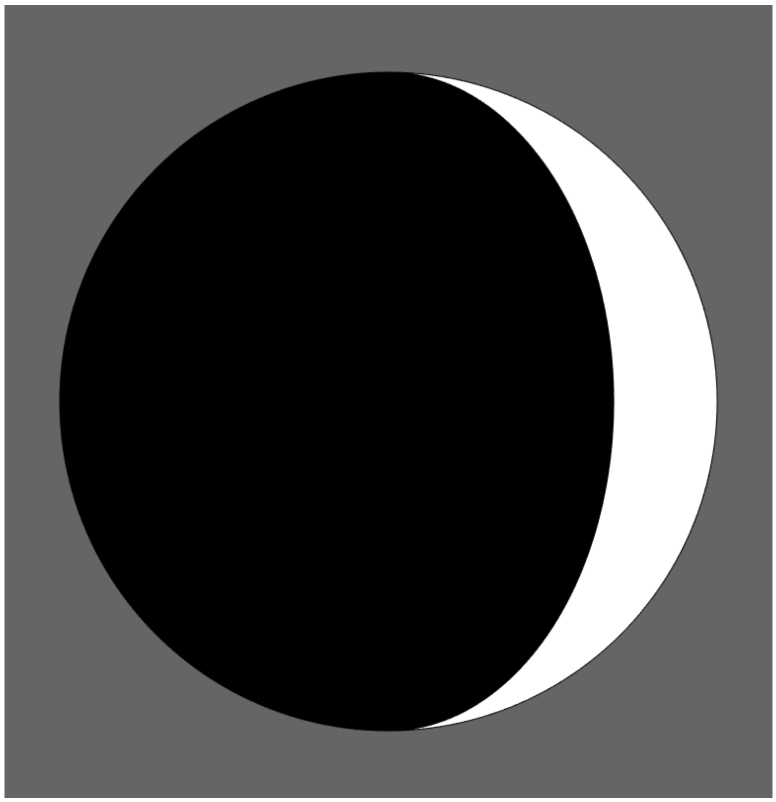
Classic Riddler

16 April 2021

Riddle:

After a new moon, the crescent appears to grow slowly at first. At some point, the moon will be one-sixth full by area, then one-quarter full, and so on. Eventually, it becomes a half-moon, at which point its growth begins to slow down. The animation below provides some insight into what's happening here:



Link

How many times faster is the area of the illuminated moon growing when it is a half-moon versus a one-sixth moon?

(Some simplifying assumptions you might make for this problem are that the moon is a perfect sphere, that its orbit around Earth is a perfect circle, that the moon orbits the Earth much faster than the Earth orbits the sun and that the sun is very, very far away. If you make additional assumptions, feel free to include them in your response.)

Solution:

The moon as viewed from Earth becomes a 2-d circle, as a projection of the 3-d surface of the half-sphere that is illuminated. The shape of the projected illumination (or it's complement) on either side (left or right) is essentially a semicircle which is compressed horizontally. The compression depends on the angle of the rotation, which I will call θ. Basically, the area A of the illuminated projection changes proportionally to $\sin\theta$, or

$$\frac{dA}{d\theta} = k \cdot \sin\theta$$

for some k. Integrating this gives:

$$A = -k \cdot \cos\theta + C$$

Letting $\theta = 0$ represent the new moon ($A = 0$), and letting the maximum area be 1, these become

$$\frac{dA}{d\theta} = \frac{\sin\theta}{2}$$
$$A = \frac{1 - \cos\theta}{2}$$

When $A = {}^1\!/_6$, $\theta = \cos^{-1}({}^2\!/_3)$, and ${}^{dA}\!/_{d\theta} = {}^{\sqrt{5}}\!/_6$. When $A = {}^1\!/_2$, $\theta = {}^{\pi}\!/_2$, and ${}^{dA}\!/_{d\theta} = {}^1\!/_2$. Taking the ratio of these leads to the solution of $\boxed{\mathbf{{}^{3\sqrt{5}}\!/_5 \approx 1.342}}$.