Express Riddler

12 November 2021

Riddle:

I have three dice (d4, d6, d8) on my desk that I fiddle with while working, much to the chagrin of my co-workers. For the uninitiated, the d4 is a tetrahedron that is equally likely to land on any of its four faces (numbered 1 through 4), the d6 is a cube that is equally likely to land on any of its six faces (numbered 1 through 6), and the d8 is an octahedron that is equally likely to land on any of its eight faces (numbered 1 through 8).

I like to play a game in which I roll all three dice in "numerical" order: d4, then d6 and then d8. I win this game when the three rolls form a strictly increasing sequence (such as 2-4-7, but *not* 2-4-4). What is my probability of winning?

Extra credit: Instead of three dice, I now have six dice: d4, d6, d8, d10, d12 and d20. If I roll all six dice in "numerical" order, what is the probability I'll get a strictly increasing sequence?

Solution:

There are $4 \times 6 \times 8 = 192$ possible ways to roll the three dice in order. That is in the range where it is possible to list out every winning combination. I've listed out those combinations here:

1-2-3	1-3-6	1-5-7	2-3-8	2-6-7	3-5-8
1-2-4	1-3-7	1-5-8	2-4-5	2-6-8	3-6-7
1-2-5	1-3-8	1-6-7	2-4-6	3-4-5	3-6-8
1-2-6	1-4-5	1-6-8	2-4-7	3-4-6	4-5-6
1-2-7	1-4-6	2-3-4	2-4-8	3-4-7	4-5-7
1-2-8	1-4-7	2-3-5	2-5-6	3-4-8	4-5-8
1-3-4	1-4-8	2-3-6	2-5-7	3-5-6	4-6-7
1-3-5	1-5-6	2-3-7	2-5-8	3-5-7	4-6-8

That is a total of 48 combinations, which means the solution is $^{48}/_{192}$, or $\boxed{\mathbf{^1/_4}}$.

Another way of looking at the first solution is that there are six combinations that begin 1-2, five that begin 1-3, four that begin 1-4, three that begin 1-5, and two that begin 1-6. Similarly, there are five combinations that begin 2-3, four that begin 2-4, three that begin 2-5, and two that begin 2-6. Then there are four that begin 3-4, three that begin 3-5, and two that begin 3-6. Finally there are three that begin 4-5 and two that begin 4-6. These can be written as a sum:

$$\sum_{j=3}^{6} \sum_{i=2}^{j} i$$

which indeed gives the solution of 48.

I starting writing out the winning combinations for the six-dice problem. It yields similar (though much larger) patterns of winning combinations. For example, there are 15 combinations that begin 1-2-3-4-5, 14 that begin 1-2-3-4-6, etc., down to eight solutions that begin 1-2-3-4-12. Extending this pattern ultimately gives the sum:

$$\sum_{m=12}^{15} \sum_{l=11}^{m} \sum_{k=10}^{l} \sum_{j=9}^{k} \sum_{i=8}^{j} i$$

This has a total sum of 5,434. I don't have an explanation for each of the indices in the sum, but it seems to work. Because there are $4 \times 6 \times 8 \times 10 \times 12 \times 20 = 460{,}800$ possible rolls, this means the solution is ${}^{5{,}434}/_{460{,}800}$, or $\boxed{\mathbf{{}^{2{,}717}/_{230{,}400} \approx 0.01179}}$.