CHAPTER
EIGHT

Arbitrage in Option Pricing

All general laws are attended with inconveniences, when applied to particular cases.

David Hume, *"Of the Rise and Progress of the Arts and Sciences"*

The **no-arbitrage principle** says there should be no free lunch. Simple as it is, this principle supplies the essential argument for option pricing. After the argument is presented in Section 8.1, several important option pricing relations will be derived.

8.1 The Arbitrage Argument

A riskless arbitrage opportunity is one that, without any initial investment, generates nonnegative returns under all circumstances and positive returns under some circumstances. In an efficient market, such opportunities should not exist. This no-arbitrage principle is behind modern theories of option pricing if not a concept that unifies all of finance [87, 303]. The related **portfolio dominance principle** says that portfolio A should be more valuable than portfolio B if A's payoff is at least as good under all circumstances and better under some circumstances.

A simple corollary of the no-arbitrage principle is that a portfolio yielding a zero return in every possible scenario must have a zero PV. Any other value would imply arbitrage opportunities, which one can realize by shorting the portfolio if its value is positive and buying it if its value is negative. The no-arbitrage principle also justifies the PV formula $P = \sum_{i=1}^{n} C_i d(i)$ for a security with known cash flow $C_1, C_2, \dots, C_n$ (recall that $d(i)$ is the price of the i-period zero-coupon bond with \$1 par value). Any price other than P will lead to riskless gains by means of trading the security and the zeros. Specifically, if the price P^* is lower than P, we short the zeros that match the security's n cash flows in both maturity and amount and use P^* of the proceeds P to buy the security. Because the cash inflows of the security will offset exactly the obligations of the zeros, a riskless profit of $P - P^*$ dollars has been realized now. See Fig. 8.1. On the other hand, if the security is priced higher than P, one can realize a riskless profit by reversing the trades.

Here are two more examples. First, an American option cannot be worth less than the intrinsic value for, otherwise, one can buy the option, promptly exercise it, and sell the stock with a profit. Second, a put or a call must have a nonnegative

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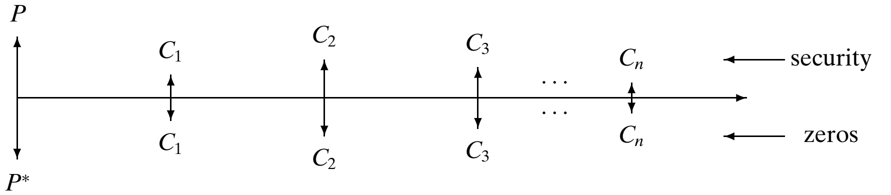

Figure 8.1: Price of fixed cash flow. Consider a security with cash flow $C_1, C_2, \ldots, C_n$ and price P^*. Assemble a portfolio of zero-coupon bonds with matching principals $C_1, C_2, \ldots, C_n$ and maturities $1, 2, \ldots, n$. Let its total price be P. Then $P = P^*$ to preclude arbitrage opportunities.

value for, otherwise, one can buy it for a positive cash flow now and end up with a nonnegative amount at expiration.

➤ **Exercise 8.1.1** Give an arbitrage argument for $d(1) \geq d(2) \geq \cdots$.

➤ **Exercise 8.1.2 (Arbitrage Theorem)**. Consider a world with m states and n securities. Denote the payoff of security j in state i by D_{ij}. Let D be the $m \times n$ matrix whose (i, j)th entry is D_{ij}. Formulate necessary conditions for arbitrage freedom.

8.2 Relative Option Prices

We derive arbitrage-free relations that option values must satisfy. These relations hold regardless of the probabilistic model for stock prices. We only assume, among other things, that there are no transactions costs or margin requirements, borrowing and lending are available at the riskless interest rate, interest rates are nonnegative, and there are no arbitrage opportunities. To simplify the presentation, let the current time be time zero. $\text{PV}(x)$ stands for the PV of x dollars at expiration; hence $\text{PV}(x) = xd(\tau)$, where τ is the time to expiration.

The following lemma shows that American option values rise with the time to expiration. This proposition is consistent with the quotations in Fig. 7.4; however, it may not hold for European options.

LEMMA 8.2.1 *An American call (put) with a longer time to expiration cannot be worth less than an otherwise identical call (put) with a shorter time to expiration.*

Proof: We prove the lemma for the call only. Suppose instead that $C_{t_1} > C_{t_2}$, where $t_1 < t_2$. We buy C_{t_2} and sell C_{t_1} to generate a net cash flow of $C_{t_1} - C_{t_2}$ at time zero. Up to the moment when the time to t_2 is τ and the short call either expires or is exercised, the position is worth $C_\tau - \max(S_\tau - X, 0)$. If this value is positive, close out the position with a profit by selling the remaining call. Otherwise, $\max(S_\tau - X, 0) > C_\tau \geq 0$, and the short call is exercised. In this case, we exercise the remaining call and have a net cash flow of zero. In both cases, the total payoff is positive without any initial investment.

LEMMA 8.2.2 *A call (put) option with a higher (lower) strike price cannot be worth more than an otherwise identical call (put) with a lower (higher) strike price.*

Proof: We prove the lemma for the call only. This proposition certainly holds at expiration; hence it is valid for European calls. Let the two strike prices be $X_1 < X_2$. Suppose that $C_{X_1} < C_{X_2}$ instead. We buy the low-priced C_{X_1} and write the high-priced

C_{X_2}, generating a positive return. If the holder of C_{X_2} exercises it before expiration, just exercise the long call to generate a positive cash flow of $X_2 - X_1$.

LEMMA 8.2.3 *The difference in the values of two otherwise identical options cannot be greater than the difference in their strike prices.*

Proof: We consider the calls only. Let the two strike prices be $X_1 < X_2$. Assume that $C_{X_1} - C_{X_2} > X_2 - X_1$ instead. We buy the lower-priced C_{X_2}, write the higher-priced C_{X_1}, generating a positive return, and deposit $X_2 - X_1$ in a riskless bank account.

Suppose that the holder of C_{X_1} exercises the option before expiration. There are two cases. If $C_{X_2} > S - X_1$, then sell C_{X_2} to realize a cash flow of $C_{X_2} - (S - X_1) > 0$. Otherwise, exercise C_{X_2} and realize a cash flow of $X_1 - X_2 < 0$. In both cases, close out the position with the money in the bank to realize a nonnegative net cash flow.

Suppose the holder of C_{X_1} does not exercise the option early. At the expiration date, our cash flow is 0, $X_1 - S < 0$, and $X_1 - X_2 < 0$, respectively, if $S \leq X_1$, $X_1 < S < X_2$, and $X_2 \leq S$. The net cash flow remains nonnegative after the money in the bank account is added, which is at least $X_2 - X_1$.

LEMMA 8.2.4 *A call is never worth more than the stock price, an American put is never worth more than the strike price, and a European put is never worth more than the present value of the strike price.*

Proof: If the call value exceeded the stock price, a covered call position could earn arbitrage profits. If the put value exceeded the strike price, writing a cash-secured put would earn arbitrage profits. The tighter bound holds for European puts because the cash can earn riskless interest until expiration.

➤ **Exercise 8.2.1** Show that Lemma 8.2.3 can be strengthened for European calls as follows: The difference in the values of two otherwise identical options cannot be greater than the present value of the difference in their strike prices.

➤ **Exercise 8.2.2** Derive a bound similar to that of Lemma 8.2.4 for European puts under negative interest rates. (This case might be relevant when inflation makes the real interest rate negative.)

8.3 Put–Call Parity and Its Consequences

Assume that either the stock pays no cash dividends or that the options are protected so that the option values are insensitive to cash dividends. Note that analysis for options on a non-dividend-paying stock holds for protected options on a dividend-paying stock by definition. Results for protected options therefore are not listed separately.

Consider the portfolio of one short European call, one long European put, one share of stock, and a loan of $\text{PV}(X)$. All options are assumed to carry the same strike price and time to expiration τ. The initial cash flow is therefore $C - P - S + \text{PV}(X)$. At expiration, if the stock price S_τ is at most X, the put will be worth $X - S_\tau$ and the call will expire worthless. On the other hand, if $S_\tau > X$, the call will be worth $S_\tau - X$ and the put will expire worthless. After the loan, now X, is repaid, the net future cash flow is zero in either case. The no-arbitrage principle implies that the initial investment to set up the portfolio must be nil as well. We have proved the following **put–call parity** for European options:

$$C = P + S - \mathrm{PV}(X). \tag{8.1}$$

This identity seems to be due to Castelli in 1877 and thence has been rediscovered many times [156].

The put–call parity implies that there is essentially only one kind of European option because the other can be **replicated** from it in combination with the underlying stock and riskless lending or borrowing. Combinations such as this create **synthetic securities**. For example, rearranging Eq. (8.1) as $S = C - P + \mathrm{PV}(X)$, we see that a stock is equivalent to a portfolio containing a long call, a short put, and lending $\mathrm{PV}(X)$. Other uses of the put–call parity are also possible. Consider $C - P = S - \mathrm{PV}(X)$, which implies that a long call and a short put amount to a long position in stock and borrowing the PV of the strike price – in other words, buying the stock on margin. This might be preferred to taking a levered long position in stock as buying stock on margin is subject to strict margin requirements.

The put–call parity implies that $C = (S - X) + [\, X - \mathrm{PV}(X)\,] + P \geq S - X$. Because $C \geq 0$, it follows that $C \geq \max(S - X, 0)$, the intrinsic value. An American call also cannot be worth less than its intrinsic value. Hence we have the following lemma.

LEMMA 8.3.1 *An American call or a European call on a non-dividend-paying stock is never worth less than its intrinsic value.*

A European put may sell below its intrinsic value. In Fig. 7.3, for example, the put value is less than its intrinsic value when the option is deep in the money. This can be verified more formally, as follows. The put–call parity implies that

$$P = (X - S) + [\, \mathrm{PV}(X) - X + C \,].$$

As the put goes deeper in the money, the call value drops toward zero and $P \approx (X - S) + \mathrm{PV}(X) - X < X - S$, its intrinsic value under positive interest rates. By the put–call parity, the following lower bound holds for European puts.

LEMMA 8.3.2 *For European puts,* $P \geq \max(\mathrm{PV}(X) - S, 0)$.

Suppose that the PV of the dividends whose ex-dividend dates occur before the expiration date is D. Then the put–call parity becomes

$$C = P + S - D - \mathrm{PV}(X). \tag{8.2}$$

➤ **Exercise 8.3.1** (1) Suppose that the time to expiration is 4 months, the strike price is \$95, the call premium is \$6, the put premium is \$3, the current stock price is \$94, and the continuously compounded annual interest rate is 10%. How to earn a riskless arbitrage profit? (2) An options market maker writes calls to a client, then immediately buys puts and the underlying stock. Argue that this portfolio, called **conversion**, should earn a riskless profit.

➤ **Exercise 8.3.2** Strengthen Lemma 8.3.1 to $C \geq \max(S - \mathrm{PV}(X), 0)$.

➤ **Exercise 8.3.3** In a certain world in which a non-dividend-paying stock's price at any time is known, a European call is worthless if its strike price is higher than the known stock price at expiration. However, Exercise 8.3.2 says that $C \geq S - \mathrm{PV}(X)$, which is positive when $S > \mathrm{PV}(X)$. Try to resolve the contradiction when $X > S > \mathrm{PV}(X)$.

➤ **Exercise 8.3.4** Prove put–call parity (8.2) for a single dividend of size D^* at some time t_1 before expiration: $C = P + S - \mathrm{PV}(X) - D^* d(t_1)$.

➤ **Exercise 8.3.5** A European **capped call option** is like a European call option except that the payoff is $H - X$ instead of $S - X$ when the terminal stock price S exceeds H. Construct a portfolio of European options with an identical payoff.

➤ **Exercise 8.3.6** Consider a European-style derivative whose payoff is a piecewise linear function passing through the origin. A security with this payoff is called a **generalized option**. Show that it can be replicated by a portfolio of European calls.

8.4 Early Exercise of American Options

Assume that interest rates are positive in this section. It turns out that it never pays to exercise an American call before expiration if the underlying stock does not pay dividends; selling is better than exercising. Here is the argument. By Exercise 8.3.2, $C \geq \max(S - \mathrm{PV}(X), 0)$. If the call is exercised, the value is the smaller $S - X$. The disparity comes from two sources: (1) the loss of the insurance against subsequent stock price once the call is exercised and (2) the time value of money as X is paid on exercise. As a consequence, every pricing relation for European calls holds for American calls when the underlying stock pays no dividends. This somewhat surprising result is due to Merton [660].

THEOREM 8.4.1 *An American call on a non-dividend-paying stock should not be exercised before expiration.*

The above theorem does not mean American calls should be kept until maturity. What it does imply is that when early exercise is being considered, a *better* alternative is to sell it. Early exercise may become optimal for American calls on a dividend-paying stock, however. The reason has to do with the fact that the stock price declines as the stock goes ex-dividend. Surprisingly, an American call should be exercised at only a few dates.

THEOREM 8.4.2 *An American call will be exercised only at expiration or just before an ex-dividend date.*

Proof: We first show that $C > S - X$ at any time other than the expiration date or just before an ex-dividend date. Assume otherwise: $C \leq S - X$. Now, buy the call, short the stock, and lend $Xd(\tau)$, where τ is time to the next dividend date. The initial cash flow is positive because $X > Xd(\tau)$. We subsequently close out the position just before the next ex-dividend date by calling the loan, worth X, and selling the call, worth at least $\max(S_\tau - X, 0)$ by Lemma 8.3.1. The proceeds are sufficient to buy the stock at S_τ. The initial cash flow thus represents an arbitrage profit. Now that the value of a call exceeds its intrinsic value between ex-dividend dates, selling it is better than exercising it.

Unlike American calls on a non-dividend-paying stock, it might be optimal to exercise an American put even if the underlying stock does not pay dividends. Part of the reason lies in the fact that the time value of money now favors early exercise: Exercising a put generates an immediate cash income X. One consequence is that early exercise becomes more profitable as the interest rate increases, other things being equal.

The existence of dividends tends to offset the benefits of early exercise in the case of American puts. Consider a stock that is currently worthless, $S = 0$. If the holder of a put exercises the option, X is tendered. If the holder sells the option, he receives $P \leq X$ by Lemma 8.2.4 and keeps the stock. Doing nothing generates no income. If the stock will remain worthless till expiration, exercising the put now is optimal. It is therefore no longer true that we consider only a few points for early exercise of the put. Consequently, concrete results regarding early exercise of American puts are scarcer and weaker.

The put–call parity holds for European options only; for American options,

$$P \geq C + \mathrm{PV}(X) - S \tag{8.3}$$

because an American call has the same value as a European call by Theorem 8.4.1 and an American put is at least as valuable as its European counterpart.

➤ **Exercise 8.4.1** Consider an investor with an American call on a stock currently trading at \$45 per share. The option's expiration date is exactly 2 months away, the strike price is \$40, and the continuously compounded rate of interest is 8%. Suppose the stock is deemed overpriced and it pays no dividends. Should the option be exercised?

➤ **Exercise 8.4.2** Prove that if at all times before expiration the PV of the interest from the strike price exceeds the PV of future dividends before the expiration date, the call should not be exercised before expiration.

➤ **Exercise 8.4.3** Why is it not optimal to exercise an American put immediately before an ex-dividend date?

➤ **Exercise 8.4.4** Argue that an American put should be exercised when $X - S > \mathrm{PV}(X)$.

➤ **Exercise 8.4.5** Assume that the underlying stock does not pay dividends. Supply arbitrage arguments for the following claims. (1) The value of a call, be it European or American, cannot exceed the price of the underlying stock. (2) The value of a European put is $\mathrm{PV}(X)$ when $S = 0$. (3) The value of an American put is X when $S = 0$.

➤ **Exercise 8.4.6** Prove that American options on a non-dividend-paying stock satisfy $C - P \geq S - X$. (This and relation (8.3) imply that American options on a non-dividend-paying stock satisfy $C - S + X \geq P \geq C - S + \mathrm{PV}(X)$.)

8.5 Convexity of Option Prices

The convexity of option price is stated and proved below.

LEMMA 8.5.1 *For three otherwise identical calls with strike prices* $X_1 < X_2 < X_3$,

$$C_{X_2} \leq \omega C_{X_1} + (1-\omega)\, C_{X_3},$$
$$P_{X_2} \leq \omega P_{X_1} + (1-\omega)\, P_{X_3}.$$

Here $\omega \equiv (X_3 - X_2)/(X_3 - X_1)$. (*Equivalently,* $X_2 = \omega X_1 + (1-\omega)\, X_3$.)

Proof: We prove the lemma for the calls only. Suppose the lemma were wrong. Write C_{X_2}, buy ωC_{X_1}, and buy $(1-\omega)\, C_{X_3}$ to generate a positive cash flow now. If the short call is not exercised before expiration, hold the calls until expiration. The cash flow is described by

	$S \leq X_1$	$X_1 < S \leq X_2$	$X_2 < S < X_3$	$X_3 \leq S$
Call written at X_2	0	0	$X_2 - S$	$X_2 - S$
ω calls bought at X_1	0	$\omega(S - X_1)$	$\omega(S - X_1)$	$\omega(S - X_1)$
$1-\omega$ calls bought at X_3	0	0	0	$(1-\omega)(S - X_3)$
Net cash flow	0	$\omega(S - X_1)$	$\omega(S - X_1) + (X_2 - S)$	0

Because the net cash flows are either nonnegative or positive, there is an arbitrage profit.

Suppose that the short call is exercised early when the stock price is S. If $\omega C_{X_1} + (1-\omega)\, C_{X_3} > S - X_2$, sell the long calls to generate a net cash flow of $\omega C_{X_1} + (1-\omega)\, C_{X_3} - (S - X_2) > 0$. Otherwise, exercise the long calls and deliver the stock. The net cash flow is $-\omega X_1 - (1-\omega)\, X_3 + X_2 = 0$. Again, there is an arbitrage profit.

By Lemma 8.2.3, we know the slope of the call (put) value, when plotted against the strike price, is at most one (minus one, respectively). Lemma 8.5.1 adds that the shape is convex.

EXAMPLE 8.5.2 The prices of the Merck July 30 call, July 35 call, and July 40 call are \$15.25, \$9.5, and \$5.5, respectively, from Fig. 7.4. These prices satisfy the convexity property because $9.5 \times 2 < 15.25 + 5.5$. Look up the prices of the Microsoft April 60 put, April 65 put, and April 70 put. The prices are \$0.125, \$0.375, and \$1.5, respectively, which again satisfy the convexity property.

8.6 The Option Portfolio Property

Stock index options are fundamentally options on a stock portfolio. The American option on the Standard & Poor's 100 (S&P 100) Composite Stock Price Index is currently the most actively traded option contract in the United States [150, 746, 865]. Options on the Standard & Poor's 500 (S&P 500) Composite Stock Price Index are also available. They are European, however. Options on the Dow Jones Industrial Average (DJIA) were introduced in 1997. The underlying index, DJX, is DJIA divided by 100. Other popular stock market indices include the Russell 2000 Index for small company stocks and the broadest based Wilshire 5000 Index. Figure 8.2 tabulates some indices as of February 7, 2000.

As the following theorem shows, an option on a portfolio of stocks is cheaper than a portfolio of options. Hence it is cheaper to hedge against market movements as a whole with index options than with options on individual stocks.

	High	*Low*	*Close*	*Net Chg.*	*From Dec. 31*	*%Chg.*
DJ Indus (DJX)	109.71	108.46	109.06	−0.58	−5.91	−5.1
S&P 100 (OEX)	778.01	768.45	774.19	−1.32	−18.64	−2.4
S&P 500 -A.M.(SPX)	1427.23	1413.33	1424.24	−0.13	−45.01	−3.1
Nasdaq 100 (NDX)	3933.75	3858.89	3933.34	+58.97	+225.51	+6.1
NYSE (NYA)	627.03	621.14	623.84	−3.06	−26.46	−4.1
Russell 2000 (RUT)	532.40	525.52	532.39	+6.87	+27.64	+5.5
Major Mkt (XMI) .	1110.00	1096.63	1098.09	−11.64	−67.89	−5.8
Value Line (VLE) . .	1011.29	1003.91	1006.99	−2.13	−18.81	−1.8

Figure 8.2: Stock index quotations. Source: *Wall Street Journal*, February 8, 2000.

THEOREM 8.6.1 *Consider a portfolio of non-dividend-paying assets with weights ω_i. Let C_i denote the price of a European call on asset i with strike price X_i. Then the index call on the portfolio with a strike price $X \equiv \sum_i \omega_i X_i$ has a value of, at most, $\sum_i \omega_i C_i$. The same result holds for European puts as well. All options expire on the same date.*

The theorem in the case of calls follows from the following inequality:

$$\max\left(\sum_{i=1}^{n} \omega_i (S_i - X_i), 0\right) \geq \sum_{i=1}^{n} \max(\omega_i (S_i - X_i), 0),$$

where S_i denote the price of asset i. It is clear that a portfolio of options and an option on a portfolio have the same payoff if the underlying stocks either all finish in the money or out of the money. Their payoffs diverge only when the underlying stocks are not perfectly correlated with each other. The degree of the divergence tends to increase the more the underlying stocks are uncorrelated.

➤ **Exercise 8.6.1** Consider the portfolio of puts and put on the portfolio in Theorem 8.6.1. Because both provide a floor of $\sum_i \omega_i X_i$, why do they not fetch the same price?

Concluding Remarks and Additional Reading

The no-arbitrage principle can be traced to Pascal (1623–1662), philosopher, theologian, and founder of probability and decision theories [409, 410]. In the 1950s, Miller and Modigliani made it a pillar of financial theory [64, 853]. Bounds in this chapter are model free and should be satisfied by any proposed model [236, 346, 470]. Observe that they are all *relative* price bounds. The next chapter presents absolute option prices based on plausible models of stock prices. Justifications for the index options can be found in [236, Section 8.3].