

Classic Riddler

9 July 2021

Riddle:

The astronomers of Planet Xiddler are back!

This time, they have identified three planets that circularly orbit a neighboring star. Planet A is three astronomical units away from its star and completes its orbit in three years. Planet B is four astronomical units away from the star and completes its orbit in four years. Finally, Planet C is five astronomical units away from the star and completes its orbit in five years. They report their findings to Xiddler's Grand Minister, along with the auspicious news that all three planets are currently lined up (i.e., they are collinear) with their star. However, the Grand Minister is far more interested in the three planets than the star and wants to know how long it will be until the planets are next aligned.

How many years will it be until the three planets are again collinear (not necessarily including the star)?

Solution:

The angular frequencies of the three planets are

$$\omega_A = 2\pi \left(\frac{1}{3} \right)$$

$$\omega_B = 2\pi \left(\frac{1}{4} \right)$$

$$\omega_C = 2\pi \left(\frac{1}{5} \right)$$

However, because it isn't specified in the problem, the planets aren't necessarily orbiting in the same direction. That means that each of the frequencies could be negative or positive, depending on whether each planet is moving in a clockwise or counterclockwise direction. There are eight possible combinations of orbits. Without loss of generality, I will assume that planet C moves counterclockwise, leaving four possibilities for A and B. The other four cases with C moving clockwise will result in the same four final answers.

To simplify the problem, I will consider the positions of A and B relative to a fixed position for C. This means that A and B will orbit with modified frequencies while C does not move. Specifically, the modified frequencies are

$$\omega'_A = 2\pi \left(\pm \frac{1}{3} - \frac{1}{5} \right)$$

$$\omega'_B = 2\pi \left(\pm \frac{1}{4} - \frac{1}{5} \right)$$

If the circular orbits are centered at the origin (and the x-axis directed at C), then the positions of the planets are

$$(x_A(t), y_A(t)) = (3 \cos(\omega'_A t), 3 \sin(\omega'_A t))$$

$$(x_B(t), y_B(t)) = (4 \cos(\omega'_B t), 4 \sin(\omega'_B t))$$

$$(x_C(t), y_C(t)) = (5, 0)$$

With these equations the planets are aligned along the x-axis at time $t = 0$, with t measured in years. From the equations I can determine the slope of the lines passing through C and either A or B. If the slopes of these lines are

the same, then they are the same line, and the planets are collinear. To determine the slope m from two points, I use the equation

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

so that the two slopes between A and C and B and C are

$$m_{AC} = \frac{-3 \sin(\omega'_A t)}{5 - 3 \cos(\omega'_A t)}$$

$$m_{BC} = \frac{-4 \sin(\omega'_B t)}{5 - 4 \cos(\omega'_B t)}$$

Setting these two equations equal to each other allows solving for t , which I did using Wolfram Alpha. As mentioned before, there are four possible solutions depending on the orbits of A and B. If both A and B are moving clockwise (ω_A and ω_B are negative), then the first solution for t after 0 is (approximately) **1.6336 years**. If A is moving clockwise and B is moving counterclockwise (ω_A is negative and ω_B is positive), then the first solution is **5.3061 years**. If A is moving counterclockwise and B is moving clockwise (ω_A is positive and ω_B is negative), then the first solution is **1.5996 years**. Finally, if both A and B are moving counterclockwise (ω_A and ω_B are positive), then the first solution is **7.7668 years**. The solutions are periodic, since all the planets will line up on the same axis with the sun again every 60 years, and there are multiple solutions within a period. But these are simply the first times that the planets line up in any orientation.