CHAPTER TWENTY-FOUR

Foundations of Term Structure Modeling

[The] foundations are the most controversial parts of many, if not all, sciences.

Leonard J. Savage (1917–1971), *The Foundations of Statistics*

This chapter introduces basic definitions and results in term structure modeling. It lays the theoretical foundations for interest rate models. A few simple models are presented at the end of the chapter.

24.1 Terminology

A period denotes a unit of elapsed time throughout this chapter. Hence, viewed at time t, the next time instant refers to time $t + dt$ in the continuous-time model and time $t + 1$ in the discrete-time case. If the discrete-time model results from dividing the time interval $[s, t]$ into n periods, then each period takes $(t - s)/n$ time. Here bonds are assumed to have a par value of one unless stated otherwise. We use the same notation for discrete-time and continuous-time models as the context is always clear. The time unit for continuous-time models is usually measured by the year. We standardize the following notation:

t: a point in time.
$r(t)$: the one-period riskless rate prevailing at time t for repayment one period later (the instantaneous spot rate, or **short rate**, at time t).
$P(t, T)$: the PV at time t of \$1 at time T.
$r(t, T)$: the $(T - t)$-period interest rate prevailing at time t stated on a per-period basis and compounded once per period – in other words, the $(T - t)$-period spot rate at time t. (This definition dictates that continuous-time models use continuous compounding and discrete-time models use periodic compounding.) The **long rate** is defined as $r(t, \infty)$, that is, the continuously compounded yield on a consol bond that pays out \$1 per unit time forever and never repays principal.
$F(t, T, M)$: the forward price at time t of a forward contract that delivers at time T a zero-coupon bond maturing at time $M \geq T$.
$f(t, T, L)$: the L-period forward rate at time T implied at time t stated on a per-period basis and compounded once per period.

$f(t, T)$: the one-period or instantaneous forward rate at time T as seen at time t stated on a per-period basis and compounded once per period. It is $f(t, T, 1)$ in the discrete-time model and $f(t, T, dt)$ in the continuous-time model. Note that $f(t, t)$ equals the short rate $r(t)$.

24.2 Basic Relations

The price of a zero-coupon bond is

$$P(t, T) = \begin{cases} [\,1 + r(t, T)\,]^{-(T-t)} & \text{in discrete time} \\ e^{-r(t,T)(T-t)} & \text{in continuous time} \end{cases}.$$

Recall that $r(t, T)$ as a function of T defines the spot rate curve at time t. By definition,

$$f(t, t) = \begin{cases} r(t, t+1) & \text{in discrete time} \\ r(t, t) & \text{in continuous time} \end{cases}.$$

Forward prices and zero-coupon bond prices are related by

$$F(t, T, M) = \frac{P(t, M)}{P(t, T)}, \quad T \leq M, \tag{24.1}$$

which says that the forward price equals the FV at time T of the underlying asset. Equation (24.1) can be verified with an arbitrage argument similar to the "locking in" of forward rates in Subsection 5.6.1 (see Exercise 24.2.1, part (1)). Equation (24.1) holds whether the model is discrete-time or continuous-time, and it implies that

$$F(t, T, M) = F(t, T, S)\, F(t, S, M), \quad T \leq S \leq M.$$

Forward rates and forward prices are related definitionally by

$$f(t, T, L) = \left[\frac{1}{F(t, T, T+L)}\right]^{1/L} - 1 = \left[\frac{P(t, T)}{P(t, T+L)}\right]^{1/L} - 1 \tag{24.2}$$

in discrete time (hence periodic compounding). In particular, $1 + f(t, T, 1) = 1/F(t, T, T+1) = P(t, T)/P(t, T+1)$. In continuous time (hence continuous compounding),

$$f(t, T, L) = -\frac{\ln F(t, T, T+L)}{L} = \frac{\ln(P(t, T)/P(t, T+L))}{L} \tag{24.3}$$

by Eq. (24.1). Furthermore, because

$$f(t, T, \Delta t) = \frac{\ln(P(t, T)/P(t, T+\Delta t))}{\Delta t} \to -\frac{\partial \ln P(t, T)}{\partial T} = -\frac{\partial P(t, T)/\partial T}{P(t, T)},$$

we conclude that

$$f(t, T) \equiv \lim_{\Delta t \to 0} f(t, T, \Delta t) = -\frac{\partial P(t, T)/\partial T}{P(t, T)}, \quad t \leq T. \tag{24.4}$$

Because Eq. (24.4) is equivalent to

$$P(t, T) = e^{-\int_t^T f(t,s)\, ds}, \tag{24.5}$$

the spot rate curve is $r(t, T) = [1/(T-t)]\int_t^T f(t, s)\, ds$. The discrete analog to Eq. (24.5) is

$$P(t, T) = \frac{1}{[1+r(t)][1+f(t, t+1)]\cdots[1+f(t, T-1)]}. \tag{24.6}$$

The liquidity premium is the difference between the forward rate and the expected spot rate, $f(t, T) - E_t[r(T) \mid r(t)]$. Finally, the short rate and the market discount function are related by

$$r(t) = -\left.\frac{\partial P(t, T)}{\partial T}\right|_{T=t}.$$

This can be verified with Eq. (24.4) and the observation that $P(t, t) = 1$ and $r(t) = f(t, t)$.

➤ **Exercise 24.2.1** (1) Supply the arbitrage argument for Eq. (24.1). (2) Generalize (1) by describing a strategy that replicates the forward contract on a coupon bond that may make payments before the delivery date.

➤ **Exercise 24.2.2** Suppose we sell one T-time zero-coupon bond and buy $P(t, T)/P(t, M)$ units of M-time zero-coupon bonds at time t. Proceed from here to justify Eq. (24.1).

➤ **Exercise 24.2.3** Prove Eq. (24.4) from Eq. (5.11).

➤ **Exercise 24.2.4** Prove Eq. (24.6) from Eq. (24.2).

➤ **Exercise 24.2.5** Show that the τ-period spot rate equals $(1/\tau)\sum_{i=0}^{\tau-1} f(t, t+i)$ (average of forward rates) if all the rates are continuously compounded.

➤ **Exercise 24.2.6** Verify that

$$f(t, T, L) = \frac{1}{L}\left[\frac{P(t, T)}{P(t, T+L)} - 1\right]$$

is the analog to Eq. (24.2) under simple compounding.

➤ **Exercise 24.2.7** Prove the following continuous-time analog to Eq. (5.9):

$$f(t, T, M-t) = \frac{(M-t)\, r(t, M) - (T-t)\, r(t, T)}{M-T}.$$

(Hint: Eq. (24.3).)

➤ **Exercise 24.2.8** Derive the liquidity premium and the forward rate for the Merton model. Verify that the forward rate goes to minus infinity as the maturity goes to infinity.

➤ **Exercise 24.2.9** Show that

$$\frac{P(t, T)}{M(t)} = \frac{1}{M(T)}$$

in a certain economy, where $M(t) \equiv e^{\int_0^t r(s)\, ds}$ is the money market account. (Hint: Exercise 5.6.6.)

24.2.1 Compounding Frequency

A rate can be expressed in different, yet equivalent, ways, depending on the desired compounding frequency (review Section 3.1). The convention in this chapter is to standardize on continuous compounding for continuous-time models and periodic compounding for discrete-time models unless stated otherwise.

The choice between continuous compounding and periodic compounding does have serious implications for interest rate models. Let r_{e} be the effective annual interest rate and let $r_{\mathrm{c}} \equiv \ln(1+r_{\mathrm{e}})$ be the equivalent continuously compounded rate. Both, we note, are instantaneous rates. When the continuously compounded interest rate is lognormally distributed, Eurodollar futures have negative infinite values [875]. However, this problem goes away if it is the effective rate that is lognormally distributed [781, 782].

➤ **Exercise 24.2.10** Suppose that the effective annual interest rate follows

$$\frac{dr_{\mathrm{e}}}{r_{\mathrm{e}}} = \mu(t)\, dt + \sigma(t)\, dW.$$

Prove that

$$\frac{dr_{\mathrm{c}}(t)}{1-e^{-r_{\mathrm{c}}(t)}} = \left\{ \mu(t) - \frac{1}{2}\left[\, 1 - e^{-r_{\mathrm{c}}(t)}\,\right]\sigma(t)^2 \right\} dt + \sigma(t)\, dW.$$

(The continuously compounded rate is approximately lognormally distributed when $r_{\mathrm{c}}(t) = o(dt)$ as $1 - e^{-r_{\mathrm{c}}(t)} \approx r_{\mathrm{c}}(t) + o(dt^2)$ and converges to a normal distribution when $r_{\mathrm{c}}(t) \to \infty$.)

24.3 **Risk-Neutral Pricing**

The local expectations theory postulates that the expected rate of return of any riskless bond over a single period equals the prevailing one-period spot rate, i.e., for all $t+1 < T$,

$$\frac{E_t[\, P(t+1, T)\,]}{P(t, T)} = 1 + r(t). \tag{24.7}$$

Relation (24.7) in fact follows from the risk-neutral valuation principle, Theorem 13.2.3, which is assumed to hold for continuous-time models. The local expectations theory is thus a consequence of the existence of a risk-neutral probability π, and we may use $E_t^{\pi}[\,\cdot\,]$ in place of $E_t[\,\cdot\,]$. Rewrite Eq. (24.7) as

$$\frac{E_t^{\pi}[\, P(t+1, T)\,]}{1+r(t)} = P(t, T),$$

which says that the current spot rate curve equals the expected spot rate curve one period from now discounted by the short rate. Apply the preceding equality iteratively to obtain

$$\begin{aligned} P(t, T) &= E_t^{\pi}\left[\frac{P(t+1, T)}{1+r(t)} \right] = E_t^{\pi}\left[\frac{E_{t+1}^{\pi}[\, P(t+2, T)\,]}{\{1+r(t)\}\{1+r(t+1)\}} \right] \\ \cdots &= E_t^{\pi}\left[\frac{1}{\{1+r(t)\}\{1+r(t+1)\}\cdots\{1+r(T-1)\}} \right]. \end{aligned} \tag{24.8}$$

Because Eq. (24.7) can also be expressed as

$$E_t[\, P(t+1, T)\,] = F(t, t+1, T),$$

the forward price for the next period is an unbiased estimator of the expected bond price.

In continuous time, the local expectations theory implies that

$$P(t, T) = E_t\big[\, e^{-\int_t^T r(s)\,ds}\,\big], \quad t < T. \tag{24.9}$$

In other words, the actual probability and the risk-neutral probability are identical. Note that $e^{\int_t^T r(s)\,ds}$ is the bank account process, which denotes the rolled-over money market account. We knew that bond prices relative to the money market account are constant in a certain economy (see Exercise 24.2.9). Equation (24.9) extends that proposition to stochastic economies. When the local expectations theory holds, riskless arbitrage opportunities are impossible [232]. The local expectations theory, however, is not the only version of expectations theory consistent with equilibrium [351].

The risk-neutral methodology can be used to price interest rate swaps. Consider an interest rate swap made at time t with payments to be exchanged at times $t_1, t_2, \ldots, t_n$. The fixed rate is c per annum. The floating-rate payments are based on the future annual rates $f_0, f_1, \ldots, f_{n-1}$ at times $t_0, t_1, \ldots, t_{n-1}$. For simplicity, assume that $t_{i+1} - t_i$ is a fixed constant Δt for all i, and that the notional principal is \$1. If $t < t_0$, we have a forward interest rate swap because the first payment is not based on the rate that exists when the agreement is reached. The ordinary swap corresponds to $t = t_0$.

The amount to be paid out at time t_{i+1} is $(f_i - c)\,\Delta t$ for the floating-rate payer. Note that simple rates are adopted here; hence f_i satisfies

$$P(t_i, t_{i+1}) = \frac{1}{1 + f_i \Delta t}.$$

The value of the swap at time t is thus

$$\begin{aligned}
&\sum_{i=1}^{n} E_t^{\pi}\big[\, e^{-\int_t^{t_i} r(s)\,ds}(f_{i-1} - c)\,\Delta t\,\big] \\
&\quad = \sum_{i=1}^{n} E_t^{\pi}\left[e^{-\int_t^{t_i} r(s)\,ds}\left\{ \frac{1}{P(t_{i-1}, t_i)} - (1 + c\Delta t) \right\} \right] \\
&\quad = \sum_{i=1}^{n} [\, P(t, t_{i-1}) - (1 + c\Delta t) \times P(t, t_i)\,] \\
&\quad = P(t, t_0) - P(t, t_n) - c\Delta t \sum_{i=1}^{n} P(t, t_i).
\end{aligned}$$

So a swap can be replicated as a portfolio of bonds. In fact, it can be priced by simple PV calculations. The swap rate, which gives the swap zero value, equals

$$\frac{P(t, t_0) - P(t, t_n)}{\sum_{i=1}^{n} P(t, t_i)\,\Delta t}.$$

The swap rate is the fixed rate that equates the PVs of the fixed payments and the floating payments. For an ordinary swap, $P(t, t_0) = 1$.

➤ **Exercise 24.3.1** Assume that the local expectations theory holds. Prove that the $(T-t)$-time spot rate at time t is less than or equal to $E_t[\int_t^T r(s)\, ds]/(T-t)$, the expected average interest rate between t and T, with equality only if there is no uncertainty about $r(s)$.

➤ **Exercise 24.3.2** Under the local expectations theory, prove that the forward rate $f(t, T)$ is less than the expected spot rate $E_t[r(T)]$ provided that interest rates tend to move together in that $E_t[r(T) \mid \int_t^T r(s)\, ds]$ is increasing in $\int_t^T r(s)\, ds$. (The unbiased expectations theory is hence inconsistent with the local expectations theory. See also Exercise 5.7.3, part (2).)

➤ **Exercise 24.3.3** Show that the calibrated binomial interest rate tree generated by the ideas enumerated in Subsection 23.2.2 (hence the slightly more general tree of Exercise 23.2.3 as well) satisfies the local expectations theory. How about the uncalibrated tree in Fig. 23.6?

➤ **Exercise 24.3.4** Show that, under the unbiased expectations theory,

$$P(t, T) = \frac{1}{[1 + r(t)]\{1 + E_t[r(t+1)]\} \cdots \{1 + E_t[r(T-1)]\}}$$

in discrete time and $P(t, T) = e^{-\int_t^T E_t[r(s)]\, ds}$ in continuous time. (The preceding equation differs from Eq. (24.8), which holds under the local expectations theory.)

➤ **Exercise 24.3.5** The price of a consol that pays dividends continuously at the rate of \$1 per annum satisfies the following expected discounted-value formula:

$$P(t) = E_t^{\pi}\left[\int_t^{\infty} e^{-\int_t^T r(s)\, ds}\, dT\right].$$

Compare this equation with Eq. (24.9) and explain the difference.

➤ **Exercise 24.3.6** Consider an amortizing swap in which the notional principal decreases by $1/n$ dollar at each of the n reset points. The initial principal is \$1. Write a formula for the swap rate.

➤ **Exercise 24.3.7** Argue that a forward interest rate swap is equivalent to a portfolio of one long payer swaption and one short receiver swaption. (The situation is similar to Exercise 12.2.4, which said that a forward contract is equivalent to a portfolio of European options.)

➤ **Exercise 24.3.8** Use the risk-neutral methodology to price interest rate caps, caplets, floors, and floorlets as fixed-income options.

24.4 The Term Structure Equation

In arbitrage pricing, we start exogenously with the bank account process and a primary set of traded securities plus their prices and stochastic processes. We then price a security not in the set by constructing a replicating portfolio consisting of only the primary assets. For fixed-income securities, the primary set of traded securities comprises the zero-coupon bonds and the money market account.

Let the zero-coupon bond price $P(r, t, T)$ follow

$$\frac{dP}{P} = \mu_p \, dt + \sigma_p \, dW.$$

Suppose that an investor at time t shorts one unit of a bond maturing at time s_1 and at the same time buys α units of a bond maturing at time s_2. The net wealth change follows

$$\begin{aligned} &-dP(r, t, s_1) + \alpha \, dP(r, t, s_2) \\ &\quad = [\, -P(r, t, s_1) \, \mu_p(r, t, s_1) + \alpha \, P(r, t, s_2) \, \mu_p(r, t, s_2) \,] \, dt \\ &\qquad + [\, -P(r, t, s_1) \, \sigma_p(r, t, s_1) + \alpha \, P(r, t, s_2) \, \sigma_p(r, t, s_2) \,] \, dW. \end{aligned}$$

Hence, if we pick

$$\alpha \equiv \frac{P(r, t, s_1) \, \sigma_p(r, t, s_1)}{P(r, t, s_2) \, \sigma_p(r, t, s_2)},$$

then the net wealth has no volatility and must earn the riskless return, that is,

$$\frac{-P(r, t, s_1) \, \mu_p(r, t, s_1) + \alpha \, P(r, t, s_2) \, \mu_p(r, t, s_2)}{-P(r, t, s_1) + \alpha \, P(r, t, s_2)} = r.$$

Simplify this equation to obtain

$$\frac{\sigma_p(r, t, s_1) \, \mu_p(r, t, s_2) - \sigma_p(r, t, s_2) \, \mu_p(r, t, s_1)}{\sigma_p(r, t, s_1) - \sigma_p(r, t, s_2)} = r,$$

which becomes

$$\frac{\mu_p(r, t, s_2) - r}{\sigma_p(r, t, s_2)} = \frac{\mu_p(r, t, s_1) - r}{\sigma_p(r, t, s_1)}$$

after rearrangement. Because this equality holds for any s_1 and s_2, we conclude that

$$\frac{\mu_p(r, t, s) - r}{\sigma_p(r, t, s)} \equiv \lambda(r, t) \tag{24.10}$$

for some λ independent of the bond maturity s. As $\mu_p = r + \lambda \sigma_p$, all assets are expected to appreciate at a rate equal to the sum of the short rate and a constant times the asset's volatility.

The term $\lambda(r, t)$ is called the market price of risk because it is the increase in the expected instantaneous rate of return on a bond per unit of risk. The term $\mu_p(r, t, s) - r$ denotes the **risk premium**. Again it is emphasized that the market price of risk must be the same for all bonds to preclude arbitrage opportunities [76].

Assume a Markovian short rate model, $dr = \mu(r, t) \, dt + \sigma(r, t) \, dW$. Then the bond price process is also Markovian. By Eqs. (14.15),

$$\mu_p = \left[-\frac{\partial P}{\partial T} + \mu(r, t) \, \frac{\partial P}{\partial r} + \frac{\sigma(r, t)^2}{2} \, \frac{\partial^2 P}{\partial r^2} \right] \bigg/ P, \quad \sigma_p = \left[\sigma(r, t) \, \frac{\partial P}{\partial r} \right] \bigg/ P \tag{24.11}$$

subject to $P(\,\cdot\,, T, T) = 1$. Note that both μ_p and σ_p depend on P. Substitute μ_p and σ_p above into Eq. (24.10) to obtain the following parabolic partial differential equation:

$$-\frac{\partial P}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = r\,P. \tag{24.12}$$

This is the **term structure equation** [660, 855]. Numerical procedures for solving partial differential equations were covered in Section 18.1. Once P is available, the spot rate curve emerges by means of

$$r(t,T) = -\frac{\ln P(t,T)}{T-t}.$$

The term structure equation actually applies to all interest rate derivatives, the difference being the terminal and the boundary conditions. The equation can also be expressed in terms of duration $D \equiv (\partial P/\partial r)\,P^{-1}$, convexity $C \equiv (\partial^2 P/\partial r^2)\,P^{-1}$, and **time value** $\Theta \equiv (\partial P/\partial t)\,P^{-1}$ as follows:

$$\Theta - [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,D + \frac{1}{2}\,\sigma(r,t)^2 C = r. \tag{24.13}$$

In sharp contrast to the Black–Scholes model, the specification of the short-rate process plus the assumption that the bond market is arbitrage free does *not* determine bond prices uniquely. The reasons are twofold: Interest rate is not a traded security and the market price of risk is not determined *within* the model.

The local expectations theory is usually imposed for convenience. In fact, a probability measure exists such that bonds can be priced as if the theory were true to preclude arbitrage opportunities [492, 493, 746]. In the world in which the local expectations theory holds, $\mu_p(r,t,s) = r$ and the market price of risk is zero (no risk adjustment is needed), and vice versa. The term structure equation becomes

$$-\frac{\partial P}{\partial T} + \mu(r,t)\,\frac{\partial P}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 P}{\partial r^2} = r\,P, \tag{24.14}$$

and bond price dynamics (24.11) is simplified to

$$dP = r\,P\,dt + \sigma(r,t)\,\frac{\partial P}{\partial r}\,dW.$$

The market price of risk is usually assumed to be zero unless stated otherwise. We can also derive the bond pricing formula under local expectations theory (24.9) by assuming that the short rate follows the risk-neutral process:

$$dr = [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,dt + \sigma(r,t)\,dW.$$

➤ **Exercise 24.4.1** Suppose a liability has been duration matched by a portfolio. What can we say about the relations among their respective time values and convexities?

➤ **Exercise 24.4.2** Argue that European options on zero-coupon bonds satisfy the term structure equation subject to appropriate boundary conditions.

➤ **Exercise 24.4.3** Describe an implicit method for term structure equation (24.14). You may simplify the short rate process to $dr = \mu(r)\,dt + \sigma(r)\,dW$. Assume that $\mu(r) \geq 0$ and $\sigma(0) = 0$ to avoid negative short rates.

➤ **Exercise 24.4.4** Consider a futures contract on a zero-coupon bond with maturity date T_1. The futures contract expires at time T. Let $F(r, t)$ denote the futures price that follows $dF/F = \mu_f \, dt + \sigma_f \, dW$. Prove that F satisfies

$$-\frac{\partial F}{\partial T} + [\,\mu(r,t) - \lambda(r,t)\,\sigma(r,t)\,]\,\frac{\partial F}{\partial r} + \frac{1}{2}\,\sigma(r,t)^2\,\frac{\partial^2 F}{\partial r^2} = 0$$

subject to $F(\cdot, T) = P(\cdot, T, T_1)$, where $\lambda \equiv \mu_f/\sigma_f$.

24.5 Forward-Rate Process

Assume that the zero-coupon bond price follows $dP(t, T)/P(t, T) = \mu_p(t, T)\,dt + \sigma_p(t, T)\,dW$ as before. Then the process followed by the instantaneous forward rate is [76, 477]

$$df(t, T) = \left[\sigma_p(t, T)\,\frac{\partial \sigma_p(t, T)}{\partial T} - \frac{\partial \mu_p(t, T)}{\partial T}\right] dt - \frac{\partial \sigma_p(t, T)}{\partial T}\,dW.$$

In a risk-neutral economy, the forward-rate process follows

$$df(t, T) = \left[\sigma(t, T)\int_t^T \sigma(t, s)\,ds\right] dt - \sigma(t, T)\,dW, \tag{24.15}$$

where

$$\sigma(t, T) \equiv \frac{\partial \sigma_p(t, T)}{\partial T},$$

because $\mu_p(t, T) = r(t)$ and

$$\sigma_p(t, T) = \int_t^T \frac{\partial \sigma_p(t, s)}{\partial s}\,ds.$$

➤ **Exercise 24.5.1** Justify Eq. (24.15) directly.

➤ **Exercise 24.5.2** What should $\sigma_p(t, T, P)$ be like for $df(t, T)$'s diffusion term to have the functional form $\psi(t)\,f(t, T)$? The dependence of σ_p on $P(t, T)$ is made explicit here.

24.6 The Binomial Model with Applications

The analytical framework can be nicely illustrated with the binomial model. Suppose the bond price P can move with probability q to Pu and probability $1 - q$ to Pd, where $u > d$:

$$\begin{tikzcd} & Pd \\ P \arrow[ur, "1-q"] \arrow[dr, "q"'] & \\ & Pu \end{tikzcd}.$$

Over the period, the bond's expected rate of return is

$$\widehat{\mu} \equiv \frac{q\,Pu + (1-q)\,Pd}{P} - 1 = qu + (1-q)\,d - 1, \tag{24.16}$$

and the variance of that return rate is

$$\widehat{\sigma}^2 \equiv q(1-q)(u-d)^2. \tag{24.17}$$

Among the bonds, the one whose maturity is only one period away will move from a price of $1/(1+r)$ to its par value \$1. This is the money market account modeled by the short rate. The market price of risk is defined as $\lambda \equiv (\widehat{\mu} - r)/\widehat{\sigma}$, analogous to Eq. (24.10). The same arbitrage argument as in the continuous-time case can be used to show that λ is independent of the maturity of the bond (see Exercise 24.6.2).

Now change the probability from q to

$$p \equiv q - \lambda\sqrt{q(1-q)} = \frac{(1+r)-d}{u-d}, \tag{24.18}$$

which is independent of bond maturity and q. The bond's expected rate of return becomes

$$\frac{pPu + (1-p)\,Pd}{P} - 1 = pu + (1-p)\,d - 1 = r.$$

The local expectations theory hence holds under the new probability measure p.[1]

➤ **Exercise 24.6.1** Verify Eq. (24.18).

➤ **Exercise 24.6.2** Prove that the market price of risk is independent of bond maturity. (Hint: Assemble two bonds in such a way that the portfolio is instantaneously riskless.)

➤ **Exercise 24.6.3** Assume in a period that the bond price can go from \$1 to P_{u} or P_{d} and that the value of a derivative can go from \$1 to V_{u} or V_{d}. (1) Show that a portfolio of \$1 worth of bonds and $(P_{\mathrm{d}} - P_{\mathrm{u}})/(V_{\mathrm{u}} - V_{\mathrm{d}})$ units of the derivative is riskless. (2) Prove that these many derivatives are worth

$$\frac{(R - P_{\mathrm{d}})\,V_{\mathrm{u}} + (P_{\mathrm{u}} - R)\,V_{\mathrm{d}}}{(P_{\mathrm{u}} - P_{\mathrm{d}})\,R}$$

in total, where $R \equiv 1 + r$ is the gross riskless return.

➤ **Exercise 24.6.4** Consider the symmetric random walk for modeling the short rate, $r(t+1) = \alpha + \rho r(t) \pm \sigma$. Let V denote the current value of an interest rate derivative, V_u its value at the next period if rates rise, and V_d its value at the next period if rates fall. Define $u \equiv e^{\alpha+\rho r(t)+\sigma}/P(t, t+2)$ and $d \equiv e^{\alpha+\rho r(t)-\sigma}/P(t, t+2)$, so they are the gross one-period returns on the two-period zero-coupon bond when rates go up and down, respectively. (1) Show that a portfolio consisting of \$$B$ worth of one-period bonds and two-period zero-coupon bonds with face value \$$\Delta$ to match the value of the derivative requires

$$\Delta = \frac{V_u - V_d}{(u-d)\,P(t, t+2)}, \qquad B = \frac{uV_d - dV_u}{(u-d)\,e^{r(t)}}.$$

(2) Prove that

$$V = \frac{pV_u + (1-p)\,V_d}{e^{r(t)}},$$

where $p \equiv (e^{r(t)} - d)/(u-d)$.

$$\begin{array}{lcl} & \nearrow & 92.593\ (=100/1.08) \\ 90.703 & & \\ & \searrow & 98.039\ (=100/1.02) \end{array} \qquad \begin{array}{lcl} & \nearrow & 100 \\ 96.154 & & \\ & \searrow & 100 \end{array}$$

Figure 24.1: Bond price processes. The price process of the 2-year zero-coupon bond is on the left and that of the 1-year zero-coupon bond is on the right.

➤ **Exercise 24.6.5** To use the objective probability q in pricing, we should discount by the risk-adjusted discount factor, $1 + r + \lambda\widehat{\sigma} = 1 + \widehat{\mu}$. Prove this claim.

24.6.1 Numerical Examples

The following numerical examples involve the pricing of fixed-income options, MBSs, and derivative MBSs under this spot rate curve:

Year	**1**	**2**
spot rate	4%	5%

Assume that the 1-year rate (short rate) can move up to 8% or down to 2% after a year:

$$\begin{array}{lcl} & \nearrow & 8\% \\ 4\% & & \\ & \searrow & 2\% \end{array}.$$

No real-world probabilities are specified. The prices of 1- and 2-year zero-coupon bonds are, respectively, $100/1.04 = 96.154$ and $100/(1.05)^2 = 90.703$. Furthermore, they follow the binomial processes in Fig. 24.1.

The pricing of derivatives can be simplified if we assume that investors are risk-neutral. If all securities have the same expected one-period rate of return, the riskless rate, then

$$(1-p) \times \frac{92.593}{90.703} + p \times \frac{98.039}{90.703} - 1 = 4\%,$$

where p denotes the risk-neutral probability of an up move in rates. Solving this equation leads to $p = 0.319$. Interest rate contingent claims can be priced under this probability.

➤ **Exercise 24.6.6** We could not have obtained the unique risk-neutral probability had we not imposed a prevailing term structure that must be matched. Explain.

➤ **Exercise 24.6.7** Verify the risk-neutral probability $p = 0.319$ with Eq. (24.18) instead.

24.6.2 Fixed-Income Options

A 1-year European call on the 2-year zero with a \$95 strike price has the payoffs

$$\begin{array}{lcl} & \nearrow & 0.000 \\ C & & \\ & \searrow & 3.039 \end{array}.$$

To solve for the option value C, we replicate the call by a portfolio of x 1-year and y 2-year zeros. This leads to the simultaneous equations,

$$x \times 100 + y \times 92.593 = 0.000,$$
$$x \times 100 + y \times 98.039 = 3.039,$$

which give $x = -0.5167$ and $y = 0.5580$. Consequently,

$$C = x \times 96.154 + y \times 90.703 \approx 0.93$$

to prevent arbitrage. Note that this price is derived without assuming any version of an expectations theory; instead, we derive the arbitrage-free price by replicating the claim with a money market instrument and the claim's underlying asset. The price of an interest rate contingent claim does not depend directly on the probabilities. In fact, the dependence holds only indirectly by means of the current bond prices (see Exercise 24.6.6).

An equivalent method is to utilize risk-neutral pricing. The preceding call option is worth

$$C = \frac{(1-p) \times 0 + p \times 3.039}{1.04} \approx 0.93,$$

the same as before. This is not surprising, as arbitrage freedom and the existence of a risk-neutral economy are equivalent.

➤ **Exercise 24.6.8 (Dynamic Immunization)** Explain why the replication idea solves the problem of arbitrage opportunities in immunization against parallel shifts raised in Subsection 5.8.2.

24.6.3 Futures and Forward Prices

A 1-year futures contract on the 1-year rate has a payoff of $100 - r$, where r is the 1-year rate at maturity, as shown below:

$$\begin{tikzcd}[column sep=small, row sep=small] & 92\ (=100-8) \\ F \arrow[ur] \arrow[dr] & \\ & 98\ (=100-2) \end{tikzcd} .$$

As the futures price F is the expected future payoff (see Exercise 13.2.11), $F = (1-p) \times 92 + p \times 98 = 93.914$. On the other hand, the forward price for a 1-year forward contract on a 1-year zero-coupon bond equals $90.703/96.154 = 94.331\%$. The forward price exceeds the futures price, as Exercise 12.3.3 predicted.

24.6.4 Mortgage-Backed Securities

Consider a 5%-coupon, 2-year MBS without amortization, prepayments, and default risk. Its cash flow and price process are illustrated in Fig. 24.2, and its fair price is

$$M = \frac{(1-p) \times 102.222 + p \times 107.941}{1.04} = 100.045.$$

Identical results could have been obtained by no-arbitrage considerations.

In reality mortgages can be prepaid. Assume that the security in question can be prepaid at par and such decisions are rational in that it will be prepaid only when

$$
\begin{array}{ccccc}
 & & & & 105 \\
 & & & \nearrow & \\
 & & 5 & & \\
 & \nearrow & & \searrow & \\
 & & & & 105 \\
0 & & & & \\
 & & & & 105 \\
 & \searrow & & \nearrow & \\
 & & 5 & & \\
 & & & \searrow & \\
 & & & & 105
\end{array}
\qquad\qquad
\begin{array}{ccl}
 & & 102.222\ (= 5 + (105/1.08)) \\
 & \nearrow & \\
M & & \\
 & \searrow & \\
 & & 107.941\ (= 5 + (105/1.02))
\end{array}
$$

Figure 24.2: MBS's cash flow and price process. The left diagram depicts the cash flow, and the right diagram illustrates the price process.

its price is higher than par. Prepayment will hence occur only in the "down" state when the security is worth 102.941 (excluding coupon). The price therefore follows the process

$$
M \begin{array}{l} \nearrow \ 102.222 \\ \searrow \ 105 \end{array},
$$

and the security is worth

$$M = \frac{(1-p) \times 102.222 + p \times 105}{1.04} = 99.142.$$

We go on to price **stripped mortgage-backed securities** (**SMBS**s) derived from the above prepayable mortgage. The cash flow of the **principal-only** (**PO**) **strip** comes from the mortgage's principal cash flow, whereas that of the **interest-only** (**IO**) **strip** comes from the interest cash flow (see Fig. 24.3(a)). Their prices hence follow the processes in Fig. 24.3(b). The fair prices are

$$\text{PO} = \frac{(1-p) \times 92.593 + p \times 100}{1.04} = 91.304,$$

$$\text{IO} = \frac{(1-p) \times 9.630 + p \times 5}{1.04} = 7.839.$$

Of course, $\text{PO} + \text{IO} = M$.[2]

The above formulas reveal that IO and PO strips react to changes in p differently. The value of the PO strip rises with increasing p, whereas that of the IO strip declines with increasing p. Suppose the market price of risk is positive so that the real-world probability q exceeds the risk-neutral probability p. Then the market value of the PO strip, like that of the zero-coupon bond, is lower than its discounted expected value under q, which compensates the investors for its riskiness by earning more than the riskless return on average. The market value of the IO strip, however, is higher than its discounted expected value under q, making the security earn *less than* the riskless rate even though it is a risky security. The reason is that the IO's price correlates negatively with the zero-coupon bond's.

Suppose the mortgage is split into half **floater** and half **inverse floater**. Let the floater (FLT) receive the 1-year rate. Then the inverse floater (INV) must have a

$$
\begin{tikzcd}[row sep=tiny, column sep=small]
 & \text{PO:} & 100 \\
 & 0 \arrow[ur] \arrow[dr] & \\
 & & 100 \\
0 \arrow[uur] \arrow[ddr] & & \\
 & & 0 \\
 & 100 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny, column sep=small]
 & \text{IO:} & 5 \\
 & 5 \arrow[ur] \arrow[dr] & \\
 & & 5 \\
0 \arrow[uur] \arrow[ddr] & & \\
 & & 0 \\
 & 5 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 92.593 \\
\text{PO} \arrow[ur] \arrow[dr] & \\
 & 100
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 9.630 \\
\text{IO} \arrow[ur] \arrow[dr] & \\
 & 5
\end{tikzcd}
$$

(b)

Figure 24.3: Cash flows and price processes of PO and IO strips. The price 9.630 is derived from $5 + (5/1.08)$.

coupon rate of $10\% - 1\text{-year rate}$ to make the overall coupon rate 5%. Their cash flows as percentages of par and values are shown in Fig. 24.4. The current prices are

$$\text{FLT} = \frac{1}{2} \times \frac{104}{1.04} = 50,$$

$$\text{INV} = \frac{1}{2} \times \frac{(1-p) \times 100.444 + p \times 106}{1.04} = 49.142.$$

➤ **Exercise 24.6.9** Explain why all the securities covered up to now have the same 1-year return of 4% in a risk-neutral economy.

$$
\begin{tikzcd}[row sep=tiny, column sep=small]
 & \text{FLT:} & 108 \\
 & 4 \arrow[ur] \arrow[dr] & \\
 & & 108 \\
0 \arrow[uur] \arrow[ddr] & & \\
 & & 0 \\
 & 104 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny, column sep=small]
 & \text{INV:} & 102 \\
 & 6 \arrow[ur] \arrow[dr] & \\
 & & 102 \\
0 \arrow[uur] \arrow[ddr] & & \\
 & & 0 \\
 & 106 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 104 \\
\text{FLT} \arrow[ur] \arrow[dr] & \\
 & 104
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny, column sep=small]
 & 100.444 \\
\text{INV} \arrow[ur] \arrow[dr] & \\
 & 106
\end{tikzcd}
$$

(b)

Figure 24.4: Cash flows and price processes of floater and inverse floater. The floater's price in the up node, 104, is derived from $4 + (108/1.08)$, and the inverse floater's price, 100.444, is derived from $6 + (102/1.08)$.

➤ **Exercise 24.6.10** Verify that the value of a European put, like that of the IO, declines with increasing p.

24.7 Black–Scholes Models

A few interest rate models are based on the Black–Scholes option pricing model or the related Black model. They differ mainly in whether it is the price or the yield that is being modeled. As simple as these models are and despite some difficulties, they usually provide adequate results for options with short maturities [305, 456].

24.7.1 Price Models

Suppose the long-term bond price follows geometric Brownian motion much like the stock price. As with stock options, options on the bond can be replicated by continuous trading of these bonds and borrowing at the prevailing short rate. Hence Black–Scholes formulas (12.16) apply.

This pricing model has several problems. It is inconsistent to assume that the short-term rate is a constant – as dictated by the Black–Scholes option pricing model – but the long bond price is uncertain. Another objection is about the volatility of bond prices. Although this volatility must first increase with the passage of time, it should eventually decrease toward zero because the bond converges to its par value at maturity. In other words, the price uncertainty is small in the immediate future and near bond maturity but large between these two extreme points (see Fig. 24.5). This unique property is not captured by the preceding model, which assumes that the variance of the bond price grows linearly in time.

The lognormal assumption for the zero-coupon bond price means that the continuously compounded interest rate is normally distributed. Three problems are associated with this distribution: the possibility of negative interest rates, the independence of interest rate volatility from the interest rate level, and the possibility of the bond price's rising above its sum of cash flows.

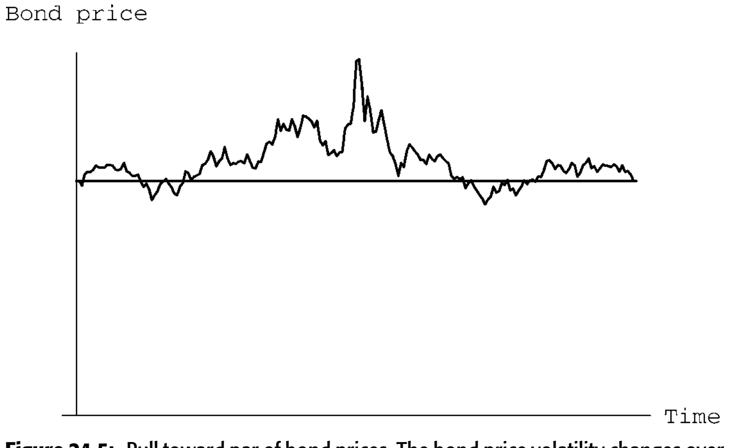

Figure 24.5: Pull toward par of bond prices. The bond price volatility changes over time.

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➤ **Exercise 24.7.1** Assume that interest rates cannot be negative. (1) Why should a call on a zero-coupon bond with a strike price of \$102 be worth zero given a par value of \$100? (2) The Black–Scholes formula gives a positive call value. Why?

➤ **Exercise 24.7.2** Consider a call on a zero-coupon bond with an expiration date that coincides with the bond's maturity. Does the call premium depend on the interest rate movements between now and the expiration date?

24.7.2 Yield Models

Consider the alternative model that models the yield to maturity, not the bond price, as geometric Brownian motion. It solves a few problems that were poisoning the preceding model. To start with, because yield to maturity now has the lognormal distribution, negative interest rates are ruled out. Furthermore, as the bond price at any time is derived from the yield's probability distribution, it will reflect both the decrease in price volatility and the pull toward par as the bond matures.

This model has its own difficulties. In the binomial setting, it is known that the one-period riskless rate must be between the one-period bond returns of up and down yield shifts to avoid arbitrage (see Exercise 9.2.1). However, if the riskless rate is a constant, preventing such opportunities may be difficult, especially in light of the fact that both the up and the down returns must eventually approach one because of the pull toward par. Another problem is that the yield volatility is constant over the life of the bond; in reality, however, it decreases as the maturity increases.

24.7.3 Models Based on the Brownian Bridge

Because zero-coupon bonds move toward par at maturity, a Brownian bridge process seems ideal for modeling their price dynamics [50]. Recall that a Brownian bridge process $\{B(t), 0 \le t \le T\}$ can be defined as

$$B(t) = W(t) - \frac{t}{T}\, W(T).$$

Note that $B(0) = B(T) = 0$. The bond price model $P(t, T) = e^{r(t-T)+\sigma B(t)}$ clearly has the desirable pull-to-par property because $P(T, T) = 1$. However, certain models based on Brownian bridge are not arbitrage free, thus not sound [193].

Additional Reading

Consult [38, 76, 290, 510, 691, 725, 731] for the theory behind the term structure models. We followed [34] in Subsection 24.6.1.

NOTES

1. Note that Eq. (24.18) is identical to risk-neutral probability (9.5) of the BOPM.
2. You can order either whole milk or skim milk plus the right amount of cream. They cost the same!