

Classic Riddler

11 September 2020

Riddle:

This month, the Tour de France is back, and so is the Tour de FiveThirtyEight!

For every mountain in the Tour de FiveThirtyEight, the first few riders to reach the summit are awarded points. The rider with the most such points at the end of the Tour is named “King of the Mountains” and gets to wear a special polka dot jersey.

At the moment, you are racing against three other riders up one of the mountains. The first rider over the top gets 5 points, the second rider gets 3, the third rider gets 2, and the fourth rider gets 1.

All four of you are of equal ability—that is, under normal circumstances, you all have an equal chance of reaching the summit first. But there’s a catch—two of your competitors are on the same team. Teammates are able to work together, drafting and setting a tempo up the mountain. Whichever teammate happens to be slower on the climb will get a boost from their faster teammate, and the two of them will both reach the summit at the faster teammate’s time.

As a lone rider, the odds may be stacked against you. In your quest for the polka dot jersey, how many points can you expect to win on this mountain, on average?

Solution:

This problem is small enough that every possibility can be listed out. First, because every rider individually has the same probability to be in any position, your own probability of being the fastest (and therefore coming in first) is simply $1/4$, so you win 5 points with probability $1/4$. Similarly, you are the slowest, and win 1 point with probability $1/4$. These are independent of the team configurations, because the teamed riders are either both behind or ahead of you together.

If you are the second-fastest rider (which happens with probability $1/4$), then you have to consider the team. There are three possibilities for the team: the fastest and third-fastest riders, the fastest and slowest riders, or the third fastest and slowest riders. In both of the possibilities with the fastest rider, you are bumped down a position, coming in third. In the other possibility, you stay in second place. So you get 2 points with probability $2/12$ and 3 points with probability $1/12$.

If you are the third-fastest rider (probability $1/4$), you must again consider the team. In one of the three possibilities, the teammates are both ahead of you, and your position does not change. In the remaining two possibilities, one teammate is ahead of you, and one teammate is behind you, in which case you are bumped down to last place. So you win 2 points with probability $1/12$ and 1 point with probability $2/12$.

Putting these together leads to an average score of $\boxed{29/12}$.