# Express Riddler

17 December 2021

Riddle:

You have been tasked with painting a modern building that is shaped like a regular tetrahedron. When the building is viewed from above, the architect wants it to appear as four congruent equilateral triangles—one central blue triangle surrounded by three white triangles.
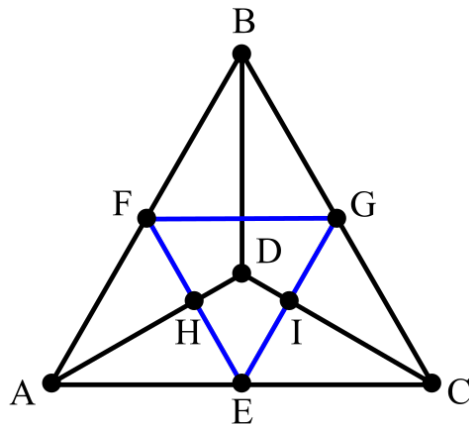
That means that three faces of the tetrahedron contain a blue kite, as shown in the animation below:
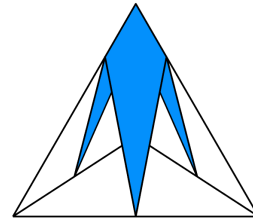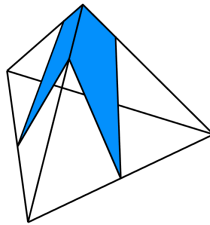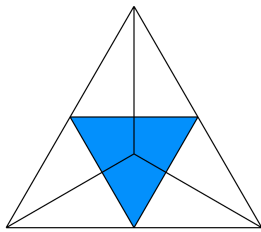
What is the measure of the smallest angle in this kite?

Solution:

To solve this, I use the diagram below, viewing the tetrahedron from directly above (like the first frame earlier). I label the nine points A–I that I use to solve the problem.

Using a side length of 1, and placing A at the origin, I can determine the three-dimensional coordinates of each point:

$$A : (0,0,0)$$
$$B : (1/2, \sqrt{3}/2, 0)$$
$$C : (1,0,0)$$
$$D : (1/2, \sqrt{3}/6, \sqrt{6}/3)$$
$$E : (1/2, 0, 0)$$
$$F : (1/4, \sqrt{3}/4, 0)$$
$$G : (3/4, \sqrt{3}/4, 0)$$
$$H : (3/8, \sqrt{3}/8, \sqrt{6}/4)$$
$$I : (5/8, \sqrt{3}/8, \sqrt{6}/4)$$

With these, I can determine vectors of the edges of the blue area on the front face:

$$\overrightarrow{EH} = (3/8, \sqrt{3}/8, \sqrt{6}/4) - (1/2, 0, 0)$$
$$= (-1/8, \sqrt{3}/8, \sqrt{6}/4)$$
$$\overrightarrow{EI} = (5/8, \sqrt{3}/8, \sqrt{6}/4) - (1/2, 0, 0)$$
$$= (1/8, \sqrt{3}/8, \sqrt{6}/4)$$

The angle $\angle HEI$ can then be determined by the formula:

$$\cos(\angle HEI) = \frac{\overrightarrow{EH} \cdot \overrightarrow{EI}}{\left|\overrightarrow{EH}\right| \left|\overrightarrow{EI}\right|}$$

This gives $\angle HEI = \arccos(13/14) \approx 21.8°$. For completeness, it needs to be shown that this is indeed the smallest angle. The top angle of the kite ($\angle HDI$) is 60° because it is the corner of an equilateral triangle. The other two side angles of the kite ($\angle DHE$ and $\angle DIE$) are both equal to $(360° - 60° - 21.8°)/2 \approx 139.1°$. So the smallest angle in the kite is (approximately) $\boxed{\mathbf{21.8°}}$.