Express Riddler

7 May 2021

Riddle:

Can you find three distinct numbers such that the second is the square of the first, the third is the square of the second, and the first is the sqaure of the third? Assuming you can, what are the three such numbers?

Extra credit: Can you find three *other* numbers with the same property?

Solution:

The only way that squaring a number (once or more) and return to the same number itself is for a number located on the unit circle in the complex plane. All such numbers have magnitude 1, so that squaring (or raising to any power) will have the effect of rotating around the circle, while maintaining the same magnitude.

In this problem, squaring a number three times is equivalent to doubling its angle from the positive real axis three times. This results in a rotation which ends up at 8 times the initial angle. If the intial angle is some fraction of the circle, say $^a/_b$, then the final angle will be $^{8a}/_b$. Removing the whole part of the fraction is equivalent to subtracting nb from the numerator for some integer n. Thus the solution(s) to the riddle comes from solving the equation $8a - nb = a$, or $7a = nb$. Thus, the solution exists for $b = 7$, and $a = 1, 2, 3, 4, 5, 6$. Letting $a = 0$ or 7 results in a solution of 1, which simply results in 1 after squaring, so doesn't meet the distinct criterion in the riddle. Other values of a outside this range result in duplicated solutions; for example the fraction $-^1/_7$ is equivalent to $^6/_7$.

Analyzing these solutions, there are two sets of three distinct numbers. $a = 1, 2, 4$ forms one loop of squares, and $a = 3, 5, 6$ forms another. Turning these fractions into riddle solutions requires using them as the exponential argument, multiplied by $2\pi i$. Thus the first set of solutions are $\boxed{\boldsymbol{e^{2\pi i*1/7},\ e^{2\pi i*2/7},\ e^{2\pi i*4/7}}}$, and the second set of solutions are $\boxed{\boldsymbol{e^{2\pi i*3/7},\ e^{2\pi i*5/7},\ e^{2\pi i*6/7}}}$.