CHAPTER THIRTEEN

Stochastic Processes and Brownian Motion

Of all the intellectual hurdles which the human mind has confronted and has overcome in the last fifteen hundred years, the one which seems to me to have been the most amazing in character and the most stupendous in the scope of its consequences is the one relating to the problem of motion.

Herbert Butterfield (1900–1979), *The Origins of Modern Science*

This chapter introduces basic ideas in stochastic processes and Brownian motion. The Brownian motion underlies the continuous-time models in this book.[1] We will often return to earlier discrete-time binomial models to mark the transition to continuous time.

13.1 Stochastic Processes

A **stochastic process** $X = \{ X(t) \}$ is a time series of random variables. In other words, $X(t)$ is a random variable for each time t and is usually called the **state** of the process at time t. For clarity, $X(t)$ is often written as X_t. A **realization** of X is called a **sample path**. Note that a sample path defines an ordinary function of t. If the times t form a countable set, X is called a discrete-time stochastic process or a **time series**. In this case, subscripts rather than parentheses are usually used, as in $X = \{ X_n \}$. If the times form a continuum, X is called a continuous-time stochastic process.

A continuous-time stochastic process $\{ X(t) \}$ is said to have **independent increments** if for all $t_0 < t_1 < \cdots < t_n$ the random variables

$$X(t_1) - X(t_0),\ X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent. It is said to possess **stationary increments** if $X(t+s) - X(t)$ has the same distribution for all t. That is, the distribution depends on only s.

The **covariance function** of a stochastic process $X = \{ X(t) \}$ is defined as

$$K_X(s, t) \equiv \text{Cov}[\, X(s), X(t) \,].$$

Note that $K_X(s,t) = K_X(t,s)$. The **mean function** is defined as $m_X(t) \equiv E[\, X(t) \,]$. A stochastic process $\{ X(t) \}$ is **strictly stationary** if for any n time points $t_1 < t_2 < \cdots < t_n$ and h, the random-variable sets $\{ X(t_1), X(t_2), \dots, X(t_n) \}$ and $\{ X(t_1+h), X(t_2+h), \dots, X(t_n+h) \}$ have the same joint probability distribution.

From this definition,

$$m_X(t) = E[\, X(t)\,] = E[\, X(t+h)\,] = m_X(t+h)$$

for any h; in other words, the mean function is a constant. Furthermore,

$$\begin{aligned} K_X(s, s+t) &= E[\,\{ X(s) - m_X \}\{ X(s+t) - m_X \}\,] \\ &= E[\,\{ X(0) - m_X \}\{ X(t) - m_X \}\,]; \end{aligned}$$

in other words, the covariance function $K_X(s,t)$ depends on only the **lag** $|s-t|$. A process $\{ X(t) \}$ is said to be **stationary** if $E[\, X(t)^2\,] < \infty$, the mean function is a constant, and the covariance function depends on only the lag.

A **Markov process** is a stochastic process for which everything that we know about its future is summarized by its current value. Formally, a continuous-time stochastic process $X = \{ X(t), t \geq 0 \}$ is Markovian if

$$\text{Prob}[\, X(t) \leq x \mid X(u), 0 \leq u \leq s\,] = \text{Prob}[\, X(t) \leq x \mid X(s)\,]$$

for $s < t$.

Random walks of various kinds are the foundations of discrete-time probabilistic models of asset prices [334]. In fact, the binomial model of stock prices is a random walk in disguise. The following examples introduce two important random walks.

EXAMPLE 13.1.1 Consider a particle on the integer line, $0, \pm 1, \pm 2, \ldots$. In each time step, this particle can make one move to the right with probability p or one move to the left with probability $1-p$ (see Fig. 13.1). Let $P_{i,j}$ represent the probability that the particle makes a transition at point j when currently in point i. Then $P_{i,i+1} = p = 1 - P_{i,i-1}$ for $i = 0, \pm 1, \pm 2, \ldots$. This random walk is **symmetric** when $p = 1/2$. The connection with the BOPM should be clear: The particle's position denotes the cumulative number of up moves minus that of down moves.

EXAMPLE 13.1.2 The **random walk with drift** is the following discrete-time process:

$$X_n = \mu + X_{n-1} + \xi_n, \tag{13.1}$$

where ξ_n are independent and identically distributed with zero mean. The drift μ is the expected change per period. This random walk is a Markov process. An

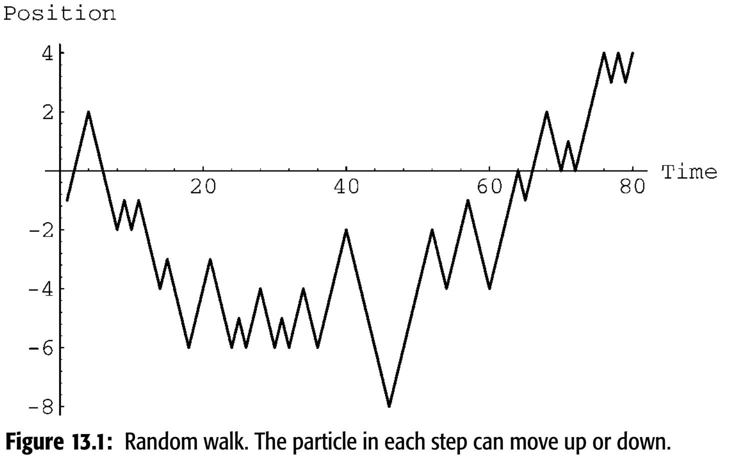

Figure 13.1: Random walk. The particle in each step can move up or down.

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alternative characterization is $\{ S_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \}$, where X_i are independent, identically distributed random variables with $E[\, X_i \,] = \mu$.

➤ **Exercise 13.1.1** Prove that

$$E[\, X(t) - X(0) \,] = t \times E[\, X(1) - X(0) \,],$$
$$\mathrm{Var}[\, X(t) \,] - \mathrm{Var}[\, X(0) \,] = t \times \{ \mathrm{Var}[\, X(1) \,] - \mathrm{Var}[\, X(0) \,] \}$$

when $\{ X(t), t \geq 0 \}$ is a process with stationary independent increments.

➤ **Exercise 13.1.2** Let $Y_1, Y_2, \ldots,$ be mutually independent random variables and X_0 an arbitrary random variable. Define $X_n \equiv X_0 + \sum_{i=1}^{n} Y_i$ for $n > 0$. Show that $\{ X_n, n \geq 0 \}$ is a stochastic process with independent increments.

➤ **Exercise 13.1.3** Let $X_1, X_2, \ldots,$ be a sequence of uncorrelated random variables with zero mean and unit variance. Prove that $\{ X_n \}$ is stationary.

➤ **Exercise 13.1.4** (1) Use Eq. (13.1) to characterize the random walk in Example 13.1.1. (2) Show that the variance of the symmetric random walk's position after n moves is n.

➤ **Exercise 13.1.5** Construct two symmetric random walks with correlation ρ.

13.2 Martingales ("Fair Games")

A stochastic process $\{ X(t), t \geq 0 \}$ is a **martingale** if $E[\, |X(t)| \,] < \infty$ for $t \geq 0$ and

$$E[\, X(t) \mid X(u), 0 \leq u \leq s \,] = X(s). \tag{13.2}$$

In the discrete-time setting, a martingale means that

$$E[\, X_{n+1} \mid X_1, X_2, \ldots, X_n \,] = X_n. \tag{13.3}$$

If X_n is interpreted as a gambler's fortune after the nth gamble, identity (13.3) says the expected fortune after the $(n+1)$th gamble equals the fortune after the nth gamble regardless of what may have occurred before. A martingale is therefore a notion of fair games. Apply the law of iterated conditional expectations to both sides of Eq. (13.3) to yield

$$E[\, X_n \,] = E[\, X_1 \,] \tag{13.4}$$

for all n. Similarly, $E[\, X(t) \,] = E[\, X(0) \,]$ in the continuous-time case.

EXAMPLE 13.2.1 Consider the stochastic process $\{ Z_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \}$, where X_i are independent random variables with zero mean. This process is a martingale because

$$\begin{aligned} E[\, Z_{n+1} \mid Z_1, Z_2, \ldots, Z_n \,] &= E[\, Z_n + X_{n+1} \mid Z_1, Z_2, \ldots, Z_n \,] \\ &= E[\, Z_n \mid Z_1, Z_2, \ldots, Z_n \,] + E[\, X_{n+1} \mid Z_1, Z_2, \ldots, Z_n \,] \\ &= Z_n + E[\, X_{n+1} \,] = Z_n. \end{aligned}$$

Note that $\{ Z_n \}$ subsumes the random walk in Example 13.1.2.

A martingale is defined with respect to a probability measure under which the conditional expectation is taken. A **probability measure** assigns probabilities to states of the world. A martingale is also defined with respect to an information set [692].

In characterizations (13.2) and (13.3), the information set contains the current and the past values of X by default. However, it need not be so. In general, a stochastic process $\{ X(t), t \geq 0 \}$ is called a martingale with respect to information sets $\{ I_t \}$ if, for all $t \geq 0$, $E[\, |X(t)|\,] < \infty$ and

$$E[\, X(u) \mid I_t \,] = X(t)$$

for all $u > t$. In the discrete-time setting, the definition becomes, for all $n > 0$,

$$E[\, X_{n+1} \mid I_n \,] = X_n,$$

given the information sets $\{ I_n \}$. The preceding definition implies that $E[\, X_{n+m} \mid I_n \,] = X_n$ for any $m > 0$ by Eq. (6.6). A typical I_n in asset pricing is the price information up to time n. Then the definition says that the future values of X will not deviate systematically from today's value given the price history; in fact, today's value is their best predictor (see Exercise 13.2.2).

EXAMPLE 13.2.2 Consider the stochastic process $\{ Z_n - n\mu, n \geq 1 \}$, where $Z_n \equiv \sum_{i=1}^{n} X_i$ and $X_1, X_2, \ldots,$ are independent random variables with mean μ. As

$$\begin{aligned} E[\, Z_{n+1} - (n+1)\,\mu \mid X_1, X_2, \ldots, X_n \,] &= E[\, Z_{n+1} \mid X_1, X_2, \ldots, X_n \,] - (n+1)\,\mu \\ &= Z_n + \mu - (n+1)\,\mu \\ &= Z_n - n\mu, \end{aligned}$$

$\{ Z_n - n\mu, n \geq 1 \}$ is a martingale with respect to $\{ I_n \}$, where $I_n \equiv \{ X_1, X_2, \ldots, X_n \}$.

➤ **Exercise 13.2.1** Let $\{ X(t), t \geq 0 \}$ be a stochastic process with independent increments. Show that $\{ X(t), t \geq 0 \}$ is a martingale if $E[\, X(t) - X(s) \,] = 0$ for any $s, t \geq 0$ and $\text{Prob}[\, X(0) = 0 \,] = 1$.

➤ **Exercise 13.2.2** If the asset return follows a martingale, then the best forecast of tomorrow's return is today's return as measured by the minimal mean-square error. Why? (Hint: see Exercise 6.4.3, part (2).)

➤ **Exercise 13.2.3** Define $Z_n \equiv \prod_{i=1}^{n} X_i, n \geq 1$, where $X_1, X_2, \ldots,$ are independent random variables with $E[\, X_i \,] = 1$. Prove that $\{ Z_n \}$ is a martingale.

➤ **Exercise 13.2.4** Consider a martingale $\{ Z_n, n \geq 1 \}$ and let $X_i \equiv Z_i - Z_{i-1}$ with $Z_0 = 0$. Prove $\text{Var}[\, Z_n \,] = \sum_{i=1}^{n} \text{Var}[\, X_i \,]$.

➤ **Exercise 13.2.5** Let $\{ S_n \equiv \sum_{i=1}^{n} X_i, n \geq 1 \}$ be a random walk, where X_i are independent random variables with $E[\, X_i \,] = 0$ and $\text{Var}[\, X_i \,] = \sigma^2$. Show that $\{ S_n^2 - n\sigma^2, n \geq 1 \}$ is a martingale.

➤ **Exercise 13.2.6** Let $\{ X_n \}$ be a martingale and let C_n denote the stake on the nth game. C_n may depend on $X_1, X_2, \ldots, X_{n-1}$ and is bounded. C_1 is a constant. Interpret $C_n(X_n - X_{n-1})$ as the gains on the nth game. The total gains up to game n are $Y_n \equiv \sum_{i=1}^{n} C_i(X_i - X_{i-1})$ with $Y_0 = 0$. Prove that $\{ Y_n \}$ is a martingale with respect to $\{ I_n \}$, where $I_n \equiv \{ X_1, X_2, \ldots, X_n \}$.

13.2.1 Martingale Pricing and Risk-Neutral Valuation

We learned in Lemma 9.2.1 that the price of a European option is the expected discounted future payoff at expiration in a risk-neutral economy. This important principle can be generalized by use of the concept of martingale. Recall the recursive valuation of a European option by means of $C = [\, pC_u + (1-p)\, C_d\,]/R$, where p is the risk-neutral probability and \$1 grows to \$$R$ in a period. Let $C(i)$ denote the value of the option at time i and consider the **discount process** $\{\, C(i)/R^i, i = 0, 1, \ldots, n\,\}$. Then,

$$E\left[\left.\frac{C(i+1)}{R^{i+1}}\,\right|\, C(i) = C\right] = \frac{pC_u + (1-p)\, C_d}{R^{i+1}} = \frac{C}{R^i}.$$

The above result can be easily generalized to

$$E\left[\left.\frac{C(k)}{R^k}\,\right|\, C(i) = C\right] = \frac{C}{R^i}, \quad i \leq k. \tag{13.5}$$

Hence the discount process is a martingale as

$$\frac{C(i)}{R^i} = E_i^{\pi}\left[\frac{C(k)}{R^k}\right], \quad i \leq k, \tag{13.6}$$

where E_i^{π} means that the expectation is taken under the risk-neutral probability conditional on the price information up to time i.[2] This risk-neutral probability is also called the **equivalent martingale probability measure** [514]. Two probability measures are said to be **equivalent** if they assign nonzero probabilities to the same set of states.

Under general discrete-time models, Eq. (13.6) holds for all assets, not just options. In the general case in which interest rates are stochastic, the equation becomes [725]

$$\frac{C(i)}{M(i)} = E_i^{\pi}\left[\frac{C(k)}{M(k)}\right], \quad i \leq k, \tag{13.7}$$

where $M(j)$ denotes the balance in the money market account at time j by use of the rollover strategy with an initial investment of \$1. For this reason, it is called the **bank account process**. If interest rates are stochastic, then $M(j)$ is a random variable. However, note that $M(0) = 1$ and $M(j)$ is known at time $j-1$. Identity (13.7) is the general formulation of risk-neutral valuation, which says the discount process is a martingale under π. We thus have the following fundamental theorem of asset pricing.

THEOREM 13.2.3 *A discrete-time model is arbitrage free if and only if there exists a probability measure such that the discount process is a martingale. This probability measure is called the risk-neutral probability measure.*

➤ **Exercise 13.2.7** Verify Eq. (13.5).

➤ **Exercise 13.2.8** Assume that one unit of domestic (foreign) currency grows to R (R_{f}, respectively) units in a period, and u and d are the up and the down moves of the domestic/foreign exchange rate, respectively. Apply identity (13.7) to derive the risk-neutral probability $[\,(R/R_{\mathrm{f}}) - d\,]/(u-d)$ for forex options under the BOPM in Exercise 11.5.4, part (1).

➤ **Exercise 13.2.9** Prove that the discounted stock price $S(i)/R^i$ follows a martingale under the risk-neutral probability; in particular, $S(0) = E^{\pi}[\, S(i)/R^i\,]$.

13.2.2 Futures Price under the Binomial Model

Futures prices form a martingale under the risk-neutral probability because the expected futures price in the next period is

$$p_{\mathrm{f}} F u + (1 - p_{\mathrm{f}})\, F d = F\left(\frac{1-d}{u-d}\, u + \frac{u-1}{u-d}\, d\right) = F$$

(review Subsection 12.4.3). The above claim can be generalized to

$$F_i = E_i^{\pi}[\, F_k \,], \quad i \leq k, \tag{13.8}$$

where F_i is the futures price at time i. This identity holds under stochastic interest rates as well (see Exercise 13.2.11).

➤ **Exercise 13.2.10** Prove that $F = E^{\pi}[\, S_n \,]$, where S_n denotes the price of the underlying non-dividend-paying stock at the delivery date of the futures contract, time n. (The futures price is thus an unbiased estimator of the expected spot price *in a risk-neutral economy*.)

➤ **Exercise 13.2.11** Show that identity (13.8) holds under stochastic interest rates.

13.2.3 Martingale Pricing and the Choice of Numeraire

Martingale pricing formula (13.7) uses the money market account as **numeraire** in that it expresses the price of any asset relative to the money market account.[3] The money market account is not the only choice for numeraire, however. If asset S, whose value is positive at all times, is chosen as numeraire, martingale pricing says there exists a risk-neutral probability π under which the relative price of any asset C is a martingale:

$$\frac{C(i)}{S(i)} = E_i^{\pi}\left[\frac{C(k)}{S(k)}\right], \quad i \leq k, \tag{13.9}$$

where $S(j)$ denotes the price of S at time j; the discount process remains a martingale.

Take the binomial model with two assets as an example. In a period, asset one's price can go from S to S_1 or S_2, whereas asset two's price can go from P to P_1 or P_2. Assume that $(S_1/P_1) < (S/P) < (S_2/P_2)$ for market completeness and to rule out arbitrage opportunities. For any derivative security, let C_1 be its price at time one if asset one's price moves to S_1 and let C_2 be its price at time one if asset one's price moves to S_2. Replicate the derivative by solving

$$\alpha S_1 + \beta P_1 = C_1,$$
$$\alpha S_2 + \beta P_2 = C_2$$

by using α units of asset one and β units of asset two. This yields

$$\alpha = \frac{P_2 C_1 - P_1 C_2}{P_2 S_1 - P_1 S_2}, \quad \beta = \frac{S_2 C_1 - S_1 C_2}{S_2 P_1 - S_1 P_2}$$

and the derivative costs

$$C = \alpha S + \beta P = \frac{P_2 S - P S_2}{P_2 S_1 - P_1 S_2} C_1 + \frac{P S_1 - P_1 S}{P_2 S_1 - P_1 S_2} C_2.$$

It is easy to verify that

$$\frac{C}{P} = p \frac{C_1}{P_1} + (1-p) \frac{C_2}{P_2}, \quad \text{where } p \equiv \frac{(S/P) - (S_2/P_2)}{(S_1/P_1) - (S_2/P_2)}. \tag{13.10}$$

The derivative's price with asset two as numeraire is thus a martingale under the risk-neutral probability p. Interestingly, the expected returns of the two assets are irrelevant.

EXAMPLE 13.2.4 For the BOPM in Section 9.2, the two assets are the money market account and the stock. Because the money market account is the numeraire, we substitute $P = 1$, $P_1 = P_2 = R$, $S_1 = Su$, and $S_2 = Sd$ into Eq. (13.10). The result,

$$p = \frac{S - (Sd/R)}{(Su/R) - (Sd/R)} = \frac{R-d}{u-d},$$

affirms the familiar risk-neutral probability. The risk-neutral probability changes if the stock is chosen as numeraire, however (see Exercise 13.2.12).

The risk-neutral probability measure therefore depends on the choice of numeraire, and switching numeraire changes the risk-neutral probability measure. Picking the "right" numeraire can simplify the task of pricing, especially for interest-rate-sensitive securities [25, 731, 783]. For the rest of the book, the money market account will continue to serve as numeraire unless stated otherwise.

➤ **Exercise 13.2.12** (1) Prove that $[\,(1/d) - (1/R)\,][\,(ud)/(u-d)\,]$ is the up-move probability for the stock price that makes the relative bond price a martingale under the binomial option pricing model in which the *stock* is chosen as numeraire. (2) Reinterpret Eq. (9.11).

➤ **Exercise 13.2.13** Show that for any $k > 0$ there exists a risk-neutral probability measure π under which the price of any asset C equals its discounted expected future price at time k, that is, $C(i) = d(k-i)\, E_i^{\pi}[\,C(k)\,]$, where $i \leq k$. Recall that $d(\cdot)$ denotes the discount function at time i. This π is called the **forward-neutral probability measure**.

13.3 Brownian Motion

Brownian motion is a stochastic process $\{\, X(t), t \geq 0\,\}$ with the following properties:

(1) $X(0) = 0$, unless stated otherwise;
(2) for any $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X(t_k) - X(t_{k-1})$ for $1 \leq k \leq n$ are independent (so $X(t) - X(s)$ is independent of $X(r)$ for $r \leq s < t$);
(3) for $0 \leq s < t$, $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $\sigma^2(t-s)$, where μ and $\sigma \neq 0$ are real numbers.

Such a process is called a (μ, σ) Brownian motion with **drift** μ and **variance** σ^2. Figure 13.2 plots a realization of a Brownian motion process. The existence and

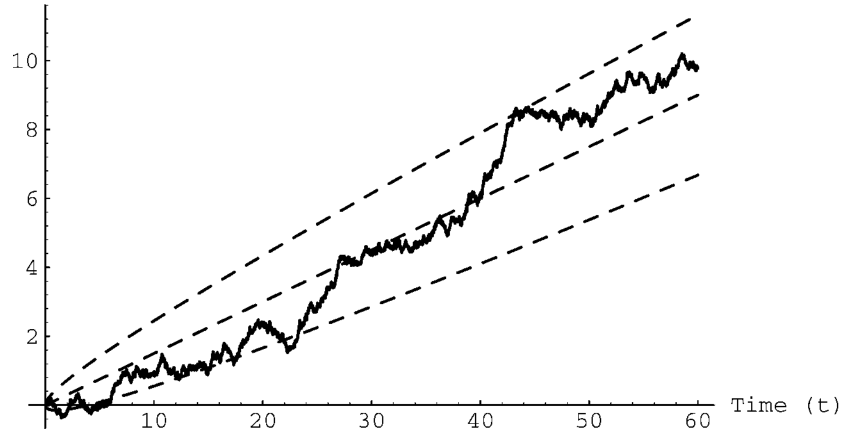

Figure 13.2: Drift and variance of Brownian motion. Shown is a sample path of a $(0.15, 0.3)$ Brownian motion. The stochastic process has volatility, as evinced by the bumpiness of the path. The envelope is for one standard deviation, or $0.3\sqrt{t}$, around the mean function, which is the deterministic process with the randomness removed.

the uniqueness of such a process are guaranteed by **Wiener's theorem** [73]. Although Brownian motion is a continuous function of t with probability one, it is almost nowhere differentiable. The $(0, 1)$ Brownian motion is also called **normalized Brownian motion** or the **Wiener process**.

Any continuous-time process with stationary independent increments can be proved to be a Brownian motion process [419]. This fact explains the significance of Brownian motion in stochastic modeling. Brownian motion also demonstrates **statistical self-similarity** in that $X(rx)/\sqrt{r}$ remains a Wiener process if X is such. This means that if we sample the process 100 times faster and then shrink the result 10 times, the path will look statistically the same as the original one. This property naturally links Brownian motion to fractals [240, 784]. Finally, Brownian motion is Markovian.

Brownian motion, named after Robert Brown (1773–1858), was first discussed mathematically by Bachelier and received rigorous treatments by Wiener (1894–1964), who came up with the above concise definition. Therefore it is sometimes called the **generalized Wiener process** or the **Wiener–Bachelier process** [343, 543].

EXAMPLE 13.3.1 Suppose that the total value of a company, measured in millions of dollars, follows a $(20, 30)$ Brownian motion (i.e., with a drift of 20 per annum and a variance of 900 per annum). The starting total value is 50. At the end of 1 year, the total value will have a normal distribution with a mean of 70 and a standard deviation of 30. At the end of 6 months, as another example, it will have a normal distribution with a mean of 60 and a standard deviation of $\sqrt{450} \approx 21.21$.

From the definition, if $\{ X(t), t \geq 0 \}$ is the Wiener process, then $X(t) - X(s) \sim N(0, t - s)$. A (μ, σ) Brownian motion $Y = \{ Y(t), t \geq 0 \}$ can be expressed in terms of the Wiener process by

$$Y(t) = \mu t + \sigma X(t). \tag{13.11}$$

As $Y(t + s) - Y(t) \sim N(\mu s, \sigma^2 s)$, our uncertainty about the future value of Y as measured by the standard deviation grows as the square root of how far we look into the future.

➤ **Exercise 13.3.1** Prove that $\{(X(t)-\mu t)/\sigma, t \geq 0\}$ is a Wiener process if $\{X(t), t \geq 0\}$ is a (μ, σ) Brownian motion.

➤ **Exercise 13.3.2** Verify that $K_X(t,s) = \sigma^2 \times \min(s,t)$ if $\{X(t), t \geq 0\}$ is a (μ, σ) Brownian motion.

➤ **Exercise 13.3.3** Let $\{X(t), t \geq 0\}$ represent the Wiener process. Show that the related process $\{X(t) - X(0), t \geq 0\}$ is a martingale. ($X(0)$ can be a random variable.)

➤ **Exercise 13.3.4** Let $p(x, y; t)$ denote the transition probability density function of a (μ, σ) Brownian motion starting at x; $p(x, y; t) = (1/\sqrt{2\pi t}\,\sigma)\exp[-(y-x-\mu t)^2/(2\sigma^2 t)]$. Show that p satisfies **Kolmogorov's backward equation** $\partial p/\partial t = (\sigma^2/2)(\partial^2 p/\partial x^2) + \mu(\partial p/\partial x)$, and **Kolmogorov's forward equation** (also called the **Fokker–Planck equation**) $\partial p/\partial t = (\sigma^2/2)(\partial^2 p/\partial y^2) - \mu(\partial p/\partial y)$.

➤ **Exercise 13.3.5** Let $\{X(t), t \geq 0\}$ be a $(0, \sigma)$ Brownian motion. Prove that the following three processes are martingales: (1) $X(t)$, (2) $X(t)^2 - \sigma^2 t$, and (3) $\exp[\,\alpha X(t) - \alpha^2\sigma^2 t/2\,]$ for $\alpha \in \boldsymbol{R}$, called **Wald's martingale**.

13.3.1 Brownian Motion as the Limit of a Random Walk

A (μ, σ) Brownian motion is the limiting case of a random walk. Suppose that a particle moves Δx either to the left with probability $1-p$ or to the right with probability p after Δt time. For simplicity assume that $n \equiv t/\Delta t$ is an integer. Its position at time t is

$$Y(t) \equiv \Delta x\,(X_1 + X_2 + \cdots + X_n),$$

where

$$X_i \equiv \begin{cases} +1, & \text{if the } i\text{th move is to the right} \\ -1, & \text{if the } i\text{th move is to the left} \end{cases},$$

and X_i are independent with $\text{Prob}[\,X_i = 1\,] = p = 1 - \text{Prob}[\,X_i = -1\,]$. Note that $E[\,X_i\,] = 2p-1$ and that $\text{Var}[\,X_i\,] = 1-(2p-1)^2$. Therefore

$$E[\,Y(t)\,] = n(\Delta x)(2p-1) \quad \text{and} \quad \text{Var}[\,Y(t)\,] = n(\Delta x)^2[\,1-(2p-1)^2\,].$$

Letting $\Delta x \equiv \sigma\sqrt{\Delta t}$ and $p \equiv (1 + (\mu/\sigma)\sqrt{\Delta t})/2$, we conclude that

$$E[\,Y(t)\,] = n\sigma\sqrt{\Delta t}\,(\mu/\sigma)\sqrt{\Delta t} = \mu t,$$
$$\text{Var}[\,Y(t)\,] = n\sigma^2\Delta t\left(1 - (\mu/\sigma)^2\Delta t\right) \to \sigma^2 t$$

as $\Delta t \to 0$. Thus $\{Y(t), t \geq 0\}$ converges to a (μ, σ) Brownian motion by the central limit theorem. In particular, Brownian motion with zero drift is the limiting case of symmetric random walk when $\mu = 0$ is chosen. Note also that

$$\text{Var}[\,Y(t+\Delta t) - Y(t)\,] = \text{Var}[\,\Delta x\, X_{n+1}\,] = (\Delta x)^2 \times \text{Var}[\,X_{n+1}\,] \to \sigma^2\Delta t.$$

The similarity to the BOPM is striking: The p above is identical to the probability in Eq. (9.15) and $\Delta x = \ln u$. This is no coincidence (see Subsection 14.4.3).

➤ **Exercise 13.3.6** Let dQ represent the probability that the random walk that converges to a $(\mu, 1)$ Brownian motion takes the moves $X_1, X_2, \ldots$. Let dP denote the probability that the symmetric random walk that converges to the Wiener process

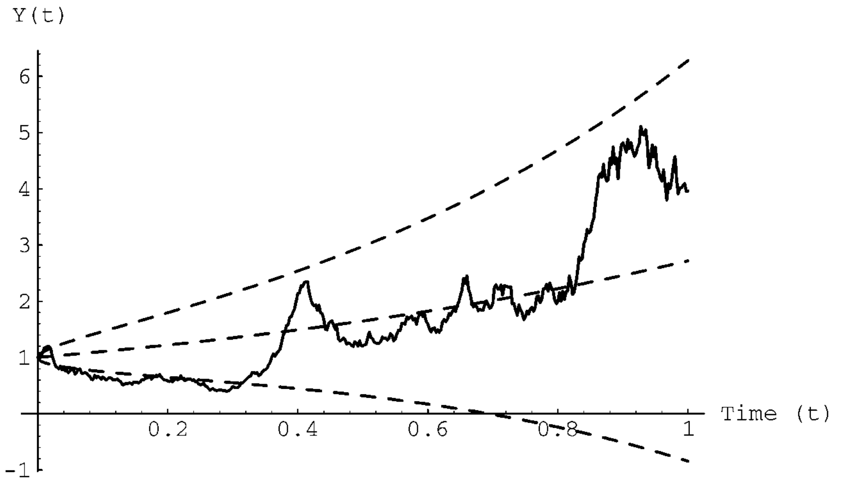

Figure 13.3: Sample path of geometric Brownian motion. The process is $Y(t) = e^{X(t)}$, where X is a $(0.5, 1)$ Brownian motion. The envelope is for one standard deviation, $\sqrt{(e^t - 1)e^{2t}}$, around the mean. Can you tell the qualitative difference between this plot and the stock price charts in Fig. 6.4?

makes identical moves. Derive dQ/dP. (The process dQ/dP in fact converges to Wald's martingale $\exp[\, \mu W(t) - \mu^2 t/2 \,]$, where $W(t)$ is the Wiener process.)

13.3.2 Geometric Brownian Motion

If $X \equiv \{ X(t), t \geq 0 \}$ is a Brownian motion process, the process $\{ Y(t) \equiv e^{X(t)}, t \geq 0 \}$ is called **geometric Brownian motion**. Its other names are **exponential Brownian motion** and **lognormal diffusion**. See Fig. 13.3 for illustration. When X is a (μ, σ) Brownian motion, we have $X(t) \sim N(\mu t, \sigma^2 t)$ and the moment generating function

$$E\left[e^{s X(t)} \right] = E[\, Y(t)^s \,] = e^{\mu t s + (\sigma^2 t s^2/2)}$$

from Eq. (6.8). Thus

$$E[\, Y(t) \,] = e^{\mu t + (\sigma^2 t/2)}, \tag{13.12}$$

$$\mathrm{Var}[\, Y(t) \,] = E[\, Y(t)^2 \,] - E[\, Y(t) \,]^2 = e^{2\mu t + \sigma^2 t}\left(e^{\sigma^2 t} - 1\right). \tag{13.12$'$}$$

Geometric Brownian motion models situations in which percentage changes are independent and identically distributed. To see this point, let Y_n denote the stock price at time n and $Y_0 = 1$. Assume that relative returns $X_i \equiv Y_i / Y_{i-1}$ are independent and identically distributed. Then $\ln Y_n = \sum_{i=1}^{n} \ln X_i$ is a sum of independent, identically distributed random variables, and $\{ \ln Y_n, n \geq 0 \}$ is approximately Brownian motion. Thus $\{ Y_n, n \geq 0 \}$ is approximately geometric Brownian motion.

➤ **Exercise 13.3.7** Let $Y(t) \equiv e^{X(t)}$, where $\{ X(t), t \geq 0 \}$ is a (μ, σ) Brownian motion. Show that $E[\, Y(t) \mid Y(s) \,] = Y(s)\, e^{(t-s)(\mu + \sigma^2/2)}$ for $s < t$.

➤ **Exercise 13.3.8** Assume that the stock price follows the geometric Brownian motion process $S(t) \equiv e^{X(t)}$, where $\{ X(t), t \geq 0 \}$ is a (μ, σ) Brownian motion. (1) Show that the stock price is growing at a rate of $\mu + \sigma^2/2$ (not μ !) on the average if by this rate we mean $R_1 \equiv (t_2 - t_1)^{-1} \ln E[\, S(t_2)/S(t_1)\,]$ for the time period $[\, t_1, t_2\,]$. (2) Show that the alternative measure for the rate of return, $R_2 \equiv (t_2 - t_1)^{-1} E[\ln(S(t_2)/S(t_1))\,]$, gives rise to μ. (3) Argue that $R_2 < R_1$ independent of any assumptions about the process $\{\, X(t)\,\}$.

13.3.3 Stationarity

The Wiener process $\{\, X(t), t \geq 0\,\}$ is not stationary (see Exercise 13.3.2). However, it can be transformed into a stationary process by

$$Y(t) \equiv e^{-t} X(e^{2t}). \tag{13.13}$$

This claim can be verified as follows. Because $Y(t) \sim N(0, 1)$, the mean function is zero, a constant. Furthermore,

$$E[\, Y(t)^2\,] = E[\, e^{-2t}\, X(e^{2t})^2\,] = e^{-2t}\, e^{2t} = 1 < \infty.$$

Finally, the covariance function $K_Y(s, t), s < t$, is

$$E[\, e^{-t} X(e^{2t}) e^{-s} X(e^{2s})\,] = e^{-s-t} E[\, X(e^{2t}) X(e^{2s})\,] = e^{-s-t} e^{2s} = e^{s-t},$$

where the next to last equality is due to Exercise 13.3.2. Therefore $\{\, Y(t), t \geq 0\,\}$ is stationary. The process Y is called the **Ornstein–Uhlenbeck process** [230, 261, 541].

13.3.4 Variations

Many formulas in standard calculus do not carry over to Brownian motion. Take the **quadratic variation** of any function $f : [\,0, \infty) \to R$ defined by[4]

$$\sum_{k=0}^{2^n-1} \left[\, f\left(\frac{(k+1)\,t}{2^n}\right) - f\left(\frac{kt}{2^n}\right)\right]^2.$$

It is not hard to see that the quadratic variation vanishes as $n \to \infty$ if f is differentiable. This conclusion no longer holds if f is Brownian motion. In fact,

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left[\, X\left(\frac{(k+1)\,t}{2^n}\right) - X\left(\frac{kt}{2^n}\right)\right]^2 = \sigma^2 t \tag{13.14}$$

with probability one, where $\{\, X(t), t \geq 0\,\}$ is a (μ, σ) Brownian motion [543]. This result informally says that $\int_0^t [\, dX(s)\,]^2 = \sigma^2 t$, which is frequently written as

$$(dX)^2 = \sigma^2\, dt. \tag{13.15}$$

It can furthermore be shown that

$$(dX)^n = 0 \quad \text{for } n > 2 \tag{13.16}$$

and $dX\, dt = 0$.

From Eq. (13.14), the **total variation** of a Brownian path is infinite with probability one:

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left|\, X\left(\frac{(k+1)\,t}{2^n}\right) - X\left(\frac{kt}{2^n}\right)\right| = \infty. \tag{13.17}$$

Brownian motion is thus continuous but with highly irregular sample paths.

➤ **Exercise 13.3.9** To see the plausibility of Eq. (13.14), take the expectation of its left-hand side and drop $\lim_{n\to\infty}$ to obtain

$$\sum_{k=0}^{2^n-1} E\left[X\left(\frac{(k+1)\,t}{2^n}\right) - X\left(\frac{kt}{2^n}\right)\right]^2.$$

Show that the preceding expression has the value $\mu^2 t^2 2^{-n} + \sigma^2 t$, which approaches $\sigma^2 t$ as $n \to \infty$.

➤ **Exercise 13.3.10** We can prove Eq. (13.17) without using Eq. (13.14). Let

$$f_n(X) \equiv \sum_{k=0}^{2^n-1} \left| X\left(\frac{(k+1)\,t}{2^n}\right) - X\left(\frac{kt}{2^n}\right)\right|.$$

(1) Prove that $|X((k+1)\,t/2^n) - X(kt/2^n)|$ has mean $2^{-n/2}\sqrt{2/\pi}$ and variance $2^{-n}(1-2/\pi)$. ($f_n(X)$ thus has mean $2^{n/2}\sqrt{2/\pi}$ and variance $1-2/\pi$.) (2) Show that $f_n(X) \to \infty$ with probability one. (Hint: $\text{Prob}[\,|X - E[\,X\,]| \geq k\,] \leq \text{Var}[\,X\,]/k^2$ by **Chebyshev's inequality**.)

13.4 Brownian Bridge

A **Brownian bridge** process $\{\, B(t), 0 \leq t \leq 1 \,\}$ is **tied-down** Brownian motion [544]. It is defined as the Wiener process plus the constraint $B(0) = B(1) = 0$. An alternative formulation is $\{\, W(t) - t\,W(1), 0 \leq t < 1 \,\}$, where $\{\, W(t), 0 \leq t \,\}$ is the Wiener process. For a general time period $[\,0, T\,]$, a Brownian bridge process can be written as

$$B(t) \equiv W(t) - \frac{t}{T}\, W(T), \quad 0 \leq t \leq T,$$

where $W(0) = 0$ and $W(T)$ is known at time zero [193]. Observe that $B(t)$ is pinned to zero at both end points, times zero and T.

➤ **Exercise 13.4.1** Prove the following identities: (1) $E[\,B(t)\,] = 0$, (2) $E[\,B(t)^2\,] = t - (t^2/T)$, and (3) $E[\,B(s)\,B(t)\,] = \min(s, t) - (st/T)$.

➤ **Exercise 13.4.2** Write the Brownian bridge process with $B(0) = x$ and $B(T) = y$.

Additional Reading

The idea of martingale is due to Lévy (1886–1974) and received thorough development by Doob [205, 280, 541, 877]. See [817] for a complete treatment of random walks. Consult [631] for a history of Brownian motion from the physicist's point of view and [277] for adding Bachelier's contribution. Reference [104] collects results and formulas in connection with Brownian motion. Advanced materials can be found in [210, 230, 364, 543]. The backward and Fokker–Planck equations mentioned in Exercise 13.3.4 describe a large class of stochastic processes with continuous sample paths [373]. The heuristic arguments in Subsection 13.3.1 showing Brownian motion as the limiting case of random walk can be made rigorous by **Donsker's theorem** [73, 289, 541, 573]. The geometric Brownian motion model for stock prices is due to Osborne [709]. Models of stock returns are surveyed in [561].

NOTES

1. Merton pioneered the alternative **jump process** in pricing [80].
2. For standard European options, price information *at* time i suffices because they are path independent.
3. Regarded by Schumpeter as the greatest economist in his monumental *History of Economic Analysis* [786], Walras (1834–1910) introduced numeraire in his equilibrium analysis, recognizing that only *relative* prices matter [31].
4. For technical reasons, the partition of $[\,0, t\,]$ is **dyadic**, i.e., at points $k(t/2^n)$ for $0 < k < 2^n$.