Real Variables

January 10, 2025

Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$

Remark. Note for real numbers, $\mathbb{Q}$ has holes in it.
Example. $\nexists p \in \mathbb{Q}\ s.t\ p^2 = 2$

Proof. Assume $\exists p \in \mathbb{Q}$ s.t $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t $p^2 = 2$. Q.E.D.

Definition 1.1 (Order)**.** An order on a set S is a relation $<$ such that:

(a) If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.

(b) If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 1.2 (Ordered Set)**.** An ordered set S is a set with an order $<$.

Definition 1.3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t $\forall x \in E : x \leq \beta$.
Similarly, a set S is bounded below if $\exists \beta \in S$ s.t $\forall x \in E : x \geq \beta$.

Definition 1.4 (LUB, GLB). Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists \alpha$ s.t. α is an upper bound for E and $\forall \gamma < \alpha$: γ is not an upper bound for E, then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists \alpha$ s.t. α is a lower bound for E and $\forall \gamma > \alpha$: γ is not a lower bound for E. Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 1.5 (LUB property). An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists \sup E \in S$; i.e., Every bounded subset of S has the least upper bound(LUB).
Example.

- $\mathbb{Z}$ *has the* LUB *property.*
- $\mathbb{Q}$ *does not have the* LUB *property.*

Theorem 1.1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof. ($\Rightarrow$)Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B. Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. ($\Leftarrow$)Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E. Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 1.6 (Fields). Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

(a) $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).

(b) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).

(d) $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.

(e) $\exists (-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.

(f) $\forall x, y \in F : xy \in E$.

(g) $\forall x, y \in F : xy = yx$.

(h) $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.

(i) If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

(j) $\forall x, y, z \in F : x(y + z) = xy + xz$

Example.

(a) $\mathbb{Q}$ *is a field, while* $\mathbb{Z}$ *is not a field.*

(b) $F_p = \{0, 1, \ldots, p - 1\}$ *with mod* p *arithmetic is a field.*

Read Text book: 114,115,116,118

Definition 1.7 (Ordered Field)**.** An ordered field F is a field that is an ordered set such that the following properties are satisfied:

(a) If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.

(b) If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

Remark. We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. $\mathbb{Q}$ *is an ordered field.*

Theorem 1.2. $\exists$ an ordered field $\mathbb{R}$ which has the LUB property and contains $\mathbb{Q}$ as a subfield.

Theorem 1.3.

(a) Arithmetic properties of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.

(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof. (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A. So, A has a least upper bound α. Since $\alpha - x < \alpha$ as $x > 0$, $\alpha - x$ is not an upper bound for A. Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A. Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. $ny - nx > 1$ and therefore, $1 + nx < ny$. Let $m \in \mathbb{Z}$ such that $(m - 1) \leq nx < m$. Such m exists by the extended version of (a). This implies there exists $m \in \mathbb{N}$ such that $nx < m \leq nx + 1 < ny$. Therefore, $x < \frac{m}{n} < y$.

(c) $\exists k \in \mathbb{Q}$ such that $k^2 = 2$; i.e., $\exists \sqrt{2} \in \mathbb{R}$. $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then $x < p < \alpha < q < y$ and $\alpha \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n-1)x \leq y < nx. \tag{1.1}$$

Proof. Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of $\mathbb{N}$ has a smallest element. Let $n =$ smallest element of A. Then the inequality holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n-1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n-1)x$. Hence, the statement holds. Q.E.D.

Lemma. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b - a)$ for some $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} b^n - a^n &= (b-a)(\underbrace{b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1}}_{n \text{ terms}}) \\ &< (b-a)nb^{n-1} \end{aligned}$$

Q.E.D.

Theorem 1.4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique})y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt{x}^n$, the n^{th} root of x).

Proof. Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x.
Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \varnothing$, E is bounded above, hence (by the least-upper-bound property) there exists a $\sup E$. Choose $y = \sup E$. Consider two cases.

(a) If $x \leq 1$, then $t_0 = \dfrac{x}{2}$ and thereby $t_0^n = \dfrac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).

(b) If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x+1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E.

(a) Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y+h)^n - y^n < x - y^n$. By the lemma , we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y+h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.

(b) Assuming that $y^n > x$, we find $k > 0$ such that $(y-k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y, which is $\sup E$. By the lemma , $y^n - (y-k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \dfrac{y^n - x}{ny^{n-1}}$. Therefore, $-(y-k)^n < -x$, or $x < (y-k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 1.8 (Cut/Dedekind Cut)**.** The set $\mathbb{R}$ elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 1.9 (Order of cuts). For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A, and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 1.5. There exists an ordered field $\mathbb{R}$ such that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ has the LUB property.

Proof. Let $\mathbb{R}$ be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 1.10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$
Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$
Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
Zero element is $(0, 0)$
One element is $(1, 0)$

Theorem 1.6. $\mathbb{C}$ is a field.

Proof. Verify the 11 field axioms. For just a few axioms:
(M3):
$x = (a, b), y = (c, d), z = (e, f)$. $x(yz) = (a, b)(ce - df, cf + de) = (a(ce-df)-b(cf+de), a(cf+de)+b(ce-df)) = (ac-bd, ad+bc)(e, f) = (xy)z$
(M4):
$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$
(M5):
$x \neq 0$ means $x = (a, b)$ with $a \neq 0$ or $b \neq 0$. That is, $a^2 + b^2 > 0$. Let $\frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$. Then $x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0)$. Q.E.D.

Identification of $\mathbb{R}$ as a subfield of $\mathbb{C}$. Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a+b = a+b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\mathrm{Re}(z) = a$,$\mathrm{Im}(z) = b$.

Definition 1.11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by $\overline{z}$

Note.

(a) $\overline{z + w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$

(c) $z + \overline{z} = 2 \cdot \mathrm{Re}(z)$

(d) $z - \overline{z} = 2i \cdot \mathrm{Im}(z)$

(e) $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$, with $=$ if any only if $z = 0$

(f) $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{a-bi}{a^2+b^2}$

Definition 1.12. $|z| = \sqrt{z\overline{z}} = \sqrt{a^2 + b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 1.7. For $z, w \in \mathbb{C}$,

(a) $|z| \geq 0$ with $=$ iff $z = 0$

(b) $|z| = |\overline{z}|$

(c) $|zw| = |z| \cdot |w|$

(d) $|\mathrm{Re}(z) \leq |z|, |\mathrm{Im}(z)| \leq |z|$

Proof. Let $z = a + bi$. Then $|\mathrm{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$ Q.E.D.

(e) $|z + w| \leq |z| + |w|$ (Triangle inequality)

Proof.

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

Theorem (Cauchy-Schwarz inequality)**.** If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$|\sum_{j=1}^{n} a_j\overline{b_j}| \leq (\sum_{j=1}^{n} |a_j|^2)^{\frac{1}{2}}(\sum_{j=1}^{n} |b_j|^2)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^{n} a_j\overline{b_j}$ defined on inner product on $\mathbb{C}^n$ and $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})}$. (Note that $\vec{a}\cdot\vec{b} = \overline{\vec{b}\cdot\vec{a}}$)

Proof. Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j\overline{b_j}$ We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0.
For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^{n} |a_j + \lambda b_j|^2 = \sum_{j=1}^{n} (a_j + \lambda b_j)(\overline{a_j} + \overline{\lambda b_j}) = \sum_{j=1}^{n} |a_j|^2 + \lambda \sum_{j=1}^{n} b_j\overline{a_j} + \overline{\lambda} \sum_{j=1}^{n} a_j \cdot \overline{b_j} + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2$. Let $\lambda = tC$ for $t \in \mathbb{R}$.
Then $0 \leq A + \lambda\overline{C} + \overline{\lambda}C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$. $p(t)$ is a quadratic function in terms of t and it must be non-negative. Regardless of the value of t. Therefore, the discriminant of $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$. Since $|C| \geq 0$, $|C|^2 \leq AB$. Q.E.D.

Definition 1.13 (Euclidean k-space)**.** For $k \in N$, $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}$ with the following properties:

Addition $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$

Scalar multiplication $\lambda\vec{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$

Inner(dot) product $(\vec{x}, \vec{y}) = \sum_{j=1}^{k} x_j y_j$, which is bilinear: $(\alpha\vec{x} + \beta\vec{y}) \cdot \vec{z} = \alpha\vec{x}\cdot\vec{z} + \beta\vec{y}\cdot\vec{z}$.

Norm $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^{k} |x_j|^{2^{1/2}}$

Remark. Addition and Scalar multiplication make $\mathbb{R}^k$ into a vector space.

Theorem 1.8. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$. Then

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0 \leftrightarrow \vec{x} = \vec{0}$

(c) $|\alpha\vec{x}| = |\alpha||\vec{x}|$

(d) $|\vec{x}\cdot\vec{y}| \leq |\vec{x}||\vec{y}|$ (special case of Cauchy-Schwarz inequality)

(e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle inequality)

Proof. $|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = |\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x}\cdot\vec{y}| + |\vec{y}|^2 \leq (|\vec{x}|+|\vec{y}|)^2$ Q.E.D.

(f) $|\vec{x}-\vec{y}| \leq |\vec{x}-\vec{z}| + |\vec{z}-\vec{y}|$

Proof. $|\vec{x}-\vec{y}| = |(\vec{x}-\vec{z}) + (\vec{z}-\vec{y})| \leq |\vec{x}-\vec{z}| + |\vec{z}-\vec{y}|$ Q.E.D.

Chapter 2

Basic Topology

Definition 2.1. Sets A and B have the same cardinality, if $\exists f : A \to B$ that is 1-1 and onto (i.e., bijective).

Theorem 2.1. Let $A \sim B$ be a relation between two sets having the same cardinality. Then is an equivalence relation. That is,

(a) $A \sim A$ (Reflexive)

(b) $A \sim B \Rightarrow B \sim A$ (Symmetry)

(c) $A \sim B \& B \sim C \Rightarrow A \sim C$ (Transitivity)

Definition 2.2. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $J_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$.

- A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$(or if $A = \emptyset$).
- A set A is countably infinite if $A \sim \mathbb{N}$.
- A set A is countable if A is finite or countably infinite.

Example. *$\mathbb{Z}$ is a countably infinite. For $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$,*

Let $f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n+1 & \text{if } n < 0 \end{cases}$

Then f is bijective and therefore $|\mathbb{Z}| = |\mathbb{N}|$

Theorem 2.8. A subset of a countably infinite set is countable.

Proof. Let A be some countably infinite set and S be a infinite subset of A.
As A is a countably infinite set, we can remove duplicates and arrange A so that $A = \{a_1, a_2, a_3, \ldots\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in S$. Let n_k be the smallest positive integer greater than n_{k-1} such that $x_{n_k} \in E$ for $k = 2, 3, \ldots$. Let $f(k) = x_{n_k}$ for $k = 1, 2, 3, \ldots$. Then this is a bijection from $\mathbb{N}$ to S. Q.E.D.

Remark. Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

Theorem 2.12. Let $E_1, E_2, \ldots$ be countably infinite sets. Then $S = \cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof. Write $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \ldots\}$
$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \ldots\}$
Form an array:

$$\begin{Bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Bmatrix}.$$

This matrix might have duplicates. Let T be a subset of $\mathbb{N}$ such that $t \in T$ if and only if t is the smallest positive integer such that $x_t \in E_1 \cup E_2 \cup \ldots \cup E_n$.
Then a set $\{x_t | t \in T \text{ and } \exists_{i \in \mathbb{N}} : x_t \in E_i\}$ is S. Clearly, $|S| = |T|$, or $S \sim T$, and T is a subset of a countably infinite set, $\mathbb{N}$. Therefore, T is also countable, implying S is also countable. As S is infinite, S is countably infinite. Q.E.D.

Corollary 2.13. If A is countable and $n \in \mathbb{N}$, then $\{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in A\}$ is countable.

Theorem 2.14. Let $A = \{(b_1, b_2, b_3 \ldots) | b_i \in \{0, 1\}\}$. I.e., A is a set of all infinite binary strings. Then A is uncountable.

Proof (Contor's Diaganalization argument,1891)**.** Let $E \subset A$ be countably infinite. $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \ |s^{(i)} \in A\}$. It suffices to find some $s \in A \setminus E$, for this shows every countably infinite subset of A is proper construction of s. Write

$$s^{(1)} = b_1^1 b_2^1 \dots \tag{2.1}$$
$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \tag{2.2}$$
$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \tag{2.3}$$
$$\vdots$$

On diagonal, flip each bit, i.e., $0 \to 1$ and $1 \to 0$ and represent the flipped bit of b_i^i by $\tilde{b_i^i}$. Let $s = \tilde{b_1^1}\tilde{b_2^2}\tilde{b_3^3}\dots$. Then $s \in A$ and $s \notin E$ as s differs from each $s^{(i)}$ in the i-th bit. Therefore, A is uncountable. Q.E.D.

Corollary 2.15. The set $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ is uncountable.

Proof. We can create $f : \mathcal{P}(\mathbb{N}) \to A$ be a bijection, where A is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \tag{2.4}$$

For example, if $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0 \dots)$. This f is a bijection, and therefore A is uncountable.

Q.E.D.

Theorem 2.16. $\mathbb{R}$ is uncountable.

Proof. This is a rough sketch of the proof:

(a) It's enough to show that $[0,1]$ is uncountable.

(b) Consider binary decimal representation of $x \in [0,1]$. For example, $x = 0.101001001\ldots$. Given x, choose maximal $b_1 \in \{0,1\}$ such that $\frac{b_1}{2} \leq x$. Then choose $b_2 \in \{0,1\}$ such that $\frac{b_1}{2} + \frac{b_2}{2} \leq x$. Continue this process to get $b_1, b_2, b_3, \ldots$. Then $x = \sup\left\{\sum_{i=1}^{n} \frac{b_i}{2^i}\right\}$. Consider any dyadic rational of the form $\frac{m}{2^n}$. Let it be $\frac{3}{2^4}$. Then this maps $\frac{3}{2^4} \to 0,0,1,1,0,0,0,\ldots$ and never produce $0,0,1,0,1,1,1,1,\ldots$, which also represents $\frac{3}{2^4}$. Let A_1 be a subset of $A = \{\text{infinite binary strings}\}$ such that A_1 does not contain any strings ending in $1,1,1,1,\ldots$. Then the decimal representation defines a bijection $f : [0) \to A \setminus A_1$.

(c) A_1 is countable because $A = (A \setminus A_1) \cup A_1$, which is uncountable.

This shows that $[0,1]$ is uncountable, and therefore $\mathbb{R}$ is uncountable.
Q.E.D.

Definition 2.3 (Metric Spaces)**.** A set X is a metric space with metric $d : X \times X \to \mathbb{R}$ if

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,q) = 0$ if $p = q$, $\forall p,q \in X$

(b) $\forall_{p,q \in X} : d(p,q) = d(q,p)$

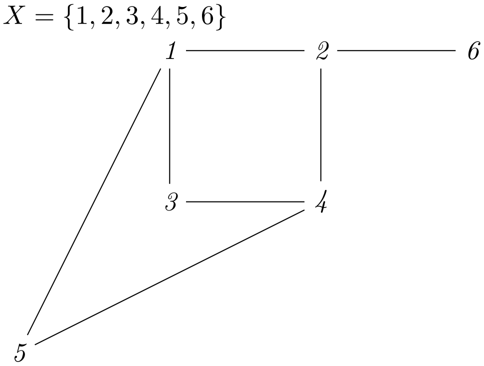
(c) $\forall_{p,q,r \in X} : d(p,q) \leq d(p,r) + d(r,q)$ (Triangle Inequality)

Remark. A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

Example (Metric Spaces)**.** *(a)* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$ *are metric spaces with* $d(p,q) = |p - q|$. *Note the meaning of* $|x|$ *depends on the context.*

(b) Every subset of a metric space is a metric space.

(c) $X = \{1,2,3,4,5,6\}$



Definition 2.4 (Neighborhood). A neighborhood in X is a set $N_r(p) := \{q : d(q,p) < r\}$, where $p \in X, r > 0$.

Remark. If $r_1 \leq r_2$, then $N_{r_1}(p) \subset N_{r_2}(p)$.

Example.

$\mathbb{R}^1$ *intervals,* $N_r(x) = \{y \in R^1 : |x-y| < r\}$

$\mathbb{R}^2$ *disks* $N_r(x) = \{y \in R^2 : |x-y| < r\}$

$\mathbb{R}^3$ *balls,* $N_r(x) = \{y \in R^3 : |x-y| < r\}$

Given example (c), $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$*,* $N_2(2) = \{1,2,4,6\}$*,* $N_3(2) = \{1,2,3,4,5,6\} = X$*.*

Definition 2.5. Let $E \subset X$. $p \in E$ is an interior point of E if $\exists r > 0$ such that $N_r(p) \subset E$.

Example.

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$.

Definition 2.6. $E \subset X$ is an open set if $\forall_{x \in E}$ is an interior point of E.

Theorem 2.19. Every neighborhood is an open set.

Proof. Let $g \in N_r(p)$. Then we must find $s > 0$, such that $N_s(g) \subset N_r(p)$. We know $d(p,q) < r$. Choose s such that $0 < s < r - d(p,q)$. Let $x \in N_s(q)$, then $d(q,x) < s < r - d(p,q)$. By triangle inequality, $d(p,x) \leq d(p,q) + d(q,x) < d(p,q) + r - d(p,q)$, so $x \in N_r(p)$, so $N_s(q) \subset N_r(p)$. Q.E.D.

Definition 2.7. Let $E \subset X$ and $p \in X$. p is a limit point of E if $\forall_{r>0} \exists_{q \in E}$ such that $q \neq p$ and $q \in N_r(p)$

Example. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ *has exactly one limit point,* 0*. note* $0 \notin E$*.*

Theorem 2.20. If p is a limit point of $E \subset S$, then every neighborhood of p contains infinitely many points of E.

Proof. Let $N_r(p)$ be a neighborhood of p. Then $N_r(p)$ contains at least one point $q_1 \in E$ such that $q_1 \neq p$. Let $r_1 = d(p, q_1)$. Then $N_{r_1}(p)$ contains some $q_2 \in E$ such that $q_2 \neq p$. Let $r_2 = d(p, q_2)$. Then $N_{r_2}(p)$ contains some $q_3 \in E$ such that $q_3 \neq p$. Continue this process to get $q_1, q_2, q_3, \ldots$. Q.E.D.

Corollary 2.21. If $E \subset X$ is finite then E has no limit points.

Definition 2.8 (Closed Set)**.** A set $E \subset X$ is closed if every limit point of E is in E.

Theorem 2.23. $E \subset X$ is open iff $E^c = \{x \in X : x \notin E\}$ is closed.

Proof.

- E is open $\Rightarrow$ E^c is closed.
  Let p be a limit point of E^c. Then every neighborhood of p contains some $q \in E^c$ such that $q \neq p$. If $p \in E$, then because E is open, p is an interior point, i.e., there exists some neighborhood of p that is a subset of E, which does not contain any points of E^c. This implies $p \notin E$ and therefore $p \in E^c$.

- E^c is closed $\Rightarrow$ E is open.
  Let $p \in E$. Then $p \notin E^c$, so p is not a limit point of E^c. Therefore, there exists some neighborhood of p that contains no points of E^c, i.e., all points of the neighborhood are in E. p Thus, Every $p \in E$ is an interior point of E, and hence E is open.

Q.E.D.

Theorem 2.24 (De Morgan's Laws)**.**

(a) $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha E_\alpha^c$

(b) $(\bigcap_\alpha E_\alpha)^c = \bigcup_\alpha E_\alpha^c$

Theorem 2.24.

(a) For all collection $\{G_\alpha\}$ of open sets : $\bigcup_\alpha G_\alpha$ is open.

(b) For all collection $\{F_\alpha\}$ of closed sets : $\bigcap_\alpha F_\alpha$ is closed.

(c) For all finite collection $\{G_1, G_2, \ldots, G_n\}$ of open sets : $\bigcap_{i=1}^n G_i$ is open.

(d) For all finite collection $\{F_1, F_2, \ldots, F_n\}$ of closed sets : $\bigcup_{i=1}^n F_i$ is closed.

Proof. (a) Let $x \in \bigcup_\alpha G_\alpha$. Then $x \in G_{\alpha_0}$ for some α_0. So there exists a neighborhood N of x such that $N \subset G_{\alpha_0} \subset \bigcup_\alpha G_\alpha$.

(b) it's suffice to prove that $(\bigcap_\alpha F_\alpha)^c$ is open. But $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$ is open by (a).

(c) Let $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for $i = 1, 2, \ldots, n$. So there exists a $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, r_2, \ldots, r_n\}$. Then $N_r(x) \subset N_{r_i} \subset G_i$ for $i = 1, 2, \ldots, n$ and therefore $N_r(x) \subset \bigcap_{i=1}^n G_i$.

(d) $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$ is open by (c).

Q.E.D.

Definition 2.9 (Closure)**.** Let $E \subset X$. Let E' be a set of limit points of E in X. The set $\overline{E} = E \cup E'$ is the closure of E.

Theorem 2.27.

(a) $\overline{E}$ is closed.

(b) $E = \overline{E} \Leftrightarrow E$ is closed.

(c) If $F \subset X$ is closed and $E \subset F$, then $\overline{E} \subset F$. (i.e., $\overline{E}$ is the smallest closed set containing E, and $\overline{E} = \bigcap_{F:\text{closed set with } F \supset E} F$.)

Proof. (a) Let p be a limit point of $\overline{E}$. It suffices to show $p \in E'$ since this implies that $p \in E' \subset E \cup E' = \overline{E}$. Let $r > 0$. $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$, i.e., $d(p,q) < \frac{r}{2}$. Since $q \in E \cup E'$, $\exists_{s \in \overline{E}}$ such that $d(q,s) < \frac{r}{2}$ (if $q \in E$, take $s = q$). But $d(p,s) \leq d(p,q) + d(q,s) < \frac{r}{2} + \frac{r}{2} = r$.

(b) $(\Rightarrow)$ by (a)

$(\Leftarrow)$ Suppose E is closed. Then $E' \subset E$, so $\overline{E} = E \cup E' = E$.

(c) Suppose F is closed. Then $F' \supset E'$ and also $F \supset F'$. So $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

Theorem 2.28. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup\{E\}$. Then $y \in \overline{E}$. Hence, $y \in E$ if E is closed.

Example. *Let $X = \mathbb{R}, d(p,q) = |p - q|$. Let $E \subset \mathbb{R}$ be nonempty and bounded above, and let $y = \sup E$. Then $y \in \overline{E}$.*

Proof. Suppose for contradiction $y \notin \overline{E}$. Then y is neither a point in E

nor a limit point of E, so $\exists$ some interval $N_r(y) = (y-r, y+r)$ such that $(y-r, y+r) \cap E = \emptyset$. However, then $y-r$ in an upper bound for E since y is a least upper bound, which is a contradiction. Therefore, $y \in \overline{E}$. Q.E.D.

Definition 2.10 (Relative Openness). Suppose X is a metric space, so $Y \in X$ is a metric space with the same metric. Let $E \subset Y$. Then E is open relative to Y if E is an open set in the metric space Y

Example. *$X = R^2 \supset \mathbb{R} = y, E = (0,1) \subset Y$. Then E is open relative to Y, but E is neither open nor closed in X.*

Theorem 2.30. A set $E \subset Y \subset X$ is open relative to $Y \Leftrightarrow \exists_{\text{open set } G \subset X} : E = G \cap Y$

Proof. ($\Rightarrow$) Suppose $E \subset Y$ is open relative to Y. Given $p \in E$, $\exists_{r_p > 0} : {N_{r_p}}^Y(p) \subset E$, where ${N_r}^Y(p) = \{q \in Y : d(p,q) < r\}$. Then $E \subset \bigcup_{p \in E} {N_{r_p}}^Y(p)$ and $\bigcup_{p \in E} {N_{r_p}}^Y(p) \subset E$. Therefore, $E = \bigcup_{p \in E} {N_{r_p}}^Y(p)$.
Let $G = \bigcup_{p \in E} {N_{r_p}}^X(p)$. This time, we are considering p's neighborhood in X, so each ${N_{r_p}}^X$ is open. Thus G is a union of open sets in X, and therefore open.
$\forall_{p \in E} : p \in N_{r_p}(p)^X$, so $E \subset G \cap Y$.
Let $p \in G \cap Y$. Then $p \in G$ and $p \in Y$. So $p \in {N_{r_p}}^X(p)$ for some $r_p > 0$. But $p \in Y$, so $p \in {N_{r_p}}^Y(p)$. Therefore, $p \in E$. This implies $G \cap Y \subset E$, and therefore $E = G \cap Y$.

($\Leftarrow$) Suppose $G \subset X$ is open and $E = G \cap Y$. Then $\forall_{p \in E} : \exists_{r_p > 0} : {N_{r_p}}^X(p) \subset G$, so ${N_{r_p}}^Y(p) = {N_{r_p}}^X(p) \cap Y \subset G \cap Y = E$.

Q.E.D.

Note: Midterm 1 material ends here.

Definition 2.11 (Open Cover). An open cover of $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t $E \subset \bigcup_\alpha G_\alpha$.

Definition 2.12 (Compact). A set $K \subset X$ is compact if every open cover has a finite subcover; i.e., $\exists_{\alpha_1, \alpha_2, \ldots \alpha_n} :$ s.t $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$

Example.

- *If E is finite, then E is compact.*
- *$(0,1) \subset \mathbb{R}$ is not compact. Bad cover: $(\frac{1}{n}, 1), n > 2$*
- *$[0, \infty] \subset \mathbb{R}$ is not compact. Bad cover: $(-1, n)$ for $n \in \mathbb{N}$.*
- *$E \subset \mathbb{R}^k$ is compact if and only if E is closed and bounded.*

Theorem 2.34. If K is compact then K is closed.

Proof. Suppose K is compact. It suffices to prove that K^c is open. Let $p \in K^c$. We need to produce $r > 0$ s.t. $N_r(p) \subset K^c$. For $q \in K$, let $W_q = N_{r_q}(q)$, where $r_q = \frac{1}{2}d(p,q) > 0$. $\forall_{x \in N_{r_q}(p)} : x \in W_q \Rightarrow d(x,p) + d(x,q) < 2r_q = d(p,q)$. However, X is a metric space and $p,q,x \in X$, so $d(p,q) \leq d(p,x) + d(x,q)$, leading to $d(p,q) \leq d(p,x) + d(x,q) < d(p,q)$, which is a contradiction. Hence, $\forall_{x \in N_{r_q}} : x \notin W_q$. $N_{r_q}(p) \subset {W_q}^c$ for $\forall_{q \in K}$. Note that $\{W_q\}_{q \in K}$ is an open cover of K. K compact $\Rightarrow \exists_{\text{finite number of open sets } W_{q_1}, W_{q_2}, \ldots W_{q_n}}$ s.t. $K \subset \bigcup_{i=1}^n W_{q_i}$. Let $r = \min\{r_{q_1}, r_{q_2}, \ldots r_{q_n}\} > 0$.

$$\therefore N_r(p) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} N_{r_p}(p) \right) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} {W_{q_i}}^c \right) = \left(\bigcup_{i \in \{1,2,\ldots \mathbb{N}\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

Theorem 2.35. If $K \subset X$ is compact then K is bounded; i.e., $\exists_{M<\infty}$ s.t. $\forall_{p,q \in K}$: $d(p,q) \leq M$

Proof. Fix $p \in K$. An open cover of K is $\{N_n(p)\}_{n \in \mathbb{N}}$. In fact, this is an open cover of X. K compact $\Rightarrow \exists_{\text{finite subcover} N_{n_1}(p), N_{n_2}(p) \ldots N_{n_m}(p)}$. Let $R = \max\{n_1, n_1, \ldots n_m\}$. $K \subset N_R(p)$. Let $M = 2R$. $\forall_{q,r \in K}$: $d(q,r) \leq d(q,p) + d(p,r) < R + R = 2R = M$. Q.E.D.

Theorem 2.35. If F is closed, K is compact, and $F \subset K$ then F is compact.

Proof. Suppose $F \subset K$. Let $\{V_\alpha\}$ be an open cover of F. It suffice to produce a finite subcover:
Consider $\{V_\alpha\}$ together with F^c. This gives an open cover of X, hence of K, so $\exists_{\text{subcover of } K}$. Drop F^c from this finite subcover. The result is a finite subcover of $\{V_\alpha\}$, which covers F Q.E.D.

Corollary 2.36. If F is closed and K is compact then $F \cap K$ is compact.

Theorem 2.33. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Note. This is not true for open sets. For instance, let $K = Y = [0,1] \subset X = \mathbb{R}$. Y is open and closed relative to Y, but Y is not open relative to X

Proof.

($\Rightarrow$) Suppose K is compact relative to X. Let $\{V_\alpha\}$ be an open cover of K relative to Y. For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then $\{V_\alpha\}$ is an open cover of K relative to X. Since K is compact relative to X, $\exists_{\text{finite subcover}}$.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K relative to X. Then $\{V_\alpha \cap Y\}$ is an open cover of K relative to Y. Since K is compact relative to Y, $\exists_{\text{finite subcover}}$.

Q.E.D.

Theorem 2.36. Suppose $\{K_\alpha\}$ is a collection of compact sets such that $\bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$. Then, $\lim\limits_{n\to\infty} \bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$, or equivalently, $\bigcap_\alpha K_\alpha \neq \emptyset$.

Example. *Let $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$. Then $\{G_j\}$ is a collection of open sets, but none of them are compact. (compact sets are closed) Then $\{G_j\}$ satisfies non-empty finite intersection property but $\bigcap_{j\in\mathbb{N}} G_j = \emptyset$.*

Proof. Suppose for contradiction $\bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$ for any $n < \infty, \alpha_i$ and $\bigcap_\alpha K_\alpha = \emptyset$. For any α_0, $K_{\alpha_0} \cap \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right) = \emptyset$. Hence, $K_{\alpha_0} \subset \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right)^c = \bigcup_{\alpha\neq\alpha_0} (K_\alpha)^c$ and $\{(K_\alpha)^c\}_{\alpha\neq\alpha_0}$ is an open cover of K_{α_0}, so $\exists$ a finite subcover of $K_{\alpha_0} \subset \bigcup_{i=1}^n {K_{\alpha_i}}^c$, which implies $K_{\alpha_0} \cap (\bigcap_{i=1}^n K_{\alpha_i}) = \emptyset$, contradiction. Q.E.D.

Corollary 2.37. If $\{K_1, K_2, \ldots\}$ are non-empty compact sets with $\forall_n :$ $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

Proof. If $n_1 < n_N$ then $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$ Q.E.D.

Theorem 2.37. If K is compact and $E \subset K$ is infinite, then E has a limit point in K.

Proof. Contrapositive of the statement is : *if $E \subset K$ has no limit point in K, then E is finite.*
Suppose every point $q \in K$ is not a limit point of E. Then

$$\exists_{V_q = N_{r_q}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}.$$

$\{V_q\}_{q \in K}$ is an open cover of K, so $\exists_{\text{finite subcover } V_{q_1} \cup V_{q_2} \cup \cdots \cup V_{q_n}}$. Then $E = E \cap K \subset (\bigcup_{i=1}^{n} V_{q_i} \cap E) \subset \{q_1, q_2, \ldots q_n\}$, so E is finite.

Q.E.D.

Theorem 2.38. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be such that $\forall_n : I_n \supset I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Since $I_n \supset I_{n+1}$, $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$. Let $E = \{a_1, a_2, \ldots\}$. Then $E \neq \emptyset$, every b_k is an upper bound for E, so $\exists x = \sup E$ and $a_k \leq x \leq b_k$ for all k. Therefore, $x \in I_k$ for all k, so $x \in \bigcap_{n=1}^{\infty} I_n$. Q.E.D.

Theorem 2.39. Let $\{I_n\}$ be a sequence of k-cells such that $i_n \supset I_{n+1}$;i.e., $I_n = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj},\ a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj}$ for $j = 1, 2, \ldots, k\}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Apply previous theorem to each component. Q.E.D.

Note. k-cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of k *closed intervals on the real line.*
Formally, Given real numbers a_i and b_i such that $a_i < b_i$ for every integer i from 1 to k,

$$I = \{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, k\}$$

Theorem 2.40. Let $I \subset \mathbb{R}^k$ be a k-cell. Then I is compact.

Proof. Let $I = I_0 = \{x = (x_1, x_2, \ldots, x_k), a_j \le x_j \le b_j\}$.
Let $\Delta = \left\{\Sigma_i^k (b_j - a_j)^2\right\}^{1/2}$. Then $|\mathbf{x} - \mathbf{y}| \le \Delta$ for $\mathbf{x}, \mathbf{y} \in I$.
Suppose for contradiction $\{G_\alpha\}$ is an open cover of I that has no finite subcover.
Let $c_j = \frac{1}{2}(a_j + b_j)$ for $j = 1, 2, \ldots, k$. Using $[a_j, c_j], [c_j, b_j]$, we get 2^k k-cells Q_i with $I = \bigcup_{i=1}^{2^k} Q_i$. At least one Q_i, call it I_1, has no finite subcover. Otherwise, every Q_i has a finite subcover, and I would have a finite subcover, namely the union of the finite subcovers of each Q_i.
Repeat this step to construct $I_0 = I, I_1, I_2, \ldots$. Then the sequence $\{I_n\}$ constructed by this process satisfies the following properties:

(a) $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \ldots$

(b) $\forall_n : I_n$ has no finite subcover from $\{G_\alpha\}$

(c) if $x, y \in I_n$ then $|x - y| \le 2^{-n}\Delta$, where $\Delta =$ diagonal of $I = \left(\sum_{j=1}^k (b_j - a_j)^2\right)^{1/2}$.

By theorem 2.38 and (a), $\exists_{x^* \in \bigcap_{n=1}^\infty I_n}$. Since $x^* \in I$, $x^* \in G_{\alpha_0}$ for some α_0, so $\exists r > 0$ such that $N_r(x^*) \subset G_{\alpha_0}$. But by (c), $I_n \subset N_{2^{-n}\Delta}(x^*)$. As soon as n is large enough that $2^{-n}\Delta < r$, we have $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$, which contradicts (b). Q.E.D.

Note. Reverse triangle inequality
$\forall_{a,b,c \in X} : d(a,b) \ge d(a,c) - d(c,b)$ because $d(a,c) \le d(a,b) + d(b,c)$.

Theorem 2.41. For $E \subset \mathbb{R}^k$, the following are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof.

$(a) \Rightarrow (b)$ Because E is bounded, i.e., $\exists_M$ s.t. $\forall_{x,y\in E} : |x-y| \leq M$, there exists a k-cell I such that $E \subset I$. Since every k-cell is compact, this implies E is a closed subset of a compact set. Hence, E is also compact.

$(b) \Rightarrow (c)$ by theorem 2.37

$(c) \Rightarrow (a)$ To see that E is bounded, suppose it were not. Then E has an infinite subset $S = \{x_1, x_2, x_3, \ldots\}$ with $\forall_n : |x_n| \geq n$. S has no limit point in $\mathbb{R}^k$ Let $S = \{(x_1, x_2, x_3, \ldots) \in E : |x_n - x_0| < \frac{1}{n}\}$. Then S is an infinite set because if S is finite, there exists a point $\mathbf{x} \in S$ such that $|\mathbf{x}| \geq |\mathbf{x}'|$ for $\mathbf{x}' \in S$. However, there exists $n \in \mathbb{N}$ such that $n > |\mathbf{x}|$ and by definition of S, there exists $x_n \in S$ such that $|x_n| \geq n > |\mathbf{x}|$, which is a contradiction. Thus, S is infinite. This S however, cannot have a limit point in E. By triangle inequality, for any $y \in R^k$, $|x_n| \leq |x_n - y| + |y|$, and from archimedean property, $\exists_{m\in\mathbb{N}}$ s.t. $m > |x_n - y| + |y|$, which implies for any $y \in R^k, r > 0$, $\exists_{m\in\mathbb{N}} : |x-y| < r < m$. However, by the definition of S, there are at most m such elements in S. Since a limit point y of E must contain an infinite number of points of E such that $d(x, y) < r$ for any $r > 0$, y cannot be a limit point, which contradicts the assumption that any infinite subset of E contains a limit point in E. Therefore, E must be bounded.
To see that E is closed, suppose it were not closed. Then $\exists_{x_0\in} : E' \setminus E$. If T has no limit point in E except $x_0 \notin E$, it contradicts (c) because T is infinite and there must be a limit point of T in E.
Therefore, we can show that E is closed by showing that T has no limit point in E except x_0. Form an infinite sequence $(x_1, x_2, x_3, \ldots), x_n \in E$ with $|x_n - x_0| < \frac{1}{n}$. Let $y \in E$, $y \neq x_0$. We'll show that y cannot be a limit point of T. $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$. Choose $n \geq \frac{2}{|y-x_0|}$, so $\frac{1}{n} \leq \frac{|y-x_0|}{2}$. Then $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$. So only finitely many x_n can lie in $N_{\frac{1}{2}|y-x_0|}(y)$. So y cannot be a limit point of S. Therefore, E is closed.

Q.E.D.

Remark. (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than $\mathbb{R}^k$.

Example. *Failure of Heine-Borel theorem in general metric spaces.*

some infinite set , discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. *Then E is bounded* *and closed but not compact.*

Theorem 2.42. [Weirstrass's theorem] Every bounded infinite subset $E \subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Choose a k-cell $I \supset E$. Since I is compact, by theorem 2.41, E has a limit point in I. Q.E.D.

Example. *Let*

$$E_0 = [0, 1] \tag{2.5}$$

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{2.6}$$

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \tag{2.7}$$

$$\vdots \tag{2.8}$$

This gives $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \ldots$, where each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 2.13 (Perfect Sets). A set P is perfect if there is no isolated point in P; i.e.,

$$P = P'.$$

Theorem 2.43. Let P be a non-empty perfect set in R^k. Then P is uncountable.

Proof. Suppose for contradiction P is countable. Since P is non-empty, there exists some $p_1 \in P$. p_1 is then also a limit point of P. Let $p_2 \in P(\neq p_1)$ be a point in $V_1 = N_{r_1}(p_1)$ for some r_1 such that $d(p_1, p_2) > r_1/2$. Let $r_2 = r_1 - d(p_1, p_2)$, $V_2 = N_{r_2} p_2$. Then $\forall_{x \in V_2} : d(p_1, x) \leq d(p_1, p_2) + d(p_2, x) < d(p_1, p_2) + r_2 = r_1$. Hence, $V_2 \subset V_1$. $\overline{V_2} \subset V_1$ as well. Also, note that $d(p_1, p_2) > r_1/2$, so $r_2 = r_1 - d(p_1, p_2) < r_1/2 < d(p_1, p_2)$. So $p_1 \notin \overline{V_2}$. Repeat this process, and let $K_n = \overline{V_n} \cap P$. $K_n \subset \overline{V_n}$. Since $\overline{V_n}$ is closed and bounded, it's compact. $\overline{V_n} \cap P$ is a closed subset of $\overline{V_n}$, so K_n is also compact. However, for any p_n, $p_n \notin K_{n+1}$, so $\bigcap_{1 \in \infty} K_n \cap P = \emptyset$. Since $K_n \subset P$, this implies $\bigcap_{1 \in \infty} K_n = \emptyset$, but each K_n is not empty, $K_n \supset K_{n+1}$, and K_n is compact. Thus, $\bigcap_{1 \in \infty} K_n \cap P$ can't be empty, so this is a contradiction. Q.E.D.

Definition 2.14 (Cantor Set). The cantor set $P := \bigcap_n^\infty E_n$.

Proposition. P is compact, non-empty and contains no open intervals (a, b) and uncountable.

Proof. **Compactness** P is compact because $P \subset E_0 = [0, 1]$ and E_0 is compact.

Non-emptiness P is non-empty because $P \subset E_0$ and E_0 is non-empty.

No open intervals P contains no open intervals (a, b) because any (a, b) contains some $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ and these are all removed.

Uncountability P is uncountable because P is a perfect set. Equivalently, P consists of points in $[0, 1]$ whose ternary, i.e., base 3, representation contains only 0's and 2's.

> **Note.** ternary representation: $0.a_1a_2a_3 \ldots = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0, 1, 2\}$.

Q.E.D.

Example (Cantor Set)**.** *Let* $E = [0, 1]$*,* $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$*.* $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$*, etc. Keep removing open middle third. This gives* $E_0 \supset E_1 \supset E_2 \ldots$*. Each* E_n *is the union of* 2^n *closed intervals of length* $\frac{1}{3^n}$*.*

Definition 2.15 (Separated Sets)**.** **Separated Sets** $A, B \subset X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Connected Sets $E \subset X$ is connected if there is no non-empty separated sets $A, B \subset E$.

Example (Separated Sets)**.** *In* $\mathbb{R}^1$*,* $[0, 1)$ *and* $(1, 2]$ *are separated so* $[0, 1) \cup (1, 2]$ *is not connected. Every interval is connected (open, closed, semi-open).*

Theorem 2.47. $E \subset \mathbb{R}^1$ is connected if and only if E is an interval; i.e., $\forall_{x,y \in E, x<y}$ s.t. $\forall_{z \in (x,y)} : z \in E$

Proof. Let $x, y \in E$. Q.E.D.

Theorem 2.48. A metric space X is connected if and only if the only nonempty subset of X which is both open and closed is X itself. $| \limsup_{n \to \infty} |\gamma_n|| \leq 0 + \varepsilon\alpha$. Since ε is arbitrary, this implies $| \limsup_{n \to \infty} |\gamma_n|| = 0$, so $\lim\limits_{n \to \infty} |\gamma_n| = 0$.

Chapter 3

Sequence and Series

3.1 Sequences

Definition 3.1. In a metric space (X, d), a sequence $\{p_n\}$ converges to p if $\forall_{\varepsilon>0}\exists_N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.
We write $\lim\limits_{n\to\infty} p_n = p$ or $p_n \to p$.

If $\{p_n\}$ does not converge to any p then it is said to diverge.

Theorem 3.3. If s_n and t_n are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$, then the following hold:

(a) $s_n + t_n \to s + t$

(b) $cs_n \to cs$, $c + s_n \to c + s$ for any $c \in \mathbb{C}$

(c) $s_n t_n \to st$

(d) $\frac{1}{s_n} \to \frac{1}{s}$ if $s \neq 0$

Lemma (Squeeze Lemma)**.** In $\mathbb{R}$, if $\forall_{n\in\mathbb{N}} : 0 \leq x_n \leq s_n$ and $\lim\limits_{n\to\infty} s_n \to 0$, then $\lim\limits_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$. Then $0 \leq x_n \leq s_n < \varepsilon$ for $n \geq N$, so $x_n \to 0$. Q.E.D.

Theorem 3.20. (a) If $p > 0$ then $\frac{1}{n^p} \to 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $\frac{1}{N^p} < \varepsilon$; i.e., $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$. Then for $n \geq N$, $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$. Q.E.D.

(b) If $p > 0$ then $\sqrt[n]{p} \to 1$.

Proof. $p = 1$ is obvious.
Suppose $p > 1$. Let $x_n = \sqrt[n]{p} - 1 > 0$. Want to show $x_n \to 0$.
Since $(x_n+1)^n$, we have $p = (x_n+1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k > \binom{n}{1} x_n' = nx_n$.
Therefore, $x_n \leq \frac{p}{n}$, so $x_n \to 0$ by the Squeeze Lemma.
Suppose $p \in (0, 1)$. Let $q = \frac{1}{p} > 1$. Then $\sqrt[n]{q} \to 1$ by the previous case. By 3.3, $\sqrt[n]{p} = \frac{1}{\sqrt[n]{q}} \to 1$. Q.E.D.

(c) $\sqrt[n]{n} \to 1$

Proof. Let $x_n = \sqrt[n]{n} - 1 > 0$, for $n \geq 2$. $n = (x_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k} (x_n)^k > \binom{n}{2} {x_n}^2 = \frac{n(n-1)}{2} x_n^2$. Therefore, $x_n \leq \sqrt{\frac{2}{n-1}}$. Q.E.D.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \to 0$; i.e., Exponentials beat powers.

Proof. We want an upper bound on $\frac{n^\alpha}{(1+p)^n}$, so seek a lower bound on $(1 + p)^n$.
$(1 + p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k} p^k$ for $k \leq n$
$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$. Then for $k \leq \frac{n}{2}$, $\binom{n}{k} p^k > (\frac{n}{2})^k \frac{p^k}{k!}$. Therefore, $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$. Let $k_0 \in \mathbb{Z}$ s.t. $k > \alpha$. Then for $n \geq 2k_0$, RHS $\to 0$ by (a).

If $|x| < 1$ then $x^n \to 0$.

Proof. $|x^n - 0| = |x|^n$, so $x^n \to 0 \Leftrightarrow |x|^n \to 0$ and $|x|^n = \frac{n_0}{(\frac{1}{|x|})^n} \to 0$ by (d) with $\alpha = 0$ and $1 + p = \frac{1}{|x|} > 1$, so $p = \frac{1}{|x|} - 1 > 0$. Q.E.D.

Q.E.D.

Theorem 3.2. (a) $p_n \to p \Leftrightarrow \forall_{r>0} : N_r(p)$ contains all but finitely many p_n.

Proof. $\forall_{n \geq N} : p_n \in N_r(p)$ Q.E.D.

(b) If $p_n \to p$ and $p_n \to p'$ then $p = p'$.

Proof. $d(p, p') \leq d(p_n, p) + d(p_n, p')$ for all n. Fix ε. Choose N such that $d(p_n, p) < \frac{\varepsilon}{2}$ and $d(p_n, p') < \frac{\varepsilon}{2}$ for $n \geq N'$. Then $d(p, p') < \varepsilon$. Then for $n \geq \max\{N, N'\}$, $d(p, p') < \varepsilon$. This is true for all $\varepsilon > 0$, so $d(p, p') = 0$. Q.E.D.

(c) If $\{p_n\}$ converges, then p_n is bounded, in a sense that $\exists_{M>0, q \in X}$ s.t. $d(p_n, q) \leq M$ for all n.

Proof. If $p_n \to p$, then $\exists N$ s.t. $d(p_n, p) < 1$ for all $n \geq N$. Thus, $\forall_{n \geq 1} : d(p_n, p) \leq \max\{1, d(p, p_1), \ldots, d(p, p_{N-1})\} = M$ Q.E.D.

(d) If $E \subset X$ has a limit point p, then $\exists_{p_n \in E}$ s.t. $p_n \to p$.

Proof. We need to choose $p_n \in E$ s.t. $d(p, p_n) < \frac{1}{n}$. Let $\varepsilon > 0$. Then $d(p, p_n) < \varepsilon$ if $n > \frac{1}{\varepsilon}$ Q.E.D.

Definition 3.2. Given $p_n, n_1 < n_2 < n_3 < \ldots$, we say $p_{n_i} = (p_{n_1}, p_{n_2}, \ldots)$ is a subsequence of p_n.

Lemma. $p_n \to p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p

Proof. Look at assignment 6 Q.E.D.

Theorem 3.6. (a) $\{p_n\}$ in X, X compact, then $\exists$ convergent subsequence.

Proof. Let $E =$ range of$\{p_n\}$. If E is finite, then $\exists p \in X$ and $n_1 < n_2 < \ldots$ s.t. $p_n = p$ for $\forall i$. This subsequence converges to p. If E is infinite then by Theorem 2.37, E has a limit point $p \in X$; i.e., every neighborhood of p contains infinitely many points of E. Choose n_1 s.t. $d(p, p_{n_1}) < 1$.

Q.E.D.

(b) $\{p_n\}$ in $\mathbb{R}^k$, bounded, then $\exists$ convergent subsequence.

Proof. Choose a k-cell I that contains $\{p_n\}$. I is compact. Apply (a).

Q.E.D.

Definition 3.3 (Cauchy Sequence)**.** $\{p_n\}$ is a Cauchy sequence in (X, d) if $\forall \varepsilon : \exists_{N \in \mathbb{N}}$ s.t. $d(p_m, p_n) < \varepsilon \forall m, n \geq N$.

Definition 3.4. For $E \subset X$, $E \neq \emptyset$, we define diam $E = \sup\{d(p,q) : p, q \in E\}$. diam $E = \infty$ if the set is not bounded above.

Example. *For a sequence p_n in X, let $E_n = \{p_N, p_{N+1}, \ldots\}$. Then $\{p_n\}$ is a Cauchy sequence iff $\lim_{N\to\infty}$ diam $E_N = 0$.*

Theorem 3.11. (a) If $p_n \to p$ then $\{p_n\}$ is a Cauchy sequence.

(b) If X is a compact metric space and $\{p_n\}$ in X is a Cauchy sequence, then $\exists_{p\in X}$ s.t. $p_n \to p$.

(c) In R^K every Cauchy sequence converges.

Remark. If a Cauchy sequence has a convergent subsequence in a metric space, then the full sequence itself converges to the same point the subsequence converges to.

Proof. Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ is Cauchy. Let $E_N = \{p_N, P_{N+1}, \ldots\}$. Then $\overline{E_N}$ is closed, hence compact. Also $\overline{E_N} \supset \overline{E_{N+1}}$ and $\lim_{N\to\infty}$ diam $\overline{E_N} = 0$ (use Theorem 3.10(a) to see diam $\overline{E_N}$ = diam E_N) By theorem 3.10(b), $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$. Claim: $p_n \to p$.
Proof of the claim: Let $\varepsilon > 0$. Choose N_0 s.t.diam $\overline{E_{N_0}} < \varepsilon$, so $d(p,q) < \varepsilon \forall g \in \overline{E_{N_0}}$, and hence $\forall g \in N_0$; i.e., $d(p, p_n) < \varepsilon$ if $n \geq N_0$.

Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ in $\mathbb{R}^k$ is Cauchy. Cauchy sequences are bounded in any metric space. Therefore, $\exists$ k-cell I, which is compact, containing $\{p_n\}$. Then (b) applies Q.E.D.

Note. The converse of Theorem 3.11(a) does not hold in general.

Example. *$X = \mathbb{Q}$ has a Cauchy sequence with no limit in $\mathbb{Q}$. (see assignment 6). Converse does hold if X is compact.*

Theorem 3.12. (a) diam $\overline{E}$ = diam E

(b) If $K_n \subset X$, $K_n \neq \emptyset$, K compact, $K_n \supset K_{n+1} \forall n$ and if $\lim_{n\to\infty}$ diam $K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a single point.

Proof. (a) $E \subset \overline{E} \Rightarrow$ diam $E \leq$ diam $\overline{E}$. For the opposite inequality, let $\varepsilon > 0, p, q \in \overline{E}$,. Choose $p', q' \in E$ s.t. $d(p,p') < \varepsilon, d(q,q') < \varepsilon$. Then $d(p,q) \leq d(p,p') + d(p',q') + d(q',q) < \varepsilon + \varepsilon = 2\varepsilon$. diam $\overline{E} \leq$ diam $E + 2\varepsilon$. Since ε is arbitrary, diam $\overline{E} \leq$ diam E.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, $K \neq \emptyset$. Since $K \subset K_n \forall n$, diam $k \leq$ diam $K_n \forall n$, so diam $K = 0$. Therefore, $d(p,q) = 0 \forall p,q \in K$, so K is a simple point.

Q.E.D.

Definition 3.5 (Complete Metric Space). A metric space (X,d) is complete if every Cauchy sequence in X converges to a point in X.

Example. *(a) X compact $\Rightarrow X$ complete.*

(b) $\mathbb{R}^k$ is complete, so is $\mathbb{C}$.

(c) $\mathbb{Q}$ is not complete. (see assignment 6)

(d) See assignment 6 for reference to completion of a metric space.

We've seen that convergent sequences are bounded. $p_n = (-1)^n$ shows the converse if false. However the converse does hold for monotonic sequences.

Definition 3.6 (Monotone).
- A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically increasing if $s_n \leq s_{n+1} \forall n$.
- A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically decreasing if $s_n \geq s_{n+1} \forall n$.

Theorem 3.14. A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.

Proof. $\Rightarrow$ all convergent sequences are bounded in any metric space.

$\Leftarrow$ **Increasing case** Let $\{s_n\}$ be monotonically increasing and $s_n \leq M \forall n$. Let $s = \sup\{s_n : n \in \mathbb{N}\}$. Then $s_n \leq s \forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $s - \varepsilon < s_N \leq s$. But then $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \ldots \leq s$, so $|s - s_n| < \varepsilon \forall n \geq N$, and therefore $s_n \to s$.

Q.E.D.

Definition 3.7 (Infinite Limits). We say

- $s_n \to \infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \geq M \forall_{n \in N}$.
- $s_n \to -\infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \leq M \forall_{n \in N}$.

Definition 3.8. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define $\limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}} s_n = \inf_{n\geq 1}\{\sup_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \sup_{m\geq n}\{s_m\}$.
$\liminf_{n\to\infty} s_n = \underline{\lim_{n\to\infty}} s_n = \sup_{n\geq 1}\{\inf_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \inf_{m\geq n}\{s_m\}$.

Note. Alternate definition; see ass 7 for equivalence

Remark. (a) If $a_n \leq b_n \forall n$ and $a_n \to a$ and $b_n \to b$, then $a \leq b$.

(b) $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$

Example. *(a)* $s_n = (-1)^n(1+\frac{1}{n^2})$ $1 \leq \sup_{m\geq n} s_m \leq 1+\frac{1}{n^2}$, *so* $\limsup_{n\to\infty} s_n = 1$. *Similarly,* $\liminf_{n\to\infty} s_n = -1$

(b) If $\{s_n\}$ *has no upper bound, then* $\sup_{m\geq n} s_m = \infty$ *and in this case we say* $\limsup_{n\to\infty} s_n = \infty$*; e.g.,*

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has $\limsup_{n\to\infty} s_n = \infty$, $\liminf_{n\to\infty} s_n = -\infty$

Lemma. $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = L \Leftrightarrow s_n \to L$.

Proof (L finite)**.**

$\Rightarrow$ This follows from $\inf_{m\geq n} s_m \leq s_n \leq \sup_{m\geq n} s_m$. $\lim_{n\to\infty} \inf_{m\geq n} s_m = \liminf_{n\to\infty} s_n$, and $\lim_{n\to\infty} \sup_{m\geq n} s_m = \limsup_{n\to\infty} s_n$. Therefore, $\lim_{n\to\infty} s_n = L$.

$\Leftarrow$ If $s_n \to L$, then $\forall_{\varepsilon>0} : \exists_N$ s.t. $s_m \in [L-\varepsilon, L+\varepsilon] \forall m \geq N$. Therefore, $\forall_{n\geq N} : L-\varepsilon \leq \inf_{m\geq N} s_m \leq \inf_{m\geq n} s_m \leq \sup_{m\geq n} s_m \leq \sup_{m\geq N} s_m \leq L+\varepsilon$. Let $n \to \infty$: $L-\varepsilon \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L+\varepsilon$. Since ε is arbitrary, so $L \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L$.

Q.E.D.

3.2 Series

Definition 3.9 (Series)**.** Let $\{a_n\}$ be a sequence in $\mathbb{C}$. Form a new sequence $\{s_n\}$, the sequence of partial sums, by $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^n a_k$. If $s_n \to s$, we say **the series** $\sum_{k=1}^\infty a_k$ **converges** and that $\sum_{k=1}^\infty a_k = s$. If $\{s_n\}$ diverges then we say $\sum_{k=1}^\infty a_k$ diverges.

Theorem 3.15. $\sum_{n\in\mathbb{N}} a_n$ converges if and only if $\forall_{\varepsilon>0} : \exists N$ s.t. $\forall n \geq m \geq N$: $|\sum_{k=m}^{n} a_k| < \varepsilon$.

Proof. $\sum_n a_n$ converges $\Leftrightarrow$ $\{s_n\}$ converges $\Leftrightarrow$ $\{s_n\}$ is a Cauchy sequence ($\because$ $\mathbb{C}$ is compact). Use $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$. Q.E.D.

Corollary 3.16. If $\sum_n a_n$ converges then $a_n \to 0$.

Proof. Take $m = n$ in Theorem 3.22. $\sum_n a_n$ converges $\Rightarrow$ $\forall_{\varepsilon>0}$: $\exists_N$ s.t. $|a_n| < \varepsilon$ if $n \geq N$. Q.E.D.

Remark. n-th term test for divergence: If $a_n \not\to 0$ then $\sum_n a_n$ diverges.
Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *diverges because* $\frac{n}{n+1} \to 1 \neq 0$.

Converse to Corollary 3.16 is false! E.g., $\sum_n \frac{1}{n}$ diverges but $\frac{1}{n} \to 0$.

Theorem 3.24. If $a_n \geq 0$, then $\sum_n a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof. $\{s_n\}$ is monotone increasing, so by Theorem 3.14, it converges if and only if it is bounded. Q.E.D.

Theorem 3.25. [Comparison Test]

(a) If $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges, then $\sum_n a_n$ converges.

Proof. Suppose $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges. Let $\varepsilon > 0$. By theorem 3.22, $\exists N$ s.t. $\sum_{k=m}^{n} c_k < \varepsilon$ if $n \geq m \geq N$. Can take $N \geq N_0$. Then $|N \geq N_0|$. $|\sum_{k=m}^{n} a_k| \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} c_k < \varepsilon$ if $n \geq m \geq N$. By theorem 3.22 again, $\sum_n a_n$ converges. Q.E.D.

(b) If $a_n \geq d_n \geq 0 \forall n \geq N_0$ and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.

Proof. This follows from (a): if $\sum_n a_n$ converges then $\sum_n d_n$ converges. Thus it's contrapositive, (b) is true. Q.E.D.

Theorem 3.26. [Geometric Series] $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$, $xS_n = x + x^2 + \cdots x^n + x^{n+1}$. Then
$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$
If $|x| < 1 (\Leftrightarrow -1 < x < 1)$, then $x^{n+1} \to 0$ and $S_n \to \frac{1}{1-x}$. If $|x| \geq 1$, then x^{n+1} does not converge to 0, so $\sum_{n=0}^{\infty} x^n$ diverges. Q.E.D.

Theorem 3.27. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof. ($\Leftarrow$) We show that if $\sum_n a_n$ diverges, then $\sum_k 2^k a_{2^k}$ diverges. For this, note that $a_1 + a_2 + \ldots + a_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$ if $2^{k+1} > n$. $a_1 + a_2 \cdots + a_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. LHS unbounded as $n \to \infty$, so RHS is also unbounded as $k \to \infty$.

($\Rightarrow$) $a_1 + a_2 + a_3 + \cdots + a_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$ if $2^k \leq n$. $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$. If $\sum_n a_n$ converges, then LHS is bounded for all n so RHS is bounded for all k. Hence RHS converges since it is monotone.

Q.E.D.

Theorem 3.28. [p-series] $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. For $p \leq 0$, $\frac{1}{n^p} \not\to 0$, so series diverges. For $p > 0$, $\frac{1}{n^p}$ is decreasing, so $\sum_{n=1} \frac{1}{n^p}$ converges iff $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$ converges. But $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$ converges iff $\frac{1}{2^{p-1}} < 1 (\Leftrightarrow p - 1 > 0)$, which is equivalent to $p > 1$. Q.E.D.

Theorem 3.29. $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (log is to base e.)

Proof. If $p \leq 0$, then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$, so $\sum_n \frac{1}{n(\log n)^p}$ diverges by the comparison test. If $p > 0$ then $\frac{1}{n(\log n)^p}$ decreases since $\log n$ increases. By theorem 3.27, $\sum_n \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k (\log 2^k)^p}$ converges $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$ Q.E.D.

Definition 3.10 (e). $e := \sum_{n=0}^{\infty} \frac{1}{n!}$.

Remark. **Convergence** $\frac{1}{n!} = \frac{1}{(n(n-1)(n-2)\cdots 3\cdot 2\cdot 1)} \leq \frac{1}{2\cdot 2\cdot 2\cdot 2\cdots 2\cdot 1} = \frac{1}{2^{n-1}}$.
Therefore, $S_n = \sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1-1/2} = 3$. Then S_n is a monotonically increasing sequence that's also bounded. Hence, $e \leq 3$

Rate of Convergence

$0 < e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n+1}} \cdot \frac{1}{(n+1)!}$

$= \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-(n+1)}}$

$= \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+1)} = \frac{1}{(n!)\cdot n}$.

Theorem 3.32. $e \notin \mathbb{Q}$.

Proof. For contradiction, suppose $e = \frac{p}{q}, p, q \in \mathbb{N}$. As $0 < e - S_q < \frac{1}{q\cdot q!}$, $0 < q!\cdot e - q!\cdot S_q < \frac{1}{q}$. Since $S_q = \sum_{k=0}^{q} \frac{1}{k!}$, $q!\cdot e$ and $S_q \cdot q!$ are both integers. However, then $q!\cdot e - q!\cdot S_q$ is an integer between 0 and $\frac{1}{q} < 1$, which is a contradiction. Q.E.D.

Theorem 3.31. $e = \lim\limits_{n\to\infty} (1 + \frac{1}{n})^n$.

Proof. Let $t_n = (1+\frac{1}{n})^n$. Then $t_n = \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot (\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}) \leq S_n$. So $\limsup_{n\to\infty} t_n \leq \limsup_{n\to\infty} S_n = \lim\limits_{n\to\infty} S_n = e$. On the other hand, for fixed m and $n \geq m$, $t_n \geq \sum_{k=0}^{m} \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^{m} \frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots\cdot(1-\frac{k-1}{n})$. Let $n \to \infty$ with m fixed. $\liminf_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!}\cdot 1 = S_m$. This is true for any m. Now let $m \to \infty$. $\liminf_{n\to\infty} t_n \geq \limsup_{m\to\infty} s_m = e$. $e \leq \liminf_{n\to\infty} t_n \leq \limsup_{n\to\infty} t_n \leq e$. Therefore, $\lim\limits_{n\to\infty} t_n$ exists and equals e. Q.E.D.

Theorem 3.33. [Root test] Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then,

$$\sum a_n \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha > 1 \\ \text{inconclusive} & \text{if } \alpha = 1 \end{cases}$$

Proof (Just outline). $\alpha < \beta < 1$ Eventually $|a_n| \leq \beta^n$, thus convergence.

$\alpha > 1$ $|a_n| > 1$ for infinitely many n, thus divergence.

$\alpha = 1$ $\frac{1}{n}$ diverges, $\frac{1}{n^2}$ converges.

Q.E.D.

Theorem 3.34. [Ratio test] The series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ converges if $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\exists_{N\in\mathbb{N}}$ s.t. $\forall_{n\geq N} : |\frac{a_{n+1}}{a_n}| \geq 1$. Otherwise, inconclusive.

Proof. (see textbook).

Convergence $\sum a_n \begin{cases} \text{converges} & \text{if } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1 \\ \text{diverges} & \text{if } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \\ \text{diverges} & \text{if } \liminf_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1 \\ \text{inconclusive} & \text{otherwise. e.g., } \lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1 \end{cases}$

Note. Note that we cannot replace $\liminf$ with $\limsup$ in the third case. For inconclusive case, check $\sum 1/n \to \infty$ and $\sum 1/n^2 \to \pi^2/6$

Q.E.D.

Example. *Let* $s_n = \sum_{n=0}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$. *First, note that* $a_{2k} = \frac{1}{2} \cdot \frac{1}{4^k}, a_{2k+1} = 2a_{2k} = \frac{1}{4^k}$ *for* $k \geq 0$.

Ratio test *Then the ratio* $\frac{a_{n+1}}{a_n}$ *is the sequence* $2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \ldots$. *Therefore,* $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}, \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$. *The ratio test is inconclusive for* s_n.

Root test $a_n = \begin{cases} \frac{1}{2} \cdot \frac{1}{2^n} & n \text{ even} \\ \frac{2}{2^n} & n \text{ odd} \end{cases}$, *so* $(\frac{1}{2})^{1/n} \cdot \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{2^{1/n}}{2}$, *thus* $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$. *Therefore,* $s_n = \sum_{n=0}^{\infty} a_n$ *converges.*

This is an example where the ratio test is inconclusive but the root test is conclusive.

Theorem 3.47. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum a_n + b_n = A + B$ and $\sum c \cdot a_n = cA$.

3.3 Power Series

Definition 3.11 (Power Series)**.** For $z \in \mathbb{C}$ and a complex sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n z^n$ is a power series.

Remark. As $z^0 = 1$ for all $z \in \mathbb{C}$, by convention we write $\sum_{n=0}^{\infty} c_n z^n = c_0 + \sum_{n=1}^{\infty} c_n z^n$.

Theorem 3.39. Let $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$, where

$$R = \begin{cases} 0 & \text{if } \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty \\ \infty & \text{if } \limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0 \end{cases}.$$

Then $\sum c_n z^n \begin{cases} \text{converges} & \text{if } |z| < R \\ \text{diverges} & \text{if } |z| > R \\ \text{inconclusive} & \text{if } |z| = R \end{cases}$. Note $R = 0$ implies the series diverges for $z \neq 0$, and $R = \infty$ implies the series converges for any $z \in \mathbb{C}$.

Proof. $\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$. By root test, the series converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$.

Note. In practice, often use the ratio test to find R.

Q.E.D.

Example. *(a)* $\sum n! \cdot z^n$ *has* $R = 0$.

By ratio test $\forall z \neq 0 : \left|\frac{(n+1)! \cdot z^{n+1}}{n! \cdot z^n}\right| = |z| \cdot (n+1) \to \infty$. *Hence, the series diverges .*

By root test *Note* $n \neq \frac{1}{2}(\frac{2}{3})^2(\frac{3}{4})^3 \cdots (\frac{n-1}{n})^{n-1} n^n$ *for* $n \geq 2$. *Then* $n \neq \frac{n^n}{(1+1)^1(1+\frac{1}{2})^2(1+\frac{1}{n-1})^{n-1}}$. *In the proof of Theorem 3.31, we saw* $(1 + \frac{1}{j}) \leq e$. *So* $n! \geq \frac{n^n}{e^{n-1}} = e \cdot (\frac{n}{e})^n$. $\sqrt[n]{n!} \geq e^{1/n} \cdot \frac{n}{e} \to \infty$ *as* $n \to \infty$. *Therefore,* $R = \frac{1}{\infty} = 0$.

Note. Cf. Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Definition 3.12 (Absolute Convergence, Conditional Convergence)**.**

(a) $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.

(b) $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Remark. All other convergence tests seen so far are actually tests for absolute convergence.

Example.
- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ *converges conditionally by the alternating series test of assignment 7's practice problem 1. See also Theorem 3.43. (MUST TAKE A LOOK!)*
- *Given* $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ *and* $a_n \to 0$, *then* $\sum(-1)^n a_n$ *converges.*
- $\sum_{n=0}^{\infty} n!2^n$ *has* $R = 0$
- $\sum_{n=0}^{\infty}\frac{z^n}{n^n}$ *has* $R = \infty$ *since* $R = \frac{1}{\limsup_{n\to\infty}\sqrt[n]{|c_n|}} = \frac{1}{\limsup_{n\to\infty}\frac{1}{n}} = 1/0 = \infty$, *or use ratio test,* $|\frac{z^{n+1}/(n+1)^{n+1}}{z^n/n^n}| = |z| = \frac{n^n}{(n+1)^{n+1}} = |z|\frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n}$. *Since* $(1+\frac{1}{n})^n \to \frac{1}{e}$ *as* $n \to \infty$, $|z|\frac{1}{n+1}\frac{1}{(1+\frac{1}{n})^n} \to 0$ *as* $n \to \infty \forall_{z\in\mathbb{C}}$ *so* $R = \infty$.

Theorem 3.45. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Theorem 3.54. Suppose $\sum a_n$ converges conditionally. Let $-\infty \leq \alpha \leq \beta \leq +\infty$. Then $\exists$bijection $f : \mathbb{N} \to \mathbb{N}$ such that with $a'_n = a_{f(n)}$ and $S'_n = \sum_{k=1}^{n} a'_k$, $\liminf_{n\to\infty} s'_n = \alpha$ and $\limsup_{n\to\infty} s'_n = \beta$. In other words, there exists a rearrangement of $\sum a_n$, say $\sum a'_n$, such that $\liminf_{n\to\infty}\sum a'_n = \alpha, \limsup_{n\to\infty}\sum a'_n = \beta$.

Proof. Take a look at the textbook Q.E.D.

Theorem 3.55. If $\sum a_n$ converges absolutely, then every rearrangement of $\sum a_n$ converges to the same sum.

Proof. Take a look at the textbook Q.E.D.

3.4 Products of Series

Motivation Consider z^N in $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n)$. Since $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n) = (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots$, z^N has coefficient $\sum_{k=0}^{N} a_k b_{N-k}$.

Definition 3.13. The product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is $\sum_{n=0}^{\infty} c_n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Note. This is a discrete convolution.

Question If $\sum a_n = A$ and $\sum b_n = B$ both converge, does $\sum c_n$ converge and if so, does it converge to AB?

Answer $\sum c_n$ converges if $\sum a_n$ and $\sum b_n$ converge absolutely. (Theorem 3.50). Moreover, if $\sum c_n$ does converge, then it must converge to AB (Theorem 3.51). Maybe no otherwise (ref: Example 3.49).

Theorem 3.50. Suppose $\sum a_n$ converges absolutely to A and $\sum b_n$ converges to B. Then $\sum c_n$ converges to AB.

Proof. Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$. Then $A_n \to A, B_n \to B$. By definition, $C_n = \sum_{k=0}^n \sum_{j=0}^k a_j b_{k-j} = \sum_{j=0}^n \sum_{k=j}^n a_j b_{k-j} = \sum_{j=0}^n a_j \sum_{k=j}^n b_{k-j} = \sum_{j=0}^n a_j B_{n-j} = \sum_{j=0}^n a_j B + \sum_{j=0}^n a_j(B_{n-j} - B)$. Let β_{n-j}, where $\beta_k = B_k - B$. Then $C_n = A_n B + \sum_{j=0}^n a_j \beta_{n-j}$. Let $\gamma_n = \sum_{j=0}^n a_j \beta_{n-j}$. Note that $A_n B \to AB, \beta_k \to 0$ as $n \to \infty$. Let $\alpha = \sum_{k=0}^{\infty} |a_k| < \infty$ ($\because a_n$ converges absolutely by assumption). Rewrite γ_n as $\gamma_n = \sum_{j=0}^n a_{n-j}\beta_j$. We know $\beta_j \to 0$ as $j \to \infty$. Let $\varepsilon > 0$. Choose N s.t. $|\beta_j| < \varepsilon$ if $j \geq N$. Then for $n \geq N+1$, $|\gamma_n| \leq |\sum_{j=0}^N a_{n-j}\beta_j| + |\sum_{j=N+1}^n a_{n-j}\beta_j|$. Note $|\sum_{j=N+1}^n a_{n-j}\beta_j| \leq \varepsilon \sum_{j=N+1}^n |a_{n-j}| \leq \varepsilon\alpha$. Let $n \to \infty$ with N fixed. Then $a_{n-j} \to 0$ for $0 \leq j \leq N$ since $|a_n| \to 0$. Q.E.D.

Theorem 3.51. If the series $\Sigma a_n, \Sigma b_n, \Sigma c_n$ converge to A, B, C respectively and $c_n = \sum_{k=0}^n a_k b_{n-k}$, then $C = AB$.

Midterm 2 covers up to this point (TB p.36-82, A.5-8)

Chapter 4

Continuity

Assume general metric spaces X, Y and $f : X \to Y$.

Definition 4.1 (4.1)**.** Suppose X, Y are metric spaces, $E \subset X$, $f : E \to Y$, $p \in E'$, where E': set of limit points in metric space X. We say $\lim_{x \to p} f(x) = q$, or $f(x) \to q$ as $x \to p$, if $\forall_{\varepsilon > 0} : \exists_{\delta > 0}$ s.t. $(0 < d_X(x, p) < \delta$ and $x \in E) \Rightarrow d_Y(f(x), q) < \varepsilon$.

Note. We don't say anything about $x = p$, $f(p)$ may not even be defined.

Theorem 4.2. $\lim_{x \to p} f(x) = q \Leftrightarrow \forall_{\{p_n\} \text{ in } E} :$ if $p_n = p$ or $p_n \to p$, then

$$\lim_{n \to \infty} f(p_n) = q,$$

where the RHS is the limit of Definition 3.1.

Note. This implies uniqueness of q in Definition 4.1.

Proof. $\Rightarrow$ Suppose $\lim_{x\to p} f(x) = q$. Let $\varepsilon > 0$. Choose $\delta > 0$ s.t. $d_Y(f(x), q)\varepsilon$ if $0 < d_X(x,p) < \delta$. Let $\{p_n\}$ be a sequence in E such that $p_n \to p$ and $p_n \neq p$. Then $\exists_N$ s.t. $0 < d_X(p_n, p) < \delta$ if $n \geq N$; i.e., $f(p_n) \to q$.

$\Leftarrow$ Consider the contrapositive of ($\Leftarrow$): $\neg(\lim_{x\to p} f(x) = q) \Rightarrow \neg(\forall_{\{p_n\} \text{ in } E} : \lim_{n\to\infty} f(p_n) = q)$. Suppose $\neg(\lim_{x\to p} f(x) = q)$. Then $\exists_{\varepsilon>0}$ s.t. $\forall_{\delta>0}$: $\exists_{x\in N_\delta^E(p)}$ s.t. $x \neq p$ and $d_Y(f(x), q) \geq \varepsilon$. Take $\delta = \delta_n = \frac{1}{n}$ and let p_n be an x as above for δ_n. Then $p_n \to p$, but $d_Y(f(p_n), q) \geq \varepsilon \forall n$, so $f(p_n) \not\to q$.

Q.E.D.

Theorem 4.4. When $Y = \mathbb{C}$, limit as defined in Definition 4.1 respects sums, products and quotients.

Proof. By Theorem 4.2, it suffices to show that the theorem holds for sequences. Q.E.D.

Definition 4.2. Suppose X, Y are metric spaces, $p \in E \subset X$, $f : E \to Y$. Then f is continuous at p if $\forall_{\varepsilon>0} : \exists_{\delta>0}$ s.t. $d_X(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon$; i.e., $f(N_\delta^E(p)) \subset N_\varepsilon^y(f(p))$. We say f is continuous if f is continuous at p for all $p \in E$.

Note. If p is an isolated point; i.e., $\exists_{\delta>0}$ s.t. $N_\delta^E(p) = \{p\}$, then every $f : E \to Y$ is continuous at p.

Theorem 4.6. Suppose $E \subset X, p \in E \cap E', f : E \to Y$. Then f is continuous at p if and only if $\lim_{x\to p} f(x) = f(p)$.

Proof. By Definition 4.1 and Definition 4.2 with $q = f(p)$. Q.E.D.

Theorem 4.7. For $E \subset X, f : E \to Y, g : f(E) \to Z$, let $h = g \circ f : E \to Z$. If f is continuous at $p \in E$ and if g is continuous at $f(p) \in Y$, then h is continuous at p.

Proof. Choose $\eta > 0$ such that $d_Y(f(p), y) < \eta \Rightarrow d_Z(g(f(p)), g(y)) < \varepsilon$ (continuity of g at $f(p)$). Choose $\delta > 0$ s.t. $d_E(x,p) < \delta \Rightarrow d_Y(f(x), f(p)) < \eta$ (continuity of f at p). Then $d_E(x,p) < \delta \Rightarrow d_Z(g(f(x)), g(f(p))) = d_Z(h(x), h(p)) < \varepsilon$. Q.E.D.

Theorem 4.8. [Topological Characterization of Continuity] $f : X \to Y$ is continuous $\Leftrightarrow$ $f^{-1}(V)$ is open for every open $V \subset Y$.

Proof. ($\Rightarrow$) Suppose f is continuous. Let $V \subset Y$ be open. Then $f^{-1}(V)$ is open. Let $p \in f^{-1}(V)$. We need to show $\exists_{\delta>0}$ s.t. $N_\delta^X(p) \subset f^{-1}(V)$. Since V is open, $\exists_{\varepsilon>0}$ s.t. $N_\varepsilon^Y(f(p)) \subset V$. Since f is continuous, $\exists_{\delta>0}$ s.t. $f(N_\delta^X(p)) \subset N_\varepsilon^Y(f(p)) \subset V$.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open for every open $V \subset Y$. Let $p \in X$ and $\varepsilon > 0$. Then $N_\varepsilon^Y(f(p))$ is open, so $f^{-1}(N_\varepsilon^Y(f(p)))$ is open. Take $V = N_\varepsilon^Y(f(p))$, which is open. Since $f^{-1}(V)$ is open and $p \in f^{-1}(V)$, there exists $\delta > 0$ such that $N_\delta^X(p) \subset f^{-1}(V)$. Then $f(N_\delta^X(p)) \subset V = N_\varepsilon^Y(f(p))$; i.e., f is continuous at p.

Q.E.D.

Remark.

(a)

(b) Continuity is determined by the open sets, not the metric. For instance, if metrics l_1, l_2, l_∞ have the same open sets in $\mathbb{R}^k$, hence the same continuous functions.

$$l_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|$$

$$l_2(x, y) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

$$l_\infty(x, y) = \max_{1 \le i \le k} |x_i - y_i|$$

(c) f with open $U \subset X \Rightarrow f(U)$ is open are called open maps. Continuous maps need not be open(e.g., $f(x) =$ some constant, $f(x) = x^2$), and open maps need not be continuous(e.g., floor function: $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$).

Corollary 4.9. $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed for every closed $F \subset Y$.

Proof. Let $V \subset Y$ be open and $F = V^c$. Then the above condition (RHS) is the same as $f^{-1}(V) = f^{-1}(F^c) = (f^{-1}(F))^c$ is open. Q.E.D.

Theorem 4.9. Let $f : X \to \mathbb{C}, g : X \to \mathbb{C}$ be continuous. Then $f+g, f \cdot g, f/g$(at points p where $g(p) \neq 0$) are also continuous.

Theorem 4.10. Given $f_i : X \to \mathbb{R}(i = 1, 2, \ldots, k)$, define $f : X \to \mathbb{R}^k$ by $f(x) = (f_1(x), \ldots, f_k(x))$. Then

(a) f is continuous if and only if each f_i is continuous.

(b) if $f, g : X \to \mathbb{R}^k$ are continuous, then so are $f + g : X \to \mathbb{R}^k, f \cdot g : X \to \mathbb{R}^1$

Example. *(a) For $i = 1, \ldots, k$, define $\varphi_i : \mathbb{R}^k \to \mathbb{R}$ by $\varphi_i(x) = x_i$, where $x = (x_1, x_2, \ldots, x_k)$. Then $|\varphi_i(x) - \varphi_i(y)| = |x_i - y_i| \leq \left(\sum_{j=1}^{k} |x_j - y_j|^2\right)^{1/2} = |x - y|$, so φ_i is continuous(take $\delta = \varepsilon$. If $|x - y| < \delta = \varepsilon$, then $|\varphi_i(x) - \varphi_i(y)| < \varepsilon$).*

(b) The functions $\mathbb{R}^k \to \mathbb{R}$ defined by $x \mapsto x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} (n_i \in \{0, 1, 2, \ldots\})$ is continuous on $\mathbb{R}^k$ and so is any polynomial $P(x) = \sum C_{n_1, n_2, n_3, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $C_{n_1, n_2, n_3, \ldots, n_k}$ is a constant (function) in $\mathbb{C}$.

(c) Rational functions $f(x) = \frac{P(x)}{Q(x)}$ are continuous at points where $Q(x) \neq 0$.

(d) The function $\mathbb{R}^k \to \mathbb{R}$ defined by $x \mapsto |x|$ is continuous.

Proof. $|x| = |y + (x - y)| \leq |y| + |x - y|$, so $|x| - |y| \leq |x - y|$. Similarly, $|y| - |x| \leq |y - x|$, so $||x| - |y|| \leq |x - y|$. Thus by taking $\delta = \varepsilon$, $|x - y| < \delta \Rightarrow ||x| - |y|| < \varepsilon$. Q.E.D.

(e) Suppose $f : X \to \mathbb{R}^k$ is continuous. Then $p \mapsto |f(p)|$ is continuous.

Proof. $p \mapsto |f(p)| = (y \mapsto |y|) \circ (p \mapsto f(p))$. Since both $(y \mapsto |y|)$, $(p \mapsto f(p))$ are continuous, $p \mapsto |f(p)|$ is continuous by Theorem 4.7. Q.E.D.

Note. A function is said to be continuous on the *domain*, not on the *range*.

Theorem 4.14. Let $f : X \to Y$ be continuous and X be compact. Then $f(X)$ is compact.

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$. We need to find a finite subcover of $f(X)$. By Theorem 4.8, each set $O_\alpha = f^{-1}(V_\alpha)$ is open and $\bigcup_\alpha O_\alpha = \bigcup_\alpha f^{-1}(V_\alpha) = f^{-1}(\bigcup_\alpha V_\alpha) = f^{-1}(f(X)) = X$. Hence, $\{O_\alpha\}$ is an open cover of X, so there exists a finite subcover $X = O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. However, then $f(x) = \bigcup_{i=1}^{n} f(O_{\alpha_i}) = \bigcup_{i=1}^{n} f(f^{-1}(V_{\alpha_i})) \subset \bigcup_{i=1}^{n} V_{\alpha_i}$. Therefore, $\{V_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $f(X)$. Q.E.D.

Definition 4.3 (4.13)**.** $f : E \to \mathbb{R}^k$ is bounded if $\exists_{M>0}$ s.t. $|f(x)| \le M \, \forall x \in E$.

Theorem 4.15. If X is compact and $f : X \to \mathbb{R}^k$, then $f(X)$ is closed and bounded (so f is bounded).

Proof. $f(X)$ is compact by Theorem 4.14, and since $f(X) \subset \mathbb{R}^k$, it is closed and bounded. Q.E.D.

Theorem 4.16. If X is compact and $f : X \to \mathbb{R}^1$ is continuous, then $\exists_{p,q\in X}$ s.t. $f(p) \le f(x) \le f(q)$ for all $x \in X$.

Proof. By Theorem 4.15, $f(X)$ is closed and bounded. By Theorem 2.28, $M \in f(X)$ and similarly $m \in f(X)$. Q.E.D.

Example. *Let $X = (0,1)$, not compact, let $f(x) = \frac{1}{x}$, continuous. However, $\nexists_{p\in X}$ s.t. $\forall_{x\in X} : f(p) \le f(x)$ and $\nexists_{q\in X}$ s.t. $\forall_{x\in X} : f(x) \le f(q)$.*

Theorem 4.17. Suppose $f : X \to Y$ is one-to-one, onto, continuous, where X is compact. Define $f^{-1} : Y \to X$ by $f^{-1}(f(x)) = x$. Then f^{-1} is continuous.

Proof. By Theorem 4.8, it suffices to prove that if $V \subset X$ is open then $(f^{-1})^{-1}(v)(= f(v))$ is open. However, $V^c \subset X$ is closed, hence V^c is compact by Theorem 4.14 and $(f(V^c))^c = f(V)$ is open. Q.E.D.

Example (Compactness is needed in Theorem 4.17)**.** *Let $X = [0, 2\pi), Y = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Define $f : X \to Y$ by $f(\theta) = (\cos\theta, \sin\theta)$. This f is 1-1, onto, and continuous, but f^{-1} is not continuous as X is not compact.*

Proof. (1) $[0, 1) \subset X$ is open but $(f^{-1})^{-1}([0, 1)) = f([0, 1))$ is not open because $(1, 0)$ is not an interior point of Y.

(2) In Y, as $(x, y) \to (1, 0)$ from above, $f((x, y)) \to 0$. As $(x, y) \to (1, 0)$ from below, $\lim f^{-1}(x, y)$ does not exist in X. (Wants to be $2\pi \notin X$), so f^{-1} is not continuous at $(1, 0) \in Y$.

Q.E.D.

Definition 4.18. Let X, Y be metric spaces and $f : X \to Y$. f is uniformly continuous on X if $\forall_{\varepsilon>0} : \exists_{\delta>0}$ s.t. for all $p, q \in X$ with $d_x(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$.

Remark. The point is for any ε, there is some δ that works for every $p, q \in X$ such that $d(p, q)$.

Example. *(a) $X = (0, 1), Y = \mathbb{R}, f(x) = \frac{1}{x}$. f is continuous on X but is not* *uniformly continuous.*

> **Proof.** For $x \in (0, \frac{1}{2})$, $|f(x) - f(2x)| = |\frac{1}{x} - \frac{1}{2x}| = \frac{1}{2x} \to \infty$ as $x \to 0$. Then for $\varepsilon = 1$, given any $\delta \in (0, \frac{1}{2})$, we can pick $x < \delta$ s.t. $d_(X)(x, 2x) = x < \delta$, but $d_Y(f(x), f(2x)) = \frac{1}{2x} > \frac{1}{2\delta} > 1$. Q.E.D.

(b) $X = [0, 5], Y = \mathbb{R}, f(x) = x^2$ *is uniformly continuous.*

> **Proof.** For $0 \le x_1 \le x_2 \le 5$ and $\varepsilon > 0$, $|f(x_1) - f(x_2)| = x_2^2 - x_1^2 = (x_2 - x_1)(x_2 + x_1) \le 10 \cdot (x_2 - x_1)$, which is less than ε if $|x_2 - x_1| < \frac{\varepsilon}{10} = \delta$ Q.E.D.

Theorem 4.19. Suppose X is a compact metric space, Y is a metric space, and $f : X \to Y$ is continuous. Then f is uniformly continuous.

> **Proof.** Fix $\varepsilon > 0$. For $p \in X$ there exists $\delta = \delta_p(\varepsilon)$ s.t. $d_X(p, q) < \delta_p \Rightarrow d_Y(f(p), f(q)) < \frac{\varepsilon}{2}$. We need to remove the p-dependence of δ_p. Let $J_p = N_{\frac{1}{2}\delta_p}(p)$. Then $\{J_p\}_{p \in X}$ is an open cover of X. Then there exists subcover $X = J_{p_1} \cup J_{p_2} \cdots \cup J_{p_n}$ (equality works as X is the whole metric space, so $X \subset J \Rightarrow X = J$). Let $\delta = \min\{\frac{1}{2}\delta_{p_1}, \frac{1}{2}\delta_{p_2}, \ldots, \frac{1}{2}\delta_{p_n}\}$. Suppose p, q with $d_X(p, q) < \delta$. Choose $m \in \{1, 2, \ldots n\}$ s.t. $p \in J_{p_m}$. Then $d_X(p, p_n) < \frac{1}{2}\delta_{p_m}$. $d_X(q, p_m) \le d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\delta p_m \le \delta p_m$. $\therefore d_Y(f(q), f(p)) \le d_Y(f(q), f(p_m)) + d_Y(f(q_m), f(p)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Q.E.D.

Theorem 4.22. If X, Y are metric spaces, $f : X \to Y$ is continuous, and $E \subset X$ is connected, then $f(E)$ is connected.

> **Proof** (By Contradiction)**.** Suppose for contradiction E is connected and there exists $A, B \subset Y$ s.t. $f(E) = A \cap B$, $f(E) \neq \emptyset$, $\overline{A} \cup B = A \cap \overline{B} = \emptyset$. Let $G = f^{-1}(A) \cap E, H = f^{-1}(B) \cap E$. Then $E = G \cup H$, G, H are nonempty. If $G \cap \overline{H} = \overline{G} \cap H = \emptyset$, it leads to a contradiction. First, $G \subset f^{-1}(A) \subset (\because A \subset \overline{A}) f^{-1}(\overline{A})$, where $f^{-1}(\overline{A})$ is closed by the corollary to Theorem 4.8, so $\overline{G} \subset f^{-1}(\overline{A})$. Second, $f(H) = B, \overline{A} \cap B = \emptyset$. Therefore, $\overline{G} \cap H = \emptyset$. WLOG, $G \cap \overline{H} = \emptyset$ as well. Hence a contradiction. Q.E.D.

Theorem 4.23. [Intermediate Value Theorem] Suppose $f : [a, b] \to \mathbb{R}$ is continuous. $\forall_{c \in (\min\{f(a), f(b)\}, \max\{f(a), f(b)\})} : \exists_{x_0 \in (a,b)}$ s.t. $f(x_0) = c$.

> **Proof.** $[a, b]$ is connected by Theorem 2.47. Hence, by Theorem 4.22, $f([a, b])$ is connected and therefore contains all points between $f(a)$ and $f(b)$. In particular, $c \in f((a, b))$ Q.E.D.

Example. *(a) there exists a continuous function called (Peano/space-filling* *curve) from* $[0,1]$ *onto the closed unit square* $S = [0,1] \times [0,1] \subset \mathbb{R}^2$.

> **Proof.** Omitted. See Rudin's problem 7.14 for an explicit example(covered in MATH-321). Q.E.D.

(b) But no such function can be one-to-one.

> **Proof.** Suppose $f : [0,1] \to S$ is 1-1, onto, continuous. Since $[0,1]$ is compact, f^{-1} is continuous by Theorem 4.17. Let $E = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then, $f(E) = S \setminus \{f(\frac{1}{2})\}$ is S minus one point, which is connected (pf omitted). But then, $f^{-1}(f(E)) = E$ must be connected by Theorem 4.22. E is not connected, so this is a contradiction. Q.E.D.

Example (18)**.** *x every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}^1$ by*

$$f(x) = \begin{cases} 0 & (x irrational) \\ \dfrac{1}{n} & (x = \dfrac{m}{n}) \end{cases}.$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point, and that f has a simple discontinuity at every rational point.

Chapter 5

Differentiation

We consider $f : [a, b] \to \mathbb{R}$.

Definition. For $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, let $f'(x) = \lim\limits_{t \to x} \frac{f(t)-f(x)}{t-x}$ if limit exists. Equivalently, $f(t) = f(x) + (t - x)[f'(x) + u(x, t)]$ with $\lim\limits_{t \to x} u(x, t) = 0$.

Example. *(a)* $f(x) = c$ *for all* $x \Rightarrow f'(x) = \lim\limits_{t \to x} \frac{c-c}{t-x} = 0$.

(b) $f(x) = x$ *for all* $x \Rightarrow f'(x) = \lim\limits_{t \to x} \frac{t-x}{t-x}$

(c) $f(x) = e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$. *Write* $t = x + h$, *so* $t \to x \Leftrightarrow h \to 0$. $\frac{e^{x+h}-e^x}{(x+h)-x} = e^x \frac{e^h-1}{h} = e^x \frac{e^h-1}{h} + e^x - e^x = e^x + e^x \frac{e^h-1-h}{h}$. *Let* $u(h) = \frac{e^h-1-h}{h}$. *Then* $u(h) = \frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}$, *so* $|u(h)| = |\frac{1}{h} \sum_{n=2}^{\infty} \frac{h^n}{n!}| \leq |h| \sum_{n=2}^{\infty} \frac{1}{n!} = (e-2)|h|$ *(note for* $n \geq 2, |h^{n-1}| \leq |h|$ *if* $|h| \leq 1$*). Hence,* $u(h) \to \infty$ *as* $h \to 0$ *and therefore* $f'(x) = e^x$.

> **Remark.** $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is well defined. $e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = e$. Regarding it as a power series, its radius of convergence is $R = \infty$. Also, $e^{x+y} = e^x e^y$ using definition 3.48 of product series (Rudin's p.178-180).

> **Note.** $f'(x)$: Lagrange's notation, $\frac{df}{dx}$: Leibnitz notation

Theorem 5.2. Suppose $f : [a, b] \to \mathbb{R}$ and $f'(x)$ exists. Then f is continuous at x.

Proof. The existence of $f'(x) \Leftrightarrow f(t) = f(x) + (t-x)[f'(x) + u(x,t)]$ with $\lim_{t\to x} u(x,t) = 0$. Let $t \to x$. $\lim_{t\to x} f(x) + (t-x)[f'(x) + u(x,t)] = f(x) + 0[f'(x) + 0] = f(x)$, so $\lim_{t\to x} f(t) = f(x)$; i.e., f is continuous at x. Q.E.D.

Remark. The converse is false; e.g., $f(x) = |x|$ is continuous for all x, but $f'(0)$ does not exist.

Theorem 5.3. If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both differentiable at x then so are $f+g, fg, \frac{f}{g}($ if $g(x) \neq 0)$, and $(f+g)'(x) = f'(x) + g'(x), (fg)'(x) = f'(x)g(x) + f(x)g'(x), (\frac{f}{g})'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$.

Proof (Only the quotient rule). $h(t) - h(x) = \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} = \frac{1}{g(t)g(x)}[(f(t)g(x) - f(x)g(x)) - (f(x)g(t) - f(x)g(x))]$.
Then $\frac{h(t)-h(x)}{t-x} = \frac{1}{g(t)g(x)}\left[\frac{f(t)-f(x)}{t-x}g(x) - f(x)\frac{g(t)-g(x)}{t-x}\right]$. Let $t \to x$. $h'(x) = \frac{1}{g(x)^2}\left[f'(x)g(x) - f(x)g'(x)\right]]$. Q.E.D.

Remark. By induction, $(f_1 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' f_3 \cdots f_n + \cdot f_1 f_2 \cdots f_n'$.

Example. *For* $n = 2, 3$, $\dfrac{d}{dx}x^n = nx^{n-1}$ *and we know this already for* $n = 0, 1$. *For* $n = -1, -2$, *let* $m = -n > 0$. *Then* $\dfrac{d}{dx}x^n = \dfrac{d}{dx}\dfrac{1}{x^m} = \dfrac{(\dfrac{d}{dx}1)x^m - (\dfrac{d}{dx}x^m)1}{(x^m)^2} = \dfrac{0x^m - mx^{m-1}1}{x^{2m}} = -mx^{m-1} = nx^{n-1}$. *Hence,* $\forall_{n\in\mathbb{Z}} : \dfrac{d}{dx}^{n} = nx^{n-1}$.

Theorem 5.5. [Chain Rule] Suppose $f : [a,b] \to \mathbb{R}$, $f'(x)$ exists for some $x \in [a,b], f([a,b]) \subset I$, where I is some interval in $\mathbb{R}$. Suppose $g : I \to \mathbb{R}$ and $g'(f(x))$ exists. Then $g \circ f$ is differentiable at x and $(g \circ f)(x)' = g'(f(x))f'(x)$

Proof. Let $h(t) = (g \circ f)(t) = g(f(t))$ for $t \in [a, b]$. Fix $x \in [a, b]$ where $f'(x)$ exists. We know:

(a) $f(t) - f(x) = (t - x)[f'(x) + u(t)]$ with $\lim_{t \to x} u(t) = 0$.

(b) With $y = f(x)$, $g(s) - g(y) = (s - y)(g'(y) + v(s))$ with $\lim_{s \to y} v(s) = 0$

As $\dfrac{h(t) - h(x)}{t - x} = \dfrac{g(f(t)) - g(f(x))}{t - x}$. By (2), $\dfrac{g(f(t)) - g(f(x))}{t - x}[g' f(x) + v(f(t))]$. Let $t \to x$. Then RHS $\to f'(x)[g'(f(x)) + 0]$ since $f(t) \to f(x)$ by continuity of f at x. Therefore, $h'(x) = f'(x)g'(f(x))$. Q.E.D.

Note. Suppose you produce $f(t)$ meters of wire by time t; i.e., rate of wire production is $f'(t)$ m/x. Also suppose you get \$ $g(l)$ for l meters of wire; rate of profit is $g'(l)$ $\$/m$. Then the rate of earning by time t is $g'(f(t))f'(t)$ $\$/m$.

Example. *(a)* $\dfrac{d}{dx} e^{x^2} = 2xe^{x^2}$

(b) $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$

Remark. (a) f is continuous on $\mathbb{R}$, including at $x = 0$.

Proof. $|f(x)| \leq |x|$, so by the Squeeze theorem, $\lim_{x \to 0} f(x) = 0 = f(0)$. Q.E.D.

(b) f is differentiable on $x \neq 0$, but not differentiable at $x = 0$. For $x \neq 0$, $f'(x) = \sin \dfrac{1}{x} + x(\cos \dfrac{1}{x})(\dfrac{-1}{x^2}) = \sin \dfrac{1}{x} - \dfrac{1}{x} \cos \dfrac{1}{x}$. For $x = 0$, $\dfrac{f(t) - f(0)}{t - 0} = \dfrac{t \sin \frac{1}{t}}{t} = \sin \dfrac{1}{t}$, which does not converge. Therefore, f not differentiable at $x = 0$.

(c) Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \textit{if } x \neq 0 \\ 0 & x = 0 \end{cases}$.

(a) f is continuous in $\mathbb{R}$ including at $x = 0$ ($\because |f(x)| \leq |x^2|$).

(b) f is differentiable in $\mathbb{R}$ including at $x = 0$.

Proof. For $x \neq 0$, $f'(x) = 2x \sin \dfrac{1}{x} + x^2(\cos \dfrac{1}{x})(\dfrac{-1}{x^2}) = 2x \sin \dfrac{1}{x} -$

$\cos\frac{1}{x}$. For $x = 0$, $\frac{f(t) - f(0)}{t - 0} = \frac{t^2 \sin\frac{1}{t}}{t} = t\sin\frac{1}{t} \to 0$ as $t \to 0$. Hence, $f'(0) = 0$. HOWEVER! f' is not continuous at $x = 0$, because $\lim_{x\to 0} f'(x)$ does not exist. Q.E.D.

Definition 5.1. Let X be a metric space, $f : X \to \mathbb{R}$. f has a *local* max at $x \in X$ if $\exists_{\delta>0}$ s.t. $f(y) \geq f(x)$ for all $y \in N_\delta(x)$.

Theorem 5.8. Let $f : [a, b] \to \mathbb{R}$. If f has a local min or a local max at $x \in (a, b)$ and if $f'(x)$ exists, then $f'(x) = 0$.

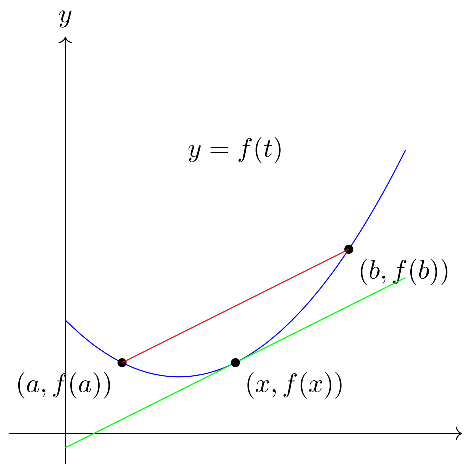
Proof (local min). Suppose f has a local min at x and $f'(x)$ exists. Choose $\delta > 0$ s.t. $N_\delta(x) \subset (a, b)$ and $f(t) \geq f(x)$ if $t \in (x - \delta, x + \delta)$. For $x < t < x+\delta$, $\frac{f(t) - f(x)}{t - x} \geq 0 (\because f(t) \geq f(x), t > x)$, so $f'(x) \geq 0$. For $x - \delta < t < x$, $\frac{f(t) - f(x)}{t - x} \leq 0 (\because f(t) - f(x) \geq 0, t < x)$, so $f'(x) \leq 0$. Hence, $f'(x) = 0$. Q.E.D.

Remark. Note that the hypothesis of the theorem requires *open* interval and existence $f'(x)$. If these conditions are not met, then $f'(x) = 0$ doesn't have to be the case.

Example. *(a) $f(x) = |x|$ has a local min at $x = 0$ but $f'(0)$ does not exist.*

(b) $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$ has a local max at $x = 1$ and local min at $x = 0$, but $f'(0) = f'(1) = 1$.

Theorem 5.10. [Mean-Value Theorem] If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then $\exists_{x\in(a,b)}$ s.t. $f(b) - f(a) = f'(x)(b - a)$.



Proof. Let $L : y = f(a) + m(t-a)$, where $m = \dfrac{y - f(a)}{t-a} = \dfrac{f(b) - f(a)}{b-a}$. Subtract L from the curve $y = f(t)$. Let $h(t) = f(t) - [f(a) + m(t-a)]$. Then $h(a) = h(b) = 0$. $h'(t) = f'(t) - m = f'(t) - \dfrac{f(b)-f(a)}{b-a}$. Therefore, it suffices to find x s.t. $h'(x) = 0$. h is continuous and $[a,b]$ is compact, so $h([a,b])$ is also compact. Hence, h attains its global max$(= \sup\{h([a,b])\})$ and global min$(= \inf\{h([a,b])\})$ on $[a,b]$. If $h(t) = 0$ for all $t \in [a,b]$ then $h'(t) = 0$ for all $t \in [a,b]$ so any $x \in (a,b)$ will do. Otherwise, h attains its global max or global min at some $x \in (a,b)$. By Theorem 5.8, $h'(x) = 0$. Q.E.D.

Theorem 5.11. If f is differentiable on (a,b) then

(a) $f'(x) \geq 0$ for all $x \in (a,b)$ implies f is monotone increasing.

(b) $f'(x) \leq 0$ for all $x \in (a,b)$ implies f is monotone decreasing.

(c) $f'(x) = 0$ for all $x \in (a,b)$ implies f is constant.

Proof ((a) only). Suppose $f'(x) \geq 0$ for all $x \in (a,b)$. For $a < x_1 < x_2 < b$, $\exists_{x \in (x_1, x_2)}$ s.t. $f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$ by Theorem 5.10. As $f'(x) \geq 0, x_2 \geq x_1$, $f(x_2) - f(x_1) \geq 0$. Q.E.D.

Definition 5.2. $f^{(0)} = f, f^{(1)} = f', f^{(2)} = f'', f^{(3)} = f'''$ and so on.

Theorem 5.15. [Taylor's Theorem] Suppose $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, and

$f^{(n+1)}(x)$ exists for all $x \in (a,b)$. Let $x, x_0 \in [a,b]$. Then $\exists_{c\in(x,x_0)}$ s.t.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{P_n(x), n^{\text{th}} \text{ Taylor polynomial of } f \text{ at } x_0} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Taylor Remainder}}.$$

Proof. If $n = 0$, the mean-value theorem guarantees existence of c. For general n, $A \in \mathbb{R}$ by $R_n(x) = f(x) - P_n(x) = \dfrac{A}{(n+1)!}(x-x_0)^{n+1}$, where A depends on f, n, x, x_0. Claim: $A = f^{(n+1)}(c)$ for some c between x and x_0.
Define $g(t) = f(t) - P_n(t) - \dfrac{A}{(n+1)!}(t-x_0)^{n+1}$ for $t \in [a,b]$. Then $g(x_0) = 0$. $g(x) = f(x) - P_n(x) - \dfrac{A}{(n+1)!}(x-x_0)^{n+1} = 0$ by the definition of A. Also for $j = 1, \ldots, n$, then $P_n^{(j)}(x_0) = f^{(j)}(x_0)$, $\dfrac{\mathrm{d}^j}{\mathrm{d}x^j}(x-x_0)^{n+1}\mid_{x=x_0} = 0$. Hence, $g^{(j)}(x_0) = f^{(j)}(x_0) - P_n^{(j)}(x_0) - 0 = 0$. $g^{(n+1)}(t) = f^{(n+1)}(t) - 0 - A$. We need to find c s.t. $g^{(n+1)}(c) = 0$.
$g(x) = g(x_0) = 0 \Rightarrow \exists_{c_1 \in (\min\{x_0,x\}, \max\{x_0,x\})}$ s.t. $g'(c_1) = 0$.
$g'(x_0) = g'(c_1) = 0 \Rightarrow \exists_{c_2 \in (\min\{x_0,x,c_1\}, \max\{x_0,x,c_1\})}$ s.t. $g''(c_2) = 0$.
$\vdots$
Finally, $\exists_{c_{n+1}=c}$ s.t. $g^{n+1}(c) = 0$ and hence $f^{(n+1)}(c) = A$. Q.E.D.

Example. *(not in Rudin) Does* $\sum_{n=1}^{\infty}\left(\sqrt{1+\frac{1}{n^2}} - 1\right)$ *converge or diverge?*

Method 1:

$$\sqrt{1+\frac{1}{n^2}} - 1 = \frac{\left(\sqrt{1+\frac{1}{n^2}} - 1\right)\left(\sqrt{1+\frac{1}{n^2}} + 1\right)}{\sqrt{1+\frac{1}{n^2}} + 1} = \frac{1+\frac{1}{n^2} - 1}{\sqrt{1+\frac{1}{n^2}} + 1} = \frac{1}{n^2}\frac{1}{\sqrt{1+\frac{1}{n^2}} + 1} \le \frac{1}{n^2},$$

so the series converges by the comparison test since $\sum \frac{1}{n^2}$ *converges.*

Method 2: Using Taylor's theorem. Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$. *Then*

$$\begin{aligned}
f'(x) &= \frac{1}{2}(1+x)^{-1/2} \\
f''(x) &= \frac{1}{2}(-\frac{1}{2})(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2} \\
f(x) &= P_1(x) + R_1(x) \\
&= f(0) + \frac{f'(0)}{1!}(x-0)^1 + \frac{f''(c)(x-0)^2}{2!} \\
&= 1 + \frac{1}{2}x + R_1(x).
\end{aligned}$$

$|R_1(x)| \le (\frac{1}{2}\cdot\frac{1}{2}\cdot 1)\frac{1}{2!}x^2 = \frac{1}{8}x^2$ *for* $x \in [0,1]$. *Therefore,* $\sqrt{1+\frac{1}{n^2}} - 1 = f(\frac{1}{n^2}) - 1 = \frac{1}{2}(\frac{1}{n^2}) + R_1(\frac{1}{n^2}) \le \frac{1}{2n^2} + \frac{1}{8}\frac{1}{n^4}$. *Since* $\sum\left(\frac{1}{2n^2} + \frac{1}{8n^4}\right)$ *converges,* $\sum\left(\sqrt{1+\frac{1}{n^2}} - 1\right)$ *converges by comparison test.*

Example. *Let* $f(x) = \sin x, x_0 = 0$.

Taylor series for $f(x)$**.** $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, \ldots,$ *so* $f^{(k)}(x) = \begin{cases} (-1)^m \sin x & (k=2m) \\ (-1)^m \cos x & (k=2m+1) \end{cases}$. *Hence* $n \ge 0, f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$, *where* c *between* 0 *and* x. *Remainder estimate:* $\left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \le \frac{|x|^{n+1}}{(n+1)!} \to 0$ *as* $n \to \infty$ *because* $\frac{|x|^{n+1}}{(n+1)!}$ *is the* $(n+1)^{th}$ *term in convergent series* $e^{|x|} = \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$.

$\sin x = \sum_{n=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$.

Taylor approximation. *Find* $\sin 0.2$ *to within an error* $\pm 10^{-6}$. *Use* $\sin 0.2 = \frac{2}{10} - \frac{1}{3!}(\frac{2}{10})^3 + \frac{1}{5!}(\frac{2}{10})^5 - \cdots$.

Method 1: Alternating Series Test. *If* $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$, *and* $a_n \to 0$, *then* $\sum_{k=1}^{\infty}(-1)^{k-1}a_k = s$ *converges and* $|s - s_n| \le a_{n+1}$. *Above series satisfies the hypotheses, so truncation error is* $\le$ *first omitted term. We look for when* $\frac{1}{(2k+1)!}(\frac{2}{10})^{2k+1} \le 10^{-6}$; *i.e.,*

$(2k+1)! \cdot \frac{10^{2k+1}}{2^{2k+1}} \ge 10^6$.

If $k = 1$, $3! \cdot \frac{10^3}{2^3} < 10^6$.

If $k = 2$, $5! \cdot \frac{10^5}{2^5} < 10^6$.

If $k = 3$, $7! \cdot \dfrac{10^7}{2^7} \geq 10^6$, *so* $k = 3$ *works. Therefore,* $\sin 0.2 = 0.2 - \dfrac{1}{3!}(0.2)^3 + \dfrac{1}{5!}(0.2)^5 \pm 10^{-6} = 0.198669 \pm 10^{-6}$.

Method 2: General Case. *If alternating series test does not apply, estimate remainder using the worst* c *for* $\left|\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right|$. *In our example,* $\left|\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| \leq \dfrac{1}{(n+1)!}(0.2)^{n+1}$, *so seek* n *s.t.* $\dfrac{1}{(n+1)!}\left(\dfrac{2}{10}\right)^{n+1} \leq 10^{-6}$. *First* n *that works is* $n = 6$, *same as before.*

Chapter 6

Riemann-Stieltjes Integral

Definition (Partition)**.** A partition P of $[a, b]$ is $\{x_0, x_1, x_2, \ldots, x_n\}$ for some $n \geq 1$, with $a = x_0 \leq x_1 \ldots \leq x_{n-1} \leq x_n = b$.

Notation. $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \ldots, n$

$f : [a, b] \to \mathbb{R}$ be bounded, which is not necessarily continuous

$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$, $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$

$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$

$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$.

> **Note.** $L(P, f) \leq U(p, f)$ always.

Definition (Riemann Integral)**.** **Upper Riemann Integral** : $\overline{\int_a^b} f(x)dx = \inf_P\{U(P, f)\} = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$.

Lower Riemann Integral : $\underline{\int_a^b} f(x)dx = \sup_P\{L(P, f)\} = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$.

Riemann Integrable : f is Riemann integrable on $[a, b]$ if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$. If f is Riemann integrable on $[a, b]$, we write $f \in \mathscr{R}[a, b]$ and

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

Note. Since f is bounded, $m = \inf\{f(x) : a \leq x \leq b\}$ and $M = \sup\{f(x) : a \leq x \leq b\}$ are both finite. Hence, for any P, $m \leq m_i \leq M_i \leq M$ and $\forall_i : m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$.

Notation. Let $\alpha : [a, b] \to \mathbb{R}$ is a monotone increasing function. Then $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

Definition 6.2. Given P, let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. (Note: $\Delta\alpha_i \geq 0$).
For bounded f, let $U(p, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$, $L(P, f, \alpha) = \sum_{i=1}^{n} m_i \delta\alpha_i$.

Upper Riemann-Stieltjes Integral $\overline{\int_a^b} f(x) d\alpha = \overline{\int_a^b} f(x) d\alpha(x) = \inf_P\{U(P, f, \alpha)\} = \inf\{U(P, f, \alpha) : P \text{ is a partition of } [a, b]\}$.

Lower Riemann-Stieltjes Integral $\underline{\int_a^b} f(x) d\alpha = \underline{\int_a^b} f(x) d\alpha(x) = \sup_P\{L(P, f, \alpha)\} = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a, b]\}$.

If $\overline{\int_a^b} f(x) d\alpha = \underline{\int_a^b} f(x) d\alpha$, then $f \in R[a, b, \alpha]$ and $\int_a^b f(x) d\alpha = \overline{\int_a^b} f(x) d\alpha = \underline{\int_a^b} f(x) d\alpha$.
If $\alpha(x) = x$, then equivalent to $\int_a^b f(x) dx$.

Definition 6.3. (a) Partition P^* is called a refinement of P if $P \subset P^*$.

(b) Partition P^* is called the common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$.

Theorem 6.4. If P^* is a refinement of P then $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$.

Proof. It's enough to consider p^* with one extra point: $x_{i-1} \leq x^* \leq x_i$.
Sketch for L:

$$
\begin{aligned}
&L(P^*, f, \alpha) - L(p, f, \alpha) \\
&= m^*[\alpha(x^*) - \alpha(x_{i-1})] + m_i[\alpha(x_i)\alpha(x^*)] - m_i[\alpha(x^*) - \alpha(x_{i-1})] - m_i[\alpha(x_i) - \alpha(x^*)] \\
&= (m^* - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (m_i - m_i)[\alpha(x_i) - \alpha(x^*)]
\end{aligned}
$$

Q.E.D.

Notation. When f, α are fixed, we write $L(P) = L(P, f, \alpha), U(P) = U(P, f, \alpha)$

Theorem 6.5. $\underline{\int_a^b} f \mathrm{d}\alpha \leq \overline{\int_a^b} f \mathrm{d}\alpha$.

Proof. For partitions P_1, P_2, let $P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \leq L(P^*) \leq U(P^*) \leq U(P_2)$. In particular, $\sup_{P_1}\{L(P_1)\} \leq U(P_2)$ for all P_2. Hence, $\sup_{P_1}\{L(P_1)\} \leq \inf_{P_2}\{U(P_2)\}$. Q.E.D.

Theorem 6.6. $f \in \mathscr{R}_\alpha[a,b] \Leftrightarrow \forall_{\varepsilon>0} : \exists P_\varepsilon$ s.t. $U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$

Proof. Let $\varepsilon > 0$.

($\Rightarrow$) By hypothesis, $\sup_P\{L(P)\} = \underline{\int_a^b} f \mathrm{d}\alpha = \overline{\int_a^b} f \mathrm{d}\alpha = \inf_P\{U(P)\}$.

$\exists P_1, P_2$ s.t. $L(P_1) > \underline{\int_a^b} f \mathrm{d}\alpha - \varepsilon/2$ and $U(P_2) < \overline{\int_a^b} f \mathrm{d}\alpha + \varepsilon/2$. Then $U(P_2) - L(P_1) < \varepsilon$. Let $P_\varepsilon = P^* = P_1 \cup P_2$. By Theorem 6.4, $L(P_1) \leq L(P^*) \leq U(P^*) \leq U(P_2)$, so $U(P_\varepsilon) - L(P_\varepsilon) \leq U(P_2) - L(P_1) < \varepsilon$.

($\Leftarrow$) $0 \leq \overline{\int_a^b} f \mathrm{d}\alpha - \underline{\int_a^b} f \mathrm{d}\alpha \leq U(P_\varepsilon) - L(P_\varepsilon) < \varepsilon$. Since ε is arbitrary, $\overline{\int_a^b} f \mathrm{d}\alpha = \underline{\int_a^b} f \mathrm{d}\alpha$.

Q.E.D.

Remark. very important

Theorem 6.7. Let $\varepsilon_0 > 0$ be fixed. Suppose there exists a partition $P = \{x_0 = a, \ldots, x_n = b\}$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon_0$. Let s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$. Then,

(a) For any refinement of P, denoted by P^*, $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \varepsilon_0$ also holds true

(b) $\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon_0$

(c) If $f \in \mathscr{R}_\alpha$, then $\left|\sum_{i=1}^n f(t_i)\Delta\alpha_i - \int_a^b f \mathrm{d}\alpha\right| < \varepsilon_0$

Theorem 6.8. If f is continuous on $[a,b]$ then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. For any P, $U(P) - L(P) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i$ Since $[a,b]$ is compact, f is uniformly continuous on $[a,b]$ (Theorem 4.19), so $\forall \eta > 0 : \exists \delta > 0$ s.t. $|x - t| < \delta \Rightarrow |f(x) - f(t)| < \eta$. Given $\varepsilon > 0$, choose η s.t. $\eta[\alpha(b) - \alpha(a)] < \varepsilon$ and choose P with $\Delta x_i < \delta = \delta(\eta)$ for all i. Then $U(P) - L(P) < \sum_{i=1}^n \eta\Delta\alpha_i = \eta[\alpha(b) - \alpha(a)] < \varepsilon$. Therefore, $f \in \mathscr{R}_\alpha[a,b]$. Q.E.D.

Theorem 6.9. If f is monotone increasing or decreasing on $[a,b]$ and α is continuous on $[a,b]$ then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. By definition, $U(P) - L(P) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i$. Given $n \in \mathbb{N}$, let P s.t. $\Delta\alpha_i = \dfrac{\alpha(b)-\alpha(a)}{n}$ for all i. Then, $U(P) - L(P) = \dfrac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n} M_i - m_i$. Suppose f is increasing, so $M_i - m_i = f(x_i) - f(x_{i-1})$. Then $U(P) - L(P) = \dfrac{\alpha(b)-\alpha(a)}{n}\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \dfrac{\alpha(b)-\alpha(a)}{n}[f(b) - f(a)]$. Given $\varepsilon > 0$, we can choose n (hence P) s.t. $U(P) - L(P) < \varepsilon$. Therefore, $f \in \mathscr{R}_\alpha[a,b]$ by Theorem 6.6. Q.E.D.

Note. We always assume α is monotone.

Theorem 6.10. If f is bounded on $[a,b]$ and has only finitely many discontinuities, and α is continuous at each point where f is not, then $f \in \mathscr{R}_\alpha[a,b]$.

Proof. We apply Theorem 6.6. Use $U(P) - L(P) = \sum_{i=1}^{n}(M_i - m_i)\Delta_i$. Let $\varepsilon > 0$ and $E = \{e_1, \dots, e_k\}$ be the set of points where f is discontinuous. α is assumed to be continuous at each e_i, which implies $\exists (u_j, v_j)$ s.t. $u_j < e_j < v_j$ and $\alpha(v_j) - \alpha(u_j) < \varepsilon$. (Relax inequality to include equality if $e_1 = a$, $e_k = b$) Let $K = [a,b] \cap \left(\bigcup_{j=1}^{k}(u_j, v_j)\right)^c$. K is compact. f is continuous on K, so f is uniformly continuous on K by Theorem 4.19. Hence, $\exists \delta > 0$ s.t. $s, t \in K$ and $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon$. Form P to consist of $\{u_1, v_1, \dots, u_k, v_k\}$ and additional points x_i in K with $\Delta x_i < \delta$. For such i, $M_i - m_i < \varepsilon$. Then $U(P) - L(P) < \varepsilon$. For $[u_j, v_j], M_j - m_j \le 2M$, where $M = \sup\{|f(x)| : x \in [a,b]\}$, and $\Delta\alpha_j < \varepsilon$. Then $0 \le U(P) - L(P) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \le \underbrace{K \cdot 2M\varepsilon}_{\text{From } [u_j, v_j] \text{ intervals}} + \underbrace{\varepsilon[\alpha(b) - \alpha(a)]}_{\text{From } K \text{ intervals}}$. RHS is as small as we want by taking ε small enough. Q.E.D.

Remark. (a) Theorem 6.10 implies part of A1.2 but do the problem from first principles. Do not apply Theorem 6.10 directly.

(b) A1.4 shows what can happen if f, α are discontinuous at the same point.

Theorem 6.11. If $f \in \mathscr{R}_\alpha[a,b], m \le f(x) \le M$ for all $x \in [a,b]$, and $\varphi : [m, M] \to \mathbb{R}$ is continuous, then $\varphi \circ f \in \mathscr{R}_\alpha[a,b]$.

Proof. See textbook. pf similar to pf of Theorem 6.10. Q.E.D.

Example. *$f \in \mathscr{R}_\alpha[a, b] \Rightarrow f^2 \in \mathscr{R}_\alpha[a, b], |f| \in \mathscr{R}_\alpha[a, b]$ where $phi(t) = t^2$ and $\varphi(t) = |t|$ respectively.*

Note. $\varphi \in \mathscr{R}_\alpha[m, M]$ does not imply $\varphi \circ f \in \mathscr{R}_\alpha[a, b]$. See A2.