Express Riddler

5 February 2021

Riddle:

For your first weekly CrossProduct, there are five three-digit numbers — each belongs in a row of the table below, with one digit per cell. The products of the three digits of each number are shown in the rightmost column. Meanwhile, the products of the digits in the hundreds, tens and ones places, respectively, are shown in the bottom row.

			135
			45
			64
			280
			70
3,000	3,969	640	

Can you find all five three-digit numbers and complete the table?

Solution:

The riddle is essentially asking for 15 digits to be placed in a 5×3 table. The first step is to decompose each number into either three or five (not-necessarily-prime) single-digit factors. For some, there are multiple sets of possible factors. I list these below:

$$
\begin{aligned}
135 &= 3 \cdot 5 \cdot 9 \\
45 &= 3 \cdot 3 \cdot 5 & &= 1 \cdot 5 \cdot 9 \\
64 &= 1 \cdot 8 \cdot 8 & &= 2 \cdot 4 \cdot 8 & &= 4 \cdot 4 \cdot 4 \\
280 &= 5 \cdot 7 \cdot 8 \\
70 &= 2 \cdot 5 \cdot 7 \\
3000 &= 4 \cdot 5 \cdot 5 \cdot 5 \cdot 6 & &= 3 \cdot 5 \cdot 5 \cdot 5 \cdot 8 \\
3969 &= 3 \cdot 3 \cdot 7 \cdot 7 \cdot 9 & &= 1 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \\
640 &= 1 \cdot 2 \cdot 5 \cdot 8 \cdot 8 & &= 1 \cdot 4 \cdot 4 \cdot 5 \cdot 8 & &= 2 \cdot 2 \cdot 4 \cdot 5 \cdot 8 & &= 2 \cdot 4 \cdot 4 \cdot 4 \cdot 5
\end{aligned}
$$

The set of factors that appear for the first five numbers must match the set of factors for the final three. Additionally, each of the three digits from the first five numbers must come from a different factorization of the bottom three numbers, and each of the five digits from the final three numbers must come from a different factorization of the first five numbers. In the factorization of 3,969, the two 7s must come from the 280 and 70. The remaining factors are either $3 \cdot 3 \cdot 9$ or $1 \cdot 9 \cdot 9$. The $3 \cdot 3 \cdot 9$ is not possible, because the factorization of 64 does not have any factor of 3. This means that the middle column must be (in order) 9, 9, 1, 7, 7.

Now that the third row has a 1, the only possible factorization of 64 is $1 \cdot 8 \cdot 8$. Since the 1 is in the middle, this row is 8, 1, 8.

The first factorization of 3,000 is not possible because a 6 does not appear in any other factorization. So the first column must use $3 \cdot 5 \cdot 5 \cdot 5 \cdot 8$. The 8 is in the third row. The 3 cannot appear in the bottom two rows, so those must be 5s. This means that the bottom two rows are 5, 7, 8 and 5, 7, 2.

The third column now has two 8s and a 2 in the last three rows. The only factorization of 640 with these is $1 \cdot 2 \cdot 5 \cdot 8 \cdot 8$. So the third column has a 1 and a 5 in either of the first two rows.

The only factors left are a 3 and 5 in the first column, and 1 and 5 in the last column. The 135 needs an additional factor of 3 with the 9 in the middle. Therefore, the first column has 3 in the first row and 5 in the second row. This means the first row is 3, 9, 5, and the second row is 5, 9, 1.

The final solution is

3	9	5
5	9	1
8	1	8
5	7	8
5	7	2