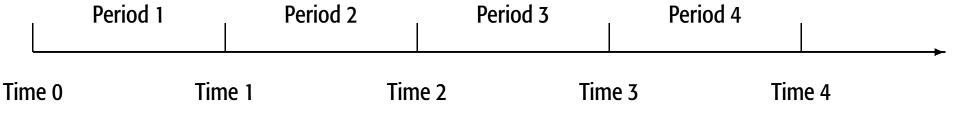
CHAPTER THREE

Basic Financial Mathematics

In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect.

Joseph Alois Schumpeter (1883–1950), *Capitalism, Socialism and Democracy*

To put a value on any financial instrument, the first step is to look at its cash flow. As we are most interested in the present value of expected cash flows, three features stand out: magnitudes and directions of the cash flows, times when the cash flows occur, and an appropriate factor to discount the cash flows. This chapter deals with elementary financial mathematics. The following convenient time line will be adopted throughout the chapter:



3.1 Time Value of Money

Interest is the cost of borrowing money [785, 787]. Let r be the annual interest rate. If the interest is **compounded** once per year, the **future value** (**FV**) of P dollars after n years is $\text{FV} = P(1+r)^n$. To look at it from another perspective, FV dollars n years from now is worth $P = \text{FV} \times (1+r)^{-n}$ today, its **present value** (**PV**).[1] The process of obtaining the present value is called **discounting**.

In general, if interest is compounded m times per annum, the future value is

$$\text{FV} = P\left(1+\frac{r}{m}\right)^{nm}. \tag{3.1}$$

Hence, $[\,1+(r/m)\,]^m - 1$ is the equivalent annual rate compounded once per annum or simply the **effective** annual interest rate. In particular, we have *annual* compounding with $m=1$, *semiannual* compounding with $m=2$, *quarterly* compounding with $m=4$, *monthly* compounding with $m=12$, *weekly* compounding with $m=52$, and *daily* compounding with $m=365$. Two widely used yields are the **bond-equivalent yield** (**BEY**) (the annualized yield with semiannual compounding) and the **mortgage-equivalent yield** (**MEY**) (the annualized yield with monthly compounding).

An interest rate of r compounded m times a year is equivalent to an interest rate of r/m per $1/m$ year by definition. If a loan asks for a return of 1% per month, for example, the annual interest rate will be 12% with monthly compounding.

EXAMPLE 3.1.1 With an annual interest rate of 10% compounded twice per annum, each dollar will grow to be $[1+(0.1/2)]^2 = 1.1025$ 1 year from now. The rate is therefore equivalent to an interest rate of 10.25% compounded once per annum.

EXAMPLE 3.1.2 An insurance company has to pay \$20 million 4 years from now to pensioners. Suppose that it can invest money at an annual rate of 7% compounded semiannually. Because the effective annual rate is $[1+(0.07/2)]^2 - 1 = 7.1225\%$, it should invest $20{,}000{,}000 \times (1.071225)^{-4} = 15{,}188{,}231$ dollars today.

As m approaches infinity and $[1+(r/m)]^m \to e^r$, we obtain **continuous compounding**:

$$\text{FV} = Pe^{rn},$$

where $e = 2.71828$. We call scheme (3.1) **periodic compounding** to differentiate it from continuous compounding. Continuous compounding is easier to work with. For instance, if the annual interest rate is r_1 for n_1 years and r_2 for the following n_2 years, the future value of \$1 will be $e^{r_1 n_1 + r_2 n_2}$.

➤ **Exercise 3.1.1** Verify that, given an annual rate, the effective annual rate is higher the higher the frequency of compounding.

➤ **Exercise 3.1.2** Below is a typical credit card statement:

NOMINAL ANNUAL PERCENTAGE RATE (%)	18.70
MONTHLY PERIODIC RATE (%)	1.5583

Figure out the frequency of compounding.

➤ **Exercise 3.1.3** (1) It was mentioned in Section 1.4 that workstations improved their performance by 54% per year between 1987 and 1992 and that the DRAM technology has quadrupled its capacity every 3 years since 1977. What are their respective annual growth rates with continuous compounding? (2) The number of requests received by the National Center for Supercomputing Applications (NCSA) WWW servers grew from ~300,000 per day in May 1994 to ~500,000 per day in September 1994. What is the growth rate per month (compounded monthly) during this period?

3.1.1 Efficient Algorithms for Present and Future Values

The PV of the cash flow $C_1, C_2, \ldots, C_n$ at times $1, 2, \ldots, n$ is

$$\frac{C_1}{1+y} + \frac{C_2}{(1+y)^2} + \cdots + \frac{C_n}{(1+y)^n}.$$

It can be computed by the algorithm in Fig. 3.1 in time $O(n)$, as the bulk of the computation lies in the four arithmetic operations during each execution of the loop that is executed n times. We can save one arithmetic operation within the loop by creating a new variable, say z, and assigning $1+y$ to it before the loop. The statement $d := d \times (1+y)$ can then be replaced with $d := d \times z$. Such optimization is often performed by modern compilers automatically behind the scene. This lends

Figure 3.1: Algorithm for PV. C_t are the cash flows, y is the interest rate, and n is the term of the investment. We can easily verify that the variable d is equal to $(1+y)^i$ at the beginning of the for loop. As a result, the variable x becomes the partial sum $\sum_{t=1}^{i} C_t(1+y)^{-t}$ at the end of each loop. This proves the correctness of the algorithm.

Algorithm for evaluating present value:

```
input:  y, n, C_t (1 ≤ t ≤ n);
real    x, d;
x := 0;
d := 1 + y;
for (i = 1 to n) {
        x := x + (C_i / d);
        d := d × (1 + y);
}
return x;
```

support to the earlier argument for asymptotic analysis: In a complex environment in which many manipulations are being done without our knowing them, the best we can do is often the asymptotics.

One further simplification is to replace the loop with the following statement:

$$\texttt{for } (i = n \texttt{ down to } 1)\ \{\, x := (x + C_i)/d; \,\}.$$

The above loop computes the PV by means of

$$\left\{ \cdots \left[\left(\frac{C_n}{1+y} + C_{n-1} \right) \frac{1}{1+y} + C_{n-2} \right] \frac{1}{1+y} + \cdots \right\} \frac{1}{1+y}.$$

This idea, which is due to Horner (1786–1837) in 1819 [582], is the most efficient possible in terms of the absolute number of arithmetic operations [103].

Computing the FV is almost identical to the algorithm in Fig. 3.1. The following changes to that algorithm are needed: (1) d is initialized to 1 instead of $1+y$, (2) i should start from n and run down to 1, and (3) $x := x + (C_i/d)$ is replaced with $x := x + (C_i \times d)$.

➤ **Exercise 3.1.4** Prove the correctness of the FV algorithm mentioned in the text.

3.1.2 Conversion between Compounding Methods

We can compare interest rates with different compounding methods by converting one into the other. Suppose that r_1 is the annual rate with continuous compounding and r_2 is the equivalent rate compounded m times per annum. Then $[\,1+(r^2/m)\,]^m = e^{r_1}$. Therefore

$$r_1 = m \ln\left(1+\frac{r_2}{m}\right), \tag{3.2}$$

$$r_2 = m\left(e^{r_1/m} - 1\right). \tag{3.3}$$

EXAMPLE 3.1.3 Consider an interest rate of 10% with quarterly compounding. The equivalent rate with continuous compounding is

$$4 \times \ln\left(1+\frac{0.1}{4}\right) = 0.09877, \quad \text{or} \quad 9.877\%,$$

derived from Eq. (3.2) with $m = 4$ and $r_2 = 0.1$.

For n compounding methods, there is a total of $n(n-1)$ possible pairwise conversions. Such potentially huge numbers of cases invite programming errors. To make that number manageable, we can fix a ground case, say continuous compounding, and then convert rates to their continuously compounded equivalents before any comparison. This cuts the number of possible conversions down to the more desirable $2(n-1)$.

3.1.3 Simple Compounding

Besides periodic compounding and continuous compounding (hence **compound interest**), there is a different scheme for computing interest called **simple compounding** (hence **simple interest**). Under this scheme, interest is computed on the original principal. Suppose that P dollars is borrowed at an annual rate of r. The simple interest each year is Pr.

3.2 Annuities

An **ordinary annuity** pays out the same C dollars at the end of each year for n years. With a rate of r, the FV at the end of the nth year is

$$\sum_{i=0}^{n-1} C(1+r)^i = C\,\frac{(1+r)^n-1}{r}. \tag{3.4}$$

For the **annuity due**, cash flows are received at the beginning of each year. The FV is

$$\sum_{i=1}^{n} C(1+r)^i = C\,\frac{(1+r)^n-1}{r}\,(1+r). \tag{3.5}$$

If m payments of C dollars each are received per year (the **general annuity**), then Eqs. (3.4) and (3.5) become

$$C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}, \qquad C\,\frac{\left(1+\frac{r}{m}\right)^{nm}-1}{\frac{r}{m}}\left(1+\frac{r}{m}\right),$$

respectively. Unless stated otherwise, an ordinary annuity is assumed from now on. The PV of a general annuity is

$$\text{PV} = \sum_{i=1}^{nm} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm}}{\frac{r}{m}}. \tag{3.6}$$

EXAMPLE 3.2.1 The PV of an annuity of \$100 per annum for 5 years at an annual interest rate of 6.25% is

$$100 \times \frac{1-(1.0625)^{-5}}{0.0625} = 418.387$$

based on Eq. (3.6) with $m=1$.

EXAMPLE 3.2.2 Suppose that an annuity pays \$5,000 per month for 9 years with an interest rate of 7.125% compounded monthly. Its PV, \$397,783, can be derived from Eq. (3.6) with $C = 5000$, $r = 0.07125$, $n = 9$, and $m = 12$.

An annuity that lasts forever is called a **perpetual annuity**. We can derive its PV from Eq. (3.6) by letting n go to infinity:

$$\mathrm{PV} = \frac{mC}{r}. \tag{3.7}$$

This formula is useful for valuing perpetual fixed-coupon debts [646]. For example, consider a financial instrument promising to pay \$100 once a year forever. If the interest rate is 10%, its PV is $100/0.1 = 1000$ dollars.

➤ **Exercise 3.2.1** Derive the PV formula for the general annuity due.

3.3 Amortization

Amortization is a method of repaying a loan through regular payments of interest *and* principal. The size of the loan – the **original balance** – is reduced by the principal part of the payment. The interest part of the payment pays the interest incurred on the **remaining principal balance**. As the principal gets paid down over the term of the loan,[2] the interest part of the payment diminishes.

Home mortgages are typically amortized. When the principal is paid down consistently, the risk to the lender is lowered. When the borrower sells the house, the remaining principal is due the lender. The rest of this section considers mainly the equal-payment case, i.e., fixed-rate **level-payment fully amortized mortgages**, commonly known as **traditional mortgages**.

EXAMPLE 3.3.1 A home buyer takes out a 15-year \$250,000 loan at an 8.0% interest rate. Solving Eq. (3.6) with $\mathrm{PV} = 250000$, $n = 15$, $m = 12$, and $r = 0.08$ gives a monthly payment of $C = 2389.13$. The amortization schedule is shown in Fig. 3.2. We can verify that in every month (1) the principal and the interest parts of the payment add up to \$2,389.13, (2) the remaining principal is reduced by the amount indicated under the Principal heading, and (3) we compute the interest by multiplying the remaining balance of the previous month by $0.08/12$.

Month	*Payment*	*Interest*	*Principal*	*Remaining principal*
				250,000.000
1	2,389.13	1,666.667	722.464	249,277.536
2	2,389.13	1,661.850	727.280	248,550.256
3	2,389.13	1,657.002	732.129	247,818.128
		...		
178	2,389.13	47.153	2,341.980	4,730.899
179	2,389.13	31.539	2,357.591	2,373.308
180	2,389.13	15.822	2,373.308	0.000
Total	430,043.438	180,043.438	250,000.000	

Figure 3.2: An amortization schedule. See Example 3.3.1.

Suppose that the amortization schedule lets the lender receive m payments a year for n years. The amount of each payment is C dollars, and the annual interest rate is r. Right after the kth payment, the remaining principal is the PV of the future $nm - k$ cash flows:

$$\sum_{i=1}^{nm-k} C\left(1+\frac{r}{m}\right)^{-i} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-nm+k}}{\frac{r}{m}}. \tag{3.8}$$

For example, Eq. (3.8) generates the same remaining principal as that in the amortization schedule of Example 3.3.1 for the third month with $C = 2389.13$, $n = 15$, $m = 12$, $r = 0.08$, and $k = 3$.

A popular mortgage is the **adjustable-rate mortgage** (**ARM**). The interest rate now is no longer fixed but is tied to some publicly available index such as the constant-maturity Treasury (CMT) rate or the Cost of Funds Index (COFI). For instance, a mortgage that calls for the interest rate to be **reset** every month requires that the monthly payment be recalculated every month based on the prevailing interest rate and the remaining principal at the beginning of the month. The attractiveness of ARMs arises from the typically lower initial rate, thus qualifying the home buyer for a bigger mortgage, and the fact that the interest rate adjustments are capped.

A common method of paying off a long-term loan is for the borrower to pay interest on the loan *and* to pay into a **sinking fund** so that the debt can be retired with proceeds from the fund. The sum of the interest payment and the sinking-fund deposit is called the **periodic expense** of the debt. In practice, sinking-fund provisions vary. Some start several years after the issuance of the debt, others allow a balloon payment at maturity, and still others use the fund to periodically purchase bonds in the market [767].

EXAMPLE 3.3.2 A company borrows \$100,000 at a semiannual interest rate of 10%. If the company pays into a sinking fund earning 8% to retire the debt in 7 years, the semiannual payment can be calculated by Eq. (3.6) as follows:

$$\frac{100000 \times 0.08/2}{1-(1+0.08/2)^{-14}} = 9466.9.$$

Interest on the loan is $100000 \times (0.1/2) = 5000$ semiannually. The periodic expense is thus $5000 + 9466.9 = 14466.9$ dollars.

➤ **Exercise 3.3.1** Explain why

$$\mathrm{PV}\left(1+\frac{r}{m}\right)^{k} - \sum_{i=1}^{k} C\left(1+\frac{r}{m}\right)^{i-1}$$

where the PV from Eq. (3.6) equals that of Eq. (3.8).

➤ **Exercise 3.3.2** Start with the cash flow of a level-payment mortgage with the lower monthly fixed interest rate $r - x$. From the monthly payment D, construct a cash flow that grows at a rate of x per month: D, De^{x}, De^{2x}, De^{3x}, ... Both x and r are continuously compounded. Verify that this new cash flow, discounted at r, has the same PV as that of the original mortgage. (This identity forms the basis of the **graduated-payment mortgages** (**GPMs**) [330].)

➢ **Programming Assignment 3.3.3** Write a program that prints out the monthly amortization schedule. The inputs are the annual interest rate and the number of payments.

3.4 Yields

The term **yield** denotes the return of investment and has many variants [284]. The **nominal yield** is the **coupon rate** of the bond. In the *Wall Street Journal* of August 26, 1997, for instance, a corporate bond issued by AT&T is quoted as follows:

Company	*Cur Yld.*	*Vol.*	*Close*	*Net chg.*
ATT85/831	8.1	162	1061/2	−3/8

This bond matures in the year 2031 and has a nominal yield of $8\frac{5}{8}\%$, which is part of the identification of the bond. In the same paper, we can find other AT&T bonds: ATT43/498, ATT6s00, ATT51/801, and ATT63/404. The **current yield** is the annual coupon interest divided by the market price. In the preceding case, the annual interest is $8\,\frac{5}{8} \times 1000/100 = 86.25$, assuming a par value of \$1,000. The closing price is $106\,\frac{1}{2} \times 1000/100 = 1065$ dollars. (Corporate bonds are quoted as a percentage of par.) Therefore $86.25/1065 \approx 8.1\%$ is the current yield at market closing. The preceding two yield measures are of little use in comparing returns. For example, the nominal yield completely ignores the market condition, whereas the current yield fails to take the future into account, even though it does depend on the current market price.

Securities such as U.S. **Treasury bills** (**T-bills**) pay interest based on the **discount method** rather than on the more common **add-on method** [95]. With the discount method, interest is subtracted from the par value of a security to derive the purchase price, and the investor receives the par value at maturity. Such a security is said to be issued **on a discount basis** and is called a **discount security**. The **discount yield** or **discount rate** is defined as

$$\frac{\text{par value} - \text{purchase price}}{\text{par value}} \times \frac{360 \text{ days}}{\text{number of days to maturity}}. \tag{3.9}$$

This yield is also called the **yield on a bank discount basis**. When the discount yield is calculated for short-term securities, a year is assumed to have 360 days [698, 827].

EXAMPLE 3.4.1 T-bills are a short-term debt instrument with maturities of 3, 6, or 12 months. They are issued in U.S.\$10,000 denominations. If an investor buys a U.S.\$10,000, 6-month T-bill for U.S.\$9,521.45 with 182 days remaining to maturity, the discount yield is

$$\frac{10000 - 9521.45}{10000} \times \frac{360}{182} = 0.0947,$$

or 9.47%. It is this annualized yield that is quoted. The equivalent effective yield with continuous compounding is

$$\frac{365}{182} \times \ln\left(\frac{10000}{9521.45}\right) = 0.09835,$$

or 9.835%.

The **CD-equivalent yield** (also called the **money-market-equivalent yield**) is a simple annualized interest rate defined by

$$\frac{\text{par value} - \text{purchase price}}{\text{purchase price}} \times \frac{360}{\text{number of days to maturity}}.$$

To make the discount yield more comparable with yield quotes of other money market instruments, we can calculate its CD-equivalent yield as

$$\frac{360 \times \text{discount yield}}{360 - (\text{number of days to maturity} \times \text{discount yield})},$$

which we can derive by plugging in discount yield formula (3.9) and simplifying. To make the discount yield more comparable with the BEY, we compute

$$\frac{\text{par value} - \text{purchase price}}{\text{purchase price}} \times \frac{365}{\text{number of days to maturity}}.$$

For example, the discount yield in Example 3.4.1 (9.47%) now becomes

$$\frac{478.55}{9521.45} \times \frac{365}{182} = 0.1008, \quad \text{or} \quad 10.08\%. \tag{3.10}$$

The T-bill's **ask yield** is computed in precisely this way [510].

3.4.1 Internal Rate of Return

For the rest of this section, the yield we are concerned with, unless stated otherwise, is the **internal rate of return** (**IRR**). The IRR is the interest rate that equates an investment's PV with its price P:

$$P = \frac{C_1}{(1+y)} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} + \cdots + \frac{C_n}{(1+y)^n}. \tag{3.11}$$

The right-hand side of Eq. (3.11) is the PV of the cash flow $C_1, C_2, \ldots, C_n$ discounted at the IRR y. Equation (3.11) and its various generalizations form the foundation upon which pricing methodologies are built.

EXAMPLE 3.4.2 A bank lent a borrower \$260,000 for 15 years to purchase a house. This 15-year mortgage has a monthly payment of \$2,000. The annual yield is 4.583% because $\sum_{i=1}^{12\times 15} 2000 \times [\, 1 + (0.04583/12)\,]^{-i} \approx 260000$.

EXAMPLE 3.4.3 A financial instrument promises to pay \$1,000 for the next 3 years and sells for \$2,500. Its yield is 9.7%, which can be verified as follows. With 0.097 as the discounting rate, the PVs of the three cash flows are $1000/(1+0.097)^t$ for $t = 1, 2, 3$. The numbers – 911.577, 830.973, and 757.5 – sum to \$2,500.

Example 3.4.3 shows that it is easy to *verify* if a number is the IRR. Finding it, however, generally requires numerical techniques because closed-form formulas in general do not exist. This issue will be picked up in Subsection 3.4.3.

EXAMPLE 3.4.4 A financial instrument can be bought for \$1,000, and the investor will end up with \$2,000 5 years from now. The yield is the y that equates 1000 with $2000 \times (1+y)^{-5}$, the present value of \$2,000. It is $(1000/2000)^{-1/5} - 1 \approx 14.87\%$.

Given the cash flow $C_1, C_2, \ldots, C_n$, its FV is

$$\text{FV} = \sum_{t=1}^{n} C_t(1+y)^{n-t}. \tag{3.12}$$

By Eq. (3.11), the yield y makes the preceding FV equal to $P(1+y)^n$. Hence, in principle, multiple cash flows can be reduced to a single cash flow $P(1+y)^n$ at maturity. In Example 3.4.4 the investor ends up with \$2,000 at the end of the fifth year one way or another. This brings us to an important point. Look at Eqs. (3.11) and (3.12) again. They mean the same thing because both implicitly assume that all cash flows are reinvested at the *same* rate as the IRR y.

Example 3.4.4 suggests a general yield measure: Calculate the FV and then find the yield that equates it with the PV. This is the **holding period return** (**HPR**) methodology.[3] With the HPR, it is no longer mandatory that all cash flows be reinvested at the same rate. Instead, explicit assumptions about the reinvestment rates must be made for the cash flows. Suppose that the reinvestment rate has been determined to be r_e. Then the FV is

$$\text{FV} = \sum_{t=1}^{n} C_t(1+r_e)^{n-t}.$$

We then solve for the **holding period yield** y such that $\text{FV} = P(1+y)^n$. Of course, if the reinvestment assumptions turn out to be wrong, the yield will not be realized. This is the **reinvestment risk**. Financial instruments without intermediate cash flows evidently do not have reinvestment risks.

EXAMPLE 3.4.5 A financial instrument promises to pay \$1,000 for the next 3 years and sells for \$2,500. If each cash flow can be put into a bank account that pays an effective rate of 5%, the FV of the security is $\sum_{t=1}^{3} 1000 \times (1+0.05)^{3-t} = 3152.5$, and the holding period yield is $(3152.5/2500)^{1/3} - 1 = 0.08037$, or 8.037%. This yield is considerably lower than the 9.7% in Example 3.4.3.

➤ **Exercise 3.4.1** A security selling for \$3,000 promises to pay \$1,000 for the next 2 years and \$1,500 for the third year. Verify that its annual yield is 7.55%.

➤ **Exercise 3.4.2** A financial instrument pays C dollars per year for n years. The investor interested in the instrument expects the cash flows to be reinvested at an annual rate of r and is asking for a yield of y. What should this instrument be selling for in order to be attractive to this investor?

3.4.2 Net Present Value

Consider an investment that has the cash flow $C_1, C_2, \ldots, C_n$ and is selling for P. For an investor who believes that this security should have a return rate of y^*, the **net present value** (**NPV**) is

$$\sum_{t=1}^{n} \frac{C_t}{(1+y^*)^t} - P.$$

The IRR is thus the return rate that nullifies the NPV. In general, the NPV is the difference between the PVs of cash inflow and cash outflow. Businesses are often assumed to maximize their assets' NPV.

EXAMPLE 3.4.6 The management is presented with the following proposals:

		Net Cash Flow at end of		
Proposal	*Investment Now*	*Year 1*	*Year 2*	*Year 3*
A	9,500	4,500	2,000	6,000
B	6,000	2,500	1,000	5,000

It believes that the company can earn 15% effective on projects of this kind. The NPV for Proposal A is

$$\frac{4500}{1.15} + \frac{2000}{(1.15)^2} + \frac{6000}{(1.15)^3} - 9500 = -129.57$$

and that for Proposal B is

$$\frac{2500}{1.15} + \frac{1000}{(1.15)^2} + \frac{5000}{(1.15)^3} - 6000 = 217.64.$$

Proposal A is therefore dropped in favor of Proposal B.

➤ **Exercise 3.4.3** Repeat the calculation for Example 3.4.6 for an expected return of 4%.

3.4.3 Numerical Methods for Finding Yields

Computing the yield amounts to solving $f(y) = 0$ for $y \geq -1$, where

$$f(y) \equiv \sum_{t=1}^{n} \frac{C_t}{(1+y)^t} - P \tag{3.13}$$

and P is the market price. (The symbol $\equiv$ introduces definitions.) The function $f(y)$ is monotonic in y if the C_ts are all positive. In this case, a simple geometric argument shows that a unique solution exists (see Fig. 3.3). Even in the general case in which

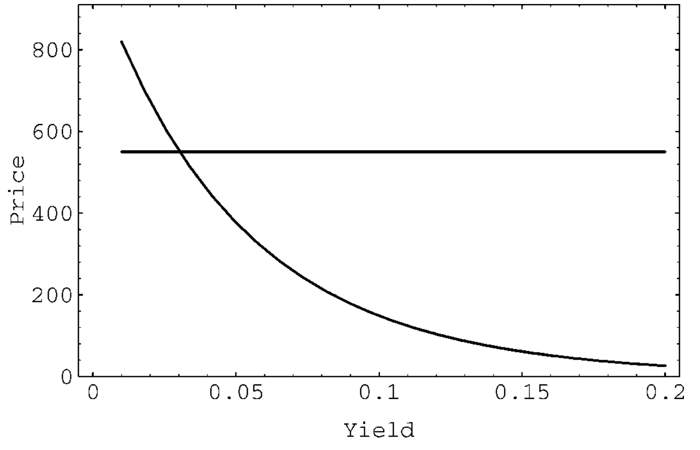


Figure 3.3: Computing yields. The current market price is represented by the horizontal line, and the PV of the future cash flow is represented by the downward-sloping curve. The desired yield is the value on the x axis at which the two curves intersect.

The bisection method for solving equations:

input: ϵ, a, and b ($b > a$ and $f(a)f(b) < 0$);
real length, c;
length $:= b - a$;
while [length $> \epsilon$] {
  $c := (b + a)/2$;
  if [$f(c) = 0$] return c;
  else if [$f(a)\ f(c) < 0$] $b := c$;
  else $a := c$;
}
return c;

Figure 3.4: Bisection method. The number ϵ is an upper bound on the absolute error of the returned value c: $|\xi - c| \leq \epsilon$. The initial bracket $[a, b]$ guarantees the existence of a root with the $f(a)\ f(b) < 0$ condition. "if [$f(c) = 0$]" may be replaced with testing if $|f(c)|$ is a very small number.

all the C_ts do not have the same sign, usually only one value makes economic sense [547]. We now turn to the algorithmic problem of finding the solution to y.

The Bisection Method

One of the simplest and failure-free methods to solve equations such as Eq. (3.13) for any well-behaved function is the **bisection method**. Start with two numbers, a and b, where $a < b$ and $f(a)f(b) < 0$. Then $f(\xi)$ must be zero for some ξ between a and b, written as $\xi \in [a, b]$.[4] If we evaluate f at the midpoint $c \equiv (a + b)/2$, then (1) $f(c) = 0$, (2) $f(a)f(c) < 0$, or (3) $f(c)f(b) < 0$. In the first case we are done, in the second case we continue the process with the new bracket $[a, c]$, and in the third case we continue with $[c, b]$. Note that the bracket is halved in the latter two cases. After n steps, we will have confined ξ within a bracket of length $(b - a)/2^n$. Figure 3.4 implements the above idea.

The complexity of the bisection algorithm can be analyzed as follows. The while loop is executed, at most, $1 + \log_2[(b - a)/\epsilon]$ times. Within the loop, the number of arithmetic operations is dominated by the evaluation of f. Denote this number by $\mathcal{C}_f$. The running time is $O(\mathcal{C}_f \log_2[(b - a)/\epsilon])$. In particular, in computing the IRR, the running time is $O(n \log_2[(b - a)/\epsilon])$ because $\mathcal{C}_f = O(n)$ by the algorithm in Fig. 3.1.

The Newton–Raphson Method

The iterative **Newton–Raphson method** converges faster than the bisection method. In **iterative methods**, we start with a first approximation x_0 to a root of $f(x) = 0$. Successive approximations are then computed by

$$x_0,\ F(x_0),\ F(F(x_0)),\ \ldots$$

for some function F. In other words, if we let x_k denote the kth approximation, then $x_k = F^{(k)}(x_0)$, where

$$F^{(k)}(x) \equiv \overbrace{F(F(\cdots(F}^{k}(x))\cdots)).$$

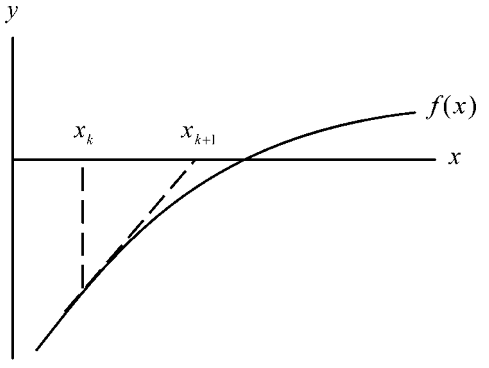


Figure 3.5: Newton–Raphson method.

In practice, we should put an upper bound on the number of iterations k. The necessary condition for the convergence of such a procedure to a root ξ is

$$|F'(\xi)| \leq 1, \tag{3.14}$$

where F' denotes the derivative of F [447].

The Newton–Raphson method picks $F(x) \equiv x - f(x)/f'(x)$; in other words,

$$x_{k+1} \equiv x_k - \frac{f(x_k)}{f'(x_k)}. \tag{3.15}$$

This is the method of choice when f' can be evaluated efficiently and is nonzero near the root [727]. See Fig. 3.5 for an illustration and Fig. 3.6 for the algorithm. When yields are being computed,

$$f'(x) = -\sum_{t=1}^{n} \frac{tC_t}{(1+x)^{t+1}}.$$

Assume that we start with an initial guess x_0 near a root ξ. It can be shown that

$$\xi - x_{k+1} \approx -(\xi - x_k)^2 \frac{f''(\xi)}{2f'(\xi)}.$$

This means that the method converges *quadratically*: Near the root, each iteration roughly doubles the number of significant digits. To achieve $|x_{k+1} - x_k| \leq \epsilon$ required by the algorithm, $O(\log\log(1/\epsilon))$ iterations suffice. The running time is thus

$$O((\mathcal{C}_f + \mathcal{C}_{f'}) \log\log(1/\epsilon)).$$

The Newton–Raphson method for solving equations:

input: $\epsilon, x_{\text{initial}}$;
real $x_{\text{new}}, x_{\text{old}}$;
$x_{\text{old}} := x_{\text{initial}}$;
$x_{\text{new}} := \infty$;
while $[\,|x_{\text{new}} - x_{\text{old}}| > \epsilon\,]$
  $x_{\text{new}} = x_{\text{old}} - f(x_{\text{old}})/f'(x_{\text{old}})$;
return x_{new};

Figure 3.6: Algorithm for the Newton–Raphson method. A good initial guess is important [727].

In particular, the running time is $O(n \log\log(1/\epsilon))$ for yields calculations. This bound compares favorably with the $O(n \log[(b-a)/\epsilon])$ bound of the bisection method.

A variant of the Newton–Raphson method that does not require differentiation is the **secant method** [35]. This method starts with two approximations, x_0 and x_1, and computes the $(k+1)$th approximation by

$$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}.$$

The secant method may be preferred when the calculation of f' is to be avoided. Its convergence rate, 1.618, is slightly worse than that of the Newton–Raphson method, 2, but better than that of the bisection method, 1.

Unlike the bisection method, neither the Newton–Raphson method nor the secant method guarantees that the root remains bracketed; as a result, they may not converge at all. The **Ridders method**, in contrast, always brackets the root. It starts with x_0 and x_1 that bracket a root and sets $x_2 = (x_0 + x_1)/2$. In general,

$$x_{k+1} = x_k + \text{sign}[f(x_{k-2}) - f(x_{k-1})] \frac{f(x_k)(x_k - x_{k-2})}{\sqrt{f(x_k)^2 - f(x_{k-2}) f(x_{k-1})}}.$$

The Ridders method has a convergence rate of $\sqrt{2}$ [727].

➤ **Exercise 3.4.4** Let $f(x) \equiv x^3 - x^2$ and start with the guess $x_0 = 2.0$ to the equation $f(x) = 0$. Iterate the Newton–Raphson method five times.

➤ **Exercise 3.4.5** Suppose that $f'(\xi) \neq 0$ and $f''(\xi)$ is bounded. Verify that condition (3.14) holds for the Newton–Raphson method.

➤ **Exercise 3.4.6** Let ξ be a root of f and J be an interval containing ξ. Suppose that $f'(x) \neq 0$ and $f''(x) \geq 0$ or $f''(x) \leq 0$ for $x \in J$. Explain why the Newton–Raphson method converges monotonically to ξ from any point $x_0 \in J$ such that $f(x_0) f''(x_0) \geq 0$.

3.4.4 Solving Systems of Nonlinear Equations

The Newton–Raphson method can be extended to higher dimensions. Consider the two-dimensional case. Let (x_k, y_k) be the kth approximation to the solution of the two simultaneous equations

$$f(x, y) = 0,$$
$$g(x, y) = 0.$$

The $(k+1)$th approximation (x_{k+1}, y_{k+1}) satisfies the following linear equations:

$$\begin{bmatrix} \partial f(x_k, y_k)/\partial x & \partial f(x_k, y_k)/\partial y \\ \partial g(x_k, y_k)/\partial x & \partial g(x_k, y_k)/\partial y \end{bmatrix} \begin{bmatrix} \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = - \begin{bmatrix} f(x_k, y_k) \\ g(x_k, y_k) \end{bmatrix}, \tag{3.16}$$

where $\Delta x_{k+1} \equiv x_{k+1} - x_k$ and $\Delta y_{k+1} \equiv y_{k+1} - y_k$. Equations (3.16) have a unique solution for $(\Delta x_{k+1}, \Delta y_{k+1})$ when the matrix is invertible. Note that the $(k+1)$th approximation is $(x_k + \Delta x_{k+1}, y_k + \Delta y_{k+1})$. Solving nonlinear equations has thus been reduced to solving a set of linear equations. Generalization to n dimensions is straightforward.

➤ **Exercise 3.4.7** Write the analogous n-dimensional formula for Eqs. (3.16).

➤ **Exercise 3.4.8** Describe a bisection method for solving systems of nonlinear equations in the two-dimensional case. (The bisection method may be applied in cases in which the Newton–Raphson method fails.)

3.5 Bonds

A bond is a contract between the issuer (borrower) and the bondholder (lender). The issuer promises to pay the bondholder interest, if any, and principal on the remaining balance. Bonds usually refer to long-term debts. A bond has a **par value**.[5] The **redemption date** or **maturity date** specifies the date on which the loan will be repaid. A bond pays interest at the coupon rate on its par value at regular time intervals until the maturity date. The payment is usually made semiannually in the United States. The **redemption value** is the amount to be paid at a redemption date. A bond is **redeemed at par** if the redemption value is the same as the par value. Redemption date and maturity date may differ.

There are several ways to **redeem** or **retire** a bond. A bond is redeemed at maturity if the principal is repaid at maturity. Most corporate bonds are **callable**, meaning that the issuer can retire some or all of the bonds before the stated maturity, usually at a price above the par value.[6]

Because this provision gives the issuer the advantage of calling a bond when the prevailing interest rate is much lower than the coupon rate, the bondholders usually demand a premium. A callable bond may also have **call protection** so that it is not callable for the first few years. **Refunding** involves using the proceeds from the issuance of new bonds to retire old ones. A corporation may deposit money into a sinking fund and use the funds to buy back some or all of the bonds. **Convertible bonds** can be converted into the issuer's common stock. A **consol** is a bond that pays interest forever. It can therefore be analyzed as a perpetual annuity whose value and yield satisfy the simple relation

$$P = c/r, \tag{3.17}$$

where c denotes the interest payout per annum.

The U.S. bond market is the largest in the world. It consists of U.S. Treasury securities, U.S. agency securities, corporate bonds, Yankee bonds, municipal securities, mortgages, and MBSs. **Agency securities** are those issued by either the U.S. Federal government agencies or U.S. Federal government-sponsored organizations. The mortgage market is usually the largest (U.S.$6,388 billion as of 1999), followed by the U.S. Treasury securities market (U.S.$3,281 billion as of 1999).

Treasury securities with maturities of 1 year or less are discount securities: the T-bills. Treasury securities with original maturities between 2 and 10 years are called **Treasury notes** (**T-notes**). Those with maturities greater than ten years are called **Treasury bonds** (**T-bonds**). Both T-notes and T-bonds are coupon securities, paying interest every 6 months.

Bonds are usually quoted as a percentage of par value. A quote of 95 therefore means 95% of par value. For T-notes and T-bonds, a quote of 100.05 means 100 5/32% of par value, not 100.05%. It is typically written as 100-05.

➤ **Exercise 3.5.1** A consol paying out continuously at a rate of c dollars per annum has value $\int_0^\infty ce^{-rt}\,dt$, where r is the continuously compounded annual yield. Justify the preceding formula. (Consistent with Eq. (3.17), this integral evaluates to c/r.)

3.5.1 Valuation

Let us begin with **pure discount bonds**, also known as **zero-coupon bonds** or simply **zeros**. They promise a single payment in the future and are sold at a discount from the par value. The price of a zero-coupon bond that pays F dollars in n periods is $F/(1+r)^n$, where r is the interest rate per period. Zero-coupon bonds can be bought to meet future obligations without reinvestment risk. They are also an important theoretical tool in the analysis of coupon bonds, which can be thought of as a package of zero-coupon bonds. Although the U.S. Treasury does not issue such bonds with maturities over 1 year, there were companies that specialized in **coupon stripping** to create stripped Treasury securities. This financial innovation became redundant when the U.S. Treasury facilitated the creation of zeros by means of the Separate Trading of Registered Interest and Principal Securities program (STRIPS) in 1985 [799]. Prices and yields of stripped Treasury securities have been published daily in the *Wall Street Journal* since 1989.

EXAMPLE 3.5.1 Suppose that the interest rate is 8% compounded semiannually. A zero-coupon bond that pays the par value 20 years from now will be priced at $1/(1.04)^{40}$, or 20.83%, of its par value and will be quoted as 20.83. If the interest rate is 9% instead, the same bond will be priced at only 17.19. If the bond matures in 10 years instead of 20, its price would be 45.64 with an 8% interest rate. Clearly both the maturity and the market interest rate have a profound impact on price.

A **level-coupon bond** pays interest based on the coupon rate and the par value, which is paid at maturity. If F denotes the par value and C denotes the coupon, then the cash flow is as shown in Fig. 3.7. Its price is therefore

$$\mathrm{PV} = \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^{i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}} = C\,\frac{1-\left(1+\frac{r}{m}\right)^{-n}}{\frac{r}{m}} + \frac{F}{\left(1+\frac{r}{m}\right)^{n}}, \tag{3.18}$$

where n is the number of cash flows, m is the number of payments per year, and r is the annual interest rate compounded m times per annum. Note that $C = Fc/m$ when c is the annual coupon rate.

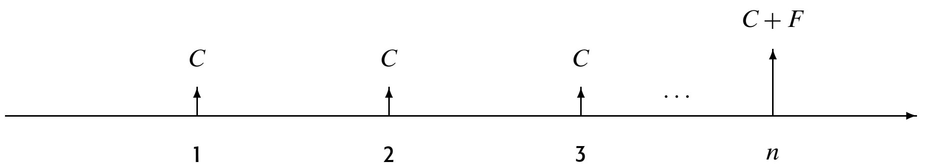


Figure 3.7: Cash flow of level-coupon bond.

EXAMPLE 3.5.2 Consider a 20-year 9% bond with the coupon paid semiannually. This means that a payment of $1000 \times 0.09/2 = 45$ dollars will be made every 6 months until maturity, and $1,000 will be paid at maturity. Its price can be computed from Eq. (3.18) with $n = 2 \times 20$, $r = 0.08$, $m = 2$, $F = 1$, and $C = 0.09/2$. The result is 1.09896, or 109.896% of par value. When the coupon rate is higher than the interest rate, as is the case here, a level-coupon bond will be selling above its par value.

The **yield to maturity** of a level-coupon bond is its IRR when the bond is held to maturity. In other words, it is the r that satisfies Eq. (3.18) with the PV being the bond price. For example, for an investor with a 15% BEY to maturity, a 10-year bond with a coupon rate of 10% paid semiannually should sell for

$$5 \times \frac{1 - [\,1 + (0.15/2)\,]^{-2\times 10}}{0.15/2} + \frac{100}{[\,1 + (0.15/2)\,]^{2\times 10}} = 74.5138$$

percent of par.

For a callable bond, the **yield to stated maturity** measures its yield to maturity as if it were not callable. The **yield to call** is the yield to maturity satisfied by Eq. (3.18), with n denoting the number of remaining coupon payments until the first call date and F replaced with the **call price**, the price at which the bond will be called. The related **yield to par call** assumes the call price is the par value. The **yield to effective maturity** replaces n with the **effective maturity date**, the redemption date when the bond is called. Of course, this date has to be estimated. The **yield to worst** is the minimum of the yields to call under all possible call dates.

➤ **Exercise 3.5.2** A company issues a 10-year bond with a coupon rate of 10%, paid semiannually. The bond is callable at par after 5 years. Find the price that guarantees a return of 12% compounded semiannually for the investor.

➤ **Exercise 3.5.3** How should pricing formula (3.18) be modified if the interest is taxed at a rate of T and capital gains are taxed at a rate of T_G?

➤ **Exercise 3.5.4** (1) Derive $\partial P/\partial n$ and $\partial P/\partial r$ for zero-coupon bonds. (2) For $r = 0.04$ and $n = 40$ as in Example 3.5.1, verify that the price will go down by approximately $d \times 8.011\%$ of par value for every $d\%$ increase in the period interest rate r for small d.

3.5.2 Price Behaviors

The price of a bond goes in the opposite direction from that of interest rate movements: Bond prices fall when interest rates rise, and vice versa. This is because the PV decreases as interest rates increase.[7] A good example is the loss of U.S.$1 trillion worldwide that was due to interest rate hikes in 1994 [312].

Equation (3.18) can be used to show that a level-coupon bond will be selling **at a premium** (above its par value) when its coupon rate is above the market interest rate, **at par** (at its par value) when its coupon rate is equal to the market interest rate, and **at a discount** (below its par value) when its coupon rate is below the market interest rate. The table in Fig. 3.8 shows the relation between the price of a bond and the required yield. Bonds selling at par are called **par bonds**.

The price/yield relation has a **convex** shape, as shown in Fig. 3.9. Convexity is attractive for bondholders because the price decrease per percent rate increase is

Yield (%)	Price (% of par)
7.5	113.37
8.0	108.65
8.5	104.19
9.0	100.00
9.5	96.04
10.0	92.31
10.5	88.79

Figure 3.8: Price/yield relations. A 15-year 9% coupon bond is assumed.

smaller than the price increase per percent rate decrease. This observation, however, may not hold for bonds with **embedded options** such as callable bonds. The convexity property has far-reaching implications for bonds and will be explored in Section 4.3.

As the maturity date draws near, a bond selling at a discount will see its price move up toward par, a bond selling at par will see its price remain at par, and a bond selling at a premium will see its price move down toward par. These phenomena are shown in Fig. 3.10. Besides the two reasons cited for causing bond prices to change (interest rate movements and a nonpar bond moving toward maturity), other reasons include changes in the yield spread to T-bonds for non-T-bonds, changes in the perceived credit quality of the issuer, and changes in the value of the embedded option.

➤ **Exercise 3.5.5** Prove that a level-coupon bond will be sold at par if its coupon rate is the same as the market interest rate.

3.5.3 Day Count Conventions

Teach us to number our days aright,
that we may gain a heart of wisdom.
—Psalms 90:12

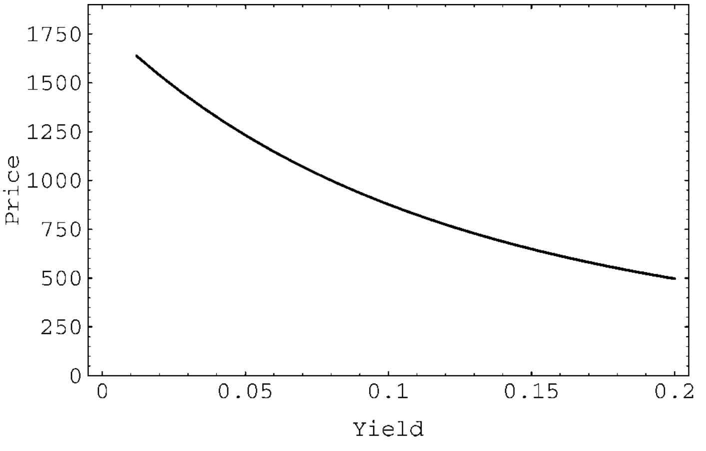


Figure 3.9: Price vs. yield. Plotted is a bond that pays 8% interest on a par value of $1,000, compounded annually. The term is 10 years.

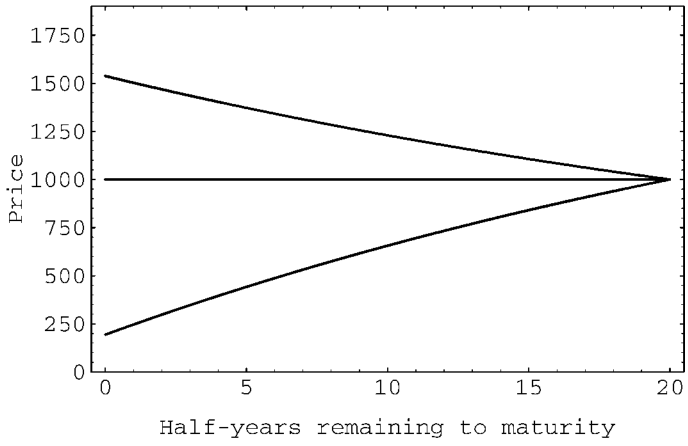


Figure 3.10: Relations between price and time to maturity. Plotted are three curves for bonds, from top to bottom, selling at a premium, at par, and at a discount, with coupon rates of 12%, 6%, and 2%, respectively. The coupons are paid semiannually. The par value is \$1,000, and the required yield is 6%. The term is 10 years (the x axis is measured in half-years).

Handling the issue of dating correctly is critical to any financial software. In the so-called **actual/actual** day count convention, the first "actual" refers to the actual number of days in a month, and the second refers to the actual number of days in a coupon period. For example, for coupon-bearing Treasury securities, the number of days between June 17, 1992, and October 1, 1992, is 106: 13 days in June, 31 days in July, 31 days in August, 30 days in September, and 1 day in October.

A convention popular with corporate and municipal bonds and agency securities is **30/360**. Here each month is assumed to have 30 days and each year 360 days. The number of days between June 17, 1992, and October 1, 1992, is now 104: 13 days in June, 30 days in July, 30 days in August, 30 days in September, and 1 day in October. In general, the number of days from date $D_1 \equiv (y_1, m_1, d_1)$ to date $D_2 \equiv (y_1, m_1, d_1)$ under the 30/360 convention can be computed by

$$360 \times (y_2 - y_1) + 30 \times (m_2 - m_1) + (d_2 - d_1),$$

where y_i denote the years, m_i the months, and d_i the days. If d_1 or d_2 is 31, we need to change it to 30 before applying the above formula.

3.5.4 Accrued Interest

Up to now, we have assumed that the next coupon payment date is exactly one period (6 months for bonds, for instance) from now. In reality, the settlement date may fall on any day between two coupon payment dates and yield measures have to be adjusted accordingly. Let

$$\omega \equiv \frac{\text{number of days between the settlement and the next coupon payment date}}{\text{number of days in the coupon period}}; \tag{3.19}$$

the day count is based on the convention applicable to the security in question. The price is now calculated by

$$\text{PV} = \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{\omega+i}} + \frac{F}{\left(1+\frac{r}{m}\right)^{\omega+n-1}}, \tag{3.20}$$

where n is the number of remaining coupon payments [328]. This price is called the **full price**, **dirty price**, or **invoice price**. Equation (3.20) reduces to Eq. (3.18) when $\omega = 1$.

As the issuer of the bond will not send the next coupon to the seller after the transaction, the buyer has to pay the seller part of the coupon during the time the bond was owned by the seller. The convention is that the buyer pays the quoted price plus the **accrued interest** calculated by

$$C \times \frac{\text{number of days from the last coupon payment to the settlement date}}{\text{number of days in the coupon period}}$$
$$= C \times (1 - \omega).$$

The yield to maturity is the r satisfying Eq. (3.20) when the PV is the invoice price, the sum of the quoted price and the accrued interest. As the quoted price in the United States does not include the accrued interest, it is also called the **clean price** or **flat price**.

EXAMPLE 3.5.3 Consider a bond with a 10% coupon rate and paying interest semiannually. The maturity date is March 1, 1995, and the settlement date is July 1, 1993. The day count is 30/360. Because there are 60 days between July 1, 1993, and the next coupon date, September 1, 1993, the accrued interest is $(10/2) \times [\,(180 - 60)/180\,] = 3.3333$ per \$100 of par value. At the clean price of 111.2891, the yield to maturity is 3%. This can be verified by Eq. (3.20) with $\omega = 60/180$, $m = 2$, $C = 5$, PV$= 111.2891 + 3.3333$, and $r = 0.03$.

➤ **Exercise 3.5.6** It has been mentioned that a bond selling at par will continue to sell at par as long as the yield to maturity is equal to the coupon rate. This conclusion rests on the assumption that the settlement date is on a coupon payment date. Suppose that the settlement date for a bond selling at par (i.e., the *quoted price* is equal to the par value) falls between two coupon payment dates. Prove that its yield to maturity is less than the coupon rate.

➤ **Exercise 3.5.7** Consider a bond with a 10% coupon rate and paying interest semiannually. The maturity date is March 1, 1995, and the settlement date is July 1, 1993. The day count used is actual/actual. Verify that there are 62 days between July 1, 1993, and the next coupon date, September 1, 1993, and that the accrued interest is 3.31522% of par value. Also verify that the yield to maturity is 3% when the bond is selling for 111.3.

➢ **Programming Assignment 3.5.8** Write a program that computes (1) the accrued interest as a percentage of par and (2) the BEY of coupon bonds. The inputs are the coupon rate as a percentage of par, the next coupon payment date, the coupon payment frequency per annum, the remaining number of coupon payments after the next coupon, and the day count convention.

3.5.5 Yield for a Portfolio of Bonds

Calculation for the yield to maturity for a portfolio of bonds is no different from that for a single bond. First, the cash flows of the individual bonds are combined. Then the yield is calculated based on the combined cash flow as if it were from a single bond.

EXAMPLE 3.5.4 A bond portfolio consists of two zero-coupon bonds. The bonds are selling at 50 and 20, respectively. The term is exactly 3 years from now. To calculate the yield, we solve

$$50 + 20 = \frac{100 + 100}{(1 + y)^6}$$

for y. Because $y = 0.19121$, the annualized yield is 38.242%. The yields to maturity for the individual bonds are 24.4924% and 61.5321%. Neither a simple average (43.01225%) nor a weighted average (35.0752%) matches 38.242%.

3.5.6 Components of Return

Recall that a bond has a price

$$P = Fc\,\frac{1 - (1 + y)^{-n}}{y} + \frac{F}{(1 + y)^n},$$

where c is the period coupon rate and y is the period interest rate. Its **total monetary return** is $P(1 + y)^n - P$, which is equal to

$$\begin{aligned} &Fc\,\frac{(1 + y)^n - 1}{y} + F - P \\ &\quad = Fc\,\frac{(1 + y)^n - 1}{y} + F - Fc\,\frac{1 - (1 + y)^{-n}}{y} - \frac{F}{(1 + y)^n}. \end{aligned}$$

This return can be broken down into three components: **capital gain/loss** $F - P$, **coupon interest** nFc, and **interest on interest** equal to

$$\begin{aligned} [\,P(1 + y)^n - P\,] - (F - P) - nFc &= P(1 + y)^n - F - nFc \\ &= Fc\,\frac{(1 + y)^n - 1}{y} - nFc. \end{aligned}$$

The interest on interest's percentage of the total monetary return can be shown to increase as c increases. This means that the higher the coupon rate, the more dependent is the total monetary return on the interest on interest. So bonds selling at a premium are more dependent on the interest-on-interest component, given the same maturity and yield to maturity. It can be verified that when the bond is selling at par ($c = y$), the longer the maturity n, the higher the proportion of the interest on interest among the total monetary return. The same claim also holds for bonds selling at a premium ($y < c$) or at a discount ($y > c$).

The above observations reveal the impact of reinvestment risk. Coupon bonds that obtain a higher percentage of their monetary return from the reinvestment of coupon interests are more vulnerable to changes in reinvestment opportunities. The yield to maturity, which assumes that all coupon payments can be reinvested at the yield to maturity, is problematic because this assumption is seldom realized in a changing environment.

Recall that the HPR measures the return by holding the security until the **horizon date**. This period of time is called the **holding period** or the **investment horizon**. The HPR is composed of (1) capital gain/loss on the horizon date, (2) cash flow income such as coupon and mortgage payments, and (3) reinvestment income from reinvesting the cash flows received between the settlement date and the horizon date. Apparently, one has to make explicit assumptions about the reinvestment rate during the holding period and the security's market price on the horizon date called the **horizon price**. Computing the HPR for each assumption is called **scenario analysis**. The scenarios may be analyzed to find the optimal solution [891]. The **value at risk** (**VaR**) methodology is a refinement of scenario analysis. It constructs a confidence interval for the dollar return at horizon based on some stochastic models (see Section 31.4).

EXAMPLE 3.5.5 Consider a 5-year bond paying semiannual interest at a coupon rate of 10%. Assume that the bond is bought for 90 and held to maturity with a reinvestment rate of 5%. The coupon interest plus the interest on interest amounts to

$$\sum_{i=1}^{2\times 5} \frac{10}{2} \times \left(1 + \frac{0.05}{2}\right)^{i-1} = 56.017 \text{ dollars.}$$

The capital gain is $100 - 90 = 10$. The HPR is therefore $56.017 + 10 = 66.017$ dollars. The holding period yield is $y = 12.767\%$ because

$$\left(1 + \frac{y}{2}\right)^{2\times 5} = \frac{100 + 56.017}{90}.$$

As a comparison, its BEY to maturity is 12.767%. Clearly, different HPRs obtain under different reinvestment rate assumptions. If the security is to be sold before it matures, its horizon price needs to be figured out as well.

➤ **Exercise 3.5.9** Prove that the holding period yield of a level-coupon bond is exactly y when the horizon is one period from now.

Additional Reading

Yield, day count, and accrued interest interact in complex ways [827]. See [244, 323, 325, 328, 895] for more information about the materials in the chapter. Consult [35, 224, 381, 417, 447, 727] for the numerical techniques on solving equations.

NOTES

1. The idea of PV is due to Irving Fisher (1867–1947) in 1896 [646].
2. There are arrangements whereby the remaining principal actually increases and then decreases over the term of the loan. The same principle applies (see Exercise 3.3.2).
3. Terms with identical connotation include **total return, horizon return, horizon total return**, and **investment horizon return** [646].
4. $[a, b]$ denotes the interval $a \le x \le b$, $[a, b)$ denotes the interval $a \le x < b$, $(a, b]$ denotes the interval $a < x \le b$, and (a, b) denotes the interval $a < x < b$.
5. Also called **denomination, face value, maturity value**, or **principal value**.
6. See [767] for the reasons why companies issue callable bonds. Callable bonds were not issued by the U.S. Treasury after February 1985 [325].
7. Reversing this basic relation is common. For example, it is written in [703] that "If Japanese banks are hit by a liquidity problem, they may have to sell U.S. Treasury bonds. A strong sell-off could have the effect of pushing down bond yields and rattling Wall Street."