Real Variables

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Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$

Remark. Note for real numbers, $\mathbb{Q}$ has holes in it.
Example. $\nexists p \in \mathbb{Q}\ s.t\ p^2 = 2$

Proof. Assume $\exists p \in \mathbb{Q}$ s.t $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t $p^2 = 2$. Q.E.D.

Definition 1.1 (Order)**.** An order on a set S is a relation $<$ such that:

(a) If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.

(b) If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 1.2 (Ordered Set)**.** An ordered set S is a set with an order $<$.

Definition 1.3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t $\forall x \in E : x \leq \beta$.
Similarly, a set S is bounded below if $\exists \beta \in S$ s.t $\forall x \in E : x \geq \beta$.

Definition 1.4 (LUB, GLB)**.** Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists \alpha$ s.t. α is an upper bound for E and $\forall \gamma < \alpha$: γ is not an upper bound for E, then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists \alpha$ s.t. α is a lower bound for E and $\forall \gamma > \alpha$: γ is not a lower bound for E. Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 1.5 (LUB property)**.** An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists \sup E \in S$; i.e., Every bounded subset of S has the least upper bound(LUB).
Example.

- *$\mathbb{Z}$ has the LUB property.*
- *$\mathbb{Q}$ does not have the LUB property.*

Theorem 1.1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof. ($\Rightarrow$)Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B. Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. ($\Leftarrow$)Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E. Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 1.6 (Fields)**.** Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

(a) $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).

(b) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).

(d) $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.

(e) $\exists (-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.

(f) $\forall x, y \in F : xy \in E$.

(g) $\forall x, y \in F : xy = yx$.

(h) $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.

(i) If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

(j) $\forall x, y, z \in F : x(y + z) = xy + xz$

Example.

(a) $\mathbb{Q}$ *is a field, while* $\mathbb{Z}$ *is not a field.*

(b) $F_p = \{0, 1, \ldots, p - 1\}$ *with mod* p *arithmetic is a field.*

Read Text book: 114,115,116,118

Definition 1.7 (Ordered Field)**.** An ordered field F is a field that is an ordered set such that the following properties are satisfied:

(a) If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.

(b) If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

Remark. We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. $\mathbb{Q}$ *is an ordered field.*

Theorem 1.2. $\exists$ an ordered field $\mathbb{R}$ which has the LUB property and contains $\mathbb{Q}$ as a subfield.

Theorem 1.3.

(a) Arithmetic properties of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.

(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof. (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A. So, A has a least upper bound α. Since $\alpha - x < \alpha$ as $x > 0$, $\alpha - x$ is not an upper bound for A. Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A. Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y-x) > 1$. $ny - nx > 1$ and therefore, $1 + nx < ny$. Let $m \in \mathbb{Z}$ such that $(m-1) \leq nx < m$. Such m exists by the extended version of (a). This implies there exists $m \in \mathbb{N}$ such that $nx < m \leq nx + 1 < ny$. Therefore, $x < \frac{m}{n} < y$.

(c) $\exists k \in \mathbb{Q}$ such that $k^2 = 2$; i.e., $\exists \sqrt{2} \in \mathbb{R}$. $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q-p)$. Then $x < p < \alpha < q < y$ and $\alpha \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n-1)x \leq y < nx. \tag{1.1}$$

Proof. Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of $\mathbb{N}$ has a smallest element. Let $n =$ smallest element of A. Then the inequality holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n-1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n-1)x$. Hence, the statement holds. Q.E.D.

Lemma. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b-a)$ for some $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} b^n - a^n &= (b-a)(\underbrace{b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1}}_{n \text{ terms}}) \\ &< (b-a)nb^{n-1} \end{aligned}$$

Q.E.D.

Theorem 1.4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique}) y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt{x}^n$, the n^{th} root of x).

Proof. Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x.
Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \varnothing$, E is bounded above, hence (by the least-upper-bound property) there exists a $\sup E$. Choose $y = \sup E$. Consider two cases.

(a) If $x \leq 1$, then $t_0 = \dfrac{x}{2}$ and thereby $t_0^n = \dfrac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).

(b) If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x+1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E.

(a) Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y+h)^n - y^n < x - y^n$. By the lemma , we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y+h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.

(b) Assuming that $y^n > x$, we find $k > 0$ such that $(y-k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y, which is $\sup E$. By the lemma , $y^n - (y-k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \dfrac{y^n - x}{ny^{n-1}}$. Therefore, $-(y-k)^n < -x$, or $x < (y-k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 1.8 (Cut/Dedekind Cut)**.** The set $\mathbb{R}$ elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 1.9 (Order of cuts). For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A, and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 1.5. There exists an ordered field $\mathbb{R}$ such that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ has the LUB property.

Proof. Let $\mathbb{R}$ be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 1.10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$
Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$
Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
Zero element is $(0, 0)$
One element is $(1, 0)$

Theorem 1.6. $\mathbb{C}$ is a field.

Proof. Verify the 11 field axioms. For just a few axioms:
(M3):
$x = (a, b), y = (c, d), z = (e, f)$. $x(yz) = (a, b)(ce - df, cf + de) = (a(ce-df)-b(cf+de), a(cf+de)+b(ce-df)) = (ac-bd, ad+bc)(e, f) = (xy)z$
(M4):
$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$
(M5):
$x \neq 0$ means $x = (a, b)$ with $a \neq 0$ or $b \neq 0$. That is, $a^2 + b^2 > 0$. Let $\frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$. Then $x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0)$. Q.E.D.

Identification of $\mathbb{R}$ as a subfield of $\mathbb{C}$. Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a+b = a+b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\text{Re}(z) = a$,$\text{Im}(z) = b$.

Definition 1.11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by $\overline{z}$

Note.

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$

(c) $z + \overline{z} = 2 \cdot \mathrm{Re}(z)$

(d) $z - \overline{z} = 2i \cdot \mathrm{Im}(z)$

(e) $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$, with $=$ if any only if $z = 0$

(f) $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{a-bi}{a^2+b^2}$

Definition 1.12. $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 1.7. For $z, w \in \mathbb{C}$,

(a) $|z| \geq 0$ with $=$ iff $z = 0$

(b) $|z| = |\overline{z}|$

(c) $|zw| = |z| \cdot |w|$

(d) $|\mathrm{Re}(z) \leq |z|, |\mathrm{Im}(z)| \leq |z|$

Proof. Let $z = a + bi$. Then $|\mathrm{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$
Q.E.D.

(e) $|z + w| \leq |z| + |w|$ (Triangle inequality)

Proof.

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

Theorem (Cauchy-Schwarz inequality)**.** If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$|\sum_{j=1}^{n} a_j \overline{b_j}| \leq (\sum_{j=1}^{n} |a_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{n} |b_j|^2)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^{n} a_j \overline{b_j}$ defined on inner product on $\mathbb{C}^n$ and $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})}$. (Note that $\vec{a} \cdot \vec{b} = \overline{\vec{b} \cdot \vec{a}}$)

Proof. Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \overline{b_j}$ We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0.
For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^{n} |a_j + \lambda b_j|^2 = \sum_{j=1}^{n} (a_j + \lambda b_j)(\overline{a_j} + \overline{\lambda b_j}) = \sum_{j=1}^{n} |a_j|^2 + \lambda \sum_{j=1}^{n} b_j \overline{a_j} + \overline{\lambda} \sum_{j=1}^{n} a_j \cdot \overline{b_j} + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2$. Let $\lambda = tC$ for $t \in \mathbb{R}$.
Then $0 \leq A + \lambda \overline{C} + \overline{\lambda} C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$. $p(t)$ is a quadratic function in terms of t and it must be non-negative. Regardless of the value of t. Therefore, the discriminant of $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$. Since $|C| \geq 0$, $|C|^2 \leq AB$. Q.E.D.

Definition 1.13 (Euclidean k-space)**.** For $k \in N$, $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}$ with the following properties:

Addition $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$

Scalar multiplication $\lambda \vec{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$

Inner(dot) product $(\vec{x}, \vec{y}) = \sum_{j=1}^{k} x_j y_j$, which is bilinear: $(\alpha \vec{x} + \beta \vec{y}) \cdot \vec{z} = \alpha \vec{x} \cdot \vec{z} + \beta \vec{y} \cdot \vec{z}$.

Norm $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^{k} |x_j|^{2^{1/2}}$

Remark. Addition and Scalar multiplication make $\mathbb{R}^k$ into a vector space.

Theorem 1.8. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$. Then

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0 \leftrightarrow \vec{x} = \vec{0}$

(c) $|\alpha \vec{x}| = |\alpha||\vec{x}|$

(d) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ (special case of Cauchy-Schwarz inequality)

(e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle inequality)

Proof. $|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = |\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x}\cdot\vec{y}| + |\vec{y}|^2 \leq (|\vec{x}|+|\vec{y}|)^2$ Q.E.D.

(f) $|\vec{x}-\vec{y}| \leq |\vec{x}-\vec{z}| + |\vec{z}-\vec{y}|$

Proof. $|\vec{x}-\vec{y}| = |(\vec{x}-\vec{z}) + (\vec{z}-\vec{y})| \leq |\vec{x}-\vec{z}| + |\vec{z}-\vec{y}|$ Q.E.D.

Chapter 2

Basic Topology

Definition 2.1. Sets A and B have the same cardinality, if $\exists f : A \to B$ that is 1-1 and onto (i.e., bijective).

Theorem 2.1. Let $A \sim B$ be a relation between two sets having the same cardinality. Then is an equivalence relation. That is,

(a) $A \sim A$ (Reflexive)

(b) $A \sim B \Rightarrow B \sim A$ (Symmetry)

(c) $A \sim B \& B \sim C \Rightarrow A \sim C$ (Transitivity)

Definition 2.2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $J_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$.

- A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$(or if $A = \emptyset$).
- A set A is countably infinite if $A \sim \mathbb{N}$.
- A set A is countable if A is finite or countably infinite.

Example. *$\mathbb{Z}$ is a countably infinite. For $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$,*

Let $f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n+1 & \text{if } n < 0 \end{cases}$

Then f is bijective and therefore $|\mathbb{Z}| = |\mathbb{N}|$

Theorem 2.8. A subset of a countably infinite set is countable.

Proof. Let A be some countably infinite set and S be a infinite subset of A.
As A is a countably infinite set, we can remove duplicates and arrange A so that $A = \{a_1, a_2, a_3, \ldots\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in S$. Let n_k be the smallest positive integer greater than n_{k-1} such that $x_{n_k} \in E$ for $k = 2, 3, \ldots$. Let $f(k) = x_{n_k}$ for $k = 1, 2, 3, \ldots$. Then this is a bijection from $\mathbb{N}$ to S. Q.E.D.

Remark. Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

Theorem 2.12. Let $E_1, E_2, \ldots$ be countably infinite sets. Then $S = \cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof. Write $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \ldots\}$
$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \ldots\}$
Form an array:

$$\begin{Bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Bmatrix}.$$

This matrix might have duplicates. Let T be a subset of $\mathbb{N}$ such that $t \in T$ if and only if t is the smallest positive integer such that $x_t \in E_1 \cup E_2 \cup \ldots \cup E_n$.
Then a set $\{x_t | t \in T \text{ and } \exists_{i\in\mathbb{N}} : x_t \in E_i\}$ is S. Clearly, $|S| = |T|$, or $S \sim T$, and T is a subset of a countably infinite set, $\mathbb{N}$. Therefore, T is also countable, implying S is also countable. As S is infinite, S is countably infinite. Q.E.D.

Corollary 2.13. If A is countable and $n \in \mathbb{N}$, then $\{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in A\}$ is countable.

Theorem 2.14. Let $A = \{(b_1, b_2, b_3 \ldots) | b_i \in \{0, 1\}\}$. I.e., A is a set of all infinite binary strings. Then A is uncountable.

Proof (Contor's Diaganalization argument,1891)**.** Let $E \subset A$ be countably infinite. $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \ |s^{(i)} \in A\}$. It suffices to find some $s \in A \setminus E$, for this shows every countably infinite subset of A is proper construction of s. Write

$$s^{(1)} = b_1^1 b_2^1 \dots \tag{2.1}$$

$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \tag{2.2}$$

$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \tag{2.3}$$

$$\vdots$$

On diagonal, flip each bit, i.e., $0 \to 1$ and $1 \to 0$ and represent the flipped bit of b_i^i by $\tilde{b_i^i}$. Let $s = \tilde{b_1^1}\tilde{b_2^2}\tilde{b_3^3}\dots$. Then $s \in A$ and $s \notin E$ as s differs from each $s^{(i)}$ in the i-th bit. Therefore, A is uncountable. Q.E.D.

Corollary 2.15. The set $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ is uncountable.

Proof. We can create $f : \mathcal{P}(\mathbb{N}) \to A$ be a bijection, where A is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \tag{2.4}$$

For example, if $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0 \dots)$. This f is a bijection, and therefore A is uncountable.

Q.E.D.

Theorem 2.16. $\mathbb{R}$ is uncountable.

Proof. This is a rough sketch of the proof:

(a) It's enough to show that $[0,1]$ is uncountable.

(b) Consider binary decimal representation of $x \in [0,1]$. For example, $x = 0.101001001\ldots$. Given x, choose maximal $b_1 \in \{0,1\}$ such that $\frac{b_1}{2} \leq x$. Then choose $b_2 \in \{0,1\}$ such that $\frac{b_1}{2} + \frac{b_2}{2} \leq x$. Continue this process to get $b_1, b_2, b_3, \ldots$. Then $x = \sup\left\{\sum_{i=1}^{n} \frac{b_i}{2^i}\right\}$. Consider any dyadic rational of the form $\frac{m}{2^n}$. Let it be $\frac{3}{2^4}$. Then this maps $\frac{3}{2^4} \to 0,0,1,1,0,0,0,\ldots$ and never produce $0,0,1,0,1,1,1,1,\ldots$, which also represents $\frac{3}{2^4}$. Let A_1 be a subset of $A = \{\text{infinite binary strings}\}$ such that A_1 does not contain any strings ending in $1,1,1,1,\ldots$. Then the decimal representation defines a bijection $f : [0) \to A \setminus A_1$.

(c) A_1 is countable because $A = (A \setminus A_1) \cup A_1$, which is uncountable.

This shows that $[0,1]$ is uncountable, and therefore $\mathbb{R}$ is uncountable.
Q.E.D.

Definition 2.3 (Metric Spaces)**.** A set X is a metric space with metric $d : X \times X \to \mathbb{R}$ if

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,q) = 0$ if $p = q$, $\forall p,q \in X$

(b) $\forall_{p,q \in X} : d(p,q) = d(q,p)$

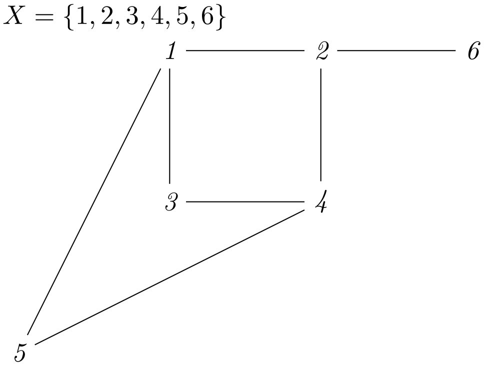
(c) $\forall_{p,q,r \in X} : d(p,q) \leq d(p,r) + d(r,q)$ (Triangle Inequality)

Remark. A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

Example (Metric Spaces)**.** *(a)* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$ *are metric spaces with* $d(p,q) = |p - q|$. *Note the meaning of* $|x|$ *depends on the context.*

(b) Every subset of a metric space is a metric space.

(c) $X = \{1,2,3,4,5,6\}$



Definition 2.4 (Neighborhood). A neighborhood in X is a set $N_r(p) := \{q : d(q,p) < r\}$, where $p \in X, r > 0$.

Remark. If $r_1 \leq r_2$, then $N_{r_1}(p) \subset N_{r_2}(p)$.

Example.

$\mathbb{R}^1$ *intervals,* $N_r(x) = \{y \in R^1 : |x - y| < r\}$

$\mathbb{R}^2$ *disks* $N_r(x) = \{y \in R^2 : |x - y| < r\}$

$\mathbb{R}^3$ *balls,* $N_r(x) = \{y \in R^3 : |x - y| < r\}$

Given example (c), $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$, $N_2(2) = \{1,2,4,6\}$, $N_3(2) = \{1,2,3,4,5,6\} = X$.

Definition 2.5. Let $E \subset X$. $p \in E$ is an interior point of E if $\exists r > 0$ such that $N_r(p) \subset E$.

Example.

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$.

Definition 2.6. $E \subset X$ is an open set if $\forall_{x \in E}$ is an interior point of E.

Theorem 2.19. Every neighborhood is an open set.

Proof. Let $g \in N_r(p)$. Then we must find $s > 0$, such that $N_s(g) \subset N_r(p)$. We know $d(p,q) < r$. Choose s such that $0 < s < r - d(p,q)$. Let $x \in N_s(q)$, then $d(q,x) < s < r - d(p,q)$. By triangle inequality, $d(p,x) \leq d(p,q) + d(q,x) < d(p,q) + r - d(p,q)$, so $x \in N_r(p)$, so $N_s(q) \subset N_r(p)$. Q.E.D.

Definition 2.7. Let $E \subset X$ and $p \in X$. p is a limit point of E if $\forall_{r>0} \exists_{q \in E}$ such that $q \neq p$ and $q \in N_r(p)$

Example. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ *has exactly one limit point,* 0. *note* $0 \notin E$.

Theorem 2.20. If p is a limit point of $E \subset S$, then every neighborhood of p contains infinitely many points of E.

Proof. Let $N_r(p)$ be a neighborhood of p. Then $N_r(p)$ contains at least one point $q_1 \in E$ such that $q_1 \neq p$. Let $r_1 = d(p, q_1)$. Then $N_{r_1}(p)$ contains some $q_2 \in E$ such that $q_2 \neq p$. Let $r_2 = d(p, q_2)$. Then $N_{r_2}(p)$ contains some $q_3 \in E$ such that $q_3 \neq p$. Continue this process to get $q_1, q_2, q_3, \ldots$. Q.E.D.

Corollary 2.21. If $E \subset X$ is finite then E has no limit points.

Definition 2.8 (Closed Set)**.** A set $E \subset X$ is closed if every limit point of E is in E.

Theorem 2.23. $E \subset X$ is open iff $E^c = \{x \in X : x \notin E\}$ is closed.

Proof.

- E is open $\Rightarrow E^c$ is closed.
  Let p be a limit point of E^c. Then every neighborhood of p contains some $q \in E^c$ such that $q \neq p$. If $p \in E$, then because E is open, p is an interior point, i.e., there exists some neighborhood of p that is a subset of E, which does not contain any points of E^c. This implies $p \notin E$ and therefore $p \in E^c$.

- E^c is closed $\Rightarrow E$ is open.
  Let $p \in E$. Then $p \notin E^c$, so p is not a limit point of E^c. Therefore, there exists some neighborhood of p that contains no points of E^c, i.e., all points of the neighborhood are in E. p Thus, Every $p \in E$ is an interior point of E, and hence E is open.

Q.E.D.

Theorem 2.24 (De Morgan's Laws)**.**

(a) $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha E_\alpha^c$

(b) $(\bigcap_\alpha E_\alpha)^c = \bigcup_\alpha E_\alpha^c$

Theorem 2.24.

(a) For all collection $\{G_\alpha\}$ of open sets : $\bigcup_\alpha G_\alpha$ is open.

(b) For all collection $\{F_\alpha\}$ of closed sets : $\bigcap_\alpha F_\alpha$ is closed.

(c) For all finite collection $\{G_1, G_2, \ldots, G_n\}$ of open sets : $\bigcap_{i=1}^n G_i$ is open.

(d) For all finite collection $\{F_1, F_2, \ldots, F_n\}$ of closed sets : $\bigcup_{i=1}^n F_i$ is closed.

Proof. (a) Let $x \in \bigcup_\alpha G_\alpha$. Then $x \in G_{\alpha_0}$ for some α_0. So there exists a neighborhood N of x such that $N \subset G_{\alpha_0} \subset \bigcup_\alpha G_\alpha$.

(b) it's suffice to prove that $(\bigcap_\alpha F_\alpha)^c$ is open. But $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$ is open by (a).

(c) Let $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for $i = 1, 2, \ldots, n$. So there exists a $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then $N_r(x) \subset N_{r_i} \subset G_i$ for $i = 1, 2, \ldots, n$ and therefore $N_r(x) \subset \bigcap_{i=1}^n G_i$.

(d) $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$ is open by (c).

Q.E.D.

Definition 2.9 (Closure)**.** Let $E \subset X$. Let E' be a set of limit points of E in X. The set $\overline{E} = E \cup E'$ is the closure of E.

Theorem 2.27.

(a) $\overline{E}$ is closed.

(b) $E = \overline{E} \Leftrightarrow E$ is closed.

(c) If $F \subset X$ is closed and $E \subset F$, then $\overline{E} \subset F$. (i.e., $\overline{E}$ is the smallest closed set containing E, and $\overline{E} = \bigcap_{F:\text{closed set with } F \supset E} F$.)

Proof. (a) Let p be a limit point of $\overline{E}$. It suffices to show $p \in E'$ since this implies that $p \in E' \subset E \cup E' = \overline{E}$. Let $r > 0$. $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$, i.e., $d(p,q) < \frac{r}{2}$. Since $q \in E \cup E'$, $\exists_{s \in \overline{E}}$ such that $d(q,s) < \frac{r}{2}$ (if $q \in E$, take $s = q$). But $d(p,s) \leq d(p,q) + d(q,s) < \frac{r}{2} + \frac{r}{2} = r$.

(b) ($\Rightarrow$) by (a)

($\Leftarrow$) Suppose E is closed. Then $E' \subset E$, so $\overline{E} = E \cup E' = E$.

(c) Suppose F is closed. Then $F' \supset E'$ and also $F \supset F'$. So $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

Theorem 2.28. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup\{E\}$. Then $y \in \overline{E}$. Hence, $y \in E$ if E is closed.

Example. *Let $X = \mathbb{R}, d(p,q) = |p-q|$. Let $E \subset \mathbb{R}$ be nonempty and bounded above, and let $y = \sup E$. Then $y \in \overline{E}$.*

Proof. Suppose for contradiction $y \notin \overline{E}$. Then y is neither a point in E

nor a limit point of E, so $\exists$ some interval $N_r(y) = (y-r, y+r)$ such that $(y-r, y+r) \cap E = \emptyset$. However, then $y-r$ in an upper bound for E since y is a least upper bound, which is a contradiction. Therefore, $y \in \overline{E}$. Q.E.D.

Definition 2.10 (Relative Openness). Suppose X is a metric space, so $Y \in X$ is a metric space with the same metric. Let $E \subset Y$. Then E is open relative to Y if E is an open set in the metric space Y

Example. *$X = R^2 \supset \mathbb{R} = y, E = (0,1) \subset Y$. Then E is open relative to Y, but E is neither open nor closed in X.*

Theorem 2.30. A set $E \subset Y \subset X$ is open relative to $Y \Leftrightarrow \exists_{\text{open set } G \subset X} : E = G \cap Y$

Proof. ($\Rightarrow$) Suppose $E \subset Y$ is open relative to Y. Given $p \in E$, $\exists_{r_p > 0} : {N_{r_p}}^Y(p) \subset E$, where ${N_r}^Y(p) = \{q \in Y : d(p,q) < r\}$. Then $E \subset \bigcup_{p \in E} {N_{r_p}}^Y(p)$ and $\bigcup_{p \in E} {N_{r_p}}^Y(p) \subset E$. Therefore, $E = \bigcup_{p \in E} {N_{r_p}}^Y(p)$.

Let $G = \bigcup_{p \in E} {N_{r_p}}^X(p)$. This time, we are considering p's neighborhood in X, so each ${N_{r_p}}^X$ is open. Thus G is a union of open sets in X, and therefore open.

$\forall_{p \in E} : p \in N_{r_p}(p)^X$, so $E \subset G \cap Y$.

Let $p \in G \cap Y$. Then $p \in G$ and $p \in Y$. So $p \in {N_{r_p}}^X(p)$ for some $r_p > 0$. But $p \in Y$, so $p \in {N_{r_p}}^Y(p)$. Therefore, $p \in E$. This implies $G \cap Y \subset E$, and therefore $E = G \cap Y$.

($\Leftarrow$) Suppose $G \subset X$ is open and $E = G \cap Y$. Then $\forall_{p \in E} : \exists_{r_p > 0} : {N_{r_p}}^X(p) \subset G$, so ${N_{r_p}}^Y(p) = {N_{r_p}}^X(p) \cap Y \subset G \cap Y = E$.

Q.E.D.

Note: Midterm 1 material ends here.

Definition 2.11 (Open Cover). An open cover of $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.12 (Compact). A set $K \subset X$ is compact if every open cover has a finite subcover; i.e., $\exists_{\alpha_1, \alpha_2, \ldots \alpha_n} :$ s.t $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$

Example.

- *If E is finite, then E is compact.*
- *$(0,1) \subset \mathbb{R}$ is not compact. Bad cover: $(\frac{1}{n}, 1), n > 2$*
- *$[0, \infty] \subset \mathbb{R}$ is not compact. Bad cover: $(-1, n)$ for $n \in \mathbb{N}$.*
- *$E \subset \mathbb{R}^k$ is compact if and only if E is closed and bounded.*

Theorem 2.34. If K is compact then K is closed.

Proof. Suppose K is compact. It suffices to prove that K^c is open. Let $p \in K^c$. We need to produce $r > 0$ s.t. $N_r(p) \subset K^c$. For $q \in K$, let $W_q = N_{r_q}(q)$, where $r_q = \frac{1}{2}d(p,q) > 0$. $\forall_{x \in N_{r_q}(p)} : x \in W_q \Rightarrow d(x,p) + d(x,q) < 2r_q = d(p,q)$. However, X is a metric space and $p, q, x \in X$, so $d(p,q) \leq d(p,x) + d(x,q)$, leading to $d(p,q) \leq d(p,x) + d(x,q) < d(p,q)$, which is a contradiction. Hence, $\forall_{x \in N_{r_q}} : x \notin W_q$. $N_{r_q}(p) \subset {W_q}^c$ for $\forall_{q \in K}$. Note that $\{W_q\}_{q \in K}$ is an open cover of K. K compact $\Rightarrow \exists_{\text{finite number of open sets } W_{q_1}, W_{q_2}, \ldots W_{q_n}}$ s.t. $K \subset \bigcup_{i=1}^{n} W_{q_i}$. Let $r = \min\{r_{q_1}, r_{q_2}, \ldots r_{q_n}\} > 0$.

$$\therefore N_r(p) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} N_{r_p}(p) \right) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} {W_{q_i}}^c \right) = \left(\bigcup_{i \in \{1,2,\ldots \mathbb{N}\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

Theorem 2.35. If $K \subset X$ is compact then K is bounded; i.e., $\exists_{M < \infty}$ s.t. $\forall_{p,q \in K}$: $d(p,q) \leq M$

Proof. Fix $p \in K$. An open cover of K is $\{N_n(p)\}_{n \in \mathbb{N}}$. In fact, this is an open cover of X. K compact $\Rightarrow \exists_{\text{finite subcover} N_{n_1}(p), N_{n_2}(p) \ldots N_{n_m}(p)}$. Let $R = \max\{n_1, n_1, \ldots n_m\}$. $K \subset N_R(p)$. Let $M = 2R$. $\forall_{q,r \in K}$: $d(q,r) \leq d(q,p) + d(p,r) < R + R = 2R = M$. Q.E.D.

Theorem 2.35. If F is closed, K is compact, and $F \subset K$ then F is compact.

Proof. Suppose $F \subset K$. Let $\{V_\alpha\}$ be an open cover of F. It suffice to produce a finite subcover:
Consider $\{V_\alpha\}$ together with F^c. This gives an open cover of X, hence of K, so $\exists_{\text{subcover of } K}$. Drop F^c from this finite subcover. The result is a finite subcover of $\{V_\alpha\}$, which covers F Q.E.D.

Corollary 2.36. If F is closed and K is compact then $F \cap K$ is compact.

Theorem 2.33. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Note. This is not true for open sets. For instance, let $K = Y = [0,1] \subset X = \mathbb{R}$. Y is open and closed relative to Y, but Y is not open relative to X

Proof.

($\Rightarrow$) Suppose K is compact relative to X. Let $\{V_\alpha\}$ be an open cover of K relative to Y. For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then $\{V_\alpha\}$ is an open cover of K relative to X. Since K is compact relative to X, $\exists_{\text{finite subcover}}$.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K relative to X. Then $\{V_\alpha \cap Y\}$ is an open cover of K relative to Y. Since K is compact relative to Y, $\exists_{\text{finite subcover}}$.

Q.E.D.

Theorem 2.36. Suppose $\{K_\alpha\}$ is a collection of compact sets such that $\bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$. Then, $\lim\limits_{n\to\infty} \bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$, or equivalently, $\bigcap_\alpha K_\alpha \neq \emptyset$.

Example. *Let $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$. Then $\{G_j\}$ is a collection of open sets, but none of them are compact. (compact sets are closed) Then $\{G_j\}$ satisfies non-empty finite intersection property but $\bigcap_{j\in\mathbb{N}} G_j = \emptyset$.*

Proof. Suppose for contradiction $\bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$ and $\bigcap_\alpha K_\alpha = \emptyset$. For any α_0, $K_{\alpha_0} \cap \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right) = \emptyset$. Hence, $K_{\alpha_0} \subset \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right)^c = \bigcup_{\alpha\neq\alpha_0} (K_\alpha)^c$ and $\{(K_\alpha)^c\}_{\alpha\neq\alpha_0}$ is an open cover of K_{α_0}, so $\exists$ a finite subcover of $K_{\alpha_0} \subset \bigcup_{i=1}^n K_{\alpha_i}{}^c$, which implies $K_{\alpha_0} \cap (\bigcap_{i=1}^n K_{\alpha_i}) = \emptyset$, contradiction. Q.E.D.

Corollary 2.37. If $\{K_1, K_2, \ldots\}$ are non-empty compact sets with $\forall_n$: $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

Proof. If $n_1 < n_N$ then $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$ Q.E.D.

Theorem 2.37. If K is compact and $E \subset K$ is infinite, then E has a limit point in K.

Proof. Contrapositive of the statement is : *if $E \subset K$ has no limit point in K, then E is finite.*
Suppose every point $q \in K$ is not a limit point of E. Then

$$\exists_{V_q = N_{r_q}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}.$$

$\{V_q\}_{q \in K}$ is an open cover of K, so $\exists_{\text{finite subcover } V_{q_1} \cup V_{q_2} \cup \cdots \cup V_{q_n}}$. Then $E = E \cap K \subset (\bigcup_{i=1}^{n} V_{q_i} \cap E) \subset \{q_1, q_2, \ldots q_n\}$, so E is finite.

Q.E.D.

Theorem 2.38. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be such that $\forall_n : I_n \supset I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Since $I_n \supset I_{n+1}$, $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$. Let $E = \{a_1, a_2, \ldots\}$. Then $E \neq \emptyset$, every b_k is an upper bound for E, so $\exists x = \sup E$ and $a_k \leq x \leq b_k$ for all k. Therefore, $x \in I_k$ for all k, so $x \in \bigcap_{n=1}^{\infty} I_n$. Q.E.D.

Theorem 2.39. Let $\{I_n\}$ be a sequence of k-cells such that $i_n \supset I_{n+1}$;i.e., $I_n = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj},\ a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj} \text{ for } j = 1, 2, \ldots, k\}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Apply previous theorem to each component. Q.E.D.

Note. k-cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of k *closed intervals on the real line.*
Formally, Given real numbers a_i and b_i such that $a_i < b_i$ for every integer i from 1 to k,

$$I = \{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, k\}$$

Theorem 2.40. Let $I \subset \mathbb{R}^k$ be a k-cell. Then I is compact.

Proof. Let $I = I_0 = \{x = (x_1, x_2, \ldots, x_k), a_j \le x_j \le b_j\}$.
Let $\Delta = \left\{\Sigma_i^k (b_j - a_j)^2\right\}^{1/2}$. Then $|\mathbf{x} - \mathbf{y}| \le \Delta$ for $\mathbf{x}, \mathbf{y} \in I$.
Suppose for contradiction $\{G_\alpha\}$ is an open cover of I that has no finite subcover.
Let $c_j = \frac{1}{2}(a_j + b_j)$ for $j = 1, 2, \ldots, k$. Using $[a_j, c_j], [c_j, b_j]$, we get 2^k k-cells Q_i with $I = \bigcup_{i=1}^{2^k} Q_i$. At least one Q_i, call it I_1, has no finite subcover. Otherwise, every Q_i has a finite subcover, and I would have a finite subcover, namely the union of the finite subcovers of each Q_i.
Repeat this step to construct $I_0 = I, I_1, I_2, \ldots$. Then the sequence $\{I_n\}$ constructed by this process satisfies the following properties:

(a) $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \ldots$

(b) $\forall_n : I_n$ has no finite subcover from $\{G_\alpha\}$

(c) if $x, y \in I_n$ then $|x - y| \le 2^{-n}\Delta$, where $\Delta =$ diagonal of $I = \left(\sum_{j=1}^{k} (b_j - a_j)^2\right)^{1/2}$.

By theorem 2.38 and (a), $\exists_{x^* \in \bigcap_{n=1}^{\infty} I_n}$. Since $x^* \in I$, $x^* \in G_{\alpha_0}$ for some α_0, so $\exists r > 0$ such that $N_r(x^*) \subset G_{\alpha_0}$. But by (c), $I_n \subset N_{2^{-n}\Delta}(x^*)$. As soon as n is large enough that $2^{-n}\Delta < r$, we have $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$, which contradicts (b). Q.E.D.

Note. Reverse triangle inequality
$\forall_{a,b,c \in X} : d(a,b) \ge d(a,c) - d(c,b)$ because $d(a,c) \le d(a,b) + d(b,c)$.

Theorem 2.41. For $E \subset \mathbb{R}^k$, the following are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof.

$(a) \Rightarrow (b)$ Because E is bounded, i.e., $\exists_M$ s.t. $\forall_{x,y \in E} : |x - y| \leq M$, there exists a k-cell I such that $E \subset I$. Since every k-cell is compact, this implies E is a closed subset of a compact set. Hence, E is also compact.

$(b) \Rightarrow (c)$ by theorem 2.37

$(c) \Rightarrow (a)$ To see that E is bounded, suppose it were not. Then E has an infinite subset $S = \{x_1, x_2, x_3, \ldots\}$ with $\forall_n : |x_n| \geq n$. S has no limit point in $\mathbb{R}^k$ Let $S = \{(x_1, x_2, x_3, \ldots) \in E : |x_n - x_0| < \frac{1}{n}\}$. Then S is an infinite set because if S is finite, there exists a point $\mathbf{x} \in S$ such that $|\mathbf{x}| \geq |\mathbf{x}'|$ for $\mathbf{x}' \in S$. However, there exists $n \in \mathbb{N}$ such that $n > |\mathbf{x}|$ and by definition of S, there exists $x_n \in S$ such that $|x_n| \geq n > |\mathbf{x}|$, which is a contradiction. Thus, S is infinite. This S however, cannot have a limit point in E. By triangle inequality, for any $y \in R^k$, $|x_n| \leq |x_n - y| + |y|$, and from archimedean property, $\exists_{m \in \mathbb{N}}$ s.t. $m > |x_n - y| + |y|$, which implies for any $y \in R^k, r > 0$, $\exists_{m \in \mathbb{N}} : |x - y| < r < m$. However, by the definition of S, there are at most m such elements in S. Since a limit point y of E must contain an infinite number of points of E such that $d(x, y) < r$ for any $r > 0$, y cannot be a limit point, which contradicts the assumption that any infinite subset of E contains a limit point in E. Therefore, E must be bounded.
To see that E is closed, suppose it were not closed. Then $\exists_{x_0 \in} : E' \setminus E$. If T has no limit point in E except $x_0 \notin E$, it contradicts (c) because T is infinite and there must be a limit point of T in E.
Therefore, we can show that E is closed by showing that T has no limit point in E except x_0. Form an infinite sequence $(x_1, x_2, x_3, \ldots), x_n \in E$ with $|x_n - x_0| < \frac{1}{n}$. Let $y \in E$, $y \neq x_0$. We'll show that y cannot be a limit point of T. $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$. Choose $n \geq \frac{2}{|y - x_0|}$, so $\frac{1}{n} \leq \frac{|y - x_0|}{2}$. Then $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$. So only finitely many x_n can lie in $N_{\frac{1}{2}|y - x_0|}(y)$. So y cannot be a limit point of S. Therefore, E is closed.

Q.E.D.

Remark. (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than $\mathbb{R}^k$.

Example. *Failure of Heine-Borel theorem in general metric spaces.*

some infinite set , discrete metric $d(x, y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y \end{cases}$. *Then E is bounded* *and closed but not compact.*

Theorem 2.42. [Weirstrass's theorem] Every bounded infinite subset $E \subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Choose a k-cell $I \supset E$. Since I is compact, by theorem 2.41, E has a limit point in I. Q.E.D.

Example. *Let*

$$E_0 = [0, 1] \tag{2.5}$$

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{2.6}$$

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \tag{2.7}$$

$$\vdots \tag{2.8}$$

This gives $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \ldots$, where each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 2.13 (Perfect Sets). A set P is perfect if there is no isolated point in P; i.e.,

$$P = P'.$$

Theorem 2.43. Let P be a non-empty perfect set in R^k. Then P is uncountable.

Proof. Suppose for contradiction P is countable. Since P is non-empty, there exists some $p_1 \in P$. p_1 is then also a limit point of P. Let $p_2 \in P (\neq p_1)$ be a point in $V_1 = N_{r_1}(p_1)$ for some r_1 such that $d(p_1, p_2) > r_1/2$. Let $r_2 = r_1 - d(p_1, p_2)$, $V_2 = N_{r_2} p_2$. Then $\forall_{x \in V_2} : d(p_1, x) \leq d(p_1, p_2) + d(p_2, x) < d(p_1, p_2) + r_2 = r_1$. Hence, $V_2 \subset V_1$. $\overline{V_2} \subset V_1$ as well. Also, note that $d(p_1, p_2) > r_1/2$, so $r_2 = r_1 - d(p_1, p_2) < r_1/2 < d(p_1, p_2)$. So $p_1 \notin \overline{V_2}$. Repeat this process, and let $K_n = \overline{V_n} \cap P$. $K_n \subset \overline{V_n}$. Since $\overline{V_n}$ is closed and bounded, it's compact. $\overline{V_n} \cap P$ is a closed subset of $\overline{V_n}$, so K_n is also compact. However, for any p_n, $p_n \notin K_{n+1}$, so $\bigcap_{1 \in \infty} K_n \cap P = \emptyset$. Since $K_n \subset P$, this implies $\bigcap_{1 \in \infty} K_n = \emptyset$, but each K_n is not empty, $K_n \supset K_{n+1}$, and K_n is compact. Thus, $\bigcap_{1 \in \infty} K_n \cap P$ can't be empty, so this is a contradiction. Q.E.D.

Definition 2.14 (Cantor Set). The cantor set $P := \bigcap_n^\infty E_n$.

Proposition. P is compact, non-empty and contains no open intervals (a, b) and uncountable.

Proof. **Compactness** P is compact because $P \subset E_0 = [0, 1]$ and E_0 is compact.

Non-emptiness P is non-empty because $P \subset E_0$ and E_0 is non-empty.

No open intervals P contains no open intervals (a, b) because any (a, b) contains some $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ and these are all removed.

Uncountability P is uncountable because P is a perfect set. Equivalently, P consists of points in $[0, 1]$ whose ternary, i.e., base 3, representation contains only 0's and 2's.

> **Note.** ternary representation: $0.a_1a_2a_3\ldots = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0, 1, 2\}$.

Q.E.D.

Example (Cantor Set)**.** *Let* $E = [0, 1]$*,* $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$*.* $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$*, etc. Keep removing open middle third. This gives* $E_0 \supset E_1 \supset E_2 \ldots$*. Each* E_n *is the union of* 2^n *closed intervals of length* $\frac{1}{3^n}$*.*

Definition 2.15 (Separated Sets)**.** **Separated Sets** $A, B \subset X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Connected Sets $E \subset X$ is connected if there is no non-empty separated sets $A, B \subset E$.

Example (Separated Sets)**.** *In* $\mathbb{R}^1$*,* $[0, 1)$ *and* $(1, 2]$ *are separated so* $[0, 1) \cup (1, 2]$ *is not connected. Every interval is connected (open, closed, semi-open).*

Theorem 2.47. $E \subset \mathbb{R}^1$ is connected if and only if E is an interval; i.e., $\forall_{x,y \in E, x<y}$ s.t. $\forall_{z \in (x,y)} : z \in E$

Proof. Let $x, y \in E$. Q.E.D.

Theorem 2.48. A metric space X is connected if and only if the only nonempty subset of X which is both open and closed is X itself. $|\limsup_{n\to\infty} |\gamma_n|| \leq 0+\varepsilon\alpha$. Since ε is arbitrary, this implies $|\limsup_{n\to\infty} |\gamma_n|| = 0$, so $\lim\limits_{n\to\infty} |\gamma_n| = 0$.

Chapter 3

Sequence and Series

3.1 Sequences

Definition 3.1. In a metric space (X, d), a sequence $\{p_n\}$ converges to p if $\forall_{\varepsilon>0}\exists_N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.
We write $\lim\limits_{n\to\infty} p_n = p$ or $p_n \to p$.

If $\{p_n\}$ does not converge to any p then it is said to diverge.

Theorem 3.3. If s_n and t_n are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$, then the following hold:

(a) $s_n + t_n \to s + t$

(b) $cs_n \to cs$, $c + s_n \to c + s$ for any $c \in \mathbb{C}$

(c) $s_n t_n \to st$

(d) $\frac{1}{s_n} \to \frac{1}{s}$ if $s \neq 0$

Lemma (Squeeze Lemma)**.** In $\mathbb{R}$, if $\forall_{n\in\mathbb{N}} : 0 \leq x_n \leq s_n$ and $\lim\limits_{n\to\infty} s_n \to 0$, then $\lim\limits_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$. Then $0 \leq x_n \leq s_n < \varepsilon$ for $n \geq N$, so $x_n \to 0$. Q.E.D.

Theorem 3.20. (a) If $p > 0$ then $\frac{1}{n^p} \to 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $\frac{1}{N^p} < \varepsilon$; i.e., $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$. Then for $n \geq N$, $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$. Q.E.D.

(b) If $p > 0$ then $\sqrt[n]{p} \to 1$.

Proof. $p = 1$ is obvious.
Suppose $p > 1$. Let $x_n = \sqrt[n]{p} - 1 > 0$. Want to show $x_n \to 0$.
Since $(x_n+1)^n$, we have $p = (x_n+1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k > \binom{n}{1} x_n' = nx_n$.
Therefore, $x_n \leq \frac{p}{n}$, so $x_n \to 0$ by the Squeeze Lemma.
Suppose $p \in (0, 1)$. Let $q = \frac{1}{p} > 1$. Then $\sqrt[n]{q} \to 1$ by the previous case. By 3.3, $\sqrt[n]{p} = \frac{1}{\sqrt[n]{q}} \to 1$. Q.E.D.

(c) $\sqrt[n]{n} \to 1$

Proof. Let $x_n = \sqrt[n]{n} - 1 > 0$, for $n \geq 2$. $n = (x_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k}(x_n)^k > \binom{n}{2}{x_n}^2 = \frac{n(n-1)}{2}x_n^2$. Therefore, $x_n \leq \sqrt{\frac{2}{n-1}}$. Q.E.D.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \to 0$; i.e., Exponentials beat powers.

Proof. We want an upper bound on $\frac{n^\alpha}{(1+p)^n}$, so seek a lower bound on $(1+p)^n$.
$(1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k} p^k$ for $k \leq n$
$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$. Then for $k \leq \frac{n}{2}$, $\binom{n}{k} p^k > (\frac{n}{2})^k \frac{p^k}{k!}$. Therefore, $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$. Let $k_0 \in \mathbb{Z}$ s.t. $k > \alpha$. Then for $n \geq 2k_0$, RHS $\to 0$ by (a).

If $|x| < 1$ then $x^n \to 0$.

Proof. $|x^n - 0| = |x|^n$, so $x^n \to 0 \Leftrightarrow |x|^n \to 0$ and $|x|^n = \frac{n_0}{(\frac{1}{|x|})^n} \to 0$ by (d) with $\alpha = 0$ and $1 + p = \frac{1}{|x|} > 1$, so $p = \frac{1}{|x|} - 1 > 0$. Q.E.D.

Q.E.D.

Theorem 3.2. (a) $p_n \to p \Leftrightarrow \forall_{r>0} : N_r(p)$ contains all but finitely many p_n.

Proof. $\forall_{n \geq N} : p_n \in N_r(p)$ Q.E.D.

(b) If $p_n \to p$ and $p_n \to p'$ then $p = p'$.

Proof. $d(p,p') \leq d(p_n,p) + d(p_n,p')$ for all n. Fix ε. Choose N such that $d(p_n,p) < \frac{\varepsilon}{2}$ and $d(p_n,p') < \frac{\varepsilon}{2}$ for $n \geq N'$. Then $d(p,p') < \varepsilon$. Then for $n \geq \max\{N,N'\}$, $d(p,p') < \varepsilon$. This is true for all $\varepsilon > 0$, so $d(p,p') = 0$. Q.E.D.

(c) If $\{p_n\}$ converges, then p_n is bounded, in a sense that $\exists_{M>0,q\in X}$ s.t. $d(p_n,q) \leq M$ for all n.

Proof. If $p_n \to p$, then $\exists N$ s.t. $d(p_n,p) < 1$ for all $n \geq N$. Thus, $\forall_{n\geq 1} : d(p_n,p) \leq \max\{1, d(p,p_1), \ldots, d(p,p_{N-1})\} = M$ Q.E.D.

(d) If $E \subset X$ has a limit point p, then $\exists_{p_n \in E}$ s.t. $p_n \to p$.

Proof. We need to choose $p_n \in E$ s.t. $d(p,p_n) < \frac{1}{n}$. Let $\varepsilon > 0$. Then $d(p,p_n) < \varepsilon$ if $n > \frac{1}{\varepsilon}$ Q.E.D.

Definition 3.2. Given $p_n, n_1 < n_2 < n_3 < \ldots$, we say $p_{n_i} = (p_{n_1}, p_{n_2}, \ldots)$ is a subsequence of p_n.

Lemma. $p_n \to p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p

Proof. Look at assignment 6 Q.E.D.

Theorem 3.6. (a) $\{p_n\}$ in X, X compact, then $\exists$ convergent subsequence.

Proof. Let $E =$ range of$\{p_n\}$. If E is finite, then $\exists p \in X$ and $n_1 < n_2 < \ldots$ s.t. $p_n = p$ for $\forall i$. This subsequence converges to p. If E is infinite then by Theorem 2.37, E has a limit point $p \in X$; i.e., every neighborhood of p contains infinitely many points of E. Choose n_1 s.t. $d(p,p_{n_1}) < 1$.

Q.E.D.

(b) $\{p_n\}$ in $\mathbb{R}^k$, bounded, then $\exists$ convergent subsequence.

Proof. Choose a k-cell I that contains $\{p_n\}$. I is compact. Apply (a).

Q.E.D.

Definition 3.3 (Cauchy Sequence)**.** $\{p_n\}$ is a Cauchy sequence in (X,d) if $\forall \varepsilon : \exists_{N\in\mathbb{N}}$ s.t. $d(p_m,p_n) < \varepsilon \forall m,n \geq N$.

Definition 3.4. For $E \subset X$, $E \neq \emptyset$, we define diam $E = \sup\{d(p,q) : p,q \in E\}$. diam $E = \infty$ if the set is not bounded above.

Example. *For a sequence p_n in X, let $E_n = \{p_N, p_{N+1}, \ldots\}$. Then $\{p_n\}$ is a Cauchy sequence iff $\lim\limits_{N\to\infty}$ diam $E_N = 0$.*

Theorem 3.11. (a) If $p_n \to p$ then $\{p_n\}$ is a Cauchy sequence.

(b) If X is a compact metric space and $\{p_n\}$ in X is a Cauchy sequence, then $\exists_{p\in X}$ s.t. $p_n \to p$.

(c) In R^K every Cauchy sequence converges.

Remark. If a Cauchy sequence has a convergent subsequence in a metric space, then the full sequence itself converges to the same point the subsequence converges to.

Proof. Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ is Cauchy. Let $E_N = \{p_N, P_{N+1}, \ldots\}$. Then $\overline{E_N}$ is closed, hence compact. Also $\overline{E_N} \supset \overline{E_{N+1}}$ and $\lim\limits_{N\to\infty}$ diam $\overline{E_N} = 0$ (use Theorem 3.10(a) to see diam $\overline{E_N}$ = diam E_N) By theorem 3.10(b), $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$. Claim: $p_n \to p$.
Proof of the claim: Let $\varepsilon > 0$. Choose N_0 s.t.diam $\overline{E_{N_0}} < \varepsilon$, so $d(p,q) < \varepsilon \forall g \in \overline{E_{N_0}}$, and hence $\forall g \in N_0$; i.e., $d(p,p_n) < \varepsilon$ if $n \geq N_0$.

Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ in $\mathbb{R}^k$ is Cauchy. Cauchy sequences are bounded in any metric space. Therefore, $\exists$ k-cell I, which is compact, containing $\{p_n\}$. Then (b) applies Q.E.D.

Note. The converse of Theorem 3.11(a) does not hold in general.

Example. *$X = \mathbb{Q}$ has a Cauchy sequence with no limit in $\mathbb{Q}$. (see assignment 6). Converse does hold if X is compact.*

Theorem 3.12. (a) diam $\overline{E}$ = diam E

(b) If $K_n \subset X$, $K_n \neq \emptyset$, K compact, $K_n \supset K_{n+1} \forall n$ and if $\lim\limits_{n\to\infty}$ diam $K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a single point.

Proof. (a) $E \subset \overline{E} \Rightarrow \text{diam } E \leq \text{diam } \overline{E}$. For the opposite inequality, let $\varepsilon > 0, p, q \in \overline{E}$,. Choose $p', q' \in E$ s.t. $d(p, p') < \varepsilon, d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < \varepsilon + \varepsilon = 2\varepsilon$. diam $\overline{E} \leq$ diam $E + 2\varepsilon$. Since ε is arbitrary, diam $\overline{E} \leq$ diam E.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, $K \neq \emptyset$. Since $K \subset K_n \forall n$, diam $k \leq$ diam $K_n \forall n$, so diam $K = 0$. Therefore, $d(p, q) = 0 \forall p, q \in K$, so K is a simple point.

Q.E.D.

Definition 3.5 (Complete Metric Space)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X.

Example. *(a) X compact $\Rightarrow X$ complete.*

(b) $\mathbb{R}^k$ is complete, so is $\mathbb{C}$.

(c) $\mathbb{Q}$ is not complete. (see assignment 6)

(d) See assignment 6 for reference to completion of a metric space.

We've seen that convergent sequences are bounded. $p_n = (-1)^n$ shows the converse if false. However the converse does hold for monotonic sequences.

Definition 3.6 (Monotone)**.**
- A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically increasing if $s_n \leq s_{n+1} \forall n$.
- A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically decreasing if $s_n \geq s_{n+1} \forall n$.

Theorem 3.14. A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.

Proof. $\Rightarrow$ all convergent sequences are bounded in any metric space.

$\Leftarrow$ **Increasing case** Let $\{s_n\}$ be monotonically increasing and $s_n \leq M \forall n$. Let $s = \sup\{s_n : n \in \mathbb{N}\}$. Then $s_n \leq s \forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $s - \varepsilon < s_N \leq s$. But then $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \ldots \leq s$, so $|s - s_n| < \varepsilon \forall n \geq N$, and therefore $s_n \to s$.

Q.E.D.

Definition 3.7 (Infinite Limits)**.** We say
- $s_n \to \infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \geq M \forall_{n \in N}$.
- $s_n \to -\infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \leq M \forall_{n \in N}$.

Definition 3.8. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define $\limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}} s_n = \inf_{n\geq 1}\{\sup_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \sup_{m\geq n}\{s_m\}$.
$\liminf_{n\to\infty} s_n = \underline{\lim_{n\to\infty}} s_n = \sup_{n\geq 1}\{\inf_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \inf_{m\geq n}\{s_m\}$.

Note. Alternate definition; see ass 7 for equivalence

Remark. (a) If $a_n \leq b_n \forall n$ and $a_n \to a$ and $b_n \to b$, then $a \leq b$.

(b) $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$

Example. *(a)* $s_n = (-1)^n(1+\frac{1}{n^2})$ $1 \leq \sup_{m\geq n} s_m \leq 1+\frac{1}{n^2}$, *so* $\limsup_{n\to\infty} s_n = 1$. *Similarly,* $\liminf_{n\to\infty} s_n = -1$

(b) If $\{s_n\}$ *has no upper bound, then* $\sup_{m\geq n} s_m = \infty$ *and in this case we say* $\limsup_{n\to\infty} s_n = \infty$*; e.g.,*

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has $\limsup_{n\to\infty} s_n = \infty$, $\liminf_{n\to\infty} s_n = -\infty$

Lemma. $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = L \Leftrightarrow s_n \to L$.

Proof (L finite)**.**

$\Rightarrow$ This follows from $\inf_{m\geq n} s_m \leq s_n \leq \sup_{m\geq n} s_m$. $\lim_{n\to\infty} \inf_{m\geq n} s_m = \liminf_{n\to\infty} s_n$, and $\lim_{n\to\infty} \sup_{m\geq n} s_m = \limsup_{n\to\infty} s_n$. Therefore, $\lim_{n\to\infty} s_n = L$.

$\Leftarrow$ If $s_n \to L$, then $\forall_{\varepsilon>0} : \exists_N$ s.t. $s_m \in [L-\varepsilon, L+\varepsilon] \forall m \geq N$. Therefore, $\forall_{n\geq N} : L-\varepsilon \leq \inf_{m\geq N} s_m \leq \inf_{m\geq n} s_m \leq \sup_{m\geq n} s_m \leq \sup_{m\geq N} s_m \leq L+\varepsilon$. Let $n \to \infty$: $L-\varepsilon \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L+\varepsilon$. Since ε is arbitrary, so $L \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L$.

Q.E.D.

3.2 Series

Definition 3.9 (Series)**.** Let $\{a_n\}$ be a sequence in $\mathbb{C}$. Form a new sequence $\{s_n\}$, the sequence of partial sums, by $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^n a_k$. If $s_n \to s$, we say **the series** $\sum_{k=1}^\infty a_k$ **converges** and that $\sum_{k=1}^\infty a_k = s$. If $\{s_n\}$ diverges then we say $\sum_{k=1}^\infty a_k$ diverges.

Theorem 3.15. $\sum_{n\in\mathbb{N}} a_n$ converges if and only if $\forall_{\varepsilon>0} : \exists N$ s.t. $\forall n \geq m \geq N$: $|\sum_{k=m}^{n} a_k| < \varepsilon$.

Proof. $\sum_n a_n$ converges $\Leftrightarrow$ $\{s_n\}$ converges $\Leftrightarrow$ $\{s_n\}$ is a Cauchy sequence ($\because$ $\mathbb{C}$ is compact). Use $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$. Q.E.D.

Corollary 3.16. If $\sum_n a_n$ converges then $a_n \to 0$.

Proof. Take $m = n$ in Theorem 3.22. $\sum_n a_n$ converges $\Rightarrow$ $\forall_{\varepsilon>0}$: $\exists_N$ s.t. $|a_n| < \varepsilon$ if $n \geq N$. Q.E.D.

Remark. n-th term test for divergence: If $a_n \not\to 0$ then $\sum_n a_n$ diverges.
Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *diverges because* $\frac{n}{n+1} \to 1 \neq 0$.

Converse to Corollary 3.16 is false! E.g., $\sum_n \frac{1}{n}$ diverges but $\frac{1}{n} \to 0$.

Theorem 3.24. If $a_n \geq 0$, then $\sum_n a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof. $\{s_n\}$ is monotone increasing, so by Theorem 3.14, it converges if and only if it is bounded. Q.E.D.

Theorem 3.25. [Comparison Test]

(a) If $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges, then $\sum_n a_n$ converges.

Proof. Suppose $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges. Let $\varepsilon > 0$. By theorem 3.22, $\exists N$ s.t. $\sum_{k=m}^{n} c_k < \varepsilon$ if $n \geq m \geq N$. Can take $N \geq N_0$. Then $|N \geq N_0|$. $|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} c_k < \varepsilon$ if $n \geq m \geq N$. By theorem 3.22 again, $\sum_n a_n$ converges. Q.E.D.

(b) If $a_n \geq d_n \geq 0 \forall n \geq N_0$ and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.

Proof. This follows from (a): if $\sum_n a_n$ converges then $\sum_n d_n$ converges. Thus it's contrapositive, (b) is true. Q.E.D.

Theorem 3.26. [Geometric Series] $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$, $xS_n = x + x^2 + \cdots x^n + x^{n+1}$. Then

$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1 (\Leftrightarrow -1 < x < 1)$, then $x^{n+1} \to 0$ and $S_n \to \frac{1}{1-x}$. If $|x| \geq 1$, then x^{n+1} does not converge to 0, so $\sum_{n=0}^{\infty} x^n$ diverges. Q.E.D.

Theorem 3.27. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof. ($\Leftarrow$) We show that if $\sum_n a_n$ diverges, then $\sum_k 2^k a_{2^k}$ diverges. For this, note that $a_1 + a_2 + \ldots + a_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$ if $2^{k+1} > n$. $a_1 + a_2 \cdots + a_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. LHS unbounded as $n \to \infty$, so RHS is also unbounded as $k \to \infty$.

($\Rightarrow$) $a_1 + a_2 + a_3 + \cdots + a_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$ if $2^k \leq n$. $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$. If $\sum_n a_n$ converges, then LHS is bounded for all n so RHS is bounded for all k. Hence RHS converges since it is monotone.

Q.E.D.

Theorem 3.28. [p-series] $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. For $p \leq 0$, $\frac{1}{n^p} \not\to 0$, so series diverges. For $p > 0$, $\frac{1}{n^p}$ is decreasing, so $\sum_{n=1} \frac{1}{n^p}$ converges iff $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$ converges. But $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$ converges iff $\frac{1}{2^{p-1}} < 1 (\Leftrightarrow p - 1 > 0)$, which is equivalent to $p > 1$. Q.E.D.

Theorem 3.29. $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (log is to base e.)

Proof. If $p \leq 0$, then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$, so $\sum_n \frac{1}{n(\log n)^p}$ diverges by the comparison test. If $p > 0$ then $\frac{1}{n(\log n)^p}$ decreases since $\log n$ increases. By theorem 3.27, $\sum_n \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k (\log 2^k)^p}$ converges $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$ Q.E.D.

Definition 3.10 (e). $e := \sum_{n=0}^{\infty} \frac{1}{n!}$.

Remark. **Convergence** $\frac{1}{n!} = \frac{1}{(n(n-1)(n-2)\cdots 3\cdot 2\cdot 1)} \le \frac{1}{2\cdot 2\cdot 2\cdot 2\cdot \cdots \cdot 2\cdot 1} = \frac{1}{2^{n-1}}$.

Therefore, $S_n = \sum_{k=0}^{n} \frac{1}{k!} \le 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1-1/2} = 3$. Then S_n is a monotonically increasing sequence that's also bounded. Hence, $e \le 3$

Rate of Convergence

$0 < e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n+1}} \cdot \frac{1}{(n+1)!}$

$= \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-(n+1)}}$

$= \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+1)} = \frac{1}{(n!)\cdot n}$.

Theorem 3.32. $e \notin \mathbb{Q}$.

Proof. For contradiction, suppose $e = \frac{p}{q}, p, q \in \mathbb{N}$. As $0 < e - S_q < \frac{1}{q \cdot q!}$, $0 < q! \cdot e - q! \cdot S_q < \frac{1}{q}$. Since $S_q = \sum_{k=0}^{q} \frac{1}{k!}$, $q! \cdot e$ and $S_q \cdot q!$ are both integers. However, then $q! \cdot e - q! \cdot S_q$ is an integer between 0 and $\frac{1}{q} < 1$, which is a contradiction. Q.E.D.

Theorem 3.31. $e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$.

Proof. Let $t_n = (1+\frac{1}{n})^n$. Then $t_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot (\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}) \le S_n$. So $\limsup_{n\to\infty} t_n \le \limsup_{n\to\infty} S_n = \lim_{n\to\infty} S_n = e$. On the other hand, for fixed m and $n \ge m$, $t_n \ge \sum_{k=0}^{m} \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^{m} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n}) \cdot \cdots \cdot (1-\frac{k-1}{n})$. Let $n \to \infty$ with m fixed. $\liminf_{n\to\infty} t_n \ge \sum_{k=0}^{m} \frac{1}{k!} \cdot 1 = S_m$. This is true for any m. Now let $m \to \infty$. $\liminf_{n\to\infty} t_n \ge \limsup_{m\to\infty} s_m = e$. $e \le \liminf_{n\to\infty} t_n \le \limsup_{n\to\infty} t_n \le e$. Therefore, $\lim_{n\to\infty} t_n$ exists and equals e. Q.E.D.

Theorem 3.33. [Root test] Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then,

$$\sum a_n \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha > 1 \\ \text{inconclusive} & \text{if } \alpha = 1 \end{cases}$$

Proof (Just outline). $\alpha < \beta < 1$ Eventually $|a_n| \leq \beta^n$, thus convergence.

$\alpha > 1$ $|a_n| > 1$ for infinitely many n, thus divergence.

$\alpha = 1$ $\frac{1}{n}$ diverges, $\frac{1}{n^2}$ converges.

Q.E.D.

Theorem 3.34. [Ratio test] The series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ converges if $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\exists_{N\in\mathbb{N}}$ s.t. $\forall_{n\geq N} : |\frac{a_{n+1}}{a_n}| \geq 1$. Otherwise, inconclusive.

Proof. (see textbook).

$$\textbf{Convergence } \sum a_n \begin{cases} \text{converges} & \text{if } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1 \\ \text{diverges} & \text{if } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \\ \text{diverges} & \text{if } \liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \\ \text{inconclusive} & \text{otherwise. e.g., } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1 \end{cases}$$

Note. Note that we cannot replace lim inf with lim sup in the third case. For inconclusive case, check $\sum 1/n \to \infty$ and $\sum 1/n^2 \to \pi^2/6$

Q.E.D.

Example. *Let* $s_n = \sum_{n=0}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$. *First, note that* $a_{2k} = \frac{1}{2} \cdot \frac{1}{4^k}, a_{2k+1} = 2a_{2k} = \frac{1}{4^k}$ *for* $k \geq 0$.

Ratio test *Then the ratio* $\frac{a_{n+1}}{a_n}$ *is the sequence* $2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \ldots$. *Therefore,* $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}, \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$. *The ratio test is inconclusive for* s_n.

Root test $a_n = \begin{cases} \frac{1}{2} \cdot \frac{1}{2^n} & n \text{ even} \\ \frac{2}{2^n} & n \text{ odd} \end{cases}$, *so* $(\frac{1}{2})^{1/n} \cdot \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{2^{1/n}}{2}$, *thus* $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$. *Therefore,* $s_n = \sum_{n=0}^{\infty} a_n$ *converges.*

This is an example where the ratio test is inconclusive but the root test is conclusive.

Theorem 3.47. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum a_n + b_n = A + B$ and $\sum c \cdot a_n = cA$.

3.3 Power Series

Definition 3.11 (Power Series)**.** For $z \in \mathbb{C}$ and a complex sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n z^n$ is a power series.

Remark. As $z^0 = 1$ for all $z \in \mathbb{C}$, by convention we write $\sum_{n=0}^{\infty} c_n z^n = c_0 + \sum_{n=1}^{\infty} c_n z^n$.

Theorem 3.39. Let $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$, where

$$R = \begin{cases} 0 & \text{if } \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty \\ \infty & \text{if } \limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0 \end{cases}.$$

Then $\sum c_n z^n \begin{cases} \text{converges} & \text{if } |z| < R \\ \text{diverges} & \text{if } |z| > R \\ \text{inconclusive} & \text{if } |z| = R \end{cases}$. Note $R = 0$ implies the series diverges for $z \neq 0$, and $R = \infty$ implies the series converges for any $z \in \mathbb{C}$.

Proof. $\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$. By root test, the series converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$.

Note. In practice, often use the ratio test to find R.

Q.E.D.

Example. *(a) $\sum n! \cdot z^n$ has $R = 0$.*

By ratio test $\forall z \neq 0 : \left|\frac{(n+1)! \cdot z^{n+1}}{n! \cdot z^n}\right| = |z| \cdot (n+1) \to \infty$. *Hence, the series diverges .*

By root test *Note $n \neq \frac{1}{2}(\frac{2}{3})^2(\frac{3}{4})^3 \cdots (\frac{n-1}{n})^{n-1} n^n$ for $n \geq 2$. Then $n \neq \frac{n^n}{(1+1)^1(1+\frac{1}{2})^2(1+\frac{1}{n-1})^{n-1}}$. In the proof of Theorem 3.31, we saw $(1 + \frac{1}{j}) \leq e$. So $n! \geq \frac{n^n}{e^{n-1}} = e \cdot (\frac{n}{e})^n$. $\sqrt[n]{n!} \geq e^{1/n} \cdot \frac{n}{e} \to \infty$ as $n \to \infty$. Therefore, $R = \frac{1}{\infty} = 0$.*

Note. Cf. Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Definition 3.12 (Absolute Convergence, Conditional Convergence)**.**

(a) $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

(b) $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

> **Remark.** All other convergence tests seen so far are actually tests for absolute convergence.

Example.
- *$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally by the alternating series test of assignment 7's practice problem 1. See also Theorem 3.43. (MUST TAKE A LOOK!)*
- *Given $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $a_n \to 0$, then $\sum(-1)^n a_n$ converges.*
- *$\sum_{n=0}^{\infty} n!2^n$ has $R = 0$*
- *$\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ has $R = \infty$ since $R = \frac{1}{\limsup_{n\to\infty}\sqrt[n]{|c_n|}} = \frac{1}{\limsup_{n\to\infty}\frac{1}{n}} = 1/0 = \infty$, or use ratio test, $|\frac{z^{n+1}/(n+1)^{n+1}}{z^n/n^n}| = |z| = \frac{n^n}{(n+1)^{n+1}} = |z|\frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n}$. Since $(1+\frac{1}{n})^n \to \frac{1}{e}$ as $n \to \infty$, $|z|\frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n} \to 0$ as $n \to \infty \forall_{z\in\mathbb{C}}$ so $R = \infty$.*

Theorem 3.45. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Theorem 3.54. Suppose $\sum a_n$ converges conditionally. Let $-\infty \leq \alpha \leq \beta \leq +\infty$. Then $\exists$bijection $f : \mathbb{N} \to \mathbb{N}$ such that with $a'_n = a_{f(n)}$ and $S'_n = \sum_{k=1}^{n} a'_k$, $\liminf_{n\to\infty} s'_n = \alpha$ and $\limsup_{n\to\infty} s'_n = \beta$. In other words, there exists a rearrangement of $\sum a_n$, say $\sum a'_n$, such that $\liminf_{n\to\infty}\sum a'_n = \alpha, \limsup_{n\to\infty}\sum a'_n = \beta$.

> **Proof.** Take a look at the textbook Q.E.D.

Theorem 3.55. If $\sum a_n$ converges absolutely, then every rearrangement of $\sum a_n$ converges to the same sum.

> **Proof.** Take a look at the textbook Q.E.D.

3.4 Products of Series

Motivation Consider z^N in $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n)$. Since $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n) = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots$, z^N has coefficient $\sum_{k=0}^{N} a_k b_{N-k}$.

Definition 3.13. The product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

> **Note.** This is a discrete convolution.

Question If $\sum a_n = A$ and $\sum b_n = B$ both converge, does $\sum c_n$ converge and if so, does it converge to AB?

Answer $\sum c_n$ converges if $\sum a_n$ and $\sum b_n$ converge absolutely. (Theorem 3.50). Moreover, if $\sum c_n$ does converge, then it must converge to AB (Theorem 3.51). Maybe no otherwise (ref: Example 3.49).

Theorem 3.50. Suppose $\sum a_n$ converges absolutely to A and $\sum b_n$ converges to B. Then $\sum c_n$ converges to AB.

Proof. Let $A_n = \sum_{k=0}^{n} a_k$, $B_n = \sum_{k=0}^{n} b_k$, $C_n = \sum_{k=0}^{n} c_k$. Then $A_n \to A, B_n \to B$. By definition, $C_n = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j} = \sum_{j=0}^{n}\sum_{k=j}^{n} a_j b_{k-j} = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} b_{k-j} = \sum_{j=0}^{n} a_j B_{n-j} = \sum_{j=0}^{n} a_j B + \sum_{j=0}^{n} a_j (B_{n-j} - B)$. Let β_{n-j}, where $\beta_k = B_k - B$. Then $C_n = A_n B + \sum_{j=0}^{n} a_j \beta_{n-j}$. Let $\gamma_n = \sum_{j=0}^{n} a_j \beta_{n-j}$. Note that $A_n B \to AB, \beta_k \to 0$ as $n \to \infty$. Let $\alpha = \sum_{k=0}^{\infty} |a_k| < \infty$ ($\because a_n$ converges absolutely by assumption). Rewrite γ_n as $\gamma_n = \sum_{j=0}^{n} a_{n-j}\beta_j$. We know $\beta_j \to 0$ as $j \to \infty$. Let $\varepsilon > 0$. Choose N s.t. $|\beta_j| < \varepsilon$ if $j \geq N$. Then for $n \geq N+1$, $|\gamma_n| \leq |\sum_{j=0}^{N} a_{n-j}\beta_j| + |\sum_{j=N+1}^{n} a_{n-j}\beta_j|$. Note $|\sum_{j=N+1}^{n} a_{n-j}\beta_j| \leq \varepsilon \sum_{j=N+1}^{n} |a_{n-j}| \leq \varepsilon\alpha$. Let $n \to \infty$ with N fixed. Then $a_{n-j} \to 0$ for $0 \leq j \leq N$ since $|a_n| \to 0$. Q.E.D.

Theorem 3.51. If the series $\Sigma a_n, \Sigma b_n, \Sigma c_n$ converge to A, B, C respectively and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$.

Midterm 2 covers up to this point (TB p.36-82, A.5-8)

Chapter 4

Continuity

Assume general metric spaces X, Y and $f : X \to Y$.

Definition 4.1 (4.1). Suppose X, Y are metric spaces, $E \subset X$, $f : E \to Y$, $p \in E'$, where E': set of limit points in metric space X. We say $\lim_{x \to p} f(x) = q$, or $f(x) \to q$ as $x \to p$, if $\forall_{\varepsilon > 0} : \exists_{\delta > 0}$ s.t. $(0 < d_X(x, p) < \delta$ and $x \in E) \Rightarrow d_Y(f(x), q) < \varepsilon$.

Note. We don't say anything about $x = p$, $f(p)$ may not even be defined.

Theorem 4.2. $\lim_{x \to p} f(x) = q \Leftrightarrow \forall_{\{p_n\} \text{ in } E} :$ if $p_n = p$ or $p_n \to p$, then

$$\lim_{n \to \infty} f(p_n) = q,$$

where the RHS is the limit of Definition 3.1.

Note. This implies uniqueness of q in Definition 4.1.

Proof. $\Rightarrow$ Suppose $\lim_{x\to p} f(x) = q$. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $d_Y(f(x), q)\varepsilon$ if $0 < d_X(x, p) < \delta$. Let $\{p_n\}$ be a sequence in E such that $p_n \to p$ and $p_n \neq p$. Then $\exists_N$ s.t. $0 < d_X(p_n, p) < \delta$ if $n \geq N$; i.e., $f(p_n) \to q$.

$\Leftarrow$ Consider the contrapositive of ($\Leftarrow$): $\neg(\lim_{x\to p} f(x) = q) \Rightarrow \neg(\forall_{\{p_n\} \text{ in } E} : \lim_{n\to\infty} f(p_n) = q)$. Suppose $\neg(\lim_{x\to p} f(x) = q)$. Then $\exists_{\varepsilon>0}$ s.t. $\forall_{\delta>0}$: $\exists_{x\in N_\delta^E(p)}$ s.t. $x \neq p$ and $d_Y(f(x), q) \geq \varepsilon$. Take $\delta = \delta_n = \frac{1}{n}$ and let p_n be an x as above for δ_n. Then $p_n \to p$, but $d_Y(f(p_n), q) \geq \varepsilon \forall n$, so $f(p_n) \not\to q$.

Q.E.D.

Theorem 4.4. When $Y = \mathbb{C}$, limit as defined in Definition 4.1 respects sums, products and quotients.

Proof. By Theorem 4.2, it suffices to show that the theorem holds for sequences. Q.E.D.

Definition 4.2. Suppose X, Y are metric spaces, $p \in E \subset X$, $f : E \to Y$. Then f is continuous at p if $\forall_{\varepsilon>0} : \exists_{\delta>0}$ s.t. $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$; i.e., $f(N_\delta^E(p)) \subset N_\varepsilon^y(f(p))$. We say f is continuous if f is continuous at p for all $p \in E$.

Note. If p is an isolated point; i.e., $\exists_{\delta>0}$ s.t. $N_\delta^E(p) = \{p\}$, then every $f : E \to Y$ is continuous at p.

Theorem 4.6. Suppose $E \subset X, p \in E \cap E', f : E \to Y$. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.

Proof. By Definition 4.1 and Definition 4.2 with $q = f(p)$. Q.E.D.

Theorem 4.7. For $E \subset X, f : E \to Y, g : f(E) \to Z$, let $h = g \circ f : E \to Z$. If f is continuous at $p \in E$ and if g is continuous at $f(p) \in Y$, then h is continuous at p.

Proof. Choose $\eta > 0$ such that $d_Y(f(p), y) < \eta \Rightarrow d_Z(g(f(p)), g(y)) < \varepsilon$ (continuity of g at $f(p)$). Choose $\delta > 0$ s.t. $d_E(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$ (continuity of f at p). Then $d_E(x, p) < \delta \Rightarrow d_Z(g(f(x)), g(f(p))) = d_Z(h(x), h(p)) < \varepsilon$. Q.E.D.

Theorem 4.8. [Topological Characterization of Continuity] $f : X \to Y$ is continuous $\Leftrightarrow f^{-1}(V)$ is open for every open $V \subset Y$.

Proof. ($\Rightarrow$) Suppose f is continuous. Let $V \subset Y$ be open. Then $f^{-1}(V)$ is open. Let $p \in f^{-1}(V)$. We need to show $\exists_{\delta>0}$ s.t. $N_\delta^X(p) \subset f^{-1}(V)$. Since V is open, $\exists_{\varepsilon>0}$ s.t. $N_\varepsilon^Y(f(p)) \subset V$. Since f is continuous, $\exists_{\delta>0}$ s.t. $f(N_\delta^X(p)) \subset N_\varepsilon^Y(f(p)) \subset V$.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open for every open $V \subset Y$. Let $p \in X$ and $\varepsilon > 0$. Then $N_\varepsilon^Y(f(p))$ is open, so $f^{-1}(N_\varepsilon^Y(f(p)))$ is open. Take $V = N_\varepsilon^Y(f(p))$, which is open. Since $f^{-1}(V)$ is open and $p \in f^{-1}(V)$, there exists $\delta > 0$ such that $N_\delta^X(p) \subset f^{-1}(V)$. Then $f(N_\delta^X(p)) \subset V = N_\varepsilon^Y(f(p))$; i.e., f is continuous at p.

Q.E.D.

Remark.

(a)

(b) Continuity is determined by the open sets, not the metric. For instance, if metrics l_1, l_2, l_∞ have the same open sets in $\mathbb{R}^k$, hence the same continuous functions.

$$l_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|$$

$$l_2(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

$$l_\infty(x, y) = \max_{1 \le i \le k} |x_i - y_i|$$

(c) f with open $U \subset X \Rightarrow f(U)$ is open are called open maps. Continuous maps need not be open(e.g., $f(x) =$ some constant, $f(x) = x^2$), and open maps need not be continuous(e.g., floor function: $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$).

Corollary 4.9. $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed for every closed $F \subset Y$.

Proof. Let $V \subset Y$ be open and $F = V^c$. Then the above condition (RHS) is the same as $f^{-1}(V) = f^{-1}(F^c) = (f^{-1}(F))^c$ is open. Q.E.D.

Theorem 4.9. Let $f : X \to \mathbb{C}, g : X \to \mathbb{C}$ be continuous. Then $f + g, f \cdot g, f/g$(at points p where $g(p) \neq 0$) are also continuous.

Theorem 4.10. Given $f_i : X \to \mathbb{R}(i = 1, 2, \ldots, k)$, define $f : X \to \mathbb{R}^k$ by $f(x) = (f_1(x), \ldots, f_k(x))$. Then

(a) f is continuous if and only if each f_i is continuous.

(b) if $f, g : X \to \mathbb{R}^k$ are continuous, then so are $f + g : X \to \mathbb{R}^k, f \cdot g : X \to \mathbb{R}^1$

Example. *(a) For $i = 1, \ldots, k$, define $\varphi_i : \mathbb{R}^k \to \mathbb{R}$ by $\varphi_i(x) = x_i$, where $x = (x_1, x_2, \ldots, x_k)$. Then $|\varphi_i(x) - \varphi_i(y)| = |x_i - y_i| \le \left(\sum_{j=1}^k |x_j - y_j|^2\right)^{1/2} = |x - y|$, so φ_i is continuous(take $\delta = \varepsilon$. If $|x - y| < \delta = \varepsilon$, then $|\varphi_i(x) - \varphi_i(y)| < \varepsilon$).*

(b) The functions $\mathbb{R}^k \to \mathbb{R}$ defined by $x \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} (n_i \in \{0, 1, 2, \ldots\})$ is continuous on $\mathbb{R}^k$ and so is any polynomial $P(x) = \sum C_{n_1,n_2,n_3,\ldots,n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $C_{n_1,n_2,n_3,\ldots,n_k}$ is a constant (function) in $\mathbb{C}$.

(c) Rational functions $f(x) = \frac{P(x)}{Q(x)}$ are continuous at points where $Q(x) \neq 0$.

(d) The function $\mathbb{R}^k \to \mathbb{R}$ defined by $x \mapsto |x|$ is continuous.

Proof. $|x| = |y + (x - y)| \le |y| + |x - y|$, so $|x| - |y| \le |x - y|$. Similarly, $|y| - |x| \le |y - x|$, so $||x| - |y|| \le |x - y|$. Thus by taking $\delta = \varepsilon$, $|x - y| < \delta \Rightarrow ||x| - |y|| < \varepsilon$. Q.E.D.

(e) Suppose $f : X \to \mathbb{R}^k$ is continuous. Then $p \mapsto |f(p)|$ is continuous.

Proof. $p \mapsto |f(p)| = (y \mapsto |y|) \circ (p \mapsto f(p))$. Since both $(y \mapsto |y|)$, $(p \mapsto f(p))$ are continuous, $p \mapsto |f(p)|$ is continuous by Theorem 4.7. Q.E.D.

Note. A function is said to be continuous on the *domain*, not on the *range*.

Theorem 4.14. Let $f : X \to Y$ be continuous and X be compact. Then $f(X)$ is compact.

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. We need to find a finite subcover of $f(X)$. By Theorem 4.8, each set $O_\alpha = f^{-1}(V_\alpha)$ is open and $\bigcup_\alpha O_\alpha = \bigcup_\alpha f^{-1}(V_\alpha) = f^{-1}(\bigcup_\alpha V_\alpha) = f^{-1}(f(X)) = X$. Hence, $\{O_\alpha\}$ is an open cover of X, so there exists a finite subcover $X = O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. However, then $f(x) = \bigcup_{i=1}^n f(O_{\alpha_i}) = \bigcup_{i=1}^n f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^n V_{\alpha_i}$. Therefore, $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $f(X)$. Q.E.D.

Definition 4.3 (4.13)**.** $f : E \to \mathbb{R}^k$ is bounded if $\exists_{M>0}$ s.t. $|f(x)| \leq M \, \forall x \in E$.

Theorem 4.15. If X is compact and $f : X \to \mathbb{R}^k$, then $f(X)$ is closed and bounded (so f is bounded).

Proof. $f(X)$ is compact by Theorem 4.14, and since $f(X) \subset \mathbb{R}^k$, it is closed and bounded. Q.E.D.

Theorem 4.16. If X is compact and $f : X \to \mathbb{R}^1$ is continuous, then $\exists_{p,q \in X}$ s.t. $f(p) \leq f(x) \leq f(q)$ for all $x \in X$.

Proof. By Theorem 4.15, $f(X)$ is closed and bounded. By Theorem 2.28, $M \in f(X)$ and similarly $m \in f(X)$. Q.E.D.

Example. *Let* $X = (0,1)$*, not compact, let* $f(x) = \frac{1}{x}$*, continuous. However,* $\nexists_{p \in X}$ *s.t.* $\forall_{x \in X} : f(p) \leq f(x)$ *and* $\nexists_{q \in X}$ *s.t.* $\forall_{x \in X} : f(x) \leq f(q)$*.*

Theorem 4.17. Suppose $f : X \to Y$ is one-to-one, onto, continuous, where X is compact. Define $f^{-1} : Y \to X$ by $f^{-1}(f(x)) = x$. Then f^{-1} is continuous.

Proof. By Theorem 4.8, it suffices to prove that if $V \subset X$ is open then $(f^{-1})^{-1}(v)(= f(v))$ is open. However, $V^c \subset X$ is closed, hence V^c is compact by Theorem 4.14 and $(f(V^c))^c = f(V)$ is open. Q.E.D.

Example (Compactness is needed in Theorem 4.17)**.** *Let* $X = [0, 2\pi), Y = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$*. Define* $f : X \to Y$ *by* $f(\theta) = (\cos\theta, \sin\theta)$*. This* f *is 1-1, onto, and continuous, but* f^{-1} *is not continuous as* X *is not compact.*

Proof.
(1) $[0,1) \subset X$ is open but $(f^{-1})^{-1}([0,1)) = f([0,1))$ is not open because $(1,0)$ is not an interior point of Y.

(2) In Y, as $(x,y) \to (1,0)$ from above, $f((x,y)) \to 0$. As $(x,y) \to (1,0)$ from below, $\lim f^{-1}(x,y)$ does not exist in X. (Wants to be $2\pi \notin X$), so f^{-1} is not continuous at $(1,0) \in Y$.

Q.E.D.

Definition 4.18. Let X, Y be metric spaces and $f : X \to Y$. f is uniformly continuous on X if $\forall_{\varepsilon>0} : \exists_{\delta>0}$ s.t. for all $p, q \in X$ with $d_x(p,q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$.

Remark. The point is for any ε, there is some δ that works for every $p, q \in X$ such that $d(p,q)$.

Example. *(a)* $X = (0,1), Y = \mathbb{R}, f(x) = \frac{1}{x}$*.* f *is continuous on* X *but is not* *uniformly continuous.*

> **Proof.** For $x \in (0, \frac{1}{2})$, $|f(x) - f(2x)| = |\frac{1}{x} - \frac{1}{2x}| = \frac{1}{2x} \to \infty$ as $x \to 0$. Then for $\varepsilon = 1$, given any $\delta \in (0, \frac{1}{2})$, we can pick $x < \delta$ s.t. $d_(X)(x, 2x) = x < \delta$, but $d_Y(f(x), f(2x)) = \frac{1}{2x} > \frac{1}{2\delta} > 1$. Q.E.D.

(b) $X = [0,5], Y = \mathbb{R}, f(x) = x^2$ *is uniformly continuous.*

> **Proof.** For $0 \le x_1 \le x_2 \le 5$ and $\varepsilon > 0$, $|f(x_1) - f(x_2)| = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) \le 10 \cdot (x_2 - x_1)$, which is less than ε if $|x_2 - x_1| < \frac{\varepsilon}{10} = \delta$ Q.E.D.

Theorem 4.19. Suppose X is a compact metric space, Y is a metric space, and $f : X \to Y$ is continuous. Then f is uniformly continuous.

> **Proof.** Fix $\varepsilon > 0$. For $p \in X$ there exists $\delta = \delta_p(\varepsilon)$ s.t. $d_X(p,q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. We need to remove the p-dependence of δ_p. Let $J_p = N_{\frac{1}{2}\delta_p}(p)$. Then $\{J_p\}_{p \in X}$ is an open cover of X. Then there exists subcover $X = J_{p_1} \cup J_{p_2} \cdots \cup J_{p_n}$ (equality works as X is the whole metric space, so $X \subset J \Rightarrow X = J$). Let $\delta = \min\{\frac{1}{2}\delta_{p_1}, \frac{1}{2}\delta_{p_2}, \dots, \frac{1}{2}\delta_{p_n}\}$. Suppose p, q with $d_X(p,q) < \delta$. Choose $m \in \{1, 2, \dots n\}$ s.t. $p \in J_{p_m}$. Then $d_X(p, p_n) < \frac{1}{2}\delta_{p_m}$. $d_X(q, p_m) \le d_X(q,p) + d_X(p, p_m) < \delta + \frac{1}{2}\delta p_m \le \delta p_m$. $\therefore d_Y(f(q), f(p)) \le d_Y(f(q), f(p_m)) + d_Y(f(q_m), f(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.

Theorem 4.22. If X, Y are metric spaces, $f : X \to Y$ is continuous, and $E \subset X$ is connected, then $f(E)$ is connected.

> **Proof (By Contradiction).** Suppose for contradiction E is connected and there exists $A, B \subset Y$ s.t. $f(E) = A \cap B$, $f(E) \neq \emptyset$, $\overline{A} \cup B = A \cap \overline{B} = \emptyset$. Let $G = f^{-1}(A) \cap E, H = f^{-1}(B) \cap E$. Then $E = G \cup H$, G, H are nonempty. If $G \cap \overline{H} = \overline{G} \cap H = \emptyset$, it leads to a contradiction. First, $G \subset f^{-1}(A) \subset (\because A \subset \overline{A}) f^{-1}(\overline{A})$, where $f^{-1}(\overline{A})$ is closed by the corollary to Theorem 4.8, so $\overline{G} \subset f^{-1}(\overline{A})$. Second, $f(H) = B, \overline{A} \cap B = \emptyset$. Therefore, $\overline{G} \cap H = \emptyset$. WLOG, $G \cap \overline{H} = \emptyset$ as well. Hence a contradiction. Q.E.D.

Theorem 4.23. [Intermediate Value Theorem] Suppose $f : [a,b] \to \mathbb{R}$ is continuous. $\forall_{c \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\})} : \exists_{x_0 \in (a,b)}$ s.t. $f(x_0) = c$.

> **Proof.** $[a,b]$ is connected by Theorem 2.47. Hence, by Theorem 4.22, $f([a,b])$ is connected and therefore contains all points between $f(a)$ and $f(b)$. In particular, $c \in f((a,b))$ Q.E.D.

Example. *(a) there exists a continuous function called (Peano/space-filling* *curve) from* $[0,1]$ *onto the closed unit square* $S = [0,1] \times [0,1] \subset \mathbb{R}^2$.

> **Proof.** Omitted. See Rudin's problem 7.14 for an explicit example(covered in MATH-321). Q.E.D.

(b) But no such function can be one-to-one.

> **Proof.** Suppose $f : [0,1] \to S$ is 1-1, onto, continuous. Since $[0,1]$ is compact, f^{-1} is continuous by Theorem 4.17. Let $E = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then, $f(E) = S \setminus \{f(\frac{1}{2})\}$ is S minus one point, which is connected (pf omitted). But then, $f^{-1}(f(E)) = E$ must be connected by Theorem 4.22. E is not connected, so this is a contradiction. Q.E.D.

Example (18)**.** *x every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$ by*

$$f(x) = \begin{cases} 0 & (x irrational) \\ \dfrac{1}{n} & (x = \dfrac{m}{n}) \end{cases}.$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point, and that f has a simple discontinuity at every rational point.

Chapter 5

Differentiation

We consider $f : [a, b] \to \mathbb{R}$.

Definition. For $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, let $f'(x) = \lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x}$ if limit exists. Equivalently, $f(t) = f(x) + (t - x)[f'(x) + u(x, t)]$ with $\lim\limits_{t\to x} u(x, t) = 0$.

Example. *(a)* $f(x) = c$ *for all* $x \Rightarrow f'(x) = \lim\limits_{t\to x} \frac{c-c}{t-x} = 0$.

(b) $f(x) = x$ *for all* $x \Rightarrow f'(x) = \lim\limits_{t\to x} \frac{t-x}{t-x}$

(c) $f(x) = e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$. *Write* $t = x + h$, *so* $t \to x \Leftrightarrow h \to 0$. $\frac{e^{x+h}-e^x}{(x+h)-x} = e^x \frac{e^h-1}{h} = e^x \frac{e^h-1}{h} + e^x - e^x = e^x + e^x \frac{e^h-1-h}{h}$. *Let* $u(h) = \frac{e^h-1-h}{h}$. *Then* $u(h) = \frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}$, *so* $|u(h)| = |\frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}| \leq |h| \sum_{n=2}^{\infty} \frac{1}{n!} = (e - 2)|h|$ *(note for* $n \geq 2, |h^{n-1}| \leq |h|$ *if* $|h| \leq 1$*). Hence,* $u(h) \to \infty$ *as* $h \to 0$ *and therefore* $f'(x) = e^x$.

> **Remark.** $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is well defined. $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = e$. Regarding it as a power series, its radius of convergence is $R = \infty$. Also, $e^{x+y} = e^x e^y$ using definition 3.48 of product series (Rudin's p.178-180).

> **Note.** $f'(x)$: Lagrange's notation, $\frac{df}{dx}$: Leibnitz notation

Theorem 5.2. Suppose $f : [a, b] \to \mathbb{R}$ and $f'(x)$ exists. Then f is continuous at x.

Proof. The existence of $f'(x) \Leftrightarrow f(t) = f(x) + (t-x)[f'(x) + u(x,t)]$ with $\lim_{t\to x} u(x,t) = 0$. Let $t \to x$. $\lim_{t\to x} f(x) + (t-x)[f'(x) + u(x,t)] = f(x) + 0[f'(x) + 0] = f(x)$, so $\lim_{t\to x} f(t) = f(x)$; i.e., f is continuous at x. Q.E.D.

Remark. The converse is false; e.g., $f(x) = |x|$ is continuous for all x, but $f'(0)$ does not exist.

Theorem 5.3. If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both differentiable at x then so are $f+g, fg, \frac{f}{g}($ if $g(x) \neq 0)$, and $(f+g)'(x) = f'(x) + g'(x), (fg)'(x) = f'(x)g(x) + f(x)g'(x), (\frac{f}{g})'(x) = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}$.

Proof (Only the quotient rule)**.** $h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{1}{g(t)g(x)}[(f(t)g(x) - f(x)g(x)) - (f(x)g(t) - f(x)g(x))]$.
Then $\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)g(x)}\left[\frac{f(t)-f(x)}{t-x}g(x) - f(x)\frac{g(t)-g(x)}{t-x}\right]$. Let $t \to x$. $h'(x) = \frac{1}{g(x)^2}[f'(x)g(x) - f(x)g'(x)]]$. Q.E.D.

Remark. By induction, $(f_1 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdot f_1 f_2 \cdots f_n'$.

Example. *For $n = 2, 3$, $\dfrac{d}{dx}x^n = nx^{n-1}$ and we know this already for $n = 0, 1$. For $n = -1, -2$, let $m = -n > 0$. Then $\dfrac{d}{dx}x^n = \dfrac{d}{dx}\dfrac{1}{x^m} = \dfrac{(\frac{d}{dx}1)x^m - (\frac{d}{dx}x^m)1}{(x^m)^2} = \dfrac{0x^m - mx^{m-1}1}{x^{2m}} = -mx^{m-1} = nx^{n-1}$. Hence, $\forall_{n\in\mathbb{Z}} : \dfrac{d}{dx}^n = nx^{n-1}$.*

Theorem 5.5. [Chain Rule] Suppose $f : [a,b] \to \mathbb{R}$, $f'(x)$ exists for some $x \in [a,b], f([a,b]) \subset I$, where I is some interval in $\mathbb{R}$. Suppose $g : I \to \mathbb{R}$ and $g'(f(x))$ exists. Then $g \circ f$ is differentiable at x and $(g \circ f)(x)' = g'(f(x))f'(x)$

Proof. Let $h(t) = (g \circ f)(t) = g(f(t))$ for $t \in [a, b]$. Fix $x \in [a, b]$ where $f'(x)$ exists. We know:

(a) $f(t) - f(x) = (t - x)[f'(x) + u(t)]$ with $\lim_{t \to x} u(t) = 0$.

(b) With $y = f(x)$, $g(s) - g(y) = (s - y)(g'(y) + v(s))$ with $\lim_{s \to y} v(s) = 0$

As $\dfrac{h(t) - h(x)}{t - x} = \dfrac{g(f(t)) - g(f(x))}{t - x}$. By (2), $\dfrac{g(f(t)) - g(f(x))}{t - x}[g' f(x) + v(f(t))]$. Let $t \to x$. Then RHS $\to f'(x)[g'(f(x)) + 0]$ since $f(t) \to f(x)$ by continuity of f at x. Therefore, $h'(x) = f'(x)g'(f(x))$. Q.E.D.

Note. Suppose you produce $f(t)$ meters of wire by time t; i.e., rate of wire production is $f'(t)$ m/x. Also suppose you get \$ $g(l)$ for l meters of wire; rate of profit is $g'(l)$ $\$/m$. Then the rate of earning by time t is $g'(f(t))f'(t)$ $\$/m$.

Example. *(a)* $\dfrac{d}{dx} e^{x^2} = 2xe^{x^2}$

(b) $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Remark. (a) f is continuous on $\mathbb{R}$, including at $x = 0$.

Proof. $|f(x)| \leq |x|$, so by the Squeeze theorem, $\lim_{x \to 0} f(x) = 0 = f(0)$. Q.E.D.

(b) f is differentiable on $x \neq 0$, but not differentiable at $x = 0$. For $x \neq 0$, $f'(x) = \sin \dfrac{1}{x} + x(\cos \dfrac{1}{x})(\dfrac{-1}{x^2}) = \sin \dfrac{1}{x} - \dfrac{1}{x} \cos \dfrac{1}{x}$. For $x = 0$, $\dfrac{f(t) - f(0)}{t - 0} = \dfrac{t \sin \frac{1}{t}}{t} = \sin \dfrac{1}{t}$, which does not converge. Therefore, f not differentiable at $x = 0$.

(c) Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) f is continuous in $\mathbb{R}$ including at $x = 0$ ($\because |f(x)| \leq |x^2|$).

(b) f is differentiable in $\mathbb{R}$ including at $x = 0$.

Proof. For $x \neq 0$, $f'(x) = 2x \sin \dfrac{1}{x} + x^2(\cos \dfrac{1}{x})(\dfrac{-1}{x^2}) = 2x \sin \dfrac{1}{x} -$

$\cos\frac{1}{x}$. For $x = 0$, $\frac{f(t)-f(0)}{t-0} = \frac{t^2 \sin\frac{1}{t}}{t} = t\sin\frac{1}{t} \to 0$ as $t \to 0$. Hence, $f'(0) = 0$. HOWEVER! f' is not continuous at $x = 0$, because $\lim_{x\to 0} f'(x)$ does not exist. Q.E.D.

Definition 5.1. Let X be a metric space, $f : X \to \mathbb{R}$. f has a *local* max at $x \in X$ if $\exists_{\delta>0}$ s.t. $f(y) \geq f(x)$ for all $y \in N_\delta(x)$.

Theorem 5.8. Let $f : [a, b] \to \mathbb{R}$. If f has a local min or a local max at $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

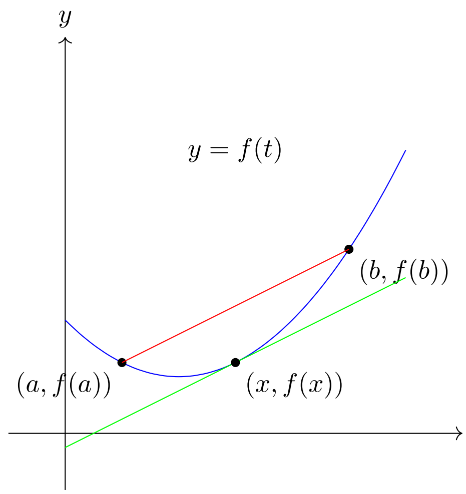
Proof (local min). Suppose f has a local min at x and $f'(x)$ exists. Choose $\delta > 0$ s.t. $N_\delta(x) \subset (a, b)$ and $f(t) \geq f(x)$ if $t \in (x - \delta, x + \delta)$. For $x < t < x+\delta$, $\frac{f(t)-f(x)}{t-x} \geq 0 (\because f(t) \geq f(x), t > x)$, so $f'(x) \geq 0$. For $x - \delta < t < x$, $\frac{f(t)-f(x)}{t-x} \leq 0 (\because f(t) - f(x) \geq 0, t < x)$, so $f'(x) \leq 0$. Hence, $f'(x) = 0$. Q.E.D.

Remark. Note that the hypothesis of the theorem requires *open* interval and existence $f'(x)$. If these conditions are not met, then $f'(x) = 0$ doesn't have to be the case.

Example. *(a) $f(x) = |x|$ has a local min at $x = 0$ but $f'(0)$ does not exist.*

(b) $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$ has a local max at $x = 1$ and local min at $x = 0$, but $f'(0) = f'(1) = 1$.

Theorem 5.10. [Mean-Value Theorem] If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then $\exists_{x\in(a,b)}$ s.t. $f(b) - f(a) = f'(x)(b - a)$.



Proof. Let $L : y = f(a) + m(t-a)$, where $m = \dfrac{y - f(a)}{t-a} = \dfrac{f(b) - f(a)}{b-a}$. Subtract L from the curve $y = f(t)$. Let $h(t) = f(t) - [f(a) + m(t-a)]$. Then $h(a) = h(b) = 0$. $h'(t) = f'(t) - m = f'(t) - \dfrac{f(b)-f(a)}{b-a}$. Therefore, it suffices to find x s.t. $h'(x) = 0$. h is continuous and $[a,b]$ is compact, so $h([a,b])$ is also compact. Hence, h attains its global max$(= \sup\{h([a,b])\})$ and global min$(= \inf\{h([a,b])\})$ on $[a,b]$. If $h(t) = 0$ for all $t \in [a,b]$ then $h'(t) = 0$ for all $t \in [a,b]$ so any $x \in (a,b)$ will do. Otherwise, h attains its global max or global min at some $x \in (a,b)$. By Theorem 5.8, $h'(x) = 0$. Q.E.D.

Theorem 5.11. If f is differentiable on (a,b) then

(a) $f'(x) \geq 0$ for all $x \in (a,b)$ implies f is monotone increasing.

(b) $f'(x) \leq 0$ for all $x \in (a,b)$ implies f is monotone decreasing.

(c) $f'(x) = 0$ for all $x \in (a,b)$ implies f is constant.

Proof ((a) only). Suppose $f'(x) \geq 0$ for all $x \in (a,b)$. For $a < x_1 < x_2 < b$, $\exists_{x \in (x_1, x_2)}$ s.t. $f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$ by Theorem 5.10. As $f'(x) \geq 0, x_2 \geq x_1$, $f(x_2) - f(x_1) \geq 0$. Q.E.D.

Definition 5.2. $f^{(0)} = f, f^{(1)} = f', f^{(2)} = f'', f^{(3)} = f'''$ and so on.

Theorem 5.15. [Taylor's Theorem] Suppose $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, and

$f^{(n+1)}(x)$ exists for all $x \in (a,b)$. Let $x, x_0 \in [a,b]$. Then $\exists_{c \in (x,x_0)}$ s.t.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{P_n(x),\, n^{\text{th}} \text{ Taylor polynomial of } f \text{ at } x_0} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Taylor Remainder}}.$$

Proof. If $n = 0$, the mean-value theorem guarantees existence of c. For general n, $A \in \mathbb{R}$ by $R_n(x) = f(x) - P_n(x) = \frac{A}{(n+1)!}(x-x_0)^{n+1}$, where A depends on f, n, x, x_0. Claim: $A = f^{(n+1)}(c)$ for some c between x and x_0.
Define $g(t) = f(t) - P_n(t) - \frac{A}{(n+1)!}(t-x_0)^{n+1}$ for $t \in [a,b]$. Then $g(x_0) = 0$. $g(x) = f(x) - P_n(x) - \frac{A}{(n+1)!}(x-x_0)^{n+1} = 0$ by the definition of A. Also for $j = 1, \ldots, n$, then $P_n^{(j)}(x_0) = f^{(j)}(x_0)$, $\frac{\mathrm{d}^j}{\mathrm{d}x^j}(x-x_0)^{n+1} \mid_{x=x_0} = 0$. Hence, $g^{(j)}(x_0) = f^{(j)}(x_0) - P_n^{(j)}(x_0) - 0 = 0$. $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - A$. We need to find c s.t. $g^{(n+1)}(c) = 0$.
$g(x) = g(x_0) = 0 \Rightarrow \exists_{c_1 \in (\min\{x_0,x\}, \max\{x_0,x\})}$ s.t. $g'(c_1) = 0$.
$g'(x_0) = g'(c_1) = 0 \Rightarrow \exists_{c_2 \in (\min\{x_0,x,c_1\}, \max\{x_0,x,c_1\})}$ s.t. $g''(c_2) = 0$.
$\vdots$
Finally, $\exists_{c_{n+1}=c}$ s.t. $g^{n+1}(c) = 0$ and hence $f^{(n+1)}(c) = A$. Q.E.D.

Example. *(not in Rudin) Does* $\sum_{n=1}^{\infty}\left(\sqrt{1+\frac{1}{n^2}} - 1\right)$ *converge or diverge?*

Method 1:

$$\sqrt{1+\frac{1}{n^2}} - 1 = \frac{\left(\sqrt{1+\frac{1}{n^2}} - 1\right)\left(\sqrt{1+\frac{1}{n^2}} + 1\right)}{\sqrt{1+\frac{1}{n^2}} + 1} = \frac{1+\frac{1}{n^2} - 1}{\sqrt{1+\frac{1}{n^2}} + 1} = \frac{1}{n^2}\frac{1}{\sqrt{1+\frac{1}{n^2}} + 1} \le \frac{1}{n^2},$$

so the series converges by the comparison test since $\sum \frac{1}{n^2}$ *converges.*

Method 2: Using Taylor's theorem. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. *Then*

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2} \\ f''(x) &= \frac{1}{2}(-\frac{1}{2})(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2} \\ f(x) &= P_1(x) + R_1(x) \\ &= f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(c)(x-0)^2}{2!} \\ &= 1 + \frac{1}{2}x + R_1(x). \end{aligned}$$

$|R_1(x)| \leq (\frac{1}{2}\cdot\frac{1}{2}\cdot 1)\frac{1}{2!}x^2 = \frac{1}{8}x^2$ *for* $x \in [0,1]$. *Therefore,* $\sqrt{1+\frac{1}{n^2}} - 1 = f(\frac{1}{n^2}) - 1 = \frac{1}{2}(\frac{1}{n^2}) + R_1(\frac{1}{n^2}) \leq \frac{1}{2n^2} + \frac{1}{8}\frac{1}{n^4}$. *Since* $\sum\left(\frac{1}{2n^2} + \frac{1}{8n^4}\right)$ *converges,* $\sum\left(\sqrt{1+\frac{1}{n^2}} - 1\right)$ *converges by comparison test.*

Example. *Let* $f(x) = \sin x, x_0 = 0$.

Taylor series for $f(x)$**.** $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, \ldots,$ *so* $f^{(k)}(x) = \begin{cases} (-1)^m \sin x & (k = 2m) \\ (-1)^m \cos x & (k = 2m+1) \end{cases}$. *Hence* $n \geq 0, f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$, *where* c *between* 0 *and* x. *Remainder estimate:* $\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$ *as* $n \to \infty$ *because* $\frac{|x|^{n+1}}{(n+1)!}$ *is the* $(n+1)^{th}$ *term in convergent series* $e^{|x|} = \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$.

$\sin x = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

Taylor approximation. *Find* $\sin 0.2$ *to within an error* $\pm 10^{-6}$. *Use* $\sin 0.2 = \frac{2}{10} - \frac{1}{3!}(\frac{2}{10})^3 + \frac{1}{5!}(\frac{2}{10})^5 - \cdots$.

Method 1: Alternating Series Test. *If* $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, *and* $a_n \to 0$, *then* $\sum_{k=1}^{\infty} (-1)^{k-1}a_k = s$ *converges and* $|s - s_n| \leq a_{n+1}$. *Above series satisfies the hypotheses, so truncation error is* $\leq$ *first omitted term. We look for when* $\frac{1}{(2k+1)!}(\frac{2}{10})^{2k+1} \leq 10^{-6}$; *i.e.,*

$(2k+1)! \cdot \frac{10^{2k+1}}{2^{2k+1}} \geq 10^6$.

If $k = 1$, $3! \cdot \frac{10^3}{2^3} < 10^6$.

If $k = 2$, $5! \cdot \frac{10^5}{2^5} < 10^6$.

If $k = 3$, $7! \cdot \dfrac{10^7}{2^7} \geq 10^6$, *so* $k = 3$ *works. Therefore,* $\sin 0.2 = 0.2 - \dfrac{1}{3!}(0.2)^3 + \dfrac{1}{5!}(0.2)^5 \pm 10^{-6} = 0.198669 \pm 10^{-6}$.

Method 2: General Case. *If alternating series test does not apply, estimate remainder using the worst* c *for* $\left|\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right|$. *In our example,* $\left|\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \leq \dfrac{1}{(n+1)!}(0.2)^{n+1}$, *so seek* n *s.t.* $\dfrac{1}{(n+1)!}\left(\dfrac{2}{10}\right)^{n+1} \leq 10^{-6}$. *First* n *that works is* $n = 6$, *same as before.*

Chapter 6

Riemann-Stieltjes Integral

Definition (Partition). A partition P of $[a, b]$ is $\{x_0, x_1, x_2, \ldots, x_n\}$ for some $n \geq 1$, with $a = x_0 \leq x_1 \ldots \leq x_{n-1} \leq x_n = b$.

Notation. $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$

$f : [a, b] \to \mathbb{R}$ be bounded, which is not necessarily continuous

$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$

$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$

$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$.

> **Note.** $L(P, f) \leq U(p, f)$ always.

Definition (Riemann Integral). **Upper Riemann Integral** : $\overline{\int_a^b} f(x)dx = \inf_P\{U(P, f)\} = \inf\{U(P, f) : P$ is a partition of $[a, b]\}$.

Lower Riemann Integral : $\underline{\int_a^b} f(x)dx = \sup_P\{L(P, f)\} = \sup\{L(P, f) : P$ is a partition of $[a, b]\}$.

Riemann Integrable : f is Riemann integrable on $[a, b]$ if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. If f is Riemann integrable on $[a, b]$, we write $f \in \mathscr{R}[a, b]$ and

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Note. Since f is bounded, $m = \inf\{f(x) : a \le x \le b\}$ and $M = \sup\{f(x) : a \le x \le b\}$ are both finite. Hence, for any P, $m \le m_i \le M_i \le M$ and $\forall_i : m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$.

Notation. Let $\alpha : [a,b] \to \mathbb{R}$ is a monotone increasing function. Then $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

Definition 6.2. Given P, let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. (Note: $\Delta\alpha_i \ge 0$).
For bounded f, let $U(P,f,\alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$, $L(P,f,\alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$.

Upper Riemann-Stieltjes Integral $\overline{\int_a^b} f(x)d\alpha = \overline{\int_a^b} f(x)d\alpha(x) = \inf_P\{U(P,f,\alpha)\} = \inf\{U(P,f,\alpha) : P \text{ is a partition of } [a,b]\}$.

Lower Riemann-Stieltjes Integral $\underline{\int_a^b} f(x)d\alpha = \underline{\int_a^b} f(x)d\alpha(x) = \sup_P\{L(P,f,\alpha)\} = \sup\{L(P,f,\alpha) : P \text{ is a partition of } [a,b]\}$.

If $\overline{\int_a^b} f(x)d\alpha = \underline{\int_a^b} f(x)d\alpha$, then $f \in R[a,b,\alpha]$ and $\int_a^b f(x)d\alpha = \overline{\int_a^b} f(x)d\alpha = \underline{\int_a^b} f(x)d\alpha$.
If $\alpha(x) = x$, then equivalent to $\int_a^b f(x)dx$.

Definition 6.3. (a) Partition P^* is called a refinement of P if $P \subset P^*$.

(b) Partition P^* is called the common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$.

Theorem 6.4. If P^* is a refinement of P then $L(P,f,\alpha) \le L(P^*,f,\alpha) \le U(P^*,f,\alpha) \le U(P,f,\alpha)$.

Proof. It's enough to consider p^* with one extra point: $x_{i-1} \le x^* \le x_i$.
Sketch for L:

$$\begin{aligned}
&L(P^*,f,\alpha) - L(p,f,\alpha) \\
&= m^*[\alpha(x^*) - \alpha(x_{i-1})] + m_i[\alpha(x_i)\alpha(x^*)] - m_i[\alpha(x^*) - \alpha(x_{i-1})] - m_i[\alpha(x_i) - \alpha(x^*)] \\
&= (m^* - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (m_i - m_i)[\alpha(x_i) - \alpha(x^*)]
\end{aligned}$$

Q.E.D.

Notation. When f, α are fixed, we write $L(P) = L(P,f,\alpha), U(P) = U(P,f,\alpha)$

Theorem 6.5. $\underline{\int_a^b} f\,\mathrm{d}\alpha \le \overline{\int_a^b} f\,\mathrm{d}\alpha$.

Proof. For partitions P_1, P_2, let $P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \le L(P^*) \le U(P^*) \le U(P_2)$. In particular, $\sup_{P_1}\{L(P_1)\} \le U(P_2)$ for all P_2. Hence, $\sup_{P_1}\{L(P_1)\} \le \inf_{P_2}\{U(P_2)\}$. Q.E.D.

Theorem 6.6. $f \in \mathscr{R}_\alpha[a,b] \Leftrightarrow \forall_{\varepsilon>0} : \exists P_\varepsilon$ s.t. $U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$

Proof. Let $\varepsilon > 0$.

($\Rightarrow$) By hypothesis, $\sup_P\{L(P)\} = \underline{\int_a^b} f\mathrm{d}\alpha = \overline{\int_a^b} f\mathrm{d}\alpha = \inf_P\{U(P)\}$.

$\exists P_1, P_2$ s.t. $L(P_1) > \underline{\int_a^b} f\mathrm{d}\alpha - \varepsilon/2$ and $U(P_2) < \overline{\int_a^b} f\mathrm{d}\alpha + \varepsilon/2$. Then $U(P_2) - L(P_1) < \varepsilon$. Let $P_\varepsilon = P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \le L(P^*) \le U(P^*) \le U(P_2)$, so $U(P_\varepsilon) - L(P_\varepsilon) \le U(P_2) - L(P_1) < \varepsilon$.

($\Leftarrow$) $0 \le \overline{\int_a^b} f\mathrm{d}\alpha - \underline{\int_a^b} f\mathrm{d}\alpha \le U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$. Since ε is arbitrary, $\overline{\int_a^b} f\mathrm{d}\alpha = \underline{\int_a^b} f\mathrm{d}\alpha$.

Q.E.D.

Remark. very important

Theorem 6.7. Let $\varepsilon_0 > 0$ be fixed. Suppose there exists a partition $P = \{x_0 = a, \ldots, x_n = b\}$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon_0$. Let s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$. Then,

(a) For any refinement of P, denoted by P^*, $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \varepsilon_0$ also holds true

(b) $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon_0$

(c) If $f \in \mathscr{R}_\alpha$, then $\left|\sum_{i=1}^n f(t_i)\Delta\alpha_i - \int_a^b f\mathrm{d}\alpha\right| < \varepsilon_0$

Theorem 6.8. If f is continuous on $[a,b]$ then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. For any P, $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i$. Since $[a,b]$ is compact, f is uniformly continuous on $[a,b]$ (Theorem 4.19), so $\forall \eta > 0 : \exists \delta > 0$ s.t. $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \eta$. Given $\varepsilon > 0$, choose η s.t. $\eta[\alpha(b) - \alpha(a)] < \varepsilon$ and choose P with $\Delta x_i < \delta = \delta(\eta)$ for all i. For such P, $M_i - m_i \le \eta$. Then $U(P) - L(P) \le \sum_{i=1}^n \eta\Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon$. Therefore, $f \in \mathscr{R}_\alpha[a,b]$. Q.E.D.

Theorem 6.9. If f is monotone increasing or decreasing on $[a,b]$ and α is continuous on $[a,b]$ then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. By definition, $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i$. Given $n \in \mathbb{N}$, let P s.t. $\Delta\alpha_i = \dfrac{\alpha(b)-\alpha(a)}{n}$ for all i. Such P exists by the intermediate value theorem (Theorem 4.23) as α is continuous. Then, $U(P) - L(P) = \dfrac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^n M_i - m_i$. Suppose f is increasing, so $M_i - m_i = f(x_i) - f(x_{i-1})$. Then $U(P) - L(P) = \dfrac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^n f(x_i) - f(x_{i-1}) = \dfrac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)]$. Given $\varepsilon > 0$, we can choose n (hence P) s.t. $U(P) - L(P) < \varepsilon$. Therefore, $f \in \mathscr{R}_\alpha[a,b]$ by Theorem 6.6. Q.E.D.

Note. We always assume α is monotone.

Theorem 6.10. If f is bounded on $[a,b]$ and has only finitely many discontinuities, and α is continuous at each point where f is not, then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. We apply Theorem 6.6. Use $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i$. Let $\varepsilon > 0$ and $E = \{e_1, \dots, e_k\}$ be the set of points where f is discontinuous. α is assumed to be continuous at each e_i, which implies $\exists (u_j, v_j)$ s.t. $u_j < e_j < v_j$ and $\alpha(v_j) - \alpha(u_j) < \varepsilon$. (Relax inequality to include equality if $e_1 = a$, $e_k = b$).
Let $K = [a,b] \cap \left(\bigcup_{j=1}^k (u_j, v_j)\right)^c$. K is compact. f is continuous on K, so f is uniformly continuous on K by Theorem 4.19. Hence, $\exists \delta > 0$ s.t. for $s, t \in K$, $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$.
Form P to consist of $\{u_1, v_1, \dots, u_k, v_k\}$ and additional points in K so that $\Delta x_i < \delta$. If x_i is in K, then $M_i - m_i < \varepsilon$. Otherwise, $x_i = u_j$ or $x_i = v_j$ for some j, so $\Delta\alpha_i \le \varepsilon$. Then

$$\begin{aligned} 0 \le U(P) - L(P) &= \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i \\ &\le \underbrace{k \cdot 2M\varepsilon}_{\text{from intervals in } \bigcup_{j=1}^k (u_j, v_j)} + \underbrace{\varepsilon[\alpha(b) - \alpha(a)]}_{\text{from intervals in } K}. \end{aligned}$$

As RHS is as small as we want by taking ε small enough, $f \in \mathscr{R}_\alpha[a,b]$. Q.E.D.

Remark. (a) Theorem 6.10 implies part of A1.2 but do the problem from first principles. Do not apply Theorem 6.10 directly.

(b) A1.4 shows what can happen if f, α are discontinuous at the same point.

Theorem 6.11. If $f \in \mathscr{R}_\alpha[a,b], m \leq f(x) \leq M$ for all $x \in [a,b]$, and $\varphi : [m, M] \to \mathbb{R}$ is continuous, then $\varphi \circ f \in \mathscr{R}_\alpha[a,b]$.

Proof. Let $\varepsilon > 0$. As φ continuous on $[m, M]$, φ is uniformly continuous on $[m, M]$ by Theorem 4.19. That is, $\exists \delta < \varepsilon$ s.t. $|\varphi(s) - \varphi(t)| < \varepsilon$ if $|s - t| < \delta$ for $s, t \in [m, M]$.
Since $f \in \mathscr{R}_\alpha$, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(P) - L(P) < \delta^2$.
Let $A = \{i \in \{1, 2, \ldots, n\} : M_i - m_i < \delta\}, B = \{i \in \{1, 2, \ldots, n\} : M_i - m_i \geq \delta\}$. Note $A \cup B = \{1, 2, \ldots, n\}$.
Let $M_i^* = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $m_i^* = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$. Suppose $i \in A$. Then $M_i - m_i < \delta$. By definition of δ, this implies $|M_i^* - m_i^*| \leq \varepsilon$.
Suppose $i \in B$. By definition of P,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \delta^2.$$

As $M_i - m_i \geq \delta$,

$$\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \delta^2.$$

Hence, $\sum_{i \in B} \Delta\alpha_i < \delta$. Then,

$$\begin{aligned}
&\sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i \\
&\leq 2 \cdot \underbrace{\sup\{|\varphi|\}}_{=\sup\{|\varphi(t)| : m \leq t \leq M\}} \cdot \left(\sum_{i \in B} \Delta\alpha_i\right) \\
&< 2 \cdot \sup\{|\varphi|\} \cdot \delta \\
&< 2 \cdot \sup\{|\varphi|\} \cdot \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
U(P, \varphi \circ f, \alpha) - L(P, \varphi \circ f, \alpha) &= \sum_{i=1}^{n} (M_i^* - m_i^*)\Delta\alpha_i \\
&= \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i \\
&< \varepsilon[(\alpha(b) - \alpha(a)) + 2 \cdot \sup\{|\varphi|\}]
\end{aligned}$$

Q.E.D.

Example. *$f \in \mathscr{R}_\alpha[a, b] \Rightarrow f^2 \in \mathscr{R}_\alpha[a, b], |f| \in \mathscr{R}_\alpha[a, b]$ where $\varphi(t) = t^2$ and $\varphi(t) = |t|$ respectively.*

Note. $\varphi \in \mathscr{R}_\alpha[m, M]$ does not imply $\varphi \circ f \in \mathscr{R}_\alpha[a, b]$. See A2.

Theorem 6.12. (Linearity and related properties)

(a) If $f, f_1, f_2 \in \mathscr{R}_\alpha[a,b]$, then $f_1 + f_2 \in \mathscr{R}_\alpha[a,b]$, and $icf \in \mathscr{R}_\alpha[a,b]$. Also $\int_a^b (f_1+f_2)\mathrm{d}\alpha = \int_a^b f_1\mathrm{d}\alpha + \int_a^b f_2\mathrm{d}\alpha$ and $\int_a^b cf\mathrm{d}\alpha c \int_a^b f\mathrm{d}\alpha$.

Proof. TEXTBOOK Q.E.D.

(b) $f_1, f_2 \in \mathscr{R}_\alpha$ and $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$, then $\int_a^b f_1\mathrm{d}\alpha \leq \int_a^b f_2\mathrm{d}\alpha$.

Proof. $L(P,f) \leq L(P,f_2)$(for all P). $L(P,f_2) \leq \sup_P L(P,f_2) = \int_a^b f_2\mathrm{d}\alpha$. $\int_a^b f_1\mathrm{d}\alpha = \sup_P L(P,f_1) \leq \int_a^b f_2\mathrm{d}\alpha$. Q.E.D.

(c) If $f \in \mathscr{R}_\alpha[a,b], c \in [a,b]$ then $f \in \mathscr{R}_\alpha][a,c]$ and $f \in \mathscr{R}_\alpha[a,b]$ and $\int_a^b f\mathrm{d}\alpha = \int_a^c f\mathrm{d}\alpha + \int_c^b f\mathrm{d}\alpha$.

(d) If $f \in \mathscr{R}_\alpha$ and $|f(x)| \leq M$ for all $x \in [a,b]$, then $|\int_a^b f\mathrm{d}\alpha| \leq M(\alpha(b) - \alpha(a))$.

Proof. Let $P = \{a,b\}$. Then $-M[\alpha(b)-\alpha(a)] \leq m_1\Delta\alpha_1 = L(P) \leq \int_a^b f\mathrm{d}\alpha \leq U(P) = M_1\Delta\alpha_1 \leq M[\alpha(b)-\alpha(a)]$. Q.E.D.

(e) If $f \in \mathscr{R}_{\alpha_1}$ and $f \in \mathscr{R}_\alpha$, then $f \in \mathscr{R}_{\alpha_1+\alpha_2}$ and

$$\int_a^b f\mathrm{d}(\alpha_1+\alpha_2) = \int_a^b f\mathrm{d}\alpha_1 + \int_a^b f\mathrm{d}\alpha_2 \tag{*}$$

.

Proof. of * Let $\varepsilon > 0$. Choose P_1, P_2 s.t. $U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \frac{\varepsilon}{2}$, where $j = 1,2$. Let $P^* = P_1 \cup P_2$. By Theorem 6.4,

$$U(P^*, f, \alpha_j) - L(P^*, f, \alpha_j) < \frac{\varepsilon}{2} \tag{**}$$

. Since $(\Delta\alpha_1)_i + (\Delta\alpha_2)_i = (\Delta(\alpha_1+\alpha_2))_i$, adding gives $U(P^*, f, \alpha_1+\alpha_2) - L(P^*, f, \alpha_1+\alpha_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. By Theorem 6.6, $f \in \mathscr{R}_{\alpha_1+\alpha_2}$. Also $\int_a^b f\mathrm{d}(\alpha_1+\alpha_2) \leq U(P^*, f, \alpha_1+\alpha_2) = U(P^*, f, \alpha_1) + U(P^*, f, \alpha_2) < \int_a^b f\mathrm{d}\alpha + \frac{\varepsilon}{2} + \int_a^b f\mathrm{d}\alpha_2 + \frac{\varepsilon}{2}$ by (**). Similarly, $\int_a^b f\mathrm{d}(\alpha_1+\alpha_2) \geq L(P^*, f, \alpha_1+\alpha_2) = L(P^*, f, \alpha_1) + L(P^*, f, \alpha_2) > \int_a^b f\mathrm{d}\alpha_1 - \frac{\varepsilon}{2} + \int_a^b f\mathrm{d}\alpha_2 - \frac{\varepsilon}{2}$ by (**). As ε is arbitrary, (*) holds. Q.E.D.

If $f \in \mathscr{R}_\alpha$ and $c \geq 0$ then $f \in \mathscr{R}_\alpha$ and $\int_a^b f \mathrm{d}c\alpha = c\int_a^b f \mathrm{d}\alpha$.

Theorem 6.13. (a) $f, g \in \mathscr{R}_\alpha \Rightarrow fg \in \mathscr{R}_\alpha$

Proof. By Theorem 6.12, $f, g \in \mathscr{R}_\alpha \Rightarrow f+g \in \mathscr{R}_\alpha$, so $f^2 \in \mathscr{R}_\alpha$. Then by Theorem 6.11, $(f \pm g)^2 \in \mathscr{R}_\alpha$. Since $(f+g)^2 - (f-g)^2 = 4fg$, so by Theorem 6.12(a), $fg \in \mathscr{R}_\alpha$. Q.E.D.

(b) If $f \in \mathscr{R}_\alpha$, then $|f| \in \mathscr{R}_\alpha$, and $\left|\int_a^b f \mathrm{d}\alpha\right| \leq \int_a^b |f| \mathrm{d}\alpha$.

Proof. By Theorem 6.11, $|f| \in \mathscr{R}_\alpha$ (take $\varphi(t) = |t|$). Let

$$c = \operatorname{sgn}\left(\int_a^b f \mathrm{d}\alpha\right) = \begin{cases} +1 & \text{if } \int_a^b f \mathrm{d}\alpha > 0 \\ 0 & \text{if } \int_a^b f \mathrm{d}\alpha = 0 \\ -1 & \text{if } \int_a^b f \mathrm{d}\alpha < 0 \end{cases}.$$

Then $\left|\int_a^b f \mathrm{d}\alpha\right| = c\int_a^b f \mathrm{d}\alpha \leq \int_a^b |f| \mathrm{d}\alpha$. Q.E.D.

Theorem 6.15. Suppose f is bounded on $[a, b]$ and $s \in (a, b)$, f continuous at s. Let

$$\alpha(x) = \begin{cases} 0 & x \leq s \\ 1 & x > s \end{cases}.$$

Then $\int_a^b f \mathrm{d}\alpha$ exists and $\int_a^b f \mathrm{d}\alpha = f(s)$.

Remark. (a) By Theorem **??**, if α is monotone-increasing, then $\alpha(x^+)$ and $\alpha(x^-)$ exist for all $x \in (a, b)$, and $\alpha(\overline{x}) \leq \alpha(x) \leq \alpha(x^+)$.

(b) In Theorem 6.15, α is left-continuous at s.

Exercise C

onclusion same for $\alpha = \begin{cases} 0 & x < s \\ 1 & x \geq s \end{cases}$