Classic Riddler

24 September 2021

Riddle:

You may recall a previous riddle about the new Olympic event, sport climbing. This week, puzzle submitter Andy Esposito takes us back to this exciting event:

The finals of the sport climbing competition has eight climbers, each of whom compete in three different events: speed climbing, bouldering and lead climbing. Based on their time and performance, each of the eight climbers is given a ranking (first through eighth, with no ties allowed) in each event, as well as a corresponding score (1 through 8, respectively).

The three scores each climber earns are then multiplied together to give a final score. For example, a climber who placed second in speed climbing, fifth in bouldering and sixth in lead climbing would receive a score of $2 \times 5 \times 6$, or 60 points. The gold medalist is whoever achieves the lowest final score among the eight finalists.

What is the highest (i.e., worst) score one could achieve in this event and still have a chance of winning (or at least tying for first place overall)?

Solution:

The product of all scores in each event is 8!, so the total product across all events is $8!^3$. If each player were to have the same score at the end (an eight-way tie), that score would be $8!^{3/8} \approx 53.3$. Of course, this is impossible, but it sets an upper limit on the winning score; If a "winning" score were 54 or higher, another score would need to be 53 or lower, which would then become the winning score.

A score of 53 is not possible, since it is prime. Similarly, 52 and 51 are not possible, since they have factors of 13 and 17, respectively.

The first possible score is 50, which can come from event scores of 2, 5, and 5. For this to be a winning score, each other climber must have scores at or above 50. Using the same logic as earlier, the average score of the remaining seven climbers must be $(8!^3/50)^{1/7} \approx 53.8$. So at least one other climber must have a score below 54. Scores of 53, 52, 51, as well as scores below 50 are excluded. So the next-best score must also be 50. However, two (or more) climbers cannot both score 50, because the total product ($8!^3$) only has three factors of 5. So 50 is not the solution.

The next possible score is 49 (from 1, 7, and 7). After a score of 49, the remaining average is $(8!^3/49)^{1/7} \approx 53.99$. Again, the next-best score must be below 54. It also cannot be 49, because the total product only has three factors of 7. If the second-best score is 50, then the remaining average is $(8!^3/49/50)^{1/6} \approx 54.7$. The third-best score again cannot be 50, but it could be 54 (from 3, 3, and 6). The remaining average is $(8!^3/49/50/54)^{1/5} \approx 54.8$. At this point, the fourth-best score would have to be 54. However, this is not possible, since it *must* be factored as 3, 3, and 6. But a third-place event finish can only happen three times, so this is not possible, and 49 is also not the solution.

The next possible score is 48. For this to be the solution, it only needs to be shown that some combination of event scores results in a lowest, winning score of 48. I show this below, using A–H as the players, ranked in order of total score.

Player	Score 1	Score 2	Score 3	Total Score
A	1	6	8	48
B	6	8	1	48
C	8	1	6	48
D	3	4	4	48
E	2	5	5	50
F	4	2	7	56
G	7	3	3	63
H	5	7	2	70

So in this example, there is a four-way tie for first place. But it is sufficient to show that the highest-possible winning score is $\boxed{\mathbf{48}}$. I don't know if this example (along with permutations) is unique.