Express Riddler

20 August 2021

Riddle:

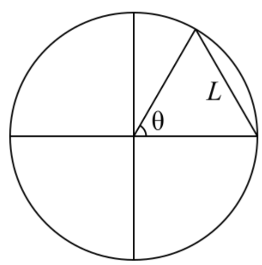
Help, there's a cricket on my floor! I want to trap it with a cup so that I can safely move it outside. But every time I get close, it hops exactly 1 foot in a random direction.

I take note of its starting position and come closer. Boom—it hops in a random direction. I get close again. Boom—it takes another hop in a random direction, independent of the direction of the first hop.

What is the *most probable* distance between the cricket's current position after two random jumps and its starting position? (Note: This puzzle is not asking for the *expected* distance, but rather the *most probable* distance. In other words, if you consider the probability distribution over all possible distances, where is the peak of this distribution?)

Solution:

We can assume that the first jump moves from the edge of a circle (with 1-ft radius) to the center of the circle. Then the second jump will land somewhere on the edge of the circle. We can call the angle between the two jumps θ, and the distance (in ft) between the initial and final positions L. This is shown below:



The distance L can be written as

$$\begin{aligned} L(\theta) &= \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} \\ &= \sqrt{2 - 2\cos\theta} \end{aligned}$$

Finding the most probably distance requires maximizing this equation. Because L is positive, it is maximized at the same points as L^2, which is easier to differentiate:

$$\begin{aligned} L^2 &= 2 - 2\cos\theta \\ \frac{d}{d\theta}L^2 &= 2\sin\theta \end{aligned}$$

Because of symmetry, this only needs to be evaluated in the range $[0, \pi]$. The zeroes are at $\theta = 0$ and π, which conveniently are also the edges of the range. Evaluating these shows that the maximum is at $\theta = \pi$, with a most-likely distance of $\boxed{\textbf{2 ft}}$.