CHAPTER
FIVE

Term Structure of Interest Rates

He pays least [. . .] who pays latest.
Charles de Montesquieu (1689–1755), *The Spirit of Laws*

The term structure of interest rates is concerned with how the interest rates change with maturity and how they may evolve in time. It is fundamental to the valuation of fixed-income securities. This subject is important also because the term structure is the starting point of any stochastic theory of interest rate movements. Interest rates in this chapter are period based unless stated otherwise. This simplifies the presentation by eliminating references to the compounding frequency per annum.

5.1 Introduction

The set of yields to maturity for bonds of equal quality and differing solely in their terms to maturity[1] forms the **term structure**. This term often refers exclusively to the yields of zero-coupon bonds. Term to maturity is the time period during which the issuer has promised to meet the conditions of the obligation. A **yield curve** plots yields to maturity against maturity and represents the prevailing interest rates for various terms. See Fig. 5.1 for a sample Treasury yield curve. A **par yield curve** is constructed from bonds trading near their par value.

At least four yield-curve shapes can be identified. A **normal** yield curve is upward sloping, an **inverted** yield curve is downward sloping, a **flat** yield curve is flat (see Fig. 5.2), and a **humped** yield curve is upward sloping at first but then turns downward sloping. We will survey the theories advanced to explain the shapes of the yield curve in Section 5.1.

The U.S. Treasury yield curve is the most widely followed yield curve for the following reasons. First, it spans a full range of maturities, from 3 months to 30 years. Second, the prices are representative because the Treasuries are extremely liquid and their market deep. Finally, as the Treasuries are backed by the full faith and credit of the U.S. government, they are perceived as having no credit risk [95]. The most recent Treasury issues for each maturity are known as the **on-the-run** or **current coupon** issues in the secondary market (see Fig. 5.3). Issues auctioned before the current coupon issues are referred to as **off-the-run** issues. On-the-run and off-the-run yield curves are based on their respective issues [325, 489].

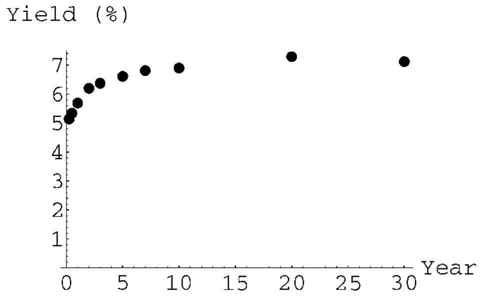


Figure 5.1: Treasury yield curve. The Treasury yield curve as of May 3, 1996, published by the U.S. Treasury and based on bid quotations on the most actively traded Treasury securities as of 3:30 PM with information from the Federal Reserve Bank of New York.

The yield on a non-Treasury security must exceed the base interest rate offered by an on-the-run Treasury security of comparable maturity by a positive spread called the **yield spread** [326]. This spread reflects the risk premium of holding securities not issued by the government. The base interest rate is also known as the **benchmark interest rate**.

5.2 Spot Rates

The i-period **spot rate** $S(i)$ is the yield to maturity of an i-period zero-coupon bond. The PV of \$1 i periods from now is therefore $[1 + S(i)]^{-i}$. The one-period spot rate – the **short rate** – will play an important role in modeling interest rate dynamics later in the book. A **spot rate curve** is a plot of spot rates against maturity. Its other names include **spot yield curve** and **zero-coupon yield curve**.

In the familiar bond price formula,

$$\sum_{i=1}^{n} \frac{C}{(1+y)^i} + \frac{F}{(1+y)^n},$$

every cash flow is discounted at the same yield to maturity, y. To see the inconsistency, consider two riskless bonds with different yields to maturity because of their different cash flow streams. The yield-to-maturity methodology discounts their contemporaneous cash flows with different rates, but common sense dictates that cash flows occurring at the same time should be discounted at the same rate. The spot-rate methodology does exactly that.

A fixed-rate bond with cash flow $C_1, C_2, \ldots, C_n$ is equivalent to a package of zero-coupon bonds, with the ith bond paying C_i dollars at time i. For example, a

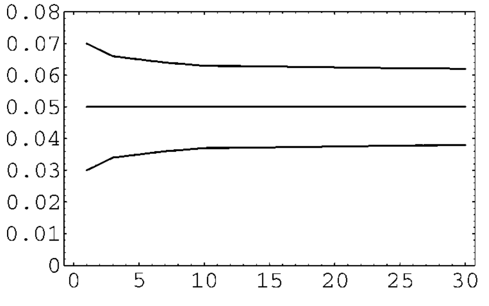


Figure 5.2: Three types of yield curves. Depicted from top to bottom are inverted, flat, and normal yield curves.

	Curr	Securities	Prev Close		9:28	
3	–	11/13/97	5.10	5.24	5.11	5.25
6	–	2/12/98	5.13	5.34	5.12	5.33
1	–	8/20/98	5.20	5.49	5.19	5.48
2	5.875	7/31/99	100-03+	5.81	100-04+	5.80
3	6.000	8/15/00	100-03+	5.96	100-04+	5.95
5	6.000	7/31/02	99-23+	6.06	99-24	6.06
10	6.125	8/15/07	99-07	6.23	99-09	6.22
30	6.375	8/15/27	97-25+	6.54	97-27+	6.54

Figure 5.3: On-the-run U.S. Treasury yield curve (Aug. 18, 1997, 9:28 AM EDT). Source: Bloomberg.

level-coupon bond has the price

$$P = \sum_{i=1}^{n} \frac{C}{[\,1+S(i)\,]^{i}} + \frac{F}{[\,1+S(n)\,]^{n}}. \tag{5.1}$$

This pricing method incorporates information from the term structure by discounting each cash flow at the corresponding spot rate. In general, any riskless security having a predetermined cash flow $C_1, C_2, \dots, C_n$ should have a market price of

$$P = \sum_{i=1}^{n} C_i\, d(i),$$

where

$$d(i) \equiv [\,1+S(i)\,]^{-i}, \quad i = 1, 2, \dots, n,$$

are called the **discount factors**. The discount factor $d(i)$ denotes the PV of \$1 i periods from now, in other words, the price of the zero-coupon bond maturing i periods from now. If the market price is less than P, it is said to be **undervalued** or **cheap**. It is said to be **overvalued** or **rich** otherwise. The discount factors are often interpolated to form a continuous function called the **discount function**. It is the discount factors, not the spot rates, that are directly observable in the market.

➤ **Exercise 5.2.1** Prove that the yield to maturity y is approximately

$$\frac{\sum_i [\,\partial C_i(y)/\partial y\,]\, S(i)}{\partial P/\partial y}$$

to the first order, where $C_i(y) \equiv C_i/(1+y)^i$ denotes the ith cash flow discounted at the rate y. Note that $\partial C_i(y)/\partial y$ is the dollar duration of the i-period zero-coupon bond. (The yield to maturity is thus roughly a weighted sum of the spot rates, with each weight proportional to the dollar duration of the cash flow.)

5.3 Extracting Spot Rates from Yield Curves

Spot rates can be extracted from the yields of coupon bonds. Start with the short rate $S(1)$, which is available because short-term Treasuries are zero-coupon bonds. Now $S(2)$ can be computed from the two-period coupon bond price P by use of Eq. (5.1),

$$P = \frac{C}{1+S(1)} + \frac{C+100}{[\,1+S(2)\,]^2}.$$

EXAMPLE 5.3.1 Suppose the 1-year T-bill has a yield of 8%. Because this security is a zero-coupon bond, the 1-year spot rate is 8%. When the 2-year 10% T-note is trading at 90, the 2-year spot rate satisfies

$$90 = \frac{10}{1.08} + \frac{110}{[\,1+S(2)\,]^2}.$$

Therefore $S(2) = 0.1672$, or 16.72%.

In general, $S(n)$ can be computed from Eq. (5.1), given the market price of the n-period coupon bond and $S(1), S(2), \ldots, S(n-1)$. The complete algorithm is given in Fig. 5.4. The correctness of the algorithm is easy to see. The initialization steps and step 3 ensure that

$$p = \sum_{j=1}^{i-1} \frac{1}{[\,1+S(j)\,]^j}$$

at the beginning of each loop. Step 1 solves for x such that

$$P_i = \sum_{j=1}^{i-1} \frac{C_i}{[\,1+S(j)\,]^j} + \frac{C_i+100}{(1+x)^i},$$

where C_i is the level-coupon payment of bond i and P_i is its price.

Each execution of step 1 requires $O(1)$ arithmetic operations because $x = [\,(C_i+100)/(P_i - C_i\,p)\,]^{1/i} - 1$ and expressions like y^z can be computed by $\exp[\,z \ln y\,]$ (note that $\exp[\,x\,] \equiv e^x$). Similarly, step 3 runs in $O(1)$ time. The total running time is hence $O(n)$.

Algorithm for extracting spot rates from coupon bonds:

```
input:  n, C[1..n], P[1..n];
real    S[1..n], p, x;
S[1] := (100/P[1]) − 1;
p := P[1]/100;
for (i = 2 to n) {
        1. Solve P[i] = C[i] × p + (C[i] + 100)/(1 + x)^i for x;
        2. S[i] := x;
        3. p := p + (1 + x)^(−i);
}
return S[ ];
```

Figure 5.4: Algorithm for extracting spot rates from a yield curve. $P[\,i\,]$ is the price (as a percentage of par) of the coupon bond maturing i periods from now, $C[\,i\,]$ is the coupon of the i-period bond expressed as a percentage of par, and n is the term of the longest maturity bond. The first bond is a zero-coupon bond. The i-period spot rate is computed and stored in $S[\,i\,]$.

In reality, computing the spot rates is not as clean-cut as the above **bootstrapping** procedure. Treasuries of the same maturity might be selling at different yields (the **multiple cash flow problem**), some maturities might be missing from the data points (the **incompleteness problem**), Treasuries might not be of the same quality, and so on. Interpolation and fitting techniques are needed in practice to create a smooth spot rate curve (see Chap. 22). Such schemes, however, usually lack economic justifications.

➤ **Exercise 5.3.1** Suppose that $S(i) = 0.10$ for $1 \leq i < 20$ and a 20-period coupon bond is selling at par, with a coupon rate of 8% paid semiannually. Calculate $S(20)$.

➢ **Programming Assignment 5.3.2** Implement the algorithm in Fig. 5.4 plus an option to return the annualized spot rates by using the user-supplied annual compounding frequency.

5.4 Static Spread

Consider a *risky* bond with the cash flow $C_1, C_2, \ldots, C_n$ and selling for P. Were this bond riskless, it would fetch

$$P^* = \sum_{t=1}^{n} \frac{C_t}{[\,1 + S(t)\,]^t}.$$

Because riskiness must be compensated for, $P < P^*$. The **static spread** is the amount s by which the spot rate curve has to shift *in parallel* in order to price the bond correctly:

$$P = \sum_{t=1}^{n} \frac{C_t}{[\,1 + s + S(t)\,]^t}.$$

It measures the spread that a risky bond would realize over the entire Treasury spot rate curve if the bond is held to maturity. Unlike the yield spread, which is the difference between the yield to maturity of the risky bond and that of a Treasury security with comparable maturity, the static spread incorporates information from the term structure. The static spread can be computed by the Newton–Raphson method.

➢ **Programming Assignment 5.4.1** Write a program to compute the static spread. The inputs are the payment frequency per annum, the annual coupon rate as a percentage of par, the market price as a percentage of par, the number of remaining coupon payments, and the discount factors. Some numerical examples are tabulated below:

Price (% of par)	*98*	*98.5*	*99*	*99.5*	*100*	*100.5*	*101*
Static spread (%)	0.435	0.375	0.316	0.258	0.200	0.142	0.085

(A 5% 15-year bond paying semiannual interest under a flat 7.8% spot rate curve is assumed.)

5.5 Spot Rate Curve and Yield Curve

Many interesting relations hold between spot rate and yield to maturity. Let y_k denote the yield to maturity for the k-period coupon bond. The spot rate dominates the yield to maturity if the yield curve is normal; in other words, $S(k) \geq y_k$ if $y_1 < y_2 < \cdots$ (see Exercise 5.5.1, statement (1)). Analogously, the spot rate is dominated by the yield to maturity if the yield curve is inverted. Moreover, the spot rate dominates the yield to maturity if the spot rate curve is normal ($S(1) < S(2) < \cdots$) and is dominated when the spot rate curve is inverted (see Exercise 5.5.1, statement (2)). Of course, if the yield curve is flat, the spot rate curve coincides with the yield curve.

These results illustrate the **coupon effect** on the yield to maturity [848]. For instance, under a normal spot rate curve, a coupon bond has a lower yield than a zero-coupon bond of equal maturity. Picking a zero-coupon bond over a coupon bond based purely on the zero's higher yield to maturity is therefore flawed.

The spot rate curve often has the same shape as the yield curve. That is, if the spot rate curve is inverted (normal, respectively), then the yield curve is inverted (normal, respectively). However, this is only a trend, not a mathematical truth. Consider a three-period coupon bond that pays \$1 per period and repays the principal of \$100 at the end of the third period. With the spot rates $S(1) = 0.1$, $S(2) = 0.9$, and $S(3) = 0.901$, the yields to maturity can be calculated as $y_1 = 0.1$, $y_2 = 0.8873$, and $y_3 = 0.8851$, clearly not strictly increasing. However, when the final principal payment is relatively insignificant, the spot rate curve and the yield curve do share the same shape. Such is the case with bonds of high coupon rates and long maturities (see Exercise 5.5.3). When we refer to the typical agreement in shape later, the above proviso will be implicit.

➤ **Exercise 5.5.1** Prove the following statements: (1) The spot rate dominates the yield to maturity when the yield curve is normal, and (2) the spot rate dominates the yield to maturity if the spot rate curve is normal, and it is smaller than the yield to maturity if the spot rate curve is inverted.

➤ **Exercise 5.5.2** Contrive an example of a normal yield curve that implies a spot rate curve that is not normal.

➤ **Exercise 5.5.3** Suppose that the bonds making up the yield curve are ordinary annuities instead of coupon bonds. (1) Prove that a yield curve is normal if the spot rate curve is normal. (2) Still, a normal yield curve does not guarantee a normal spot rate curve. Verify this claim with this normal yield curve: $y_1 = 0.1$, $y_2 = 0.43$, $y_3 = 0.456$.

5.6 Forward Rates

The yield curve contains not only the prevailing interest rates but also information regarding future interest rates currently "expected" by the market, the **forward rates**. By definition, investing \$1 for j periods will end up with $[\,1 + S(j)\,]^j$ dollars at time j. Call it the **maturity strategy**. Alternatively, suppose we invest \$1 in bonds for i periods and at time i invest the proceeds in bonds for another $j - i$ periods, where $j > i$. Clearly we will have $[\,1 + S(i)\,]^i [\,1 + S(i, j)\,]^{j-i}$ dollars at time j, where $S(i, j)$ denotes the $(j - i)$-period spot rate i periods from now, which is unknown today.

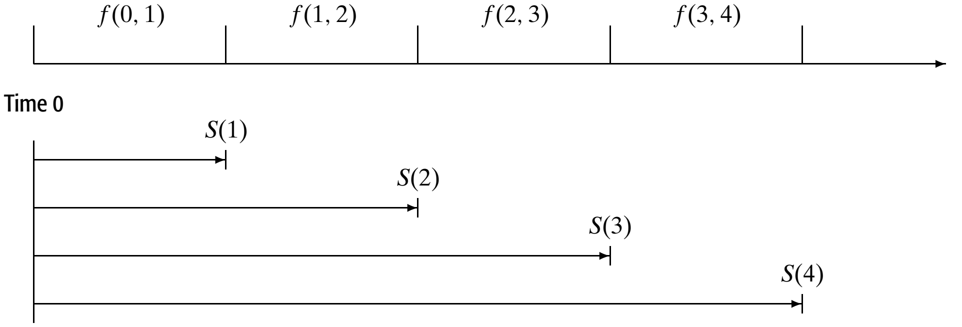


Figure 5.5: Time line for spot and forward rates.

Call it the **rollover strategy**. When $S(i, j)$ equals

$$f(i, j) \equiv \left\{ \frac{[\,1 + S(j)\,]^j}{[\,1 + S(i)\,]^i} \right\}^{1/(j-i)} - 1, \tag{5.2}$$

we will end up with $[\,1 + S(j)\,]^j$ dollars again. (By definition, $f(0, j) = S(j)$.) The rates computed by Eq. (5.2) are called the (**implied**) **forward rates** or, more precisely, the $(j - i)$-**period forward rate** i **periods from now**. Figure 5.5 illustrates the time lines for spot rates and forward rates.

In the above argument, we were not assuming any a priori relation between the implied forward rate $f(i, j)$ and the future spot rate $S(i, j)$. This is the subject of the term structure theories to which we will turn shortly. Rather, we were merely looking for the future spot rate that, *if realized*, would equate the two investment strategies. Forward rates with a duration of a single period are called **instantaneous forward rates** or **one-period forward rates**.

When the spot rate curve is normal, the forward rate dominates the spot rates:

$$f(i, j) > S(j) > \cdots > S(i). \tag{5.3}$$

This claim can be easily extracted from Eq. (5.2). When the spot rate curve is inverted, the forward rate is in turn dominated by the spot rates:

$$f(i, j) < S(j) < \cdots < S(i). \tag{5.4}$$

See Fig. 5.6 for illustration.

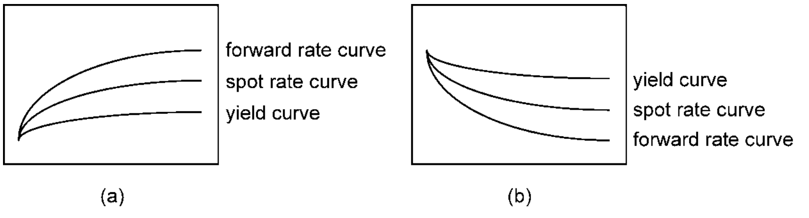


Figure 5.6: Yield curve, spot rate curve, and forward rate curve. When the yield curve is normal, it is dominated by the spot rate curve, which in turn is dominated by the forward rate curve (if the spot rate curve is also normal). When the yield curve is inverted, on the other hand, it dominates the spot rate curve, which in turn dominates the forward rate curve (if the spot rate curve is also inverted). The forward rate curve here is a plot of one-period forward rates.

Forward rates, spot rates, and the yield curve[2] can be derived from each other. For example, the future value of \$1 at time n can be derived in two ways. We can buy n-period zero-coupon bonds and receive $[\,1+S(n)\,]^n$ or we can buy one-period zero-coupon bonds today and then a series of such bonds at the forward rates as they mature. The future value of this approach is $[\,1+S(1)\,][\,1+f(1,2)\,]\cdots[\,1+f(n-1,n)\,]$. Because they are identical,

$$S(n) = \{\,[\,1+S(1)\,][\,1+f(1,2)\,]\cdots[\,1+f(n-1,n)\,]\,\}^{1/n} - 1. \tag{5.5}$$

Hence, the forward rates, specifically the one-period forward rates $f(s, s+1)$, determine the spot rate curve.

EXAMPLE 5.6.1 Suppose that the following 10 spot rates are extracted from the yield curve:

Period	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Rate (%)	4.00	4.20	4.30	4.50	4.70	4.85	5.00	5.25	5.40	5.50

The following are the 9 one-period forward rates, starting one period from now.

Period	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Rate (%)	4.40	4.50	5.10	5.50	5.60	5.91	7.02	6.61	6.40

If \$1 is invested in a 10-period zero-coupon bond, it will grow to be $(1+0.055)^{10} = 1.708$. An alternative strategy is to invest \$1 in one-period zero-coupon bonds at 4% and reinvest at the one-period forward rates. The final result,

$$\begin{aligned}&1.04 \times 1.044 \times 1.045 \times 1.051 \times 1.055 \times 1.056\\&\quad \times 1.0591 \times 1.0702 \times 1.0661 \times 1.064 = 1.708,\end{aligned}$$

is exactly the same as expected.

➤ **Exercise 5.6.1** Assume that all coupon bonds are par bonds. Extract the spot rates and the forward rates from the following yields to maturity: $y_1 = 0.03$, $y_2 = 0.04$, and $y_3 = 0.045$.

➤ **Exercise 5.6.2** Argue that $[\,1+f(a, a+b+c)\,]^{b+c} = [\,1+f(a, a+b)\,]^b[\,1+f(a+b, a+b+c)\,]^c$.

➤ **Exercise 5.6.3** Show that $f(T, T+1) = d(T)/d(T+1) - 1$ (to be generalized in Eq. (24.2)).

➤ **Exercise 5.6.4** Let the price of a 10-year zero-coupon bond be quoted at 60 and that of a 9.5-year zero-coupon bond be quoted at 62. Calculate the percentage changes in the 10-year spot rate and the 9.5-year forward rate if the 10-year bond price moves up by 1%. (All rates are bond equivalent.)

➤ **Exercise 5.6.5** Prove that the forward rate curve lies above the spot rate curve when the spot rate curve is normal, below it when the spot rate curve is inverted, and that they cross where the spot rate curve is instantaneously flat (see Fig. 5.7).

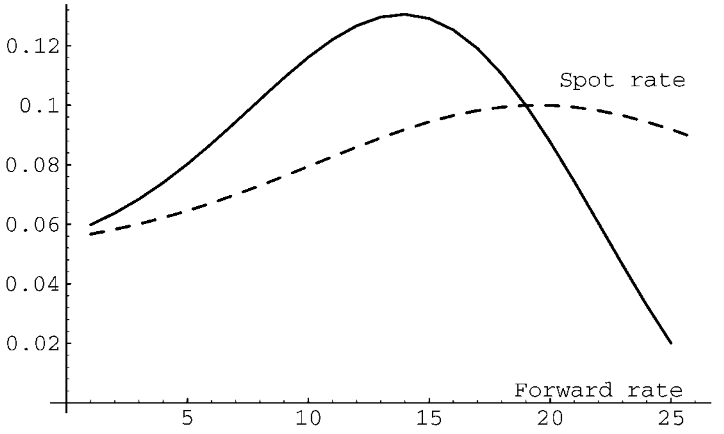


Figure 5.7: Spot rate curve and forward rate curve. The forward rate curve is built by one-period forward rates.

5.6.1 Locking in the Forward Rate

Although forward rates may or may not be realized in the future, we can lock in any forward rate $f(n, m)$ today by buying one n-period zero-coupon bond for $1/[\,1 + S(n)\,]^n$ and selling $[\,1 + S(m)\,]^m/[\,1 + S(n)\,]^n$ m-year zero-coupon bonds. Here is the analysis. There is no net initial investment because the cash inflow and the cash outflow, both at $1/[\,1 + S(n)\,]^n$ dollars, cancel out. At time n there will be a cash inflow of \$1, and at time m there will be a cash outflow of $[\,1 + S(m)\,]^m/[\,1 + S(n)\,]^n$ dollars. This cash flow stream implies the rate $f(n, m)$ between times n and m (see Fig. 5.8).

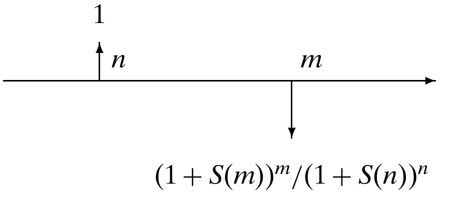
The above transactions generate the cash flow of an important kind of financial instrument called a **forward contract**. In our particular case, this forward contract, agreed on *today*, enables us to borrow money at time n *in the future* and repay the loan at time $m > n$ with an interest rate equal to the forward rate $f(n, m)$.

Now that forward rates can be locked in, clearly they should not be negative. However, forward rates derived by Eq. (5.2) may be negative if the spot rate curve is steeply downward sloping. It must be concluded that the spot rate curve cannot be arbitrarily specified.

➤ **Exercise 5.6.6** (1) The fact that the forward rate can be locked in today means that future spot rates must equal today's forward rates, or $S(a, b) = f(a, b)$, in a certain economy. Why? How about an uncertain economy? (2) Verify that forward rates covering the same time period will not change over time in a certain economy.

➤ **Exercise 5.6.7** (1) Confirm that a 50-year bond selling at par (\$1,000) with a semiannual coupon rate of 2.55% is equivalent to a 50-year bond selling for \$1,000 with a semiannual coupon rate of 2.7% and a par value of \$329.1686. (2) Argue that a

Figure 5.8: Locking in the forward rate. By trading zero-coupon bonds of maturities n and m in the right proportion, the forward rate $f(n, m)$ can be locked in today.



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100-year bond selling at par with a semiannual coupon rate of 2.7% is equivalent to a portfolio of the above-mentioned 50-year bond and a contract to buy 50 years from now a 50-year bond at a price of \$329.1686 with a semiannual coupon rate of 2.7%. (3) Verify that the bond to be bought 50 years hence has a semiannual yield of 8.209%. (Therefore we should not underestimate the importance of later forward rates on long-term coupon bonds' prices as even a small increase in yields between two long-term coupon bonds could imply an unreasonably high forward rate.)

5.6.2 Term Structure of Credit Spreads

Static spread can be interpreted as the constant **credit spread** to the Treasury spot rate curve that reflects the risk premium of a corporate bond. However, an identical credit spread at all maturities runs counter to the common sense that the credit spread should rise with maturity; a corporation is more likely to fail in, say, 10 years rather than in 1.

One theory of term structure of credit spreads postulates that the price of a corporate bond equals that of the Treasury times the probability of solvency. Furthermore, once default occurs, a corporation remains in that state and pays zero dollar. Because the probability of default is one minus the probability of solvency,

$$1 - \text{probability of default (1 period)} = \frac{\text{price of 1-period corporate zero}}{\text{price of 1-period Treasury zero}}.$$

After using the above equation to compute the probability of default for corporate bonds with one period to maturity, we can calculate the **forward probability of default**, the conditional probability of default in the second period given that default has not occurred in the first period. This forward probability of default clearly satisfies

$$\begin{aligned}&[\,1 - \text{probability of default (1 period)}\,]\\ &\quad\times [\,1 - \text{forward probability of default (period 2)}\,]\\ &\quad= \frac{\text{price of 2-period corporate zero}}{\text{price of 2-period Treasury zero}}.\end{aligned}$$

In general, the equation satisfied by the forward probability is

$$\begin{aligned}&[\,1 - \text{probability of default } (i-1 \text{ periods})\,]\\ &\quad\times [\,1 - \text{forward probability of default (period } i)\,]\\ &\quad= \text{probability the corporate bond survives past time } i\\ &\quad= \frac{\text{price of } i\text{-period corporate zero}}{\text{price of } i\text{-period Treasury zero}}.\end{aligned} \tag{5.6}$$

The algorithm for computing the forward probabilities of default is shown in Fig. 5.9.

➤ **Exercise 5.6.8** Consider the following four zero-coupon bonds:

Type	*Maturity*	*Price*	*Yield*	*Type*	*Maturity*	*Price*	*Yield*
Treasury	1 year	94	6.28%	Treasury	2 year	87	7.09%
Corporate	1 year	92	8.51%	Corporate	2 year	84	8.91%

Compute the probabilities of default and the forward probabilities of default.

Algorithm for forward probabilities of default:

```
input:  n, P[1..n], Q[1..n];
real    f[1..n], p[1..n];
p[1] := 1 − (Q[1]/P[1]);
f[1] := p[1];
for  (i = 2 to n) {
        f[i] := 1 − (1 − p[i − 1])^(−1) × (Q[i]/P[i]);
        p[i] := p[i − 1] × f[i];
}
return  f[ ];
```

Figure 5.9: Algorithm for forward probabilities of default. $P[\,i\,]$ is the price of the riskless i-period zero, $Q[\,i\,]$ is the price of the risky i-period zero, $p[\,i\,]$ stores the probability of default during period one to i, and the forward probability of default for the ith period is calculated in $f[\,i\,]$.

➤ **Exercise 5.6.9** (1) Prove Eq. (5.6). (2) Define the **forward spread** for period i, $s(i)$, as the difference between the instantaneous period-i forward rate $f(i-1, i)$ obtained by riskless bonds and the instantaneous period-i forward rate $f_c(i-1, i)$ obtained by corporate bonds. Prove that $s(i)$ roughly equals the forward probability of default for period i.

5.6.3 Spot and Forward Rates under Continuous Compounding

Under continuous compounding, the pricing formula becomes

$$P = \sum_{i=1}^{n} Ce^{-iS(i)} + Fe^{-nS(n)}.$$

In particular, the market discount function is

$$d(n) = e^{-nS(n)}. \tag{5.7}$$

A bootstrapping procedure similar to the one in Fig. 5.4 can be used to calculate the spot rates under continuous compounding. The spot rate is now an arithmetic average of forward rates:

$$S(n) = \frac{f(0,1) + f(1,2) + \cdots + f(n-1,n)}{n}. \tag{5.8}$$

The formula for the forward rate is also very simple:

$$f(i, j) = \frac{jS(j) - iS(i)}{j - i}. \tag{5.9}$$

In particular, the one-period forward rate equals

$$f(j, j+1) = (j+1)\,S(j+1) - jS(j) = -\ln\frac{d(j+1)}{d(j)} \tag{5.10}$$

(compare it with Exercise 5.6.3).

Rewrite Eq. (5.9) as

$$f(i, j) = S(j) + [\, S(j) - S(i)\,]\, \frac{i}{j-i}.$$

Then under continuous time instead of discrete time, Eq. (5.10) becomes

$$f(T, T+\Delta T) = S(T+\Delta T) + [\, S(T+\Delta T) - S(T)\,]\, \frac{T}{\Delta T},$$

and the instantaneous forward rate at time T equals

$$f(T) \equiv \lim_{\Delta T \to 0} f(T, T+\Delta T) = S(T) + T\, \frac{\partial S}{\partial T}. \tag{5.11}$$

Note that $f(T) > S(T)$ if and only if $\partial S/\partial T > 0$.

➤ **Exercise 5.6.10** Derive Eqs. (5.8) and (5.9).

➤ **Exercise 5.6.11** Compute the one-period forward rates from this spot rate curve: $S(1) = 2.0\%$, $S(2) = 2.5\%$, $S(3) = 3.0\%$, $S(4) = 3.5\%$, and $S(5) = 4.0\%$.

➤ **Exercise 5.6.12** (1) Figure out a case in which a change in the spot rate curve leaves all forward rates unaffected. (2) Derive the duration $-(\partial P/\partial y)/P$ under the shape change in (1), where y is the short rate $S(1)$.

5.6.4 Spot and Forward Rates under Simple Compounding

This is just a brief subsection because the basic principles are similar. The pricing formula becomes

$$P = \sum_{i=1}^{n} \frac{C}{1 + i\,S(i)} + \frac{F}{1 + nS(n)}.$$

The market discount function is

$$d(n) = [\, 1 + nS(n)\,]^{-1}, \tag{5.12}$$

and the $(i - j)$-period forward rate j periods from now is

$$f(i, j) = \frac{[\, 1 + j\,S(j)\,][\, 1 + i\,S(i)\,]^{-1} - 1}{j - i}. \tag{5.13}$$

To annualize the rates, multiply them by the number of periods per annum.

➤ **Exercise 5.6.13** Derive Eq. (5.13).

5.7 Term Structure Theories

Term structure theories attempt to explain the relations among interest rates of various maturities. As the spot rate curve is most critical for the purpose of valuation, the term structure theories discussed below will be about the spot rate curve.

5.7.1 Expectations Theory

Unbiased Expectations Theory

According to the **unbiased expectations theory** attributed to Irving Fisher, forward rate equals the average future spot rate:

$$f(a,b) = E[\,S(a,b)\,], \tag{5.14}$$

where $E[\,\cdot\,]$ denotes mathematical expectation [653, 799]. Note that this theory does not imply that the forward rate is an accurate predictor for the future spot rate. It merely asserts that it does not deviate from the future spot rate systematically. The theory also implies that the maturity strategy and the rollover strategy produce the same result at the horizon on the average (see Exercise 5.7.2). A normal spot rate curve, according to the theory, is due to the fact that the market expects the future spot rate to rise. Formally, because $f(j, j+1) > S(j+1)$ if and only if $S(j+1) > S(j)$ from Eq. (5.2), it follows that

$$E[\,S(j, j+1)\,] > S(j+1) \quad \text{if and only if } S(j+1) > S(j)$$

when the theory holds. Conversely, the theory implies that the spot rate is expected to fall if and only if the spot rate curve is inverted [750].

The unbiased expectations theory, however, has been rejected by most empirical studies dating back at least to Macaulay [627, 633, 767], with the possible exception of the period before the founding of the Federal Reserve System in 1915 [639, 751]. Because the term structure has been upward sloping ~80% of the time, the unbiased expectations theory would imply that investors have expected interest rates to rise 80% of the time. This does not seem plausible. It also implies that riskless bonds, regardless of their different maturities, earn the same return on the average (see Exercise 5.7.1) [489, 568]. This is not credible either, because that would mean investors are indifferent to risk.

➤ **Exercise 5.7.1** Prove that an n-period zero-coupon bond sold at time $k < n$ has a holding period return of exactly $S(k)$ if the forward rates are realized.

➤ **Exercise 5.7.2** Show that

$$[\,1+S(n)\,]^n = E[\,1+S(1)\,]\,E[\,1+S(1,2)\,]\cdots E[\,1+S(n-1,n)\,]$$

under the unbiased expectations theory.

Other Versions of the Expectations Theory

At least four other versions of the expectations theory have been proposed, but they are inconsistent with each other for subtle reasons [232]. Expectation also plays a critical role in other theories, which differ by how risks are treated [492].

Consider a theory that says the expected returns on all possible riskless bond strategies are equal for all holding periods. In particular,

$$[\,1+S(2)\,]^2 = [\,1+S(1)\,]\,E[\,1+S(1,2)\,] \tag{5.15}$$

because of the equivalency between buying a two-period bond and rolling over one-period bonds. After rearrangement, $E[\,1+S(1,2)\,] = [\,1+S(2)\,]^2/[\,1+S(1)\,]$. Now consider the following two one-period strategies. The first strategy buys a two-period bond and sells it after one period. The expected return is $E[\{1+S(1,2)\}^{-1}][1+S(2)]^2$. The second strategy buys a one-period bond with a return of $1+S(1)$. The theory says they are equal: $E[\{1+S(1,2)\}^{-1}][1+S(2)]^2 = 1+S(1)$, which implies that

$$\frac{[1+S(2)]^2}{1+S(1)} = \frac{1}{E[\{1+S(1,2)\}^{-1}]}.$$

Combining this equation with Eq. (5.15), we conclude that

$$E\left[\frac{1}{1+S(1,2)}\right] = \frac{1}{E[1+S(1,2)]}.$$

However, this is impossible, save for a certain economy. The reason is **Jensen's inequality**, which states that $E[g(X)] > g(E[X])$ for any nondegenerate **random variable** X and strictly convex function g (i.e., $g''(x) > 0$). Use $g(x) \equiv (1+x)^{-1}$ to prove our point. So this version of the expectations theory is untenable.

Another version of the expectations theory is the **local expectations theory** [232, 385]. It postulates that the expected rate of return of any bond over a single period equals the prevailing one-period spot rate:

$$\frac{E[\{1+S(1,n)\}^{-(n-1)}]}{[1+S(n)]^{-n}} = 1+S(1) \quad \text{for } all\ n>1. \tag{5.16}$$

This theory will form the basis of many stochastic interest rate models later. We call

$$\frac{E[\{1+S(1,n)\}^{-(n-1)}]}{[1+S(n)]^{-n}} - [1+S(1)]$$

the **holding premium**, which is zero under the local expectations theory.

Each version of the expectations theory postulates that a certain expected difference, called the **liquidity premium** or the **term premium**, is zero. For instance, the liquidity premium is $f(a,b) - E[S(a,b)]$ under the unbiased expectations theory and it is the holding premium under the local expectations theory [694]. The incompatibility between versions of the expectations theory alluded to earlier would disappear, had they postulated *nonzero* liquidity premiums [143]. For example, the **biased expectations theory** says that

$$f(a,b) - E[S(a,b)] = p(a,b),$$

where the liquidity premium p is not zero [39, 653]. A nonzero liquidity premium is reasonably supported by evidence. There is also evidence that p is neither constant nor time-independent [43, 335].

➤ **Exercise 5.7.3** (1) Prove that

$$E\left[\frac{1}{\{1+S(1)\}\{1+S(1,2)\}\cdots\{1+S(n-1,n)\}}\right] = \frac{1}{[1+S(n)]^n}$$

under the local expectations theory. (2) Show that the local expectations theory is inconsistent with the unbiased expectations theory.

➤ **Exercise 5.7.4** The **return-to-maturity expectations theory** postulates that the maturity strategy earns the same return as the rollover strategy with one-period bonds, i.e.,

$$[\,1+S(n)\,]^n = E[\,\{\,1+S(1)\,\}\,\{\,1+S(1,2)\,\}\cdots\{\,1+S(n-1,n)\,\}\,], \quad n>1.$$

Show that it is inconsistent with the local expectations theory.

5.7.2 Liquidity Preference Theory

The **liquidity preference theory** holds that investors demand a risk premium for holding long-term bonds [492]. The liquidity preference theory is attributed to Hicks [445]. Consider an investor with a holding period of two. If the investor chooses the maturity strategy and is forced to sell the two-period bonds because of an unexpected need for cash, he would face the **interest rate risk** and the ensuing **price risk** because bond prices depend on the prevailing interest rates at the time of the sale. This risk is absent from the rollover strategy. As a consequence, the investor demands a higher return for longer-term bonds. This implies that $f(a,b) > E[\,S(a,b)\,]$. When the spot rate curve is inverted,

$$\begin{aligned}
&[\,1+S(i)\,]^{1/(i+1)}\{\,1+E[\,S(1,i+1)\,]\,\}^{i/(i+1)}\\
&\quad< [\,1+S(1)\,]^{1/(i+1)}\{\,1+E[\,S(1,i+1)\,]\,\}^{i/(i+1)}\\
&\quad< 1+S(i+1)\\
&\quad< 1+S(i).
\end{aligned}$$

Thus $E[\,S(1,i+1)\,] < S(i)$. The market therefore has to expect the interest rate to decline in order for an inverted spot rate curve to be observed.

The liquidity preference theory seems to be consistent with the typically upward-sloping yield curve. Even if people expect the interest rate to decline and rise equally frequently, the theory asserts that the curve is upward sloping more often. This is because a rising expected interest rate is associated with only a normal spot rate curve, and a declining expected interest rate can sometimes be associated with a normal spot rate curve. Only when the interest rate is expected to fall below a threshold does the spot rate curve become inverted. The unbiased expectations theory, we recall, is not consistent with this case.

➤ **Exercise 5.7.5** Show that the market has to expect the interest rate to decline in order for a flat spot rate curve to occur under the liquidity preference theory.

5.7.3 Market Segmentation Theory

The **market segmentation theory** holds that investors are restricted to bonds of certain maturities either by law, preferences, or customs. For instance, life insurance companies generally prefer long-term bonds, whereas commercial banks favor shorter-term ones. The spot rates are determined within each maturity sector separately [653, 799].

The market segmentation theory is closely related to the **preferred habitats theory** of Culbertson, Modigliani, and Sutch [674]. This theory holds that the investor's horizon determines the riskiness of bonds. A horizon of 5 years will prefer a 5-year zero-coupon bond, demanding higher returns from both 2- and 7-year bonds, for example, because the former choice has reinvestment risk and the latter has price risk. Hence, in contrast to the liquidity preference theory, $f(a,b) < E[\,S(a,b)\,]$ can happen if the market is dominated by long-term investors.

5.8 Duration and Immunization Revisited

Rate changes considered before for duration were parallel shifts under flat spot rate curves. We now study duration and immunization under more general spot rate curves and movements.

5.8.1 Duration Measures

Let $S(1), S(2), \ldots$, be the spot rate curve and $P(y) \equiv \sum_i C_i/[\,1 + S(i) + y\,]^i$ be the price associated with the cash flow $C_1, C_2 \ldots$. Define duration as

$$-\frac{\partial P(y)/P(0)}{\partial y}\bigg|_{y=0} = \frac{\sum_i \frac{iC_i}{[1+S(i)]^{i+1}}}{\sum_i \frac{C_i}{[1+S(i)]^i}}.$$

Note that the curve is shifted in parallel to $S(1) + \Delta y, S(2) + \Delta y, \ldots$, before letting Δy go to zero. As before, the percentage price change roughly equals duration multiplied by the size of the parallel shift in the spot rate curve. But the simple linear relation between duration and MD (4.4) breaks down. One way to regain it is to resort to a different kind of shift, the **proportional shift**, defined as

$$\frac{\Delta[\,1 + S(i)\,]}{1 + S(i)} = \frac{\Delta[\,1 + S(1)\,]}{1 + S(1)}$$

for all i [317]. Here, Δx denotes the change in x when the short-term rate is shifted by Δy. Duration now becomes

$$\frac{1}{1 + S(1)} \left\{ \frac{\sum_i \frac{iC_i}{[1+S(i)]^i}}{\sum_i \frac{C_i}{[1+S(i)]^i}} \right\}. \tag{5.17}$$

If we define **Macaulay's second duration** to be the number within the braces in Eq. (5.17):

$$\text{duration} = \frac{\text{Macaulay's second duration}}{[\,1 + S(1)\,]}.$$

This measure is also called **Bierwag's duration** [71, 496].

Parallel shift does not reflect market reality. For example, long-term rates do not correlate perfectly with short-term rates; in fact, the two rates often move in opposite directions. Short-term rates are also historically more volatile. Practitioners sometimes break the spot rate curve into segments and measure the duration in each segment [470].

Duration can also be defined under custom changes of the yield curve. For example, we may define the **short-end duration** as the effective duration under the following shifts. The 1-year yield is changed by ± 50 basis points ($\pm 0.5\%$). The amounts of yield changes for maturities $1 \leq i \leq 10$ are $\pm 50 \times (11 - i)/10$ basis points. Yields of maturities longer than 10 remain intact. If the yield curve is normal, the $+50$ basis-point change corresponds to **flattening** of the yield curve, whereas the -50 basis-point change corresponds to **steepening** of the yield curve. **Long-end duration** can be specified similarly. Two custom shifts are behind nonproportional shifts (see Exercise 5.8.3) and Ho's **key rate durations** (see Section 27.5).

Although durations have many variants, the one feature that all share is that the term structure can shift in only a fixed pattern. Despite its theoretical limitations, duration seems to provide as good an estimate for price volatility as more sophisticated measures [348]. Furthermore, immunization with the MD, still widely used [91], is as effective as alternative duration measures [424]. One explanation is that, although long-term rates and short-term rates do not in general move by the same amount or even in the same direction, roughly parallel shifts in the spot rate curve are responsible for more than 80% of the movements in interest rates [607].

➤ **Exercise 5.8.1** Assume continuous compounding. Show that if the yields to maturity of all fixed-rate bonds change by the same amount, then (1) the spot rate curve must be flat and (2) the spot rate curve shift must be parallel. (Hint: The yields of zero-coupon bonds of various maturities change by the same amount.)

➤ **Exercise 5.8.2** Verify duration (5.17).

➤ **Exercise 5.8.3** Empirically, long-term rates change less than short-term ones. To incorporate this fact into duration, we may postulate **nonproportional shifts** as

$$\frac{\Delta[\,1+S(i)\,]}{1+S(i)} = K^{i-1}\frac{\Delta[\,1+S(1)\,]}{1+S(1)} \quad \text{for some } K<1.$$

Show that a t-period zero-coupon bond's price sensitivity satisfies

$$\frac{\Delta P}{P} = -t\,K^{t-1}\frac{\Delta[\,1+S(1)\,]}{1+S(1)}$$

under nonproportional shifts.

5.8.2 Immunization

The Case of NO Rate Changes

Recall that in the absence of interest rate changes and assuming a flat spot rate curve, it suffices to match the PVs of the future liability and the asset to achieve immunization (see Exercise 4.2.7). This conclusion continues to hold even if the spot rate curve is not flat, as long as the future spot rates equal the forward rates. Here is the analysis. Let L be the liability at time m. Then

$$P=\sum_{i=1}^{n}\frac{C}{[\,1+S(i)\,]^{i}}+\frac{F}{[\,1+S(n)\,]^{n}}=\frac{L}{[\,1+S(m)\,]^{m}}.$$

The PV of the liability at any time $k\leq m$ is hence

$$\frac{L}{[\,1+S(k,m)\,]^{m-k}}=P[\,1+S(k)\,]^{k}$$

by Eq. (5.2) and the premise that $f(a,b)=S(a,b)$. The PV of the bond plus the reinvestments of the coupon payments at the same time is

$$\begin{aligned}
&\sum_{i=1}^{k}C[\,1+S(i,k)\,]^{k-i}+\sum_{i=1}^{n-k}\frac{C}{[\,1+S(k,i+k)\,]^{i}}+\frac{F}{[\,1+S(k,n)\,]^{n-k}}\\
&\quad=\sum_{i=1}^{k}\frac{C[\,1+S(k)\,]^{k}}{[\,1+S(i)\,]^{i}}+\sum_{i=1}^{n-k}\frac{C[\,1+S(k)\,]^{k}}{[\,1+S(i+k)\,]^{i+k}}+\frac{F[\,1+S(k)\,]^{k}}{[\,1+S(n)\,]^{n}}\\
&\quad=P[\,1+S(k)\,]^{k},
\end{aligned}$$

which matches the liability precisely. Therefore, in the absence of unpredictable interest rate changes, duration matching and rebalancing are not needed for immunization.

The Case of Certain Rate Movements

Recall that a future liability can be immunized by a portfolio of bonds with the same PV and MD under flat spot rate curves (see Subsection 4.2.2). If only parallel shifts are allowed, this conclusion can be extended to general spot rate curves. Here is the analysis. We are working with continuous compounding. The liability L is T periods from now. Without loss of generality, assume that the portfolio consists of only zero-coupon bonds maturing at t_1 and t_2 with $t_1 < T < t_2$. Let there be n_i bonds maturing at time $t_i, i = 1, 2$. Assume that $L = 1$ for simplicity. The portfolio's PV is

$$V \equiv n_1 e^{-S(t_1)\, t_1} + n_2 e^{-S(t_2)\, t_2} = e^{-S(T)\, T},$$

and its MD is

$$\frac{n_1 t_1 e^{-S(t_1)\, t_1} + n_2 t_2 e^{-S(t_2)\, t_2}}{V} = T.$$

These two equations imply that

$$n_1 e^{-S(t_1)\, t_1} = \frac{V(t_2 - T)}{t_2 - t_1}, \quad n_2 e^{-S(t_2)\, t_2} = \frac{V(t_1 - T)}{t_1 - t_2}.$$

Now shift the spot rate curve uniformly by $\delta \neq 0$. The portfolio's PV becomes

$$\begin{aligned} n_1 e^{-[\, S(t_1)+\delta\,]\, t_1} + n_2 e^{-[\, S(t_2)+\delta\,]\, t_2} &= e^{-\delta t_1} \frac{V(t_2 - T)}{t_2 - t_1} + e^{-\delta t_2} \frac{V(t_1 - T)}{t_1 - t_2} \\ &= \frac{V}{t_2 - t_1} [e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1)], \end{aligned}$$

whereas the liability's PV after the parallel shift is $e^{-[\, S(T)+\delta\,]\, T} = e^{-\delta T} V$. As

$$\frac{V}{t_2 - t_1} [e^{-\delta t_1}(t_2 - T) + e^{-\delta t_2}(T - t_1)] > e^{-\delta T} V,$$

immunization is established. See Fig. 5.10 for illustration.

Intriguingly, we just demonstrated that (1) a duration-matched position under parallel shifts in the spot rate curve implies a free lunch as any interest rate change generates profits and (2) no investors would hold the T-period bond because a portfolio of t_1- and t_2-period bonds has a higher return for any interest rate shock (in fact, they would own bonds of only the shortest and the longest maturities). Implausible as the assertions may be, the reasoning seems impeccable. The way to resolve the conundrum lies in observing that rate changes were assumed to be *instantaneous*. The problem disappears when price changes occur *after* rate changes [207, 848].[3]

A barbell portfolio often arises from maximizing the portfolio convexity, as argued in Section 4.3. Higher convexity may be undesirable, however, when it comes to immunization. Recall that convexity assumes parallel shifts in the term structure. The moment this condition is compromised, as is often the case in reality, the more

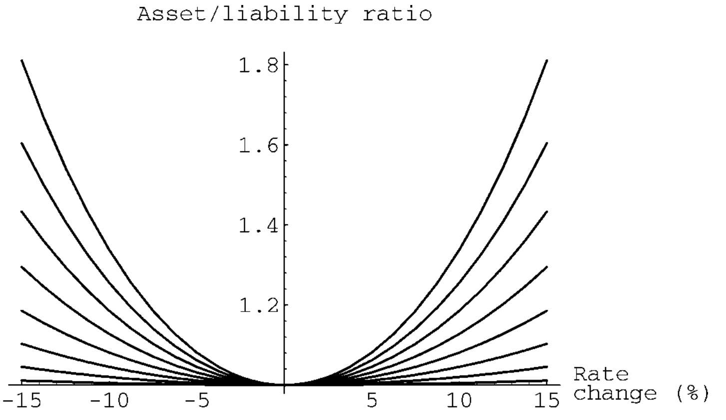

Figure 5.10: Asset/liability ratios under parallel shifts in the spot rate curve. Each curve is the result of a pair of zero-coupon bonds with maturities (t_1, t_2) to immunize a liability 10 periods away. All curves have a minimum value of one when there are no shifts. Interest rate changes move the portfolio value ahead of the liability, and the effects are more pronounced the more t_1 and t_2 are away from 10.

dispersed the cash flows, the more exposed the portfolio is to the **shape risk** (or the **twist risk**) [206, 246].

➤ **Exercise 5.8.4** Repeat the above two-bond argument to prove that the claims in Exercise 4.2.8 remain valid under the more general setting here.

Additional Reading

Consult [325, 514, 583, 629] for more information on the term structure of credit spread. Pointers to empirical studies of the expectations theory can be found in [144, 147]. Also called the **Fisher–Weil duration** [424], Macaulay's second duration is proposed in [350]. See [239, 245, 267, 318] for alternative approaches to immunization and [367] for immunization under stochastic interest rates.

NOTES

1. **"Maturity"** and **"term"** are usually used in place of "term to maturity."
2. The coupon rates of the coupon bonds making up the yield curve need to be specified.
3. We return to this issue in Exercises 14.4.4 and 24.6.8.