Estimation of the Distortion Risk Measure*

Yujian Gao
The Chinese University of Hong Kong, Shenzhen
School of Science and Engineering
Shenzhen, China
yujiangao@link.cuhk.edu.cn

***Abstract*—This paper explores the asymptotic behavior of risk measures under regular and second-order regular variation conditions in insurance risk models. Specifically, it focuses on the distortion risk measure. We provide mathematical formulations for distortion functions to capture the impact of tail dependencies and extreme values. Theoretical insights into their behavior as risk thresholds approach infinity are offered, with proofs based on existing probabilistic methods. Our findings contribute to the understanding of systemic risks and offer a foundation for more robust risk management strategies in financial and actuarial contexts.**

***Index Terms*—Risk Measures, Regular Variation, Asymptotic Approximation, Quantile Estimation of Conditional Random Variables**

I. Introduction

In the complex landscape of financial risk management, accurately quantifying the potential extreme losses is critical. Traditional risk measures like Value-at-Risk (VaR) and Expected Shortfall (ES) have been widely adopted in industry practices. However, their effectiveness in capturing tail dependencies and systemic risks remains a subject for ongoing research. This paper delves into advanced risk measures, particularly focusing on the distortion risk measure for the conditional random variables under models exhibiting regular and second-order regular variations.

The initial sections of this paper review the fundamental concepts of regular variation (RV) and second-order regular variation (2RV), which are essential for understanding the scaling behaviors of loss distributions. Subsequent sections develop the asymptotic approximations of the distortion risk measure along with the proof of its characteristics. This analytical approach is supported by theoretical proofs that underline the coherence and asymptotic normality of the distortion risk measure in extreme scenarios.

By extending the conventional frameworks to include conditional and distorted risk measures, this study aims to offer a nuanced view of risk quantification that aligns with the practical needs of risk managers in financial institutions and insurance companies. The outcomes are expected to enhance the predictive power of risk models and inform the development of more effective risk management policies.

II. Definitions and Background

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L_+(\mathbb{P})$ the set of non-negative random variables. Consider $X, Y \in L_+(\mathbb{P})$ two random insurance risks possessing distribution function (df) F_i, $i = 1, 2$, respectively. The corresponding survival functions are $\bar{F}_i(x) = 1 - F_i(x)$, $i = 1, 2$. It is assumed that the decision-maker orders its preferences via a risk measure ρ, which is simply a functional, i.e. $\rho : L_+(\mathbb{P}) \to \mathcal{R}$.

In this section, we first review the definitions and basic properties of regular variation (RV) and the second-order regular variation (2RV). Next, we

A measure function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be regular varying at infinity with index α, $\alpha \in \mathbb{R}$, denoted by $f \in RV_\alpha$, if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^\alpha.$$

If $\alpha = 0$, then f is said to be slowly varying at infinity.

A measure function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be second-order regular varying with the first-order index $\alpha \in \mathbb{R}$ and second-order index $\rho \le 0$, denoted by $f \in 2RV_{\alpha,\rho}$, if there exist some eventually positive or negative measurable function $A(t)$ with $A(t) \to 0$ as $t \to \infty$ such that

$$\lim_{t\to\infty} \frac{\frac{f(tx)}{f(t)} - x^\alpha}{A(t)} = x^\alpha \frac{x^\rho - 1}{\rho}$$

$\frac{(x^\rho-1)}{\rho}$ is interpreted as $\log x$ when $\rho=0$, and $A(t)$ is called an auxiliary function of f.

Now we recall the definition of two popular scalar risk measures: the Value-at-Risk (VaR) and Expected Shortfall (ES). The VaR risk measure is defined as the α quantile of loss distribution of X, a generic loss or risk r.v.s under consideration:

$$\mathrm{VaR}_\alpha(X)=F_X^{-1}(\alpha), \quad 0<\alpha<1$$

where $F_X^{-1}(\alpha)$ is the inverse of the distribution function, and the corresponding ES risk measure (which is more sensitive to the losses in the tail of the distribution) is given by

$$\mathrm{ES}_\alpha(X)=\frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t(X)\,dt, \quad 0<\alpha<1.$$

The ES (sometimes called Tail-Value-at-Risk(TVaR)) is a coherent risk measure in the sense of Artzner et al. (1999). If F is continuous, it represents the conditional expected loss given that the loss exceeds its VaR:

$$\mathrm{ES}_\alpha(X)=\mathbb{E}[X|X>\mathrm{VaR}_\alpha(X)]$$

VaR and ES occur as special case of distortion risk measures. In full generality, a distortion function $g:[0,1]\to[0,1]$ is an non-decreasing function such that $g(0)=0$ and $g(1)=1$. The set of all distortion is denoted by $\mathcal{G}$. A distortion risk measure which has the following mathematical formulation:

$$\rho(X;g)=\int_0^\infty g(F(x))\,dx, \tag{2.3}$$

The definition of CoVaR is given as follows: Let $\alpha,\beta\in(0,1)$, then,

$$\mathrm{CoVaR}_{\alpha,\beta}[X|Y]=\mathrm{VaR}_\beta[X|Y>\mathrm{VaR}_\alpha[Y]]$$

The above definition is adapted by Mainik and Schaanning (2014) to the case of CoES, under which the coherence of ES is inherited by CoES: Let $\alpha,\beta\in(0,1)$, then,

$$\mathrm{CoES}_{\alpha,\beta}[X|Y]=\frac{1}{1-\beta}\int_\beta^1 \mathrm{CoVaR}_{\alpha,t}[X|Y]\,dt$$

One easily verifies that, with continuous marginals, the CoES can be represented through a conditional expectation of Y, in a manner similar to the familiar representation of ES:

$$\mathrm{CoES}_{\alpha,\beta}[X|Y]=\mathbb{E}[X|Y>\mathrm{VaR}_\alpha(Y),X>\mathrm{CoVaR}_{\alpha,\beta}[X|Y]]$$

The definition of MES is given as follows: Let $\alpha\in(0,1)$, then,

$$\mathrm{MES}_\alpha[X|Y]=\mathbb{E}[X|Y>\mathrm{VaR}_\alpha(Y)]$$

we define the risk measure is given in Asimit and Li (2016) and is as follows:

$$\rho(t;g)=\rho(X|Y>t;g)$$

our main aim is to approximate $\rho(\cdot;g)$ for the conditional r.v. $X|Y>t$ for large t. Therefore, we shall investigate the asymptotic approximation as $t\to\infty$ for

$$\rho(t;g)=\int_0^\infty g\left(F_{X|Y>t}(x)\right)dx$$

According to Dhaene et al. (2022), if $t=\rho(Y;h)$, then $\rho(t;g)$ is the conditional distortion (CoD) risk measure. Specifically, let $t=\mathrm{VaR}_\alpha(Y)$,

(i) if $g(x)=I_{(1-\beta,1]}(x)$, then $\rho(t;g)=CoVaR_{\alpha,\beta}[X|Y]$
(ii) if $g(x)=\min\{1,\frac{x}{1-\beta}\}$, then $\rho(t;g)=CoES_{\alpha,\beta}[X|Y]$
(iii) if $g(x)=x$, then $\rho(t;g)=MES_\alpha[X|Y]$

There are many choices for the distortion function (for example, see Jones and Zitikis (2003, 2007)) and some of the well-known examples are as follows:

(i) Dual-power: $g(s)=1-(1-s)^\beta$, $\beta>1$;
(ii) TVaR: $g(s)=\min(s/(1-\beta),1)$, $0<\beta<1$;
(iii) Gini: $g(s)=(1+\beta)s-\beta s^2$;
(iv) Proportional hazard transform (PHT): $g(s)=s^{1-\beta}$, $0\le\beta<1$;
(v) Wang Transform: $g(s)=F_N(F_N^{-1}+\lambda)$, $\lambda>0$, where $F_N(\cdot)$ and $F_N^{-1}(\cdot)$ represent the standard normal df and its inverse, respectively.

Definition 2.1. The right-hand upper tail dependence between X and Y is described by the following joint convergence condition $JC(R)$: for all $(x,y)\in[0,\infty)^2$, such that at least x or y is finite, the limit

$$\lim_{t\to\infty} tP(F_1(X)\le x/t, F_2(Y)\le y/t)=:R(x,y)$$

The function R completely determines the so-called stable tail dependence function l as, for all $x,y\ge 0$,

$$l(x,y)=x+y-R(x,y);$$

see Drees and Huang (1998), Beirlant et al. (2004), section 8.2., Cai et al. (2015), Daouia et al. (2018).

Definition 2.2. Condition $JC(R)$ holds and there exist $\beta > \alpha$ and $\kappa < 0$ such that

$$\lim_{t\to\infty} \sup_{x\in(0,\infty),y\in[\frac{1}{2},2]} \left| \frac{tP(F_1(X) \le \frac{x}{t}, F_2(Y) \le \frac{y}{t}) - R(x,y)}{\min(x^\beta,1)} \right|$$

$$= O(t^\kappa)$$

Condition $JC_2(R,\beta,\kappa)$

III. Asymptotic Estimation and Asymptotic Normality

First, we get first-order asymptotics for the distortion risk measure $\rho(t;g)$

Theorem 2.1. Suppose that condition $JC(R)$ and that $F_i \in RV_{-\gamma_i}$, $i = 1,2$, then

$$\rho(t;g) = F_1^{-1}\left(F_2(t)\right) \int_0^\infty g\left(R(x^{-\gamma_1},1)\right) dx$$

Proof.

$$\begin{aligned}
\rho(t;g) &= \int_0^\infty g\left(F_{X|Y>t}(x)\right) dx \\
&= \int_0^\infty g\left(\frac{P(X>x,Y>t)}{P(Y>t)}\right) dx \\
&= \int_0^\infty g\left(\frac{P(F_1(X)\le F_1(x), F_2(Y)\le F_2(t))}{F_2(t)}\right) dx \\
&= F_1^{-1}\left(F_2(t)\right) \int_0^\infty g\Bigg(\frac{P(F_1(X)\le F_1^{-1}(F_2^{-1}(t)x),}{} \\
&\qquad \frac{F_2(Y)\le F_2(t))}{F_2(t)}\Bigg) dx \\
&= F_1^{-1}\left(F_2(t)\right) \int_0^\infty g\left(R(x^{-\gamma_1},1)\right) dx \\
&= \left(\frac{F_2(t)}{F_1(t)}\right)^{1/\gamma_1} \int_0^\infty g\left(R(x^{-\gamma_1},1)\right) dx
\end{aligned}$$

where in the last step we used $F_1 \in RV_{-\gamma_1}$, then $F_1\left(F_1^{-1}\left(F_2(t)\right)x\right) \sim x^{-\gamma_1} F_2(t)$

Corollary 2.1. Under the condition of Theorem 2.1, if $F_1(t) \sim cF_2(t)$, then

$$\rho(t;g) = t\int_0^\infty g\left(R(x^{-\gamma_1},1)\right) dx$$

We therefore proceed to prove its asymptotic normality. The first method we are going to try is the delta method inspired by Zhao, Mao and Fan (2020). We continue from the proof of Theorem 2.1. and can get the following derivation:

$$\begin{aligned}
\frac{\hat\rho(t;g)}{\rho(t;g)} &= \frac{\left(\frac{\bar F_2(t)}{\bar F_1(t)}\right)^{\frac{1}{\hat\gamma_1}} \int_0^\infty g(R(x^{-\hat\gamma_1},1))dx}{\left(\frac{\bar F_2(t)}{\bar F_1(t)}\right)^{\frac{1}{\gamma_1}} \int_0^\infty g(R(x^{-\gamma_1},1))dx} \\
&= \left(\frac{\bar F_2(t)}{\bar F_1(t)}\right)^{\frac{1}{\hat\gamma_1}-\frac{1}{\gamma_1}} \cdot \frac{\int_0^\infty g(R(x^{-\hat\gamma_1},1))dx}{\int_0^\infty g(R(x^{-\gamma_1},1))dx} \\
\log\left(\frac{\hat\rho(t;g)}{\rho(t;g)}\right) &= \left(\frac{1}{\hat\gamma_1}-\frac{1}{\gamma_1}\right)\log\left(\frac{\bar F_2(t)}{\bar F_1(t)}\right) \\
&\quad + \left[\log\left(\int_0^\infty g(R(x^{-\hat\gamma_1},1))dx\right)\right. \\
&\quad \left. -\log\left(\int_0^\infty g(R(x^{-\gamma_1},1))dx\right)\right] \\
&= I_1 + I_2
\end{aligned}$$

For I_1, note that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\text{thus} \frac{I_1}{\hat\gamma_1-\gamma_1} = \log\left(\frac{\bar F_2(t)}{\bar F_1(t)}\right)\left(-\frac{1}{\hat\gamma_1^2}\right)$$

$$\Rightarrow \sqrt{n(1-q_n)}I_1 \Rightarrow d\left[\log\left(\frac{\bar F_2(t)}{\bar F_1(t)}\right)\left(-\frac{1}{\hat\gamma_1^2}\right)\right]\Gamma$$

$$\begin{aligned}
\text{For } I_2\text{, let} \quad & f(t) = \log\int_0^\infty g(R(x^{-t},1))dx \\
&\Rightarrow \frac{f(\hat\gamma)-f(\gamma)}{\hat\gamma-\gamma} = \frac{I_2}{\hat\gamma-\gamma}\cdot\frac{df(t)}{dt} \\
&\Rightarrow \sqrt{n(1-q_n)}I_2 \underset{d}{\to} \left[\frac{df(t)}{dt}\right]\Gamma \\
\frac{df(t)}{dt} &= \frac{\frac{d}{dt}\int_0^\infty g(R(x^{-t},1))dx}{\int_0^\infty g(R(x^{-t},1))dx} \\
\text{The nominator} &= \frac{d}{dt}\int_0^\infty g(R(x^{-t},1))dx \\
&= \int_0^\infty \frac{\partial}{\partial t} g(R(x^{-t},1))dx \\
&= \int_0^\infty g'(R(x^{-t},1))\cdot\frac{\partial}{\partial t}R(x^{-t},1)dx
\end{aligned}$$

Nevertheless, pursuing this line of inquiry reaches an impasse due to the inaccessibility of the partial derivative of the R function; that is to say, the computation of the partial derivative of the R function is unfeasible with the data at hand.

In light of this challenge, the methodology proposed by Jonathan El Methni in 2016 offers a viable alternative. El Methni introduces an innovative approach in his publication for estimating the distortion risk measure using a singular random variable, X. This method is delineated as follows, providing a clear framework for such estimation.

$$\rho(X; g) = R_g(X) = \int_0^\infty g(1 - F(x))dx$$

Accordingly, this report is devoted to accomplishing two primary objectives: First, it substantiates the asymptotic normality of the distortion risk measure associated with a singular random variable, X. Second, it integrates this finding with the scenario involving our conditional random variable, culminating in an asymptotic approximation and validation of the asymptotic normality for the conditional random variable in question. We then proceed to prove the asymptotic normality of the distortion risk measure associated with a singular random variable by offering an alternative and interpretable expression of $R_g(X)$. We do a classical change-of-variable, namely setting $\alpha = 1 - F(x)$ with an integration by part and we can get:

$$R_g(X) = \int_0^1 g(\alpha)dq(1-\alpha) = \int_0^1 q(1-\alpha)dg(\alpha) \quad (1)$$

Here we introduce a function g_β which is defined by

$$\forall y \in [0,1],\ g_\beta(y) := g\left(\min\left[1, \frac{y}{1-\beta}\right]\right) = \begin{cases} g\left(\frac{y}{1-\beta}\right) & \text{if } y \leq 1-\beta \\ 1 & \text{otherwise.} \end{cases}$$

which is deduced from g by a piecewise linear transformation. It is obvious that if g is concave then so is g_β. In other words, if g gives rise to a coherent distortion risk measure, so does g_β. Ans we now can consider the distortion risk measure with the $'$new$'$ distortion function g_β

$$R_{g,\beta}(X) := \int_0^\infty g_\beta(1 - F(x))\, dx.$$

According to the proposition 1 in Wang:

Assume that for some $t > 0$, the function q is continuous and strictly increasing on $[t, 1]$. Then for all $\beta > t$ and any strictly increasing and continuously differentiable function h on $(0, \infty)$,it holds that

$$R_{g,\beta}(h(X)) = R_g(h(X_\beta))$$
$$with\, P(X_\beta \leq x) = P(X \leq x \mid X > q(\beta)).$$

when $\beta \uparrow 1$, we may then think of this construction as a way to consider Distortion Risk Measures of the extremes of X. With the change of variable result (1), we can rewrite the distortion risk measure with the function g_β as

$$\begin{aligned} R_{g,\beta_n}(X) &= \int_0^1 q(1-\alpha)\, dg_{\beta_n}(\alpha) \\ &= \int_0^1 q(1-(1-\beta_n)s)\, dg(s) \\ &= \int_0^1 (1-\beta_n)^{-\gamma} s^{-\gamma}\, dg(s) \\ &= \int_0^1 q(\beta_n)s^{-\gamma}\, dg(s). \qquad (2) \end{aligned}$$

In this case, an estimator of $R_{g,\beta}(X)$ would then be obtained by plugging estimators of $q(\beta_n)$ and γ in the right-hand side of (2).

We should note that (2) is a strong relationship and cannot be expected to hold under all scenarios, but it shall stay true when X has a heavy-tailed distribution, the rigorous definition has already been presented in the Definition and Background section. We then denote $\hat{F}_n(x)$ as the empirical cdf related to this sample and $\hat{q}_n(\alpha)$ as the empirical quantile function:

$$\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^n \mathbb{I}\{X_i \leq x\}$$

$$\hat{q}_n(\alpha) = \inf\{t \in \mathbb{R} | \hat{F}_n(t) \geq \alpha\} = X_{\lceil n\alpha \rceil, n}$$

in which $X_{1,n} \leq \ldots \leq X_{n,n}$ are the order statistics of the sample $(X_1, \ldots, X_n)$ and $\lceil \cdot \rceil$ is the ceiling function. We then can finally represent our distortion estimator:

$$\hat{R}_{g,\beta_n}(X) := X_{\lceil n\beta_n \rceil, n} \int_0^1 s^{-\hat{\gamma}_n} dg(s) \quad (4)$$

or

$$\hat{R}_{g,\beta_n}(X) := \hat{q}_n(\beta_n) \int_0^1 s^{-\hat{\gamma}_n} dg(s) \quad (5)$$

Here, for the estimation of the index γ, we can use the widely used and popular Hill estimator (Hill 1975):

$$\hat{\gamma}_H = \frac{1}{k}\sum_{i=0}^{k-1} \log X_{n-i,n} - \log X_{n-k,n}.$$

What's more, notice that the integrability condition $\int_0^1 s^{-\gamma-\eta}\, dg(s) < \infty$ should be thought of as a condition that guarantees the existence of the distortion

risk measure. This condition directly leads to the theorem 1 in Wang, which discussed the asymptotic properties. Theorem 1 consists of 2 parts. The first part is about the consistency of the estimator and the second part is about the asymptotic behavior of the estimator. The first part is as follows:(Assume that U is regularly varying with index γ>0. Assume further that $\beta_n \to 1$ and $n(1-\beta_n) \to \infty$)
1. Pick a distortion function g and a > 0. If there is some η>0 such that:

$$\int_0^1 s^{-\gamma-\eta} dg(s) < \infty$$

and $\hat{\gamma_n}$ is a consistent estimator of γ, then

$$\frac{\hat{R}_{g,\beta_n}(X^a)}{R_{g,\beta_n}(X)} - 1 \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Proof of Theorem 1. Write for any j:

$$\frac{\hat{R}_{g_j,\beta_n}(X)}{R_{g_j,\beta_n}(X)} = \frac{X_{\lceil n\beta_n \rceil,n}}{q(\beta_n)} \frac{\int_0^1 s^{-\hat{\gamma}_n} dg_j(s)}{\int_0^1 s^{-\gamma} dg_j(s)}$$

We start by showing the consistency statement: from Lemma 3(i) in Wang we can obtain

$$\frac{\hat{R}_{g_j,\beta_n}(X)}{R_{g_j,\beta_n}(X)} = \frac{X_{\lceil n\beta_n \rceil,n}}{q(\beta_n)}(1+o_p(1))$$

Write now $X_{\lfloor n\beta_n \rfloor,n} = U(Y_{\lfloor n\beta_n \rfloor,n})$ where Y has a standard Pareto distribution, and use Corollary 2.2.2 in de Haan and Ferreira (2006) together with the regular variation property of U to get

$$\frac{\hat{R}_{g_j,\beta_n}(X)}{R_{g_j,\beta_n}(X)} \xrightarrow{P} 1.$$

Now we introduced the second part of the theorem.
2. Assume moreover that U satisfies 2RV condition with the parameter (γ, ρ, A) and

$$\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda \in \mathbb{R}.$$

Pick a d-tuple of distortion functions $(g_1, \ldots, g_d)$. If for some $\eta > 0$,

$$\forall j \in \{1, \ldots, d\}, \int_0^1 s^{\gamma-1/2-\eta} dg_j(s) < \infty,$$

then, provided we have the joint convergence:

$$\sqrt{n(1-\beta_n)} \left(\hat{\gamma}_n - \gamma, \quad \frac{X_{\lceil n\beta_n \rceil,n}}{q(\beta_n)} - 1 \right) \xrightarrow{d} (\Gamma, \Theta)$$

Specifically, if the $\hat{\gamma_n}$ is the Hill estimator we mentioned earlier with:

$$\hat{\gamma}_H = \frac{1}{k} \sum_{i=0}^{k-1} \log X_{n-i,n} - \log X_{n-k,n}.$$

then (see Theorem 2.4.8 and the proof of Theorem 3.2.5 in de Haan and Ferreira, 2006) we have the following joint convergence in distribution, under the bias condition $\sqrt{n(1-\beta_n)}A((1-\beta_n)^{-1}) \to \lambda$:

$$\sqrt{n(1-\beta_n)} \left(\hat{\gamma}_n - \gamma, \frac{X_{\lceil n\beta_n \rceil,n}}{q(\beta_n)} - 1 \right) \xrightarrow{d}$$
$$\left(\gamma \int_0^1 [s^{-1}W(s) - W(1)]ds + \frac{\lambda}{1-\rho}, \gamma W(1) \right)$$

Note that the W here is a standard Brownian motion, and therefore the right-hand side is a Gaussian random pair.
The proof of this asymptotic normality can be found at the appendix of Wang's paper.

With all the length, we have successfully derived the approximation and proved the asymptotic normality of the distortion measure, but with only one variable X. However, this paper aims to derive the same things but with the conditional random variable $X|Y>t$. Compare the expression of $X|Y>t$ and X and the final expression of our approximation, which we recall:

$$\hat{R}_{g,\beta_n}(X) := \hat{q}_n(\beta_n) \int_0^1 s^{-\hat{\gamma}_n} dg(s)$$

The only thing left is to derive the estimation of our conditional random variable's quantile function. My initial approach is to describe the quantile function of the conditional random variable from the sample point of view:
Given only a random variable X, the estimation of its quantile function is given by

$$\hat{F}(q_n) = X_{n-\lfloor n(1-q_n) \rfloor,n}.$$

And now we want to derive the estimation quantile function of the conditional random variable $X|Y>Q(1-p)$ out of the samples $(X_1, Y_1), (X_2, Y_2) \cdots (X_n, Y_n)$.

In the initial phase of our analysis, we arrange the Y_i, in ascending order and select the subset within the interval $[Q(1-p), Y_n]$ and proceed to find the corresponding quantile function of X. It is important to note that after ordering and selecting Y_i, we are left with $n - \lfloor n(1-q_n) \rfloor$ samples for further examination. Subsequently, we organize the remaining $n-\lfloor n(1-q_n)\rfloor$ samples of X to obtain the order statistics $X_1, X_2, \cdots X_{n-\lfloor n(1-q_n)\rfloor}$. We then estimate their q_n quantile and yield the following expression:

$$X_{n-\lfloor n(1-q_n)\rfloor-\lfloor (n-\lfloor n(1-q_n)\rfloor)\cdot(1-q_n)\rfloor, n-\lfloor n(1-q_n)\rfloor}$$

However, there is an underlying issue with this final expression; it fails to capture the dependency between Y and X.

Consequently, we adopt the second method to estimate the conditional random variables quantile function. It is a novel approach proposed by Jeffrey S.Racine and Kevin Li (2017) Before we dive into Racine and Li's estimation method, we first recall the definition of conditional quantile function: A conditional τ-th quantile associated with a conditional distribution function $F(y|x)$ is defined by $\tau \in (0,1)$ as

$$\hat{q}_\tau(y) = \inf\{x : \hat{F}(x|y) \geq \tau\} = \hat{F}^{-1}(\tau|y). \quad (6)$$

Then we consider a smooth location-scale model of the form:

$$X_i = a(Y_i) + b(Y_i)\epsilon_i$$

where a(y) and b(y) ≥ 0 are unknown smooth location and scale functions, y is a predictor, and ϵ has zero mean, unit variance, and is otherwise an unknown error process with distribution F_ϵ that is independent of y. And we are interested in estimating the τ th conditional quantile of X, $q_\tau(y) = a(y) + b(y)Q_{\epsilon,\tau}$ as defined in (6), where $Q_{\epsilon,\tau}$ is the τ th quantile of ϵ.

Consider $QN_{(\mu,\sigma),\delta}$ as representing the δth quantile for a Gaussian random variable with an average of μ and a variation characterized by σ, that is,

$$Q_{N(\mu,\sigma),\delta} = \mu + \sigma\sqrt{2}\text{erf}^{-1}(2\delta - 1) = \mu + \sigma Q_{N,\delta}, \quad (7)$$

where $\delta \in (0,1)$ stands as a parameter whose value is influenced by empirical methods. Here, $Q_{N,\delta}$ is defined as the δth quantile for a normalized Gaussian variable, and $\text{erf}^{-1}(\cdot)$ is the inverse error function.

According to Racine and Li, assuming that ϵ is a continuous random variable, and acknowledging that the quantile functions for ϵ, $Q_{\epsilon,\tau}$, and for a normalized Gaussian variable, $Q_{N,\delta}$, are monotonically increasing with respect to τ and δ, respectively. Therefore it is known that for each $\tau \in (0,1)$, there is a unique $\delta = \delta(\tau)$ such that $Q_{\epsilon,\tau} = Q_{N,\delta}$. We then use $\delta_0 = \delta_0(\tau)$ to denote the value τ. i.e.,

$$Q_{N,\delta_0} = Q_{\epsilon,\tau}.$$

Finally, we present the estimation of the τ quantile of X given Y :

$$\tilde{q}_{\delta_0}(y) = \frac{\sum_{i=1}^n Q_{N(X_i,b(Y_i)),\delta_0} K_h(Y_i,y)}{\sum_{i=1}^n K_h(Y_i,y)}.$$

IV. Conclusion

In this research, we have methodically derived the asymptotic estimation of the distortion risk measure for conditional random variables and established its asymptotic normality, by starting with a single random variable and then deriving the quantile function in the form of conditional random variables. The study focused on the asymptotic behaviors of risk measures under regular and second-order regular variation conditions, which are pivotal in the realms of insurance and financial risk modeling and, therefore lays a foundational understanding that is critical for advanced risk management applications. The combined insights from deriving the asymptotic normality of the distortion risk measure and estimating the quantile function of conditional random variables provide a robust toolkit for financial analysts. These tools are expected to improve the accuracy of risk assessments in financial markets, thereby informing more effective risk management strategies.

Acknowledgment

I would like to express my heartfelt gratitude to Professor Liu Yang for providing me with this research opportunity. His meticulous care and concern have been invaluable throughout the entire process, including topic selection, project advancement, and monitoring the progress of the work. I am also profoundly grateful to Professor Geng Bingzhen for his countless insights and guidance on this project. He has led me through various problem-solving approaches and provided the initial draft of this report. It can be said that without these two professors, this research report would not have been possible. Lastly, I would like to thank my dear friend Deng as well as my classmates in the research group, Weng and Chen. Without their support, care, and assistance, I could not have successfully completed this semester's research. Once again, thank you all.

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