Hierarchical Bayesian Model for Hospital Length of Stay

1 Observed and unobserved variables

Observed variables

$$\begin{aligned} A_t &:= \text{number of COVID-19 hospital admissions on day } t \\ C_t &:= \text{COVID-19 hospital midnight census count on day } t \end{aligned}$$

Unobserved (but unambiguously deducible) variables

$$D_t := \text{number of COVID-19 hospital discharges on day } t$$

Relations

$$\begin{aligned} C_t &= \sum_{\tau=0}^{t} A_\tau - \sum_{\tau=0}^{t} D_\tau \\ D_t &= \sum_{\tau=0}^{t} D_\tau - \sum_{\tau=0}^{t-1} D_\tau \end{aligned}$$

COVID-19 hospital length of stay distribution: Weibull

$$D_t \sim \text{NegativeBinomial}\left(d_t\,,\, d_t + \frac{d_t^2}{\psi} \right)$$

$$d_t = \sum_{\tau=0}^{t-1} A_\tau \cdot \pi_{t-\tau}$$

$$\pi_\tau := P\left(\begin{array}{c} \text{die or discharged} \\ \text{on the } \tau^{\text{th}} \text{ day} \\ \text{after admission} \end{array} \middle| \begin{array}{c} \text{COVID-19} \\ \text{hospital} \\ \text{admission} \end{array} \right) = \left\{ \begin{array}{ll} \displaystyle\int_0^{3/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau = 1, \\ \displaystyle\int_{\tau-1/2}^{\tau+1/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau = 2,3,\ldots \end{array} \right.$$

Impose prior distribution on, and compute/estimate posterior distribution for:

$$(\alpha,\sigma,\psi) \in (0,\infty)\times(0,\infty)\times(0,\infty).$$

Derivations

$$\begin{aligned} f_{\text{Wb}}(\,s\,;\alpha,\sigma) &= \left(\begin{array}{c} \text{probability density function of the Weibull distribution} \\ \text{with shape parameter } \alpha > 0 \text{, scale parameter } \sigma > 0 \end{array} \right) \\ &= \frac{\alpha}{\sigma}\cdot\left(\frac{s}{\sigma}\right)^{\alpha-1}\cdot\exp\left(\left(-\frac{s}{\sigma}\right)^{\alpha}\right), \quad \text{for } s \geq 0 \end{aligned}$$

$$\begin{aligned} F_{\text{Wb}}(\,s\,;\alpha,\sigma) &= \left(\begin{array}{c} \text{cumulative distribution function of the Weibull distribution} \\ \text{with shape parameter } \alpha > 0 \text{, scale parameter } \sigma > 0 \end{array} \right) \\ &= 1 - \exp\left(-(s/\sigma)^{\alpha} \right), \quad \text{for } s \geq 0 \end{aligned}$$

Hence,

$$\pi_\tau \;:=\; P\left(\begin{array}{c}\text{die or discharged}\\ \text{on the } \tau^{\text{th}} \text{ day}\\ \text{after admission}\end{array}\;\middle|\;\begin{array}{c}\text{COVID-19}\\ \text{hospital}\\ \text{admission}\end{array}\right) \;=\; \left\{\begin{array}{ll} \displaystyle\int_0^{3/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau=1,\\ \displaystyle\int_{\tau-1/2}^{\tau+1/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau=2,3,\ldots \end{array}\right.$$

$$= \left\{\begin{array}{ll} F_{\text{Wb}}\!\left(\dfrac{3}{2}\,;\,\alpha,\sigma\right), & \text{for } \tau=1,\\[2ex] F_{\text{Wb}}\!\left(\dfrac{\tau+1/2}{2}\,;\,\alpha,\sigma\right) \;-\; F_{\text{Wb}}\!\left(\dfrac{\tau-1/2}{2}\,;\,\alpha,\sigma\right), & \text{for } \tau=2,3,\ldots \end{array}\right.$$

$$= \left\{\begin{array}{ll} 1 \;-\; \exp\!\left(-\left(\dfrac{3/2}{\sigma}\right)^{\alpha}\right), & \text{for } \tau=1,\\[2ex] \exp\!\left(-\left(\dfrac{\tau-1/2}{\sigma}\right)^{\alpha}\right) \;-\; \exp\!\left(-\left(\dfrac{\tau+1/2}{\sigma}\right)^{\alpha}\right), & \text{for } \tau=2,3,\ldots \end{array}\right.$$