CHAPTER FOUR

Bond Price Volatility

Can anyone measure the ocean by handfuls or measure the sky with his hands?

Isaiah 40:12

Understanding how interest rates affect bond prices is key to risk management of interest-rate-sensitive securities. This chapter focuses on bond price volatility or the extent of price movements when interest rates move. Two classic notions, **duration** and **convexity**, are introduced for this purpose with a few applications in risk management. Coupon bonds mean level-coupon bonds for the rest of the book.

4.1 Price Volatility

The sensitivity of the percentage price change to changes in interest rates measures price volatility. We define **price volatility** by $-(\partial P/P)/\partial y$. The price volatility of a coupon bond is

$$-\frac{\partial P/P}{\partial y} = -\frac{(C/y)\,n - (C/y^2)((1+y)^{n+1} - (1+y)) - nF}{(C/y)\,[\,(1+y)^{n+1} - (1+y)\,] + F(1+y)}, \tag{4.1}$$

where n is the number of periods before maturity, y is the period yield, F is the par value, and C is the coupon payment per period. For bonds without embedded options, $-(\partial P/P)/\partial y > 0$ for obvious reasons.

Price volatility increases as the coupon rate decreases, other things being equal (see Exercise 4.1.2). Consequently zero-coupon bonds are the most volatile, and bonds selling at a deep discount are more volatile than those selling near or above par. Price volatility also increases as the required yield decreases, other things being equal (see Exercise 4.1.3). So bonds traded with higher yields are less volatile.

For bonds selling above or at par, price volatility increases, but at a decreasing rate, as the **term to maturity** lengthens (see Fig. 4.1). Bonds with a longer maturity are therefore more volatile. This is consistent with the preference for liquidity and with the empirical fact that long-term bond prices are more volatile than short-term ones. (The *yields* of long-term bonds, however, are less volatile than those of short-term bonds [217].) For bonds selling below par, price volatility first increases, then decreases, as shown in Fig. 4.2 [425]. Longer maturity here can no longer be equated with higher price volatility.

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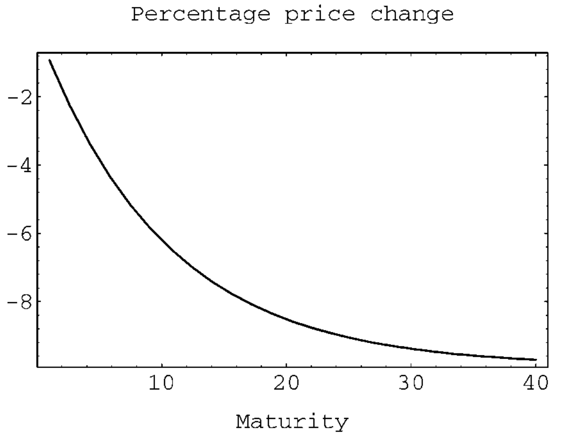

Figure 4.1: Volatility with respect to terms to maturity: par bonds. Plotted is the percentage bond price change per percentage change in the required yield at various terms to maturity. The annual coupon rate is 10% with semiannual coupons. The yield to maturity is identical to the coupon rate.

➤ **Exercise 4.1.1** Verify Eq. (4.1).

➤ **Exercise 4.1.2** Show that price volatility never decreases as the coupon rate decreases when yields are positive.

➤ **Exercise 4.1.3** (1) Prove that price volatility always decreases as the yield increases when the yield equals the coupon rate. (2) Prove that price volatility always decreases as the yield increases, generalizing (1).

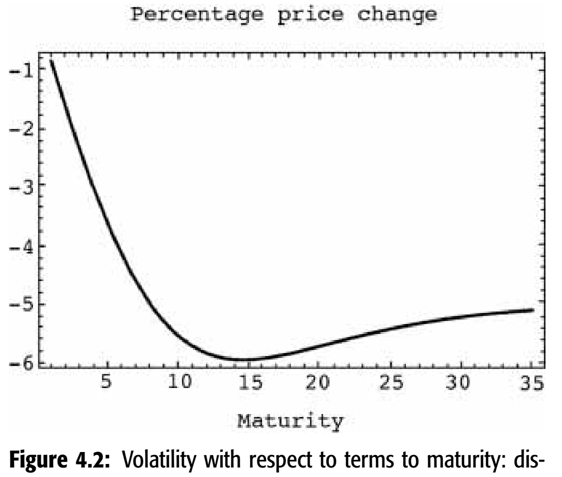

Figure 4.2: Volatility with respect to terms to maturity: discount bonds. The annual coupon rate is 10% with semiannual coupons, and the yield to maturity is 40% (a deep discount bond). The terms to maturity are measured in half-years. The rest follows Fig. 4.1.

4.2 Duration

The **Macaulay duration** (**MD**), first proposed in 1938 by Macaulay, is defined as the weighted average of the times to an asset's cash flows [627]. The weights are the cash flows' PVs divided by the asset's price. Formally,

$$\text{MD} \equiv \frac{1}{P}\sum_{i=1}^{n}\frac{iC_i}{(1+y)^i},$$

where n is the number of periods before maturity, y is the required yield, C_i is the cash flow at time i, and P is the price. Clearly, the MD, in periods, is equal to

$$\text{MD} = -(1+y)\,\frac{\partial P/P}{\partial y}. \tag{4.2}$$

This simple relation was discovered by Hicks (1904–1989) in 1939 [231, 496]. In particular, the MD of a coupon bond is

$$\text{MD} = \frac{1}{P}\left[\sum_{i=1}^{n}\frac{iC}{(1+y)^i} + \frac{nF}{(1+y)^n}\right]. \tag{4.3}$$

The above equation can be simplified to

$$\text{MD} = \frac{c(1+y)\,[\,(1+y)^n - 1\,] + ny(y-c)}{cy\,[\,(1+y)^n - 1\,] + y^2},$$

where c is the period coupon rate. The MD of a zero-coupon bond (corresponding to $c = 0$) is n, its term to maturity. In general, the Macaulay duration of a coupon bond is less than its maturity. The MD of a coupon bond approaches $(1+y)/y$ as the maturity increases, independent of the coupon rate.

Equations (4.2) and (4.3) hold only if the coupon C, the par value F, and the maturity n are all independent of the yield y, in other words, if the cash flow is independent of yields. When the cash flow is sensitive to interest rate movements, the MD is no longer inappropriate. To see this point, suppose that the market yield declines. The MD will be lengthened by Exercise 4.1.3, Part (2). However, for securities whose maturity actually decreases as a result, the MD may decrease.

Although the MD has its origin in measuring the length of time a bond investment is outstanding, it should be seen mainly as measuring the sensitivity of price to market yield changes, that is, as price volatility [348]. As a matter of fact, many, if not most, duration-related terminologies cannot be comprehended otherwise.

To convert the MD to be year based, modify (4.3) as follows:

$$\frac{1}{P}\left[\sum_{i=1}^{n}\frac{i}{k}\,\frac{C}{\left(1+\frac{y}{k}\right)^i} + \frac{n}{k}\,\frac{F}{\left(1+\frac{y}{k}\right)^n}\right],$$

where y is the *annual* yield and k is the compounding frequency per annum. Equation (4.2) also becomes

$$\text{MD} = -\left(1+\frac{y}{k}\right)\frac{\partial P/P}{\partial y}.$$

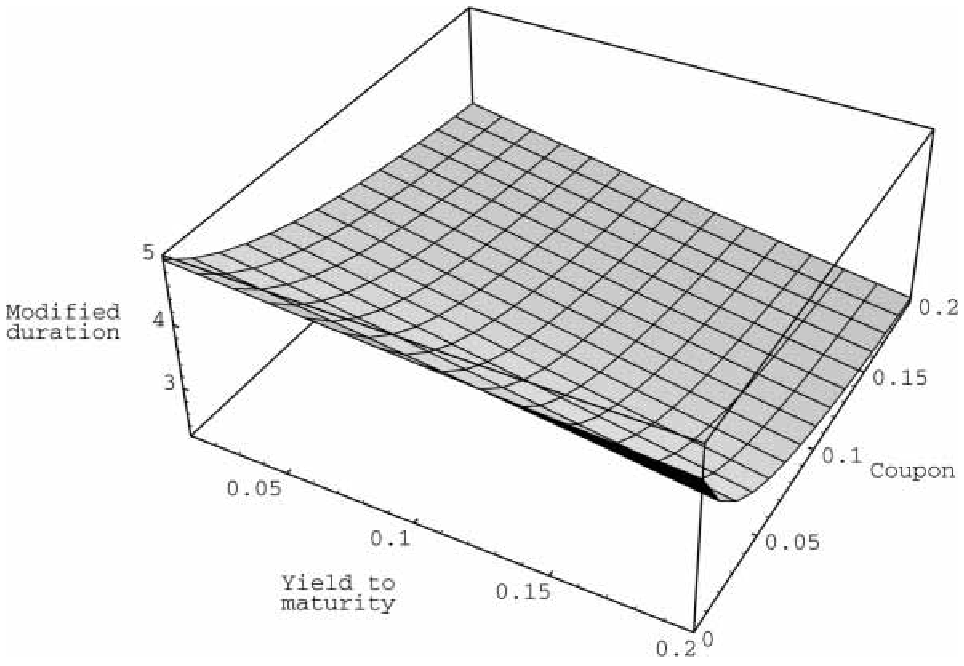

Figure 4.3: Modified duration with respect to coupon rate and yield. Bonds are assumed to pay semiannual coupon payments with a maturity date of September 15, 2000. The settlement date is September 15, 1995.

Note from the definition that

$$\text{MD (in years)} = \frac{\text{MD (in periods)}}{k}.$$

A related measure is the **modified duration**, defined as

$$\text{modified duration} \equiv -\frac{\partial P/P}{\partial y} = \frac{\text{MD}}{(1+y)}. \tag{4.4}$$

The modified duration of a coupon bond is Eq. (4.1), for example (see Fig. 4.3). By Taylor expansion,

$$\text{percentage price change} \approx -\text{modified duration} \times \text{yield change}.$$

The modified duration of a portfolio equals $\sum_i \omega_i D_i$, where D_i is the modified duration of the ith asset and ω_i is the market value of that asset expressed as a percentage of the market value of the portfolio. Modified duration equals MD (in periods)/$(1+y)$ or MD (in years)/$(1+y/k)$ if the cash flow is independent of changes in interest rates.

EXAMPLE 4.2.1 Consider a bond whose modified duration is 11.54 with a yield of 10%. This means if the yield increases instantaneously from 10% to 10.1%, the approximate percentage price change would be $-11.54 \times 0.001 = -0.01154$, or -1.154%.

A general numerical formula for volatility is the **effective duration**, defined as

$$\frac{P_- - P_+}{P_0(y_+ - y_-)}, \tag{4.5}$$

where P_- is the price if the yield is decreased by Δy, P_+ is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, $y_+ \equiv y + \Delta y$, $y_- \equiv y - \Delta y$, and Δy is sufficiently small. In principle, we can compute the effective duration of just about any financial instrument. A less accurate, albeit computationally economical formula for effective duration is to use **forward difference**,

$$\frac{P_0 - P_+}{P_0 \, \Delta y} \tag{4.6}$$

instead of the **central difference** in (4.5).

Effective duration is most useful in cases in which yield changes alter the cash flow or securities whose cash flow is so complex that simple formulas are unavailable. This measure strengthens the contention that duration should be looked on as a measure of volatility and not average term to maturity. In fact, it is possible for the duration of a security to be longer than its maturity or even to go negative [321]! Neither can be understood under the maturity interpretation.

For the rest of the book, duration means the mathematical expression $-(\partial P/P)/\partial y$ or its approximation, effective duration. As a consequence,

$$\text{percentage price change} \approx -\text{duration} \times \text{yield change}.$$

The principal applications of duration are in hedging and asset/liability management [55].

➤ **Exercise 4.2.1** Assume that 9% is the annual yield to maturity compounded semiannually. Calculate the MD of a 3-year bond paying semiannual coupons at an annual coupon rate of 10%.

➤ **Exercise 4.2.2** Duration is usually expressed in percentage terms for quick mental calculation: Given duration $D_{\%}$, the percentage price change expressed in percentage terms is approximated by $-D_{\%} \times \Delta r$ when the yield increases instantaneously by $\Delta r\%$. For instance, the price will drop by 20% if $D_{\%} = 10$ and $\Delta r = 2$ because $10 \times 2 = 20$. Show that $D_{\%}$ equals modified duration.

➤ **Exercise 4.2.3** Consider a coupon bond and a traditional mortgage with the same maturity and payment frequency. Show that the mortgage has a smaller MD than the bond when both provide the same yield to maturity.[1] For simplicity, assume that both instruments have the same market price.

➤ **Exercise 4.2.4** Verify that the MD of a traditional mortgage is $(1+y)/y - n/((1+y)^n - 1)$.

4.2.1 Continuous Compounding

Under continuous compounding, the formula for duration is slightly changed. The price of a bond is now $P = \sum_i C_i e^{-yt_i}$, and

$$\text{duration (continuous compounding)} \equiv \frac{\sum_i t_i C_i e^{-yt_i}}{P} = -\frac{\partial P/P}{\partial y}. \tag{4.7}$$

Unlike the MD in Eq. (4.2), the extra $1 + y$ term disappears.

➤ **Exercise 4.2.5** Show that the duration of an n-period zero-coupon bond is n.

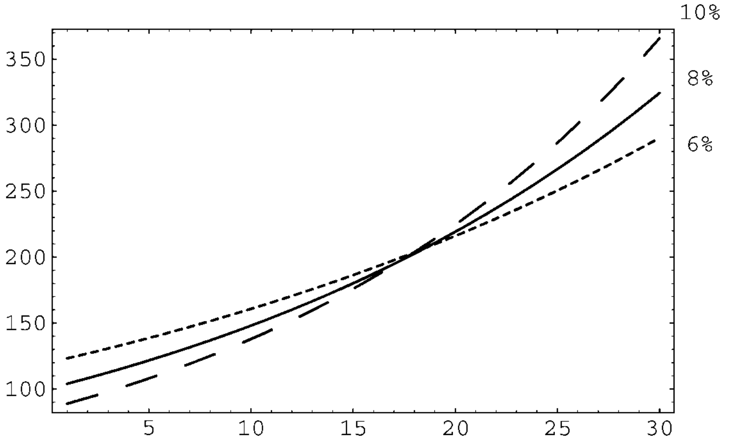

Figure 4.4: Bond value under three rate scenarios. Plotted is the value of an 8% 15-year bond from now to maturity if the interest rate is unchanged at 8% (solid curve), increased to 10% (dashed curve), and decreased to 6% (dotted curve). At the MD $m = 17.9837$ (half-years), the curves roughly meet [98].

4.2.2 Immunization

Buying coupon bonds to meet a future liability incurs some risks. Assume that we are at the horizon date when the liability is due. If interest rates rise subsequent to the bond purchase, the interest on interest from the reinvestment of the coupon payments will increase, and a capital loss will occur for the sale of the bonds. The reverse is true if interest rates fall. The results are uncertainties in meeting the liability.

A portfolio is said to **immunize** a liability if its value at the horizon date covers the liability for small rate changes now. How do we find such a bond portfolio? Amazingly, the answer is as elegant as it is simple: We construct a bond portfolio whose MD is equal to the horizon and whose PV is equal to the PV of the single future liability [350]. Then, at the horizon date, losses from the interest on interest will be compensated for by gains in the sale price when interest rates fall, and losses from the sale price will be compensated for by the gains in the interest on interest when interest rates rise (see Fig. 4.4). For example, a \$100,000 liability 12 years from now should be matched by a portfolio with an MD of 12 years and a future value of \$100,000.

The proof is straightforward. Assume that the liability is a certain L at time m and the current interest rate is y. We are looking for a portfolio such that

(1) its FV is L at the horizon m,
(2) $\partial \text{FV}/\partial y = 0$,
(3) FV is convex around y.

Condition (1) says the obligation is met. Conditions (2) and (3) together mean that L is the portfolio's minimum FV at the horizon for small rate changes.

Let $\text{FV} \equiv (1+y)^m P$, where P is the PV of the portfolio. Now,

$$\frac{\partial \text{FV}}{\partial y} = m(1+y)^{m-1} P + (1+y)^m \frac{\partial P}{\partial y}. \tag{4.8}$$

Imposing Condition (2) leads to

$$m = -(1+y)\,\frac{\partial P/P}{\partial y}. \tag{4.9}$$

This identity is what we were after: the MD is equal to the horizon m.

Suppose that we use a coupon bond for immunization. Because

$$\mathrm{FV} = \sum_{i=1}^{n} \frac{C}{(1+y)^{i-m}} + \frac{F}{(1+y)^{n-m}},$$

it follows that

$$\frac{\partial^2 \mathrm{FV}}{\partial y^2} = \sum_{i=1}^{n} \frac{(m-i)(m-i-1)\,C}{(1+y)^{i-m+2}} + \frac{(m-n)(m-n-1)F}{(1+y)^{n-m+2}} > 0 \tag{4.10}$$

for $y > -1$ because $(m-i)(m-i+1)$ is either zero or positive. Because the FV is convex for $y > -1$, the minimum value of the FV is indeed L (see Fig. 4.5).

If there is no single bond whose MD matches the horizon, a portfolio of two bonds, A and B, can be assembled by the solution of

$$\begin{aligned} 1 &= \omega_{\mathrm{A}} + \omega_{\mathrm{B}}, \\ D &= \omega_{\mathrm{A}} D_{\mathrm{A}} + \omega_{\mathrm{B}} D_{\mathrm{B}} \end{aligned} \tag{4.11}$$

for ω_{A} and ω_{B}. Here, D_i is the MD of bond i and ω_i is the weight of bond i in the portfolio. Make sure that D falls between D_{A} and D_{B} to guarantee $\omega_{\mathrm{A}} > 0$, $\omega_{\mathrm{B}} > 0$, and positive portfolio convexity.

Although we have been dealing with immunizing a single liability, the extension to multiple liabilities can be carried out along the same line. Let there be a liability of size L_i at time i and a cash inflow A_i at time i. The NPV of these cash flows at

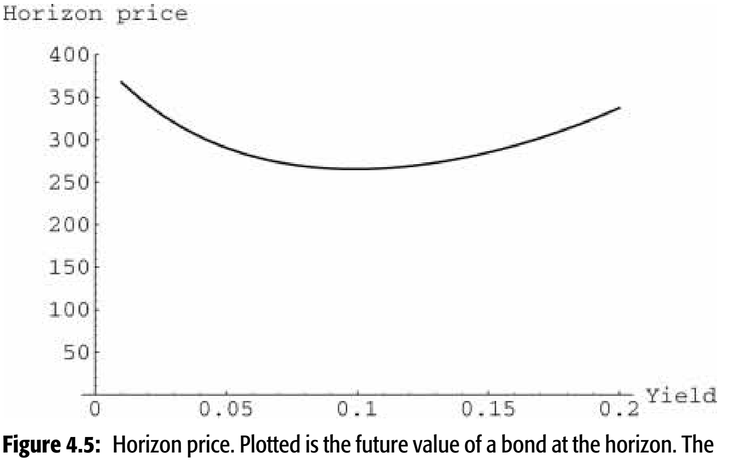

Figure 4.5: Horizon price. Plotted is the future value of a bond at the horizon. The yield at which the graph is minimized equates the bond's MD with the horizon. In this example, the bond pays semiannual coupons at an annual rate of 10% for 30 years and the horizon is 10 years from now. The FV is minimized at $y = 9.91\%$, and the MD at y is exactly 10 years. The bond's FV at the horizon will increase if the rate moves.

the horizon is

$$\text{FV} = \sum_i (A_i - L_i)(1+y)^{m-i}.$$

Conditions (1)–(3) require that $\text{FV} = 0$, $\partial \text{FV}/\partial y = 0$, and $\partial^2 \text{FV}/\partial y^2 > 0$ around the current rate y. Together, they guarantee that the cash inflows suffice to cover the liabilities for small instantaneous rate movements now. In this more general setting, the distribution of individual assets' durations must have a wider range than that of the liabilities to achieve immunization (see Exercise 4.2.11).

Of course, a stream of liabilities can always be immunized with a matching stream of zero-coupon bonds. This is called **cash matching**, and the bond portfolio is called a **dedicated portfolio** [799]. Two problems with this approach are that (1) zero-coupon bonds may be missing for certain maturities and (2) they typically carry lower yields.

Immunization is a dynamic process. It has to be **rebalanced** constantly to ensure that the MD remains matched to the horizon for the following reasons. The MD decreases as time passes, and, except for zero-coupon bonds, the decrement is not identical to the decrement in the time to maturity [217]. This phenomenon is called **duration drift** [246]. This point can be easily confirmed by a coupon bond whose MD matches the horizon. Because the bond's maturity date lies beyond the horizon date, its duration will remain positive at the horizon instead of zero. Therefore immunization needs to be reestablished even if interest rates never change. Interest rates will fluctuate during the holding period, but it was assumed that interest rates change *instantaneously* after immunization has been established and then stay there. Finally, the durations of assets and liabilities may not change at the same rate [689].

When liabilities and assets are mismatched in terms of duration, adverse interest rate movements can quickly wipe out the equity. A bank that finances long-term mortgage investments with short-term credit from the savings accounts or certificates of deposit (CDs) runs such a risk. Other institutions that worry about duration matching are pension funds and life insurance companies [767].

➤ **Exercise 4.2.6** In setting up the two-bond immunization in Eqs. (4.11), we did not bother to check the convexity condition. Justify this omission.

➤ **Exercise 4.2.7** Show that, in the absence of interest rate changes, it suffices to match the PVs of the liability and the asset.

➤ **Exercise 4.2.8** Start with a bond whose PV is equal to the PV of a future liability and whose MD exceeds the horizon. Show that, at the horizon, the bond will fall short of the liability if interest rates rise and more than meet the goal if interest rates fall. The reverse is true if the MD falls short of the horizon.

➤ **Exercise 4.2.9** Consider a liability currently immunized by a coupon bond. Suppose that the interest rate changes instantaneously. Prove that profits will be generated when rebalancing is performed at time Δt from now (but before the maturity).

➤ **Exercise 4.2.10** The liability has an MD of 3 years, but the money manager has access to only two kinds of bonds with MDs of 1 year and 4 years. What is the right proportion of each bond in the portfolio in order to match the liability's MD?

➤ **Exercise 4.2.11** (1) To achieve **full immunization**, we set up cash inflows at more points in time than liabilities as follows. Consider a single-liability cash outflow L_t at time t. Assemble a portfolio with a cash inflow A_1 at time $t - a_1$ and a cash inflow A_2 at time $t + a_2$ with $a_1, a_2 > 0$ and $a_1 \leq t$. Conditions (1) and (2) demand that

$$P(y) = A_1 e^{a_1 y} + A_2 e^{-a_2 y} - L_t = 0,$$

$$\frac{dP(y)}{dy} = A_1 a_1 e^{a_1 y} - A_2 a_2 e^{-a_2 y} = 0$$

under continuous compounding. Solve the two equations for any two unknows of your choice, say A_1 and A_2, and prove that it achieves immunization for any changes in y. (2) Generalize the result to more than two cash inflows.

4.2.3 Macaulay Duration of Floating-Rate Instruments

A **floating-rate instrument** makes interest rate payments based on some publicized index such as the prime rate, the London Interbank Offered Rate (LIBOR), the U.S. T-bill rate, the CMT rate, or the COFI [348]. Instead of being locked into a number, the coupon rate is reset periodically to reflect the prevailing interest rate.

Assume that the coupon rate c equals the market yield y and that the bond is priced at par. The first reset date is j periods from now, and resets will be performed thereafter. Let the principal be \$1 for simplicity. The cash flow of the floating-rate instrument is thus

$$\overbrace{c, c, \ldots, c}^{j}, \overbrace{y, \ldots, y, y+1}^{n-j},$$

where c is a constant and $y = c$. So the coupon payment at time $j + 1$ starts to reflect the market yield. For example, when $j = 0$, every coupon payment reflects the prevailing market yield, and when $j = 1$, which is more typical, interest rate movements during the first period will not affect the first coupon payment. The MD is

$$\begin{aligned} & -(1+y) \left. \frac{\partial P/P}{\partial y} \right|_{c=y} \\ &= \sum_{i=1}^{j} i \frac{y}{(1+y)^i} + \sum_{i=j+1}^{n} \left[i \frac{y}{(1+y)^i} - \frac{1}{(1+y)^{i-1}} \right] + n \frac{1}{(1+y)^n} \\ &= \text{MD} - \sum_{i=j+1}^{n} \frac{1}{(1+y)^{i-1}} = \frac{(1+y)[1-(1+y)^{-j}]}{y}, \end{aligned} \tag{4.12}$$

where MD denotes the MD of an otherwise identical fixed-rate bond. Interestingly, the MD is independent of the maturity of the bond, n. Formulas for nonpar bonds are more complex but do not involve any new ideas [306, 348].

The attractiveness of floating-rate instruments is not hard to explain. Floating-rate instruments are typically less sensitive to interest rate changes than are fixed-rate instruments. In fact, the less distant the first reset date, the less volatile the instrument. And when every coupon is adjusted to reflect the market yield, there is no more interest rate risk. Indeed, the MD is zero when $j = 0$. In the typical case of $j = 1$, the MD is one period. By contrast, a bond that pays 5% per period for 30 periods has an MD of 16.14 periods.

➤ **Exercise 4.2.12** Show that the MD of a floating-rate instrument cannot exceed the first reset date.

4.2.4 Hedging

Hedging aims at offsetting the price fluctuations of the position to be hedged by the hedging instrument in the opposite direction, leaving the total wealth unchanged [222]. Define **dollar duration** as

$$\text{dollar duration} \equiv \text{modified duration} \times \text{price} \,(\%\text{ of par}) = -\frac{\partial P}{\partial y},$$

where P is the price as a percentage of par. It is the tangent on the price/yield curve such as the one in Fig. 3.9. The approximate dollar price change per \$100 of par value is

$$\text{price change} \approx -\text{dollar duration} \times \text{yield change}.$$

The related **price value of a basis point**, or simply **basis-point value** (**BPV**), defined as the dollar duration divided by 10,000, measures the price change for a one basis-point change in the interest rate. One **basis point** equals 0.01%.

Because securities may react to interest rate changes differently, we define **yield beta** as

$$\text{yield beta} \equiv \frac{\text{change in yield for the hedged security}}{\text{change in yield for the hedging security}},$$

which measures relative yield changes. If we let the **hedge ratio** be

$$h \equiv \frac{\text{dollar duration of the hedged security}}{\text{dollar duration of the hedging security}} \times \text{yield beta}, \tag{4.13}$$

then hedging is accomplished when the value of the hedging security is h times that of the hedged security because

$$\begin{aligned}&\text{dollar price change of the hedged security}\\ &\quad = -h \times \text{dollar price change of the hedging security}.\end{aligned}$$

EXAMPLE 4.2.2 Suppose we want to hedge bond A with a duration of seven by using bond B with a duration of eight. Under the assumption that the yield beta is one and both bonds are selling at par, the hedge ratio is 7/8. This means that an investor who is long \$1 million of bond A should short \$7/8 million of bond B.

4.3 Convexity

The important notion of **convexity** is defined as

$$\text{convexity (in periods)} \equiv \frac{\partial^2 P}{\partial y^2}\,\frac{1}{P}. \tag{4.14}$$

It measures the curvature of the price/yield relation. The convexity of a coupon bond is

$$\frac{1}{P}\left[\sum_{i=1}^{n} i(i+1)\frac{C}{(1+y)^{i+2}} + n(n+1)\frac{F}{(1+y)^{n+2}}\right]$$
$$= \frac{1}{P}\left\{\frac{2C}{y^3}\left[1-\frac{1}{(1+y)^n}\right] - \frac{2Cn}{y^2(1+y)^{n+1}} + \frac{n(n+1)\,[F-(C/y)]}{(1+y)^{n+2}}\right\}, \tag{4.15}$$

which is positive. For a bond with positive convexity, the price rises more for a rate decline than it falls for a rate increase of equal magnitude. Hence between two bonds with the same duration, the one with a higher convexity is more valuable, other things being equal. Convexity measured in periods and convexity measured in years are related by

$$\text{convexity (in years)} = \frac{\text{convexity (in periods)}}{k^2}$$

when there are k periods per annum. It can be shown that the convexity of a coupon bond increases as its coupon rate decreases (see Exercise 4.3.4). Furthermore, for a given yield and duration, the convexity decreases as the coupon decreases [325]. In analogy with Eq. (4.7), the convexity under continuous compounding is

$$\text{convexity (continuous compounding)} \equiv \frac{\sum_i t_i^2 C_i e^{-y t_i}}{P} = \frac{\partial^2 P/P}{\partial y^2}.$$

The approximation $\Delta P/P \approx -\text{duration} \times \text{yield change}$ we saw in Section 4.2 works for small yield changes. To improve on it for larger yield changes, second-order terms are helpful:

$$\frac{\Delta P}{P} \approx \frac{\partial P}{\partial y}\frac{1}{P}\Delta y + \frac{1}{2}\frac{\partial^2 P}{\partial y^2}\frac{1}{P}(\Delta y)^2$$
$$= -\text{duration} \times \Delta y + \frac{1}{2} \times \text{convexity} \times (\Delta y)^2.$$

See Fig. 4.6 for illustration.

A more general notion of convexity is the **effective convexity** defined as

$$\frac{P_+ + P_- - 2 \times P_0}{P_0\,[\,0.5 \times (y_+ - y_-)\,]^2}, \tag{4.16}$$

where P_- is the price if the yield is decreased by Δy, P_+ is the price if the yield is increased by Δy, P_0 is the initial price, y is the initial yield, $y_+ \equiv y + \Delta y$, $y_- \equiv y - \Delta y$, and Δy is sufficiently small. Note that $\Delta y = (y_+ - y_-)/2$. Effective convexity is most relevant when a bond's cash flow is interest rate sensitive.

The two-bond immunization scheme in Subsection 4.2.2 shows that countless two-bond portfolios with varying duration pairs $(D_{\mathrm{A}}, D_{\mathrm{B}})$ can be assembled to satisfy Eqs. (4.11). However, which one is to be preferred? As convexity is a desirable feature, we phrase this question as one of maximizing the portfolio convexity among all the portfolios with identical duration. Let there be n kinds of bonds, with bond i having duration D_i and convexity C_i, where $D_1 < D_2 < \cdots < D_n$. Typically,

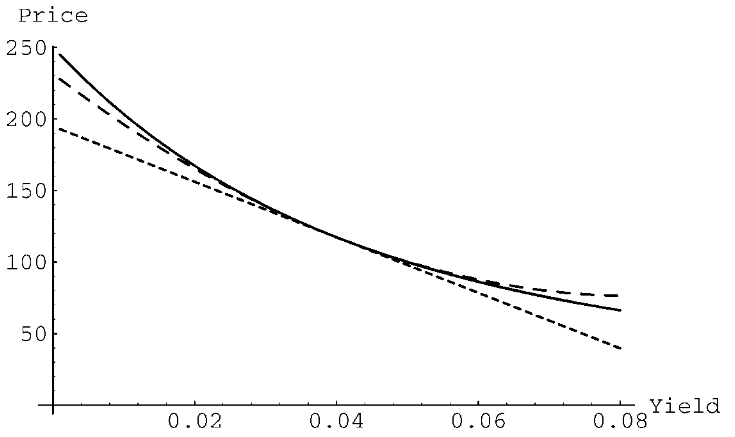

Figure 4.6: Linear and quadratic approximations to bond price changes. The dotted curve is the result of a duration-based approximation, whereas the dashed curve, which fits better, utilizes the convexity information. The bond in question has 30 periods to maturity with a period coupon rate of 5%. The current yield is 4% per period.

$D_1 = 0.25$ (3-month discount instruments) and $D_n = 30$ (30-year zeros). We then solve the following constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & \omega_1 C_1 + \omega_2 C_2 + \cdots + \omega_n C_n, \\ \text{subject to} & 1 = \omega_1 + \omega_2 + \cdots + \omega_n, \\ & D = \omega_1 D_1 + \omega_2 D_2 + \cdots + \omega_n D_n, \\ & 0 \leq \omega_i \leq 1. \end{array}$$

The function to be optimized, $\omega_1 C_1 + \omega_2 C_2 + \cdots + \omega_n C_n$, is called the **objective function**. The equalities or inequalities make up the **constraints**. The preceding optimization problem is a **linear programming problem** because all the functions are linear. The solution usually implies a **barbell portfolio**, so called because the portfolio contains bonds at the two extreme ends of the duration spectrum (see Exercise 4.3.6).

➤ **Exercise 4.3.1** In practice, convexity should be expressed in percentage terms, call it $C_{\%}$, for quick mental calculation. The percentage price change in percentage terms is then approximated by $-D_{\%} \times \Delta r + C_{\%} \times (\Delta r)^2/2$ when the yield increases instantaneously by $\Delta r\,\%$. For example, if $D_{\%} = 10$, $C_{\%} = 1.5$, and $\Delta r = 2$, the price will drop by 17% because

$$-D_{\%} \times \Delta r + \frac{1}{2} \times C_{\%} \times (\Delta r)^2 = -10 \times 2 + \frac{1}{2} \times 1.5 \times 2^2 = -17.$$

Show that $C_{\%}$ equals convexity divided by 100.

➤ **Exercise 4.3.2** Prove that $\partial(\text{duration})/\partial y = (\text{duration})^2 - \text{convexity}$.

➤ **Exercise 4.3.3** Show that the convexity of a zero-coupon bond is $n(n+1)/(1+y)^2$.

➤ **Exercise 4.3.4** Verify that convexity (4.15) increases as the coupon rate decreases.

➤ **Exercise 4.3.5** Prove that the barbell portfolio has the highest convexity for $n = 3$.

➤ **Exercise 4.3.6** Generalize Exercise 4.3.5: Prove that a barbell portfolio achieves immunization with maximum convexity given $n > 3$ kinds of zero-coupon bonds.

Additional Reading

Duration and convexity measure only the risk of changes in interest rate levels. Other types of risks, such as the frequency of large movements in interest rates, are ignored [618]. They furthermore assume parallel shifts in the yield curve, whereas yield changes are not always parallel in reality (more is said about yield curves in Chap. 5). Closed-form formulas for duration and convexity can be found in [89, 209]. See [496] for a penetrating review. Additional immunization techniques can be found in [206, 325, 547]. The idea of immunization is due to Redington in 1952 [732]. Consult [213, 281, 545] for more information on linear programming. Many fundamental problems in finance and economics are best cast as optimization problems [247, 278, 281, 891].

NOTE

1. The bond was the standard design for mortgages, called **balloon mortgages**, before the Federal Housing Administration introduced fully amortized mortgages [330]. Balloon mortgages are more prone to default because the borrower may not have the funds for the balloon payment due. This exercise shows that fully amortized mortgages are less volatile than balloon mortgages if prepayments are nonexisting.