Classic Riddler

22 October 2021

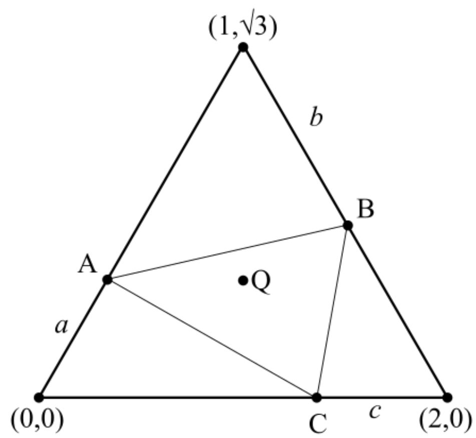
Riddle:

Suppose you have an equilateral triangle. You pick three random points, one along each of its three edges, uniformly along the length of each edge—that is, each point along each edge has the same probability of being selected.

With those three randomly selected points, you can form a new triangle inside the original one. What is the probability that the center of the larger triangle also lies inside the smaller one?

Solution:

To solve this problem, I set up the diagram below. I set the side length of the triangle to be 2, with vertices located at $(0,0)$, $(2,0)$, and $(1,\sqrt{3})$. The center point Q of the triangle is at $(1, \sqrt{3}/3)$. I designate the three points along the sides as A, B, and C; a, b, and c are the variables that represent the proportion along the side length where A, B, and C are located, respectively.

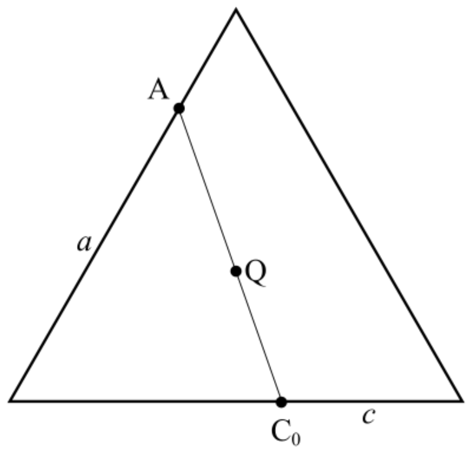


Based on this, point A is at $(a, \sqrt{3}a)$, B is at $(1+b, \sqrt{3}(1-b))$, and C is at $(2(1-c), 0)$.

There are eight scenarios for the variables. First, if each of a, b, and c is greater than $1/2$, then Q is guaranteed to be inside the smaller triangle. This happens because if a is greater than $1/2$, the line $\overline{\text{AB}}$ lies above Q no matter the value of b. Similarly, if b is greater than $1/2$, $\overline{\text{BC}}$ lies to the right of Q, and if c is greater than $1/2$, $\overline{\text{AC}}$ lies to the left of Q. The same reasoning applies when a, b, and c are each less than $1/2$. Because a, b, and c are evenly distributed between 0 and 1, these two scenarios each happen with probability $(1/2)^3 = 1/8$.

Each of the other six scenarios involves two variables on the same side of $1/2$ and the remaining variable on the other side of $1/2$. Because of symmetry, each of these will have the same probability of having Q inside the smaller triangle. So without loss of generality, I consider the case with a greater than $1/2$ and c less than $1/2$. As mentioned above, it doesn't matter what b is when a is greater than $1/2$, so I actually consider both scenarios together for b less than and greater than $1/2$.

For a given value of a, I draw the line $\overline{\text{AQ}}$ and extend it to the C side of the equilateral triangle. I designate the point of intersection as C_0. This is shown in the second diagram below.



I can determine that C_0 is at $({}^{2a}/_{(3a-1)}, 0)$. Because this is an absolute point on the triangle, I can convert this to the proportion along the axis, which I designate c_0. The value of c_0 is ${}^{(2a-1)}/_{(3a-1)}$. This becomes a lower limit for integration along c. The upper limit within this scenario is $^1/_2$. The variable a is integrated between $^1/_2$ and 1, and b is ignored in the scenario. The probability p of having Q inside the smaller triangle in this scenario is

$$p = \int_{\frac{1}{2}}^{1} \int_{\frac{2a-1}{3a-1}}^{\frac{1}{2}} dc\, da$$

which has a result of approximately 0.0707. Because this represents two of six equivalent scenarios, this must be tripled and added to the probabilities above. The total probability is (approximately) $\boxed{\mathbf{0.4621}}$.