Express Riddler

6 November 2020

Riddle:

Last weekend's New York City Marathon was canceled. But runners from Des Linden—one of the top American marathoners—to FiveThirtyEight's own Santul Nerkar—my number one editor—still went out there and braved the course. Santul finished in a time of 3:41:43 (3 hours, 41 minutes, 43 seconds), which averaged to 8 minutes, 27 seconds per mile.

Suppose, while training, Santul completed two 20-mile runs on a treadmill. For the first run, he set the treadmill to a constant speed so that he ran every mile in 9 minutes.

The second run was a little different. He started at a 10 minute-per-mile pace and accelerated continuously until he was running at an 8-minute-per mile pace at the end. Moreover, Santul's minutes-per-mile pace (i.e., *not* his speed) changed linearly over time. So a quarter of the way through the duration (in time, not distance) of the run, his pace was 9 minutes, 30 seconds per mile, halfway through it was 9 minutes per mile, etc.

Which training run was faster (i.e., took less time) overall? And what were Santul's times for the two runs?

Solution:

I will label the constant-speed time t_1 and the accelerating-speed time t_2. The first time is easy to calculate:

$$\begin{aligned} t_1 &= 9 \text{ min/mile} * 20 \text{ mile} \\ &= 180 \text{ min} \end{aligned}$$

The second time is more complicated. The pace is given as an inverse rate ${}^1/_v(t)$. Based on the parameters given, the inverse rate can be written as

$$\frac{1}{v}(t) = 10 - \frac{2}{t_2} t$$

so that the pace is 10 min/mile at $t = 0$ and 8 min/mile at $t = t_2$. The total distance of 20 miles is the integral of the velocity with respect to time, the pace must be inverted and integrated.

$$\begin{aligned} 20 &= \int v(t)\, dt \\ &= \int_0^{t_2} \frac{1}{10 - \frac{2}{t_2} t}\, dt \\ &= -\frac{t_2}{2} \ln\left(10 - \frac{2}{t_2}\right)\Bigg|_0^{t_2} \\ &= -\frac{t_2}{2} \left[\ln(8) - \ln(10)\right] \\ &= \frac{t_2}{2} \ln\left(\frac{5}{4}\right) \end{aligned}$$

$$\begin{aligned} t_2 &= \frac{40}{\ln\left(\frac{5}{4}\right)} \\ &= 179.2568\ldots \text{ min} \end{aligned}$$

So the solution is that $\boxed{\textbf{the second run is faster by about 45 seconds}}$.