

Express Riddler

2 April 2021

Riddle:

You and Wenjun are playing a game in which you alternate taking turns, removing pennies from a pile. On your turn, you can remove either one or two pennies from the pile. On Wenjun's turn, he can remove either two or three pennies. Whoever takes the last penny *loses*. (If there is only one penny left and it's Wenjun's turn, then he skips his turn, which means you will lose). Suppose both you and Wenjun play optimally.

- 1) If you go first, then what initial numbers of pennies mean *you* will win the game?
- 2) If Wenjun goes first, then what initial numbers of pennies mean *he* will win the game?

Solution:

Like other NIM games, the way to solve is starting from the end of the game when somebody loses. In order for Wenjun to lose, he has to start a turn with exactly 2 pennies. Since he can't remove 3 coins, he must remove both of the remaining pennies. If there is 1 penny left, he gets to pass, and you lose by removing the last penny. (If there are 0 pennies left, that means you have already lost.) With 3 pennies left, he can choose to remove only 2, leaving you to remove the last penny. Similarly, with 4 pennies left, he can choose to remove 3, also leaving you to remove the last penny.

So in order to make sure that Wenjun loses, you must finish a turn with 2 pennies left. This can be done if you start your turn with 3 (and remove 1) or start your turn with 4 (and remove 2). Thus, if Wenjun finishes a turn with 3 or 4 pennies left, you can win. If Wenjun starts a turn with 6 pennies, then he can only finish his turn with 3 or 4 pennies, and he will lose in the end.

This is where the pattern starts to repeat. Wenjun will lose (and you will win) if his turn starts with 2, 6, 10, 14... pennies, or more generally, any number of the form $4n + 2$. Following the same pattern, Wenjun will also lose if your turn starts with 3, 4, 7, 8, 11, 12..., which are numbers of the form $4n$ and $4n + 3$.

So to answer the specific questions in the riddle: When you go first, you will win if the game starts with **$4n$ or $4n + 3$ pennies**.

When Wenjun goes first, Wenjun will win if the game starts with **$4n, 4n + 1$, or $4n + 3$ pennies**.