Real Variables I

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Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\right\}$

Remark. Note for real numbers, $\mathbb{Q}$ has holes in it.
Example. $\nexists p \in \mathbb{Q}\ s.t\ p^2 = 2$

Proof (Proof)**.** Assume $\exists p \in \mathbb{Q}$ s.t $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t $p^2 = 2$. Q.E.D.

Definition 1 (Order)**.** An order on a set S is a relation $<$ such that:

1. If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.
2. If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 2 (Ordered Set)**.** An ordered set S is a set with an order $<$.

Definition 3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t $\forall x \in E : x \leq \beta$.
Similarly, a set S is bounded below if $\exists \beta \in S$ s.t $\forall x \in E : x \geq \beta$.

Definition 4 (LUB, GLB). Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists\alpha$ s.t. α is an upper bound for E and $\forall\gamma < \alpha$: γ is not an upper bound for E. Then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists\alpha$ s.t. α is a lower bound for E and $\forall\gamma > \alpha$: γ is not a lower bound for E. Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 5 (LUB property). An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists(\sup E) \in S$

Example.

- $\mathbb{Z}$ *has the LUB property.*
- $\mathbb{Q}$ *does not have the LUB property.*

Theorem 1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof (Proof). $(\Rightarrow)$Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B. Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. $(\Leftarrow)$Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E. Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 6 (Fields). Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

1. $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).
2. $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).
4. $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.
5. $\exists(-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.
6. $\forall x, y \in F : xy \in E$.
7. $\forall x, y \in F : xy = yx0$.
8. $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.
9. If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

10. $\forall x, y, z \in F : x(y + z) = xy + xz$

Example.

1. *$\mathbb{Q}$ is a field, while $\mathbb{Z}$ is not a field.*
2. *$F_p = \{0, 1, \ldots, p - 1\}$ with mod p arithmetic is a field.*

Read Text book: 114,115,116,118

Definition 7 (Ordered Field)**.** An ordered field F is a field that is an ordered set such that the following properties are satisfied:

1. If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.
2. If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

> **Remark.** We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. *$\mathbb{Q}$ is an ordered field.*

Theorem 2. $\exists$ an ordered field $\mathbb{R}$ which has the LUB property and contains $\mathbb{Q}$ as a subfield.

Theorem 3.

(a) Arithmetic properties of $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) $\mathbb{Q}$ is dense in $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.

(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof (Proof). (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A. So, A has a least upper bound α. So, $\alpha - x < \alpha$. $\therefore \alpha - x$ is not an upper bound for A. Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A. Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. So, $ny - nx > 1$. So, $nx < y$. Let $m \in \mathbb{Z}$ such that $m > nx$. So, $nx < m$, $x < \frac{m}{n}$. Let $p = \frac{m}{n}$. So, $x < p < y$. **TODO!!**

(c) We know that $\exists p \in \mathbb{Q}$ such that $p^2 = 2$. Show that $\exists \sqrt{2} \in \mathbb{R}$ such that $(\sqrt{2})^2 = 2$. It obeys $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then $x < p < \alpha < q < y$ and $x \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n-1)x \leq y < nx. \tag{1.1}$$

Proof (Proof). Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of $\mathbb{N}$ has a smallest element. Let n = smallest element of A. Then the equation holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n-1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n-1)x$. Hence, the statement holds. Q.E.D.

Lemma 1. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b-n)$ for some $n \in \mathbb{N}$.

Proof (Proof).

$$b^n - a^n = (b-a)(\underbrace{b^{n-1} + ab^{n-2} + \ldots + a^{n-2}b + a^{n-1}}_{n \text{ terms}}) \tag{1.2}$$

$$< nb^{n-1} \tag{1.3}$$

Q.E.D.

Theorem 4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique}) y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt{x}^n$, the n^{th} root of x).

Proof (Proof). Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x.
Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \varnothing$, E is bounded above, hence (by the least-upper-bound property) there exists a $\sup E$. Choose $y = \sup E$. Consider two cases.

1. If $x \leq 1$, then $t_0 = \dfrac{x}{2}$ and thereby $t_0^n = \dfrac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).
2. If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x+1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E.

1. Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y+h)^n - y^n < x - y^n$. By the lemma 1, we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y+h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.
2. Assuming that $y^n > x$, we find $k > 0$ such that $(y-k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y, which is $\sup E$. By the lemma 1, $y^n - (y-k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \dfrac{y^n - x}{ny^{n-1}}$. Therefore, $-(y-k)^n < -x$, or $x < (y-k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 8 (Cut/Dedekind Cut). The set $\mathbb{R}$ elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 9 (Order of cuts). For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A, and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 5. There exists an ordered field $\mathbb{R}$ such that $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R}$ has the LUB property.

Proof (Proof). Let $\mathbb{R}$ be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) | a \in \mathbb{R}, b \in \mathbb{R}\}$
Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$
Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
Zero element is $(0, 0)$
One element is $(1, 0)$

Theorem 6. $\mathbb{C}$ is a field.

Proof (Proof). Verify the 11 field axioms. For just a few axioms:
(M3):
$x = (a, b), y = (c, d), z = (e, f)$. $x(yz) = (a, b)(ce - df, cf + de) = (a(ce-df)-b(cf+de), a(cf+de)+b(ce-df)) = (ac-bd, ad+bc)(e, f) = (xy)z$
(M4):
$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$
(M5):
$x \neq 0$ means $x = (a, b)$ with $a \neq 0$ or $b \neq 0$. That is, $a^2 + b^2 > 0$. Let $\frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$. Then $x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0)$. Q.E.D.

Identification of $\mathbb{R}$ as a subfield of $\mathbb{C}$. Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a+b = a+b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\mathrm{Re}(z) = a$,$\mathrm{Im}(z) = b$.

Definition 11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by $\overline{z}$

Note.

1. $\overline{z + w} = \overline{z} + \overline{w}$

2. $\overline{zw} = \overline{z} \cdot \overline{w}$

3. $z + \overline{z} = 2 \cdot \text{Re}(z)$

4. $z - \overline{z} = 2i \cdot \text{Im}(z)$

5. $z\overline{z} = (a+bi)(a-bi) = a^2 + b^2 \geq 0$, with $=$ if any only if $z = 0$

6. $\frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{a-bi}{a^2+b^2}$

Definition 12. $|z| = \sqrt{z\overline{z}} = \sqrt{a^2+b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 7. For $z, w \in \mathbb{C}$,

1. $|z| \geq 0$ with $=$ iff $z = 0$

2. $|z| = |\overline{z}|$

3. $|zw| = |z| \cdot |w|$

4. $|\text{Re}(z) \leq |z|, |\text{Im}(z)| \leq |z|$

   Proof (Proof)**.** Let $z = a+bi$. Then $|\text{Re}(z)| = |a| \leq \sqrt{a^2+b^2} = |z|$ Q.E.D.

5. $|z+w| \leq |z| + |w|$ (Triangle inequality)

   Proof (Proof)**.**

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) \\ &= z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 \\ &= |z|^2 + 2\text{Re}(z\overline{w}) + |w|^2 \\ &<= (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

Theorem 8 (Cauchy-Schwarz inequality)**.** If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$|\sum_{j=1}^{n} a_j \overline{b_j}| \leq (\sum_{j=1}^{n} |a_j|^2)^{\frac{1}{2}} (\sum_{j=1}^{n} |b_j|^2)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^{n} a_j \overline{b_j}$ defined on inner product on $\mathbb{C}^n$ and $|(\vec{a}, \vec{b}) \leq \sqrt{(\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{b})}$.

Proof (Proof). Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \overline{b_j}$ We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0.
For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^{n} |a_j + \lambda b_j|^2 = \sum_{j=1}^{n} (a_j + \lambda b_j)(\overline{a_j} + \overline{\lambda b_j}) = \sum_{j=1}^{n} |a_j|^2 + \lambda \sum_{j=1}^{n} b_j \overline{a_j} + \overline{\lambda} \sum_{j=1}^{n} a_j \cdot \overline{b_j} + |\lambda|^2 \sum_{j=1}^{n} |b_j|^2$. So $0 \leq A + \lambda \overline{C} + \overline{\lambda} C + |\lambda|^2 lhiB$ Q.E.D.