Mathematical Finance Guide

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1 Preliminaries

1.1 Interest rates and compounding

The formula for compounding is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where:

- A is the final amount that will be achieved after interest
- P is the principal amount (initial amount)
- r is the annual interest rate (in decimal)
- n is the number of compounding periods per year
- t is the time the money is invested for, in years

1.2 Zero coupon bonds and discounting

A zero coupon bond is a bond that pays its holder its face value at maturity. The price of a zero coupon bond is given by:

$$P = \frac{F}{(1+r)^t}$$

where:

- P is the price of the bond
- F is the face value of the bond
- r is the yield or rate of return of the bond
- t is the time to maturity, in years

1.3 Annuities

An annuity is a series of equal payments made at equal intervals of time. The present value (PV) of an annuity is given by:

$$PV = PMT \times \frac{1-(1+r)^{-n}}{r}$$

where:

- PV is the present value of the annuity
- PMT is the amount of each annuity payment
- r is the interest rate per period
- n is the number of periods

1.4 Daycount conventions

Daycount conventions are methods for calculating the number of days between two dates and the number of days in a year. Here are some commonly used conventions:

- Actual/Actual (ISMA/ISO): Number of actual days in period / 365 or 366
- Actual/360: Number of actual days in period / 360
- Actual/365 Fixed: Number of actual days in period / 365
- 30/360 (or Bond Basis): 30 days in a month and 360 days in a year

2 Forwards, Swaps, and Options

2.1 Derivative contracts

A derivative is a financial instrument whose value is dependent on the value of another asset, known as the underlying asset. The derivative itself is a contract between two or more parties, and its price is determined by fluctuations in the underlying asset. Examples of derivatives include options, futures, and swaps. Derivatives are typically used for hedging risk or for speculation.

2.2 Forward contracts

A forward contract is a private agreement between two parties to buy or sell an asset at a specified future date for a price agreed upon today. Unlike standardized futures contracts, which trade on an exchange, forward contracts are private agreements that trade over-the-counter. They are customized to fit the specific requirements of the buyer and seller.

2.3 Forward on asset paying no income

The price of a forward contract on an asset that pays no income (like a non-dividend paying stock or a commodity) is calculated using the formula:

$$F = S_0 e^{rt}$$

where:

- F is the forward price
- S_0 is the spot price of the asset
- r is the risk-free rate
- t is the time to delivery

2.4 Forward on asset paying known income

If the underlying asset provides a known yield (like a bond or a stock that pays a known dividend), the forward price formula adjusts to:

$$F = S_0 e^{(r-q)t}$$

where q is the yield of the asset, to reflect that the holder of the asset receives income during the holding period.

2.5 Review of assumptions

The forward pricing models assume the absence of arbitrage opportunities, the ability to borrow and lend at the risk-free rate, and the ability to buy or sell any quantity of the asset at the spot price. These assumptions are common in financial models and reflect a perfectly efficient market.

2.6 Value of forward contract

At any point before the maturity of the forward contract, the value of the contract is given by:

$$V_t = S_t - Ke^{-r(T-t)}$$

where:

- V_t is the value of the contract at time t
- S_t is the spot price at time t
- K is the agreed-upon delivery price from the contract
- r is the risk-free rate
- T is the time to maturity from contract initiation

This represents the net payment that would be made if the contract were settled at that time.

2.7 Forward on stock paying dividends and on currency

When dealing with stocks that pay dividends or currencies, the forward price formula adjusts to:

$$F = S_0 e^{(r-q)t}$$

where q now represents the dividend yield or foreign interest rate.

2.8 Physical versus cash settlement

Forward contracts can be settled in two ways at maturity:

- Physical delivery: The seller gives the underlying asset to the buyer in exchange for the forward price
- Cash settlement: The seller pays the buyer the difference between the market price of the underlying asset and the forward price (if positive), or the buyer pays the seller (if negative)

3 Forward Rates and Libor

3.1 Forward zero coupon bond prices

Zero coupon bonds pay no coupons and are issued at a discount to face value. The forward price of a zero coupon bond that matures at time T and is bought at time $t < T$ is given by:

$$F(t,T) = \frac{B(t)}{B(T)}$$

where $B(t)$ is the price of a zero-coupon bond that matures at time t.

3.2 Forward interest rates

The forward interest rate is the future yield on a bond or loan agreed upon now. For a zero coupon bond, the forward interest rate from time t_1 to t_2 is given by:

$$f(t_1,t_2) = \frac{1}{t_2 - t_1}\left(\frac{B(t_1)}{B(t_2)} - 1\right)$$

where $B(t)$ is the price of a zero-coupon bond that matures at time t.

3.3 Libor

LIBOR (London Interbank Offered Rate) is a benchmark interest rate at which major global banks lend to one another in the international interbank market for short-term loans. It is a key reference rate for a vast amount of financial contracts worldwide.

3.4 Forward rate agreements and forward Libor

A Forward Rate Agreement (FRA) is a financial contract that allows the buyer to lock in a future interest rate. The payoff at maturity is given by:

$$N(L(T_1,T_2) - K)(T_2 - T_1)$$

where:

- N is the notional amount
- $L(T_1,T_2)$ is the realized LIBOR from time T_1 to T_2
- K is the FRA rate
- T_1 and T_2 are the start and end times of the FRA

3.5 Valuing floating and fixed cashflows

In financial mathematics, floating cashflows are typically valued by discounting each cashflow at the appropriate zero-coupon bond rate, and fixed cashflows are valued by discounting at the forward rate.

4 Interest Rate Swaps

4.1 Swap definition

An interest rate swap is a financial derivative in which two parties agree to exchange interest rate cash flows, based on a specified notional amount from a fixed rate to a floating rate (or vice versa) within a particular period.

4.2 Forward swap rate and swap value

The forward swap rate is the fixed rate that would make the swap have a zero value at initiation. For an interest rate swap, it is the rate that equates the present value of fixed and floating cashflows. The value of the swap at time t is given by:

$$V(t) = N\,(S(t) - K)\,B(T - t)$$

where:

- $V(t)$ is the value of the swap at time t
- N is the notional amount
- $S(t)$ is the floating rate at time t
- K is the fixed swap rate
- $B(T - t)$ is the discount factor for the remaining life of the swap

4.3 Spot-starting swaps

A spot-starting swap is an interest rate swap that begins at the current spot date. It is the most common type of swap and is used to transform the nature of cash flows from a loan, bond or other financial instrument.

4.4 Swaps as difference between bonds

A swap can also be thought of as the difference between two bonds. One can view an interest rate swap as a portfolio consisting of a long position in a fixed rate bond and a short position in a floating rate bond (or vice versa).

5 Futures Contracts

5.1 Futures definition

A futures contract is a standardized legal agreement to buy or sell something at a predetermined price at a specified time in the future, between parties not known to each other. The asset transacted is usually a commodity or financial instrument, and the contract is facilitated through a futures exchange.

5.2 Futures versus forward prices

While futures and forwards both allow the purchase or sale of an asset at a specific time for a specific price, there are some significant differences. Futures are standardized contracts traded on an exchange, while forwards are private agreements between two parties. In addition, futures contracts are usually closed out before delivery, whereas forward contracts typically involve delivery of the asset. The price of a futures contract tends to be higher than the forward price due to the daily settling (mark-to-market) feature of futures.

5.3 Futures on Libor rates

Futures on Libor rates are a type of interest rate future. The underlying asset is the interest rate, as it is based on the London Interbank Offered Rate (Libor). These are used to hedge against or speculate on changes in future interest rates.

The price of a Libor futures contract is usually quoted as 100 minus the implied forward Libor rate. For example, if the quoted price is 98.00, the implied forward Libor rate is $100 - 98.00 = 2.00\%$.

6 No-Arbitrage Principle

6.1 Assumption of no-arbitrage

The no-arbitrage principle is a fundamental concept in financial economics. It states that in a frictionless, informationally efficient market, there is no way to make riskless profits. In other words, any investment strategy that generates a sure profit without any investment and without incurring risk is not possible.

6.2 Monotonicity theorem

The monotonicity theorem states that if an asset or portfolio has a higher payoff than another in every state of the world, then it should have a higher price. If it does not, then there would exist an arbitrage opportunity.

6.3 Arbitrage violations

Arbitrage violations occur when there exist assets or portfolios that, despite having the same risk and expected return, are priced differently in the market. This discrepancy creates an opportunity for riskless profit, contradicting the no-arbitrage principle. In an efficient market, arbitrageurs would quickly take advantage of these opportunities, which would realign the prices and eliminate the arbitrage.

7 Options

7.1 Option definitions

An option is a type of derivative that represents a contract sold by one party (the option writer) to another party (the option holder). The contract offers the buyer the right, but not the obligation, to buy (call option) or sell (put option) a security or other financial asset at an agreed-upon price during a certain period or on a specific date.

7.2 Put-call parity

Put-call parity is a principle that defines the relationship between the price of European put and call options of the same class. It states that the simultaneous holding of a long European call option and a short European put option yields the same return as holding one forward contract on the same underlying asset with the same expiration and a forward price equal to the option's strike price.

$$C - P = S_0 - Ke^{-rT}$$

where C and P are the call and put prices, S_0 is the initial price of the underlying, K is the strike price, r is the risk-free rate, and T is the time to maturity.

7.3 Bounds on call prices

The price of a European call option C is always bounded by the present value of the strike price and the spot price of the underlying:

$$S_0 \geq C \geq \max(0, S_0 - Ke^{-rT})$$

7.4 Call and put spreads

A spread is an options strategy in which a trader simultaneously purchases and sells two options that are identical in every way but have different strike prices.

7.5 Butterflies and convexity of option prices

A butterfly spread is an options strategy combining bull and bear spreads, involving either four calls or four puts (each with the same expiration date and underlying asset). The trader sells two option contracts at the middle strike price and buys one option contract at a lower strike price and one option contract at a higher strike price.

7.6 Digital options

A digital (or binary) option pays a fixed amount or nothing at all, depending on the price of the underlying asset at the option's expiration. For a digital call with strike price K, the payoff is given by:

$$\text{Payoff} = \begin{cases} 1 & \text{if } S_T > K \\ 0 & \text{otherwise} \end{cases}$$

where S_T is the price of the underlying at time T.

7.7 Options on forward contracts

An option on a forward contract gives the holder the right, but not the obligation, to enter into a forward contract. The payoff from a call option on a forward with strike price K is given by:

$$\text{Payoff} = \max(0, F_T - K)$$

where F_T is the price of the forward contract at time T.

STILL WORK IN PROGRESS...