Real Variables I

October 28, 2024

Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$

Remark. Note for real numbers, $\mathbb{Q}$ has holes in it.
Example. $\nexists p \in \mathbb{Q}\ s.t\ p^2 = 2$

Proof. Assume $\exists p \in \mathbb{Q}$ s.t $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t $p^2 = 2$. Q.E.D.

Definition 1 (Order)**.** An order on a set S is a relation $<$ such that:

(a) If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.

(b) If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 2 (Ordered Set)**.** An ordered set S is a set with an order $<$.

Definition 3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t $\forall x \in E : x \leq \beta$.
Similarly, a set S is bounded below if $\exists \beta \in S$ s.t $\forall x \in E : x \geq \beta$.

Definition 4 (LUB, GLB). Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists \alpha$ s.t. α is an upper bound for E and $\forall \gamma < \alpha$: γ is not an upper bound for E, then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists \alpha$ s.t. α is a lower bound for E and $\forall \gamma > \alpha$: γ is not a lower bound for E. Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 5 (LUB property). An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists \sup E \in S$; i.e., Every bounded subset of S has the least upper bound(LUB).
Example.

- $\mathbb{Z}$ *has the LUB property.*
- $\mathbb{Q}$ *does not have the LUB property.*

Theorem 1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof. ($\Rightarrow$)Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B. Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. ($\Leftarrow$)Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E. Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 6 (Fields). Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

(a) $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).

(b) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).

(d) $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.

(e) $\exists(-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.

(f) $\forall x, y \in F : xy \in E$.

(g) $\forall x, y \in F : xy = yx$.

(h) $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.

(i) If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

(j) $\forall x, y, z \in F : x(y+z) = xy + xz$

Example.

(a) $\mathbb{Q}$ *is a field, while* $\mathbb{Z}$ *is not a field.*

(b) $F_p = \{0, 1, \ldots, p-1\}$ *with mod* p *arithmetic is a field.*

Read Text book: 114,115,116,118

Definition 7 (Ordered Field)**.** An ordered field F is a field that is an ordered set such that the following properties are satisfied:

(a) If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.

(b) If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

> **Remark.** We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. $\mathbb{Q}$ *is an ordered field.*

Theorem 2. $\exists$ an ordered field $\mathbb{R}$ which has the LUB property and contains $\mathbb{Q}$ as a subfield.

Theorem 3.

(a) Arithmetic properties of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.

(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof. (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A. So, A has a least upper bound α. Since $\alpha - x < \alpha$ as $x > 0$, $\alpha - x$ is not an upper bound for A. Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A. Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. $ny - nx > 1$ and therefore, $1 + nx < ny$. Let $m \in \mathbb{Z}$ such that $(m - 1) \leq nx < m$. Such m exists by the extended version of (a). This implies there exists $m \in \mathbb{N}$ such that $nx < m \leq nx + 1 < ny$. Therefore, $x < \frac{m}{n} < y$.

(c) $\exists k \in \mathbb{Q}$ such that $k^2 = 2$; i.e., $\exists \sqrt{2} \in \mathbb{R}$. $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then $x < p < \alpha < q < y$ and $\alpha \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n-1)x \leq y < nx. \tag{1.1}$$

Proof. Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of $\mathbb{N}$ has a smallest element. Let $n =$ smallest element of A. Then the inequality holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n-1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n-1)x$. Hence, the statement holds. Q.E.D.

Lemma 1. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b-a)$ for some $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} b^n - a^n &= (b-a)(\underbrace{b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1}}_{n \text{ terms}}) \\ &< (b-a)nb^{n-1} \end{aligned}$$

Q.E.D.

Theorem 4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique})y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt{x}^n$, the n^{th} root of x).

Proof. Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x.
Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \varnothing$, E is bounded above, hence (by the least-upper-bound property) there exists a $\sup E$. Choose $y = \sup E$. Consider two cases.

(a) If $x \leq 1$, then $t_0 = \dfrac{x}{2}$ and thereby $t_0^n = \dfrac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).

(b) If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x+1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E.

(a) Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y + h)^n - y^n < x - y^n$. By the lemma 1, we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y + h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.

(b) Assuming that $y^n > x$, we find $k > 0$ such that $(y - k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y, which is $\sup E$. By the lemma 1, $y^n - (y - k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \dfrac{y^n - x}{ny^{n-1}}$. Therefore, $-(y - k)^n < -x$, or $x < (y - k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 8 (Cut/Dedekind Cut)**.** The set $\mathbb{R}$ elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 9 (Order of cuts)**.** For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A, and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 5. There exists an ordered field $\mathbb{R}$ such that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ has the LUB property.

Proof. Let $\mathbb{R}$ be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$
Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$
Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
Zero element is $(0, 0)$
One element is $(1, 0)$

Theorem 6. $\mathbb{C}$ is a field.

Proof. Verify the 11 field axioms. For just a few axioms:
(M3):
$x = (a, b), y = (c, d), z = (e, f)$. $x(yz) = (a, b)(ce - df, cf + de) = (a(ce-df)-b(cf+de), a(cf+de)+b(ce-df)) = (ac-bd, ad+bc)(e, f) = (xy)z$
(M4):
$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$
(M5):
$x \neq 0$ means $x = (a, b)$ with $a \neq 0$ or $b \neq 0$. That is, $a^2 + b^2 > 0$. Let $\frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$. Then $x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0)$. Q.E.D.

Identification of $\mathbb{R}$ as a subfield of $\mathbb{C}$. Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a+b = a+b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\mathrm{Re}(z) = a$,$\mathrm{Im}(z) = b$.

Definition 11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by $\overline{z}$

Note.

(a) $\overline{z+w} = \overline{z} + \overline{w}$

(b) $\overline{zw} = \overline{z} \cdot \overline{w}$

(c) $z + \overline{z} = 2 \cdot \mathrm{Re}(z)$

(d) $z - \overline{z} = 2i \cdot \mathrm{Im}(z)$

(e) $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2 \geq 0$, with $=$ if any only if $z = 0$

(f) $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{a-bi}{a^2+b^2}$

Definition 12. $|z| = \sqrt{z\overline{z}} = \sqrt{a^2+b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 7. For $z, w \in \mathbb{C}$,

(a) $|z| \geq 0$ with $=$ iff $z = 0$

(b) $|z| = |\overline{z}|$

(c) $|zw| = |z| \cdot |w|$

(d) $|\mathrm{Re}(z) \leq |z|, |\mathrm{Im}(z)| \leq |z|$

Proof. Let $z = a + bi$. Then $|\mathrm{Re}(z)| = |a| \leq \sqrt{a^2+b^2} = |z|$ Q.E.D.

(e) $|z+w| \leq |z| + |w|$ (Triangle inequality)

Proof.

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

Theorem 8 (Cauchy-Schwarz inequality)**.** If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$|\sum_{j=1}^{n} a_j \overline{b_j}| \leq (\sum_{j=1}^{n} |a_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{n} |b_j|^2)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^{n} a_j \overline{b_j}$ defined on inner product on $\mathbb{C}^n$ and $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})}$. (Note that $\vec{a} \cdot \vec{b} = \overline{\vec{b} \cdot \vec{a}}$)

Proof. Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \overline{b_j}$ We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0.
For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^{n} |a_j + \lambda b_j|^2 = \sum_{j=1}^{n} (a_j + \lambda b_j)(\overline{a_j} + \overline{\lambda b_j}) = \sum_{j=1}^{n} |a_j|^2 + \lambda \sum_{j=1}^{n} b_j \overline{a_j} + \overline{\lambda} \sum_{j=1}^{n} a_j \cdot \overline{b_j} + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2$. Let $\lambda = tC$ for $t \in \mathbb{R}$.
Then $0 \leq A + \lambda\overline{C} + \overline{\lambda}C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$. $p(t)$ is a quadratic function in terms of t and it must be non-negative. Regardless of the value of t. Therefore, the discriminant of $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$. Since $|C| \geq 0$, $|C|^2 \leq AB$. Q.E.D.

Definition 13 (Euclidean k-space)**.** For $k \in N$, $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_k \in \mathbb{R}\}$ with the following properties:

Addition $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$

Scalar multiplication $\lambda\vec{x} = (\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$

Inner(dot) product $(\vec{x}, \vec{y}) = \sum_{j=1}^{k} x_j y_j$, which is bilinear: $(\alpha\vec{x} + \beta\vec{y}) \cdot \vec{z} = \alpha\vec{x} \cdot \vec{z} + \beta\vec{y} \cdot \vec{z}$.

Norm $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^{k} |x_j|^{2^{1/2}}$

Remark. Addition and Scalar multiplication make $\mathbb{R}^k$ into a vector space.

Theorem 9. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$. Then

(a) $|\vec{x}| \geq 0$

(b) $|\vec{x}| = 0 \leftrightarrow \vec{x} = \vec{0}$

(c) $|\alpha\vec{x}| = |\alpha||\vec{x}|$

(d) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ (special case of Cauchy-Schwarz inequality)

(e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle inequality)

Proof. $|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x} \cdot \vec{y}| + |\vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2$ Q.E.D.

(f) $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$

Proof. $|\vec{x} - \vec{y}| = |(\vec{x} - \vec{z}) + (\vec{z} - \vec{y})| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$ Q.E.D.

Chapter 2

Basic Topology

Definition 14. Sets A and B have the same cardinality, if $\exists f : A \to B$ that is 1-1 and onto (i.e., bijective).

Theorem 10. Let $A \sim B$ be a relation between two sets having the same cardinality. Then is an equivalence relation. That is,

(a) $A \sim A$ (Reflexive)

(b) $A \sim B \Rightarrow B \sim A$ (Symmetry)

(c) $A \sim B \& B \sim C \Rightarrow A \sim C$ (Transitivity)

Definition 15. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $J_n = \{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$.

- A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$(or if $A = \emptyset$).
- A set A is countably infinite if $A \sim \mathbb{N}$.
- A set A is countable if A is finite or countably infinite.

Example. *$\mathbb{Z}$ is a countably infinite. For $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$,*

Let $f(n) = \begin{cases} 1 & \textit{if } n = 0 \\ 2n & \textit{if } n > 0 \\ -2n + 1 & \textit{if } n < 0 \end{cases}$

Then f is bijective and therefore $|\mathbb{Z}| = |\mathbb{N}|$

Theorem 11. A subset of a countably infinite set is countable.

Proof. Let A be some countably infinite set and S be a infinite subset of A.
As A is a countably infinite set, we can remove duplicates and arrange A so that $A = \{a_1, a_2, a_3, \ldots\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in S$. Let n_k be the smallest positive integer greater than n_{k-1} such that $x_{n_k} \in E$ for $k = 2, 3, \ldots$. Let $f(k) = x_{n_k}$ for $k = 1, 2, 3, \ldots$. Then this is a bijection from $\mathbb{N}$ to S. Q.E.D.

Remark. Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

Theorem 12. Let $E_1, E_2, \ldots$ be countably infinite sets. Then $S = \cup_{n=1}^{\infty} E_n$ is countably infinite.

Proof. Write $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \ldots\}$
$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \ldots\}$
Form an array:

$$\begin{Bmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \cdots \\ x_{21} & x_{22} & x_{23} & x_{24} & \cdots \\ x_{31} & x_{32} & x_{33} & x_{34} & \cdots \\ x_{41} & x_{42} & x_{43} & x_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{Bmatrix}.$$

This matrix might have duplicates. Let T be a subset of $\mathbb{N}$ such that $t \in T$ if and only if t is the smallest positive integer such that $x_t \in E_1 \cup E_2 \cup \ldots \cup E_n$.
Then a set $\{x_t | t \in T \text{ and } \exists_{i \in \mathbb{N}} : x_t \in E_i\}$ is S. Clearly, $|S| = |T|$, or $S \sim T$, and T is a subset of a countably infinite set, $\mathbb{N}$. Therefore, T is also countable, implying S is also countable. As S is infinite, S is countably infinite. Q.E.D.

Corollary. If A is countable and $n \in \mathbb{N}$, then $\{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in A\}$ is countable.

Theorem 13. Let $A = \{(b_1, b_2, b_3 \ldots) | b_i \in \{0, 1\}\}$. I.e., A is a set of all infinite binary strings. Then A is uncountable.

Proof (Contor's Diaganalization argument,1891)**.** Let $E \subset A$ be countably infinite. $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \ | s^{(i)} \in A\}$. It suffices to find some $s \in A \setminus E$, for this shows every countably infinite subset of A is proper construction of s. Write

$$s^{(1)} = b_1^1 b_2^1 \dots \tag{2.1}$$
$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \tag{2.2}$$
$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \tag{2.3}$$
$$\vdots$$

On diagonal, flip each bit, i.e., $0 \to 1$ and $1 \to 0$ and represent the flipped bit of b_i^i by $\tilde{b_i^i}$. Let $s = \tilde{b_1^1}\tilde{b_2^2}\tilde{b_3^3}\dots$. Then $s \in A$ and $s \notin E$ as s differs from each $s^{(i)}$ in the i-th bit. Therefore, A is uncountable. Q.E.D.

Corollary. The set $\mathcal{P}(\mathbb{N})$ of subsets of $\mathbb{N}$ is uncountable.

Proof. We can create $f : \mathcal{P}(\mathbb{N}) \to A$ be a bijection, where A is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \tag{2.4}$$

For example, if $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0 \dots)$. This f is a bijection, and therefore A is uncountable.

Q.E.D.

Theorem 14. $\mathbb{R}$ is uncountable.

Proof. This is a rough sketch of the proof:

(a) It's enough to show that $[0,1]$ is uncountable.

(b) Consider binary decimal representation of $x \in [0,1]$. For example, $x = 0.101001001\ldots$. Given x, choose maximal $b_1 \in \{0,1\}$ such that $\frac{b_1}{2} \leq x$. Then choose $b_2 \in \{0,1\}$ such that $\frac{b_1}{2} + \frac{b_2}{2} \leq x$. Continue this process to get $b_1, b_2, b_3, \ldots$. Then $x = \sup\left\{\sum_{i=1}^{n} \frac{b_i}{2^i}\right\}$. Consider any dyadic rational of the form $\frac{m}{2^n}$. Let it be $\frac{3}{2^4}$. Then this maps $\frac{3}{2^4} \to 0,0,1,1,0,0,0,\ldots$ and never produce $0,0,1,0,1,1,1,1,\ldots$, which also represents $\frac{3}{2^4}$. Let A_1 be a subset of $A = \{\text{infinite binary strings}\}$ such that A_1 does not contain any strings ending in $1,1,1,1,\ldots$. Then the decimal representation defines a bijection $f : [0) \to A \setminus A_1$.

(c) A_1 is countable because $A = (A \setminus A_1) \cup A_1$, which is uncountable.

This shows that $[0,1]$ is uncountable, and therefore $\mathbb{R}$ is uncountable.

Q.E.D.

Definition 16 (Metric Spaces)**.** A set X is a metric space with metric $d : X \times X \to \mathbb{R}$ if

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,q) = 0$ if $p = q$, $\forall p,q \in X$

(b) $\forall_{p,q \in X} : d(p,q) = d(q,p)$

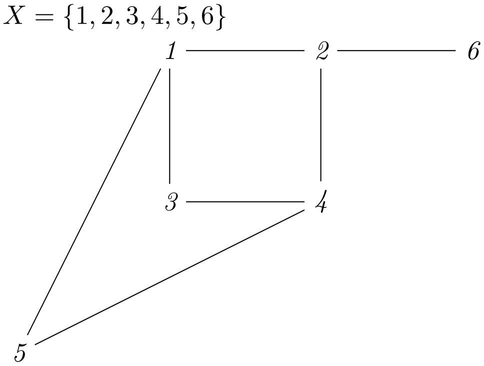
(c) $\forall_{p,q,r \in X} : d(p,q) \leq d(p,r) + d(r,q)$ (Triangle Inequality)

Remark. A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

Example (Metric Spaces)**.** *(a)* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$ *are metric spaces with* $d(p,q) = |p - q|$*. Note the meaning of* $|x|$ *depends on the context.*

(b) Every subset of a metric space is a metric space.

(c) $X = \{1,2,3,4,5,6\}$



Definition 17 (Neighborhood). A neighborhood in X is a set $N_r(p) := \{q : d(q,p) < r\}$, where $p \in X, r > 0$.

Remark. If $r_1 \leq r_2$, then $N_{r_1}(p) \subset N_{r_2}(p)$.

Example.

$\mathbb{R}^1$ *intervals,* $N_r(x) = \{y \in R^1 : |x-y| < r\}$

$\mathbb{R}^2$ *disks* $N_r(x) = \{y \in R^2 : |x-y| < r\}$

$\mathbb{R}^3$ *balls,* $N_r(x) = \{y \in R^3 : |x-y| < r\}$

Given example (c), $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$, $N_2(2) = \{1,2,4,6\}$, $N_3(2) = \{1,2,3,4,5,6\} = X$.

Definition 18. Let $E \subset X$. $p \in E$ is an interior point of E if $\exists r > 0$ such that $N_r(p) \subset E$.

Example.

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$.

Definition 19. $E \subset X$ is an open set if $\forall_{x \in E}$ is an interior point of E.

Theorem 15. Every neighborhood is an open set.

Proof. Let $g \in N_r(p)$. Then we must find $s > 0$, such that $N_s(g) \subset N_r(p)$. We know $d(p,q) < r$. Choose s such that $0 < s < r - d(p,q)$. Let $x \in N_s(q)$, then $d(q,x) < s < r - d(p,q)$. By triangle inequality, $d(p,x) \leq d(p,q) + d(q,x) < d(p,q) + r - d(p,q)$, so $x \in N_r(p)$, so $N_s(q) \subset N_r(p)$. Q.E.D.

Definition 20. Let $E \subset X$ and $p \in X$. p is a limit point of E if $\forall_{r>0}\exists_{q \in E}$ such that $q \neq p$ and $q \in N_r(p)$

Example. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ *has exactly one limit point,* 0. *note* $0 \notin E$.

Theorem 16. If p is a limit point of $E \subset S$, then every neighborhood of p contains infinitely many points of E.

Proof. Let $N_r(p)$ be a neighborhood of p. Then $N_r(p)$ contains at least one point $q_1 \in E$ such that $q_1 \neq p$. Let $r_1 = d(p,q_1)$. Then $N_{r_1}(p)$ contains some $q_2 \in E$ such that $q_2 \neq p$. Let $r_2 = d(p,q_2)$. Then $N_{r_2}(p)$ contains some $q_3 \in E$ such that $q_3 \neq p$. Continue this process to get $q_1, q_2, q_3, \ldots$. Q.E.D.

Corollary. If $E \subset X$ is finite then E has no limit points.

Definition 21 (Closed Set). A set $E \subset X$ is closed if every limit point of E is in E.

Theorem 17. $E \subset X$ is open iff $E^c = \{x \in X : x \notin E\}$ is closed.

Proof.

- E is open $\Rightarrow E^c$ is closed.
  Let p be a limit point of E^c. Then every neighborhood of p contains some $q \in E^c$ such that $q \neq p$. If $p \in E$, then because E is open, p is an interior point, i.e., there exists some neighborhood of p that is a subset of E, which does not contain any points of E^c. This implies $p \notin E$ and therefore $p \in E^c$.

- E^c is closed $\Rightarrow E$ is open.
  Let $p \in E$. Then $p \notin E^c$, so p is not a limit point of E^c. Therefore, there exists some neighborhood of p that contains no points of E^c, i.e., all points of the neighborhood are in E. p Thus, Every $p \in E$ is an interior point of E, and hence E is open.

Q.E.D.

Theorem 18 (De Morgan's Laws).

(a) $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha E_\alpha^c$

(b) $(\bigcap_\alpha E_\alpha)^c = \bigcup_\alpha E_\alpha^c$

Theorem 19.

(a) For all collection $\{G_\alpha\}$ of open sets : $\bigcup_\alpha G_\alpha$ is open.

(b) For all collection $\{F_\alpha\}$ of closed sets : $\bigcap_\alpha F_\alpha$ is closed.

(c) For all finite collection $\{G_1, G_2, \ldots, G_n\}$ of open sets : $\bigcap_{i=1}^n G_i$ is open.

(d) For all finite collection $\{F_1, F_2, \ldots, F_n\}$ of closed sets : $\bigcup_{i=1}^n F_i$ is closed.

Proof. (a) Let $x \in \bigcup_\alpha G_\alpha$. Then $x \in G_{\alpha_0}$ for some α_0. So there exists a neighborhood N of x such that $N \subset G_{\alpha_0} \subset \bigcup_\alpha G_\alpha$.

(b) it's suffice to prove that $(\bigcap_\alpha F_\alpha)^c$ is open. But $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$ is open by (a).

(c) Let $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for $i = 1, 2, \dots, n$. So there exists a $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $N_r(x) \subset N_{r_i} \subset G_i$ for $i = 1, 2, \dots, n$ and therefore $N_r(x) \subset \bigcap_{i=1}^n G_i$.

(d) $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$ is open by (c).

Q.E.D.

Definition 22 (Closure)**.** Let $E \subset X$. Let E' be a set of limit points of E in X. The set $\overline{E} = E \cup E'$ is the closure of E.

Theorem 20.

(a) $\overline{E}$ is closed.

(b) $E = \overline{E} \Leftrightarrow E$ is closed.

(c) If $F \subset X$ is closed and $E \subset F$, then $\overline{E} \subset F$. (i.e., $\overline{E}$ is the smallest closed set containing E, and $\overline{E} = \bigcap_{F:\text{closed set with } F \supset E} F$.)

Proof. (a) Let p be a limit point of $\overline{E}$. It suffices to show $p \in E'$ since this implies that $p \in E' \subset E \cup E' = \overline{E}$. Let $r > 0$. $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$, i.e., $d(p,q) < \frac{r}{2}$. Since $q \in E \cup E'$, $\exists_{s \in E}$ such that $d(q,s) < \frac{r}{2}$ (if $q \in E$, take $s = q$). But $d(p,s) \leq d(p,q) + d(q,s) < \frac{r}{2} + \frac{r}{2} = r$.

(b) $(\Rightarrow)$ by (a)

$(\Leftarrow)$ Suppose E is closed. Then $E' \subset E$, so $\overline{E} = E \cup E' = E$.

(c) Suppose F is closed. Then $F' \supset E'$ and also $F \supset F'$. So $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

Example. *Let $X = \mathbb{R}, d(p,q) = |p - q|$. Let $E \subset \mathbb{R}$ be nonempty and bounded above, and let $y = \sup E$. Then $y \in \overline{E}$.*

Proof. Suppose for contradiction $y \notin \overline{E}$. Then y is neither a point in E nor a limit point of E, so $\exists$ some interval $N_r(y) = (y - r, y + r)$ such that $(y-r, y+r) \cap E = \emptyset$. However, then $y - r$ in an upper bound for E since y is a least upper bound, which is a contradiction. Therefore, $y \in \overline{E}$. Q.E.D.

Definition 23 (Relative Openness)**.** Suppose X is a metric space, so $Y \in X$ is a metric space with the same metric. Let $E \subset Y$. Then E is open relative to Y if E is an open set in the metric space Y

Example. *$X = R^2 \supset \mathbb{R} = y, E = (0,1) \subset Y$. Then E is open relative to Y, but E is neither open nor closed in X.*

Theorem 21. A set $E \subset Y \subset X$ is open relative to $Y \Leftrightarrow \exists_{\text{open set } G \subset X} : E = G \cap Y$

Proof. ($\Rightarrow$) Suppose $E \subset Y$ is open relative to Y. Given $p \in E$, $\exists_{r_p > 0} : {N_{r_p}}^Y(p) \subset E$, where ${N_r}^Y(p) = \{q \in Y : d(p,q) < r\}$. Then $E \subset \bigcup_{p \in E} {N_{r_p}}^Y(p)$ and $\bigcup_{p \in E} {N_{r_p}}^Y(p) \subset E$. Therefore, $E = \bigcup_{p \in E} {N_{r_p}}^Y(p)$.
Let $G = \bigcup_{p \in E} {N_{r_p}}^X(p)$. This time, we are considering p's neighborhood in X, so each ${N_{r_p}}^X$ is open. Thus G is a union of open sets in X, and therefore open.
$\forall_{p \in E} : p \in N_{r_p}(p)^X$, so $E \subset G \cap Y$.
Let $p \in G \cap Y$. Then $p \in G$ and $p \in Y$. So $p \in {N_{r_p}}^X(p)$ for some $r_p > 0$. But $p \in Y$, so $p \in {N_{r_p}}^Y(p)$. Therefore, $p \in E$. This implies $G \cap Y \subset E$, and therefore $E = G \cap Y$.

($\Leftarrow$) Suppose $G \subset X$ is open and $E = G \cap Y$. Then $\forall_{p \in E} : \exists_{r_p > 0} : {N_{r_p}}^X(p) \subset G$, so ${N_{r_p}}^Y(p) = {N_{r_p}}^X(p) \cap Y \subset G \cap Y = E$.

Q.E.D.

Note: Midterm 1 material ends here.

Definition 24 (Open Cover)**.** An open cover of $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t $E \subset \bigcup_\alpha G_\alpha$.

Definition 25 (Compact)**.** A set $K \subset X$ is compact if every open cover has a finite subcover; i.e., $\exists_{\alpha_1, \alpha_2, \ldots \alpha_n}$: s.t $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \ldots \cup G_{\alpha_n}$

Example.

- *If E is finite, then E is compact.*
- *$(0,1) \subset \mathbb{R}$ is not compact. Bad cover: $(\frac{1}{n}, 1), n > 2$*
- *$[0, \infty] \subset \mathbb{R}$ is not compact. Bad cover: $(-1, n)$ for $n \in \mathbb{N}$.*
- *$E \subset \mathbb{R}^k$ is compact if and only if E is closed and bounded.*

Theorem 22. If K is compact then K is closed.

Proof. Suppose K is compact. It suffices to prove that K^c is open. Let $p \in K^c$. We need to produce $r > 0$ s.t. $N_r(p) \subset K^c$. For $q \in K$, let $W_q = N_{r_q}(q)$, where $r_q = \frac{1}{2}d(p,q) > 0$. $\forall_{x \in N_{r_q}(p)} : x \in W_q \Rightarrow d(x,p) + d(x,q) < 2r_q = d(p,q)$. However, X is a metric space and $p, q, x \in X$, so $d(p,q) \leq d(p,x) + d(x,q)$, leading to $d(p,q) \leq d(p,x) + d(x,q) < d(p,q)$, which is a contradiction. Hence, $\forall_{x \in N_{r_q}} : x \notin W_q$. $N_{r_q}(p) \subset {W_q}^c$ for $\forall_{q \in K}$. Note that $\{W_q\}_{q \in K}$ is an open cover of K. K compact $\Rightarrow \exists_{\text{finite number of open sets } W_{q_1}, W_{q_2}, \ldots W_{q_n}}$ s.t. $K \subset \bigcup_{i=1}^n W_{q_i}$. Let $r = \min\{r_{q_1}, r_{q_2}, \ldots r_{q_n}\} > 0$.

$$\therefore N_r(p) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} N_{r_p}(p) \right) \subset \left(\bigcap_{i \in \{1,2,\ldots n\}} {W_{q_i}}^c \right) = \left(\bigcup_{i \in \{1,2,\ldots \mathbb{N}\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

Theorem 23. If $K \subset X$ is compact then K is bounded; i.e., $\exists_{M < \infty}$ s.t. $\forall_{p,q \in K} : d(p,q) \leq M$

Proof. Fix $p \in K$. An open cover of K is $\{N_n(p)\}_{n \in \mathbb{N}}$. In fact, this is an open cover of X. K compact $\Rightarrow \exists_{\text{finite subcover} N_{n_1}(p), N_{n_2}(p) \ldots N_{n_m}(p)}$. Let $R = \max\{n_1, n_1, \ldots n_m\}$. $K \subset N_R(p)$. Let $M = 2R$. $\forall_{q,r \in K} : d(q,r) \leq d(q,p) + d(p,r) < R + R = 2R = M$. Q.E.D.

Theorem 24. If F is closed, K is compact, and $F \subset K$ then F is compact.

Proof. Suppose $F \subset K$. let $\{V_\alpha\}$ be an open cover of F. It suffice to produce a finite subcover:
Consider $\{V_\alpha\}$ together with F^c. This gives an open cover of X, hence of K, so $\exists_{\text{subcover of } K}$. Drop F^c from this finite subcover. The result is a finite subcover of $\{V_\alpha\}$, which covers F Q.E.D.

Corollary. If F is closed and K is compact then $F \cap K$ is compact.

Theorem 25. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y.

Note. This is not true for open sets. For instance, let $K = Y = [0,1] \subset X = \mathbb{R}$. Y is open and closed relative to Y, but Y is not open relative to X

Proof.

($\Rightarrow$) Suppose K is compact relative to X. Let $\{V_\alpha\}$ be an open cover of K relative to Y. For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then $\{V_\alpha\}$ is an open cover of K relative to X. Since K is compact relative to X, $\exists_{\text{finite subcover}}$.

($\Leftarrow$) Suppose K is compact relative to Y. Let $\{V_\alpha\}$ be an open cover of K relative to X. Then $\{V_\alpha \cap Y\}$ is an open cover of K relative to Y. Since K is compact relative to Y, $\exists_{\text{finite subcover}}$.

Q.E.D.

Theorem 26. Suppose $\{K_\alpha\}$ is a collection of compact sets such that $\bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$. Then $\lim\limits_{n\to\infty} \bigcap_{i\in\{1,2,\ldots,n\}} K_{\alpha_i} \neq \emptyset$, or equivalently, $\bigcap_\alpha K_\alpha \neq \emptyset$

Example. *Let $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$. Then $\{G_j\}$ is a collection of open sets, but none of them are compact. (compact sets are closed) Then $\{G_j\}$ satisfies non-empty finite intersection property but $\bigcap_{j\in\mathbb{N}} G_j = \emptyset$.*

Proof. Then $K_{\alpha_0} \cap \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right) = \emptyset$, so $K_{\alpha_0} \subset \left(\bigcap_{\alpha\neq\alpha_0} K_\alpha\right)^c = \bigcup_{\alpha\neq\alpha_0} (K_\alpha)^c$ and $\{(K_\alpha)^c\}_{\alpha\neq\alpha_0}$ is an open cover of K_{α_0}, so $\exists$ a finite subcover $K_{\alpha_0} \subset \bigcup_{i=1}^n {K_{\alpha_i}}^c$. But then, $K_{\alpha_0} \cap (\bigcap_{i=1}^n K_{\alpha_i}) = \emptyset$, contradiction. Q.E.D.

Corollary. If $\{K_1, K_2, \ldots\}$ are non-empty compact sets with $\forall_n : K_n \supset K_{n+1}$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

Proof. If $n_1 < n_N$ then $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$ Q.E.D.

Theorem 27. If K is compact and $E \subset K$ is infinite, then E has a limit point in K.

Proof. Contrapositive of the statement is : *if $E \subset K$ has no limit point in K, then E is finite.*
Suppose every point $g \in K$ is not a limit point of E. Then

$$\exists_{V_q = N_{r_g}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}$$

$\{V_q\}_{q\in K}$ is an open cover of K, so $\exists_{\text{finite subcover } V_{q_1} \cup V_{q_2}, \ldots V_{q_n}}$. Then $E = E \cap K \subset (\bigcup_{i=1}^n V_{q_i} \cap E) \subset \{q_1, q_2, \ldots q_n\}$, so E is finite.

Q.E.D.

Theorem 28. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be such that $\forall_n : I_n \supset I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Since $I_n \supset I_{n+1}$, $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$. Let $E = \{a_1, a_2, \ldots\}$. Then $E \neq \emptyset$, every b_k is an upper bound for E, so $\exists x = \sup E$ and $a_k \leq x \leq b_k$ for all k. Therefore, $x \in I_k$ for all k, so $x \in \bigcap_{n=1}^{\infty} I_n$. Q.E.D.

Theorem 29. Let $\{I_n\}$ be a sequence of k-cells such that $i_n \supset I_{n+1}$;i.e., $I_n = \{\mathbf{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj},\ a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj}$ for $j = 1, 2, \ldots, k\}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Apply previous theorem to each component. Q.E.D.

Note. k-cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of k *closed intervals on the real line.*
Formally, Given real numbers a_i and b_i such that $a_i < b_i$ for every integer i from 1 to k,

$$I = \{x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, k\}$$

Theorem 30. Let $I \subset \mathbb{R}^k$ be a k-cell. Then I is compact.

Proof. Let $I = I_0 = \{x = (x_1, x_2, \ldots, x_k), a_j \leq x_j \leq b_j\}$.
Let $\Delta = \left\{\Sigma_i^k (b_j - a_j)^2\right\}^{1/2}$. Then $|\mathbf{x} - \mathbf{y}| \leq \Delta$ for $\mathbf{x}, \mathbf{y} \in I$.
Suppose for contradiction $\{G_\alpha\}$ is an open cover of I that has no finite subcover.
Let $c_j = \frac{1}{2}(a_j + b_j)$ for $j = 1, 2, \ldots, k$. Using $[a_j, c_j], [c_j, b_j]$, we get 2^k k-cells Q_i with $I = \bigcup_{i=1}^{2^k} Q_i$. At least one Q_i, call it I_1, has no finite subcover. Otherwise, every Q_i has a finite subcover, and I would have a finite subcover, namely the union of the finite subcovers of each Q_i.
Repeat this step to construct $I_0 = I, I_1, I_2, \ldots$. Then the sequence $\{I_n\}$ constructed by this process satisfies the following properties:

(a) $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \ldots$

(b) $\forall_n : I_n$ has no finite subcover from $\{G_\alpha\}$

(c) if $x, y \in I_n$ then $|x - y| \leq 2^{-n}\Delta$, where $\Delta =$ diagonal of $I = \left(\sum_{j=1}^{k} (b_j - a_j)^2\right)^{1/2}$.

By theorem 29 and (a), $\exists_{x^* \in \bigcap_{n=1}^{\infty} I_n}$. Since $x^* \in I$, $x^* \in G_{\alpha_0}$ for some α_0, so $\exists r > 0$ such that $N_r(x^*) \subset G_{\alpha_0}$. But by (c), $I_n \subset N_{2^{-n}\Delta}(x^*)$. As soon as n is large enough that $2^{-n}\Delta < r$, we have $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$, which contradicts (b). Q.E.D.

Note. Reverse triangle inequality
$\forall_{a,b,c\in X} : d(a,b) \geq d(a,c) - d(c,b)$ because $d(a,c) \leq d(a,b) + d(b,c)$.

Theorem 31. For $E \subset \mathbb{R}^k$, the following are equivalent:

(a) E is closed and bounded.

(b) E is compact.

(c) Every infinite subset of E has a limit point in E.

Proof.

$(a) \Rightarrow (b)$ Because E is bounded, there exists a k-cell I such that $E \subset I$; i.e., $\exists_M$ s.t. $\forall_{x,y \in E} : |x - y| \leq M$. Therefore, E is compact.

$(b) \Rightarrow (c)$ by theorem 28

$(c) \Rightarrow (a)$ To see that E is bounded, suppose it were not. Then E has an infinite subset $S = \{x_1, x_2, x_3, \ldots\}$ with $\forall_n : |x_n| \geq n$. S has no limit point in $\mathbb{R}^k$ Let $S = \{(x_1, x_2, x_3, \ldots) \in E : |x_n - x_0| < \frac{1}{n}\}$. Then S is an infinite set because if S is finite, there exists a point $\mathbf{x} \in S$ such that $|\mathbf{x}| \geq |\mathbf{x}'|$ for $\mathbf{x}' \in S$. However, there exists $n \in \mathbb{N}$ such that $n > |\mathbf{x}|$ and by definition of S, there exists $x_n \in S$ such that $|x_n| \geq n > |\mathbf{x}|$, which is a contradiction. Thus, S is infinite. This S however, cannot have a limit point in E. By triangle inequality, for any $y \in R^k$, $|x_n| \leq |x_n - y| + |y|$, and from archimedean property, $\exists_{m \in \mathbb{N}}$ s.t. $m > |x_n - y| + |y|$, which implies for any $y \in R^k, r > 0$, $\exists_{m \in \mathbb{N}} : |x - y| < r < m$. However, by the definition of S, there are at most m such elements in S. Since a limit point y of E must contain an infinite number of points of E such that $d(x, y) < r$ for any $r > 0$, y cannot be a limit point, which contradicts the assumption that any infinite subset of E contains a limit point in E. Therefore, E must be bounded.
To see that E is closed, suppose it were not closed. Then $\exists_{x_0 \in} : E' \setminus E$. If T has no limit point in E except $x_0 \notin E$, it contradicts (c) because T is infinite and there must be a limit point of T in E.
Therefore, we can show that E is closed by showing that T has no limit point in E except x_0. Form an infinite sequence $(x_1, x_2, x_3, \ldots), x_n \in E$ with $|x_n - x_0| < \frac{1}{n}$. Let $y \in E$, $y \neq x_0$. We'll show that y cannot be a limit point of T. $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$. Choose $n \geq \frac{2}{|y - x_0|}$, so $\frac{1}{n} \leq \frac{|y - x_0|}{2}$. Then $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$. So only finitely many x_n can lie in $N_{\frac{1}{2}|y - x_0|}(y)$. So y cannot be a limit point of S. Therefore, E is closed.

Q.E.D.

Remark. (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than $\mathbb{R}^k$.

Example. *Failure of Heine-Borel theorem in general metric spaces.*

some infinite set , discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. *Then E is bounded and closed but not compact.*

Theorem 32 (Weirstrass's theorem)**.** Every bounded infinite subset $E \subset \mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Choose a k-cell $I \supset E$. Since I is compact, by theorem 31, E has a limit point in I. Q.E.D.

Example. *Let*

$$E_0 = [0, 1] \tag{2.5}$$
$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{2.6}$$
$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \tag{2.7}$$
$$\vdots \tag{2.8}$$

This gives $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \ldots$, *where each* E_n *is the union of* 2^n *closed intervals of length* $\frac{1}{3^n}$.

Definition 26 (Cantor Set)**.** The cantor set $P := \bigcap_n^\infty E_n$.

Proposition 1. P is compact, non-empty and contains no open intervals (a, b) and uncountable.

Proof. **Compactness** P is compact because $P \subset E_0 = [0, 1]$ and E_0 is compact.

Non-emptiness P is non-empty because $P \subset E_0$ and E_0 is non-empty.

No open intervals P contains no open intervals (a, b) because any (a, b) contains some $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ and these are all removed.

Uncountability P is uncountable because P is a perfect set. Equivalently, P consists of points in $[0, 1]$ whose ternary, i.e., base 3, representation contains only 0's and 2's.

> **Note.** ternary representation: $0.a_1a_2a_3\ldots = \sum_{n=1}^\infty \frac{a_n}{3^n}$ where $a_n \in \{0, 1, 2\}$.

Q.E.D.

Example (Cantor Set)**.** *Let* $E = [0, 1]$, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, *etc. Keep removing open middle third. This gives* $E_0 \supset E_1 \supset E_2 \ldots$. *Each* E_n *is the union of* 2^n *closed intervals of length* $\frac{1}{3^n}$.

Definition 27 (Separated Sets)**.** **Separated Sets** $A, B \subset X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Connected Sets $E \subset X$ is connected if there is no non-empty separated sets $A, B \subset E$.

Example (Separated Sets). *In $\mathbb{R}^1$, $[0,1)$ and $(1,2]$ are separated so $[0,1)U(1,2]$ is not connected. Every interval is connected (open, closed, semi-open).*

Theorem 33. $E \subset \mathbb{R}^1$ is connected if and only if E is an interval; i.e., $\forall_{x,y \in E, x<y}$ s.t. $\forall_{z \in (x,y)} : z \in E$

Theorem 34. A metric space X is connected if and only if the only nonempty subset of X which is both open and closed is X itself.

Chapter 3

Sequence and Series

3.1 Sequences

Definition 28. In a metric space (X, d), a sequence $\{p_n\}$ converges to p if $\forall_{\varepsilon>0}\exists_N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.
We write $\lim_{n\to\infty} p_n = p$ or $p_n \to p$.

If $\{p_n\}$ does not converge to any p then it is said to diverge.

Theorem 35. If s_n and t_n are sequences in $\mathbb{C}$ with $s_n \to s$ and $t_n \to t$, then the following hold:

(a) $s_n + t_n \to s + t$

(b) $cs_n \to cs$, $c + s_n \to c + s$ for any $c \in \mathbb{C}$

(c) $s_n t_n \to st$

(d) $\frac{1}{s_n} \to \frac{1}{s}$ if $s \neq 0$

Lemma 2 (Squeeze Lemma)**.** In $\mathbb{R}$, if $\forall_{n\in\mathbb{N}} : 0 \leq x_n \leq s_n$ and $\lim_{n\to\infty} s_n \to 0$, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$. Then $0 \leq x_n \leq s_n < \varepsilon$ for $n \geq N$, so $x_n \to 0$. Q.E.D.

Theorem 36. (a) If $p > 0$ then $\frac{1}{n^p} \to 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $\frac{1}{N^p} < \varepsilon$; i.e., $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$.
Then for $n \geq N$, $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$. Q.E.D.

(b) If $p > 0$ then $\sqrt[n]{p} \to 1$.

Proof. $p = 1$ is obvious.
Suppose $p > 1$. Let $x_n = \sqrt[n]{p} - 1 > 0$. Want to show $x_n \to 0$.
Since $(x_n+1)^n$, we have $p = (x_n+1)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k > \binom{(}{n}, 1)x_n' = nx_n$.
Therefore, $x_n \le \frac{p}{n}$, so $x_n \to 0$ by the Squeeze Lemma.
Suppose $p \in (0,1)$. Let $q = \frac{1}{p} > 1$. Then $\sqrt[n]{q} \to 1$ by the previous case. By 35, $\sqrt[n]{p} = \frac{1}{\sqrt[n]{q}} \to 1$. Q.E.D.

(c) $\sqrt[n]{n} \to 1$

Proof. Let $x_n = \sqrt[n]{n} - 1 > 0$, for $n \ge 2$. $n = (x_n + 1)^n = \sum_{k=0}^{n} \binom{n}{k}(x_k)^k > \binom{(}{n}, 2)x_n{}^2 = \frac{n(n-1)}{2}x_n^2$. Therefore, $x_n \le \sqrt{\frac{2}{n-1}}$. Q.E.D.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \to 0$; i.e., Exponentials beat powers.

Proof. We want an upper bound on $\frac{n^\alpha}{(1+p)^n}$, so seek a lower bound on $(1+p)^n$.
$(1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k > \binom{n}{k} p^k$ for $k \le n$
$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$. Then for $k \le \frac{n}{2}$, $\binom{n}{k} p^k > (\frac{n}{2})^k \frac{p^k}{k!}$. Therefore, $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$. Let $k_0 \in \mathbb{Z}$ s.t. $k > \alpha$. Then for $n \ge 2k_0$, RHS $\to 0$ by (a).

If $|x| < 1$ then $x^n \to 0$.

Proof. $|x^n - 0| = |x|^n$, so $x^n \to 0 \Leftrightarrow |x|^n \to 0$ and $|x|^n = \frac{n_0}{(\frac{1}{|x|})^n} \to 0$ by (d) with $\alpha = 0$ and $1 + p = \frac{1}{|x|} > 1$, so $p = \frac{1}{|x|} - 1 > 0$. Q.E.D.

Q.E.D.

Theorem 37. . Let $\{p_n\}$ be a sequence in (X, d).

(a) $p_n \to p \Leftrightarrow \forall_{r>0} : N_r(p)$ contains all but finitely many p_n.

Proof. $\forall_{n \ge N} : p_n \in N_r(p)$ Q.E.D.

(b) If $p_n \to p$ and $p_n \to p'$ then $p = p'$.

Proof. $d(p, p') \le d(p_n, p) + d(p_n, p')$ for all n. Fix ε. Choose N such that $d(p_n, p) < \frac{\varepsilon}{2}$ and $d(p_n, p') < \frac{\varepsilon}{2}$ for $n \ge N'$. Then $d(p, p') < \varepsilon$. Then for $n \ge \max\{N, N'\}$, $d(p, p') < \varepsilon$. This is true for all $\varepsilon > 0$, so $d(p, p') = 0$. Q.E.D.

(c) If $\{p_n\}$ converges, then p_n is bounded, in a sense that $\exists_{M>0, q \in X}$ s.t. $d(p_n, q) \le M$ for all n.

Proof. If $p_n \to p$, then $\exists N$ s.t. $d(p_n, p) < 1$ for all $n \geq N$. Thus, $\forall_{n\geq 1} : d(p_n, p) \leq \max\{1, d(p, p_1), \ldots, d(p, p_{N-1})\} = M$ Q.E.D.

(d) If $E \subset X$ has a limit point p, then $\exists_{p_n \in E}$ s.t. $p_n \to p$.

Proof. We need to choose $p_n \in E$ s.t. $d(p, p_n) < \frac{1}{n}$. Let $\varepsilon > 0$. Then $d(p, p_n) < \varepsilon$ if $n > \frac{1}{\varepsilon}$ Q.E.D.

Definition 29. Given $p_n, n_1 < n_2 < n_3 < \ldots$, we say $p_{n_i} = (p_{n_1}, p_{n_2}, \ldots)$ is a subsequence of p_n.

Lemma 3. $p_n \to p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p

Proof. Look at assignment 6 Q.E.D.

Theorem 38. (a) $\{p_n\}$ in X, X compact, then $\exists$ convergent subsequence.

Proof. Let $E =$ range of$\{p_n\}$. If E is finite, then $\exists p \in X$ and $n_1 < n_2 < \ldots$ s.t. $p_n = p$ for $\forall i$. This subsequence converges to p. If E is infinite then by Theorem 28, E has a limit point $p \in X$; i.e., every neighborhood of p contains infinitely many points of E. Choose n_1 s.t. $d(p, p_{n_1}) < 1$.

Q.E.D.

(b) $\{p_n\}$ in $\mathbb{R}^k$, bounded, then $\exists$ convergent subsequence.

Proof. Choose a k-cell I that contains $\{p_n\}$. I is compact. Apply (a).

Q.E.D.

Definition 30 (Cauchy Sequence)**.** $\{p_n\}$ is a Cauchy sequence in (X, d) if $\forall \varepsilon : \exists_{N \in \mathbb{N}}$ s.t. $d(p_m, p_n) < \varepsilon \forall m, n \geq N$.

Definition 31. For $E \subset X$, $E \neq \emptyset$, we define diam $E = \sup\{d(p, q) : p, q \in E\}$. diam $E = \infty$ if the set is not bounded above.

Example. *For a sequence p_n in X, let $E_n = \{p_N, p_{N+1}, \ldots\}$. Then $\{p_n\}$ is a Cauchy sequence iff $\lim_{N\to\infty}$ diam $E_N = 0$.*

Theorem 39. (a) If $p_n \to p$ then $\{p_n\}$ is a Cauchy sequence.

(b) If X is a compact metric space and $\{p_n\}$ in X is a Cauchy sequence, then $\exists_{p\in X}$ s.t. $p_n \to p$.

(c) In R^K every Cauchy sequence converges.

Proof. Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ is Cauchy. Let $E_N = \{p_N, P_{N+1}, \ldots\}$. Then $\overline{E_N}$ is closed, hence compact. Also $\overline{E_N} \supset \overline{E_{N+1}}$ and $\lim\limits_{N\to\infty}$ diam $\overline{E_N} = 0$ (use Theorem 3.10(a) to see diam $\overline{E_N}$ = diam E_N) By theorem 3.10(b), $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$. Claim: $p_n \to p$.
Proof of the claim: Let $\varepsilon > 0$. Choose N_0 s.t.diam $\overline{E_{N_0}} < \varepsilon$, so $d(p, q) < \varepsilon \forall g \in \overline{E_{N_0}}$, and hence $\forall g \in N_0$; i.e., $d(p, p_n) < \varepsilon$ if $n \geq N_0$.

Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ in $\mathbb{R}^k$ is Cauchy. Cauchy sequences are bounded in any metric space. Therefore, $\exists$ k-cell I, which is compact, containing $\{p_n\}$. Then (b) applies Q.E.D.

Note. The converse of Theorem 39(a) does not hold in general.
Example. *$X = \mathbb{Q}$ has a Cauchy sequence with no limit in $\mathbb{Q}$. (see assignment 6). Converse does hold if X is compact.*

Theorem 40. (a) diam $\overline{E}$ = diam E

(b) If $K_n \subset X$, $K_n \neq \emptyset$, K compact, $K_n \supset K_{n+1} \forall n$ and if $\lim\limits_{n\to\infty}$ diam $K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a single point.

Proof. (a) $E \subset \overline{E} \Rightarrow$ diam $E \leq$ diam $\overline{E}$. For the opposite inequality, let $\varepsilon > 0, p, q \in \overline{E}$,. Choose $p', q' \in E$ s.t. $d(p, p') < \varepsilon, d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < \varepsilon + \varepsilon = 2\varepsilon$. diam $\overline{E} \leq$ diam $E + 2\varepsilon$. Since ε is arbitrary, diam $\overline{E} \leq$ diam E.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, $K \neq \emptyset$. Since $K \subset K_n \forall n$, diam $k \leq$ diam $K_n \forall n$, so diam $K = 0$. Therefore, $d(p, q) = 0 \forall p, q \in K$, so K is a simple point.

Q.E.D.

Definition 32 (Complete Metric Space)**.** A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X.

Example. *(a) X compact $\Rightarrow$ X complete.*

(b) $\mathbb{R}^k$ is complete, so is $\mathbb{C}$.

(c) $\mathbb{Q}$ is not complete. (see assignment 6)

(d) *See assignment 6 for reference to completion of a metric space.*

We've seen that convergent sequences are bounded. $p_n = (-1)^n$ shows the converse if false. However the converse does hold for monotonic sequences.

Definition 33 (Monotone). • A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically increasing if $s_n \leq s_{n+1} \forall n$.

- A sequence $\{s_n\}$ in $\mathbb{R}$ is monotonically decreasing if $s_n \geq s_{n+1} \forall n$.

Theorem 41. A monotone sequence in $\mathbb{R}$ converges if and only if it is bounded.

Proof. $\Rightarrow$ all convergent sequences are bounded in any metric space.

$\Leftarrow$ **Increasing case** Let $\{s_n\}$ be monotonically increasing and $s_n \leq M \forall n$. Let $s = \sup\{s_n : n \in \mathbb{N}\}$. Then $s_n \leq s \forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $s - \varepsilon < s_N \leq s$. But then $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \ldots \leq s$, so $|s - s_n| < \varepsilon \forall n \geq N$, and therefore $s_n \to s$.

Q.E.D.

Definition 34 (Infinite Limits). We say

- $s_n \to \infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \geq M \forall_{n \in N}$.
- $s_n \to -\infty$ if $\forall_{M \in \mathbb{R}} : \exists_N$ s.t. $s_n \leq M \forall_{n \in N}$.

Definition 35. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define $\limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}} s_n = \inf_{n\geq 1}\{\sup_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \sup_{m\geq n}\{s_m\}$.
$\liminf_{n\to\infty} s_n = \underline{\lim_{n\to\infty}} s_n = \sup_{n\geq 1}\{\inf_{m\geq n}\{s_m\}\} = \lim_{n\to\infty} \inf_{m\geq n}\{s_m\}$.

Note. Alternate definition; see ass 7 for equivalence

Remark. (a) If $a_n \leq b_n \forall n$ and $a_n \to a$ and $b_n \to b$, then $a \leq b$.

(b) $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$

Example. (a) $s_n = (-1)^n(1+\frac{1}{n^2})$ $1 \leq \sup_{m\geq n} s_m \leq 1+\frac{1}{n^2}$, *so* $\limsup_{n\to\infty} s_n = 1$. *Similarly,* $\liminf_{n\to\infty} s_n = -1$

(b) *If* $\{s_n\}$ *has no upper bound, then* $\sup_{m\geq n} s_m = \infty$ *and in this case we say* $\limsup_{n\to\infty} s_n = \infty$*; e.g.,*

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has $\limsup_{n\to\infty} s_n = \infty$, $\liminf_{n\to\infty} s_n = -\infty$

Lemma 4. $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = L \Leftrightarrow s_n \to L$.

Proof (L finite)**.**

$\Rightarrow$ This follows from $\inf_{m\geq n} s_m \leq s_n \leq \sup_{m\geq n} s_m$. $\lim\limits_{n\to\infty} \inf_{m\geq n} s_m = \liminf_{n\to\infty} s_n$, and $\lim\limits_{n\to\infty} \sup_{m\geq n} s_m = \limsup_{n\to\infty} s_n$. Therefore, $\lim\limits_{n\to\infty} s_n = L$.

$\Leftarrow$ If $s_n \to L$, then $\forall_{\varepsilon>0} : \exists_N$ s.t. $s_m \in [L-\varepsilon, L+\varepsilon] \forall m \geq N$. Therefore, $\forall_{n\geq N} : L - \varepsilon \leq \inf_{m\geq N} s_m \leq \inf_{m\geq n} s_m \leq \sup_{m\geq n} s_m \leq \sup_{m\geq N} s_m \leq L + \varepsilon$. Let $n \to \infty$: $L - \varepsilon \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L+\varepsilon$. Since ε is arbitrary, so $L \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq L$.

Q.E.D.

3.2 Series

Definition 36 (Series)**.** Let $\{a_n\}$ be a sequence in $\mathbb{C}$. Form a new sequence $\{s_n\}$, the sequence of partial sums, by $s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k$. If $s_n \to s$, we say **the series** $\sum_{k=1}^{\infty} a_k$ **converges** and that $\sum_{k=1}^{\infty} a_k = s$. If $\{s_n\}$ diverges then we say $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 42. $\sum_{n\in\mathbb{N}} a_n$ converges if and only if $\exists_{\varepsilon>0} : \exists N$ s.t. $|\sum_{k=m}^{n} a_k| < \varepsilon \forall n \geq m \geq N$.

Proof. $\sum_n a_n$ converges $\Leftrightarrow$ $\{s_n\}$ converges $\Rightarrow$ $\{s_n\}$ is a Cauchy sequence. Use $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$. Q.E.D.

Corollary. If $\sum_n a_n$ converges then $a_n \to 0$.

Proof. Take $m = n$ in Theorem 3.22. $\sum_n a_n$ converges $\Rightarrow$ $\forall_{\varepsilon>0} : \exists_N$ s.t. $|a_n| < \varepsilon$ if $n \geq N$. Q.E.D.

Remark. n-th term test for divergence: If $a_n \not\to 0$ then $\sum_n a_n$ diverges.

Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ *diverges because* $\frac{n}{n+1} \to 1 \neq 0$.

Converse to Corollary is false! E.g., $\sum_n \frac{1}{n}$ diverges but $\frac{1}{n} \to 0$.

Theorem 43. If $a_n \geq 0$, then $\sum_n a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof. $\{s_n\}$ is monotone increasing, so by Theorem 41, it converges if and only if it is bounded. Q.E.D.

Theorem 44 (Comparison Test). (a) If $|a_n| \le c_n \forall n \ge N_0$ and $\sum_n c_n$ converges, then $\sum_n a_n$ converges.

Proof. Suppose $|a_n| \le c_n \forall n \ge N_0$ and $\sum_n c_n$ converges. Let $\varepsilon > 0$. By theorem 3.22, $\exists N$ s.t. $\sum_{k=m}^{n} c_k < \varepsilon$ if $n \ge m \ge N$. Can take $N \ge N_0$. Then $|N \ge N_0|$. $|\sum_{k=m}^{n} a_k| \le \sum_{k=m}^{n} |a_k| \le \sum_{k=m}^{n} c_k < \varepsilon$ if $n \ge m \ge N$. By theorem 3.22 again, $\sum_n a_n$ converges. Q.E.D.

(b) If $a_n \ge d_n \ge 0 \forall n \ge N_0$ and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.

Proof. This follows from (a): if $\sum_n a_n$ converges then $\sum_n d_n$ converges. Thus it's contrapositive, (b) is true. Q.E.D.

Theorem 45 (Geometric Series). $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$, $xS_n = x + x^2 + \cdots x^n + x^{n+1}$. Then

$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1 (\Leftrightarrow -1 < x < 1)$, then $x^{n+1} \to 0$ and $S_n \to \frac{1}{1-x}$. If $|x| \ge 1$, then x^{n+1} does not converge to 0, so $\sum_{n=0}^{\infty} x^n$ diverges. Q.E.D.

Theorem 46. Suppose $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof. ($\Leftarrow$) We show that if $\sum_n a_n$ then $\sum_k 2^k a_{2^k}$ diverges. For this, note that $a_1 + a_2 + \ldots + a_n \le a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$ if $2^{k+1} > n$.
$a_1 + a_2 \cdots + a_n \le a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. LHS unbounded as $n \to \infty$, so RHS is also unbounded as $k \to \infty$.

($\Rightarrow$) $a_1 + a_2 + a_3 + \cdots + a_n \ge a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$ if $2^k \le n$. $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \ge a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \ge \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$. If $\sum_n a_n$ converges, then LHS is bounded for all n so RHS is bounded for all k. Hence RHS converges since it is monotone.

Q.E.D.

Theorem 47 (p-series). $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Proof. For $p \leq 0$, $\frac{1}{n^p} \not\to 0$, so series diverges. For $p > 0$, $\frac{1}{n^p}$ is decreasing, so $\sum_{n=1} \frac{1}{n^p}$ converges iff $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$ converges. But $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$ converges iff $\frac{1}{2^{p-1}} < 1 (\Leftrightarrow p - 1 > 0)$, which is equivalent to $p > 1$. Q.E.D.

Theorem 48. $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (log is to base e.)

Proof. If $p \leq 0$, then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$, so $\sum_n \frac{1}{n(\log n)^p}$ diverges by the comparison test. If $p > 0$ then $\frac{1}{n(\log n)^p}$ decreases since $\log n$ increases. By theorem 46, $\sum_n \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k (\log 2^k)^p}$ converges $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$ Q.E.D.