1. Prove that we can transform a segment by transforming its endpoints and then constructing a new segment with the transformed end points.

   Solution:

   Let $\mathbf{u}$ and $\mathbf{v}$ be the endpoints of segment S. The points in S can be expressed by $(1-t)\mathbf{u}+t\mathbf{v}$, $t \in [0,1]$. Given a transformation matrix $\mathbf{M}$, the transformed S will be:
   $\mathbf{M}((1-t)\mathbf{u}+t\mathbf{v}) =$
   $\mathbf{M}((1-t)\mathbf{u})+\mathbf{M}(t\mathbf{v}) =$
   $(1-t)\mathbf{M}\mathbf{u}+t\mathbf{M}\mathbf{v}$,
   which denotes a new segment defined by endpoints $\mathbf{Mu}$ and $\mathbf{Mv}$. □

2. Prove that two successive 2D rotations are additive: $\mathbf{R}(\alpha_1)\mathbf{R}(\alpha_2) = \mathbf{R}(\alpha_1+\alpha_2)$, where $\mathbf{R}(\alpha)$ denotes a 2x2 rotation matrix of rotation angle α.
   (Recall that $\sin(\alpha_1+\alpha_2) = \sin\alpha_1\cos\alpha_2+\cos\alpha_1\sin\alpha_2$, and $\cos(\alpha_1+\alpha_2) = \cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2$.)

   Solution:

   $$\mathbf{R}(\alpha_1)\mathbf{R}(\alpha_2) =$$
   $$\begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1 \end{pmatrix}\begin{pmatrix} \cos\alpha_2 & -\sin\alpha_2 \\ \sin\alpha_2 & \cos\alpha_2 \end{pmatrix} =$$
   $$\begin{pmatrix} \cos\alpha_1\cos\alpha_2-\sin\alpha_1\sin\alpha_2 & -\cos\alpha_1\sin\alpha_2-\sin\alpha_1\cos\alpha_2 \\ \sin\alpha_1\cos\alpha_2+\cos\alpha_1\sin\alpha_2 & -\sin\alpha_1\sin\alpha_2+\cos\alpha_1\cos\alpha_2 \end{pmatrix} =$$
   $$\begin{pmatrix} \cos(\alpha_1+\alpha_2) & -\sin(\alpha_1+\alpha_2) \\ \sin(\alpha_1+\alpha_2) & \cos(\alpha_1+\alpha_2) \end{pmatrix} = \mathbf{R}(\alpha_1+\alpha_2).$$
   □

3. Let $\mathbf{R}$ be a 2D homogeneous rotation matrix of α angle and $\mathbf{T}$ a 2D homogeneous translation matrix of (x, y). a) Write the matrices corresponding to $\mathbf{RT}$ and $\mathbf{TR}$. b) Are they the same? If not, why not? c) Write the transformations equivalent to the inverse of $\mathbf{RT}$.

   Solution:

   a) $$\mathbf{RT} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & x\cos\alpha-y\sin\alpha \\ \sin\alpha & \cos\alpha & x\sin\alpha+y\cos\alpha \\ 0 & 0 & 1 \end{pmatrix},$$
   $$\mathbf{TR} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & x \\ \sin\alpha & \cos\alpha & y \\ 0 & 0 & 1 \end{pmatrix}.$$
   b) No. Multiplication of transformations is not commutative: in the first case a rotation around center of rotation (0,0) is applied after the translation, and in the second case it is applied before.
   c) $$(\mathbf{RT})^{-1} = \mathbf{T^{-1}}\mathbf{R}^{-1} = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & -x \\ \sin(-\alpha) & \cos(-\alpha) & -y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & -x \\ -\sin(\alpha) & \cos(\alpha) & -y \\ 0 & 0 & 1 \end{pmatrix}.$$
   □

4. Derive the 2D homogeneous rotation matrix $\mathbf{R}(\mathbf{p}, \alpha)$, which rotates of α degrees around center of rotation $\mathbf{p}$.

   Solution: The desired matrix is a composition of 3 transformations:
   $\mathbf{R}(\mathbf{p}, \alpha) = \mathbf{T}(\mathbf{p})\mathbf{R}(\alpha)\mathbf{T}(-\mathbf{p}) =$

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & x \\ \sin\alpha & \cos\alpha & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & -x\cos\alpha + y\sin\alpha + x \\ \sin\alpha & \cos\alpha & -x\sin\alpha - y\cos\alpha + y \\ 0 & 0 & 1 \end{pmatrix}.$$

□

5. Windowing Transformations. a) Derive a transformation matrix $\mathbf{T}$ that transforms points in the rectangle defined by lower-left corner (a, b) and upper-right corner (c, d), to another rectangle defined by corners (a', b') and (c', d'). b) What is the result of applying $\mathbf{T}$ to transform points (a, b) and (c, d)? c) Considering $a = b = a' = b' = 0$, what is the result of transforming point $(c/2, d/2)$?

   Solution: a)Just combine 3 transformations: a translation to move point (a, b) to the origin, a scaling to bring the rectangle to the target size, and then another translation to move the origin to point (c, d) :

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & a' \\ 0 & 1 & b' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{c'-a'}{c-a} & 0 & 0 \\ 0 & \frac{d'-b'}{d-b} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{c'-a'}{c-a} & 0 & -a\frac{c'-a'}{c-a} + a' \\ 0 & \frac{d'-b'}{d-b} & -b\frac{d'-b'}{d-b} + b' \\ 0 & 0 & 1 \end{pmatrix}.$$

   b) Multiply $\mathbf{T}$ to points (a, b) and (c, d) to compute the answers (a', b') and (c', d').

   c) $(c'/2, d'/2)$. □

6. Projection Transformations. a) What is the perspective transformation matrix $\mathbf{P}$ which projects points with center of projection $(0, 0, 0)$ on plane $P(\mathbf{q}, \mathbf{n})$, where $\mathbf{q}$ is a point in P, and $\mathbf{n}$ is the normal vector of P? b) What is the matrix when the projection plane is $P((3, 3, 5), (1, 0, 0))$? c) What is the result of projecting point $(2, 2, 0)$ using the matrix of item b?

   Solution: a) As seen in class, $\mathbf{P} = \begin{pmatrix} \mathbf{I}(\mathbf{q}\cdot\mathbf{n}) & \mathbf{0} \\ \mathbf{n}^T & 0 \end{pmatrix}$.

   b) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

   c) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 0 \\ 2 \end{pmatrix} = (3, 3, 0).$

   □

7. Give the following transformation matrices: a) 2D reflection across the line $y = x$, b) 2D dilation by a factor of 3 centered at the origin, c) 3D rotation around the x-axis, sending the positive z-axis to the positive y-axis. d) 3D Reflection across the xy-plane.

   Solution: a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

   □

8. When a given point $\mathbf{p}$ is transformed by a generic transformation matrix $\mathbf{M}$, can its corresponding normal vector $\mathbf{n}$ also always be transformed with a multiplication by $\mathbf{M}$? Why? If your answer was no, what matrix should you use?

   Solution:

   No. Because a generic matrix may incorrectly scale the normal while the correct normal should be the unit normal vector to the transformed surface. $(\mathbf{M}^{-1})^T$.

   $\square$