coins, with p being the probability of getting heads, and assign

$$\max(0, Su^j d^{n-j} - X)/R^n$$

to the experiment if it generates j heads. Repeat the procedure m times and take the average. This average clearly has the right expected value $E[\boldsymbol{Z}]$. Furthermore, its variance $\text{Var}[\boldsymbol{Z}]/m$ converges to zero as m increases.

Pricing European options may be too trivial a problem to which to apply the Monte Carlo method. We will see in Section 18.2 that the Monte Carlo method is an invaluable tool in pricing European-style derivative securities and MBSs.

➢ **Programming Assignment 9.2.15** Implement the Monte Carlo method. Observe its convergence rate as the sampling size m increases.

The Recursive Formulation and Its Algorithms

Most derivative pricing problems have a concise and natural *recursive* expression familiar to programmers. Yet a brute-force implementation should be resisted. For example, the recursive implementation of the binomial option pricing problem for

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the call is as follows:

$$
\begin{aligned}
&\texttt{Price}(S, u, d, X, n, \widehat{r}) \quad \{ \text{// Pricing European calls recursively.} \\
&\qquad p := (e^{\widehat{r}} - d)/(u - d); \\
&\qquad \text{if } [\, n = 0 \,] \text{ return } \max(S - X, 0); \\
&\qquad \text{else return } [\, p \times \text{Price}(Su, u, d, X, n-1, \widehat{r}) + (1 - p) \\
&\qquad\qquad\qquad \times \text{Price}(Sd, u, d, X, n-1, \widehat{r}) \,]/e^{\widehat{r}}; \\
&\}
\end{aligned}
$$

If every possible stock price sequence of length n is traced, the algorithm's running time is $O(n2^n)$, which is not practical.

9.3 The Black–Scholes Formula

On the surface, the binomial model suffers from two unrealistic assumptions: (1) The stock price takes on only two values in a period and (2) trading occurs at discrete points in time. These shortcomings are more apparent than real. As the number of periods increases, the stock price ranges over ever-larger numbers of possible values, and trading takes place nearly continuously. What needs to be done is proper calibration of the model parameters so that the model converges to the continuous-time model in the limit.

9.3.1 Distribution of the Rate of Return

Let τ denote the time to expiration of the option measured in years and r be the continuously compounded annual rate. With n periods during the option's life, each period therefore represents a time interval of τ/n. Our job is to adjust the period-based u, d, and interest rate represented by $\widehat{r}$ to match the empirical results as n goes to infinity. Clearly $\widehat{r} = r\tau/n$. As before, let R denote the period gross return $e^{\widehat{r}}$.

We proceed to derive u and d. Under the binomial model, $\ln u$ and $\ln d$ denote the stock's two possible continuously compounded rates of return per period. The rate of return in each period is characterized by the following **Bernoulli random variable**:

$$B = \begin{cases} \ln u, & \text{with probability } q \\ \ln d, & \text{with probability } 1 - q \end{cases}.$$

Let S_τ denote the stock price at expiration. The stock's continuously compounded rate of return, $\ln(S_\tau/S)$, is the sum of n independent Bernoulli random variables above, and

$$\ln \frac{S_\tau}{S} = \ln \frac{Su^j d^{n-j}}{S} = j \ln(u/d) + n \ln d, \tag{9.12}$$

where the stock price makes j upward movements in n periods. Because each upward price movement occurs with probability q, the expected number of upward price movements in n periods is $E[\, j\,] = nq$ with variance

$$\mathrm{Var}[\, j\,] = n[\, q(1-q)^2 + (1-q)(0-q)^2\,] = nq(1-q).$$

We use

$$\widehat{\mu} \equiv \frac{1}{n} E\left[\ln \frac{S_\tau}{S}\right], \qquad \widehat{\sigma}^2 \equiv \frac{1}{n} \operatorname{Var}\left[\ln \frac{S_\tau}{S}\right]$$

to denote, respectively, the expected value and the variance of the period continuously compounded rate of return. From the above,

$$\widehat{\mu} = \frac{E[\,j\,] \times \ln(u/d) + n \ln d}{n} = q \ln(u/d) + \ln d,$$

$$\widehat{\sigma}^2 = \frac{\operatorname{Var}[\,j\,] \times \ln^2(u/d)}{n} = q(1-q) \ln^2(u/d).$$

For the binomial model to converge to the expectation $\mu\tau$ and variance $\sigma^2\tau$ of the stock's true continuously compounded rate of return over τ years, the requirements are

$$n\widehat{\mu} = n(q \ln(u/d) + \ln d) \to \mu\tau, \tag{9.13}$$

$$n\widehat{\sigma}^2 = nq(1-q) \ln^2(u/d) \to \sigma^2\tau. \tag{9.14}$$

We call σ the stock's (annualized) **volatility**. Add $ud = 1$, which makes the nodes at the same horizontal level of the tree have an identical price (review Fig. 9.8). Then the above requirements can be satisfied by

$$u = e^{\sigma\sqrt{\tau/n}}, \qquad d = e^{-\sigma\sqrt{\tau/n}}, \qquad q = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{\tau}{n}}. \tag{9.15}$$

(See Exercises 9.3.1 and 9.3.8 for alternative choices of u, d, and q.) With Eqs. (9.15),

$$n\widehat{\mu} = \mu\tau,$$

$$n\widehat{\sigma}^2 = \left[1 - \left(\frac{\mu}{\sigma}\right)^2 \frac{\tau}{n}\right] \sigma^2\tau \to \sigma^2\tau.$$

We remark that the no-arbitrage inequalities $u > R > d$ may not hold under Eqs. (9.15), and the risk-neutral probability may lie outside $[\,0, 1\,]$. One solution can be found in Exercise 9.3.1 and another in Subsection 12.4.3. In any case, the problems disappear when n is suitably large.

What emerges as the limiting probabilistic distribution of the continuously compounded rate of return $\ln(S_\tau/S)$? The central limit theorem says that, under certain weak conditions, sums of independent random variables such as $\ln(S_\tau/S)$ converge to the normal distribution, i.e.,

$$\operatorname{Prob}\left[\frac{\ln(S_\tau/S) - n\widehat{\mu}}{\sqrt{n}\,\widehat{\sigma}} \leq z\right] \to N(z).$$

A simple condition for the central limit theorem to hold is the **Lyapunov condition** [100],

$$\frac{q\,|\ln u - \widehat{\mu}\,|^3 + (1-q)|\ln d - \widehat{\mu}\,|^3}{n\widehat{\sigma}^3} \to 0.$$

After substitutions, the condition becomes

$$\frac{(1-q)^2+q^2}{n\sqrt{q(1-q)}} \to 0,$$

which is true. So the continuously compounded rate of return approaches the normal distribution with mean $\mu\tau$ and variance $\sigma^2\tau$. As a result, $\ln S_\tau$ approaches the normal distribution with mean $\mu\tau + \ln S$ and variance $\sigma^2\tau$. S_τ thus has a lognormal distribution in the limit. The significance of using the continuously compounded rate is now clear: to make the rate of return normally distributed.

The lognormality of a stock price has several consequences. It implies that the stock price stays positive if it starts positive. Furthermore, although there is no upper bound on the stock price, large increases or decreases are unlikely. Finally, equal movements in the rate of return about the mean are equally likely because of the symmetry of the normal distribution: S_1 and S_2 are equally likely if $S_1/S = S/S_2$.

➤ **Exercise 9.3.1** The price volatility of the binomial model should match that of the actual stock in the limit. As q does not play a direct role in the BOPM, there is more than one way to assign u and d. Suppose we require that $q = 0.5$ instead of $ud = 1$. (1) Show that

$$u = \exp\left[\frac{\mu\tau}{n} + \sigma\sqrt{\frac{\tau}{n}}\right], \qquad d = \exp\left[\frac{\mu\tau}{n} - \sigma\sqrt{\frac{\tau}{n}}\right]$$

satisfy requirements (9.13) and (9.14) as *equalities*. (2) Is it valid to use the probability 0.5 during backward induction under these new assignments?

Comment 9.3.1 Recall that the Monte Carlo method in Subsection 9.2.4 used a biased coin. The scheme in Exercise 9.3.1, in contrast, used a *fair* coin, which may be easier to program. The choice in Eqs. (9.15) nevertheless has the advantage that $ud = 1$, which is often easier to work with algorithmically. Alternative choices of u and d are expected to have only slight, if any, impacts on the convergence of binomial tree algorithms [110].

➤ **Exercise 9.3.2** Show that

$$\frac{E[\,(S_{\Delta t} - S)/S\,]}{\Delta t} \to \mu + \frac{\sigma^2}{2}, \tag{9.16}$$

where $\Delta t \equiv \tau/n$.

Comment 9.3.2 Note the distinction between Eq. (9.13) and convergence (9.16). The former says that the annual continuously compounded rate of return over τ years, $\ln(S_\tau/S)/\tau$, has mean μ, whereas the latter says that the instantaneous rate of return, $\lim_{\Delta t\to 0}(S_{\Delta t} - S)/S)/\Delta t$, has a larger mean of $\mu + \sigma^2/2$.

9.3.2 Toward the Black–Scholes Formula

We now take the final steps toward the Black–Scholes formula as $n \to \infty$ and q equals the risk-neutral probability $p \equiv (e^{r\tau/n} - d)/(u-d)$.

LEMMA 9.3.3 *The continuously compounded rate of return* $\ln(S_\tau/S)$ *approaches the normal distribution with mean* $(r - \sigma^2/2)\,\tau$ *and variance* $\sigma^2\tau$ *in a risk-neutral economy.*

Proof: Applying $e^y = 1 + y + (y^2/2!) + \cdots$ to p, we obtain

$$p \to \frac{1}{2} + \frac{1}{2}\,\frac{r - \sigma^2/2}{\sigma}\sqrt{\frac{\tau}{n}}. \tag{9.17}$$

So the q in Eq. (9.15) implies that $\mu = r - \sigma^2/2$ and

$$n\widehat{\mu} = \left(r - \frac{\sigma^2}{2}\right)\tau$$

$$n\,\widehat{\sigma}^2 = \left[1 - \left(\frac{r - \sigma^2/2}{\sigma}\right)^2 \frac{\tau}{n}\right]\sigma^2\tau \to \sigma^2\tau.$$

Because

$$\frac{(1-p)^2 + p^2}{n\sqrt{p(1-p)}} \to 0,$$

the Lyapunov condition is satisfied and the central limit theorem is applicable.

Lemma 9.3.3 and Eqs. (6.11) imply that the expected stock price at expiration in a risk-neutral economy is $Se^{r\tau}$. The stock's expected annual rate of return is thus the riskless rate r.

THEOREM 9.3.4 (The Black-Scholes Formula):

$$C = SN(x) - Xe^{-r\tau}N(x - \sigma\sqrt{\tau}),$$
$$P = Xe^{-r\tau}N(-x + \sigma\sqrt{\tau}) - SN(-x),$$

where

$$x \equiv \frac{\ln(S/X) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

Proof: As the put–call parity can be used to prove the formula for a European put from that for a call, we prove the formula for the call only. The binomial option pricing formula in Theorem 9.2.2 is similar to the Black–Scholes formula. Clearly, we are done if

$$\Phi(a; n, pue^{-\widehat{r}}) \to N(x), \qquad \Phi(a; n, p) \to N(x - \sigma\sqrt{\tau}). \tag{9.18}$$

We prove only $\Phi(a; n, p) \to N(x - \sigma\sqrt{\tau})$; the other part can be verified analogously.

Recall that $\Phi(a; n, p)$ is the probability of at least a successes in n independent trials with success probability p for each trial. Let j denote the number of successes (upward price movements) in n such trials. This random variable, a sum of n Bernoulli variables, has mean np and variance $np(1-p)$ and satisfies

$$1 - \Phi(a; n, p) = \text{Prob}[\, j \leq a - 1\,] = \text{Prob}\left[\frac{j - np}{\sqrt{np(1-p)}} \leq \frac{a - 1 - np}{\sqrt{np(1-p)}}\right]. \tag{9.19}$$

It is easy to verify that

$$\frac{j - np}{\sqrt{np(1-p)}} = \frac{\ln(S_\tau/S) - n\widehat{\mu}}{\sqrt{n}\,\widehat{\sigma}}.$$

Now,

$$a - 1 = \frac{\ln(X/Sd^n)}{\ln(u/d)} - \epsilon$$

for some $0 < \epsilon \leq 1$. Combine the preceding equality with the definitions for $\widehat{\mu}$ and $\widehat{\sigma}$ to obtain

$$\frac{a - 1 - np}{\sqrt{np(1-p)}} = \frac{\ln(X/S) - n\widehat{\mu} - \epsilon \ln(u/d)}{\sqrt{n}\,\widehat{\sigma}}.$$

So Eq. (9.19) becomes

$$1 - \Phi(a; n, p) = \text{Prob}\left[\frac{\ln(S_\tau/S) - n\widehat{\mu}}{\sqrt{n}\,\widehat{\sigma}} \leq \frac{\ln(X/S) - n\widehat{\mu} - \epsilon \ln(u/d)}{\sqrt{n}\,\widehat{\sigma}}\right].$$

Because $\ln(u/d) = 2\sigma\sqrt{\tau/n} \to 0$,

$$\frac{\ln(X/S) - n\widehat{\mu} - \epsilon \ln(u/d)}{\sqrt{n}\,\widehat{\sigma}} \to z \equiv \frac{\ln(X/S) - \tau\left(r - \sigma^2/2\right)}{\sigma\sqrt{\tau}}.$$

Hence $1 - \Phi(a; n, p) \to N(z)$, which implies that

$$\Phi(a; n, p) \to N(-z) = N\left(\frac{\ln(S/X) + r\tau}{\sigma\sqrt{\tau}} - \frac{1}{2}\sigma\sqrt{\tau}\right) = N(x - \sigma\sqrt{\tau}),$$

as desired.

We plot the call and put values as a function of the current stock price, time to expiration, volatility, and interest rate in Fig. 9.10. Note particularly that the option value for at-the-money options is essentially a linear function of volatility.

➤ **Exercise 9.3.3** Verify the following with the Black–Scholes formula and give heuristic arguments as to why they should hold without invoking the formula. (1) $C \approx S - Xe^{-r\tau}$ if $S \gg X$. (2) $C \to S$ as $\tau \to \infty$. (3) $C \to 0$ as $\sigma \to 0$ if $S < Xe^{-r\tau}$. (4) $C \to S - Xe^{-r\tau}$ as $\sigma \to 0$ if $S > Xe^{-r\tau}$. (5) $C \to S$ as $r \to \infty$.

➤ **Exercise 9.3.4** Verify convergence (9.17).

➤ **Exercise 9.3.5** A **binary call** pays off \$1 if the underlying asset finishes above the strike price and nothing otherwise.[3] Show that its price equals $e^{-r\tau} N(x - \sigma\sqrt{\tau})$.

➤ **Exercise 9.3.6** Prove $\partial^2 P/\partial X^2 = \partial^2 C/\partial X^2$ (see Fig. 9.11 for illustration).

➤ **Exercise 9.3.7** Derive Theorem 9.3.4 from Lemma 9.3.3 and Exercise 6.1.6.

Tabulating Option Values

Rewrite the Black–Scholes formula for the European call as follows:

$$C = Xe^{-r\tau}\left[\frac{S}{Xe^{-r\tau}} N(x) - N(x - \sigma\sqrt{\tau})\right],$$

where

$$x \equiv \frac{\ln(S/(Xe^{-r\tau}))}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}.$$

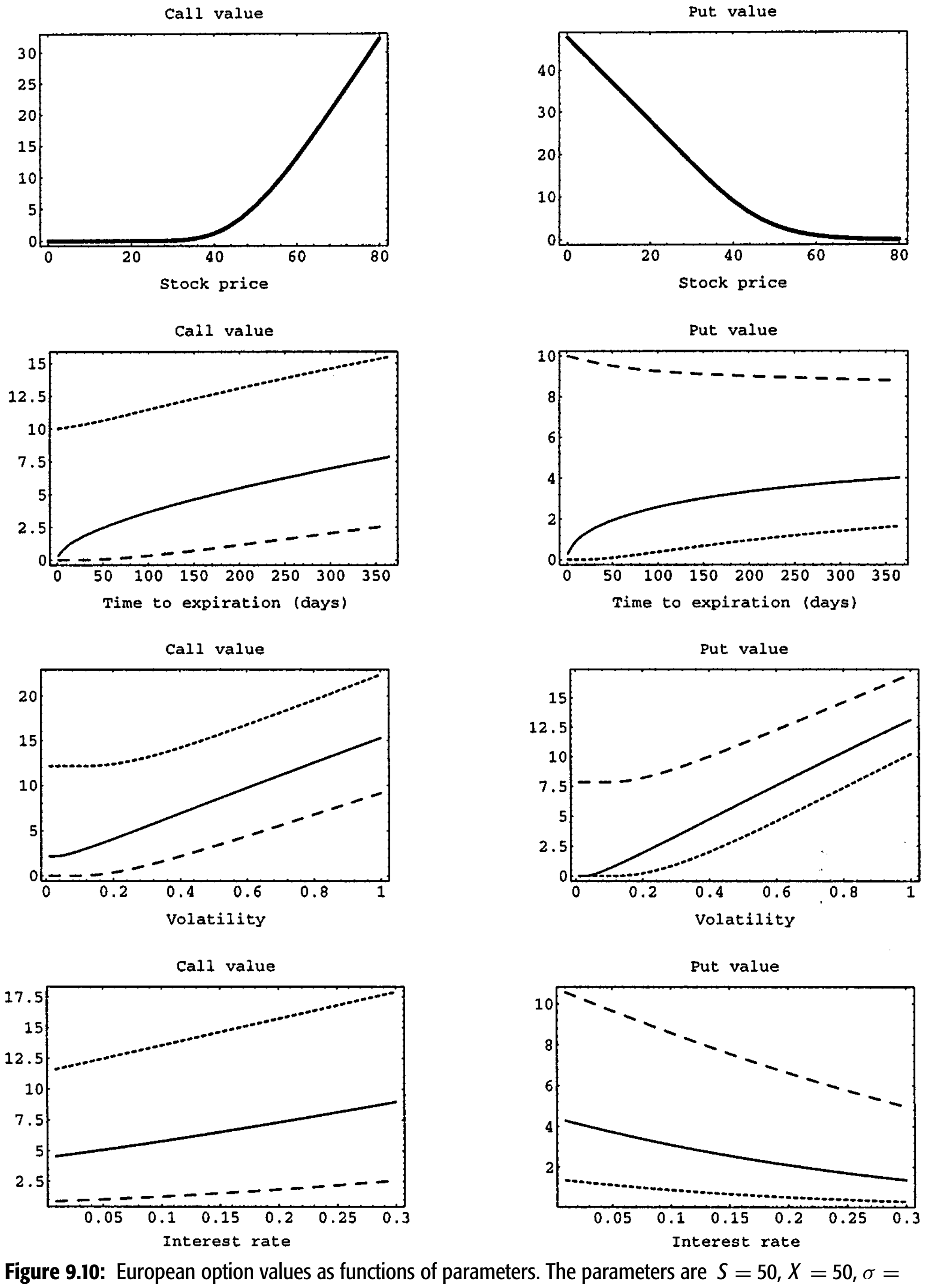

Figure 9.10: European option values as functions of parameters. The parameters are $S = 50$, $X = 50$, $\sigma = 0.3$, $\tau = 201$ (days), and $r = 8\%$. When three curves are plotted together, the dashed curve uses $S = 40$ (out-of-the-money call or in-the-money put), the solid curve uses $S = 50$ (at the money), and the dotted curve uses $S = 60$ (in-the-money call or out-of-the-money put).

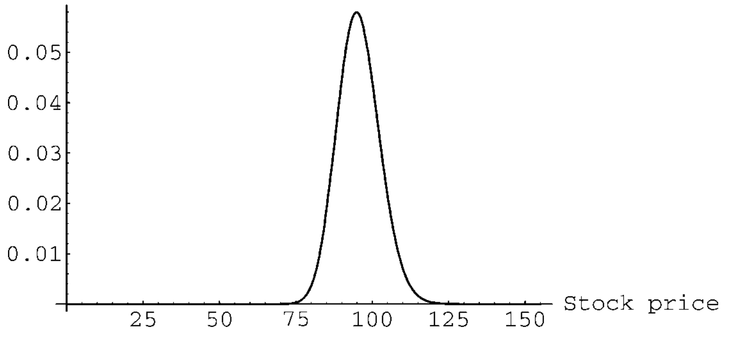

Figure 9.11: Value of state contingent claim. Exercise 7.4.5 says that $\partial^2 C/\partial X^2$, plotted here for the strike price of \$95, is the value of a state contingent claim. The fundamental identity in Exercise 9.3.6 has applications in asset pricing.

A table containing entries

$$\frac{S}{Xe^{-r\tau}}\, N(x) - N(x - \sigma\sqrt{\tau})$$

indexed by $S/(Xe^{-r\tau})$ and $\sigma\sqrt{\tau}$ allows a person to look up option values based on S, X, r, τ, and σ. The call value is then a simple multiplication of the looked-up value by $Xe^{-r\tau}$. A precomputed table of judiciously selected option values can actually be used to price options by means of interpolation [529].

9.3.3 The Black–Scholes Model and the BOPM

The Black–Scholes formula needs five parameters: S, X, σ, τ, and r. However, binomial tree algorithms take six inputs: S, X, u, d, $\widehat{r}$, and n. The connections are

$$u = e^{\sigma\sqrt{\tau/n}}, \qquad d = e^{-\sigma\sqrt{\tau/n}}, \qquad \widehat{r} = r\tau/n.$$

The resulting binomial tree algorithms converge reasonably fast, but oscillations, as displayed in Fig. 9.12, are inherent [704]. Oscillations can be eliminated by the judicious choices of u and d (see Exercise 9.3.8).

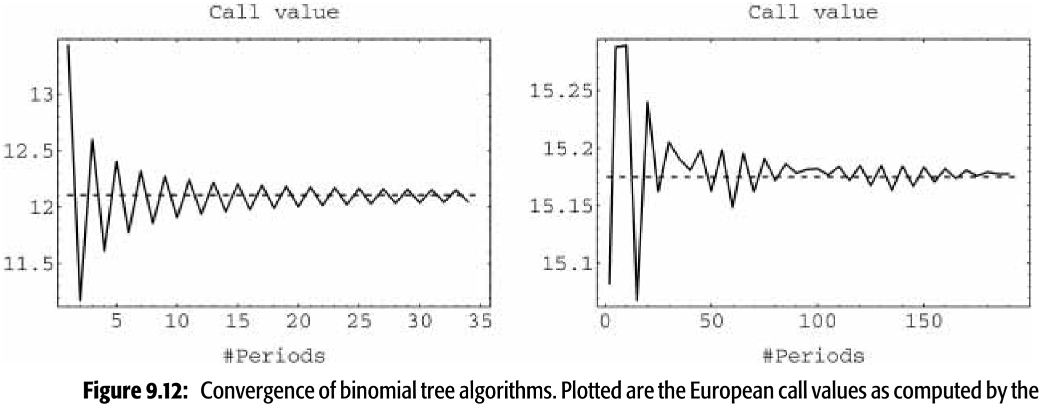

Figure 9.12: Convergence of binomial tree algorithms. Plotted are the European call values as computed by the binomial tree algorithm against the number of time partitions, n. The parameters used are $S = 100$, $X = 100$ (left) and 95 (right), $r = 8\%$, $\sigma = 0.2$, and $\tau = 1$. The analytical values, 12.1058 (left) and 15.1749 (right), are displayed for reference.

EXAMPLE 9.3.5 Consider a 3-month option when the interest rate is 8% per annum and the volatility is 30% per annum. This means that $\tau = 0.25$, $r = 0.08$, and $\sigma = 0.3$. If the binomial tree algorithm uses $n = 5$, it should use $u = e^{0.3\sqrt{0.25/5}} = 1.0694$ and $d = e^{-0.3\sqrt{0.25/5}} = 0.9351$.

➤ **Exercise 9.3.8** Here is yet another way to assign u and d:

$$u = e^{\sigma\sqrt{\tau/n}+(1/n)\ln(X/S)}, \qquad d = e^{-\sigma\sqrt{\tau/n}+(1/n)\ln(X/S)}, \qquad q = \frac{e^{r\tau/n} - d}{u - d}.$$

(1) Show that it works. (2) What is special about this choice?

9.4 Using the Black–Scholes Formula

9.4.1 Interest Rate

The riskless rate r should be the spot rate with a maturity near the option's expiration date (in practice, the specific rate depends on the investor [228]). The choice can be justified as follows. Let r_i denote the continuously compounded one-period interest rate measured in periods for period i. The bond maturing at the option's expiration date is worth $\exp[-\sum_{i=1}^{n} r_i]$ per dollar of face value. This implies that $r\tau = \sum_{i=1}^{n} r_i$. Hence a single discount bond price with maturity at time n (equivalently, the n-period spot rate) encompasses all the information needed for interest rates. In the limit, $\sum_{i=0}^{n-1} r_i \to \int_0^{\tau} r(t)\,dt$, where $r(t)$ is the short rate at time t. The relevant annualized interest rate is thus $r = (1/\tau)\int_0^{\tau} r(t)\,dt$.

Interest rate uncertainty may not be very critical for options with lives under 1 year. Plots in Fig. 9.10 also suggest that small changes in interest rates, other things being equal, do not move the option value significantly.

9.4.2 Estimating the Volatility from Historical Data

The volatility parameter σ is the sole parameter not directly observable and has to be estimated. The Black–Scholes formula assumes that stock prices are lognormally distributed. In other words, the n continuously compounded rates of return per period,

$$u_i \equiv \ln \frac{S_i}{S_{i-1}}, \quad i = 1, 2, \ldots, n,$$

are independent samples from a normal distribution with mean $\mu\tau/n$ and variance $\sigma^2\tau/n$, where S_i denotes the stock price at time i. A good estimate of the standard deviation of the per-period rate of return is

$$s \equiv \sqrt{\frac{\sum_{i=1}^{n}(u_i - \overline{u})^2}{n-1}},$$

where $\overline{u} \equiv (1/n)\sum_{i=1}^{n} u_i = (1/n)\ln(S_n/S_0)$. The preceding estimator may be biased in practice, however, notably because of the bid–ask spreads and the discreteness of stock prices [48, 201]. Estimators that utilize high and low prices can be superior theoretically in terms of lower variance [374]. We note that $\overline{u}$ and $s^2(n-1)/n$ are the ML estimators of μ and σ^2, respectively (see Section 20.1).

The **simple rate of return**, $(S_i - S_{i-1})/S_{i-1}$, is sometimes used in place of u_i to avoid logarithms. This is not entirely correct because $\ln x \approx x - 1$ only when x is small, and a small error here can mean huge differences in the option value [147].

If a period contains an ex-dividend date, its sample rate of return should be modified to

$$u_i = \ln \frac{S_i + D}{S_{i-1}},$$

where D is the amount of the dividend. If an n-for-m split occurs in a period, the sample rate of return should be modified to

$$u_i = \ln \frac{nS_i}{mS_{i-1}}.$$

Because the standard deviation of the rate of return equals $\sigma\sqrt{\tau/n}$, the estimate for σ is $s/\sqrt{\tau/n}$. This value is called **historical volatility**. Empirical evidence suggests that days when stocks were not traded should be excluded from the calculation. Some even count only trading days in the time to expiration τ [514].

Like interest rate, volatility is allowed to change over time as long as it is predictable. In the context of the binomial model, this means that u and d now depend on time. The variance of $\ln(S_\tau/S)$ is now $\int_0^\tau \sigma^2(t)\, dt$ rather than $\sigma^2\tau$, and the volatility becomes $[\int_0^\tau \sigma^2(t)\, dt/\tau\,]^{1/2}$. A word of caution here: There is evidence suggesting that volatility is stochastic (see Section 15.5).

9.4.3 Implied Volatility

The Black–Scholes formula can be used to compute the market's opinion of the volatility. This is achieved by the solution of σ given the option price, S, X, τ, and r with the numerical methods in Subsection 3.4.3. The volatility thus obtained is called the **implied volatility** – the volatility implied by the market price of the option. Volatility numbers are often stored in a table indexed by maturities and strike prices [470, 482].

Implied volatility is often preferred to historical volatility in practice, but it is not perfect. Options written on the same underlying asset usually do not produce the same implied volatility. A typical pattern is a "**smile**" in relation to the strike price: The implied volatility is lowest for at-the-money options and becomes higher the further the option is in or out of the money [150]. This pattern is especially strong for short-term options [44] and cannot be accounted for by the early exercise feature of American options [97]. To address this issue, volatilities are often combined to produce a composite implied volatility. This practice is not sound theoretically. In fact, the existence of different implied volatilities for options on the same underlying asset shows that the Black–Scholes option pricing model cannot be literally true. Section 15.5 will survey approaches that try to explain the smile.

➤ **Exercise 9.4.1** Calculating the implied volatility from the option price can be facilitated if the option price is a monotonic function of volatility. Show that this is true of the Black–Scholes formula.

➤ **Exercise 9.4.2** Solving for the implied volatility of American options as if they were European overestimates the true volatility. Discuss.

➤ **Exercise 9.4.3 (Implied Binomial Tree).** Suppose that we are given m different European options prices, their identical maturity, their strike prices, their underlying asset's current price, the underlying asset's σ, and the riskless rate. (1) What should n be? (2) Assume that the path probabilities for all paths reaching the same node are equal. How do we compute the (implied) branching probabilities at each node of the binomial tree so that these options are all priced correctly?

➤ **Programming Assignment 9.4.4** Write a program to compute the implied volatility of American options.

9.5 American Puts on a Non-Dividend-Paying Stock

Early exercise has to be considered when pricing American puts. Because the person who exercises a put receives the strike price and earns the time value of money, there is incentive for early exercise. On the other hand, early exercise may render the put holder worse off if the stock subsequently increases in value.

The binomial tree algorithm starts with the terminal payoffs $\max(0, X - Su^j d^{n-j})$ and applies backward induction. At each intermediate node, it checks for early exercise by comparing the payoff if exercised with continuation. The complete quadratic-time algorithm appears in Fig. 9.13. Figure 9.14 compares an American put with its European counterpart.

Let us go through a numerical example. Assume that $S = 160$, $X = 130$, $n = 3$, $u = 1.5$, $d = 0.5$, and $R = e^{0.18232} = 1.2$. We can verify that $p = (R - d)/(u - d) = 0.7$, $h = (P_u - P_d)/S(u - d) = (P_u - P_d)/S$, and $P = [\, pP_u + (1 - p)\, P_d\,]/R = (0.7 \times P_u + 0.3 \times P_d)/1.2$. Consider node A in Fig. 9.15. The continuation value is

$$\frac{(0.7 \times 0) + (0.3 \times 70)}{1.2} = 17.5,$$

greater than the intrinsic value $130 - 120 = 10$. Hence the option should not be exercised even if it is in the money and the put value is 17.5. As for node B, the continuation value is

$$\frac{(0.7 \times 70) + (0.3 \times 110)}{1.2} = 68.33,$$

lower than the intrinsic value $130 - 40 = 90$. The option should be exercised, and the put value is 90.

Binomial tree algorithm for pricing American puts on a non-dividend-paying stock:

input: $S, u, d, X, n, \widehat{r}$ ($u > e^{\widehat{r}} > d$ and $\widehat{r} > 0$);
real $R, p, P[n+1]$;
integer i, j;
$R := e^{\widehat{r}}$;
$p := (R - d)/(u - d)$;
for ($i = 0$ to n) { $P[i] := \max(0, X - Su^{n-i} d^i)$; }
for ($j = n - 1$ down to 0)
    for ($i = 0$ to j)
        $P[i] := \max((p \times P[i] + (1 - p) \times P[i + 1])/R, X - Su^{j-i} d^i)$;
return $P[0]$;

Figure 9.13: Binomial tree algorithm for American puts on a non-dividend-paying stock.

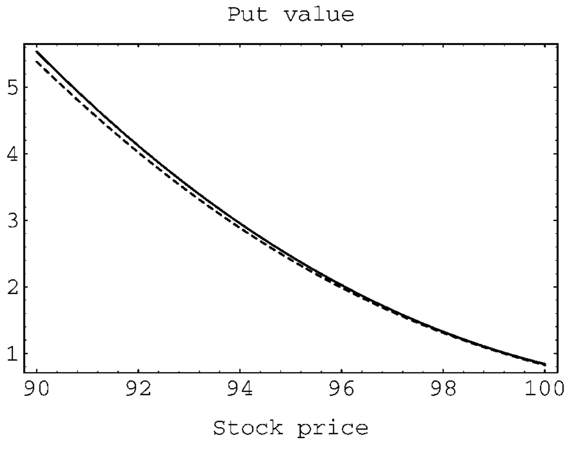

Figure 9.14: American put vs. European put. Plotted is the American put price at 1 month before expiration. The strike price is \$95, and the riskless rate is 8%. The volatility of the stock is assumed to be 0.25. The corresponding European put is also plotted (dotted curve) for comparison.

➢ **Programming Assignment 9.5.1** Implement the algorithm in Fig. 9.13 for American puts.

9.6 Options on a Stock that Pays Dividends

9.6.1 European Options on a Stock that Pays a Known Dividend Yield

The BOPM remains valid if dividends are predictable. A known **dividend yield** means that the dividend income forms a constant percentage of the stock price. For a dividend yield of δ, the stock pays out $S\delta$ on each ex-dividend date. Therefore the

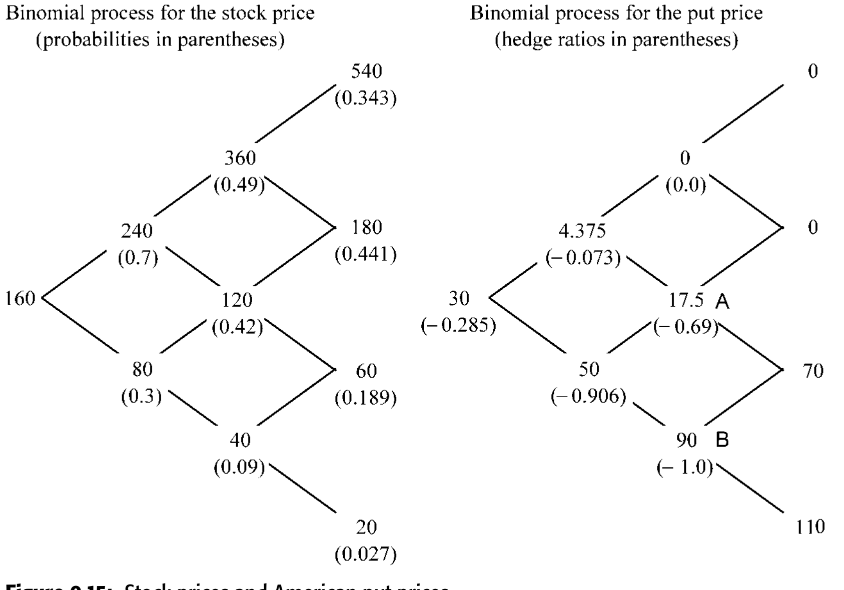

Figure 9.15: Stock prices and American put prices.

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$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & Suuu(1-\delta) \\
 & & Suu(1-\delta) \arrow[ur] \arrow[dr] & \\
 & Su \arrow[ur] \arrow[dr] & & Suud(1-\delta) \\
S \arrow[ur] \arrow[dr] & & Sud(1-\delta) \arrow[ur] \arrow[dr] & \\
 & Sd \arrow[ur] \arrow[dr] & & Sudd(1-\delta) \\
 & & Sdd(1-\delta) \arrow[ur] \arrow[dr] & \\
 & & & Sddd(1-\delta)
\end{tikzcd}$$

Figure 9.16: Binomial model for a stock that pays a known dividend yield. The ex-dividend date occurs in the second period.

stock price goes from S to $Su(1-\delta)$ or $Sd(1-\delta)$ in a period that includes an ex-dividend date. If a period does not contain an ex-dividend date, the binomial model is unchanged. See Fig. 9.16 for illustration.

For European options, only the number of ex-dividend dates matters, not their specific dates. This can be seen as follows. Let m denote the number of ex-dividend dates before expiration. The stock price at expiration is then of the form $(1-\delta)^m Su^j d^{n-j}$, independent of the timing of the dividends. Consequently we can use binomial tree algorithms for options on a non-dividend-paying stock but with the current stock price S replaced with $(1-\delta)^m S$. Pricing can thus be achieved in linear time and constant space.

➤ **Exercise 9.6.1** Argue that the value of a European option under the case of known dividend yields equals $(1-\delta)^m$ European option on a non-dividend-paying stock with the strike price $(1-\delta)^{-m} X$.

9.6.2 American Options on a Stock that Pays a Known Dividend Yield

The algorithm for American calls applies backward induction and pays attention to each ex-dividend date (see Fig. 9.17). It can be easily modified to value American puts. Early exercise might be optimal when the period contains an ex-dividend date. Suppose that $Sd(1-\delta) > X$. Then $C_u = Su(1-\delta) - X$ and $C_d = Sd(1-\delta) - X$. Therefore

$$\frac{pC_u + (1-p)\,C_d}{R} = (1-\delta)\,S - \frac{X}{R},$$

which is exceeded by $S - X$ for sufficiently large S. This proves that early exercise before expiration might be optimal.

➤ **Exercise 9.6.2** Start with an American call on a stock that pays d dividends. Consider a package of $d+1$ European calls with the same strike price as the American call such that there is a European call expiring just before each ex-dividend date and a European call expiring at the same date as the American call. In light of Theorem 8.4.2, is the American call equivalent to this package of European calls?

Binomial tree algorithm for pricing American calls on a stock that pays a known dividend yield:

input: $S, u, d, X, n, \delta\,(1 > \delta > 0), m, \widehat{r}\,(u > e^{\widehat{r}} > d$ and $\widehat{r} > 0)$;
real $R, p, C[\,n+1\,]$;
integer i, j;
$R := e^{\widehat{r}}$;
$p := (R-d)/(u-d)$;
for $(i = 0$ to $n)$ { $C[\,i\,] := \max(0, Su^{n-i} d^{i} (1-\delta)^{m} - X)$; }
for $(j = n-1$ down to $0)$
  for $(i = 0$ to $j)$ {
    if [the period $(j, j+1\,]$ contains an ex-dividend date] $m := m-1$;
    $C[\,i\,] := \max((p \times C[\,i\,] + (1-p) \times C[\,i+1\,])/R, Su^{j-i} d^{i} (1-\delta)^{m} - X)$;
  }
return $C[\,0\,]$;

Figure 9.17: Binomial tree algorithm for American calls on a stock paying a dividend yield. Recall that m initially stores the total number of ex-dividend dates at or before expiration.

➢ **Programming Assignment 9.6.3** Implement binomial tree algorithms for American options on a stock that pays a known dividend yield.

9.6.3 Options on a Stock that Pays Known Dividends

Although companies may try to maintain a constant dividend yield in the long run, a constant dividend is satisfactory in the short run. Unlike constant dividend yields, constant dividends introduce complications. Use D to denote the amount of the dividend. Suppose an ex-dividend date falls in the first period. At the end of that period, the possible stock prices are $Su - D$ and $Sd - D$. Follow the stock price one more period. It is clear that the number of possible stock prices is not three but four: $(Su - D)\,u$, $(Su - D)\,d$, $(Sd - D)\,u$, and $(Sd - D)\,d$. In other words, the binomial tree no longer combines (see Fig. 9.18). The fundamental reason is that timing of the dividends now becomes important; for example, $(Su - D)\,u$ is different from $Suu - D$. It is not hard to see that m ex-dividend dates will give rise to at least 2^m terminal nodes. The known dividends case thus consumes tremendous computation time and memory.

$$
\begin{tikzcd}[row sep=small]
 & & (Su-D)\,u \\
 & Su-D \arrow[ur] \arrow[dr] & \\
 & & (Su-D)\,d \\
S \arrow[uur] \arrow[ddr] & & \\
 & & (Sd-D)\,u \\
 & Sd-D \arrow[ur] \arrow[dr] & \\
 & & (Sd-D)\,d
\end{tikzcd}
$$

Figure 9.18: Binomial model for a stock that pays known dividends. The amount of the dividend is D, and the ex-dividend date occurs in the first period.

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A Simplifying Assumption

One way to adjust for dividends is to use the Black–Scholes formula with the stock price reduced by the present value of the anticipated dividends. This procedure is valid if the stock price can be decomposed into a sum of two components, a riskless one paying known dividends during the life of the option and a risky one. The riskless component at any time is the PV of future dividends during the life of the option. The Black–Scholes formula is then applicable with S equal to the risky component of the stock price and σ equal to the volatility of the process followed by the risky component. The stock price, between two adjacent ex-dividend dates, follows the same lognormal distribution. This means that the Black–Scholes formula can be used provided the stock price is reduced by the PV of future dividends during the life of the option. We note that uncertainty about dividends is rarely important for options lasting less than 1 year.

With the above assumption, we can start with the current stock price minus the PV of future dividends before the expiration date and develop the binomial tree for the new stock price as if there were no dividends. Then we add to each stock price on the tree the PV of all future dividends before expiration. European option prices can be computed as before on this tree of stock prices. As for American options, the same procedure applies except for the need to test for early exercises at each node.

➢ **Programming Assignment 9.6.4** Implement the ideas described in this subsection.

9.6.4 Options on a Stock that Pays a Continuous Dividend Yield

In the **continuous-payout model**, dividends are paid continuously. Such a model approximates a broad-based stock market portfolio in which some company will pay a dividend nearly every day. The payment of a **continuous dividend yield** at rate q reduces the growth rate of the stock price by q. In other words, a stock that grows from S to S_τ with a continuous dividend yield of q would grow from S to $S_\tau e^{q\tau}$ without the dividends. Hence a European option on a stock with price S paying a continuous dividend yield of q has the same value as a European option on a stock with price $Se^{-q\tau}$ that pays no dividends. The Black–Scholes formulas thus hold, with S replaced with $Se^{-q\tau}$:

$$C = Se^{-q\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{9.20}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-q\tau} N(-x), \tag{9.20$'$}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}.$$

Formulas (9.20) and (9.20′), which are due to Merton [660], remain valid even if the dividend yield is not a constant as long as it is predictable, in which case q is replaced with the average annualized dividend yield during the life of the option [470, 746].

To run binomial tree algorithms, pick the risk-neutral probability as

$$\frac{e^{(r-q)\Delta t} - d}{u - d}, \tag{9.21}$$

where $\Delta t \equiv \tau/n$. The quick reason is that the stock price grows at an expected rate of $r - q$ in a risk-neutral economy. Note that the u and d in Eqs. (9.15) now stand for stock price movements as if there were no dividends Other than the change in probability (9.21), binomial tree algorithms are identical to the no-dividend case.

➤ **Exercise 9.6.5** Prove that the put–call parity becomes $C = P + Se^{-q\tau} - \text{PV}(X)$ under the continuous-payout model.

➤ **Exercise 9.6.6** Derive probability (9.21) rigorously by an arbitrage argument.

➤ **Exercise 9.6.7** (1) Someone argues that we should use $[\,(e^{r\Delta t} - d)/(u - d)\,]$ as the risk-neutral probability thus: Because the option value is independent of the stock's expected return $\mu - q$, it can be replaced with r. Show him the mistakes. (2) Suppose that we are asked to use the original risk-neutral probability $[\,(e^{r\Delta t} - d)/(u - d)\,]$. Describe the needed changes in the binomial tree algorithm.

➤ **Exercise 9.6.8** Give an example whereby the use of risk-neutral probability (9.21) makes early exercise for American calls optimal.

➤ **Programming Assignment 9.6.9** Implement the binomial tree algorithms for American options on a stock that pays a continuous dividend yield.

9.7 Traversing the Tree Diagonally

Can the standard quadratic-time backward-induction algorithm for American options be improved? Here an algorithm, which is due to Curran, is sketched that usually skips many nodes, saving time in the process [242]. Although only American puts are considered in what follows, the parity result in Exercise 9.7.1 can be used to price American calls as well.

Figure 9.19 mentions two properties in connection with the propagation of early-exercise nodes and non-early-exercise nodes during backward induction. The first property says that a node is an early-exercise node if both its successor nodes are exercised early. A terminal node that is in the money is considered an early-exercise node for convenience. The second property says if a node is a non-early-exercise node, then all the earlier nodes at the same horizontal level are also non-early-exercise nodes. An early-exercise node, once identified, is trivial to evaluate; it is just the difference of the strike price and the stock price. A non-early-exercise node, however, must be evaluated by backward induction.

Curran's algorithm adopts a nonconventional way of traversing the tree, as shown in Fig. 9.20. Evaluation at each node is the same as backward induction

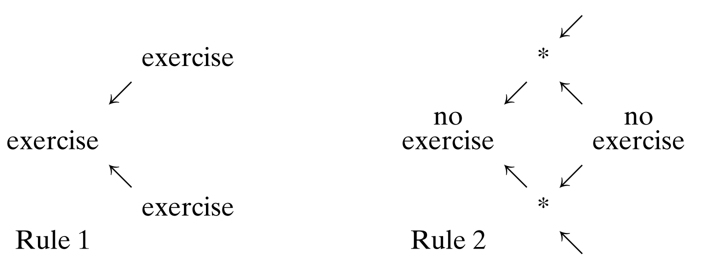


Figure 9.19: Two exercise rules. Rule 2 requires that $ud = 1$.

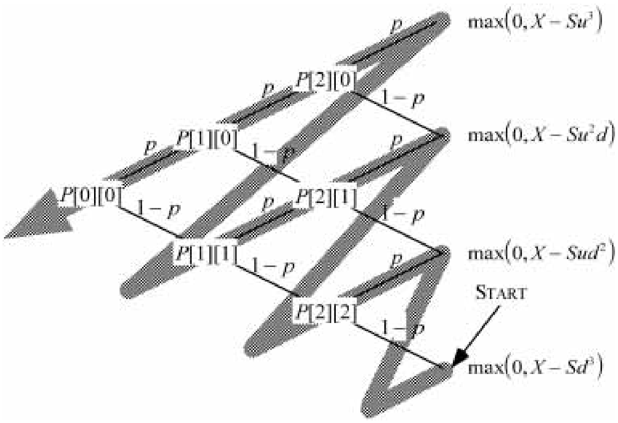


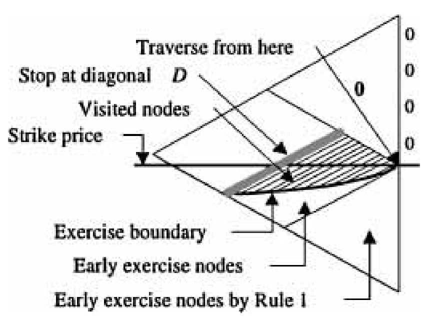
Figure 9.20: The diagonal method for binomial tree algorithms. In contrast to the vertical sweeps across time in the standard backward-induction algorithm such as Fig. 9.7, the new method traces the diagonals in the sequence shown.

before. Note that a node under evaluation always has both of its successors evaluated earlier.

Nothing would be gained if the whole tree needs to be explored. Enter the stopping rule. The process stops when a diagonal D consisting *entirely* of non-early-exercise nodes has been encountered. By Rule 2 of Fig. 9.19, all early-exercise nodes have been accounted for. When the algorithm finds an early-exercise node in traversing a diagonal, it can stop immediately and move on to the next diagonal. This is because, by Rule 1 and the sequence by which the nodes on the diagonals are traversed, the rest of the nodes on the current diagonal must all be early-exercise nodes, hence computable on the fly when needed. Also by Rule 1, the traversal can start from the zero-valued terminal node just above the strike price. Clearly $Su^a d^{n-a}$ is the stock price at that node, where a is defined in Eq. (9.9). The upper triangle above the strike price can be skipped because its nodes are all zero valued. See Fig. 9.21 for the overall strategy. The diagonal method typically skips many nodes (see Fig. 9.22).

The option value equals the sum of the discounted option values of the nodes on D, each multiplied by the probability that the stock price hits the diagonal for the *first* time at that node. It is impossible to go from the root to a node at which the option will be exercised without passing through D. For a node on D, which is the result of i up moves and j down moves from the root, the above-mentioned probability is

Figure 9.21: Search for a diagonal of non-early-exercise nodes. Only those nodes indicated will be visited. The **(optimal) exercise boundary** specifies the stock price at each time when it becomes optimal to exercise the option.



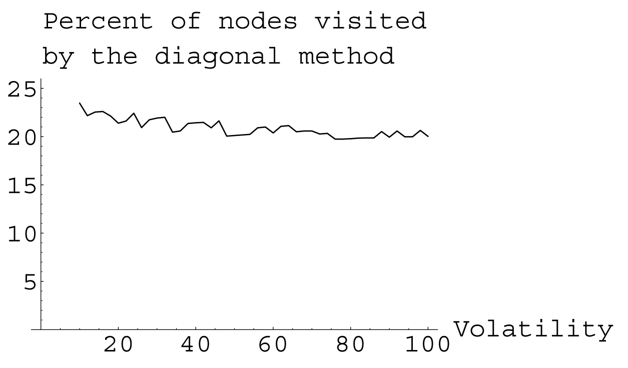


Figure 9.22: Performance of the diagonal method. The ratio of the number of nodes visited by the diagonal method to the total number of nodes is plotted. The parameters are $S = 50$, $X = 50$, $n = 100$, $r = 5\%$, $q = 1\%$, and $\tau = 1/3$. The volatility σ is in percentage terms.

$\binom{i+j-1}{i} p^i (1-p)^j$. This is because a valid path must pass through the node that is the result of i up moves and $j-1$ down moves. Call the option value on this node P_i. The desired option value then equals $\sum_{i=0}^{a-1} \binom{i+j-1}{i} p^i (1-p)^j P_i e^{-(i+j) r \Delta t}$. Because each node on D has been evaluated by that time, this part of the computation consumes $O(n)$ time. The space requirement is also linear in n because only the diagonal has to be allocated space. This idea can save computation time when D does not take long to find. See Fig. 9.23 for the algorithm.

It has been assumed up to now that the stock pays no dividends. Suppose now that the stock pays a continuous dividend yield $q \le r$ (or $r \le q$ for calls by parity). Therefore

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}.$$

Rule 1 of Fig. 9.19 continues to hold because, for a current stock price of $Su^i d^j$,

$$\begin{aligned}
[\, pP_u + (1-p)\, P_d \,]\, e^{-r\Delta t} &= [\, p(X - Su^{i+1} d^j) + (1-p)(X - Su^i d^{j+1}) \,] e^{-r\Delta t} \\
&= Xe^{-r\Delta t} - Su^i d^j [\, pu + (1-p)\, d \,]\, e^{-r\Delta t} \\
&= Xe^{-r\Delta t} - Su^i d^j e^{-q\Delta t} \\
&\le Xe^{-r\Delta t} - Su^i d^j e^{-r\Delta t} \\
&\le X - Su^i d^j.
\end{aligned}$$

Rule 2 is true in general, with or without dividends.

➤ **Exercise 9.7.1** Prove that an American call fetches the same price as an American put after swapping the current stock price with the strike price and the riskless rate with the continuous dividend yield.

➤ **Exercise 9.7.2** Verify the validity of Rule 2 under the binomial model.

➢ **Programming Assignment 9.7.3** Carefully implement the diagonal method and benchmark its efficiency against the standard backward-induction algorithm.

The diagonal method for American puts:

input: $S, u, d, \tau, X, n, r, q\ (r \geq q > 0)$;
real $p, P[\,n+1\,]$, cont, down;
integer i, j, a;
$p := (e^{(r-q)(\tau/n)} - d)/(u - d)$;
$a := \lceil \ln(X/Sd^n)/\ln(u/d) \rceil$;
$P[\,a\,] := 0$; // Collect lower boundary of zero-valued nodes in one entry.
for $(i = 0$ to $a - 1)$ $P[\,i\,] := -1$; //Upper boundary of early-exercise nodes.
for $(j = n - a$ down to $0)$
    for $(i = a - 1$ down to $0)$ {
        down $:= P[\,i\,]$; // Down move (computed in previous scan).
        if [down < 0] down $:= X - Su^i d^{j+1}$;
        cont $:= (p \times P[\,i+1\,] + (1-p) \times \text{down})/R$;
        if [cont $\geq X - Su^i d^j$ and $i > 0$]
            $P[\,i\,] :=$ cont; // No early exercise.
        else if [cont $\geq X - Su^i d^j$ and $i = 0$] { // Found D.
            $P[\,i\,] :=$ cont;
            if [$j = 0$] return $P[\,0\,]$;
            else return $\sum_{k=0}^{a-1} \binom{k+j-1}{k} p^k (1-p)^j \times P[\,k\,] \times e^{-(k+j)r(\tau/n)}$;
        }
        else break; // Early-exercise node; exit the current loop.
    }
if [$P[\,0\,] < 0$] $P[\,0\,] := X - S$;
return $P[\,0\,]$;

Figure 9.23: The diagonal method for American puts. $P\,[\,i\,]$ stores the put value when the stock price equals $Su^i d^j$, where j is the loop variable. As an early-exercide node's option value can be computed on the fly, -1 is used to state the fact that a node is an early-exercise node.

Additional Reading

The basic Black–Scholes model makes several assumptions. For example, margin requirements, taxes, and transactions costs are ignored, and only small changes in the stock price are allowed for a short period of time. See [86] for an early empirical work. Consult [236, 470] for various extensions to the basic model and [154, 423, 531, 683] for analytical results concerning American options. The Black–Scholes formula can be derived in at least four other ways [289]. A wealth of options formulas are available in [344, 423, 894]. Reference [613] considers predictable returns. Consult [48, 201, 346] for more information regarding estimating volatility from historical data. See [147, Subsection 9.3.5] and [514, Subsection 8.7.2] for more discussions on the "smile." To tackle multiple implied volatilities such as the smile, a generalized tree called the implied binomial tree may be used to price all options on the same underlying asset exactly (see Exercise 9.4.3) [215, 269, 299, 502, 503, 685]. Implied binomial trees are due to Rubinstein [770]. Wrong option pricing models and inaccurate volatility forecasts create great risk exposures for option writers [400]. See [56, 378, 420, 681, 753] for more information on the fundamental theorems of asset pricing.

The BOPM is generally attributed to Sharpe in 1975 [768] and appeared in his popular 1978 textbook, *Investments*. We followed the ideas put forth in [235, 738]. For American options, the BOPM offers a correct solution although its justification is delicate [18, 243, 576]. Several numerical methods for valuing American options are benchmarked in [127, 531, 834]. Convergences of binomial models for European and American options are investigated in [589, 590].

Many excellent textbooks cover options [236, 317, 346, 470, 878]. Read [494, 811] for intellectual developments that came before the breakthrough of Black and Scholes. To learn more about Black as a scientist, financial practitioner, and person, consult [345, 662].

NOTES

1. Their paper, "The Pricing of Options [and Corporate Liabilities]," was sent in 1970 to the *Journal of Political Economy* and was rejected immediately by the editors [64, 65].
2. Specifically, the **weak form of efficient markets hypothesis**, which says that current prices fully embody all information contained in historical prices [317]. This form of market efficiency implies that technical analysts cannot make above-average returns by reading charts of historical stock prices. It has stood up rather well [635].
3. A "clever" candidate once bought votes by issuing similar options, which paid off only when he was elected. Here is his reasoning: The option holders would not only vote for him but would also campaign hard for him, and in any case he kept the option premium if he lost the election, which he did.