Classic Riddler

10 September 2021

Riddle:

One morning, Phil was playing with his daughter, who loves to cut paper with her safety scissors. She especially likes cutting paper into "strips," which are rectangular pieces of paper whose shorter sides are at most 1 inch long.

Whenever Phil gives her a piece of standard printer paper (8.5 inches by 11 inches), she picks one of the four sides at random and then cuts a 1-inch wide strip parallel to that side. Next, she discards the strip and repeats the process, picking another side at random and cutting the strip. Eventually, she is left with nothing but strips.

On average, how many cuts will she make before she is left only with strips?

Extra credit: Instead of 8.5 by 11-inch paper, what if the paper measures m by n inches? (And for a special case of this, what if the paper is square?)

Solution:

I note that the paper size of 8.5×11 produces the same result as 9×11, since it will take 8 cuts along a side with any length greater than 8 and at most 9 inches. This allows the problem to be generalized to any two integers m and n.

Because the two opposite sides of paper are the same, any cut on those two sides results in the same new leftover paper. Therefore, there are only two options for cutting: the m side and the n side. Each side has a 50% probability of being cut. I will label the expected number of remaining cuts from an $m \times n$ paper as $C(m, n)$. If either m or n is 1, then $C = 0$. The minimum number of actual cuts starts from $m = n = 2$, with $C(2, 2) = 1$, since a cut from either side leads to a single leftover strip with width 1.

After a few steps, it is clear that there is a recursive formula for C:

$$C(m, n) = 1 + \left(C(m, n-1) + C(m-1, n)\right)/2, \quad m, n \geq 2$$

I don't know how to turn this into a general formula for $C(m, n)$. For the specific case of $m = 9$, $n = 11$, I generated a table in Excel and got a result of $\boxed{\mathbf{936{,}548/65{,}536 \approx 14.29}}$.