Classic Riddler

6 November 2020

Riddle:

Mathematician John von Neumann is credited with figuring out how to take a biased coin (whose probability of coming up heads is p, not necessarily equal to 0.5) and "simulate" a fair coin. Simply flip the coin twice. If it comes up heads both times or tails both times, then flip it twice again. Eventually, you'll get two different flips—either a heads and then a tails, or a tails and then a heads, with each of these two cases equally likely. Once you get two different flips, you can call the second of those flips the outcome of your "simulation."

For any value of p between zero and one, this procedure will always return heads half the time and tails half the time. This is pretty remarkable! But there's a downside to von Neumann's approach—you don't know how long (i.e., how many flips) the simulation will last.

Suppose I want to simulate a fair coin in at most *three* flips. For which values of p is this possible?

Extra credit: Suppose I want to simulate a fair coin in at most N flips. For how many values of p is this possible?

Solution:

We can imagine each coin flip as dividing a unit cube along one axis, with each slice separating the cube into regions with volumes p and $1-p$. After three flips there are 8 sections of the cube, with 4 distinct volumes; each section is a rectangular prism.

Letting section A represent the sequence HHH, it has volume p^3. Sections B, C, and D represent HHT, HTH, THH (in any order) and have volume $p^2(1-p)$. Sections E, F, and G represent HTT, THT, TTH (in any order) and have volume $p(1-p)^2$. Finally, section H represents TTT and has volume $(1-p)^3$.

Solving the riddle involves finding some combination of these sections whose total volume is $1/2$, meaning that after three flips, there is some specific combination of flip sequences whose total probability is $1/2$. Each combination creates a (potentially) cubic equation which gives a solution to the riddle if there is a root for $0 < p < 1$. Specifically, the equation is

$$\sum V_i = \frac{1}{2}$$

where the sum is over the volumes V_i of the specific combination of sections. In principle, there are up to $256 = 2^8$ possible combinations to check. However, the empty combination and the combination with every section cannot be solutions. Further, only half of the combinations need to be checked anyway, since the reverse combination will necessarily have the same area of $1/2$. Additionally, many distinct combinations represent the same volume. I list all of the combinations below and the numerical solutions below. Because of the symmetry of the problem, I have only searched for solutions involving section A. Most solutions have closed forms when solved in Wolfram Alpha, but not all of them; I have not bothered to manually find all closed form solutions, so I only list the numerical solutions. Combinations without valid solutions are marked with "–".

Combination	p
A	0.7937
AH	0.7887 0.2113
ABH ACH ADH	0.6846 0.2373
ABCD	0.5
ABEF ABEG ABFG ACEF ACEG ACFG ADEF ADEG ADFG	0.5
AEFH AEGH AFGH	0.5 0.7071
ABCEF ABCEG ABCFG ABDEF ABDEG ABDFG ACDEF ACDEG ACDFG	0.3522
ABEFH ABEGH ABFGH ACEFH ACEGH ACFGH ADEFH ADEGH ADFGH	–

Combination	p
AB AC AD	0.7071
ABC ABD ACD	0.5970
AEF AEG AFG	0.7347
ABCE ABCF ABCG ABDE ABDF ABDG ACDE ACDF ACDG	0.5
ABEH ABFH ABGH ACEH ACFH ACGH ADEH ADFH ADGH	0.5
ABCDE ABCDF ABCDG	0.4030
ABCEH ABCFH ABCGH ABDEH ABDFH ABDGH ACDEH ACDFH ACDGH	–
AEFGH	–

Combination	p
AE AF AG	0.7718
ABE ABF ABG ACE ACF ACG ADE ADF ADG	0.6478
AEH AFH AGH	0.7627 0.3154
ABCH ABDH ACDH	0.5 0.2929
AEFG	0.5
ABCDH	–
ABEFG ACEFG ADEFG	0.2653
ABCDEF ABCDEG ABCDFG	0.2929

Combination	p
ABCDEH ABCDFH ABCDGH	–
ABEFGH ACEFGH ADEFGH	–
ABCEFGH ABDEFGH ACDEFGH	–

Combination	p
ABCEFG ABDEFG ACDEFG	0.2282
ABCDEFG	0.2063

Combination	p
ABCEFH ABCEGH ABCFGH ABDEFH ABDEGH ABDFGH ACDEFH ACDEGH ACDFGH	–
ABCDEFH ABCDEGH ABCDFGH	–

In total, there are $\boxed{\mathbf{19}}$ possibilities for p. In an easier-to-read list, they are:

0.2063	0.2113	0.2282	0.2373	0.2653
0.2929	0.3154	0.3522	0.4030	0.5
0.5970	0.6478	0.6846	0.7071	0.7347
0.7627	0.7718	0.7887	0.7937	