Express Riddler

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Riddle:

You're playing a game of cornhole with your friends, and it's your turn to toss the four bean bags. For every bean bag you toss onto your opponents' board, you get 1 point. For every bag that goes through the hole on their board, you get 3 points. And for any bags that don't land on the board or through the hole, you get 0 points.

Your opponents had a terrible round, missing the board with all their throws. Meanwhile, your team currently has 18 points—just 3 points away from victory at 21. You're also playing with a special house rule: To win, you must score *exactly* 21 points, without going over.

Based on your history, you know there are three kinds of throws you can make:

- An *aggressive* throw, which has a 40 percent chance of going in the hole, a 30 percent chance of landing on the board and a 30 percent chance of missing the board and hole.
- A *conservative* throw, which has a 10 percent chance of going in the hole, an 80 percent chance of landing on the board and a 10 percent chance of missing the board and hole.
- A *wasted* throw, which has a 100 percent chance of missing the board and hole.

For each bean bag, you can choose any of these three throws. Your goal is to maximize your chances of scoring exactly 3 points with your four tosses. What is the probability that your team will finish the round with exactly 21 points and declare victory?

Solution:

For this problem I define a state (M, N) as having M moves left ($M = 0, 1, 2, 4$) and N points left to win ($N = 0, 1, 2, 3$ or X for an overshoot). Each state has an associated probability $P(M, N)$ that is the probability to win using the best strategy. The simplest probabilities to define are for $M = 0$ (after the fourth throw). For $N = 0$, $P = 1$, and for all other N, $P = 0$. For $N = X$ and any M, clearly $P = 0$. It is also obvious that from any state $(M > 0, 0)$, the best strategy is wasted throws until $M = 0$, which ensures that $P = 1$.

From these base set of states, I built up the other probabilities one move at a time, considering the outcomes of either the aggressive or conservative throws. For example, starting at $(1, 1)$ and throwing aggressively ends at the winning state (0,0) 30% of the time and loses otherwise, but throwing conservatively ends at the winning state 80% of the time. So from this state the best strategy is the conservative throw, and $P(1, 1) = 0.8$. Conversely, from the state (1,2), the only possible throws lead to (0,1), (0,2), and $(0, X)$, which are all losing positions, so $P(1, 2) = 0$.

It turns out that the ideal strategy is pretty intuitive, as follows:

- If exactly 0 points are still needed, throw wastes.
- If exactly 3 points are still needed, throw aggressively.
- Otherwise, throw conservatively.

This results in the following probabilities for all the relevant M and N:

		N				
		0	1	2	3	X
	0	1	0	0	0	0
	1	1	0.8	0	0.4	0
M	2	1	0.88	0.64	0.52	0
	3	1	0.888	0.768	0.748	0
	4	1	0.8888	0.7872	0.8548	0

So the final chance of winning 3 points in 4 throws is $\boxed{\textbf{85.48\%}}$.