CHAPTER ELEVEN

Extensions of Options Theory

As I never learnt mathematics, so I have had to think.
Joan Robinson (1903–1983)

This chapter samples various option instruments and presents important applications of the option pricing theory. Algorithms are described for a few nontrivial options.

11.1 Corporate Securities

With the underlying asset interpretated as the total value of the firm, Black and Scholes observed that the option pricing methodology can be applied to pricing corporate securities [87, 236, 658]. In the following analysis, it is assumed that (1) a firm can finance payouts by the sale of assets and (2) if a promised payment to an obligation other than stock is missed, the claim holders take ownership of the firm and the stockholders get nothing.

11.1.1 Risky Zero-Coupon Bonds and Stock

Consider a firm called XYZ.com. It has a simple capital structure: n shares of its own common stock S and zero-coupon bonds with an aggregate par value of X. The fundamental question is, what are the values of the bonds B and the stock?

On the bonds' maturity date, if the total value of the firm V^* is less than the bondholders' collective claim X, the firm declares bankruptcy and the stock becomes worthless. On the other hand, if $V^* > X$, then the bondholders obtain X and the stockholders $V^* - X$. The following table shows their respective payoffs:

	$V^* \leq X$	$V^* > X$
Bonds	V^*	X
Stock	0	$V^* - X$

The stock is therefore a call on the total value of the firm with a strike price of X and an expiration date equal to the maturity date of the bonds. It is this call that provides the limited liability for the stockholders. The bonds are a covered call on the total value of the firm.

Let C stand for this call and V stand for the total value of the firm. Then $nS = C$ and $B = V - C$. Knowing C thus amounts to knowing how the value of the firm is distributed between the stockholders and the bondholders. Whatever the value of C, the total value of the stock and bonds at maturity remains V^*. Hence the relative size of debt and equity is irrelevant to the firm's current value V, which is the expected PV of V^*.

From Theorem 9.3.4 and the put–call parity

$$nS = VN(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}),$$
$$B = VN(-x) + Xe^{-r\tau} N(x - \sigma\sqrt{\tau}),$$

where

$$x \equiv \frac{\ln(V/X) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}.$$

The continuously compounded yield to maturity of the firm's bond is hence $(1/\tau)\ln(X/B)$. The **default premium** is defined as the yield difference between risky and riskless bonds:

$$(1/\tau)\ln(X/B) - r = -\frac{1}{\tau}\ln\left(N(-z) + \frac{1}{\omega}N(z - \sigma\sqrt{\tau})\right), \tag{11.1}$$

where $\omega \equiv Xe^{-r\tau}/V$ and $z \equiv (\ln\omega)/(\sigma\sqrt{\tau}) + (1/2)\sigma\sqrt{\tau} = -x + \sigma\sqrt{\tau}$. Note that ω is the **debt-to-total-value ratio**. The volatility of the value of the firm, σ, can be looked on as a measure of **operating risk**. The default premium depends on only the firm's capital structure, operating risk, and debt maturity. The concept of default premium is a special case of static spread in Subsection 5.6.2.

➤ **Exercise 11.1.1** Argue that a loan guarantee that makes up any shortfalls in payments to the bondholders is a put with a strike price of B. The tacit assumption here is that the guarantor does not default.

➤ **Exercise 11.1.2** Prove Eq. (11.1).

➤ **Exercise 11.1.3** Verify the following claims and explain them in simple English: (1) $\partial B/\partial V > 0$, (2) $\partial B/\partial X > 0$, and (3) $\partial B/\partial\tau < 0$.

Numerical Illustrations

Suppose that XYZ.com's assets consist of 1,000 shares of Merck as of March 20, 1995, when Merck's market value per share is \$44.5 (see Fig. 11.1). XYZ.com's securities consist of 1,000 shares of common stock and 30 zero-coupon bonds maturing on July 21, 1995. Each bond promises to pay \$1,000 at maturity. Therefore $n = 1000$, $V = 44.5 \times n = 44500$, and $X = 30 \times n = 30000$. The Merck option relevant to our question is the July call with a strike price of $X/n = 30$ dollars. Such an option exists and is selling for \$15.25. So XYZ.com's stock is worth $15.25 \times n = 15250$ dollars, and the entire bond issue is worth $B = 44500 - 15250 = 29250$ dollars, or \$975 per bond.

The XYZ.com bonds are equivalent to a default-free zero-coupon bond with \$$X$ par value plus n written European puts on Merck at a strike price of \$30 by the put–call parity. The difference between B and the price of the default-free bond is precisely the value of these puts. Figure 11.2 shows the total market values of

			–Call–		–Put–	
Option	*Strike*	*Exp.*	*Vol.*	*Last*	*Vol.*	*Last*
Merck	30	Jul	328	15 1/4	...	...
44 1/2	35	Jul	150	9 1/2	10	1/16
44 1/2	40	Apr	887	4 3/4	136	1/16
44 1/2	40	Jul	220	5 1/2	297	1/4
44 1/2	40	Oct	58	6	10	1/2
44 1/2	45	Apr	3050	7/8	100	1 1/8
44 1/2	45	May	462	1 3/8	50	1 3/8
44 1/2	45	Jul	883	1 15/16	147	1 3/4
44 1/2	45	Oct	367	2 3/4	188	2 1/16

Figure 11.1: Merck option quotations. Source: Fig. 7.4.

the XYZ.com stock and bonds under various debt amounts X. For example, if the promised payment to bondholders is \$45,000, the relevant option is the July call with a strike price of $45000/n = 45$ dollars. Because that option is selling for \$1 15/16, the market value of the XYZ.com stock is $(1 + 15/16) \times n = 1937.5$ dollars. The market value of the stock decreases as the debt–equity ratio increases.

Conflicts between Stockholders and Bondholders

Options and corporate securities have one important difference: A firm can change its capital structure, but an option's terms cannot be changed after it is issued. This means that parameters such volatility, dividend, and strike price are under partial control of the stockholders.

Suppose XYZ.com issues 15 more bonds with the same terms in order to buy back stock. The total debt is now $X = 45{,}000$ dollars. Figure 11.2 says that the total market value of the bonds should be \$42,562.5. The *new* bondholders therefore pay $42562.5 \times (15/45) = 14187.5$ dollars, which is used by XYZ.com to buy back shares. The remaining stock is worth \$1,937.5. The stockholders therefore gain

$$(14187.5 + 1937.5) - 15250 = 875$$

dollars. The *original* bondholders lose an equal amount:

$$29250 - \left(\frac{30}{45} \times 42562.5\right) = 875. \tag{11.2}$$

This simple calculation illustrates the inherent conflicts of interest between stockholders and bondholders.

Promised payment to bondholders X	*Current market value of bonds B*	*Current market value of stock nS*	*Current total value of firm V*
30,000	29,250.0	15,250.0	44,500
35,000	35,000.0	9,500.0	44,500
40,000	39,000.0	5,500.0	44,500
45,000	42,562.5	1,937.5	44,500

Figure 11.2: Distribution of corporate value under alternative capital structures. Numbers are based on Fig. 11.1.

As another example, suppose the stockholders distribute $14,833.3 cash dividends by selling $(1/3) \times n$ Merck shares. They now have $14,833.3 in cash plus a call on $(2/3) \times n$ Merck shares. The strike price remains $X = 30000$. This is equivalent to owning two-thirds of a call on n Merck shares with a total strike price of $45,000. Because n such calls are worth $1,937.5 from Fig. 11.1, the total market value of the XYZ.com stock is $(2/3) \times 1937.5 = 1291.67$ dollars. The market value of the XYZ.com bonds is hence $(2/3) \times n \times 44.5 - 1291.67 = 28375$ dollars. As a result, the stockholders gain

$$(14833.3 + 1291.67) - 15250 \approx 875$$

dollars, and the bondholders watch their value drop from $29,250 to $28,375, a loss of $875.

Bondholders usually loathe the stockholders' taking unduly risky investments. The option theory explains it by pointing out that higher volatility increases the likelihood that the call will be exercised, to the financial detriment of the bondholders.

➤ **Exercise 11.1.4** If the bondholders can lose money in Eq. (11.2), why do they not demand lower bond prices?

➤ **Exercise 11.1.5** Repeat the steps leading to Eq. (11.2) except that, this time, the firm issues only five bonds instead of fifteen.

➤ **Exercise 11.1.6** Suppose that a holding company's securities consist of 1,000 shares of Microsoft common stock and 55 zero-coupon bonds maturing on the same date as the Microsoft April calls. Figure out the stockholders' gains (hence the original bondholders' losses) if the firm issues 5, 10, and 15 more bonds, respectively. Consult Fig. 7.4 for the market quotes.

➤ **Exercise 11.1.7** Why are dividends bad for the bondholders?

Subordinated Debts

Suppose that XYZ.com adds a **subordinated** (or **junior**) **debt** with a face value X_j to its capital structure. The original debt, with a face value of X_s, then becomes the **senior debt** and takes priority over the subordinated debt in case of default. Let both debts have the same maturity. The following table shows the payoffs of the various securities:

	$V^* \leq X_s$	$X_s < V^* \leq X_s + X_j$	$X_s + X_j < V^*$
Senior debt	V^*	X_s	X_s
Junior debt	0	$V^* - X_s$	X_j
Stock	0	0	$V^* - X_s - X_j$

The subordinated debt has the same payoff as a portfolio of a long X_s call and a short $X_s + X_j$ call – a bull call spread, in other words.

11.1.2 Warrants

Warrants represent the right to buy shares from the corporation. Unlike a call, a corporation issues warrants against its own stock, and *new* shares are issued when

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a warrant is exercised. Most warrants have terms between 5 and 10 years, although perpetual warrants exist. Warrants are typically protected against stock splits and cash dividends.

Consider a corporation with n shares of stock and m European warrants. Each warrant can be converted into one share on payment of the strike price X. The total value of the corporation is therefore $V = nS + mW$, where W denotes the current value of each warrant. At expiration, if it becomes profitable to exercise the warrants, the value of each warrant should be equal to

$$W^* = \frac{1}{n+m}(V^* + mX) - X = \frac{1}{n+m}(V^* - nX),$$

where V^* denotes the total value of the corporation just before the conversion. It will be optimal to exercise the warrants if and only if $V^* > nX$. A European warrant is therefore a European call on one $(n+m)$th of the total value of the corporation with a strike price of $Xn/(n+m)$ – equivalently, $n/(n+m)$ European call on one nth of the total value of the corporation (or $S + (m/n)\,W$) with a strike price of X. Hence

$$W = \frac{n}{n+m}\,C(W), \tag{11.3}$$

where $C(W)$ is the Black–Scholes formula for the European call but with the stock price replaced with $S + (m/n)\,W$. The value of W can be solved numerically given S.

➢ **Programming Assignment 11.1.8** (1) Write a program to solve Eq. (11.3). (2) Write a binomial tree algorithm to price American warrants.

11.1.3 Callable Bonds

Corporations issue callable bonds so that the debts can be refinanced under better terms if future interest rates fall or the corporation's financial situation improves. Consider a corporation with two classes of obligations: n shares of common stock and a single issue of callable bonds. The bonds have an aggregate face value of X, and the stockholders have the right to call the bonds at any time for a total price of X_c. Whenever the bonds are called before they mature, the payoff to the stockholders is $V - X_c$. The stock is therefore equivalent to an American call on the total value of the firm with a strike price of X_c before expiration and X at expiration.

11.1.4 Convertible Bonds

A **convertible bond (CB)** is like an ordinary bond except that it can be converted into new shares at the discretion of its owner. Consider a non-dividend-paying corporation with two classes of obligations: n shares of common stock and m zero-coupon CBs. Each CB can be converted into k newly issued shares at maturity (k is called the **conversion ratio**). If the bonds are not converted, their holders will receive X in aggregate at maturity.

The bondholders will own a fraction $\lambda \equiv (mk)/(n+mk)$ of the firm if conversion is chosen (λ is called the **dilution factor**). It makes sense to convert only if the part of the total market value of the corporation that is due the bondholders after conversion, or λV^*, exceeds X, i.e., $V^* > X/\lambda$. The payoff of the bond at maturity is (1) V^* if $V^* \leq X$, (2) X if $X < V^* \leq X/\lambda$ because it will not be converted, or (3) λV^* if $X/\lambda < V^*$.

It is in the interest of stockholders to pursue risky projects because higher volatility increases the stock price. The corporation may also withhold positive inside information from the bondholders; once the favorable information is released, the corporation calls the bonds. CBs solve both problems by giving the bondholders the option to take equity positions [837].

➤ **Exercise 11.1.9** Replicate the zero-coupon CB with the total value of the corporation and European calls on that value.

➤ **Exercise 11.1.10** (1) Replicate the zero-coupon CB with zero-coupon bonds and European calls on a fraction of the total value of the corporation. (2) Replicate the zero-coupon CB with zero-coupon bonds and warrants. (3) Show that early conversion is not optimal.

Convertible Bonds with Call Provisions

Many CBs contain call provisions. When the CBs are called, their holders can either convert the CB or redeem it at the call price. The call strategy is intended to minimize the value of the CBs. In the following analysis, assume that the CBs can be called any time before their maturity and that the corporation's value follows a continuous path without jumps.

Consider the same corporation again. In particular, the aggregate face value of the CBs is X, the aggregate call price is P, and $P \geq X$. The bondholders will own a fraction λ of the firm on conversion. We first argue that it is not optimal to call the CBs when $\lambda V < P$. As the following table shows, not calling the CBs leaves the bondholders at maturity with a value of V^*, X, or λV^* if the holders choose not to convert them earlier.

	$V^* \leq X$	$X < V^* \leq X/\lambda$	$X/\lambda < V^*$
Immediate call (PV)	P	P	P
No call throughout (FV at maturity)	V^*	X	λV^*
No call throughout (PV)	V	$\text{PV}(X)$	λV

The PVs in all three cases are either less than or equal to P. Calling the CBs immediately is hence not optimal.

We now argue that it is not optimal to call the CBs after $\lambda V = P$ happens. Calling the CBs when $\lambda V = P$ leaves the bondholders with λV^* at maturity. The bondholders' terminal wealth if the CBs are not called is tabulated in the following table.

	$V^* < X$	$X \leq V^* < X/\lambda$	$X/\lambda \leq V^*$
No call throughout	V^*	X	λV^*
Call sometime *in the future*	λV^*	λV^*	λV^*

Not calling the CBs hence may result in a higher terminal value for the bondholders than calling them. In summary, the optimal call strategy is to call the CBs the first time $\lambda V = P$ happens. More general settings will be covered in Subsection 15.3.7.

➤ **Exercise 11.1.11** Complete the proof by showing that it is not optimal to call the CBs when $\lambda V > P$.

11.2 Barrier Options

Options whose payoff depends on whether the underlying asset's price reaches a certain price level H are called **barrier options**. For example, a **knock-out option** is like an ordinary European option except that it ceases to exist if the barrier H is reached by the price of its underlying asset. A call knock-out option is sometimes called a **down-and-out option** if $H < X$. A put knock-out option is sometimes called an **up-and-out option** when $H > X$. A **knock-in option**, in contrast, comes into existence if a certain barrier is reached. A **down-and-in option** is a call knock-in option that comes into existence only when the barrier is reached and $H < X$. An **up-and-in option** is a put knock-in option that comes into existence only when the barrier is reached and $H > X$. Barrier options have been traded in the United States since 1967 and are probably the most popular among the over-the-counter options [370, 740, 894].

The value of a European down-and-in call on a stock paying a dividend yield of q is

$$Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(x) - Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(x-\sigma\sqrt{\tau}), \quad S \geq H, \tag{11.4}$$

where

$$x \equiv \frac{\ln(H^2/(SX)) + (r-q+\sigma^2/2)\,\tau}{\sigma\sqrt{\tau}}$$

and $\lambda \equiv (r-q+\sigma^2/2)/\sigma^2$ (see Fig. 11.3). A European down-and-out call can be priced by means of the **in–out parity** (see Comment 11.2.1). The value of a European

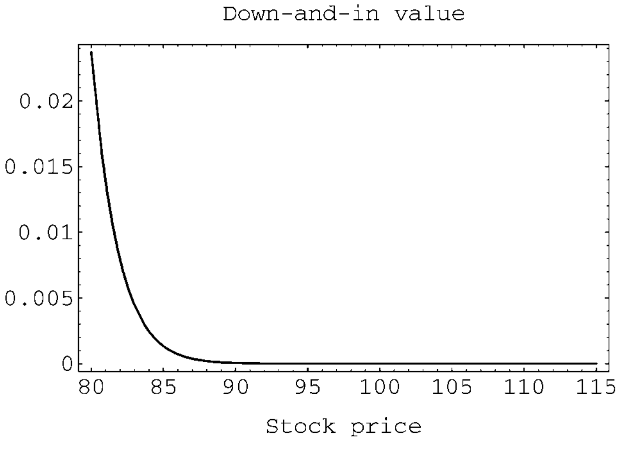

Figure 11.3: Value of down-and-in option. Plotted is the down-and-in option value as a function of the stock price with barrier $H = 80$. The other parameters are identical to those for the call in Fig. 7.3: $X = 95$, $\sigma = 0.25$, $\tau = 1/12$, and $r = 0.08$. Note the dramatic difference between the two plots.

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up-and-in put is

$$Xe^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(-x+\sigma\sqrt{\tau}) - Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(-x),$$

where $S \leq H$. A European up-and-out call can be priced by means of the in–out parity (see Exercise 11.2.1, part(2)). The formulas are due to Merton [660]. See [660] or Exercise 17.1.6 for proofs.

Backward induction can be used to price barrier options on a binomial tree. As the binomial tree algorithm works backward in time, it checks if the barrier price is reached by the underlying asset and, if so, replaces the option value with an appropriate value (see Fig. 11.4). In practice, the barrier is often monitored discretely, say at the end of the trading day, and the algorithm should reflect that.

➤ **Exercise 11.2.1** (1) Prove that a European call is equivalent to a portfolio of a European down-and-out option and a European down-and-in option with an identical barrier. (2) Prove that a European put is equivalent to a portfolio of a European up-and-out option and a European up-and-in option with an identical barrier.

Comment 11.2.1 The equivalence results in Exercise 11.2.1 are called the in–out parity [271]. Note that these results do not depend on the barrier's being a constant.

➤ **Exercise 11.2.2** Does the in–out parity apply to American-style options?

➤ **Exercise 11.2.3** Check that the formulas for the up-and-in and down-and-in options become the Black–Scholes formulas for standard European options when $S = H$.

Binomial tree algorithm for pricing down-and-out calls on a non-dividend-paying stock:

```
input:    S, u, d, X, H (H < X, H < S), n, r̂;
real      R, p, C[n+1];
integer   i, j, h;
R := e^r̂; p := (R − d)/(u − d);
h := ⌊ln(H/S)/ln u⌋; H := S u^h;
for (i = 0 to n) { C[i] := max(0, S u^(n−i) d^i − X); }
if [ n − h is even and 0 ≤ (n − h)/2 ≤ n ]
          C[(n − h)/2] := 0; // A hit.
for (j = n − 1 down to 0) {
          for (i = 0 to j)
             C[i] := (p × C[i] + (1 − p) × C[i+1])/R;
          if [ j − h is even and 0 ≤ (j − h)/2 ≤ j ]
             C[(j − h)/2] := 0; // A hit.
}
return C[0];
```

Figure 11.4: Binomial tree algorithm for down-and-out calls on a non-dividend-paying stock. Because H may not correspond to a legal stock price, we lower it to Su^h, the highest stock price not exceeding H. The new barrier corresponds to $C[(j-h)/2]$ at times $j = n, n-1, \ldots, h$. If the option provides a **rebate** K when the barrier is hit, simply change the assignment of zero to that of K.

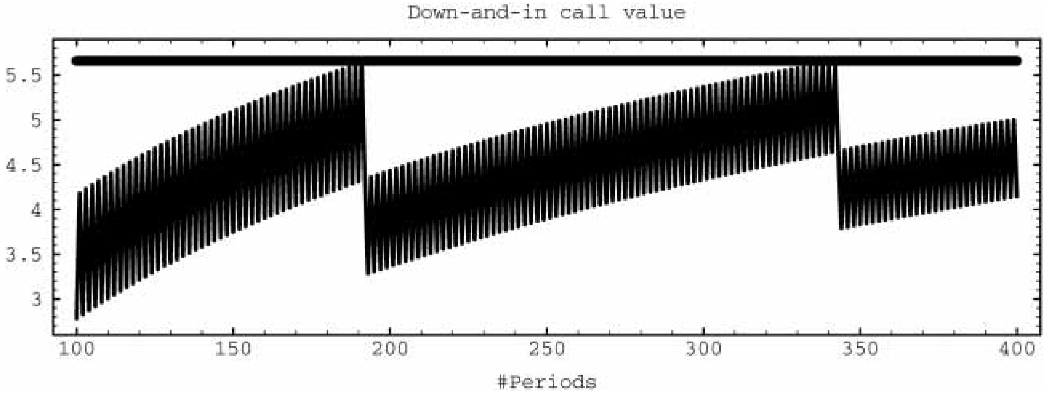

Figure 11.5: Convergence of binomial model for down-and-in calls. Plotted are the option values against the number of time periods. The option's parameters are $S = 95$, $X = 100$, $H = 90$, $r = 10\%$ (continuously compounded), $\sigma = 0.25$, and $\tau = 1$ (year). The analytical value 5.6605 is also plotted for reference.

➤ **Exercise 11.2.4** A **reset option** is like an ordinary option except that the strike price is set to H when the stock price hits H. Assume that $H < X$. Create a synthetic reset option with a portfolio of barrier options.

➢ **Programming Assignment 11.2.5** (1) Implement binomial tree algorithms for European knock-in and knock-out options with rebates. Pay special attention to convergence (see Fig. 11.5). Here is a solved problem: A down-and-in European call without rebates has a value of \$5.6605 given $S = 95$, $X = 100$, $H = 90$, $r = 10\%$ (continuously compounded), $\sigma = 0.25$, and $\tau = 1$ (year). (2) Extend the algorithms to handle American barrier options.

11.2.1 Bonds with Safety Covenants

Bonds with safety covenants can be evaluated with the help of knock-out options. Suppose that a firm is required to pass its ownership to the bondholders if its value falls below a specified barrier H, which may be a function of time. The bondholders therefore receive V the first time the firm's value falls below H. At maturity, the bondholders receive X if $V > X$ and V if $V < X$, where X is the aggregate par value. The value of the bonds therefore equals that of the firm minus a down-and-out option with a strike price X and barrier H.

11.2.2 Nonconstant Barrier

Consider the generalized barrier option with the barrier $H(t) = He^{-\rho\tau}$, where $\rho \geq 0$ and $H \leq X$. The standard barrier option corresponds to $\rho = 0$. The value of a European down-and-in call is

$$S\left[\frac{H(t)}{S}\right]^{2\lambda} N(x) - Xe^{-r\tau}\left[\frac{H(t)}{S}\right]^{2\lambda-2} N(x - \sigma\sqrt{\tau}),$$

where

$$x \equiv \frac{\ln(H(t)^2/(SX)) + (r + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}},$$

$\lambda \equiv (r - \rho + \sigma^2/2)/\sigma^2$, and $S \geq H(t)$. This result is due to Merton [660]. American options can be viewed as barrier options whose barrier is the exercise boundary that, instead of being given in advance, must be calculated.

11.2.3 Other Types of Barrier Options

Barrier options have many variations [158]. If the barrier is active during only an initial period, the option is called a **partial-barrier option**, and if the barrier is active during only the latter part of the option's life, it is called a **forward-starting-barrier option**. If the barrier must be breached for a particular length of time, we have a **Parisian option**. **Double-barrier options** have two barriers. In **rolling options**, a sequence of barriers is specified. For calls (puts), the strike price is lowered (raised, respectively) each time a barrier is hit, and the option is knocked out at the last barrier.

➤ **Exercise 11.2.6** A rolling call comes with barriers $H_1 > H_2 > \cdots > H_n$ (all below the initial stock price) and strike prices $X_0 > X_1 > \cdots > X_{n-1}$. This option starts as a European call with a strike price of X_0. When the first barrier H_1 is hit, the strike price is rolled down to X_1. In general, on hitting each barrier H_i, the strike is rolled down to X_i. The option knocks out when the last barrier H_n is hit. Replicate this option with a portfolio of down-and-out options.

11.3 Interest Rate Caps and Floors

In floating-rate debts, the borrower is concerned with rate rises and the lender is concerned with rate declines. They can seek protection in **interest rate caps** and **floors**, respectively. The writer of a cap pays the purchaser each time the contract's **reference rate** is above the contract's **cap rate** (or **ceiling rate**) on each settlement date. The writer of a floor pays the holder each time the contract's reference rate is below the contract's **floor rate** on each settlement date. The net effect is that a cap places a ceiling on the interest rate cost of a floating-rate debt, and a floor places a floor on the interest rate income of a floating-rate asset. The predetermined interest rate level such as the cap rate and the floor rate is called the **strike rate** [325]. One can also buy an interest rate cap and simultaneously sell an interest rate floor to create an **interest rate collar**. With a collar, the interest cost is bounded between the floor rate and the cap rate: When the reference rate rises above the cap rate, one is compensated by the cap seller, and when the reference rate dips below the floor rate, one pays the floor purchaser.

More formally, at each settlement date of a cap, the cap holder receives

$$\max(\text{reference rate} - \text{cap rate}, 0) \times \text{notional principal} \times t, \tag{11.5}$$

where t is the length of the payment period. Similarly, the payoff of the floor is

$$\max(\text{floor rate} - \text{reference rate}, 0) \times \text{notional principal} \times t.$$

	One month	*Three months*	*Six months*	*One year*
\$ LIBOR FT London Interbank Fixing	6 1/16	6 1/8	6 1/8	6 3/16

Figure 11.6: Sample LIBOR rate quotations. Source: *Financial Times*, May 19, 1995.

Payoff (11.5) denotes a European call on the interest rate with a strike price equal to the cap rate. Hence a cap can be seen as a package of European calls (or **caplets**) on the underlying interest rate. Similarly, a floor can be seen as a package of European puts (or **floorlets**) on the underlying interest rate.

For example, if the reference rate is the 6-month LIBOR, t is typically either 181/360 or 184/360 as LIBOR uses the actual/360 day count convention. LIBOR refers to the lending rates on U.S. dollar deposits (**Eurodollars**) between large banks in London. Many short-term debts and floating-rate loans are priced off LIBOR in that the interest rate is quoted at a fixed margin above LIBOR. Differences between the LIBOR rate and the domestic rate are due to risk, government regulations, and taxes [767]. Non-U.S.-dollar LIBORs such as the German mark LIBOR are also quoted [646]. See Fig. 11.6 for sample quotations on LIBOR rates.

Unlike stock options, caps and floors are settled in cash. The premium is expressed as a percentage of the notional principal on which the cap or floor is written. For example, for a notional principal of \$10 million, a premium of 20 basis points translates into $10 \times 20/10000 = 0.02$ million. The full premium is usually paid up front.

As a concrete example, suppose a firm issues a floating-rate note, paying the 6-month LIBOR plus 90 basis points. The firm's financial situation cannot allow paying an annual rate beyond 11%. It can purchase an interest rate cap with a cap rate of 10.1%. Thereafter, every time the rate moves above 10.1% and the firm pays more than 11%, the excess will be compensated for exactly by the dealer who sells the cap.

11.4 Stock Index Options

A stock index is a mathematical expression of the value of a portfolio of stocks. The New York Stock Exchange (NYSE) Composite Index (ticker symbol NYA), for example, is a weighted average of the prices of all the stocks traded on the NYSE; the weights are proportional to the total market values of their outstanding shares. Buying the index is thus equivalent to buying a portfolio of all the common stocks traded on the NYSE. This kind of average is called **capitalization weighted**, in which heavily capitalized companies carry more weights. The DJIA, S&P 100 Index (ticker symbol OEX), S&P 500 Index (ticker symbol SPX), and Major Market Index (ticker symbol XMI) are four more examples of stock indices. The SPX and OEX are capitalization-weighted averages, whereas the DJIA and XMI are **price-weighted** averages. A price-weighted index is calculated as $\sum_i P_i/\alpha$, where P_i is the price of stock i in the index and α is an adjustment factor that takes care of stock splits, stock dividends, bankruptcies, mergers, etc., so that the index is comparable over time. For example, $\alpha = 0.19740463$ for the DJIA as of October 26, 1999. A third weighting method is **geometric weighting**, in which every stock has the same influence on the index. The Value Line Index (ticker symbol VLE) for example is a geometrically

RANGES FOR UNDERLYING INDEXES
Monday, March 20, 1995

	High	*Low*	*Close*	*Net Chg.*	*From Dec. 31*	*%Chg.*
S&P 100 (OEX)	465.88	464.31	465.42	+0.57	+36.79	+8.6
S&P 500 -A.M.(SPX) .	496.61	495.27	496.14	+0.62	+36.87	+8.0
		...				
Nasdaq 100 (NDX) .	447.67	442.83	446.61	+2.67	+42.34	+10.5
		...				
Russell 2000 (RUT)	257.87	257.28	257.83	+0.51	+7.47	+3.0
		...				
Major Mkt (XMI) . .	435.14	432.82	434.90	+1.93	+34.95	+8.7
		...				
NYSE (NYA)	268.36	267.68	268.05	+0.21	+17.11	+6.8
Wilshire S-C (WSX)	332.50	331.69	332.11	+0.16	+11.05	+3.4
		...				
Value Line (VLE) . .	474.41	473.46	474.12	+0.58	+21.59	+4.8
		...				

Figure 11.7: Index quotations. The stock market's spectacular rise between 1995 and early 2000 is evident if we compare this table with the one in Fig. 8.2. Source: *Wall Street Journal*, March 21, 1995.

weighted index [646]. A geometrically weighted index is calculated as

$$I(t) = \prod_{i=1}^{n} \left[\frac{P_i(t)}{P_i(t-1)} \right]^{1/n} I(t-1),$$

where n is the number of stocks in the index, $I(t)$ is the index value on day t, and $P_i(t)$ is the price of stock i on day t. Stock indices are usually not adjusted for cash dividends [317]. See Fig. 11.7. Additional stock indices can be found in [95, 346, 470].

Stock indices differ also in their stock composition. For instance, the DJIA is an index of 30 blue-chip stocks,[1] whereas the S&P 500 is an index of 500 listed stocks from three exchanges. Nevertheless, their returns are usually highly correlated.

An index option is an option on an index value. Options on stock market portfolios were first offered by insurance companies in 1977. Exchange-traded index options started in March 1983 with the trading of the OEX option on the CBOE [346]. See Fig. 11.8 for sample quotations. Stock index options are settled in cash: Only the cash difference between the index's current market value and the strike price is exchanged when the option is exercised. Options on the SPX, XMI, and DJIA are European, whereas those on the OEX and NYA are American.

The cash settlement feature poses some risks to American option holders and writers. Because the exact amount to be paid when an option is exercised is determined by the closing price, there is an uncertainty called the **exercise risk**. Consider an investor who is short an OEX call and long the appropriate amounts of common stocks that comprise the index. If the call is exercised today, the writer will be notified the next business day and pay cash based on today's closing price. Because the index may open at a price different from today's closing price, the stocks may not fetch the same value as today's closing index value. Were the option settled in stocks, the writer would simply deliver the stocks the following business day without worrying about the change in the index value.

	Strike	Vol.	Last	Net Chg.	Open Int.
CHICAGO					
. . .					
S&P 100 INDEX(OEX)					
May	380p	8	5/16	. . .	897
Apr	385p	50	1/8	. . .	1,672
Apr	390p	335	1/8	. . .	4,406
. . .					
Apr	410c	12	57	+12	37
. . .					
Jun	490c	42	1 3/4	−1/4	719
Call vol. 75,513			Open Int. 415,627		
Put vol. 104,773			Open Int. 447,094		
S&P 500 INDEX-AM(SPX)					
Jun	350p	15	1/8	−1/8	2,027
Jun	375p	8	1/4	−1/8	5,307
Apr	400p	50	1/16	. . .	5,949
. . .					
Apr	405c	2	91 3/4	. . .	. . .
. . .					

	Strike	Vol.	Last	Net Chg.	Open Int.
Apr	490c	470	11	+3/8	23,903
Apr	490p	1,142	3	−3/8	14,476
May	490c	103	13 7/8	−5/8	3,086
May	490p	136	5 3/8	. . .	13,869
Jun	490c	2,560	16 7/8	+3/4	9,464
Jun	490p	333	6 7/8	−3/8	8,288
. . .					
Jun	550p	10	49 3/4	−1	575
Call vol. 34,079			Open Int. 656,653		
Put vol. 59,582			Open Int. 806,961		
. . .					
AMERICAN					
MAJOR MARKET(XMI)					
Jun	325p	20	1/16	−3/16	45
Apr	350p	300	1/16	. . .	400
. . .					
Call vol. 1,913			Open Int. 6,011		
Put vol. 4,071			Open Int. 27,791		
. . .					

Figure 11.8: Index options quotations. Source: *Wall Street Journal*, March 21, 1995.

The **size** of a stock index option contract is the dollar amount equal to \$100 times the index. A May OEX put with a strike price of \$380 costs $100 \times (5/16) = 31.25$ dollars from the data in Fig. 11.8. This particular put is out of the money.

The valuation of stock index options usually relies on the Black–Scholes option pricing model with continuous dividend yields. Hence European stock index options can be priced by Eq. (9.20). This model actually approximates a broad-based stock market index better than it approximates individual stocks.[2]

One of the primary uses of stock index options is hedging large diversified portfolios. NYA puts, for example, can be used to protect a portfolio composed primarily of NYSE-listed securities against market declines. The alternative approach of buying puts for individual stocks would be cumbersome and more expensive by Theorem 8.6.1. Stock index options can also be used to create a long position in a portfolio of stocks by the put–call parity. It is easier to implement this synthetic security than it is to buy individual stocks.

➤ **Exercise 11.4.1** Verify the following claims: (1) A 1% price change in a lower-priced stock causes a smaller movement in the price-weighted index than that in a higher-priced stock. (2) A 1% price change in a lower capitalization issue has less of an impact on the capitalization-weighted index than that in a larger capitalization issue. (3) A 1% price change in a lower-priced stock has the same impact on the geometrically weighted index as that in a higher-priced stock.

11.5 Foreign Exchange Options

In the **spot** (or **cash**) **market** in which prices are for immediate payment and delivery, exchange rates between the U.S. dollar and foreign currencies are generally quoted

EXCHANGE RATES
Thursday, January 7, 1999

Country	*U.S. $ equiv.* *Thu.*	*Wed.*	*Currency per U.S. $* *Thu.*	*Wed.*
	...			
Britain (Pound).....	1.6508	1.6548	.6058	.6043
1-month Forward.	1.6493	1.6533	.6063	.6049
3-months Forward	1.6473	1.6344	.6071	.6119
6-months Forward	1.6461	1.6489	.6075	.6065
	...			
Germany (Mark).....	.5989	.5944	1.6698	1.6823
1-month Forward.	.5998	.5962	1.6673	1.6771
3-months Forward	.6016	.5972	1.6623	1.6746
6-months Forward	.6044	.6000	1.6544	1.6666
	...			
Japan (Yen)........	.009007	.008853	111.03	112.95
1-month Forward	.009007	.008889	111.03	112.50
3-months Forward	.009008	.008958	111.02	111.63
6-months Forward	.009009	.009062	111.00	110.34
	...			
SDR..............	1.4106	1.4137	0.7089	0.7074
Euro	1.1713	1.1626	0.8538	0.8601

Figure 11.9: Exchange rate quotations. Source: *Wall Street Journal*, January 8, 1999.

with the **European terms**. This method measures the amount of foreign currency needed to buy one U.S. dollar, i.e., foreign currency units per dollar. The **reciprocal of European terms**, on the other hand, measures the U.S. dollar value of one foreign currency unit. For example, if the European-terms quote is .63 British pounds per $1 (£.63/$1), then the reciprocal-of-European-terms quote is $1.587 per British pound ($1/£.63 or $1.587/£1). The reciprocal of European terms is also called the **American terms**.

Figure 11.9 shows the spot exchange rates as of January 7, 1999. The **spot exchange rate** is the rate at which one currency can be exchanged for another, typically for settlement in 2 days. Note that the German mark is a **premium currency** because the 3-month forward exchange rate, $.6016/DEM1, exceeds the spot exchange rate, $.5989/DEM1; the mark is more valuable in the forward market than in the spot market. In contrast, the British pound is a **discount currency**. The **forward exchange rate** is the exchange rate for deferred delivery of a currency.

Foreign exchange (forex) options are settled by delivery of the underlying currency. A primary use of forex options is to hedge **currency risk**. Consider a U.S. company expecting to receive 100 million Japanese yen in March 2000. Because this company wants U.S. dollars, not Japanese yen, those 100 million Japanese yen will be exchanged for U.S. dollars. Although 100 million Japanese yen are worth 0.9007 million U.S. dollars as of January 7, 1999, they may be worth less or more in March 2000. The company decides to use options to hedge against the depreciation of the yen against the dollar. From Fig. 11.10, because the **contract size** for the Japanese yen option is JPY6,250,000, the company decides to purchase $100{,}000{,}000/6{,}250{,}000 = 16$ puts on the Japanese yen with a strike price of

		–Call–		–Put–	
		Vol.	Last	Vol.	Last
		...			
German Mark					59.31
62,500 German Marks-European Style.					
58 1/2	Jan	...	0.01	27	0.06
59	Jan	...	0.01	210	0.13
61	Jan	27	0.07	...	0.01
61 1/2	Jan	210	0.02	...	0.01

		–Call–		–Put–	
		Vol.	Last	Vol.	Last
		...			
Japanese Yen					88.46
6,250,000 J. Yen-100ths of a cent per unit.					
66 1/2	Mar	...	0.01	1	2.53
73	Mar	...	...	10	0.04
75	Mar	...	0.01	137	0.06
76	Mar	...	...	9	0.09
77	Mar	...	...	17	0.09
78	Mar	...	...	185	0.18
79	Mar	...	...	10	0.16
80	Mar	...	...	77	0.40
81	Mar	...	...	60	0.36
86	Jan	...	0.01	5	0.14
88	Mar	...	...	10	2.14
89	Mar	...	...	10	2.51
90	Feb	...	0.01	12	2.30
91	Feb	...	0.01	5	2.50
100	Mar	2	0.86	...	0.01
		...			

Figure 11.10: Forex option quotations. Source: *Wall Street Journal*, January 8, 1999.

\$.0088 and an exercise month in March 2000. This gives the company the right to sell 100,000,000 Japanese yen for $100{,}000{,}000 \times .0088 = 880{,}000$ U.S. dollars. The options command $100{,}000{,}000 \times 0.000214 = 21{,}400$ U.S. dollars in premium. The net proceeds per Japanese yen are hence $.88 - .0214 = 0.8586$ cent at the minimum.

A call to buy a currency is an insurance against the relative appreciation of that currency, whereas a put on a currency is an insurance against the relative depreciation of that currency. Put and call on a currency are therefore identical: A put to sell X_{A} units of currency A for X_{B} units of currency B is the same as a call to buy X_{B} units of currency B for X_{A} units of currency A. The above-mentioned option, for example, can be seen as the right to buy $6{,}250{,}000 \times 0.0088 = 55{,}000$ U.S. dollars for 6,250,000 Japanese yen, or equivalently a call on 55000 U.S. dollars with a strike price of $1/0.0088 \approx 113.6$ yen per dollar.

It is important to note that a DEM/\$ call (an option to buy German marks for U.S. dollars) and a \$/£ call do not a DEM/£ call make. This point is illustrated by an example. Consider a call on DEM1 for \$.71, a call on \$.71 for £.452, and a call on DEM1 for £.452. Suppose that the U.S. dollar falls to DEM1/\$.72 and £1/\$1.60. The first option nets a profit of \$.01, but the second option expires worthless. Because $0.72/1.6 = 0.45$, we know that DEM1/£.45. The DEM/£ option therefore also expires worthless. Hence the portfolio of DEM/\$ and \$/£ calls is worth more than the DEM/£ call. This conclusion can be shown to hold in general (see Exercise 15.3.16). Options such as DEM/£ calls are called **cross-currency options** from the dollar's point of view.

Many forex options are over-the-counter options. One possible reason is that the homogeneity and liquidity of the underlying asset make it easy to structure custom-made deals. Exchange-traded forex options are available on the PHLX and the Chicago Mercantile Exchange (CME) as well as on many other exchanges [346]. Most exchange-traded forex options are denominated in the U.S. dollar.

➤ **Exercise 11.5.1** A **range forward contract** has the following payoff at expiration:

$$\begin{array}{rl} 0 & \text{if the exchange rate } S \text{ lies within } [\,X_L, X_H\,], \\ S - X_H & \text{if } S > X_H, \\ S - X_L & \text{if } S < X_L. \end{array}$$

It guarantees that the effective future exchange rate will lie within X_L and X_H. Replicate this contract with standard options.

➤ **Exercise 11.5.2** In a **conditional forward contract**, the premium p is paid at expiration and only if the exchange rate is below a specified level X. The payoff at expiration is thus

$$\begin{array}{rl} S - X - p & \text{if the exchange rate } S \text{ exceeds } X, \\ -p & \text{if } S \leq X. \end{array}$$

It guarantees that the effective future exchange rate will be, at most, $X + p$. Replicate this contract with standard options.

➤ **Exercise 11.5.3** A **participating forward contract** pays off at expiration

$$\begin{array}{rl} S - X & \text{if the exchange rate } S \text{ exceeds } X, \\ \alpha(S - X) & \text{if } S \leq X. \end{array}$$

The purchaser is guaranteed an upper bound on the exchange rate at X and pays a proportion α of the decrease below X. Replicate this contract with standard options.

11.5.1 The Black–Scholes Model for Forex Options

Let S denote the spot exchange rate in domestic/foreign terms, σ the volatility of the exchange rate, r the domestic interest rate, and $\widehat{r}$ the foreign interest rate. A foreign currency is analogous to a stock's paying a known dividend yield because the owner of foreign currencies receives a continuous dividend yield equal to $\widehat{r}$ in the foreign currency. The formulas derived for stock index options in Eq. (9.20) hence apply with the dividend yield equal to $\widehat{r}$:

$$C = Se^{-\widehat{r}\tau} N(x) - Xe^{-r\tau} N(x - \sigma\sqrt{\tau}), \tag{11.6}$$

$$P = Xe^{-r\tau} N(-x + \sigma\sqrt{\tau}) - Se^{-\widehat{r}\tau} N(-x), \tag{11.6$'$}$$

where

$$x \equiv \frac{\ln(S/X) + (r - \widehat{r} + \sigma^2/2)\,\tau}{\sigma\sqrt{\tau}.}$$

The deltas of calls and puts are $e^{-\widehat{r}\tau} N(x) \geq 0$ and $-e^{-\widehat{r}\tau} N(-x) \leq 0$, respectively. The Black–Scholes model produces acceptable results for major currencies such as the German mark, the Japanese yen, and the British pound [346].

➤ **Exercise 11.5.4** Assume the BOPM. (1) Verify that the risk-neutral probability for forex options is $[\,e^{(r-\widehat{r})\,\Delta t} - d\,]/(u - d)$, where u and d denote the up and the down moves, respectively, of the domestic/foreign exchange rate in Δt time. (2) Let S be the domestic/foreign exchange rate. Show that the delta of the forex call equals

$$h \equiv e^{-\widehat{r}\Delta t} \frac{C_u - C_d}{Su - Sd}$$

if we use the foreign riskless asset (riskless in terms of foreign currency) and the domestic riskless asset to replicate the option. Above, h is the price of the foreign riskless asset held in terms of the foreign currency. (Hint: Review Eq. (9.1).)

➤ **Exercise 11.5.5** Prove that the European forex call and put are worth the same if $S = X$ and $r = \widehat{r}$ under the Black–Scholes model.

11.5.2 Some Pricing Relations

Many of the relations in Section 8.2 continue to hold for forex options after the modifications required by the existence of the continuous dividend yield (equal to the foreign interest rate). To show that a European call satisfies

$$C \geq \max(Se^{-\widehat{r}\tau} - Xe^{-r\tau}, 0), \tag{11.7}$$

consider the following strategies:

		Value at expiration	
	Initial investment	$S_\tau > X$	$S_\tau \leq X$
Buy a call	C	$S_\tau - X$	0
Buy domestic bonds (face value \$$X$)	$X e^{-r\tau}$	X	X
Total	$C + X e^{-r\tau}$	S_τ	X
Buy foreign bonds (face value 1 in foreign currency)	$Se^{-\widehat{r}\tau}$	S_τ	S_τ

Hence the first portfolio is worth at least as much as the second in every scenario and therefore cannot cost less. Bound (11.7) incidentally generalizes Exercise 8.3.2.

A European call's price may approach the lower bound in bound (11.7) as closely as may be desired (see Exercise 11.5.6). As the intrinsic value $S - X$ of an American call can exceed the lower bound in bound (11.7), early exercise may be optimal. Thus we have the following theorem.

THEOREM 11.5.1 *American forex calls may be exercised before expiration.*

➤ **Exercise 11.5.6** Show how bound (11.7) may be approximated by the C in Eq. (11.6).

➤ **Exercise 11.5.7 (Put–Call Parity).** Prove that (1) $C = P + Se^{-\widehat{r}\tau} - Xe^{-r\tau}$ for European options and (2) $P \geq \max(Xe^{-r\tau} - Se^{-\widehat{r}\tau}, 0)$ for European puts.

11.6 Compound Options

Compound options are options on options. There are four basic types of compound options: a call on a call, a call on a put, a put on a call, and a put on a put. Formulas for compound options can be found in [470].

If stock is considered a call on the total value of the firm as in Subsection 11.1.1, a stock option becomes a compound option. Renewable term life insurance offers another example: Paying a premium confers the right to renew the contract for the next term. Thus the decision to pay a premium is an option on an option [646]. A **split-fee option** provides a window on the market at the end of which the buyer can decide whether to extend it up to the notification date or to let it expire worthless [54, 346]. If the split-free option is extended up to the notification date; the second option can either expire or be exercised. The name comes from the fact that the user has to pay two fees to exercise the underlying asset [746].

Compound options are appropriate for situations in which a bid, denominated in foreign currency, is submitted for the sale of equipment. There are two levels of uncertainty at work here: the winning of the bid and the currency risk (even if the bid is won, a depreciated foreign currency may make the deal unattractive). What is needed is an arrangement whereby the company can secure a foreign currency option against foreign currency depreciation *if* the bid is won. This is an example of **contingent foreign exchange option**. A compound put option that grants the holder the right to purchase a put option in the future at prices that are agreed on today solves the problem. Forex options are not ideal because they turn the bidder into a speculator if the bid is lost.

➤ **Exercise 11.6.1** Recall our firm XYZ.com in Subsection 11.1.1. It had only two kinds of securities outstanding, shares of its own common stock and bonds. Argue that the stock becomes a compound option if the bonds pay interests before maturity.

➤ **Exercise 11.6.2** Why is a contingent forex put cheaper than a standard put?

➤ **Exercise 11.6.3** A **chooser option** (or an **as-you-like-it option**) gives its holder the right to buy for X_1 at time τ_1 either a call or a put with strike price X_2 at time τ_2. Describe the binomial tree algorithm for this option.

➢ **Programming Assignment 11.6.4** Implement binomial tree algorithms for the four compound options.

11.7 Path-Dependent Derivatives

Let $S_0, S_1, \dots, S_n$ denote the prices of the underlying asset over the life of the option. S_0 is the known price at time zero and S_n is the price at expiration. The standard European call has a terminal value that depends on only the last price, $\max(S_n - X, 0)$. Its value thus depends on only the underlying asset's terminal price regardless of how it gets there; it is **path independent**. In contrast, some derivatives are **path dependent** in that their terminal payoffs depend critically on the paths. The (arithmetic) **average-rate call** has a terminal value given by

$$\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right),$$

and the **average-rate put's** terminal value is given by

$$\max\left(X - \frac{1}{n+1}\sum_{i=0}^{n} S_i, 0\right).$$

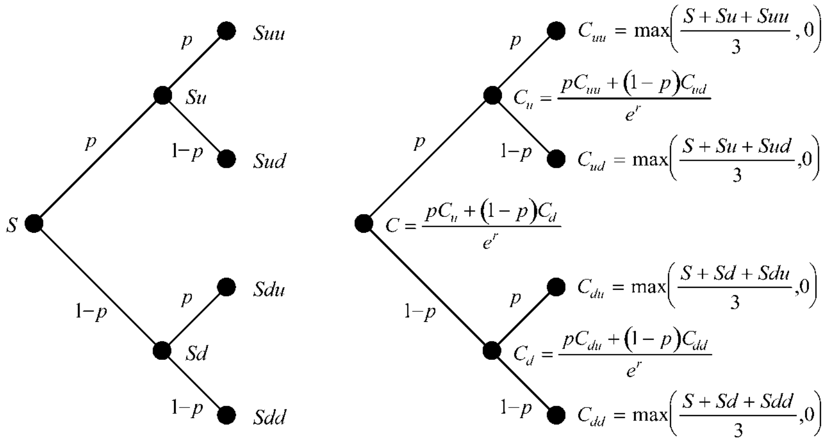

Figure 11.11: Binomial tree for average-rate call. The pricing tree on the right grows exponentially. Here a period spans one year.

Both are path dependent. At initiation, average-rate options cannot be more expensive than standard European options under the Black–Scholes option pricing model [379, 548]. Average-rate options also satisfy certain put–call parity identities [105, 598]. Average-rate options, also called **Asian options**, are useful hedging tools for firms that will make a stream of purchases over a time period because the costs are likely to be linked to the average price. They are mostly European for the same reason.

Average-rate options are notoriously hard to price. Take the terminal price S_0u^2d. Different paths to it such as $(S_0, S_0u, S_0u^2, S_0u^2d)$ and $(S_0, S_0d, S_0du, S_0du^2)$ may lead to different averages and hence payoffs: $\max((S_0 + S_0u + S_0u^2 + S_0u^2d)/4 - X, 0)$ and $\max((S_0 + S_0d + S_0du + S_0du^2)/4 - X, 0)$, respectively (see Fig. 11.11). The binomial tree for the averages therefore does not combine. A straightforward algorithm is to enumerate the 2^n price paths for an n-period binomial tree and then average the payoffs. However, the exponential complexity makes this naive algorithm impractical. As a result, the Monte Carlo method and **approximation algorithms** are the few alternatives left.

Not all path-dependent derivatives are hard to price, however. Barrier options, for example, are easy to price. When averaging is done *geometrically*, the option payoffs are

$$\max\left((S_0S_1\cdots S_n)^{1/(n+1)} - X, 0\right), \quad \max\left(X - (S_0S_1\cdots S_n)^{1/(n+1)}, 0\right).$$

For European geometric average-rate options, the limiting analytical solutions are the Black–Scholes formulas with the volatility set to $\sigma_{\rm a} \equiv \sigma/\sqrt{3}$ and the dividend yield set to $q_{\rm a} \equiv (r + q + \sigma^2/6)/2$, that is,

$$C = Se^{-q_{\rm a}\tau} N(x) - Xe^{-r\tau} N(x - \sigma_{\rm a}\sqrt{\tau}), \tag{11.8}$$

$$P = Xe^{-r\tau} N(-x + \sigma_{\rm a}\sqrt{\tau}) - Se^{-q_{\rm a}\tau} N(-x), \tag{11.8′}$$

where

$$x \equiv \frac{\ln(S/X) + \left(r - q_{\rm a} + \sigma_{\rm a}^2/2\right)\tau}{\sigma_{\rm a}\sqrt{\tau}}.$$

In practice, average-rate options almost exclusively utilize arithmetic averages [894].

Another class of options, (**floating-strike**) **lookback options**, let the stock prices determine the **strike price** [225, 388, 833]. A lookback call option on the minimum has a terminal payoff of $S_n - \min_{0 \le i \le n} S_i$, and a lookback put option on the maximum has a terminal payoff of $\max_{0 \le i \le n} S_i - S_n$. The related **fixed-strike lookback option** provides a payoff of $\max(\max_{0 \le i \le n} S_i - X, 0)$ for the call and $\max(X - \min_{0 \le i \le n} S_i, 0)$ for the put. A perpetual American lookback option is called a **Russian option** [575]. One can also define lookback call and put options on the average. Such options are also called **average-strike options** [470].

An approximation algorithm for pricing arithmetic average-rate options, which is due to Hull and White [478], is described below. This algorithm is based on the binomial tree. Consider a node at time j with the underlying asset price equal to $S_0 u^{j-i} d^i$. Name such a node $N(j, i)$. The running sum $\sum_{m=0}^{j} S_m$ at this node has a maximum value of

$$S_0(1 + \overbrace{u + u^2 + \cdots + u^{j-i} + u^{j-i} d + \cdots + u^{j-i} d^i}^{j}) = S_0 \, \frac{1 - u^{j-i+1}}{1-u} + S_0 u^{j-i} d \, \frac{1-d^i}{1-d}.$$

Divide this value by $j+1$ and call it $A_{\max}(j, i)$. Similarly, the running sum has a minimum value of

$$S_0(1 + \overbrace{d + d^2 + \cdots + d^i + d^i u + \cdots + d^i u^{j-i}}^{j}) = S_0 \, \frac{1 - d^{i+1}}{1-d} + S_0 d^i u \, \frac{1-u^{j-i}}{1-u}.$$

Divide this value by $j+1$ and call it $A_{\min}(j, i)$. Both $A_{\min}$ and $A_{\max}$ are running averages (see Fig. 11.12).

Although the possible running averages at $N(j, i)$ are far too many $\left(\binom{j}{i}\text{ of them}\right)$, all lie between $A_{\min}(j, i)$ and $A_{\max}(j, i)$. Pick $k+1$ equally spaced values in this range and treat them as the true and only running averages, which are

$$A_m(j, i) \equiv \left(\frac{k-m}{k}\right) A_{\min}(j, i) + \left(\frac{m}{k}\right) A_{\max}(j, i), \quad m = 0, 1, \ldots, k.$$

Such "bucketing" introduces errors, but it works well in practice [366]. An alternative is to pick values whose logarithms are equally spaced [478, 555].

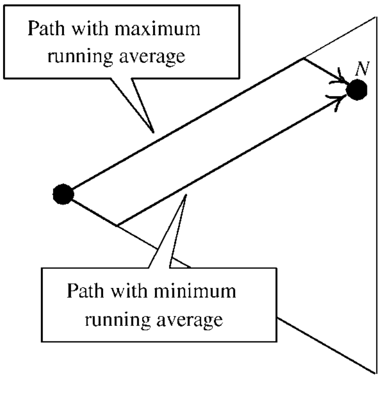

Figure 11.12: Paths with maximum and minimum running averages. Plotted are the paths with the maximum and minimum running averages from the root to node N.

During backward induction, we calculate the option values at each node for the $k+1$ running averages as follows. Suppose that the current node is $N(j, i)$ and the running average is a. Assume that the next node is $N(j+1, i)$, the result of an up move. Because the asset price there is $S_0 u^{j+1-i} d^i$, we seek the option value corresponding to the running average:

$$A_{\mathrm{u}} \equiv \frac{(j+1)\,a + S_0 u^{j+1-i} d^i}{j+2}.$$

To be sure, A_{u} is not likely to be one of the $k+1$ running averages at $N(j+1, i)$. Hence we find the running averages that bracket it, that is,

$$A_\ell(j+1, i) \le A_{\mathrm{u}} \le A_{\ell+1}(j+1, i).$$

Express A_{u} as a linearly interpolated value of the two running averages:

$$A_{\mathrm{u}} = x\,A_\ell(j+1, i) + (1-x)\,A_{\ell+1}(j+1, i), \quad 0 \le x \le 1.$$

(An alternative is the quadratic interpolation [276, 555]; see Exercise 11.7.7.) Now, obtain the approximate option value given the running average A_{u} by means of

$$C_{\mathrm{u}} \equiv x\,C_\ell(j+1, i) + (1-x)\,C_{\ell+1}(j+1, i),$$

where $C_\ell(t, s)$ denotes the option value at node $N(t, s)$ with running average $A_l(t, s)$. This interpolation introduces the second source of error. The same steps are repeated for the down node $N(j+1, i+1)$ to obtain another approximate option value C_{d}. We finally obtain the option value as $[\,pC_{\mathrm{u}} + (1-p)\,C_{\mathrm{d}}\,]\,e^{-r\Delta t}$. See Fig. 11.13 for the idea and Fig. 11.14 for the $O(kn^2)$-time algorithm.

➤ **Exercise 11.7.1** Achieve "selling at the high and buying at the low" with lookback options.

➤ **Exercise 11.7.2** Verify that the number of geometric averages at time n is $n(n+1)/2$.

➤ **Exercise 11.7.3** Explain why average-rate options are harder to manipulate. (They were first written on stocks traded on Asian exchanges, hence the name "Asian options," presumably because of their lighter trading volumes.)

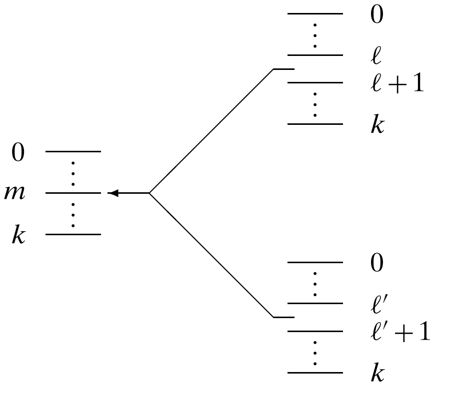

Figure 11.13: Backward induction for arithmetic average-rate options. The $k+1$ possible option values at $N(j, i)$ are stored in an array indexed by $0, 1, \ldots, k$, each corresponding to a specific arithmetic average. Each of the $k+1$ option values depends on two option values at node $N(j+1, i)$ and two option values at node $N(j+1, i+1)$.

Algorithm for pricing American arithmetic average-rate calls on a non-dividend-paying stock:

```
input:   S, u, d, X, n, r̂ (u > e^r̂ > d, r̂ > 0), k;
real     p, a, A_u, x, A_d, C_u, C_d, C[n+1][k+1], D[k+1]; // C[·][m] stores m th average.
integer i, j, m, ℓ;
p := (e^r̂ − d)/(u − d);
for (i = 0 to n) // Terminal price is S u^{n−i} d^i.
        for (m = 0 to k)
            C[i][m] := max(0, A_m(n, i) − X);
for (j = n − 1 down to 0) // Backward induction.
      for (i = 0 to j) {//Price is S u^{j−i} d^i.
            for (m = 0 to k) {
                 a := A_m(j, i); // "Average."
                 A_u := ((j + 1) a + S u^{j+1−i} d^i)/(j + 2);
                 Let ℓ be such that A_ℓ(j + 1, i) ≤ A_u ≤ A_{ℓ+1}(j + 1, i);
                 Let x be such that A_u = x A_ℓ(j + 1, i) + (1 − x) A_{ℓ+1}(j + 1, i);
                 C_u := x C[i][ℓ] + (1 − x) C[i][ℓ + 1]; // Linear interpolation.
                 A_d := ((j + 1) a + S u^{j−i} d^{i+1})/(j + 2);
                 Let ℓ be such that A_ℓ(j + 1, i + 1) ≤ A_d ≤ A_{ℓ+1}(j + 1, i + 1);
                 Let x be such that A_d = x A_ℓ(j + 1, i + 1) + (1 − x) A_{ℓ+1}(j + 1, i + 1);
                 C_d := x C[i + 1][ℓ] + (1 − x) C[i + 1][ℓ + 1]; // Linear interpolation.
                 D[m] := max(a − X, (p C_u + (1 − p) C_d) e^{−r̂});
            }
            Copy D[0..k] to C[i][0..k];
      }
return C[0][0];
```

Figure 11.14: Approximation algorithm for American arithmetic average-rate calls on a non-dividend-paying stock. The ideas in Exercise 11.7.5 can reduce computation time. For a fixed k, this algorithm may not converge as n increases [249, 251]; in fact, k may have to scale with n [555]. Note that $C[0][0], C[0][1], \ldots, C[0][k]$ are identical in value absent the rounding errors.

➤ **Exercise 11.7.4** Arithmetic average-rate options were assumed to be newly issued, and there was no historical average to deal with. Argue that no generality was lost in doing so.

➤ **Exercise 11.7.5** Let $\widehat{r} \neq 0$ denote the continuously compounded riskless rate per period. (1) The future value of the average-rate call,

$$E\left[\max\left(\frac{1}{n+1}\sum_{i=0}^{n} S_i - X, 0\right)\right],$$

is probably hard to evaluate. But show that

$$E\left[\frac{1}{n+1}\sum_{i=0}^{n} S_i\right] = \frac{S_0}{n+1}\,\frac{1-e^{\widehat{r}(n+1)}}{1-e^{\widehat{r}}}.$$

(2) If a path $S_0, S_1, \ldots, S_k$ has a running sum $\sum_{i=0}^{k-1} S_i$ equal to $(n+1)X + a$, $a \geq 0$, show that the expected terminal value of the average-rate call given this initial path is

$$\frac{a}{n+1} + \frac{S_k}{n+1}\,\frac{1-e^{\widehat{r}(n-k+1)}}{1-e^{\widehat{r}}}.$$

(So the tree growing out of node S_k can be pruned when pricing calls, leading to better efficiency.)

➤ **Exercise 11.7.6** Assume that $ud = 1$. Prove that the difference between the maximum running sum and the minimum running sum at a node with an asset price of $S_0 u^i d^{n-i}$ is an increasing function of i for $i < n/2$ and a decreasing function of i for $i > n/2$.

➤ **Exercise 11.7.7** Derive the quadratic polynomial $y = a + bx + cx^2$ that passes through three points (x_0, y_0), (x_1, y_1), and (x_2, y_2).

➤ **Exercise 11.7.8** Two sources of error were mentioned for the approximation algorithm. Argue that they disappear if all the asset prices on the tree are integers.

➤ **Exercise 11.7.9** Suppose that there is an algorithm that generates an upper bound on the true option value and an algorithm that generates a lower bound on the true option value. How do we design an approximate option pricing scheme with information on the pricing error?

➢ **Programming Assignment 11.7.10** Implement $O(n^4)$-time binomial tree algorithms for European and American geometric average-rate options.

➢ **Programming Assignment 11.7.11** Implement the algorithm in Fig. 11.14.

➢ **Programming Assignment 11.7.12** (1) Implement binomial tree algorithms for European and American lookback options. The time complexity should be, at most, cubic. (2) Improve the running time to quadratic time for newly issued floating-strike lookback options, which of course have no price history.

Additional Reading

Consult [565] for alternative models for warrants and [370] for an analytical approach to pricing American barrier options. See [346, Chap. 7] and [514, Chap. 11] for additional forex options and [886] for the term structure of exchange rate volatility. An $O(kn^2)$-time approximation algorithm with ideas similar to those in Fig. 11.14 can be found in [215, 470, 478]. More path-dependent derivatives are discussed in [377, 449, 470, 485, 746, 812]. There is a vast literature on average-rate option pricing [25, 105, 168, 169, 170, 241, 366, 379, 478, 530, 548, 555, 700, 755] as well as analytical approximations for such options [423, 596, 597, 664, 748, 851]. The $O(kn^2)$-time approximation algorithm for average-rate options in [10] is similar to, but slightly simpler than, the algorithm in Fig. 11.14. More important is its guarantee not to deviate from the naive $O(2^n)$-time binomial tree algorithm by more than $O(Xn/k)$ in the case of European average-rate options. The number k here is a parameter that can be varied for trade-off between running time and accuracy. The error bound can be further reduced to $O(X\sqrt{n}/k)$ [250]. An efficient convergent approximation algorithm that is due to Dai and Lyuu is based on the insight of Exercise 11.7.8 [251]. See [459] for a general treatment of approximation algorithms. The option pricing theory has applications in capital investment decisions [279, 672].

NOTES

1. As of February 2001, the 30 stocks were Allied Signal, Alcoa, American Express, AT&T, Boeing, Caterpillar, Citigroup, Coca-Cola, DuPont, Eastman Kodak, Exxon, General Electric, General Motors, Hewlett-Packard, Home Depot, Intel, IBM, International Paper, J.P. Morgan Chase,

Johnson & Johnson, McDonald's, Merck, Microsoft, Minnesota Mining & Manufacturing, Philip Morris, Procter & Gamble, SBC Communications, United Technologies, Wal-Mart Stores, and Walt Disney.

2. Strictly speaking, it is inconsistent to assume that both the stock index and its individual stock prices satisfy the Black–Scholes option pricing model because the sum of lognormal random variables is not lognormally distributed.