Hierarchical Bayesian Model for Hospital Length of Stay

1 Observed and unobserved variables

Observed variables

$$\begin{aligned} A_t &:= \text{number of COVID-19 hospital admissions on day } t \\ C_t &:= \text{COVID-19 hospital midnight census count on day } t \end{aligned}$$

Unobserved (but unambiguously deducible) variables

$$D_t := \text{number of COVID-19 hospital discharges on day } t$$

Relations

$$\begin{aligned} C_t &= \sum_{\tau=0}^{t} A_\tau - \sum_{\tau=0}^{t} D_\tau \\ D_t &= \sum_{\tau=0}^{t} D_\tau - \sum_{\tau=0}^{t-1} D_\tau \end{aligned}$$

Likelihood assumption #1: Daily death/discharge count $\sim$ Negative Binomial

$$D_t \sim \text{NegativeBinomial}\left(d_t \,,\, d_t + \frac{d_t^2}{\psi} \right)$$

Likelihood assumption #2: Decomposition of d_t by day of admission

$$\begin{aligned} d_t &:= \text{expected number of deaths/discharges on day } t \\ &= \sum_{\tau=0}^{t-1} \left(\begin{array}{c} \text{expected number of deaths/discharges on day } t \\ \text{among COVID-19 patients admitted on day } 0 \leq \tau < t \end{array} \right) \\ &= \sum_{\tau=0}^{t-1} \left(\begin{array}{c} \text{number of} \\ \text{admissions} \\ \text{on day } \tau \end{array} \right) \cdot \left(\begin{array}{c} \text{proportion of} \\ \text{death/discharge} \\ \text{after } t-\tau \text{ days} \end{array} \right) \\ &= \sum_{\tau=0}^{t-1} A_\tau \cdot \pi_{t-\tau} \end{aligned}$$

Likelihood assumption #3: Random admission-to-discharge delay (i.e. length of stay) $\sim$ Weibull

$$\pi_\tau := P\left(\begin{array}{c} \text{die or discharged} \\ \text{on the } \tau^{\text{th}} \text{ day} \\ \text{after admission} \end{array} \,\middle|\, \begin{array}{c} \text{COVID-19} \\ \text{hospital} \\ \text{admission} \end{array} \right) = \left\{ \begin{array}{ll} \displaystyle\int_0^{3/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau = 1, \\ \displaystyle\int_{\tau-1/2}^{\tau+1/2} f_{\text{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau = 2,3,\ldots \end{array} \right.$$

Impose prior distribution on, and compute/estimate posterior distribution for:

$$(\alpha, \sigma, \psi) \;\in\; (0,\infty)\times(0,\infty)\times(0,\infty).$$

Derivations

$$\begin{aligned} f_{\mathrm{Wb}}(\,s\,;\alpha,\sigma) &= \left(\begin{array}{c}\text{probability density function of the Weibull distribution}\\ \text{with shape parameter } \alpha>0\text{, scale parameter } \sigma>0\end{array}\right) \\ &= \frac{\alpha}{\sigma}\cdot\left(\frac{s}{\sigma}\right)^{\alpha-1}\cdot\exp\!\left(\left(-\frac{s}{\sigma}\right)^{\alpha}\right), \quad \text{for } s\geq 0 \end{aligned}$$

$$\begin{aligned} F_{\mathrm{Wb}}(\,s\,;\alpha,\sigma) &= \left(\begin{array}{c}\text{cumulative distribution function of the Weibull distribution}\\ \text{with shape parameter } \alpha>0\text{, scale parameter } \sigma>0\end{array}\right) \\ &= 1-\exp\!\left(-(s/\sigma)^{\alpha}\right), \quad \text{for } s\geq 0 \end{aligned}$$

Hence,

$$\pi_\tau \;:=\; P\left(\begin{array}{c|c}\text{die or discharged} & \text{COVID-19}\\ \text{on the }\tau^{\text{th}}\text{ day} & \text{hospital}\\ \text{after admission} & \text{admission}\end{array}\right) \;=\; \begin{cases} \displaystyle\int_0^{3/2} f_{\mathrm{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau=1, \\ \displaystyle\int_{\tau-1/2}^{\tau+1/2} f_{\mathrm{Wb}}(\,s\,;\alpha,\sigma)\,\mathrm{d}s, & \text{for } \tau=2,3,\ldots \end{cases}$$

$$= \begin{cases} F_{\mathrm{Wb}}\!\left(\dfrac{3}{2}\,;\alpha,\sigma\right), & \text{for } \tau=1, \\ F_{\mathrm{Wb}}\!\left(\dfrac{\tau+1/2}{2}\,;\alpha,\sigma\right) \;-\; F_{\mathrm{Wb}}\!\left(\dfrac{\tau-1/2}{2}\,;\alpha,\sigma\right), & \text{for } \tau=2,3,\ldots \end{cases}$$

$$= \begin{cases} 1 \;-\; \exp\!\left(-\left(\dfrac{3/2}{\sigma}\right)^{\alpha}\right), & \text{for } \tau=1, \\ \exp\!\left(-\left(\dfrac{\tau-1/2}{\sigma}\right)^{\alpha}\right) \;-\; \exp\!\left(-\left(\dfrac{\tau+1/2}{\sigma}\right)^{\alpha}\right), & \text{for } \tau=2,3,\ldots \end{cases}$$