Classic Riddler

1 October 2021

Riddle:

You are responsible for setting the ranger schedule at Riddler River National Park. Four rangers are assigned to two locations: the mountain lookout in the north and the lakeside campground in the south. Each assignment lasts one week (Monday through Friday), and every week two rangers should be in the north and two should be in the south.

Your task is to set an assignment schedule that lasts a certain number of weeks and then repeats indefinitely.

In the spirit of fairness, the rangers propose the following conditions for the schedule:

- Each ranger should spend as many weeks in the north as they do in the south.
- Each ranger should spend the same number of weeks paired with each other ranger.
- All rangers should move the same number of times over the course of the schedule. This includes potentially moving back to their starting assignment after the last week of the schedule.
- Exactly two rangers should switch locations each week.

What is the shortest possible repeating schedule that meets the rangers' conditions?

Solution:

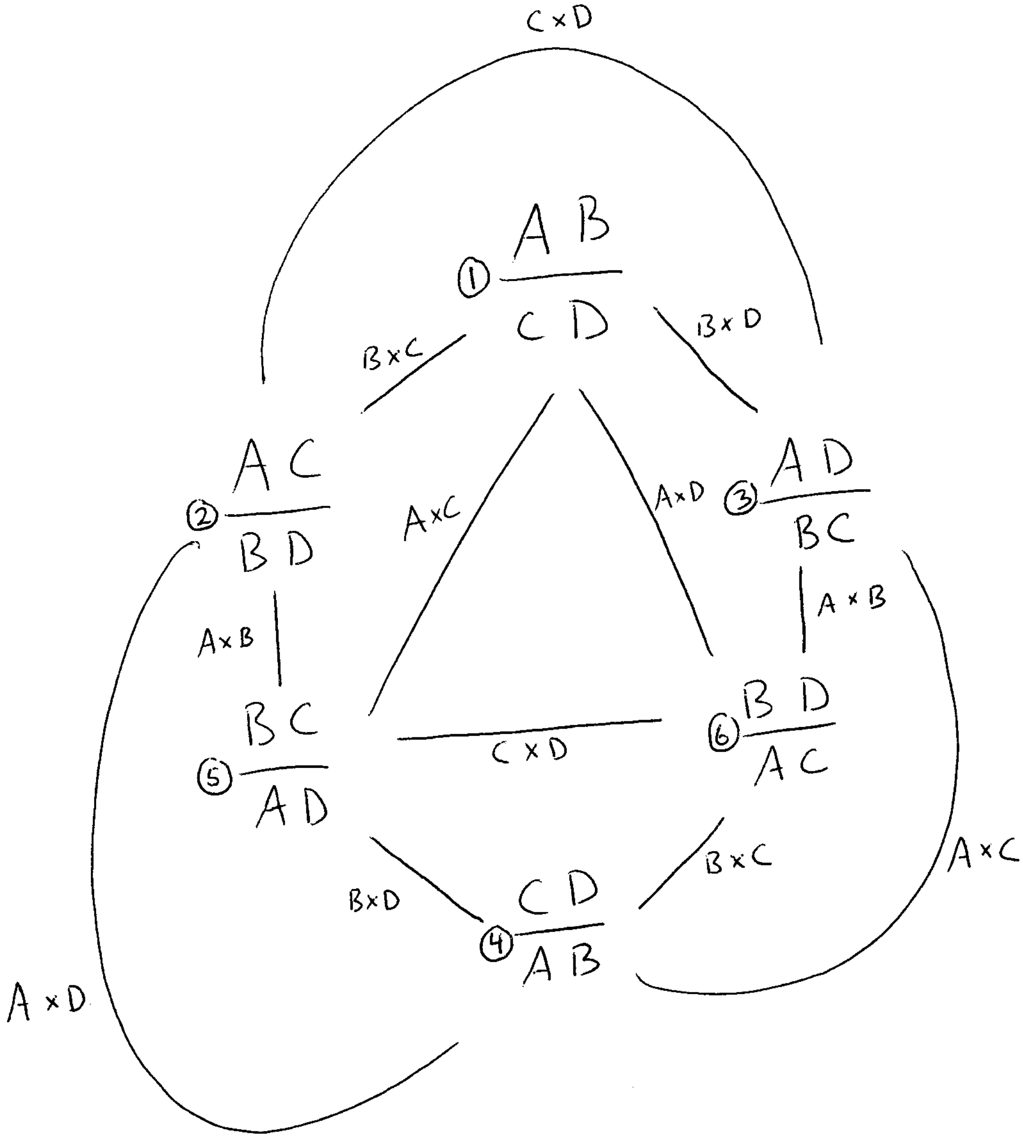
I use the letters A, B, C, and D to represent each of the four rangers, and designate the location of the rangers as a fraction. For example, $\frac{\mathrm{AB}}{\mathrm{CD}}$ indicates A and B in the north location, and C and D in the south location. There are six possible arrangements of rangers: $\frac{\mathrm{AB}}{\mathrm{CD}}$, $\frac{\mathrm{AC}}{\mathrm{BD}}$, $\frac{\mathrm{AD}}{\mathrm{BC}}$, $\frac{\mathrm{CD}}{\mathrm{AB}}$, $\frac{\mathrm{BC}}{\mathrm{AD}}$, and $\frac{\mathrm{BD}}{\mathrm{AC}}$. I number these arrangements as 1, 2, 3, 4, 5, and 6, respectively. For each arrangement, there are four possible ways for exactly two rangers to switch, leading to another arrangement. For example, to move between arrangements 1 and 2, B and C must switch. I drew out all possible arrangements and movements between them on the attached page.

Based on that drawing, it is possible to visualize each arrangement as a vertex of an octohedron, with each possible movement as an edge. Vertices on opposite ends of the octohedron (1 and 4, 2 and 6, 3 and 5) do not have an edge between them, because moving between those arrangements would require all four rangers to switch at the same time.

With this image, I could imagine paths along the edges of the octohedron. By the way the paths are defined, all paths meet the fourth condition. In order to meet the repeating requirement, a path must end on the same vertex it started. Visiting every vertex along the path further ensures it meets the first and second conditions. It is possible to visit every vertex once using six steps. For example, both the paths 1-2-3-4-5-6-1 and 1-3-6-4-2-5-1 (among numerous others) use six movements to visit all six vertices. However, with the first path, A and B each switch twice, while C and D each switch four times. Similarly, with the second path, A and B each switch 4 times, while C and D each switch twice. In fact, all paths of six movements have this phenomenon. If the above two paths are combined into a 12-movement sequence, though, the original conditions are still met, and each ranger switches six times, meeting the third condition.

Thus the solution is $\boxed{\textbf{12 weeks}}$. Here is my full solution:

Week	North	South
1	AB	CD
2	AC	BD
3	AD	BC
4	CD	AB
5	BC	AD
6	BD	AC
7	AB	CD
8	AD	BC
9	BD	AC
10	CD	AB
11	AC	BD
12	BC	AD


C × D
① AB / CD
B × C
B × D
② AC / BD
A × C
A × D
③ AD / BC
A × B
A × B
⑤ BC / AD
C × D
⑥ BD / AC
B × D
B × C
A × C
④ CD / AB
A × D