Express Riddler

11 December 2020

Riddle:

There are many ways to slice a big square into smaller squares (not necessarily of equal size), so that the smaller squares don't overlap, while still making up the entire area of the big square.

For example, you can slice the big square into four smaller squares, each a quarter of the area of the big square. Or you could slice it into seven squares, if you take one of those four squares and slice it into four yet smaller squares.

What whole numbers of squares can you *not* slice the big square into?

Solution:

Each slicing operation changes a single square into another number of squares, which is always a square number (of the form n^2). In other words, each operation adds $n^2 - 1$ squares to the total. The final number will be obtained by a sum of consecutive slicing operations.

The first few numbers of the form $n^2 - 1$ for $n \geq 2$ are 3, 8, 15, 24, and so on. I will call these the slice numbers. So the riddle is essentially asking for what numbers cannot be written as a sum of slice numbers. In fact, many larger numbers have multiple ways to be written as a sum of those numbers, but only the numbers that have no sum contribute to the solution. This question is essentially the Frobenius problem, which asks specifically for the largest number which cannot be written as a sum of numbers. In this case, just using the first two slice numbers (3 and 8), the solution to the Frobenius problem is 13 ($= 3 \times 8 - 3 - 8$). So the 15, 24, and higher slice numbers don't even matter.

Besides 13, the other (positive) numbers that aren't a sum of slice numbers are 1, 2, 4, 5, 7, and 10. Taking into account the original square as part of the riddle and shifting these numbers by 1, the solution is $\boxed{\textbf{2, 3, 5, 6, 8, 11, and 14}}$.