Express Riddler

9 July 2021

Riddle:

Earlier this year, a new generation of Brood X cicadas had emerged in many parts of the U.S. This particular brood emerges every 17 years, while some other cicada broods emerge every 13 years. Both 13 and 17 are prime numbers—and relatively prime with one another—which means these broods are rarely in phase with other predators or each other. In fact, cicadas following a 13-year cycle and cicadas following a 17-year cycle will only emerge in the same season once every 221 (i.e., 13 times 17) years!

Now, suppose there are two broods of cicadas, with periods of A and B years, that have just emerged in the same season. However, these two broods can also interfere with each other one year *after* they emerge due to a resulting lack of available food. For example, if A is 5 and B is 7, then B's emergence in year 14 (i.e., 2 times 7) means that when A emerges in year 15 (i.e., 3 times 5) there won't be enough food to go around.

If both A and B are relatively prime and are both less than or equal to 20, what is the longest stretch these two broods can go without interfering with one another's cycle? (Remember, both broods just emerged this year.) For example, if A is 5 and B is 7, then the interference would occur in year 15.

Solution:

I basically solved this problem by hand. I listed out all of the multiples of the numbers 11 through 20, up to 400 (20^2). Then I just looked for interfering multiples for pairs of distinct numbers. Numbers less than 11 would have the same (or earlier) interferences because they already divide at least one number in the range 11–20, so it wasn't necessary to consider them.

It wasn't actually necessary to look for these multiples for all pairs of numbers. For pairs of adjacent numbers of the form n-$(n+1)$, the first interference will always be after $n+1$ years, of which the maximum for this problem is 20 (for the pair 19-20). For pairs of numbers that differ by two (or four, six, or eight), pairs of even numbers didn't need to be considered, since they are not co-prime. Likewise for pairs of numbers (in the range 11–20) which are both multiples of three, four, five, and six.

It turns out the largest possible interference year occurs for the pair 17-19. This interference year is 153, which is 17×9, and occurs immediately after $19 \times 8 = 152$. So the solution is $\boxed{\textbf{153}}$.