Express Riddler

29 January 2021

Riddle:

I recently found four cubic blocks in a peculiar arrangement. Three of them were flat on the ground, with their corners touching and enclosing an equilateral triangle. Meanwhile, the fourth cube was above the other three, filling in the gap between them in a surprisingly snug manner. Here's a photo I took of this arrangement:
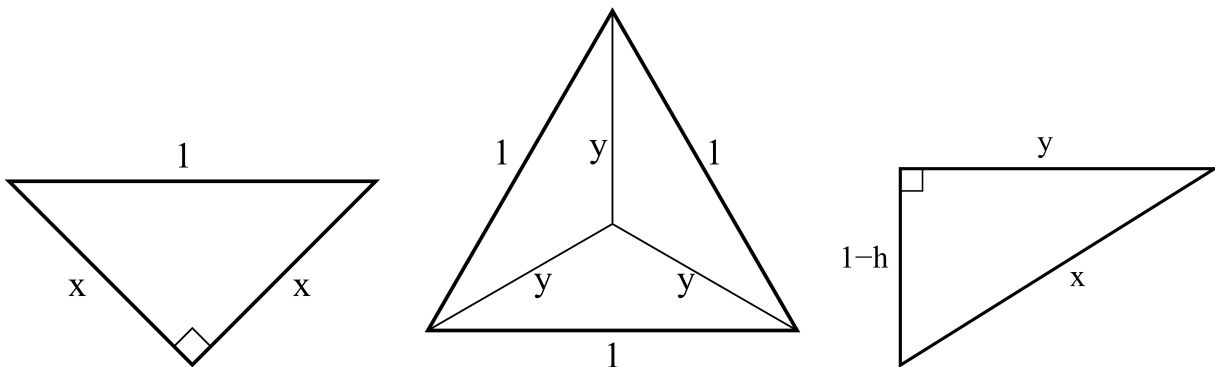

If you too have blocks at home (I mean, of course you do), see if you can make the same arrangement.

Now, if each of the four cubes has side length 1, then how far above the ground is the bottommost corner of the cube on top?

Solution:

Here are the three (not-quite-to-scale) diagrams I used to solve this geometric problem:

The left-most triangle is the section of each of the three faces of the upper cube which dips down into the other cubes. It is a right triangle pointing downward and sitting at the bottom vertex. The hypotenuse of length 1 is flush with an edge of an outer cube. The length x is the length of one of the upper cube's edges which lies inside the other cubes. It is almost trivial to show that $x = \sqrt{2}/2$.

The middle triangle is either a cross-section or a "shadow" of the upper cube along the horizontal plane. It is an equilateral triangle whose three sides of length 1 are the edges of each of the outer cubes. The lengths y are formed by the shadows of each x which meet in the center of the triangle. With a bit more geometrical calculations, it can be shown that $y = \sqrt{3}/3$.

Finally, the right-most triangle shows how each edge x dips down inside the outer cubes. From the top plane of ther outer cubes, the cube's bottom vertex is a distance $1 - h$ down, leaving the remaining h as the distance from the bottom plane. Based on the other values determined above, the value of h, and the riddle's solution, is

$$\boxed{1 - \sqrt{6}/6 \approx 0.592}.$$