

Real Variables

January 20, 2025

Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$

Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers: $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

Remark. Note for real numbers, \mathbb{Q} has holes in it.

Example. $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$

Proof. Assume $\exists p \in \mathbb{Q}$ s.t. $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$. Q.E.D.

Definition 1.1 (Order). An order on a set S is a relation $<$ such that:

- (a) If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.
- (b) If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 1.2 (Ordered Set). An ordered set S is a set with an order $<$.

Definition 1.3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t. $\forall x \in E : x \leq \beta$.

Similarly, a set S is bounded below if $\exists \beta \in S$ s.t. $\forall x \in E : x \geq \beta$.

Definition 1.4 (LUB, GLB). Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists \alpha$ s.t. α is an upper bound for E and $\forall \gamma < \alpha$: γ is not an upper bound for E , then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists \alpha$ s.t. α is a lower bound for E and $\forall \gamma > \alpha$: γ is not a lower bound for E . Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 1.5 (LUB property). An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists \sup E \in S$; i.e., Every bounded subset of S has the least upper bound (LUB).

Example.

- \mathbb{Z} has the LUB property.
- \mathbb{Q} does not have the LUB property.

Theorem 1.1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof. (\Rightarrow) Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B . Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. (\Leftarrow) Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E . Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 1.6 (Fields). Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

- (a) $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).
- (b) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).
- (c) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).
- (d) $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.
- (e) $\exists (-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.
- (f) $\forall x, y \in F : xy \in F$.
- (g) $\forall x, y \in F : xy = yx$.
- (h) $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.
- (i) If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

-
- (j) $\forall x, y, z \in F : x(y + z) = xy + xz$

Example.

- (a) \mathbb{Q} is a field, while \mathbb{Z} is not a field.
(b) $F_p = \{0, 1, \dots, p-1\}$ with mod p arithmetic is a field.

Read Text book: 114, 115, 116, 118

Definition 1.7 (Ordered Field). An ordered field F is a field that is an ordered set such that the following properties are satisfied:

- (a) If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.
(b) If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

Remark. We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. \mathbb{Q} is an ordered field.

Theorem 1.2. \exists an ordered field \mathbb{R} which has the LUB property and contains \mathbb{Q} as a subfield.

Theorem 1.3.

- (a) Arithmetic properties of \mathbb{R} : If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.
(b) \mathbb{Q} is dense in \mathbb{R} : If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.
(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof. (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A . So, A has a least upper bound α . Since $\alpha - x < \alpha$ as $x > 0$, $\alpha - x$ is not an upper bound for A . Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A . Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. $ny - nx > 1$ and therefore, $1 + nx < ny$. Let $m \in \mathbb{Z}$ such that $(m - 1) \leq nx < m$. Such m exists by the extended version of (a). This implies there exists $m \in \mathbb{N}$ such that $nx < m \leq nx + 1 < ny$. Therefore, $x < \frac{m}{n} < y$.

(c) $\exists k \in \mathbb{Q}$ such that $k^2 = 2$; i.e., $\exists \sqrt{2} \in \mathbb{R}$. $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then $x < p < \alpha < q < y$ and $\alpha \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n - 1)x \leq y < nx. \quad (1.1)$$

Proof. Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of \mathbb{N} has a smallest element. Let $n = \text{smallest element of } A$. Then the inequality holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n - 1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n - 1)x$. Hence, the statement holds.

Q.E.D.

Lemma. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b - a)$ for some $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} b^n - a^n &= (b - a) \underbrace{(b^{n-1} + ab^{n-2} + \dots + a^{n-2}b + a^{n-1})}_{n \text{ terms}} \\ &< (b - a)nb^{n-1} \end{aligned}$$

Q.E.D.

Theorem 1.4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists! (\text{unique}) y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt[n]{x}$, the n^{th} root of x).

Proof. Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x .

Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \emptyset$, E is bounded above, hence (by the least-upper-bound property) there exists a $\sup E$. Choose $y = \sup E$. Consider two cases.

- (a) If $x \leq 1$, then $t_0 = \frac{x}{2}$ and thereby $t_0^n = \frac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).
- (b) If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x + 1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E .

- (a) Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y + h)^n - y^n < x - y^n$. By the lemma, we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y + h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.
- (b) Assuming that $y^n > x$, we find $k > 0$ such that $(y - k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y , which is $\sup E$. By the lemma, $y^n - (y - k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \frac{y^n - x}{ny^{n-1}}$. Therefore, $-(y - k)^n < -x$, or $x < (y - k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 1.8 (Cut/Dedekind Cut). The set \mathbb{R} elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 1.9 (Order of cuts). For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A , and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 1.5. There exists an ordered field \mathbb{R} such that $\mathbb{Q} \subset \mathbb{R}$ and \mathbb{R} has the LUB property.

Proof. Let \mathbb{R} be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 1.10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$

Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$

Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

Zero element is $(0, 0)$

One element is $(1, 0)$

Theorem 1.6. \mathbb{C} is a field.

Proof. Verify the 11 field axioms. For just a few axioms:

(M3):

$$x = (a, b), y = (c, d), z = (e, f). \quad x(yz) = (a, b)(ce - df, cf + de) = (a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)) = (ac - bd, ad + bc)(e, f) = (xy)z$$

(M4):

$$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$$

(M5):

$$x \neq 0 \text{ means } x = (a, b) \text{ with } a \neq 0 \text{ or } b \neq 0. \text{ That is, } a^2 + b^2 > 0. \text{ Let } \frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}). \text{ Then } x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0). \quad \text{Q.E.D.}$$

Identification of \mathbb{R} as a subfield of \mathbb{C} . Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a + b = a + b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\text{Re}(z) = a, \text{Im}(z) = b$.

Definition 1.11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by \bar{z}

Note.

- (a) $\overline{z + w} = \bar{z} + \bar{w}$
- (b) $\overline{zw} = \bar{z} \cdot \bar{w}$
- (c) $z + \bar{z} = 2 \cdot \operatorname{Re}(z)$
- (d) $z - \bar{z} = 2i \cdot \operatorname{Im}(z)$
- (e) $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$, with $=$ if and only if $z = 0$
- (f) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{a^2+b^2}$

Definition 1.12. $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 1.7. For $z, w \in \mathbb{C}$,

- (a) $|z| \geq 0$ with $=$ iff $z = 0$
- (b) $|z| = |\bar{z}|$
- (c) $|zw| = |z| \cdot |w|$
- (d) $|\operatorname{Re}(z)| \leq |z|, |\operatorname{Im}(z)| \leq |z|$

Proof. Let $z = a + bi$. Then $|\operatorname{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$
Q.E.D.

- (e) $|z + w| \leq |z| + |w|$ (Triangle inequality)

Proof.

$$\begin{aligned}
 |z + w|^2 &= (z + w)(\overline{z + w}) \\
 &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
 &= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\
 &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \\
 &\leq (|z| + |w|)^2
 \end{aligned}$$

Q.E.D.

Theorem (Cauchy-Schwarz inequality). If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$\left| \sum_{j=1}^n a_j \overline{b_j} \right| \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n |b_j|^2 \right)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^n a_j \overline{b_j}$ defined on inner product on \mathbb{C}^n and $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a}, \vec{a})(\vec{b}, \vec{b})}$. (Note that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$)

Proof. Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \overline{b_j}$. We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0. For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^n |a_j + \lambda b_j|^2 = \sum_{j=1}^n (a_j + \lambda b_j)(\overline{a_j} + \overline{\lambda b_j}) = \sum_{j=1}^n |a_j|^2 + \lambda \sum_{j=1}^n \overline{b_j} \overline{a_j} + \overline{\lambda} \sum_{j=1}^n a_j \overline{b_j} + |\lambda|^2 \sum_{j=1}^n |b_j|^2$. Let $\lambda = tC$ for $t \in \mathbb{R}$. Then $0 \leq A + \lambda \overline{C} + \overline{\lambda} C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$. $p(t)$ is a quadratic function in terms of t and it must be non-negative. Regardless of the value of t . Therefore, the discriminant of $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$. Since $|C| \geq 0$, $|C|^2 \leq AB$. Q.E.D.

Definition 1.13 (Euclidean k -space). For $k \in \mathbb{N}$, $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \dots, x_k) : x_1, x_2, \dots, x_k \in \mathbb{R}\}$ with the following properties:

Addition $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_k + y_k)$

Scalar multiplication $\lambda \vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_k)$

Inner(dot) product $(\vec{x}, \vec{y}) = \sum_{j=1}^k x_j y_j$, which is bilinear: $(\alpha \vec{x} + \beta \vec{y}) \cdot \vec{z} = \alpha \vec{x} \cdot \vec{z} + \beta \vec{y} \cdot \vec{z}$.

Norm $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^k |x_j|^2^{1/2}$

Remark. Addition and Scalar multiplication make \mathbb{R}^k into a vector space.

Theorem 1.8. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$. Then

- (a) $|\vec{x}| \geq 0$
- (b) $|\vec{x}| = 0 \Leftrightarrow \vec{x} = \vec{0}$
- (c) $|\alpha \vec{x}| = |\alpha| |\vec{x}|$
- (d) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| |\vec{y}|$ (special case of Cauchy-Schwarz inequality)
- (e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle inequality)

$$\begin{aligned} \text{Proof. } |\vec{x} + \vec{y}|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq \\ &|\vec{x}|^2 + 2|\vec{x} \cdot \vec{y}| + |\vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2 \quad \text{Q.E.D.} \end{aligned}$$

(f) $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$

$$\text{Proof. } |\vec{x} - \vec{y}| = |(\vec{x} - \vec{z}) + (\vec{z} - \vec{y})| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}| \quad \text{Q.E.D.}$$

Chapter 2

Basic Topology

Definition 2.1. Sets A and B have the same cardinality, if $\exists f : A \rightarrow B$ that is 1-1 and onto (i.e., bijective).

Theorem 2.1. Let $A \sim B$ be a relation between two sets having the same cardinality. Then \sim is an equivalence relation. That is,

- (a) $A \sim A$ (Reflexive)
- (b) $A \sim B \Rightarrow B \sim A$ (Symmetry)
- (c) $A \sim B \& B \sim C \Rightarrow A \sim C$ (Transitivity)

Definition 2.2. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $J_n = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$.

- A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$ (or if $A = \emptyset$).
- A set A is countably infinite if $A \sim \mathbb{N}$.
- A set A is countable if A is finite or countably infinite.

Example. \mathbb{Z} is a countably infinite. For $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$,

$$\text{Let } f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n + 1 & \text{if } n < 0 \end{cases}$$

Then f is bijective and therefore $|\mathbb{Z}| = |\mathbb{N}|$

Theorem 2.8. A subset of a countably infinite set is countable.

Proof. Let A be some countably infinite set and S be a infinite subset of A .

As A is a countably infinite set, we can remove duplicates and arrange A so that $A = \{a_1, a_2, a_3, \dots\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in S$. Let n_k be the smallest positive integer greater than n_{k-1} such that $x_{n_k} \in S$ for $k = 2, 3, \dots$. Let $f(k) = x_{n_k}$ for $k = 1, 2, 3, \dots$. Then this is a bijection from \mathbb{N} to S . Q.E.D.

Remark. Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

Theorem 2.12. Let E_1, E_2, \dots be countably infinite sets. Then $S = \cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof. Write $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \dots\}$

$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \dots\}$

Form an array:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \dots \\ x_{21} & x_{22} & x_{23} & x_{24} & \dots \\ x_{31} & x_{32} & x_{33} & x_{34} & \dots \\ x_{41} & x_{42} & x_{43} & x_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix might have duplicates. Let T be a subset of \mathbb{N} such that $t \in T$ if and only if t is the smallest positive integer such that $x_t \in E_1 \cup E_2 \cup \dots \cup E_n$.

Then a set $\{x_t | t \in T \text{ and } \exists i \in \mathbb{N} : x_t \in E_i\}$ is S . Clearly, $|S| = |T|$, or $S \sim T$, and T is a subset of a countably infinite set, \mathbb{N} . Therefore, T is also countable, implying S is also countable. As S is infinite, S is countably infinite. Q.E.D.

Corollary 2.13. If A is countable and $n \in \mathbb{N}$, then $\{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in A\}$ is countable.

Theorem 2.14. Let $A = \{(b_1, b_2, b_3 \dots) | b_i \in \{0, 1\}\}$. I.e., A is a set of all infinite binary strings. Then A is uncountable.

Proof (Cantor's Diagonalization argument, 1891). Let $E \subset A$ be countably infinite. $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \mid s^{(i)} \in A\}$. It suffices to find some $s \in A \setminus E$, for this shows every countably infinite subset of A is proper construction of s . Write

$$s^{(1)} = b_1^1 b_2^1 \dots \quad (2.1)$$

$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \quad (2.2)$$

$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \quad (2.3)$$

$$\vdots$$

On diagonal, flip each bit, i.e., $0 \rightarrow 1$ and $1 \rightarrow 0$ and represent the flipped bit of b_i^i by \tilde{b}_i^i . Let $s = \tilde{b}_1^1 \tilde{b}_2^2 \tilde{b}_3^3 \dots$. Then $s \in A$ and $s \notin E$ as s differs from each $s^{(i)}$ in the i -th bit. Therefore, A is uncountable. Q.E.D.

Corollary 2.15. The set $\mathcal{P}(\mathbb{N})$ of subsets of \mathbb{N} is uncountable.

Proof. We can create $f : \mathcal{P}(\mathbb{N}) \rightarrow A$ be a bijection, where A is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \quad (2.4)$$

For example, if $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0 \dots)$. This f is a bijection, and therefore A is uncountable.

Q.E.D.

Theorem 2.16. \mathbb{R} is uncountable.

Proof. This is a rough sketch of the proof:

- (a) It's enough to show that $[0, 1]$ is uncountable.
- (b) Consider binary decimal representation of $x \in [0, 1]$. For example, $x = 0.101001001\dots$. Given x , choose maximal $b_1 \in \{0, 1\}$ such that $\frac{b_1}{2} \leq x$. Then choose $b_2 \in \{0, 1\}$ such that $\frac{b_1}{2} + \frac{b_2}{2} \leq x$. Continue this process to get b_1, b_2, b_3, \dots . Then $x = \sup \left\{ \sum_{i=1}^n \frac{b_i}{2^i} \right\}$. Consider any dyadic rational of the form $\frac{m}{2^n}$. Let it be $\frac{3}{2^4}$. Then this maps $\frac{3}{2^4} \rightarrow 0, 0, 1, 1, 0, 0, \dots$ and never produce $0, 0, 1, 0, 1, 1, 1, \dots$, which also represents $\frac{3}{2^4}$. Let A_1 be a subset of $A = \{\text{infinite binary strings}\}$ such that A_1 does not contain any strings ending in $1, 1, 1, 1, \dots$. Then the decimal representation defines a bijection $f : [0] \rightarrow A \setminus A_1$.
- (c) A_1 is countable because $A = (A \setminus A_1) \cup A_1$, which is uncountable.

This shows that $[0, 1]$ is uncountable, and therefore \mathbb{R} is uncountable.
Q.E.D.

Definition 2.3 (Metric Spaces). A set X is a metric space with metric $d : X \times X \rightarrow \mathbb{R}$ if

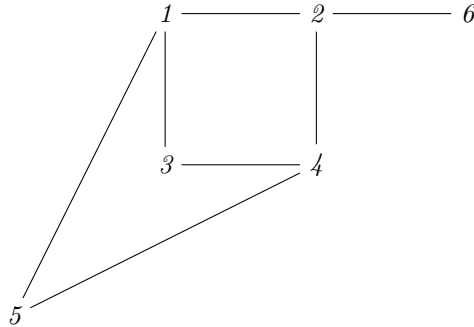
- (a) $d(p, q) > 0$ if $p \neq q$ and $d(p, q) = 0$ if $p = q$, $\forall p, q \in X$
- (b) $\forall p, q \in X : d(p, q) = d(q, p)$
- (c) $\forall p, q, r \in X : d(p, q) \leq d(p, r) + d(r, q)$ (Triangle Inequality)

Remark. A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

Example (Metric Spaces). (a) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$ are metric spaces with $d(p, q) = |p - q|$. Note the meaning of $|x|$ depends on the context.

(b) Every subset of a metric space is a metric space.

(c) $X = \{1, 2, 3, 4, 5, 6\}$



Definition 2.4 (Neighborhood). A neighborhood in X is a set $N_r(p) := \{q : d(q, p) < r\}$, where $p \in X, r > 0$.

Remark. If $r_1 \leq r_2$, then $N_{r_1}(p) \subset N_{r_2}(p)$.

Example.

\mathbb{R}^1 intervals, $N_r(x) = \{y \in \mathbb{R}^1 : |x - y| < r\}$

\mathbb{R}^2 disks $N_r(x) = \{y \in \mathbb{R}^2 : |x - y| < r\}$

\mathbb{R}^3 balls, $N_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}$

Given example (c), $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$, $N_2(2) = \{1, 2, 4, 6\}$, $N_3(2) = \{1, 2, 3, 4, 5, 6\} = X$.

Definition 2.5. Let $E \subset X$. $p \in E$ is an interior point of E if $\exists r > 0$ such that $N_r(p) \subset E$.

Example.

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$.

Definition 2.6. $E \subset X$ is an open set if $\forall x \in E$ is an interior point of E .

Theorem 2.19. Every neighborhood is an open set.

Proof. Let $g \in N_r(p)$. Then we must find $s > 0$, such that $N_s(g) \subset N_r(p)$. We know $d(p, q) < r$. Choose s such that $0 < s < r - d(p, q)$. Let $x \in N_s(g)$, then $d(q, x) < s < r - d(p, q)$. By triangle inequality, $d(p, x) \leq d(p, q) + d(q, x) < d(p, q) + r - d(p, q)$, so $x \in N_r(p)$, so $N_s(g) \subset N_r(p)$. Q.E.D.

Definition 2.7. Let $E \subset X$ and $p \in X$. p is a limit point of E if $\forall r > 0 \exists q \in E$ such that $q \neq p$ and $q \in N_r(p)$

Example. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ has exactly one limit point, 0. note $0 \notin E$.

Theorem 2.20. If p is a limit point of $E \subset S$, then every neighborhood of p contains infinitely many points of E .

Proof. Let $N_r(p)$ be a neighborhood of p . Then $N_r(p)$ contains at least one point $q_1 \in E$ such that $q_1 \neq p$. Let $r_1 = d(p, q_1)$. Then $N_{r_1}(p)$ contains some $q_2 \in E$ such that $q_2 \neq p$. Let $r_2 = d(p, q_2)$. Then $N_{r_2}(p)$ contains some $q_3 \in E$ such that $q_3 \neq p$. Continue this process to get q_1, q_2, q_3, \dots Q.E.D.

Corollary 2.21. If $E \subset X$ is finite then E has no limit points.

Definition 2.8 (Closed Set). A set $E \subset X$ is closed if every limit point of E is in E .

Theorem 2.23. $E \subset X$ is open iff $E^c = \{x \in X : x \notin E\}$ is closed.

Proof.

- E is open $\Rightarrow E^c$ is closed.
Let p be a limit point of E^c . Then every neighborhood of p contains some $q \in E^c$ such that $q \neq p$. If $p \in E$, then because E is open, p is an interior point, i.e., there exists some neighborhood of p that is a subset of E , which does not contain any points of E^c . This implies $p \notin E$ and therefore $p \in E^c$.
- E^c is closed $\Rightarrow E$ is open.
Let $p \in E$. Then $p \notin E^c$, so p is not a limit point of E^c . Therefore, there exists some neighborhood of p that contains no points of E^c , i.e., all points of the neighborhood are in E . Thus, Every $p \in E$ is an interior point of E , and hence E is open.

Q.E.D.

Theorem 2.24 (De Morgan's Laws).

- (a) $(\bigcup_{\alpha} E_{\alpha})^c = \bigcap_{\alpha} E_{\alpha}^c$
- (b) $(\bigcap_{\alpha} E_{\alpha})^c = \bigcup_{\alpha} E_{\alpha}^c$

Theorem 2.24.

- (a) For all collection $\{G_{\alpha}\}$ of open sets : $\bigcup_{\alpha} G_{\alpha}$ is open.
- (b) For all collection $\{F_{\alpha}\}$ of closed sets : $\bigcap_{\alpha} F_{\alpha}$ is closed.
- (c) For all finite collection $\{G_1, G_2, \dots, G_n\}$ of open sets : $\bigcap_{i=1}^n G_i$ is open.
- (d) For all finite collection $\{F_1, F_2, \dots, F_n\}$ of closed sets : $\bigcup_{i=1}^n F_i$ is closed.

-
- Proof.** (a) Let $x \in \bigcup_{\alpha} G_{\alpha}$. Then $x \in G_{\alpha_0}$ for some α_0 . So there exists a neighborhood N of x such that $N \subset G_{\alpha_0} \subset \bigcup_{\alpha} G_{\alpha}$.
- (b) it's suffice to prove that $(\bigcap_{\alpha} F_{\alpha})^c$ is open. But $(\bigcap_{\alpha} F_{\alpha})^c = \bigcup_{\alpha} F_{\alpha}^c$ is open by (a).
- (c) Let $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for $i = 1, 2, \dots, n$. So there exists a $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $N_r(x) \subset N_{r_i}(x) \subset G_i$ for $i = 1, 2, \dots, n$ and therefore $N_r(x) \subset \bigcap_{i=1}^n G_i$.
- (d) $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$ is open by (c).

Q.E.D.

Definition 2.9 (Closure). Let $E \subset X$. Let E' be a set of limit points of E in X . The set $\overline{E} = E \cup E'$ is the closure of E .

Theorem 2.27.

- (a) \overline{E} is closed.
- (b) $E = \overline{E} \Leftrightarrow E$ is closed.
- (c) If $F \subset X$ is closed and $E \subset F$, then $\overline{E} \subset F$. (i.e., \overline{E} is the smallest closed set containing E , and $\overline{E} = \bigcap_{F: \text{closed set with } E \subset F} F$.)

- Proof.** (a) Let p be a limit point of \overline{E} . It suffices to show $p \in E'$ since this implies that $p \in E' \subset E \cup E' = \overline{E}$. Let $r > 0$. $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$, i.e., $d(p, q) < \frac{r}{2}$. Since $q \in E \cup E'$, $\exists_{s \in \overline{E}}$ such that $d(q, s) < \frac{r}{2}$ (if $q \in E$, take $s = q$). But $d(p, s) \leq d(p, q) + d(q, s) < \frac{r}{2} + \frac{r}{2} = r$.
- (b) (\Rightarrow) by (a)
- (\Leftarrow) Suppose E is closed. Then $E' \subset E$, so $\overline{E} = E \cup E' = E$.
- (c) Suppose F is closed. Then $F' \subset F$ and also $F \supset F'$. So $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

Theorem 2.28. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup\{E\}$. Then $y \in \overline{E}$. Hence, $y \in E$ if E is closed.

Example. Let $X = \mathbb{R}$, $d(p, q) = |p - q|$. Let $E \subset \mathbb{R}$ be nonempty and bounded above, and let $y = \sup E$. Then $y \in \overline{E}$.

Proof. Suppose for contradiction $y \notin \overline{E}$. Then y is neither a point in E

nor a limit point of E , so \exists some interval $N_r(y) = (y - r, y + r)$ such that $(y - r, y + r) \cap E = \emptyset$. However, then $y - r$ is an upper bound for E since y is a least upper bound, which is a contradiction. Therefore, $y \in \overline{E}$. Q.E.D.

Definition 2.10 (Relative Openness). Suppose X is a metric space, so $Y \subset X$ is a metric space with the same metric. Let $E \subset Y$. Then E is open relative to Y if E is an open set in the metric space Y .

Example. $X = \mathbb{R}^2 \supset \mathbb{R} = Y$, $E = (0, 1) \subset Y$. Then E is open relative to Y , but E is neither open nor closed in X .

Theorem 2.30. A set $E \subset Y \subset X$ is open relative to $Y \Leftrightarrow \exists$ open set $G \subset X : E = G \cap Y$.

Proof. (\Rightarrow) Suppose $E \subset Y$ is open relative to Y . Given $p \in E$, $\exists_{r_p > 0} : N_{r_p}^Y(p) \subset E$, where $N_r^Y(p) = \{q \in Y : d(p, q) < r\}$. Then $E \subset \bigcup_{p \in E} N_{r_p}^Y(p)$ and $\bigcup_{p \in E} N_{r_p}^Y(p) \subset E$. Therefore, $E = \bigcup_{p \in E} N_{r_p}^Y(p)$.

Let $G = \bigcup_{p \in E} N_{r_p}^X(p)$. This time, we are considering p 's neighborhood in X , so each $N_{r_p}^X$ is open. Thus G is a union of open sets in X , and therefore open.

$\forall_{p \in E} : p \in N_{r_p}^X(p)$, so $E \subset G \cap Y$.

Let $p \in G \cap Y$. Then $p \in G$ and $p \in Y$. So $p \in N_{r_p}^X(p)$ for some $r_p > 0$. But $p \in Y$, so $p \in N_{r_p}^Y(p)$. Therefore, $p \in E$. This implies $G \cap Y \subset E$, and therefore $E = G \cap Y$.

(\Leftarrow) Suppose $G \subset X$ is open and $E = G \cap Y$. Then $\forall_{p \in E} : \exists_{r_p > 0} : N_{r_p}^X(p) \subset G$, so $N_{r_p}^Y(p) = N_{r_p}^X(p) \cap Y \subset G \cap Y = E$.

Q.E.D.

Note: Midterm 1 material ends here.

Definition 2.11 (Open Cover). An open cover of $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t. $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.12 (Compact). A set $K \subset X$ is compact if every open cover has a finite subcover; i.e., $\exists_{\alpha_1, \alpha_2, \dots, \alpha_n} : \text{s.t. } K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$.

Example.

- If E is finite, then E is compact.
- $(0, 1) \subset \mathbb{R}$ is not compact. Bad cover: $(\frac{1}{n}, 1), n > 2$
- $[0, \infty) \subset \mathbb{R}$ is not compact. Bad cover: $(-1, n)$ for $n \in \mathbb{N}$.
- $E \subset \mathbb{R}^k$ is compact if and only if E is closed and bounded.

Theorem 2.34. If K is compact then K is closed.

Proof. Suppose K is compact. It suffices to prove that K^c is open. Let $p \in K^c$. We need to produce $r > 0$ s.t. $N_r(p) \subset K^c$. For $q \in K$, let $W_q = N_{r_q}(q)$, where $r_q = \frac{1}{2}d(p, q) > 0$. $\forall x \in N_{r_q}(p) : x \in W_q \Rightarrow d(x, p) + d(x, q) < 2r_q = d(p, q)$. However, X is a metric space and $p, q, x \in X$, so $d(p, q) \leq d(p, x) + d(x, q)$, leading to $d(p, q) \leq d(p, x) + d(x, q) < d(p, q)$, which is a contradiction. Hence, $\forall x \in N_{r_q}(p) : x \notin W_q$. $N_{r_q}(p) \subset W_q^c$ for $\forall q \in K$. Note that $\{W_q\}_{q \in K}$ is an open cover of K . K compact $\Rightarrow \exists$ finite number of open sets $W_{q_1}, W_{q_2}, \dots, W_{q_n}$ s.t. $K \subset \bigcup_{i=1}^n W_{q_i}$. Let $r = \min\{r_{q_1}, r_{q_2}, \dots, r_{q_n}\} > 0$.

$$\therefore N_r(p) \subset \left(\bigcap_{i \in \{1, 2, \dots, n\}} N_{r_{q_i}}(p) \right) \subset \left(\bigcap_{i \in \{1, 2, \dots, n\}} W_{q_i}^c \right) = \left(\bigcup_{i \in \{1, 2, \dots, n\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

Theorem 2.35. If $K \subset X$ is compact then K is bounded; i.e., $\exists M < \infty$ s.t. $\forall p, q \in K : d(p, q) \leq M$

Proof. Fix $p \in K$. An open cover of K is $\{N_n(p)\}_{n \in \mathbb{N}}$. In fact, this is an open cover of X . K compact $\Rightarrow \exists$ finite subcover $N_{n_1}(p), N_{n_2}(p), \dots, N_{n_m}(p)$. Let $R = \max\{n_1, n_2, \dots, n_m\}$. $K \subset N_R(p)$. Let $M = 2R$. $\forall q, r \in K : d(q, r) \leq d(q, p) + d(p, r) < R + R = 2R = M$. Q.E.D.

Theorem 2.35. If F is closed, K is compact, and $F \subset K$ then F is compact.

Proof. Suppose $F \subset K$. Let $\{V_\alpha\}$ be an open cover of F . It suffices to produce a finite subcover: Consider $\{V_\alpha\}$ together with F^c . This gives an open cover of X , hence of K , so \exists subcover of K . Drop F^c from this finite subcover. The result is a finite subcover of $\{V_\alpha\}$, which covers F . Q.E.D.

Corollary 2.36. If F is closed and K is compact then $F \cap K$ is compact.

Theorem 2.33. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y .

Note. This is not true for open sets. For instance, let $K = Y = [0, 1] \subset X = \mathbb{R}$. Y is open and closed relative to Y , but Y is not open relative to X .

Proof.

- (\Rightarrow) Suppose K is compact relative to X . Let $\{V_\alpha\}$ be an open cover of K relative to Y . For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then $\{V_\alpha\}$ is an open cover of K relative to X . Since K is compact relative to X , \exists finite subcover.
- (\Leftarrow) Suppose K is compact relative to Y . Let $\{V_\alpha\}$ be an open cover of K relative to X . Then $\{V_\alpha \cap Y\}$ is an open cover of K relative to Y . Since K is compact relative to Y , \exists finite subcover.

Q.E.D.

Theorem 2.36. Suppose $\{K_\alpha\}$ is a collection of compact sets such that $\bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$. Then, $\lim_{n \rightarrow \infty} \bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$, or equivalently, $\bigcap_\alpha K_\alpha \neq \emptyset$.

Example. Let $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$. Then $\{G_j\}$ is a collection of open sets, but none of them are compact. (compact sets are closed) Then $\{G_j\}$ satisfies non-empty finite intersection property but $\bigcap_{j \in \mathbb{N}} G_j = \emptyset$.

Proof. Suppose for contradiction $\bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$ and $\bigcap_\alpha K_\alpha = \emptyset$. For any α_0 , $K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right) = \emptyset$. Hence, $K_{\alpha_0} \subset \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_0} (K_\alpha)^c$ and $\{(K_\alpha)^c\}_{\alpha \neq \alpha_0}$ is an open cover of K_{α_0} , so \exists a finite subcover of $K_{\alpha_0} \subset \bigcup_{i=1}^n K_{\alpha_i}^c$, which implies $K_{\alpha_0} \cap \left(\bigcap_{i=1}^n K_{\alpha_i}\right) = \emptyset$, contradiction. Q.E.D.

Corollary 2.37. If $\{K_1, K_2, \dots\}$ are non-empty compact sets with $\forall_n : K_n \supset K_{n+1}$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

Proof. If $n_1 < n_N$ then $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$ Q.E.D.

Theorem 2.37. If K is compact and $E \subset K$ is infinite, then E has a limit point in K .

Proof. Contrapositive of the statement is : *if $E \subset K$ has no limit point in K , then E is finite.*

Suppose every point $q \in K$ is not a limit point of E . Then

$$\exists_{V_q = N_{r_q}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}.$$

$\{V_q\}_{q \in K}$ is an open cover of K , so \exists finite subcover $V_{q_1} \cup V_{q_2} \cup \dots \cup V_{q_n}$. Then $E = E \cap K \subset (\bigcup_{i=1}^n V_{q_i} \cap E) \subset \{q_1, q_2, \dots, q_n\}$, so E is finite.

Q.E.D.

Theorem 2.38. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be such that $\forall_n : I_n \supset I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Since $I_n \supset I_{n+1}$, $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$. Let $E = \{a_1, a_2, \dots\}$. Then $E \neq \emptyset$, every b_k is an upper bound for E , so $\exists x = \sup E$ and $a_k \leq x \leq b_k$ for all k . Therefore, $x \in I_k$ for all k , so $x \in \bigcap_{n=1}^{\infty} I_n$. Q.E.D.

Theorem 2.39. Let $\{I_n\}$ be a sequence of k -cells such that $i_n \supset I_{n+1}$; i.e., $I_n = \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj}, a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj} \text{ for } j = 1, 2, \dots, k\}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Apply previous theorem to each component. Q.E.D.

Note. k -cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of k closed intervals on the real line.

Formally, Given real numbers a_i and b_i such that $a_i < b_i$ for every integer i from 1 to k ,

$$I = \{x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, k\}$$

Theorem 2.40. Let $I \subset \mathbb{R}^k$ be a k -cell. Then I is compact.

Proof. Let $I = I_0 = \{x = (x_1, x_2, \dots, x_k), a_j \leq x_j \leq b_j\}$.

Let $\Delta = \{\sum_{i=1}^k (b_j - a_j)^2\}^{1/2}$. Then $|\mathbf{x} - \mathbf{y}| \leq \Delta$ for $\mathbf{x}, \mathbf{y} \in I$.

Suppose for contradiction $\{G_\alpha\}$ is an open cover of I that has no finite subcover.

Let $c_j = \frac{1}{2}(a_j + b_j)$ for $j = 1, 2, \dots, k$. Using $[a_j, c_j], [c_j, b_j]$, we get 2^k k -cells Q_i with $I = \bigcup_{i=1}^{2^k} Q_i$. At least one Q_i , call it I_1 , has no finite subcover. Otherwise, every Q_i has a finite subcover, and I would have a finite subcover, namely the union of the finite subcovers of each Q_i . Repeat this step to construct $I_0 = I, I_1, I_2, \dots$. Then the sequence $\{I_n\}$ constructed by this process satisfies the following properties:

- (a) $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \dots$
- (b) $\forall n : I_n$ has no finite subcover from $\{G_\alpha\}$
- (c) if $x, y \in I_n$ then $|x - y| \leq 2^{-n}\Delta$, where $\Delta = \text{diagonal of } I = \left(\sum_{j=1}^k (b_j - a_j)^2\right)^{1/2}$.

By theorem 2.38 and (a), $\exists x^* \in \bigcap_{n=1}^\infty I_n$. Since $x^* \in I$, $x^* \in G_{\alpha_0}$ for some α_0 , so $\exists r > 0$ such that $N_r(x^*) \subset G_{\alpha_0}$. But by (c), $I_n \subset N_{2^{-n}\Delta}(x^*)$. As soon as n is large enough that $2^{-n}\Delta < r$, we have $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$, which contradicts (b). Q.E.D.



Note. Reverse triangle inequality

$\forall a, b, c \in X : d(a, b) \geq d(a, c) - d(c, b)$ because $d(a, c) \leq d(a, b) + d(b, c)$.

Theorem 2.41. For $E \subset \mathbb{R}^k$, the following are equivalent:

- (a) E is closed and bounded.
- (b) E is compact.
- (c) Every infinite subset of E has a limit point in E .

Proof.

(a) \Rightarrow (b) Because E is bounded, i.e., \exists_M s.t. $\forall_{x,y \in E} : |x - y| \leq M$, there exists a k -cell I such that $E \subset I$. Since every k -cell is compact, this implies E is a closed subset of a compact set. Hence, E is also compact.

(b) \Rightarrow (c) by theorem 2.37

(c) \Rightarrow (a) To see that E is bounded, suppose it were not. Then E has an infinite subset $S = \{x_1, x_2, x_3, \dots\}$ with $\forall_n : |x_n| \geq n$. S has no limit point in \mathbb{R}^k . Let $S = \{(x_1, x_2, x_3, \dots) \in E : |x_n - x_0| < \frac{1}{n}\}$. Then S is an infinite set because if S is finite, there exists a point $\mathbf{x} \in S$ such that $|\mathbf{x}| \geq |\mathbf{x}'|$ for $\mathbf{x}' \in S$. However, there exists $n \in \mathbb{N}$ such that $n > |\mathbf{x}|$ and by definition of S , there exists $x_n \in S$ such that $|x_n| \geq n > |\mathbf{x}|$, which is a contradiction. Thus, S is infinite. This S however, cannot have a limit point in E . By triangle inequality, for any $y \in \mathbb{R}^k$, $|x_n| \leq |x_n - y| + |y|$, and from archimedean property, $\exists_{m \in \mathbb{N}}$ s.t. $m > |x_n - y| + |y|$, which implies for any $y \in \mathbb{R}^k$, $r > 0$, $\exists_{m \in \mathbb{N}} : |x - y| < r < m$. However, by the definition of S , there are at most m such elements in S . Since a limit point y of E must contain an infinite number of points of E such that $d(x, y) < r$ for any $r > 0$, y cannot be a limit point, which contradicts the assumption that any infinite subset of E contains a limit point in E . Therefore, E must be bounded.

To see that E is closed, suppose it were not closed. Then $\exists_{x_0 \in E' \setminus E}$. If T has no limit point in E except $x_0 \notin E$, it contradicts (c) because T is infinite and there must be a limit point of T in E .

Therefore, we can show that E is closed by showing that T has no limit point in E except x_0 . Form an infinite sequence $(x_1, x_2, x_3, \dots), x_n \in E$ with $|x_n - x_0| < \frac{1}{n}$. Let $y \in E$, $y \neq x_0$. We'll show that y cannot be a limit point of T . $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$. Choose $n \geq \frac{2}{|y - x_0|}$, so $\frac{1}{n} \leq \frac{|y - x_0|}{2}$. Then $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$. So only finitely many x_n can lie in $N_{\frac{1}{2}|y - x_0|}(y)$. So y cannot be a limit point of S . Therefore, E is closed.

Q.E.D.

Remark. (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than \mathbb{R}^k .

Example. Failure of Heine-Borel theorem in general metric spaces.

some infinite set, discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Then E is bounded

and closed but not compact.

Theorem 2.42. [Weirstrass's theorem] Every bounded infinite subset $E \subset \mathbb{R}^k$ has a limit point in \mathbb{R}^k .

Proof. Choose a k -cell $I \supset E$. Since I is compact, by theorem 2.41, E has a limit point in I . Q.E.D.

Example. Let

$$E_0 = [0, 1] \quad (2.5)$$

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \quad (2.6)$$

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \quad (2.7)$$

$$\vdots \quad (2.8)$$

This gives $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \dots$, where each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 2.13 (Perfect Sets). A set P is perfect if there is no isolated point in P ; i.e.,

$$P = P'.$$

Theorem 2.43. Let P be a non-empty perfect set in \mathbb{R}^k . Then P is uncountable.

Proof. Suppose for contradiction P is countable. Since P is non-empty, there exists some $p_1 \in P$. p_1 is then also a limit point of P . Let $p_2 \in P (\neq p_1)$ be a point in $V_1 = N_{r_1}(p_1)$ for some r_1 such that $d(p_1, p_2) > r_1/2$. Let $r_2 = r_1 - d(p_1, p_2)$, $V_2 = N_{r_2}p_2$. Then $\forall x \in V_2 : d(p_1, x) \leq d(p_1, p_2) + d(p_2, x) < d(p_1, p_2) + r_2 = r_1$. Hence, $V_2 \subset V_1$. $\overline{V_2} \subset V_1$ as well. Also, note that $d(p_1, p_2) > r_1/2$, so $r_2 = r_1 - d(p_1, p_2) < r_1/2 < d(p_1, p_2)$. So $p_1 \notin V_2$. Repeat this process, and let $K_n = \overline{V_n} \cap P$. $K_n \subset \overline{V_n}$. Since $\overline{V_n}$ is closed and bounded, it's compact. $\overline{V_n} \cap P$ is a closed subset of $\overline{V_n}$, so K_n is also compact. However, for any p_n , $p_n \notin K_{n+1}$, so $\bigcap_{1 \in \mathbb{N}} K_n \cap P = \emptyset$. Since $K_n \subset P$, this implies $\bigcap_{1 \in \mathbb{N}} K_n = \emptyset$, but each K_n is not empty, $K_n \supset K_{n+1}$, and K_n is compact. Thus, $\bigcap_{1 \in \mathbb{N}} K_n \cap P$ can't be empty, so this is a contradiction. Q.E.D.

Definition 2.14 (Cantor Set). The cantor set $P := \bigcap_{n=1}^{\infty} E_n$.

Proposition. P is compact, non-empty and contains no open intervals (a, b) and uncountable.

Proof. Compactness P is compact because $P \subset E_0 = [0, 1]$ and E_0 is compact.

Non-emptiness P is non-empty because $P \subset E_0$ and E_0 is non-empty.

No open intervals P contains no open intervals (a, b) because any (a, b) contains some $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ and these are all removed.

Uncountability P is uncountable because P is a perfect set. Equivalently, P consists of points in $[0, 1]$ whose ternary, i.e., base 3, representation contains only 0's and 2's.

Note. ternary representation: $0.a_1a_2a_3\dots = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0, 1, 2\}$.

Q.E.D.

Example (Cantor Set). Let $E = [0, 1]$, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. Keep removing open middle third. This gives $E_0 \supset E_1 \supset E_2 \dots$. Each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 2.15 (Separated Sets). Separated Sets $A, B \subset X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Connected Sets $E \subset X$ is connected if there is no non-empty separated sets $A, B \subset E$.

Example (Separated Sets). In \mathbb{R}^1 , $[0, 1)$ and $(1, 2]$ are separated so $[0, 1) \cup (1, 2]$ is not connected. Every interval is connected (open, closed, semi-open).

Theorem 2.47. $E \subset \mathbb{R}^1$ is connected if and only if E is an interval; i.e., $\forall x, y \in E, x < y$ s.t. $\forall z \in (x, y) : z \in E$

Proof. Let $x, y \in E$. Q.E.D.

Theorem 2.48. A metric space X is connected if and only if the only nonempty subset of X which is both open and closed is X itself. $|\limsup_{n \rightarrow \infty} |\gamma_n|| \leq 0 + \varepsilon\alpha$. Since ε is arbitrary, this implies $|\limsup_{n \rightarrow \infty} |\gamma_n|| = 0$, so $\lim_{n \rightarrow \infty} |\gamma_n| = 0$.

Chapter 3

Sequence and Series

3.1 Sequences

Definition 3.1. In a metric space (X, d) , a sequence $\{p_n\}$ converges to p if $\forall \varepsilon > 0 \exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.
We write $\lim_{n \rightarrow \infty} p_n = p$ or $p_n \rightarrow p$.

If $\{p_n\}$ does not converge to any p then it is said to diverge.

Theorem 3.3. If s_n and t_n are sequences in \mathbb{C} with $s_n \rightarrow s$ and $t_n \rightarrow t$, then the following hold:

- (a) $s_n + t_n \rightarrow s + t$
- (b) $cs_n \rightarrow cs$, $c + s_n \rightarrow c + s$ for any $c \in \mathbb{C}$
- (c) $s_n t_n \rightarrow st$
- (d) $\frac{1}{s_n} \rightarrow \frac{1}{s}$ if $s \neq 0$

Lemma (Squeeze Lemma). In \mathbb{R} , if $\forall n \in \mathbb{N} : 0 \leq x_n \leq s_n$ and $\lim_{n \rightarrow \infty} s_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$. Then $0 \leq x_n \leq s_n < \varepsilon$ for $n \geq N$, so $x_n \rightarrow 0$. Q.E.D.

Theorem 3.20. (a) If $p > 0$ then $\frac{1}{n^p} \rightarrow 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $\frac{1}{N^p} < \varepsilon$; i.e., $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$.
Then for $n \geq N$, $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$. Q.E.D.

(b) If $p > 0$ then $\sqrt[p]{p} \rightarrow 1$.

Proof. $p = 1$ is obvious.
Suppose $p > 1$. Let $x_n = \sqrt[p]{p} - 1 > 0$. Want to show $x_n \rightarrow 0$.
Since $(x_n + 1)^n$, we have $p = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} x_n^k > \binom{n}{1} x_n' = n x_n$.
Therefore, $x_n \leq \frac{p}{n}$, so $x_n \rightarrow 0$ by the Squeeze Lemma.
Suppose $p \in (0, 1)$. Let $q = \frac{1}{p} > 1$. Then $\sqrt[q]{q} \rightarrow 1$ by the previous case. By 3.3, $\sqrt[p]{p} = \frac{1}{\sqrt[q]{q}} \rightarrow 1$. Q.E.D.

(c) $\sqrt[n]{n} \rightarrow 1$

Proof. Let $x_n = \sqrt[n]{n} - 1 > 0$, for $n \geq 2$. $n = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} (x_n)^k > \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$. Therefore, $x_n \leq \sqrt{\frac{2}{n-1}}$.
Q.E.D.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$; i.e., Exponentials beat powers.

Proof. We want an upper bound on $\frac{n^\alpha}{(1+p)^n}$, so seek a lower bound on $(1+p)^n$.
 $(1+p)^n = \sum_{k=0}^n \binom{n}{k} p^k > \binom{n}{k} p^k$ for $k \leq n$
 $(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$. Then for $k \leq \frac{n}{2}$, $\binom{n}{k} p^k > (\frac{n}{2})^k \frac{p^k}{k!}$. Therefore, $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$. Let $k_0 \in \mathbb{Z}$ s.t. $k > \alpha$. Then for $n \geq 2k_0$, RHS $\rightarrow 0$ by (a).

If $|x| < 1$ then $x^n \rightarrow 0$.

Proof. $|x^n - 0| = |x|^n$, so $x^n \rightarrow 0 \Leftrightarrow |x|^n \rightarrow 0$ and $|x|^n = \frac{n_0}{(\frac{1}{|x|})^n} \rightarrow 0$ by (d) with $\alpha = 0$ and $1 + p = \frac{1}{|x|} > 1$, so
 $p = \frac{1}{|x|} - 1 > 0$. Q.E.D.

Q.E.D.

Theorem 3.2. (a) $p_n \rightarrow p \Leftrightarrow \forall_{r>0} : N_r(p)$ contains all but finitely many p_n .

Proof. $\forall_{n \geq N} : p_n \in N_r(p)$ Q.E.D.

(b) If $p_n \rightarrow p$ and $p_n \rightarrow p'$ then $p = p'$.

Proof. $d(p, p') \leq d(p_n, p) + d(p_n, p')$ for all n . Fix ε . Choose N such that $d(p_n, p) < \frac{\varepsilon}{2}$ and $d(p_n, p') < \frac{\varepsilon}{2}$ for $n \geq N$. Then $d(p, p') < \varepsilon$. Then for $n \geq \max\{N, N'\}$, $d(p, p') < \varepsilon$. This is true for all $\varepsilon > 0$, so $d(p, p') = 0$. Q.E.D.

(c) If $\{p_n\}$ converges, then p_n is bounded, in a sense that $\exists_{M>0, q \in X}$ s.t. $d(p_n, q) \leq M$ for all n .

Proof. If $p_n \rightarrow p$, then $\exists N$ s.t. $d(p_n, p) < 1$ for all $n \geq N$. Thus, $\forall_{n \geq 1} : d(p_n, p) \leq \max\{1, d(p, p_1), \dots, d(p, p_{N-1})\} = M$ Q.E.D.

(d) If $E \subset X$ has a limit point p , then $\exists_{p_n \in E}$ s.t. $p_n \rightarrow p$.

Proof. We need to choose $p_n \in E$ s.t. $d(p, p_n) < \frac{1}{n}$. Let $\varepsilon > 0$. Then $d(p, p_n) < \varepsilon$ if $n > \frac{1}{\varepsilon}$ Q.E.D.

Definition 3.2. Given $p_n, n_1 < n_2 < n_3 < \dots$, we say $p_{n_i} = (p_{n_1}, p_{n_2}, \dots)$ is a subsequence of p_n .

Lemma. $p_n \rightarrow p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p

Proof. Look at assignment 6 Q.E.D.

Theorem 3.6. (a) $\{p_n\}$ in X , X compact, then \exists convergent subsequence.

Proof. Let $E = \text{range of } \{p_n\}$. If E is finite, then $\exists p \in X$ and $n_1 < n_2 < \dots$ s.t. $p_n = p$ for $\forall i$. This subsequence converges to p . If E is infinite then by Theorem 2.37, E has a limit point $p \in X$; i.e., every neighborhood of p contains infinitely many points of E . Choose n_1 s.t. $d(p, p_{n_1}) < 1$.

Q.E.D.

(b) $\{p_n\}$ in \mathbb{R}^k , bounded, then \exists convergent subsequence.

Proof. Choose a k -cell I that contains $\{p_n\}$. I is compact. Apply (a).

Q.E.D.

Definition 3.3 (Cauchy Sequence). $\{p_n\}$ is a Cauchy sequence in (X, d) if $\forall \varepsilon : \exists_{N \in \mathbb{N}}$ s.t. $d(p_m, p_n) < \varepsilon \forall m, n \geq N$.

Definition 3.4. For $E \subset X$, $E \neq \emptyset$, we define $\text{diam } E = \sup \{d(p, q) : p, q \in E\}$. $\text{diam } E = \infty$ if the set is not bounded above.

Example. For a sequence p_n in X , let $E_n = \{p_n, p_{n+1}, \dots\}$. Then $\{p_n\}$ is a Cauchy sequence iff $\lim_{N \rightarrow \infty} \text{diam } E_N = 0$.

- Theorem 3.11.** (a) If $p_n \rightarrow p$ then $\{p_n\}$ is a Cauchy sequence.
 (b) If X is a compact metric space and $\{p_n\}$ in X is a Cauchy sequence, then $\exists p \in X$ s.t. $p_n \rightarrow p$.
 (c) In \mathbb{R}^K every Cauchy sequence converges.

Remark. If a Cauchy sequence has a convergent subsequence in a metric space, then the full sequence itself converges to the same point the subsequence converges to.

Proof. Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ is Cauchy. Let $E_N = \{p_N, p_{N+1}, \dots\}$. Then $\overline{E_N}$ is closed, hence compact. Also $\overline{E_N} \supset \overline{E_{N+1}}$ and $\lim_{N \rightarrow \infty} \text{diam } \overline{E_N} = 0$ (use Theorem 3.10(a) to see $\text{diam } \overline{E_N} = \text{diam } E_N$) By theorem 3.10(b), $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$. Claim: $p_n \rightarrow p$.

Proof of the claim: Let $\varepsilon > 0$. Choose N_0 s.t. $\text{diam } \overline{E_{N_0}} < \varepsilon$, so $d(p, q) < \varepsilon \forall q \in \overline{E_{N_0}}$, and hence $\forall q \in E_{N_0}$; i.e., $d(p, p_n) < \varepsilon$ if $n \geq N_0$.

Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_m, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ in \mathbb{R}^k is Cauchy. Cauchy sequences are bounded in any metric space. Therefore, $\exists k$ -cell I , which is compact, containing $\{p_n\}$. Then (b) applies Q.E.D.

Note. The converse of Theorem 3.11(a) does not hold in general.

Example. $X = \mathbb{Q}$ has a Cauchy sequence with no limit in \mathbb{Q} . (see assignment 6). Converse does hold if X is compact.

- Theorem 3.12.** (a) $\text{diam } \overline{E} = \text{diam } E$
 (b) If $K_n \subset X$, $K_n \neq \emptyset$, K compact, $K_n \supset K_{n+1} \forall n$ and if $\lim_{n \rightarrow \infty} \text{diam } K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a single point.

Proof. (a) $E \subset \overline{E} \Rightarrow \text{diam } E \leq \text{diam } \overline{E}$. For the opposite inequality, let $\varepsilon > 0, p, q \in \overline{E}$. Choose $p', q' \in E$ s.t. $d(p, p') < \varepsilon, d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < \varepsilon + \varepsilon = 2\varepsilon$. $\text{diam } \overline{E} \leq \text{diam } E + 2\varepsilon$. Since ε is arbitrary, $\text{diam } \overline{E} \leq \text{diam } E$.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, $K \neq \emptyset$. Since $K \subset K_n \forall n$, $\text{diam } K \leq \text{diam } K_n \forall n$, so $\text{diam } K = 0$. Therefore, $d(p, q) = 0 \forall p, q \in K$, so K is a simple point.

Q.E.D.

Definition 3.5 (Complete Metric Space). A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X .

Example. (a) X compact $\Rightarrow X$ complete.

(b) \mathbb{R}^k is complete, so is \mathbb{C} .

(c) \mathbb{Q} is not complete. (see assignment 6)

(d) See assignment 6 for reference to completion of a metric space.

We've seen that convergent sequences are bounded. $p_n = (-1)^n$ shows the converse is false. However the converse does hold for monotonic sequences.

Definition 3.6 (Monotone). • A sequence $\{s_n\}$ in \mathbb{R} is monotonically increasing if $s_n \leq s_{n+1} \forall n$.

• A sequence $\{s_n\}$ in \mathbb{R} is monotonically decreasing if $s_n \geq s_{n+1} \forall n$.

Theorem 3.14. A monotone sequence in \mathbb{R} converges if and only if it is bounded.

Proof. \Rightarrow all convergent sequences are bounded in any metric space.

\Leftarrow **Increasing case** Let $\{s_n\}$ be monotonically increasing and $s_n \leq M \forall n$. Let $s = \sup\{s_n : n \in \mathbb{N}\}$. Then $s_n \leq s \forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $s - \varepsilon < s_N \leq s$. But then $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \leq \dots \leq s$, so $|s - s_n| < \varepsilon \forall n \geq N$, and therefore $s_n \rightarrow s$.

Q.E.D.

Definition 3.7 (Infinite Limits). We say

- $s_n \rightarrow \infty$ if $\forall M \in \mathbb{R} : \exists N$ s.t. $s_n \geq M \forall n \in \mathbb{N}$.
- $s_n \rightarrow -\infty$ if $\forall M \in \mathbb{R} : \exists N$ s.t. $s_n \leq M \forall n \in \mathbb{N}$.

Definition 3.8. Let $\{s_n\}$ be a sequence in \mathbb{R} . We define $\limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{\sup_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \sup_{m \geq n} \{s_m\}$.
 $\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{\inf_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{s_m\}$.

Note. Alternate definition; see ass 7 for equivalence

Remark. (a) If $a_n \leq b_n \forall n$ and $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a \leq b$.

(b) $\liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n$

Example. (a) $s_n = (-1)^n(1 + \frac{1}{n^2})$ $1 \leq \sup_{m \geq n} s_m \leq 1 + \frac{1}{n^2}$, so $\limsup_{n \rightarrow \infty} s_n = 1$. Similarly, $\liminf_{n \rightarrow \infty} s_n = -1$

(b) If $\{s_n\}$ has no upper bound, then $\sup_{m \geq n} s_m = \infty$ and in this case we say $\limsup_{n \rightarrow \infty} s_n = \infty$; e.g.,

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has $\limsup_{n \rightarrow \infty} s_n = \infty$, $\liminf_{n \rightarrow \infty} s_n = -\infty$

Lemma. $\limsup_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = L \Leftrightarrow s_n \rightarrow L$.

Proof (L finite).

\Rightarrow This follows from $\inf_{m \geq n} s_m \leq s_n \leq \sup_{m \geq n} s_m$. $\lim_{n \rightarrow \infty} \inf_{m \geq n} s_m = \liminf_{n \rightarrow \infty} s_n$, and $\lim_{n \rightarrow \infty} \sup_{m \geq n} s_m = \limsup_{n \rightarrow \infty} s_n$. Therefore, $\lim_{n \rightarrow \infty} s_n = L$.

\Leftarrow If $s_n \rightarrow L$, then $\forall \varepsilon > 0 : \exists N$ s.t. $s_m \in [L - \varepsilon, L + \varepsilon] \forall m \geq N$. Therefore, $\forall n \geq N : L - \varepsilon \leq \inf_{m \geq n} s_m \leq \inf_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq L + \varepsilon$. Let $n \rightarrow \infty$: $L - \varepsilon \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L + \varepsilon$. Since ε is arbitrary, so $L \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L$.

Q.E.D.

3.2 Series

Definition 3.9 (Series). Let $\{a_n\}$ be a sequence in \mathbb{C} . Form a new sequence $\{s_n\}$, the sequence of partial sums, by $s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$. If $s_n \rightarrow s$, we say **the series** $\sum_{k=1}^{\infty} a_k$ **converges** and that $\sum_{k=1}^{\infty} a_k = s$. If $\{s_n\}$ diverges then we say $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 3.15. $\sum_{n \in \mathbb{N}} a_n$ converges if and only if $\forall \varepsilon > 0 : \exists N$ s.t. $\forall n \geq m \geq N : |\sum_{k=m}^n a_k| < \varepsilon$.

Proof. $\sum_n a_n$ converges $\Leftrightarrow \{s_n\}$ converges $\Leftrightarrow \{s_n\}$ is a Cauchy sequence ($\because \mathbb{C}$ is compact). Use $s_n - s_{m-1} = \sum_{k=m}^n a_k$. Q.E.D.

Corollary 3.16. If $\sum_n a_n$ converges then $a_n \rightarrow 0$.

Proof. Take $m = n$ in Theorem 3.22. $\sum_n a_n$ converges $\Rightarrow \forall \varepsilon > 0 : \exists N$ s.t. $|a_n| < \varepsilon$ if $n \geq N$. Q.E.D.

Remark. n -th term test for divergence: If $a_n \not\rightarrow 0$ then $\sum_n a_n$ diverges.

Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges because $\frac{n}{n+1} \rightarrow 1 \neq 0$.

Converse to Corollary 3.16 is false! E.g., $\sum_n \frac{1}{n}$ diverges but $\frac{1}{n} \rightarrow 0$.

Theorem 3.24. If $a_n \geq 0$, then $\sum_n a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof. $\{s_n\}$ is monotone increasing, so by Theorem 3.14, it converges if and only if it is bounded. Q.E.D.

Theorem 3.25. [Comparison Test]

(a) If $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges, then $\sum_n a_n$ converges.

Proof. Suppose $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges. Let $\varepsilon > 0$. By theorem 3.22, $\exists N$ s.t. $\sum_{k=m}^n c_k < \varepsilon$ if $n \geq m \geq N$. Can take $N \geq N_0$. Then $|N \geq N_0| \cdot |\sum_{k=m}^n a_k| \leq \sum_{k=m}^n |a_k| \leq \sum_{k=m}^n c_k < \varepsilon$ if $n \geq m \geq N$. By theorem 3.22 again, $\sum_n a_n$ converges. Q.E.D.

(b) If $a_n \geq d_n \geq 0 \forall n \geq N_0$ and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.

Proof. This follows from (a): if $\sum_n a_n$ converges then $\sum_n d_n$ converges. Thus it's contrapositive, (b) is true. Q.E.D.

Theorem 3.26. [Geometric Series] $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$, $xS_n = x + x^2 + \cdots + x^n + x^{n+1}$. Then

$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1$ ($\Leftrightarrow -1 < x < 1$), then $x^{n+1} \rightarrow 0$ and $S_n \rightarrow \frac{1}{1-x}$. If $|x| \geq 1$, then x^{n+1} does not converge to 0, so $\sum_{n=0}^{\infty} x^n$ diverges. Q.E.D.

Theorem 3.27. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof. (\Leftarrow) We show that if $\sum_n a_n$ diverges, then $\sum_k 2^k a_{2^k}$ diverges.

For this, note that $a_1 + a_2 + \cdots + a_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$ if $2^{k+1} > n$.

$a_1 + a_2 + \cdots + a_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. LHS unbounded as $n \rightarrow \infty$, so RHS is also unbounded as $k \rightarrow \infty$.

(\Rightarrow) $a_1 + a_2 + a_3 + \cdots + a_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$ if $2^k \leq n$. $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$.

If $\sum_n a_n$ converges, then LHS is bounded for all n so RHS is bounded for all k . Hence RHS converges since it is monotone.

Q.E.D.

Theorem 3.28. [p -series] $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. For $p \leq 0$, $\frac{1}{n^p} \not\rightarrow 0$, so series diverges. For $p > 0$, $\frac{1}{n^p}$ is decreasing, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$ converges. But $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$ converges iff $\frac{1}{2^{p-1}} < 1$ ($\Leftrightarrow p - 1 > 0$), which is equivalent to $p > 1$. Q.E.D.

Theorem 3.29. $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (log is to base e .)

Proof. If $p \leq 0$, then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$, so $\sum_n \frac{1}{n(\log n)^p}$ diverges by the comparison test. If $p > 0$ then $\frac{1}{n(\log n)^p}$ decreases since $\log n$ increases. By theorem 3.27, $\sum_n \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k(\log 2^k)^p}$ converges $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$ Q.E.D.

Definition 3.10 (e). $e := \sum_{n=0}^{\infty} \frac{1}{n!}$.

Remark. Convergence $\frac{1}{n!} = \frac{1}{(n(n-1)(n-2)\dots 3\cdot 2\cdot 1)} \leq \frac{1}{2\cdot 2\cdot 2\dots 2\cdot 1} = \frac{1}{2^{n-1}}$.
Therefore, $S_n = \sum_{k=0}^n \frac{1}{k!} \leq 1 + \sum_{k=1}^n \frac{1}{2^{k-1}} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1-1/2} = 3$. Then S_n is a monotonically increasing sequence that's also bounded. Hence, $e \leq 3$

Rate of Convergence

$$\begin{aligned} 0 < e - S_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n+1}} \cdot \frac{1}{(n+1)!} \\ &= \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-(n+1)}} \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+1)} = \frac{1}{(n!) \cdot n}. \end{aligned}$$

Theorem 3.32. $e \notin \mathbb{Q}$.

Proof. For contradiction, suppose $e = \frac{p}{q}, p, q \in \mathbb{N}$. As $0 < e - S_q < \frac{1}{q \cdot q!}$, $0 < q! \cdot e - q! \cdot S_q < \frac{1}{q}$. Since $S_q = \sum_{k=0}^q \frac{1}{k!}$, $q! \cdot e$ and $S_q \cdot q!$ are both integers. However, then $q! \cdot e - q! \cdot S_q$ is an integer between 0 and $\frac{1}{q} < 1$, which is a contradiction. Q.E.D.

Theorem 3.31. $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.

Proof. Let $t_n = (1 + \frac{1}{n})^n$. Then $t_n = \sum_{k=0}^n \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \cdot (\frac{n}{n} \cdot \frac{n-1}{n} \dots \frac{n-k+1}{n}) \leq S_n$. So $\limsup_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_n = e$. On the other hand, for fixed m and $n \geq m$, $t_n \geq \sum_{k=0}^m \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^m \frac{1}{k!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{k-1}{n})$. Let $n \rightarrow \infty$ with m fixed. $\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!} \cdot 1 = S_m$. This is true for any m . Now let $m \rightarrow \infty$. $\liminf_{n \rightarrow \infty} t_n \geq \limsup_{m \rightarrow \infty} S_m = e$. $e \leq \liminf_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} t_n \leq e$. Therefore, $\lim_{n \rightarrow \infty} t_n$ exists and equals e . Q.E.D.

Theorem 3.33. [Root test] Let $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then,

$$\sum a_n \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha > 1 \\ \text{inconclusive} & \text{if } \alpha = 1 \end{cases}$$

Proof (Just outline). $\alpha < \beta < 1$ Eventually $|a_n| \leq \beta^n$, thus convergence.

$\alpha > 1$ $|a_n| > 1$ for infinitely many n , thus divergence.

$\alpha = 1$ $\frac{1}{n}$ diverges, $\frac{1}{n^2}$ converges.

Q.E.D.

Theorem 3.34. [Ratio test] The series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ converges if $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and diverges if $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : \left| \frac{a_{n+1}}{a_n} \right| \geq 1$. Otherwise, inconclusive.

Proof. (see textbook).

$$\text{Convergence } \sum a_n \begin{cases} \text{converges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \\ \text{diverges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \\ \text{diverges} & \text{if } \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \\ \text{inconclusive} & \text{otherwise. e.g., } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \end{cases}$$

Note. Note that we cannot replace \liminf with \limsup in the third case. For inconclusive case, check $\sum 1/n \rightarrow \infty$ and $\sum 1/n^2 \rightarrow \pi^2/6$

Q.E.D.

Example. Let $s_n = \sum_{n=0}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \dots$. First, note that $a_{2k} = \frac{1}{2} \cdot \frac{1}{4^k}$, $a_{2k+1} = 2a_{2k} = \frac{1}{4^k}$ for $k \geq 0$.

Ratio test Then the ratio $\frac{a_{n+1}}{a_n}$ is the sequence $2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \dots$. Therefore, $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}$, $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$. The ratio test is inconclusive for s_n .

Root test $a_n = \begin{cases} \frac{1}{2} \cdot \frac{1}{2^n} & n \text{ even} \\ \frac{2}{2^n} & n \text{ odd} \end{cases}$, so $(\frac{1}{2})^{1/n} \cdot \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{2^{1/n}}{2}$, thus $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$. Therefore, $s_n = \sum_{n=0}^{\infty} a_n$ converges.

This is an example where the ratio test is inconclusive but the root test is conclusive.

Theorem 3.47. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum a_n + b_n = A + B$ and $\sum c \cdot a_n = cA$.

3.3 Power Series

Definition 3.11 (Power Series). For $z \in \mathbb{C}$ and a complex sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n z^n$ is a power series.

Remark. As $z^0 = 1$ for all $z \in \mathbb{C}$, by convention we write $\sum_{n=0}^{\infty} c_n z^n = c_0 + \sum_{n=1}^{\infty} c_n z^n$.

Theorem 3.39. Let $R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}}$, where

$$R = \begin{cases} 0 & \text{if } \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \infty \\ \infty & \text{if } \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = 0 \end{cases}.$$

Then $\sum c_n z^n$ $\begin{cases} \text{converges} & \text{if } |z| < R \\ \text{diverges} & \text{if } |z| > R. \\ \text{inconclusive} & \text{if } |z| = R \end{cases}$. Note $R = 0$ implies the series diverges for $z \neq 0$, and $R = \infty$ implies the series converges for any $z \in \mathbb{C}$.

Proof. $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$. By root test, the series converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$.

Note. In practice, often use the ratio test to find R .

Q.E.D.

Example. (a) $\sum n! \cdot z^n$ has $R = 0$.

By ratio test $\forall z \neq 0 : \left| \frac{(n+1)! \cdot z^{n+1}}{n! \cdot z^n} \right| = |z| \cdot (n+1) \rightarrow \infty$. Hence, the series diverges.

By root test Note $n \neq \frac{1}{2} \left(\frac{2}{3} \right)^2 \left(\frac{3}{4} \right)^3 \dots \left(\frac{n-1}{n} \right)^{n-1} n^n$ for $n \geq 2$. Then $n \neq \frac{n^n}{(1+1)^1 (1+\frac{1}{2})^2 (1+\frac{1}{n-1})^{n-1}}$. In the proof of Theorem 3.31, we saw $(1 + \frac{1}{j}) \leq e$. So $n! \geq \frac{n^n}{e^{n-1}} = e \cdot \left(\frac{n}{e} \right)^n$. $\sqrt[n]{n!} \geq e^{1/n} \cdot \frac{n}{e} \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, $R = \frac{1}{\infty} = 0$.

Note. Cf. Stirling's formula: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$.

Definition 3.12 (Absolute Convergence, Conditional Convergence).

- (a) $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
- (b) $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Remark. All other convergence tests seen so far are actually tests for absolute convergence.

Example. • $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges conditionally by the alternating series test of assignment 7's practice problem 1. See also Theorem 3.43. (MUST TAKE A LOOK!)

- Given $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $a_n \rightarrow 0$, then $\sum (-1)^n a_n$ converges.
- $\sum_{n=0}^{\infty} n! 2^n$ has $R = 0$
- $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$ has $R = \infty$ since $R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}} = \frac{1}{\limsup_{n \rightarrow \infty} \frac{1}{n}} = 1/0 = \infty$, or use ratio test, $|\frac{z^{n+1}/(n+1)^{n+1}}{z^n/n^n}| = |z| = \frac{n^n}{(n+1)^{n+1}} = |z| \frac{1}{n+1} \frac{1}{(1+\frac{1}{n})^n}$. Since $(1 + \frac{1}{n})^n \rightarrow \frac{1}{e}$ as $n \rightarrow \infty$, $|z| \frac{1}{n+1} \frac{1}{(1+\frac{1}{n})^n} \rightarrow 0$ as $n \rightarrow \infty \forall z \in \mathbb{C}$ so $R = \infty$.

Theorem 3.45. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Theorem 3.54. Suppose $\sum a_n$ converges conditionally. Let $-\infty \leq \alpha \leq \beta \leq +\infty$. Then \exists bijection $f: \mathbb{N} \rightarrow \mathbb{N}$ such that with $a'_n = a_{f(n)}$ and $S'_n = \sum_{k=1}^n a'_k$, $\liminf_{n \rightarrow \infty} s'_n = \alpha$ and $\limsup_{n \rightarrow \infty} s'_n = \beta$. In other words, there exists a rearrangement of $\sum a_n$, say $\sum a'_n$, such that $\liminf_{n \rightarrow \infty} \sum a'_n = \alpha$, $\limsup_{n \rightarrow \infty} \sum a'_n = \beta$.

Proof. Take a look at the textbook Q.E.D.

Theorem 3.55. If $\sum a_n$ converges absolutely, then every rearrangement of $\sum a_n$ converges to the same sum.

Proof. Take a look at the textbook Q.E.D.

3.4 Products of Series

Motivation Consider z^N in $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n)$. Since $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n) = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots$, z^N has coefficient $\sum_{k=0}^N a_k b_{N-k}$.

Definition 3.13. The product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^n a_k b_{n-k}$.

Note. This is a discrete convolution.

Question If $\sum a_n = A$ and $\sum b_n = B$ both converge, does $\sum c_n$ converge and if so, does it converge to AB ?

Answer $\sum c_n$ converges if $\sum a_n$ and $\sum b_n$ converge absolutely. (Theorem 3.50). Moreover, if $\sum c_n$ does converge, then it must converge to AB (Theorem 3.51). Maybe no otherwise (ref: Example 3.49).

Theorem 3.50. Suppose $\sum a_n$ converges absolutely to A and $\sum b_n$ converges to B . Then $\sum c_n$ converges to AB .

Proof. Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$. Then $A_n \rightarrow A, B_n \rightarrow B$. By definition, $C_n = \sum_{k=0}^n \sum_{j=0}^k a_j b_{k-j} = \sum_{j=0}^n \sum_{k=j}^n a_j b_{k-j} = \sum_{j=0}^n a_j \sum_{k=j}^n b_{k-j} = \sum_{j=0}^n a_j B_{n-j} = \sum_{j=0}^n a_j B + \sum_{j=0}^n a_j (B_{n-j} - B)$. Let β_{n-j} , where $\beta_k = B_k - B$. Then $C_n = A_n B + \sum_{j=0}^n a_j \beta_{n-j}$. Let $\gamma_n = \sum_{j=0}^n a_j \beta_{n-j}$. Note that $A_n B \rightarrow AB, \beta_k \rightarrow 0$ as $n \rightarrow \infty$. Let $\alpha = \sum_{k=0}^{\infty} |a_k| < \infty$ ($\because a_n$ converges absolutely by assumption). Rewrite γ_n as $\gamma_n = \sum_{j=0}^n a_{n-j} \beta_j$. We know $\beta_j \rightarrow 0$ as $j \rightarrow \infty$. Let $\varepsilon > 0$. Choose N s.t. $|\beta_j| < \varepsilon$ if $j \geq N$. Then for $n \geq N+1$, $|\gamma_n| \leq |\sum_{j=0}^N a_{n-j} \beta_j| + |\sum_{j=N+1}^n a_{n-j} \beta_j|$. Note $|\sum_{j=N+1}^n a_{n-j} \beta_j| \leq \varepsilon \sum_{j=N+1}^n |a_{n-j}| \leq \varepsilon \alpha$. Let $n \rightarrow \infty$ with N fixed. Then $a_{n-j} \rightarrow 0$ for $0 \leq j \leq N$ since $|a_n| \rightarrow 0$. Q.E.D.

Theorem 3.51. If the series $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C respectively and $c_n = \sum_{k=0}^n a_k b_{n-k}$, then $C = AB$.

Midterm 2 covers up to this point (TB p.36-82, A.5-8)

Chapter 4

Continuity

Assume general metric spaces X, Y and $f : X \rightarrow Y$.

Definition 4.1. Suppose X, Y are metric spaces, $E \subset X$, $f : E \rightarrow Y$, $p \in E'$, where E' : set of limit points in metric space X . We say $\lim_{x \rightarrow p} f(x) = q$, or $f(x) \rightarrow q$ as $x \rightarrow p$, if $\forall \varepsilon > 0 : \exists \delta > 0$ s.t. $(0 < d_X(x, p) < \delta \text{ and } x \in E) \Rightarrow d_Y(f(x), q) < \varepsilon$.

Note. We don't say anything about $x = p$, $f(p)$ may not even be defined.

Theorem 4.2. $\lim_{x \rightarrow p} f(x) = q \Leftrightarrow \forall \{p_n\} \text{ in } E : \text{ if } p_n = p \text{ or } p_n \rightarrow p, \text{ then}$

$$\lim_{n \rightarrow \infty} f(p_n) = q,$$

where the RHS is the limit of Definition 3.1.

Note. This implies uniqueness of q in Definition 4.1.

Proof. \Rightarrow Suppose $\lim_{x \rightarrow p} f(x) = q$. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $d_Y(f(x), q) < \varepsilon$ if $0 < d_X(x, p) < \delta$. Let $\{p_n\}$ be a sequence in E such that $p_n \rightarrow p$ and $p_n \neq p$. Then \exists_N s.t. $0 < d_X(p_n, p) < \delta$ if $n \geq N$; i.e., $f(p_n) \rightarrow q$.

\Leftarrow Consider the contrapositive of (\Leftarrow) : $\neg(\lim_{x \rightarrow p} f(x) = q) \Rightarrow \neg(\forall_{\{p_n\} \text{ in } E} : \lim_{n \rightarrow \infty} f(p_n) = q)$. Suppose $\neg(\lim_{x \rightarrow p} f(x) = q)$. Then $\exists_{\varepsilon > 0}$ s.t. $\forall_{\delta > 0} : \exists_{x \in N_\delta^E(p)}$ s.t. $x \neq p$ and $d_Y(f(x), q) \geq \varepsilon$. Take $\delta = \delta_n = \frac{1}{n}$ and let p_n be an x as above for δ_n . Then $p_n \rightarrow p$, but $d_Y(f(p_n), q) \geq \varepsilon \forall n$, so $f(p_n) \not\rightarrow q$.

Q.E.D.

Theorem 4.4. When $Y = \mathbb{C}$, limit as defined in Definition 4.1 respects sums, products and quotients.

Proof. By Theorem 4.2, it suffices to show that the theorem holds for sequences. Q.E.D.

Definition 4.2. Suppose X, Y are metric spaces, $p \in E \subset X$, $f : E \rightarrow Y$. Then f is continuous at p if $\forall_{\varepsilon > 0} : \exists_{\delta > 0}$ s.t. $d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$; i.e., $f(N_\delta^E(p)) \subset N_\varepsilon^Y(f(p))$. We say f is continuous if f is continuous at p for all $p \in E$.

Note. If p is an isolated point; i.e., $\exists_{\delta > 0}$ s.t. $N_\delta^E(p) = \{p\}$, then every $f : E \rightarrow Y$ is continuous at p .

Theorem 4.6. Suppose $E \subset X, p \in E \cap E', f : E \rightarrow Y$. Then f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.

Proof. By Definition 4.1 and Definition 4.2 with $q = f(p)$. Q.E.D.

Theorem 4.7. For $E \subset X, f : E \rightarrow Y, g : f(E) \rightarrow Z$, let $h = g \circ f : E \rightarrow Z$. If f is continuous at $p \in E$ and if g is continuous at $f(p) \in Y$, then h is continuous at p .

Proof. Choose $\eta > 0$ such that $d_Y(f(p), y) < \eta \Rightarrow d_Z(g(f(p)), g(y)) < \varepsilon$ (continuity of g at $f(p)$). Choose $\delta > 0$ s.t. $d_E(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$ (continuity of f at p). Then $d_E(x, p) < \delta \Rightarrow d_Z(g(f(x)), g(f(p))) = d_Z(h(x), h(p)) < \varepsilon$. Q.E.D.

Theorem 4.8. [Topological Characterization of Continuity] $f : X \rightarrow Y$ is continuous $\Leftrightarrow f^{-1}(V)$ is open for every open $V \subset Y$.

Proof. (\Rightarrow) Suppose f is continuous. Let $V \subset Y$ be open. Then $f^{-1}(V)$ is open. Let $p \in f^{-1}(V)$. We need to show $\exists_{\delta>0}$ s.t. $N_{\delta}^X(p) \subset f^{-1}(V)$. Since V is open, $\exists_{\varepsilon>0}$ s.t. $N_{\varepsilon}^Y(f(p)) \subset V$. Since f is continuous, $\exists_{\delta>0}$ s.t. $f(N_{\delta}^X(p)) \subset N_{\varepsilon}^Y(f(p)) \subset V$.

(\Leftarrow) Suppose $f^{-1}(V)$ is open for every open $V \subset Y$. Let $p \in X$ and $\varepsilon > 0$. Then $N_{\varepsilon}^Y(f(p))$ is open, so $f^{-1}(N_{\varepsilon}^Y(f(p)))$ is open. Take $V = N_{\varepsilon}^Y(f(p))$, which is open. Since $f^{-1}(V)$ is open and $p \in f^{-1}(V)$, there exists $\delta > 0$ such that $N_{\delta}^X(p) \subset f^{-1}(V)$. Then $f(N_{\delta}^X(p)) \subset V = N_{\varepsilon}^Y(f(p))$; i.e., f is continuous at p .

Q.E.D.

Remark.

- (a)
- (b) Continuity is determined by the open sets, not the metric. For instance, if metrics l_1, l_2, l_{∞} have the same open sets in \mathbb{R}^k , hence the same continuous functions.

$$l_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

$$l_2(x, y) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2}$$

$$l_{\infty}(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$$

- (c) f with open $U \subset X \Rightarrow f(U)$ is open are called open maps. Continuous maps need not be open (e.g., $f(x) = \text{some constant}$, $f(x) = x^2$), and open maps need not be continuous (e.g., floor function: $\lfloor \cdot \rfloor : \mathbb{R} \rightarrow \mathbb{Z}$).

Corollary 4.9. $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(F)$ is closed for every closed $F \subset Y$.

Proof. Let $V \subset Y$ be open and $F = V^c$. Then the above condition (RHS) is the same as $f^{-1}(V) = f^{-1}(F^c) = (f^{-1}(F))^c$ is open. Q.E.D.

Theorem 4.9. Let $f : X \rightarrow \mathbb{C}, g : X \rightarrow \mathbb{C}$ be continuous. Then $f + g, f \cdot g, f/g$ (at points p where $g(p) \neq 0$) are also continuous.

Theorem 4.10. Given $f_i : X \rightarrow \mathbb{R} (i = 1, 2, \dots, k)$, define $f : X \rightarrow \mathbb{R}^k$ by $f(x) = (f_1(x), \dots, f_k(x))$. Then

- (a) f is continuous if and only if each f_i is continuous.
- (b) if $f, g : X \rightarrow \mathbb{R}^k$ are continuous, then so are $f + g : X \rightarrow \mathbb{R}^k, f \cdot g : X \rightarrow \mathbb{R}^1$

Example. (a) For $i = 1, \dots, k$, define $\varphi_i : \mathbb{R}^k \rightarrow \mathbb{R}$ by $\varphi_i(x) = x_i$, where $x = (x_1, x_2, \dots, x_k)$. Then $|\varphi_i(x) - \varphi_i(y)| = |x_i - y_i| \leq \left(\sum_{j=1}^k |x_j - y_j|^2 \right)^{1/2} = |x - y|$, so φ_i is continuous (take $\delta = \varepsilon$. If $|x - y| < \delta = \varepsilon$, then $|\varphi_i(x) - \varphi_i(y)| < \varepsilon$).

(b) The functions $\mathbb{R}^k \rightarrow \mathbb{R}$ defined by $x \mapsto x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} (n_i \in \{0, 1, 2, \dots\})$ is continuous on \mathbb{R}^k and so is any polynomial $P(x) = \sum C_{n_1, n_2, n_3, \dots, n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$, where $C_{n_1, n_2, n_3, \dots, n_k}$ is a constant (function) in \mathbb{C} .

(c) Rational functions $f(x) = \frac{P(x)}{Q(x)}$ are continuous at points where $Q(x) \neq 0$.

(d) The function $\mathbb{R}^k \rightarrow \mathbb{R}$ defined by $x \mapsto |x|$ is continuous.

Proof. $|x| = |y + (x - y)| \leq |y| + |x - y|$, so $|x| - |y| \leq |x - y|$. Similarly, $|y| - |x| \leq |y - x|$, so $||x| - |y|| \leq |x - y|$. Thus by taking $\delta = \varepsilon$, $|x - y| < \delta \Rightarrow ||x| - |y|| < \varepsilon$. Q.E.D.

(e) Suppose $f : X \rightarrow \mathbb{R}^k$ is continuous. Then $p \mapsto |f(p)|$ is continuous.

Proof. $p \mapsto |f(p)| = (y \mapsto |y|) \circ (p \mapsto f(p))$. Since both $(y \mapsto |y|)$, $(p \mapsto f(p))$ are continuous, $p \mapsto |f(p)|$ is continuous by Theorem 4.7. Q.E.D.

Note. A function is said to be continuous on the *domain*, not on the *range*.

Theorem 4.14. Let $f : X \rightarrow Y$ be continuous and X be compact. Then $f(X)$ is compact.

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. We need to find a finite subcover of $f(X)$. By Theorem 4.8, each set $O_\alpha = f^{-1}(V_\alpha)$ is open and $\bigcup_\alpha O_\alpha = \bigcup_\alpha f^{-1}(V_\alpha) = f^{-1}(\bigcup_\alpha V_\alpha) = f^{-1}(f(X)) = X$. Hence, $\{O_\alpha\}$ is an open cover of X , so there exists a finite subcover $X = O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$. However, then $f(X) = \bigcup_{i=1}^n f(O_{\alpha_i}) = \bigcup_{i=1}^n f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^n V_{\alpha_i}$. Therefore, $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $f(X)$. Q.E.D.

Definition 4.3 (4.13). $f : E \rightarrow \mathbb{R}^k$ is bounded if $\exists_{M>0}$ s.t. $|f(x)| \leq M \forall x \in E$.

Theorem 4.15. If X is compact and $f : X \rightarrow \mathbb{R}^k$, then $f(X)$ is closed and bounded (so f is bounded).

Proof. $f(X)$ is compact by Theorem 4.14, and since $f(X) \subset \mathbb{R}^k$, it is closed and bounded. Q.E.D.

Theorem 4.16. If X is compact and $f : X \rightarrow \mathbb{R}^1$ is continuous, then $\exists_{p,q \in X}$ s.t. $f(p) \leq f(x) \leq f(q)$ for all $x \in X$.

Proof. By Theorem 4.15, $f(X)$ is closed and bounded. By Theorem 2.28, $M \in f(X)$ and similarly $m \in f(X)$. Q.E.D.

Example. Let $X = (0, 1)$, not compact, let $f(x) = \frac{1}{x}$, continuous. However, $\nexists_{p \in X}$ s.t. $\forall_{x \in X} : f(p) \leq f(x)$ and $\nexists_{q \in X}$ s.t. $\forall_{x \in X} : f(x) \leq f(q)$.

Theorem 4.17. Suppose $f : X \rightarrow Y$ is one-to-one, onto, continuous, where X is compact. Define $f^{-1} : Y \rightarrow X$ by $f^{-1}(f(x)) = x$. Then f^{-1} is continuous.

Proof. By Theorem 4.8, it suffices to prove that if $V \subset X$ is open then $(f^{-1})^{-1}(V) (= f(V))$ is open. However, $V^c \subset X$ is closed, hence V^c is compact by Theorem 4.14 and $(f(V^c))^c = f(V)$ is open. Q.E.D.

Example (Compactness is needed in Theorem 4.17). Let $X = [0, 2\pi)$, $Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Define $f : X \rightarrow Y$ by $f(\theta) = (\cos \theta, \sin \theta)$. This f is 1-1, onto, and continuous, but f^{-1} is not continuous as X is not compact.

Proof. (1) $[0, 1) \subset X$ is open but $(f^{-1})^{-1}([0, 1)) = f([0, 1))$ is not open because $(1, 0)$ is not an interior point of Y .

(2) In Y , as $(x, y) \rightarrow (1, 0)$ from above, $f((x, y)) \rightarrow 0$. As $(x, y) \rightarrow (1, 0)$ from below, $\lim f^{-1}(x, y)$ does not exist in X . (Wants to be $2\pi \notin X$), so f^{-1} is not continuous at $(1, 0) \in Y$.

Q.E.D.

Definition 4.18. Let X, Y be metric spaces and $f : X \rightarrow Y$. f is uniformly continuous on X if $\forall_{\varepsilon>0} : \exists_{\delta>0}$ s.t. for all $p, q \in X$ with $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$.

Remark. The point is for any ε , there is some δ that works for every $p, q \in X$ such that $d(p, q) < \delta$.

Example. (a) $X = (0, 1)$, $Y = \mathbb{R}$, $f(x) = \frac{1}{x}$. f is continuous on X but is not

uniformly continuous.

Proof. For $x \in (0, \frac{1}{2})$, $|f(x) - f(2x)| = |\frac{1}{x} - \frac{1}{2x}| = \frac{1}{2x} \rightarrow \infty$ as $x \rightarrow 0$. Then for $\varepsilon = 1$, given any $\delta \in (0, \frac{1}{2})$, we can pick $x < \delta$ s.t. $d_X(x, 2x) = x < \delta$, but $d_Y(f(x), f(2x)) = \frac{1}{2x} > \frac{1}{2\delta} > 1$. Q.E.D.

(b) $X = [0, 5], Y = \mathbb{R}, f(x) = x^2$ is uniformly continuous.

Proof. For $0 \leq x_1 \leq x_2 \leq 5$ and $\varepsilon > 0$, $|f(x_1) - f(x_2)| = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) \leq 10 \cdot (x_2 - x_1)$, which is less than ε if $|x_2 - x_1| < \frac{\varepsilon}{10} = \delta$ Q.E.D.

Theorem 4.19. Suppose X is a compact metric space, Y is a metric space, and $f : X \rightarrow Y$ is continuous. Then f is uniformly continuous.

Proof. Fix $\varepsilon > 0$. For $p \in X$ there exists $\delta = \delta_p(\varepsilon)$ s.t. $d_X(p, q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. We need to remove the p -dependence of δ_p . Let $J_p = N_{\frac{1}{2}\delta_p}(p)$. Then $\{J_p\}_{p \in X}$ is an open cover of X . Then there exists subcover $X = J_{p_1} \cup J_{p_2} \cdots \cup J_{p_n}$ (equality works as X is the whole metric space, so $X \subset J \Rightarrow X = J$). Let $\delta = \min\{\frac{1}{2}\delta_{p_1}, \frac{1}{2}\delta_{p_2}, \dots, \frac{1}{2}\delta_{p_n}\}$. Suppose p, q with $d_X(p, q) < \delta$. Choose $m \in \{1, 2, \dots, n\}$ s.t. $p \in J_{p_m}$. Then $d_X(p, p_m) < \frac{1}{2}\delta_{p_m}$. $d_X(q, p_m) \leq d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\delta_{p_m} \leq \delta_{p_m}$. $\therefore d_Y(f(q), f(p)) \leq d_Y(f(q), f(p_m)) + d_Y(f(p_m), f(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.

Theorem 4.22. If X, Y are metric spaces, $f : X \rightarrow Y$ is continuous, and $E \subset X$ is connected, then $f(E)$ is connected.

Proof (By Contradiction). Suppose for contradiction E is connected and there exists $A, B \subset Y$ s.t. $f(E) = A \cap B$, $f(E) \neq \emptyset$, $\overline{A} \cup B = A \cap \overline{B} = \emptyset$. Let $G = f^{-1}(A) \cap E, H = f^{-1}(B) \cap E$. Then $E = G \cup H$, G, H are nonempty. If $G \cap \overline{H} = \overline{G} \cap H = \emptyset$, it leads to a contradiction. First, $G \subset f^{-1}(A) \subset (\because A \subset \overline{A}) f^{-1}(\overline{A})$, where $f^{-1}(\overline{A})$ is closed by the corollary to Theorem 4.8, so $\overline{G} \subset f^{-1}(\overline{A})$. Second, $f(H) = B, \overline{A} \cap B = \emptyset$. Therefore, $\overline{G} \cap H = \emptyset$. WLOG, $G \cap \overline{H} = \emptyset$ as well. Hence a contradiction. Q.E.D.

Theorem 4.23. [Intermediate Value Theorem] Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. $\forall c \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\}) : \exists x_0 \in (a, b)$ s.t. $f(x_0) = c$.

Proof. $[a, b]$ is connected by Theorem 2.47. Hence, by Theorem 4.22, $f([a, b])$ is connected and therefore contains all points between $f(a)$ and $f(b)$. In particular, $c \in f((a, b))$ Q.E.D.

Example. (a) there exists a continuous function called (Peano/space-filling

curve) from $[0, 1]$ onto the closed unit square $S = [0, 1] \times [0, 1] \subset \mathbb{R}^2$.

Proof. Omitted. See Rudin's problem 7.14 for an explicit example (covered in MATH-321). Q.E.D.

(b) But no such function can be one-to-one.

Proof. Suppose $f : [0, 1] \rightarrow S$ is 1-1, onto, continuous. Since $[0, 1]$ is compact, f^{-1} is continuous by Theorem 4.17. Let $E = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then, $f(E) = S \setminus \{f(\frac{1}{2})\}$ is S minus one point, which is connected (pf omitted). But then, $f^{-1}(f(E)) = E$ must be connected by Theorem 4.22. E is not connected, so this is a contradiction. Q.E.D.

Theorem 4.29. Let f be a monotonically increasing function on (a, b) . Then $f(x+)$ and $f(x-)$ exist at every $x \in (a, b)$; i.e.,

$$\sup_{a < t < x} f(t) = f(x^-) \leq f(x) \leq f(x^+) = \inf_{x < t < b} f(t).$$

Moreover, if $a < x < y < b$, then

$$f(x^+) \leq f(y^-).$$

Analogous results hold for monotonically decreasing functions.

Example (18). Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on \mathbb{R}^1 by

$$f(x) = \begin{cases} 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \\ \frac{1}{n} & (x = \frac{m}{n}) \end{cases}.$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point, and that f has a simple discontinuity at every rational point.

Chapter 5

Differentiation

We consider $f : [a, b] \rightarrow \mathbb{R}$.

Definition. For $f : [a, b] \rightarrow \mathbb{R}$ and $x \in [a, b]$, let $f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$ if limit exists. Equivalently, $f(t) = f(x) + (t - x)[f'(x) + u(x, t)]$ with $\lim_{t \rightarrow x} u(x, t) = 0$.

Example. (a) $f(x) = c$ for all $x \Rightarrow f'(x) = \lim_{t \rightarrow x} \frac{c - c}{t - x} = 0$.

(b) $f(x) = x$ for all $x \Rightarrow f'(x) = \lim_{t \rightarrow x} \frac{t - x}{t - x}$

(c) $f(x) = e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Write $t = x + h$, so $t \rightarrow x \Leftrightarrow h \rightarrow 0$. $\frac{e^{x+h} - e^x}{(x+h) - x} = e^x \frac{e^h - 1}{h} = e^x \frac{e^h - 1}{h} + e^x - e^x = e^x + e^x \frac{e^h - 1 - h}{h}$. Let $u(h) = \frac{e^h - 1 - h}{h}$. Then $u(h) = \frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}$, so $|u(h)| = |\frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}| \leq |h| \sum_{n=2}^{\infty} \frac{1}{n!} = (e - 2)|h|$ (note for $n \geq 2, |h^{n-1}| \leq |h|$ if $|h| \leq 1$). Hence, $u(h) \rightarrow 0$ as $h \rightarrow 0$ and therefore $f'(x) = e^x$.

Remark. $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is well defined. $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = e$. Regarding it as a power series, its radius of convergence is $R = \infty$. Also, $e^{x+y} = e^x e^y$ using definition 3.48 of product series (Rudin's p.178-180).

Note. $f'(x)$: Lagrange's notation, $\frac{df}{dx}$: Leibnitz notation

Theorem 5.2. Suppose $f : [a, b] \rightarrow \mathbb{R}$ and $f'(x)$ exists. Then f is continuous at x .

Proof. The existence of $f'(x) \Leftrightarrow f(t) = f(x) + (t-x)[f'(x) + u(x, t)]$ with $\lim_{t \rightarrow x} u(x, t) = 0$. Let $t \rightarrow x$. $\lim_{t \rightarrow x} f(x) + (t-x)[f'(x) + u(x, t)] = f(x) + 0[f'(x) + 0] = f(x)$, so $\lim_{t \rightarrow x} f(t) = f(x)$; i.e., f is continuous at x . Q.E.D.

Remark. The converse is false; e.g., $f(x) = |x|$ is continuous for all x , but $f'(0)$ does not exist.

Theorem 5.3. If $f : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \rightarrow \mathbb{R}$ are both differentiable at x then so are $f + g, fg, \frac{f}{g}$ (if $g(x) \neq 0$), and $(f + g)'(x) = f'(x) + g'(x)$, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$, $(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$.

Proof (Only the quotient rule). $h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{1}{g(t)g(x)} [(f(t)g(x) - f(x)g(t)) - (f(x)g(t) - f(x)g(x))]$. Then $\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)} \left[\frac{f(t) - f(x)}{t - x} g(x) - f(x) \frac{g(t) - g(x)}{t - x} \right]$. Let $t \rightarrow x$. $h'(x) = \frac{1}{g(x)^2} [f'(x)g(x) - f(x)g'(x)]$. Q.E.D.

Remark. By induction, $(f_1 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdots + f_1 f_2 \cdots f_n'$.

Example. For $n = 2, 3$, $\frac{d}{dx} x^n = nx^{n-1}$ and we know this already for $n = 0, 1$. For $n = -1, -2$, let $m = -n > 0$. Then $\frac{d}{dx} x^n = \frac{d}{dx} \frac{1}{x^m} = \frac{(\frac{d}{dx} 1)x^m - (\frac{d}{dx} x^m)1}{(x^m)^2} = \frac{0x^m - mx^{m-1}1}{x^{2m}} = -mx^{m-1} = nx^{n-1}$. Hence, $\forall n \in \mathbb{Z} : \frac{d}{dx} x^n = nx^{n-1}$.

Theorem 5.5. [Chain Rule] Suppose $f : [a, b] \rightarrow \mathbb{R}$, $f'(x)$ exists for some $x \in [a, b]$, $f([a, b]) \subset I$, where I is some interval in \mathbb{R} . Suppose $g : I \rightarrow \mathbb{R}$ and $g'(f(x))$ exists. Then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

Proof. Let $h(t) = (g \circ f)(t) = g(f(t))$ for $t \in [a, b]$. Fix $x \in [a, b]$ where $f'(x)$ exists. We know:

(a) $f(t) - f(x) = (t - x)[f'(x) + u(t)]$ with $\lim_{t \rightarrow x} u(t) = 0$.

(b) With $y = f(x)$, $g(s) - g(y) = (s - y)(g'(y) + v(s))$ with $\lim_{s \rightarrow y} v(s) = 0$

As $\frac{h(t) - h(x)}{t - x} = \frac{g(f(t)) - g(f(x))}{t - x}$. By (2), $\frac{g(f(t)) - g(f(x))}{t - x} = [g'(f(x)) + v(f(t))]$. Let $t \rightarrow x$. Then RHS $\rightarrow f'(x)[g'(f(x)) + 0]$ since $f(t) \rightarrow f(x)$ by continuity of f at x . Therefore, $h'(x) = f'(x)g'(f(x))$.
Q.E.D.

Note. Suppose you produce $f(t)$ meters of wire by time t ; i.e., rate of wire production is $f'(t)$ m/x. Also suppose you get \$ $g(l)$ for l meters of wire; rate of profit is $g'(l)$ \$/m. Then the rate of earning by time t is $g'(f(t))f'(t)$ \$/m.

Example. (a) $\frac{d}{dx} e^{x^2} = 2xe^{x^2}$

(b) $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Remark. (a) f is continuous on \mathbb{R} , including at $x = 0$.

Proof. $|f(x)| \leq |x|$, so by the Squeeze theorem, $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$.
Q.E.D.

(b) f is differentiable on $x \neq 0$, but not differentiable at $x = 0$. For $x \neq 0$, $f'(x) = \sin \frac{1}{x} + x(\cos \frac{1}{x})(\frac{-1}{x^2}) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$. For

$x = 0$, $\frac{f(t) - f(0)}{t - 0} = \frac{t \sin \frac{1}{t}}{t} = \sin \frac{1}{t}$, which does not converge. Therefore, f not differentiable at $x = 0$.

(c) Let $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) f is continuous in \mathbb{R} including at $x = 0$ ($\because |f(x)| \leq |x^2|$).

(b) f is differentiable in \mathbb{R} including at $x = 0$.

Proof. For $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} + x^2(\cos \frac{1}{x})(\frac{-1}{x^2}) = 2x \sin \frac{1}{x} -$

$\cos \frac{1}{x}$. For $x = 0$, $\frac{f(t) - f(0)}{t - 0} = \frac{t^2 \sin \frac{1}{t}}{t} = t \sin \frac{1}{t} \rightarrow 0$ as $t \rightarrow 0$.
 Hence, $f'(0) = 0$. HOWEVER! f' is not continuous at $x = 0$,
 because $\lim_{x \rightarrow 0} f'(x)$ does not exist. Q.E.D.

Definition 5.1. Let X be a metric space, $f : X \rightarrow \mathbb{R}$. f has a *local max* at $x \in X$ if $\exists_{\delta > 0}$ s.t. $f(y) \leq f(x)$ for all $y \in N_\delta(x)$.

Theorem 5.8. Let $f : [a, b] \rightarrow \mathbb{R}$. If f has a local min or a local max at $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

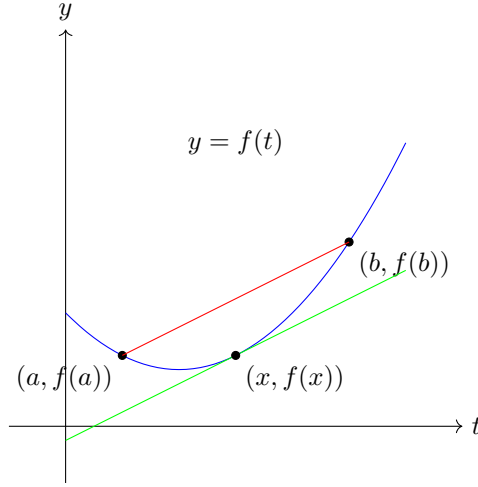
Proof (local min). Suppose f has a local min at x and $f'(x)$ exists. Choose $\delta > 0$ s.t. $N_\delta(x) \subset (a, b)$ and $f(t) \geq f(x)$ if $t \in (x - \delta, x + \delta)$.
 For $x < t < x + \delta$, $\frac{f(t) - f(x)}{t - x} \geq 0$ ($\because f(t) \geq f(x), t > x$), so $f'(x) \geq 0$.
 For $x - \delta < t < x$, $\frac{f(t) - f(x)}{t - x} \leq 0$ ($\because f(t) \geq f(x), t < x$), so $f'(x) \leq 0$. Hence, $f'(x) = 0$. Q.E.D.

Remark. Note that the hypothesis of the theorem requires *open* interval and existence $f'(x)$. If these conditions are not met, then $f'(x) = 0$ doesn't have to be the case.

Example. (a) $f(x) = |x|$ has a local min at $x = 0$ but $f'(0)$ does not exist.

(b) $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x$ has a local max at $x = 1$ and local min at $x = 0$, but $f'(0) = f'(1) = 1$.

Theorem 5.10. [Mean-Value Theorem] If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and differentiable on (a, b) , then $\exists_{x \in (a, b)}$ s.t. $f(b) - f(a) = f'(x)(b - a)$.



Proof.

Let $L : y = f(a) + m(t - a)$, where $m = \frac{y - f(a)}{t - a} = \frac{f(b) - f(a)}{b - a}$. Subtract L from the curve $y = f(t)$. Let $h(t) = f(t) - [f(a) + m(t - a)]$. Then $h(a) = h(b) = 0$. $h'(t) = f'(t) - m = f'(t) - \frac{f(b) - f(a)}{b - a}$. Therefore, it suffices to find x s.t. $h'(x) = 0$. h is continuous and $[a, b]$ is compact, so $h([a, b])$ is also compact. Hence, h attains its global max(= $\sup\{h([a, b])\}$) and global min(= $\inf\{h([a, b])\}$) on $[a, b]$. If $h(t) = 0$ for all $t \in [a, b]$ then $h'(t) = 0$ for all $t \in [a, b]$ so any $x \in (a, b)$ will do. Otherwise, h attains its global max or global min at some $x \in (a, b)$. By Theorem 5.8, $h'(x) = 0$. Q.E.D.

Theorem 5.11. If f is differentiable on (a, b) then

- (a) $f'(x) \geq 0$ for all $x \in (a, b)$ implies f is monotone increasing.
- (b) $f'(x) \leq 0$ for all $x \in (a, b)$ implies f is monotone decreasing.
- (c) $f'(x) = 0$ for all $x \in (a, b)$ implies f is constant.

Proof ((a) only). Suppose $f'(x) \geq 0$ for all $x \in (a, b)$. For $a < x_1 < x_2 < b$, $\exists_{x \in (x_1, x_2)}$ s.t. $f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$ by Theorem 5.10. As $f'(x) \geq 0$, $x_2 \geq x_1$, $f(x_2) - f(x_1) \geq 0$. Q.E.D.

Definition 5.2. $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$, $f^{(3)} = f'''$ and so on.

Theorem 5.15. [Taylor's Theorem] Suppose $f : [a, b] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, and

$f^{(n+1)}(x)$ exists for all $x \in (a, b)$. Let $x, x_0 \in [a, b]$. Then $\exists_{c \in (x, x_0)}$ s.t.

$$f(x) = \underbrace{\sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k}_{P_n(x), n^{\text{th}} \text{ Taylor polynomial of } f \text{ at } x_0} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}}_{\text{Taylor Remainder}}.$$

Proof. If $n = 0$, the mean-value theorem guarantees existence of c . For general n , $A \in \mathbb{R}$ by $R_n(x) = f(x) - P_n(x) = \frac{A}{(n+1)!} (x - x_0)^{n+1}$, where A depends on f, n, x, x_0 . Claim: $A = f^{(n+1)}(c)$ for some c between x and x_0 .

Define $g(t) = f(t) - P_n(t) - \frac{A}{(n+1)!} (t - x_0)^{n+1}$ for $t \in [a, b]$. Then $g(x_0) = 0$. $g(x) = f(x) - P_n(x) - \frac{A}{(n+1)!} (x - x_0)^{n+1} = 0$ by the definition of A . Also for $j = 1, \dots, n$, then $P_n^{(j)}(x_0) = f^{(j)}(x_0)$, $\frac{d^j}{dx^j} (x - x_0)^{n+1} \big|_{x=x_0} = 0$. Hence, $g^{(j)}(x_0) = f^{(j)}(x_0) - P_n^{(j)}(x_0) - 0 = 0$. $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - A$. We need to find c s.t. $g^{(n+1)}(c) = 0$.
 $g(x) = g(x_0) = 0 \Rightarrow \exists_{c_1 \in (\min\{x_0, x\}, \max\{x_0, x\})}$ s.t. $g'(c_1) = 0$.
 $g'(x_0) = g'(c_1) = 0 \Rightarrow \exists_{c_2 \in (\min\{x_0, x, c_1\}, \max\{x_0, x, c_1\})}$ s.t. $g''(c_2) = 0$.
 \vdots
Finally, $\exists_{c_{n+1}=c}$ s.t. $g^{(n+1)}(c) = 0$ and hence $f^{(n+1)}(c) = A$. Q.E.D.

Example. (not in Rudin) Does $\sum_{n=1}^{\infty} \left(\sqrt{1 + \frac{1}{n^2}} - 1 \right)$ converge or diverge?

Method 1:

$$\sqrt{1 + \frac{1}{n^2}} - 1 = \frac{\left(\sqrt{1 + \frac{1}{n^2}} - 1 \right) \left(\sqrt{1 + \frac{1}{n^2}} + 1 \right)}{\sqrt{1 + \frac{1}{n^2}} + 1} = \frac{1 + \frac{1}{n^2} - 1}{\sqrt{1 + \frac{1}{n^2}} + 1} = \frac{1}{n^2} \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1} \leq \frac{1}{n^2},$$

so the series converges by the comparison test since $\sum \frac{1}{n^2}$ converges.

Method 2: Using Taylor's theorem. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. Then

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{-1/2} \\ f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2} \\ f(x) &= P_1(x) + R_1(x) \\ &= f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(c)(x-0)^2}{2!} \\ &= 1 + \frac{1}{2}x + R_1(x). \end{aligned}$$

$|R_1(x)| \leq \left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1\right) \frac{1}{2!} x^2 = \frac{1}{8} x^2$ for $x \in [0, 1]$. Therefore, $\sqrt{1 + \frac{1}{n^2}} - 1 = f\left(\frac{1}{n^2}\right) - 1 = \frac{1}{2}\left(\frac{1}{n^2}\right) + R_1\left(\frac{1}{n^2}\right) \leq \frac{1}{2n^2} + \frac{1}{8} \frac{1}{n^4}$. Since $\sum \left(\frac{1}{2n^2} + \frac{1}{8n^4}\right)$ converges, $\sum \left(\sqrt{1 + \frac{1}{n^2}} - 1\right)$ converges by comparison test.

Example. Let $f(x) = \sin x$, $x_0 = 0$.

Taylor series for $f(x)$. $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \dots$,

$$\text{so } f^{(k)}(x) = \begin{cases} (-1)^m \sin x & (k = 2m) \\ (-1)^m \cos x & (k = 2m+1) \end{cases}. \text{ Hence } n \geq 0, f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k +$$

$$\frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}, \text{ where } c \text{ between } 0 \text{ and } x. \text{ Remainder estimate: } \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq$$

$$\frac{|x|^{n+1}}{(n+1)!} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ because } \frac{|x|^{n+1}}{(n+1)!} \text{ is the } (n+1)^{\text{th}} \text{ term in conver-}$$

$$\text{gent series } e^{|x|} = \sum_{n=0}^{\infty} \frac{|x|^n}{n!}.$$

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

Taylor approximation. Find $\sin 0.2$ to within an error $\pm 10^{-6}$. Use $\sin 0.2 =$

$$\frac{2}{10} - \frac{1}{3!} \left(\frac{2}{10}\right)^3 + \frac{1}{5!} \left(\frac{2}{10}\right)^5 - \dots.$$

Method 1: Alternating Series Test. If $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$, and $a_n \rightarrow 0$, then $\sum_{k=1}^{\infty} (-1)^{k-1} a_k = s$ converges and $|s - s_n| \leq a_{n+1}$. Above series satisfies the hypotheses, so truncation error is \leq first omitted term. We look for when $\frac{1}{(2k+1)!} \left(\frac{2}{10}\right)^{2k+1} \leq 10^{-6}$; i.e.,

$$(2k+1)! \cdot \frac{10^{2k+1}}{2^{2k+1}} \geq 10^6.$$

$$\text{If } k = 1, 3! \cdot \frac{10^3}{2^3} < 10^6.$$

$$\text{If } k = 2, 5! \cdot \frac{10^5}{2^5} < 10^6.$$

If $k = 3$, $7! \cdot \frac{10^7}{2^7} \geq 10^6$, so $k = 3$ works. Therefore, $\sin 0.2 = 0.2 - \frac{1}{3!}(0.2)^3 + \frac{1}{5!}(0.2)^5 \pm 10^{-6} = 0.198669 \pm 10^{-6}$.

Method 2: General Case. If alternating series test does not apply, estimate remainder using the worst c for $\left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right|$. In our example, $\left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right| \leq \frac{1}{(n+1)!} (0.2)^{n+1}$, so seek n s.t. $\frac{1}{(n+1)!} \left(\frac{2}{10} \right)^{n+1} \leq 10^{-6}$. First n that works is $n = 6$, same as before.

Chapter 6

Riemann-Stieltjes Integral

Definition (Partition). A partition P of $[a, b]$ is $\{x_0, x_1, x_2, \dots, x_n\}$ for some $n \geq 1$, with $a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b$.

Notation. $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \dots, n$

$f : [a, b] \rightarrow \mathbb{R}$ be bounded, which is not necessarily continuous

$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$

$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$

$L(P, f) = \sum_{i=1}^n m_i \Delta x_i$.

Note. $L(P, f) \leq U(P, f)$ always.

Definition (Riemann Integral). **Upper Riemann Integral :** $\overline{\int_a^b} f(x) dx = \inf_P \{U(P, f)\} = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$.

Lower Riemann Integral : $\underline{\int_a^b} f(x) dx = \sup_P \{L(P, f)\} = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$.

Riemann Integrable : f is Riemann integrable on $[a, b]$ if $\overline{\int_a^b} f(x) dx = \underline{\int_a^b} f(x) dx$. If f is Riemann integrable on $[a, b]$, we write $f \in \mathcal{R}[a, b]$ and

$$\int_a^b f(x) dx = \overline{\int_a^b} f(x) dx = \underline{\int_a^b} f(x) dx.$$

Note. Since f is bounded, $m = \inf\{f(x) : a \leq x \leq b\}$ and $M = \sup\{f(x) : a \leq x \leq b\}$ are both finite. Hence, for any P , $m \leq m_i \leq M_i \leq M$ and $\forall_i : m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$.

Notation. Let $\alpha : [a, b] \rightarrow \mathbb{R}$ is a monotone increasing function. Then $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

Definition 6.2. Given P , let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. (Note: $\Delta\alpha_i \geq 0$). For bounded f , let $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$, $L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$.

Upper Riemann-Stieltjes Integral $\int_a^b f(x) d\alpha = \overline{\int_a^b f(x) d\alpha} = \inf_P \{U(P, f, \alpha)\} = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\}.$

Lower Riemann-Stieltjes Integral $\int_a^b f(x) d\alpha = \underline{\int_a^b f(x) d\alpha} = \sup_P \{L(P, f, \alpha)\} = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\}.$

If $\int_a^b f(x) d\alpha = \underline{\int_a^b f(x) d\alpha}$, then $f \in R[a, b, \alpha]$ and $\int_a^b f(x) d\alpha = \overline{\int_a^b f(x) d\alpha} = \underline{\int_a^b f(x) d\alpha}$.

If $\alpha(x) = x$, then equivalent to $\int_a^b f(x) dx$.

Definition 6.3. (a) Partition P^* is called a refinement of P if $P \subset P^*$.

(b) Partition P^* is called the common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$.

Theorem 6.4. If P^* is a refinement of P then $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

Proof. It's enough to consider p^* with one extra point: $x_{i-1} \leq x^* \leq x_i$.

Sketch for L :

$$\begin{aligned} & L(P^*, f, \alpha) - L(P, f, \alpha) \\ &= m^*[\alpha(x^*) - \alpha(x_{i-1})] + m_i[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})] \\ &= (m^* - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (m_i - m_i)[\alpha(x_i) - \alpha(x^*)] \end{aligned}$$

Q.E.D.

Notation. When f, α are fixed, we write $L(P) = L(P, f, \alpha)$, $U(P) = U(P, f, \alpha)$

Theorem 6.5. $\int_a^b f d\alpha \leq \overline{\int_a^b f d\alpha}$.

Proof. For partitions P_1, P_2 , let $P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \leq L(P^*) \leq U(P^*) \leq U(P_2)$. In particular, $\sup_{P_1} \{L(P_1)\} \leq U(P_2)$ for all P_2 . Hence, $\sup_{P_1} \{L(P_1)\} \leq \inf_{P_2} \{U(P_2)\}$. Q.E.D.

Theorem 6.6. $f \in \mathcal{R}_\alpha[a, b] \Leftrightarrow \forall \varepsilon > 0 : \exists P_\varepsilon$ s.t. $U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$

Proof. Let $\varepsilon > 0$.

(\Rightarrow) By hypothesis, $\sup_P \{L(P)\} = \int_a^b f d\alpha = \overline{\int_a^b f d\alpha} = \inf_P \{U(P)\}$.

$\exists P_1, P_2$ s.t. $L(P_1) > \int_a^b f d\alpha - \varepsilon/2$ and $U(P_2) < \overline{\int_a^b f d\alpha} + \varepsilon/2$.

Then $U(P_2) - L(P_1) < \varepsilon$. Let $P_\varepsilon = P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \leq L(P^*) \leq U(P^*) \leq U(P_2)$, so $U(P_\varepsilon) - L(P_\varepsilon) \leq U(P_2) - L(P_1) < \varepsilon$.

(\Leftarrow) $0 \leq \overline{\int_a^b f d\alpha} - \int_a^b f d\alpha \leq U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$. Since ε is arbitrary, $\overline{\int_a^b f d\alpha} = \int_a^b f d\alpha$.

Q.E.D.

Remark. very important

Theorem 6.7. Let $\varepsilon_0 > 0$ be fixed. Suppose there exists a partition $P = \{x_0 = a, \dots, x_n = b\}$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon_0$. Let s_i, t_i be arbitrary points in $[x_{i-1}, x_i]$. Then,

- (a) For any refinement of P , denoted by P^* , $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \varepsilon_0$ also holds true
- (b) $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon_0$
- (c) If $f \in \mathcal{R}_\alpha$, then $\left| \sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon_0$

Theorem 6.8. If f is continuous on $[a, b]$ then $f \in \mathcal{R}_\alpha[a, b]$.

Proof. For any P , $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i) \Delta\alpha_i$. Since $[a, b]$ is compact, f is uniformly continuous on $[a, b]$ (Theorem 4.19), so $\forall \eta > 0 : \exists \delta > 0$ s.t. $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \eta$. Given $\varepsilon > 0$, choose η s.t. $\eta[\alpha(b) - \alpha(a)] < \varepsilon$ and choose P with $\Delta x_i < \delta = \delta(\eta)$ for all i . For such P , $M_i - m_i \leq \eta$. Then $U(P) - L(P) \leq \sum_{i=1}^n \eta \Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon$. Therefore, $f \in \mathcal{R}_\alpha[a, b]$. Q.E.D.

Theorem 6.9. If f is monotone increasing or decreasing on $[a, b]$ and α is continuous on $[a, b]$ then $f \in \mathcal{R}_\alpha[a, b]$.

Proof. By definition, $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i) \Delta\alpha_i$. Given $n \in \mathbb{N}$, let P s.t. $\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$ for all i . Such P exists by the intermediate value theorem (Theorem 4.23) as α is continuous. Then, $U(P) - L(P) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n M_i - m_i$. Suppose f is increasing, so $M_i - m_i = f(x_i) - f(x_{i-1})$. Then $U(P) - L(P) = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n f(x_i) - f(x_{i-1}) = \frac{\alpha(b) - \alpha(a)}{n} [f(b) - f(a)]$. Given $\varepsilon > 0$, we can choose n (hence P) s.t. $U(P) - L(P) < \varepsilon$. Therefore, $f \in \mathcal{R}_\alpha[a, b]$ by Theorem 6.6. Q.E.D.

Note. We always assume α is monotone.

Theorem 6.10. If f is bounded on $[a, b]$ and has only finitely many discontinuities, and α is continuous at each point where f is not, then $f \in \mathcal{R}_\alpha[a, b]$.

Proof. We apply Theorem 6.6. Use $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i) \Delta\alpha_i$. Let $\varepsilon > 0$ and $E = \{e_1, \dots, e_k\}$ be the set of points where f is discontinuous. α is assumed to be continuous at each e_i , which implies $\exists(u_j, v_j)$ s.t. $u_j < e_j < v_j$ and $\alpha(v_j) - \alpha(u_j) < \varepsilon$. (Relax inequality to include equality if $e_1 = a, e_k = b$).

Let $K = [a, b] \cap \left(\bigcup_{j=1}^k (u_j, v_j) \right)^c$. K is compact. f is continuous on K , so f is uniformly continuous on K by Theorem 4.19. Hence, $\exists \delta > 0$ s.t. for $s, t \in K$, $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$.

Form P to consist of $\{u_1, v_1, \dots, u_k, v_k\}$ and additional points in K so that $\Delta x_i < \delta$. If x_i is in K , then $M_i - m_i < \varepsilon$. Otherwise, $x_i = u_j$ or $x_i = v_j$ for some j , so $\Delta\alpha_i \leq \varepsilon$. Then

$$\begin{aligned} 0 \leq U(P) - L(P) &= \sum_{i=1}^n (M_i - m_i) \Delta\alpha_i \\ &\leq \underbrace{k \cdot 2M\varepsilon}_{\text{from intervals in } \bigcup_{j=1}^k (u_j, v_j)} + \underbrace{\varepsilon[\alpha(b) - \alpha(a)]}_{\text{from intervals in } K}. \end{aligned}$$

As RHS is as small as we want by taking ε small enough, $f \in \mathcal{R}_\alpha[a, b]$. Q.E.D.

Remark. (a) Theorem 6.10 implies part of A1.2 but do the problem from first principles. Do not apply Theorem 6.10 directly.

(b) A1.4 shows what can happen if f, α are discontinuous at the same point.

Theorem 6.11. If $f \in \mathcal{R}_\alpha[a, b]$, $m \leq f(x) \leq M$ for all $x \in [a, b]$, and $\varphi : [m, M] \rightarrow \mathbb{R}$ is continuous, then $\varphi \circ f \in \mathcal{R}_\alpha[a, b]$.

Proof. Let $\varepsilon > 0$. As φ continuous on $[m, M]$, φ is uniformly continuous on $[m, M]$ by Theorem 4.19. That is, $\exists \delta < \varepsilon$ s.t. $|\varphi(s) - \varphi(t)| < \varepsilon$ if $|s - t| < \delta$ for $s, t \in [m, M]$.

Since $f \in \mathcal{R}_\alpha$, there exists a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that $U(P) - L(P) < \delta^2$.

Let $A = \{i \in \{1, 2, \dots, n\} : M_i - m_i < \delta\}$, $B = \{i \in \{1, 2, \dots, n\} : M_i - m_i \geq \delta\}$. Note $A \cup B = \{1, 2, \dots, n\}$.

Let $M_i^* = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $m_i^* = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$. Suppose $i \in A$. Then $M_i - m_i < \delta$. By definition of δ , this implies $|M_i^* - m_i^*| \leq \varepsilon$.

Suppose $i \in B$. By definition of P ,

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2.$$

As $M_i - m_i \geq \delta$,

$$\delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2.$$

Hence, $\sum_{i \in B} \Delta \alpha_i < \delta$. Then,

$$\begin{aligned} & \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\ & \leq 2 \cdot \underbrace{\sup\{|\varphi|\}}_{=\sup\{|\varphi(t)| : m \leq t \leq M\}} \cdot \left(\sum_{i \in B} \Delta \alpha_i \right) \\ & < 2 \cdot \sup\{|\varphi|\} \cdot \delta \\ & < 2 \cdot \sup\{|\varphi|\} \cdot \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} U(P, \varphi \circ f, \alpha) - L(P, \varphi \circ f, \alpha) &= \sum_{i=1}^n (M_i^* - m_i^*) \Delta \alpha_i \\ &= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\ &< \varepsilon[(\alpha(b) - \alpha(a)) + 2 \cdot \sup\{|\varphi|\}] \end{aligned}$$

Q.E.D.

Example. $f \in \mathcal{R}_\alpha[a, b] \Rightarrow f^2 \in \mathcal{R}_\alpha[a, b], |f| \in \mathcal{R}_\alpha[a, b]$ where $\varphi(t) = t^2$ and $\varphi(t) = |t|$ respectively.

Note. $\varphi \in \mathcal{R}_\alpha[m, M]$ does not imply $\varphi \circ f \in \mathcal{R}_\alpha[a, b]$. See A2.

Theorem 6.12. (Linearity and related properties)

(a) If $f, f_1, f_2 \in \mathcal{R}_\alpha[a, b]$, then $f_1 + f_2 \in \mathcal{R}_\alpha[a, b]$, $cf \in \mathcal{R}_\alpha[a, b]$, and

$$\begin{aligned}\int_a^b (f_1 + f_2) d\alpha &= \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \\ \int_a^b cf d\alpha &= c \int_a^b f d\alpha.\end{aligned}$$

Proof. TEXTBOOK

Q.E.D.

(b) $f_1, f_2 \in \mathcal{R}_\alpha$ and $f_1(x) \leq f_2(x)$ for all $x \in [a, b]$, then $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$.

Proof. $L(P, f_1) \leq L(P, f_2) \leq \sup_P L(P, f_2) = \int_a^b f_2 d\alpha$. $\int_a^b f_1 d\alpha = \sup_P L(P, f_1) \leq \int_a^b f_2 d\alpha$. Q.E.D.

(c) If $f \in \mathcal{R}_\alpha[a, b]$, $c \in [a, b]$ then $f \in \mathcal{R}_\alpha[a, c]$ and $f \in \mathcal{R}_\alpha[a, b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.

(d) If $f \in \mathcal{R}_\alpha$ and $|f(x)| \leq M$ for all $x \in [a, b]$, then $|\int_a^b f d\alpha| \leq M(\alpha(b) - \alpha(a))$.

Proof. Let $P = \{a, b\}$. Then $-M[\alpha(b) - \alpha(a)] \leq m_1 \Delta \alpha_1 = L(P) \leq \int_a^b f d\alpha \leq U(P) = M_1 \Delta \alpha_1 \leq M[\alpha(b) - \alpha(a)]$. Q.E.D.

(e) If $f \in \mathcal{R}_{\alpha_1}$ and $f \in \mathcal{R}_{\alpha_2}$, then $f \in \mathcal{R}_{\alpha_1 + \alpha_2}$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2. \quad (*)$$

If $f \in \mathcal{R}_\alpha$ and $c \geq 0$ then $f \in \mathcal{R}_\alpha$ and $\int_a^b f d(c\alpha) = c \int_a^b f d\alpha$.

Proof (*). Let $\varepsilon > 0$. Choose P_1, P_2 s.t. $U(P_j, f, \alpha_j) - L(P_j, f, \alpha_j) < \frac{\varepsilon}{2}$, where $j = 1, 2$. Let $P^* = P_1 \cup P_2$. By Theorem 6.4,

$$U(P^*, f, \alpha_j) - L(P^*, f, \alpha_j) < \frac{\varepsilon}{2}. \quad (**)$$

Since $(\Delta\alpha_1)_i + (\Delta\alpha_2)_i = (\Delta(\alpha_1 + \alpha_2))_i$,

$$\begin{aligned} & U(P^*, f, (\alpha_1 + \alpha_2)) - L(P^*, f, (\alpha_1 + \alpha_2)) \\ &= \sum_{i=1}^n (M_i - m_i)(\Delta(\alpha_1 + \alpha_2))_i \\ &= \sum_{i=1}^n (M_i - m_i)[(\Delta\alpha_1)_i + (\Delta\alpha_2)_i] \\ &= \sum_{i=1}^n (M_i - m_i)(\Delta\alpha_1)_i + \sum_{i=1}^n (M_i - m_i)(\Delta\alpha_2)_i \\ &= U(P^*, f, \alpha_1) - L(P^*, f, \alpha_1) + U(P^*, f, \alpha_2) - L(P^*, f, \alpha_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By Theorem 6.6, $f \in \mathcal{R}_{\alpha_1 + \alpha_2}$. Also $\int_a^b f d(\alpha_1 + \alpha_2) \leq U(P^*, f, \alpha_1 + \alpha_2) = U(P^*, f, \alpha_1) + U(P^*, f, \alpha_2) < \int_a^b f d\alpha_1 + \frac{\varepsilon}{2} + \int_a^b f d\alpha_2 + \frac{\varepsilon}{2}$ by (**). Similarly, $\int_a^b f d(\alpha_1 + \alpha_2) \geq L(P^*, f, \alpha_1 + \alpha_2) = L(P^*, f, \alpha_1) + L(P^*, f, \alpha_2) > \int_a^b f d\alpha_1 - \frac{\varepsilon}{2} + \int_a^b f d\alpha_2 - \frac{\varepsilon}{2}$ by (**). As ε is arbitrary, (*) holds. Q.E.D.

Theorem 6.13. (a) $f, g \in \mathcal{R}_\alpha \Rightarrow fg \in \mathcal{R}_\alpha$

Proof. By Theorem 6.11 with $\varphi(t) = t^2$, $h \in \mathcal{R}_\alpha \Rightarrow h^2 \in \mathcal{R}_\alpha$. By Theorem 6.12(a), $f, g \in \mathcal{R}_\alpha \Rightarrow f + g \in \mathcal{R}_\alpha$, so $(f \pm g)^2 \in \mathcal{R}_\alpha$. Since $(f + g)^2 - (f - g)^2 = 4fg$, by Theorem 6.12(a), $fg \in \mathcal{R}_\alpha$. Q.E.D.

(b) If $f \in \mathcal{R}_\alpha$, then $|f| \in \mathcal{R}_\alpha$, and $\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$.

Proof. By Theorem 6.11, $|f| \in \mathcal{R}_\alpha$ (take $\varphi(t) = |t|$). Let

$$c = \operatorname{sgn} \left(\int_a^b f d\alpha \right) = \begin{cases} +1 & \text{if } \int_a^b f d\alpha > 0 \\ 0 & \text{if } \int_a^b f d\alpha = 0 \\ -1 & \text{if } \int_a^b f d\alpha < 0 \end{cases}.$$

As $cf \leq |f|$, $\left| \int_a^b f d\alpha \right| = c \int_a^b f d\alpha = \int_a^b cf d\alpha \leq \int_a^b |f| d\alpha$. Q.E.D.

Definition 6.14. [Unit Step Function]

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Theorem 6.15. Suppose f is bounded on $[a, b]$ and continuous at $s \in (a, b)$.

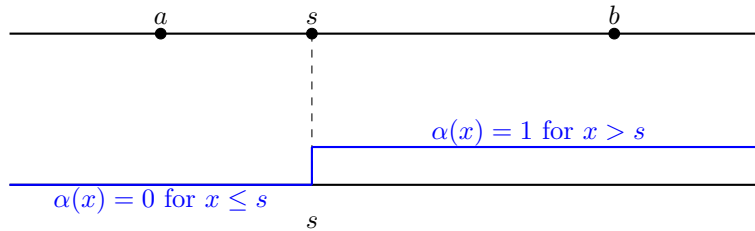
Let

$$\alpha(x) = \begin{cases} 0 & x \leq s \\ 1 & x > s \end{cases}.$$

Then $\int_a^b f d\alpha$ exists, and

$$\int_a^b f d\alpha = f(s).$$

Proof. Let $P = \{a, s, x, b\}$ with $x \in (s, b)$. $U(P) = \sum_{i=1}^3 M_i \Delta\alpha_i = 0 + M_2 \Delta\alpha_1 = M_2$. $L(P) = \sum_{i=1}^3 m_i \Delta\alpha_i = 0 + m_2 \Delta\alpha_1 = m_2$. Hence, $m_2 \leq \int_a^b f d\alpha \leq \overline{\int_a^b f d\alpha} \leq M_2$. Let x approach s . By continuity of f at s , M_2 approaches $f(s)$ from above and m_2 approaches $f(s)$ from below. Hence, $\int_a^b f d\alpha = f(s) = \overline{\int_a^b f d\alpha} = \int_a^b f d\alpha = f(s)$.



Q.E.D.

Remark. (a) By Theorem 4.29, if α is monotone-increasing, then $\alpha(x^+)$ and $\alpha(x^-)$ exist for all $x \in (a, b)$, and $\alpha(x^-) \leq \alpha(x) \leq \alpha(x^+)$.

(b) In Theorem 6.15, α is left-continuous at s .

Exercise

Prove the same conclusion for $\alpha(x) = \begin{cases} 0 & x < s \\ 1 & x \geq s \end{cases}$.

(c) This α plays the role of the Dirac delta function:

$$\int_a^b f(x)\delta(x-s)dx = f(s)$$

where $\delta(x-s) = \begin{cases} 0 & x \neq s \\ \infty & x = s \end{cases}$ is the Dirac delta function. Technically, there is no such function as $\delta(x-s)$, but it is a useful concept in physics. Note δ is kind of like α' in Theorem 6.15. (See A2).

(d) Theorem 6.15 has $\alpha(x) = I(x-s)$.

Theorem 6.16. Let $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$, $s_n \in (a, b)$, where $s_i \neq s_j$ if $i \neq j$.

Let $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x-s_n)$, a monotone increasing function. Let f be continuous on $[a, b]$. Then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

Note.

(a) $\alpha(x)$ converges for any $x \in (a, b)$ by the comparison test, with

$$0 \leq \sum_{i=1}^{\infty} c_n I(x-s_n) \leq \sum_{n=1}^{\infty} c_n < \infty$$

(b) $\sum_{n=1}^{\infty} c_n f(s_n)$ converges by the comparison test, with

$$|c_n f(s_n)| \leq M \cdot c_n$$

Proof. Let $R_N = \int_a^b f d\alpha - \sum_{i=1}^N c_n f(s_n)$.

Claim: $R_N \rightarrow 0$ as $N \rightarrow \infty$; i.e., given $\varepsilon > 0$, $\exists N_0$ s.t. $|R_N| < \varepsilon$ for all $N \geq N_0$.

Let

$$\alpha_1(x) = \sum_{n=1}^N c_n I(x - s_n)$$

$$\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n).$$

By Theorem 6.12 and Theorem 6.15,

$$\int_a^b f d\alpha_1 = \sum_{n=1}^N c_n \int_a^b f(x) dI(x - s_n) = \sum_{n=1}^N c_n f(s_n)$$

$$\int_a^b f d\alpha_2 = \sum_{n=N+1}^{\infty} c_n f(s_n).$$

By Theorem 6.12,

$$\int_a^b f d\alpha = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.$$

Then $R_N = \int_a^b f d\alpha_2$. Choose N_0 s.t. $\sum_{n=N_0+1}^{\infty} c_n < \varepsilon$. By Theorem 6.12(d),

$$|R_N| \leq M \cdot [\alpha_2(b) - \underbrace{\alpha_2(a)}_{=0}] = M \cdot \sum_{n=N+1}^{\infty} c_n < M \cdot \varepsilon$$

for all $N \geq N_0$.

Q.E.D.

Theorem 6.17. Suppose

- (a) $|f(x)| \leq M$ for all $x \in [a, b]$,
- (b) α is differentiable on $[a, b]$ and increasing on $[a, b]$.
- (c) $\alpha'(x) \in \mathcal{R}[a, b]$

Then

$$f \in \mathcal{R}_\alpha[a, b] \Leftrightarrow f\alpha' \in \mathcal{R}[a, b] \quad (*)$$

If $(*)$ holds, then

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx \quad (**)$$

I

Proof. (i) It suffices to show that $\overline{\int_a^b f d\alpha} = \overline{\int_a^b f \alpha' dx}$ and $\underline{\int_a^b f d\alpha} = \underline{\int_a^b f \alpha' dx}$.

(ii) Let $\varepsilon > 0$. Since $\alpha' \in \mathcal{R}$, there exists P s.t. $U(P, \alpha') - L(P, \alpha') < \varepsilon$ by Theorem 6.6.

Note this and the rest of the proof hold for any refinement of P . $U(P, \alpha') - L(P, \alpha') = \sum_{i=1}^n (A_i - a_i) \Delta x_i$, where $A_i = \sup\{\alpha'(x) : x \in [x_{i-1}, x_i]\}$ and $a_i = \inf\{\alpha'(x) : x_{i-1} \leq x \leq x_i\}$. By the mean value theorem,

$$\exists t_i \in [x_{i-1}, x_i] \text{ s.t. } \Delta \alpha_i = \alpha'(t_i) \cdot \Delta x_i.$$

For any $s_i \in [x_{i-1}, x_i]$, $|\alpha'(s_i) - \alpha'(t_i)| \leq A_i - a_i$, so

$$\sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon.$$

[cf. Theorem 6.7(b)]

(iii) For any $s_i \in [x_{i-1}, x_i]$,

$$\left| \sum_{i=1}^n f(s_i) \cdot \underbrace{\Delta \alpha_i}_{\alpha'(t_i) \Delta x_i} - \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i \right| = \left| \sum_{i=1}^n f(s_i) [\alpha'(t_i) - \alpha'(s_i)] \Delta x_i \right| < M\varepsilon.$$

Therefore,

$$\sum_{i=1}^n f(s_i) \Delta \alpha_i \leq \sum_{i=1}^n f(s_i) \alpha'(s_i) \Delta x_i + M\varepsilon \leq U(P, f\alpha') + M\varepsilon.$$

Taking supremum over s_i 's, $U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon$, so

$$\begin{aligned} \overline{\int_a^b f d\alpha} &\leq U(P, f, \alpha) \\ &\leq U(P, f\alpha') + M\varepsilon. \end{aligned}$$

As $\inf_P U(P, f, \alpha) \leq \inf_P U(P, f\alpha')$,

$$\overline{\int_a^b f d\alpha} \leq \overline{\int_a^b f \alpha' dx} + M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\int_a^b f d\alpha} \leq \overline{\int_a^b f \alpha' dx}.$$

Similarly, $\underline{\int_a^b f d\alpha} \geq \underline{\int_a^b f \alpha' dx}$.

Theorem 6.19. Suppose $\varphi : [A, B] \rightarrow [a, b]$ is strictly increasing, continuous, and onto. Suppose α is increasing on $[a, b]$ and $f \in \mathcal{R}_\alpha[a, b]$. Let $g = f \circ \varphi : [A, B] \rightarrow \mathbb{R}$, $\beta = \alpha \circ \varphi : [A, B] \rightarrow \mathbb{R}$. Then $g \in \mathcal{R}_\beta[A, B]$ and $\int_A^B g d\beta = \int_a^b f d\alpha$.

Note. This is the change of variables formula for Riemann-Stieltjes integrals. It generalizes the calculus formula. $\int_a^b f(x) dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(y)) \varphi'(y) dy$. Here $\alpha(x) = x$, so $\beta = \varphi$ and $d\beta = \varphi'(y) dy$.

Proof. Partition P of $[a, b]$ and Q of $[A, B]$ are in one-to-one correspondence via $x_i = \varphi(y_i)$. $g([y_{i-1}, y_i]) = f([x_{i-1}, x_i])$ and $\alpha(x_i) = (\alpha \circ \varphi)(y_i) = \beta(y_i)$, so $U(P, f, \alpha) = U(Q, g, \beta)$, and $L(P, f, \alpha) = L(Q, g, \beta)$. Let $\varepsilon > 0$. Since $f \in \mathcal{R}_\alpha[a, b]$, there exists P s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$, so $U(Q, g, \beta) - L(Q, g, \beta) < \varepsilon$, and $g \in \mathcal{R}_\beta[A, B]$. Also, $\int_A^B g d\beta = \inf_Q U(Q, g, \beta) = \inf_P U(P, f, \alpha) = \int_a^b f d\alpha$. Q.E.D.

Example. $\int_a^b \sin x^2 dx$ for $0 \leq a < b$. Here, $f(x) = \sin x^2$, $\alpha(x) = x$. Let $x^2 = y$, so $x = \varphi(y) = \sqrt{y}$, $\varphi^{-1}(y) = y^2$. Then $\varphi : \underbrace{[a^2, b^2]}_{=[A, B]} \rightarrow [a, b]$

is continuous, strictly increasing, and onto. $g(y) = (f \circ \varphi)(y) = \sin y$. $\beta(y) = (\alpha \circ \varphi)(y) = \sqrt{y}$. Theorem 6.19 gives

$$\int_a^b \sin x^2 dx = \int_{a^2}^{b^2} \sin y d\beta = \int_{a^2}^{b^2} \sin y \frac{1}{2\sqrt{y}} dy,$$

where the last equality follows from the Theorem 6.17 as $\beta' = \frac{1}{2\sqrt{y}} \in \mathcal{R}[a^2, b^2]$. Hence,

$$\int_a^b \sin x^2 dx = \int_{a^2}^{b^2} \frac{\sin y}{2\sqrt{y}} dy.$$

Theorem 6.20. Let $f \in \mathcal{R}[a, b]$ and for $x \in [a, b]$, define $F(x) = \int_a^x f(t) dt$. Then F is continuous on $[a, b]$, and if f is continuous at $x_0 \in [a, b]$, then $F'(x_0)$ exists and $F'(x_0) = f(x_0)$.

Proof. Continuity of F : Choose M s.t. $|f(t)| \leq M$ for all $t \in [a, b]$.
 For $a \leq x < y \leq b$, $|F(y) - F(x)| = \left| \int_a^y f(t)dt - \int_a^x f(t)dt \right| = \left| \int_x^y f(t)dt \right| \leq M |y - x|$. Let $\varepsilon > 0$. If $|y - x| < \delta = \frac{\varepsilon}{M}$, then $|F(y) - F(x)| < M \cdot \frac{\varepsilon}{M} = \varepsilon$. Therefore, F is continuous on $[a, b]$.

Differentiability of F and $F'(x_0) = f(x_0)$: Let $h > 0$.

$$\frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) = \frac{1}{h} \left[\int_{x_0}^{x_0+h} f(t)dt \right] - f(x_0) = \frac{1}{h} \int_{x_0}^{x_0+h} [f(t) - f(x_0)]dt.$$

As f is continuous at x_0 , $\forall \varepsilon > 0 : \exists \delta > 0$ s.t. $|t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon$. Thus, if $h < \delta$,

$$\left| \frac{F(x_0 + h) - F(x_0)}{h} - f(x_0) \right| = \left| \frac{1}{h} \int_{x_0}^{x_0+h} \underbrace{[f(t) - f(x_0)]}_{\in (-\varepsilon, \varepsilon)} dt \right| \leq \frac{1}{h} \cdot h\varepsilon = \varepsilon$$

Q.E.D.

Theorem 6.21. [Fundamental Theorem of Calculus] If $f \in \mathcal{R}[a, b]$ and if \exists differentiable function F on $[a, b]$ s.t. $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.

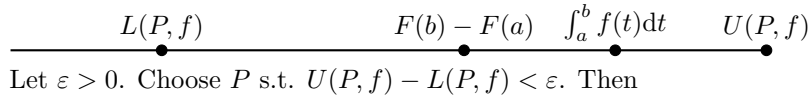
Proof. For any partition P , $F(b) - F(a) = \sum_{i=1}^n [F(x_i) - F(x_{i-1})]$, and by the mean value theorem 5.10,

$$\forall i \in \{1, 2, \dots, n\} : \exists t_i \in [x_{i-1}, x_i] \text{ s.t. } F(x_i) - F(x_{i-1}) = F'(t_i) \Delta x_i = f(t_i) \Delta x_i.$$

Therefore,

$$F(b) - F(a) = \sum_{i=1}^n F'(t_i) \Delta x_i = \left[\sum_{i=1}^n \underbrace{f(t_i)}_{\in [m_i, M_i]} \Delta x_i \right] \in [L(P, f), U(P, f)].$$

Also, $\int_a^b f(t) dt \in [L(P, f), U(P, f)]$.



$$\left| [F(b) - F(a)] - \int_a^b f(t) dt \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\left| [F(b) - F(a)] - \int_a^b f(t) dt \right| = 0$. **Q.E.D.**

Theorem 6.22. [Integration by Parts] If F, G are differentiable on $[a, b]$ and $F' = f, G' = g \in \mathcal{R}[a, b]$, then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Moreover, if F, G are (monotone) increasing, on $[a, b]$, then

$$\int_a^b F dG = FG|_a^b - \int_a^b G dF. \quad (**)$$

Proof. Let $H(x) = F(x)G(x)$. Then $H' = F'G + FG' = fG + Fg \in \mathcal{R}[a, b]$, so

$$\int_a^b H'(x)dx = \int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx.$$

By Theorem 6.21 to H ,

$$\begin{aligned} \int_a^b H'(x)dx &= H(b) - H(a) = F(b)G(b) - F(a)G(a) \\ &= \int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx \\ \therefore \int_a^b F(x)g(x)dx &= F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx. \end{aligned}$$

Q.E.D.

Remark. See problem 6.17 for a version with α .