CHAPTER TWENTY-TWO

Term Structure Fitting

That's an old besetting sin; they think calculating is inventing.
Johann Wolfgang Goethe (1749–1832), *Der Pantheist*

Fixed-income analysis starts with the yield curve. This chapter reviews **term structure fitting**, which means generating a curve to represent the yield curve, the spot rate curve, the forward rate curve, or the discount function. The constructed curve should fit the data reasonably well and be sufficiently smooth. The data are either bond prices or yields, and may be raw or synthetic as prepared by reputable firms such as Salomon Brothers (now part of Citigroup).

22.1 Introduction

The yield curve consists of hundreds of dots. Because bonds may have distinct qualities in terms of tax treatment, callability, and so on, more than one yield can appear at the same maturity. Certain maturities may also lack data points. These two problems were referred to in Section 5.3 as the multiple cash flow problem and the incompleteness problem. As a result, both regression (for the first problem) and interpolation (for the second problem) are needed for constructing a continuous curve from the data.

A functional form is first postulated, and its parameters are then estimated based on bond data. Two examples are the exponential function for the discount function and polynomials for the spot rate curve [7, 317]. The resulting curve is further required to be continuous or even differentiable as the relation between yield and maturity is expected to be fairly smooth. Although functional forms with more parameters often describe the data better, they are also more likely to **overfit** the data. An economically sensible curve that fits the data relatively well should be preferred to an economically unreasonable curve that fits the data extremely well.

Whether we fit the discount function $d(t)$, the spot rate function $s(t)$, or the forward rate curve $f(t)$ makes no difference theoretically. They carry the same information because

$$d(t) = e^{-ts(t)} = e^{-\int_0^t f(s)\,ds},$$

$$f(t) = s(t) + ts'(t)$$

under continuous compounding. Knowing any one of the three therefore suffices to infer the other two. The reality is more complicated, however. Empirically speaking, after the fitting, the smoothest curve is the discount function, followed by the spot rate curve, followed by the forward rate curve [848].

Using stripped Treasury security yields for spot rate curve fitting, albeit sensible, can be misleading. The major potential problem is taxation. The accrued compound interest of a stripped security is taxed annually, rendering its yield after-tax spot rate unless tax-exempt investors dominate the market. The liquidity of these securities is also not as great as that of coupon Treasuries. Finally, stripped Treasuries of certain maturities may attract investors willing to trade yield for a desirable feature associated with that particular maturity sector [326, 653]. This may happen because of dedicated portfolios set up for immunization.

Tax is a general problem, not just for stripped securities. Suppose capital gains are taxed more favorably than coupon income. Then a bond selling at a discount, generating capital gains at maturity, should have a lower yield to maturity than a similar par bond in order to produce comparable after-tax returns [568]. As a result, these two bonds will have distinct yields to maturity. This is the coupon effect at play.

➤ **Exercise 22.1.1** Verify the relation between $s(t)$ and $d(t)$ given above.

22.2 Linear Interpolation

A simple fitting method to handle the incompleteness problem is **linear yield interpolation** [335]. This technique starts with a list of bonds, preferably those selling near par and whose prices are both available and accurate. Usually only the on-the-run issues satisfy the criteria.[1] The scheme constructs a yield curve by connecting the yields with straight lines (see Figs. 22.1 and 22.2). The yield curve alone does not contain enough information to derive spot rates or, for that matter, discount factors and forward rates. This problem disappears for the par yield curve as the yield of a par bond equals its coupon rate.

The spot rate curve implied by the linearly interpolated yield curve is usually unsatisfactory in terms of shape. It may contain convex segments, for instance. The forward rate curve also behaves badly: It is extremely bumpy, with each bump corresponding to a specific bond in the data set and may be convex where it should be concave [848]. Despite these reservations, this scheme enjoys better statistical properties than many others [90].

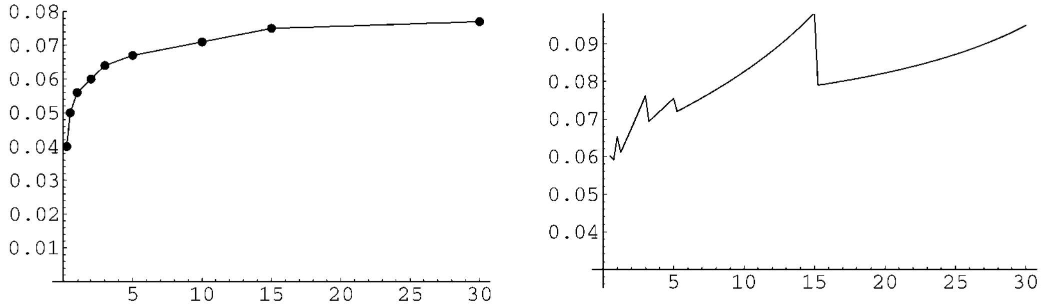

Figure 22.1: Linear interpolation of yield curve and forward rate curve. The par yield curve, linearly interpolated, is on the left, and the corresponding forward rate curve is on the right.

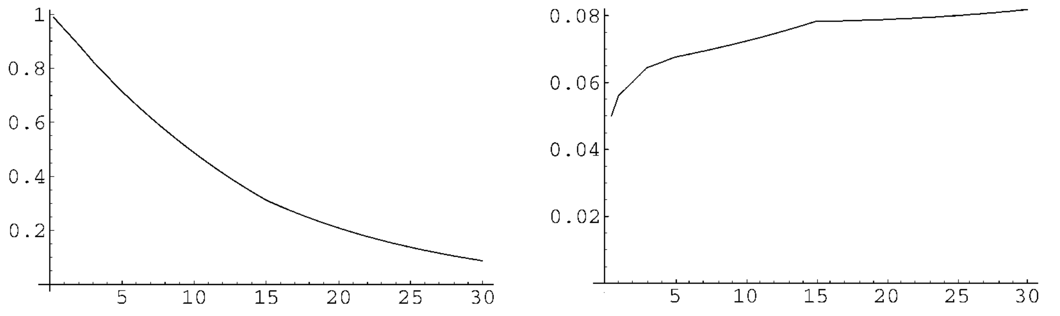

Figure 22.2: Discount function and spot rate curve. Plotted are the discount function and the spot rate curve as implied by the linearly interpolated par yield curve of Fig. 22.1.

A related scheme starts with the observation that the discount function is exponential in nature. It interpolates between known discount factors to obtain the discount function as follows. Let $t_1 < t < t_2$ and suppose that both $d(t_1)$ and $d(t_2)$ are available. The intermediate discount factor $d(t)$ is then interpolated by

$$d(t) = d(t_1)^{\frac{t(t_2-t)}{t_1(t_2-t_1)}} d(t_2)^{\frac{t(t-t_1)}{t_2(t_2-t_1)}}. \tag{22.1}$$

➤ **Exercise 22.2.1** Show that exponential interpolation scheme (22.1) for the discount function is equivalent to the linear interpolation scheme for the spot rate curve when spot rates are continuously compounded.

22.3 Ordinary Least Squares

Absent arbitrage opportunities, coupon bond prices and discount factors must satisfy

$$\begin{aligned}
P_1 &= (C_1+1)\, d(1),\\
P_2 &= C_2 d(1) + (C_2+1)\, d(2),\\
P_3 &= C_3 d(1) + C_3 d(2) + (C_3+1)\, d(3),\\
&\ \vdots\\
P_n &= C_n d(1) + C_n d(2) + \cdots + (C_n+1)\, d(n).
\end{aligned}$$

In the preceding equations, the ith coupon bond has a coupon of C_i, a maturity of i periods, and a price of P_i. Once the discount factors $d(i)$ are solved for, the i-period spot rate $S(i)$ is simply $S(i) = d(i)^{-1/i} - 1$. This formulation makes clear what can derail in practice the bootstrapping procedure to extract spot rates from coupon bond prices in Section 5.2. A first case in point is when there are more data points P_i than variables $d(i)$. This scenario results in an overdetermined linear system – the multiple cash flow problem. A second case in point is when bonds of certain maturities are missing. Then we have an underdetermined linear system, which corresponds to the incompleteness problem.

The above formulation suggests that solutions be based on the principle of least squares. Suppose there are m bonds and the ith coupon bond has n_i periods to maturity, where $n_1 \le n_2 \le \cdots \le n_m = n$. Then we have the following system of m equations:

$$P_i = C_i d(1) + C_i d(2) + \cdots + (C_i+1)\, d(n_i), \quad 1 \le i \le m. \tag{22.2}$$

If $m \geq n$, an overdetermined system results. This system can be solved for the n unknowns $d(1), d(2), \ldots, d(n)$ by use of the LS algorithm in Section 19.2 with minimizing the mean-square error as the objective. Certain equations may also be given more weights. For instance, each equation may be weighted proportional to the inverse of the bid–ask spread or the inverse of the duration [90, 147]. The computational problem is then equivalent to the weighted LS problem. If a few special bonds affect the result too much, the least absolute deviation can be adopted as the objective function. Finally, we can impose the conditions $d(1) \geq d(2) \geq \cdots \geq d(n) > 0$ (see Exercise 8.1.1), and the computational problem becomes that of optimizing a quadratic objective function under linear constraints – a **quadratic programming problem** [153].

We can handle the tax issue as follows. For each coupon bond, deduct the tax rate from the coupon rate. For each discount bond, reduce the principal repayment by the capital gains tax. For each premium bond, assume the loss will be amortized linearly over the life of the bond. For each zero-coupon bond, treat the income tax on imputed interest in each period as a negative cash flow. Finally, apply the methodology to the tax-adjusted cash flows to obtain the after-tax discount function [652].

Multiple-regression scheme (22.2) is called the McCulloch scheme. Other functional forms and target curves are clearly possible [653]. The Bradley–Crane scheme for example takes the form $\ln(1 + S(n)) = a + b_1 n + b_2 \ln n + \epsilon_n$ with three parameters, a, b_1, and b_2. The ϵ terms as usual represent errors. The Elliott–Echols scheme, as another example, adopts the form $\ln(1 + S(n_i)) = a + b_1/n_i + b_2 n_i + c_3 C_i + \epsilon_i$, where n_i is the term to maturity of the ith coupon bond. The explicit incorporation of the coupon rates is intended to take care of the coupon effect on yields due to tax considerations. Both schemes target the spot rate curve. See Fig. 22.3 for illustration.

Term structure fitting can also be model driven. Suppose we accept the Merton model for interest rate dynamics. It follows that the spot rate curve is a degree-two polynomial of the form $r + (\mu/2)\, t - (\sigma^2/6)\, t^2$ (see Fig. 22.4). This paradigm derives the model parameters – μ and σ in the Merton model – from regression [256, 511].

➤ **Exercise 22.3.1** Show how to fit a quadratic function $d(t) = a_0 + a_1 t + a_2 t^2$ to the discount factors by using multiple regression.

➤ **Exercise 22.3.2** Suppose we want to fit an exponential curve $y = ae^{bx}$ to the data, but we have only a linear-regression solver. How do we proceed?

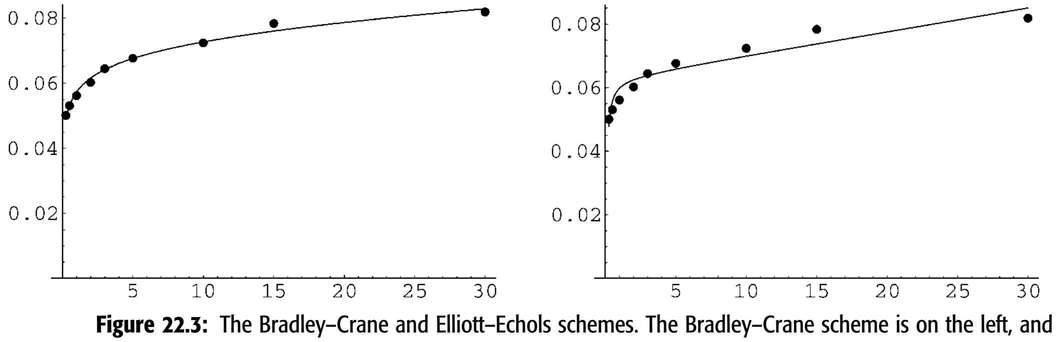

Figure 22.3: The Bradley–Crane and Elliott–Echols schemes. The Bradley–Crane scheme is on the left, and the Elliott–Echols scheme with the C_i set to zero is on the right.

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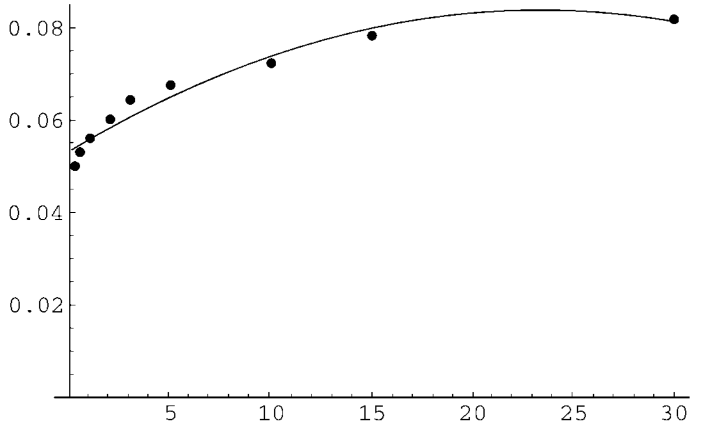

Figure 22.4: Polynomial regression.

22.4 Splines

The LS scheme of McCulloch, unlike those of Bradley–Crane and Elliott–Echols, cannot handle the incompleteness problem without imposing further restrictions. One proposal is to regard $d(i)$ as a linear combination of certain prespecified functions:

$$d(i) = 1 + \sum_{j=1}^{\ell} a_j f_j(i), \quad 1 \le i \le m, \tag{22.3}$$

where $f_j(i), 1 \le j \le \ell$, are known functions of maturity i and $a_1, a_2, \ldots, a_\ell$ are the parameters to be estimated [652]. Because $P(0) = 1$, $f_j(0)$ equals zero for every j. Substitute Eq. (22.3) into Eq. (22.2) to obtain the following overdetermined system:

$$P_i - 1 - C_i n_i = \sum_{k=1}^{\ell} a_k \left[f_k(n_i) + C_i \sum_{j=1}^{n_i} f_k(j) \right], \quad 1 \le i \le m. \tag{22.4}$$

There are now only ℓ unknown coefficients rather than n.

A lot of choices are open for the basis functions $f_j(\cdot)$. For instance, letting $f_j(x) = x^j$ makes the discount function a sum of polynomials. In this case, the problem is reduced to polynomial regression and is solved in the least-squares context (see Example 19.2.1). McCulloch suggested that the discount function be a cubic spline [652]. He also recommended picking the breakpoints that make each subinterval contain an equal number of maturity dates. This is probably the most well-known method.

➤ **Exercise 22.4.1** Verify Eq. (22.4).

➤ **Exercise 22.4.2** Assume continuous compounding. Justify the following claims. (1) If the forward rate curve should be a continuous function, a quadratic spline is the lowest-order spline that can fit the discount function. (2) If the forward rate curve should be continuously differentiable, a spline of at least cubic order is needed for the same purpose.

22.5 The Nelson–Siegel Scheme

A functional form that can be described by only a few parameters has obvious advantages. The Bradley–Crane, Elliott–Echols, and polynomial schemes fall into this category. The cubic-spline scheme does not, however. Its further problem, shared by others as well, is that the function tends to bend sharply toward the end of the maturity range. This does not seem to be representative of a true yield curve and suggests that predictions outside the sample maturity range are suspect [800, 801]. Finally, the spline scheme has difficulties producing well-behaved forward rates [352].

A good forward rate curve is important because many important interest rate models are based on forward rates. Nelson and Siegel proposed a parsimonious scheme for the forward rate curve [695]. This scheme is the most well known among families of smooth forward rate curves [77, 133]. Let the instantaneous forward rate curve be described by

$$f(\tau) = \beta_0 + \beta_1 e^{-\tau/\alpha} + \beta_2 \,\frac{\tau}{\alpha}\, e^{-\tau/\alpha},$$

where α is a constant (all rates are continuously compounded). The intent is to be able to measure the strengths of the short-, medium-, and long-term components of the forward rate curve. Specifically, the contribution of the long-term component is β_0, that of the short-term component is β_1, and that of the medium-term component is β_2. We can then find the quadruple $(\beta_0, \beta_1, \beta_2, \alpha)$ that minimizes the mean-square error between f and the data (see Fig. 22.5). The spot rate curve is

$$S(\tau) = \frac{\int_0^{\tau} f(s)\, ds}{\tau} = \beta_0 + (\beta_1 + \beta_2)\left(1 - e^{-\tau/\alpha}\right)\frac{\alpha}{\tau} - \beta_2 e^{-\tau/\alpha},$$

which is linear in the coefficients, given α. Both the forward rate curve and the spot rate curve converge to a constant, which has some appeal. All other functional forms seen so far have unbounded magnitude in the long end of the yield curve. The Nelson–Siegel scheme does not rule out negative forward rates.

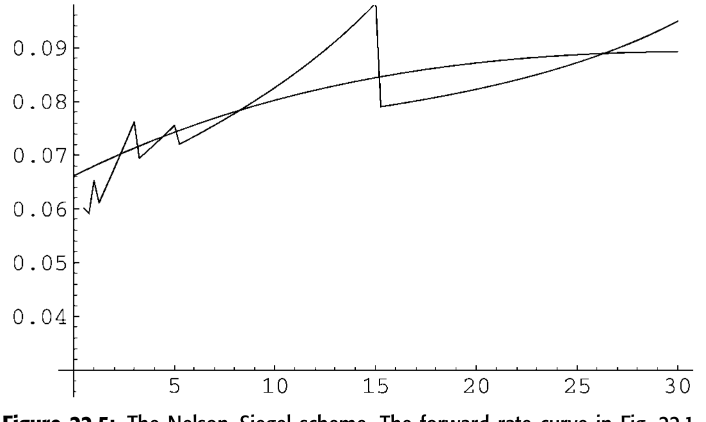

Figure 22.5: The Nelson–Siegel scheme. The forward rate curve in Fig. 22.1 (repeated here) is fitted by the Nelson–Siegel scheme.

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Additional Reading

In reality we have bid and asked quotes. Sometimes the asked quotes are used to construct the curve, whereas the mean prices may be preferred in other times. Some schemes treat the fitting error as zero as long as the fitted value lies within the bid–ask spread [90]. See [551] for term structure modeling in Japan. Splines, being piecewise polynomials, seem ill-suited for the the discount function, which is exponential in nature. To tackle the mismatch, Fong and Vasicek proposed **exponential splines** [856], but the results seem little different from those of polynomial splines [801]. In [171] polynomials are proposed to fit each spot rate with the degrees of the polynomials chosen heuristically. Consult [39, 219, 352, 609, 801] for more fitting ideas and [135] for a linear programming approach to fitting the spot rate curve. Many fitting schemes are compared in [90].

NOTE

1. However, on-the-run bonds may be overvalued for a variety of reasons [890].