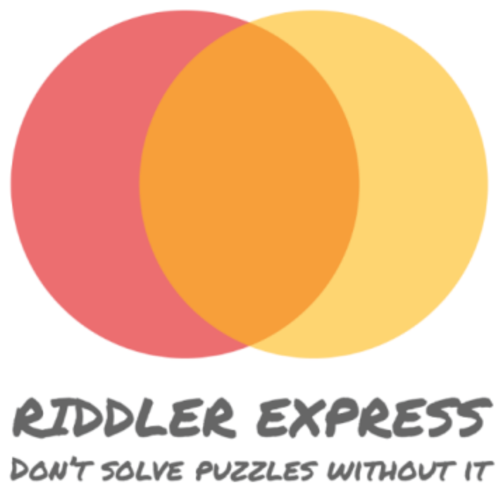
Express Riddler

20 September 2019

Riddle:

This week, Riddler Nation needs your help designing its new credit card, appropriately named Riddler Express™—don't solve puzzles without it!

The logo consists of two overlapping circles with radius 1 inch, creating three distinct regions: one region that's shared between the two circles, and two regions that are part of one circle, but not the other.

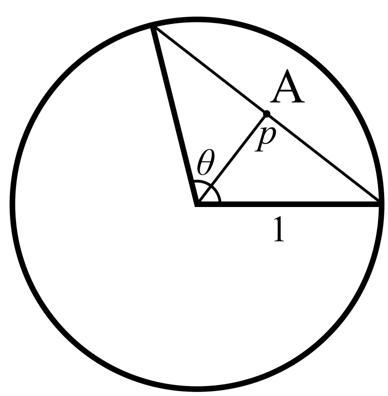


Because the Riddler Express™ card is for the mathematically inclined, its parent company has issued a corporate mandate regarding symmetry. In particular, the areas of all three regions must be *exactly* the same. If that's the case, how far apart must the centers of the two circles be?

Extra Credit: If you symmetrically arrange *three* circles of radius 1, you have seven distinct regions. How far apart should the centers of the circles be such that the areas of the largest and smallest of the seven regions are as close to equivalent as possible?

Solution:

The following diagram illustrates the quantities used to solve this riddle.



The quasi-lune area A represents half of the overlapping region. With a radius of 1, the area of each circle is $\pi 1^2 = \pi$. Then the area of the overlapping region is $\pi/2$, and the area of A is $\pi/4$. The area can be solved from by subtracting the area of the triangle A_t from the area of the arc A_a, both determined by the angle θ:

$$
\begin{aligned}
A = \frac{\pi}{4} &= A_a - A_t \\
&= \pi\left(\frac{\theta}{2\pi}\right) - \frac{\sin\theta}{2}
\end{aligned}
$$

which gives

$$
\theta - \sin\theta = \frac{\pi}{2}
$$

This has the unique solution $\theta \approx 2.3099$. Now with the angle determined, the distance between the circles' centers can be calculated. With the point p at the midpoint of the connecting line, it is also located halfway between the two circles' centers. Therefore the centers are separated by twice the distance $\overline{op}$ (with point o as the origin of the unit circle):

$$
\begin{aligned}
2\overline{op} &= 2\sqrt{\left(0 - \frac{1+\cos\theta}{2}\right)^2 + \left(0 - \frac{\sin\theta}{2}\right)^2} \\
&= \sqrt{2 + 2\cos\theta}
\end{aligned}
$$

This has the (approximate) solution $\boxed{\mathbf{0.8079}}$.