Express Riddler

26 June 2020

Riddle:

In Riddler City, the city streets follow a grid layout, running north-south and east-west. You're driving north when you decide to play a little game. Every time you reach an intersection, you randomly turn left or right, each with a 50 percent chance.

After driving through 10 intersections, what is the probability that you are still driving north?

Extra credit: Now suppose that at every intersection, there's a one-third chance you turn left, a one-third chance you turn right and a one-third chance you drive straight. After driving through 10 intersections, *now* what's the probability that you are still driving north?

Solution:

This riddle is really straightforward. The outcome of turning from north is the same as turning from south; both lead to driving east or west, each with 50% probability. Similarly, turning from east or west both result in driving north or south with 50% probability. Therefore, the total trip is just a pattern, alternating between north/south and east/west each turn. So if you start driving north, after 10 or any number of even intersections, you are driving north or south, each still with 50% probability. So the solution is $\boxed{\textbf{50\%}}$.

The extra credit requires actual math. From each direction, we can easily figure out the probabilities to go in any direction. From north, each of west, north, and east has a $1/3$ probability, and so on. We can turn that into a matrix, with each row corresponding to an incoming direction, and each column an outgoing direction. The value at a given position is the probability of turning to an outgoing direction given an incoming direction. Calling the matrix A, and letting the rows and columns be arranged in the order north, east, south, and west, we have:

$$A = \begin{bmatrix} 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

To find the probabilities for a final direction, we need to multiply this by a 4×1 vector that describes the initial distributions of driving directions. In this case, the direction is only north, so the vector v becomes

$$v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Multiplying this by the matrix (Av) gives us another 4×1 vector that lists the probabilities for each direction after a single turn. The probability distrubtion after n turns just means A gets multiplied n times, so the calculation becomes $A^n v$. We can separate these terms and calculate A^n before applying it to v. Luckily, there is an easy general form for A^n in this case.

$$A^n = \begin{cases} \frac{1}{3^n}\begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix} & n \text{ even} \\ \frac{1}{3^n}\begin{bmatrix} a & a & b & a \\ a & a & a & b \\ b & a & a & a \\ a & b & a & a \end{bmatrix} & n \text{ odd} \end{cases}$$

where

$$a = \left\lceil \frac{3^n}{4} \right\rceil$$

$$b = \left\lfloor \frac{3^n}{4} \right\rfloor$$

For $n = 10$, the final probability for starting north and ending north $\left({}^{a(10)}/_{3^{10}}\right)$ is ${}^{14763}/_{59049} \approx 0.25001$, which is very close to ${}^{1}/_{4}$. Of course, as the number of turns gets large the probabilities all start to smear together and approach ${}^{1}/_{4}$, but the initial direction always keeps a very slight advantage. So the solution to the extra credit is $\boxed{\mathbf{{}^{14763}/_{59049}}}$.