

Real Variables I

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Chapter 1

Number Systems

Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$

Integers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers: $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

Remark. Note for real numbers, \mathbb{Q} has holes in it.

Example. $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$

Proof. Assume $\exists p \in \mathbb{Q}$ s.t. $p^2 = 2$. Then $p = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $b \neq 0$. So, $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$. So, a^2 is even $\Rightarrow a$ is even. So, $a = 2k$ for some $k \in \mathbb{Z}$. So, $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$. So, b^2 is even $\Rightarrow b$ is even. So, $b = 2l$ for some $l \in \mathbb{Z}$. So, a and b are both even, which contradicts the fact that a and b are coprime. So, $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$. Q.E.D.

Definition 1 (Order). An order on a set S is a relation $<$ such that:

- (a) If $a, b \in S$, then exactly one of $a < b$, $a = b$, or $b < a$ is true.
- (b) If $a, b, c \in S$ and $a < b$ and $b < c$, then $a < c$.

Definition 2 (Ordered Set). An ordered set S is a set with an order $<$.

Definition 3. Let S be an ordered set. A set $E \subset S$ is bounded above if $\exists \beta \in S$ s.t. $\forall x \in E : x \leq \beta$.
Similarly, a set S is bounded below if $\exists \beta \in S$ s.t. $\forall x \in E : x \geq \beta$.

Definition 4 (LUB, GLB). Let S be an ordered set and $E \subset S$, $E \neq \emptyset$, with E bounded above. If $\exists \alpha$ s.t. α is an upper bound for E and $\forall \gamma < \alpha$: γ is not an upper bound for E , then such α is called least upper bound (LUB), or *Supremum*. Similarly, if $\exists \alpha$ s.t. α is a lower bound for E and $\forall \gamma > \alpha$: γ is not a lower bound for E . Then such α is called greatest lower bound (GLB), or *Infimum*.

Definition 5 (LUB property). An ordered set S has the least upper bound (LUB) property if $\forall E \subset S$ if $E \neq \emptyset$ and E bounded above implies $\exists \sup E \in S$; i.e., Every bounded subset of S has the least upper bound(LUB).

Example.

- \mathbb{Z} has the LUB property.
- \mathbb{Q} does not have the LUB property.

Theorem 1. Let S be an ordered set. Then S has the LUB property if and only if S has the GLB property.

Proof. (\Rightarrow) Suppose S has the LUB property. Let $B \subset S$ be non-empty and bounded below. Let L be the set of all lower bounds of B . Then L is non-empty and bounded above. Let $\alpha = \sup L$. We claim that $\alpha = \inf B$. (\Leftarrow) Suppose S has the GLB property. Let $E \subset S$ be non-empty and bounded above. Let U be the set of all upper bounds of E . Then U is non-empty and bounded below. Let $\beta = \inf U$. We claim that $\beta = \sup E$. Q.E.D.

Definition 6 (Fields). Let F be a set with two operations, addition and multiplication. Then F is a field if the following axioms are satisfied:

- $a + b = b + a$ and $a \cdot b = b \cdot a$ for all $a, b \in F$ (Commutative laws).
- $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in F$ (Associative laws).
- $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in F$ (Distributive law).
- $\exists 0 \in F$ s.t. $a + 0 = a$ for all $a \in F$.
- $\exists (-a) \in F$ s.t. $a + (-a) = 0$ for all $a \in F$.
- $\forall x, y \in F : xy \in F$.
- $\forall x, y \in F : xy = yx$.
- $\exists 1 \in F$ s.t. $a \cdot 1 = a$ for all $a \in F$.
- If $a \neq 0$, then $\exists a^{-1} \in F$ s.t. $a \cdot a^{-1} = 1$.

(j) $\forall x, y, z \in F : x(y + z) = xy + xz$

Example.

(a) \mathbb{Q} is a field, while \mathbb{Z} is not a field.

(b) $F_p = \{0, 1, \dots, p-1\}$ with mod p arithmetic is a field.

Read Text book: 114, 115, 116, 118

Definition 7 (Ordered Field). An ordered field F is a field that is an ordered set such that the following properties are satisfied:

(a) If $a, b, c \in F$ and $a < b$, then $a + c < b + c$.

(b) If $a, b \in F$ and $0 < a$ and $0 < b$, then $0 < ab$.

Remark. We say x is positive if $x > 0$ and x is negative if $x < 0$.

Example. \mathbb{Q} is an ordered field.

Theorem 2. \exists an ordered field \mathbb{R} which has the LUB property and contains \mathbb{Q} as a subfield.

Theorem 3.

(a) Arithmetic properties of \mathbb{R} : If $x, y \in \mathbb{R}$ and $x > 0$ then $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) \mathbb{Q} is dense in \mathbb{R} : If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.

(c) $x, y \in \mathbb{R}$ then $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < \alpha < y$.

Proof. (a) Let $A = \{nx \mid n \in \mathbb{N}\}$. Suppose $\forall nx \in A : nx \leq y$. Then y is an upper bound for A . So, A has a least upper bound α . Since $\alpha - x < \alpha$ as $x > 0$, $\alpha - x$ is not an upper bound for A . Thus, $\exists m \in \mathbb{N} : mx > \alpha - x$, so $\alpha < (m+1)x \in A$, contradicting the fact that α is a supremum of A . Therefore, $\exists n \in \mathbb{N}$ such that $nx > y$.

(b) Since $y - x > 0$, by (a), $\exists n \in \mathbb{N}$ such that $n(y - x) > 1$. $ny - nx > 1$ and therefore, $1 + nx < ny$. Let $m \in \mathbb{Z}$ such that $(m - 1) \leq nx < m$. Such m exists by the extended version of (a). This implies there exists $m \in \mathbb{N}$ such that $nx < m \leq nx + 1 < ny$. Therefore, $x < \frac{m}{n} < y$.

(c) $\exists k \in \mathbb{Q}$ such that $k^2 = 2$; i.e., $\exists \sqrt{2} \in \mathbb{R}$. $0 < \sqrt{2} < 2$ because if $\sqrt{2} \geq 2$ then $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$, which is a contradiction. By (b), $\exists p \in \mathbb{Q}$ such that $x < p < y$ and $\exists q \in \mathbb{Q}$ such that $x < p < q < y$. Let $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$. Then $x < p < \alpha < q < y$ and $\alpha \notin \mathbb{Q}$ since otherwise $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$ would be rational

Q.E.D.

Note. (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n - 1)x \leq y < nx. \quad (1.1)$$

Proof. Case 1: $y \geq 0$. Let $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$. By (a), $A \neq \emptyset$. Every non-empty subset of \mathbb{N} has a smallest element. Let $n = \text{smallest element of } A$. Then the inequality holds true. Case 2: Let $y < 0$, then there exists $n \in \mathbb{N}$ such that $(n - 1)x \leq -y < nx$, which implies that (by changing sign for all terms) $-nx < y \leq -(n - 1)x$. Hence, the statement holds.

Q.E.D.

Lemma 1. Let $a, b \in \mathbb{R}$ such that $0 < a < b$, then $0 < b^n - a^n \leq nb^{n-1}(b - a)$ for some $n \in \mathbb{N}$.

Proof.

$$\begin{aligned} b^n - a^n &= (b - a) \underbrace{(b^{n-1} + ab^{n-2} + \dots + a^{n-2}b + a^{n-1})}_{n \text{ terms}} \\ &< (b - a)nb^{n-1} \end{aligned}$$

Q.E.D.

Theorem 4. $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique}) y > 0 : y^n = x$ (we write $y = x^{1/n} = \sqrt[n]{x}$, the n^{th} root of x).

Proof. Uniqueness: For any $y_1, y_2 \in \mathbb{R}$, if $0 < y_1 < y_2$, then $0 < y_1^n < y_2^n$, hence y_1^n and y_2^n cannot both be equal to x .

Existence: Let $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$. If $E \neq \emptyset$, E is bounded above, hence (by the least-upper-bound property) there exists a sup E . Choose $y = \sup E$. Consider two cases.

- (a) If $x \leq 1$, then $t_0 = \frac{x}{2}$ and thereby $t_0^n = \frac{x^n}{2^n} < x^n \leq x$ (by assumption that $x \leq 1$).
- (b) If $x > 1$, then let $t_0 = 1$, leading to $t_0^n = 1 < x$.

In either case, $t_0 \in E$, and hence E is not empty. 1(a) (E is bounded above) Let $\beta = x + 1$. Then, $\beta^n = (x + 1)^n > x + 1 > x$. Then, for any $t \in E$, we have that $t^n < x < \beta^n$, hence $t < \beta$, making t an upper bound of E .

- (a) Assuming that $y^n < x$, we find $0 < h < 1$ such that $(y+h)^n < x$, which leads to $y + h \in E$, something that contradicts with the fact that $y = \sup E$. This is equivalent to finding an $0 < h < 1$ such that $(y + h)^n - y^n < x - y^n$. By the lemma 1, we have $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$ for any $0 < h < 1$. Choose h so that $\frac{(x-y)^n}{n(y+1)^{n-1}}$. Then $0 < h < 1$ still holds and $hn(y+1)^{n-1} < x - y^n$, leading to $(y + h)^n < x$, and therefore $y + h \in E$. However, this contradicts the fact that $y = \sup E$ as $y + h > y$.
- (b) Assuming that $y^n > x$, we find $k > 0$ such that $(y-k)^n > x$, which leads to a contradiction since otherwise $y - k$ would be an upper bound for E that's smaller than y , which is $\sup E$. By the lemma 1, $y^n - (y-k)^n \leq ny^{n-1}k < y^n - x$ for any $h < \frac{y^n - x}{ny^{n-1}}$. Therefore, $-(y-k)^n < -x$, or $x < (y-k)^n$. Thus, $y - k$ is also an upper bound of E and $y - k < y = \sup E$, which is a contradiction.

Since $y^n < x$ and $y^n > x$ are both contradictions, $y^n = x$. Q.E.D.

Definition 8 (Cut/Dedekind Cut). The set \mathbb{R} elements are (Dedekind) cuts, which are sets $\alpha \subset \mathbb{Q}$ such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in α

Example. $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$, $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

Definition 9 (Order of cuts). For $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta := \alpha \subset \beta$

Proof (test). Let γ be set of cuts A , and show that γ is a cut and that $\gamma = \sup A$. Q.E.D.

Theorem 5. There exists an ordered field \mathbb{R} such that $\mathbb{Q} \subset \mathbb{R}$ and \mathbb{R} has the LUB property.

Proof. Let \mathbb{R} be the set of all cuts with:

order $a < b := a \subset b$.

addition $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$.

multiplication $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$.

Q.E.D.

Complex Numbers

Definition 10 (Complex Field). The underlying set is $\mathbb{C} = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$

Addition is defined as $(a, b) + (c, d) = (a + c, b + d)$

Multiplication is defined as $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

Zero element is $(0, 0)$

One element is $(1, 0)$

Theorem 6. \mathbb{C} is a field.

Proof. Verify the 11 field axioms. For just a few axioms:

(M3):

$$x = (a, b), y = (c, d), z = (e, f). \quad x(yz) = (a, b)(ce - df, cf + de) = (a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)) = (ac - bd, ad + bc)(e, f) = (xy)z$$

(M4):

$$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$$

(M5):

$$x \neq 0 \text{ means } x = (a, b) \text{ with } a \neq 0 \text{ or } b \neq 0. \text{ That is, } a^2 + b^2 > 0. \text{ Let } \frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}). \text{ Then } x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0). \quad \text{Q.E.D.}$$

Identification of \mathbb{R} as a subfield of \mathbb{C} . Identify $(a, 0) \in \mathbb{C}$ with $a \in \mathbb{R}$. Then $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 0)(b, 0) = (ab, 0)$, so we can represent them by $a + b = a + b$, $a \cdot b = a \cdot b$. Write $i = (0, 1)$. $i^2 = (0, 1)(0, 1) = (-1, 0)$. So, $i^2 = -1$. $(a, b) \leftrightarrow a + bi$. Usually write $z = a + bi$ for $z \in \mathbb{C}$. $\text{Re}(z) = a, \text{Im}(z) = b$.

Definition 11. Complex conjugate of $z = a + bi$ is defined as $a - bi$ and denoted by \bar{z}

Note.

- (a) $\overline{z + w} = \bar{z} + \bar{w}$
- (b) $\overline{zw} = \bar{z} \cdot \bar{w}$
- (c) $z + \bar{z} = 2 \cdot \operatorname{Re}(z)$
- (d) $z - \bar{z} = 2i \cdot \operatorname{Im}(z)$
- (e) $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$, with $=$ if and only if $z = 0$
- (f) $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{a^2+b^2}$

Definition 12. $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$

In particular, if $z = a \in \mathbb{R}$ then $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Theorem 7. For $z, w \in \mathbb{C}$,

- (a) $|z| \geq 0$ with $=$ iff $z = 0$
- (b) $|z| = |\bar{z}|$
- (c) $|zw| = |z| \cdot |w|$
- (d) $|\operatorname{Re}(z)| \leq |z|, |\operatorname{Im}(z)| \leq |z|$

Proof. Let $z = a + bi$. Then $|\operatorname{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$
Q.E.D.

- (e) $|z + w| \leq |z| + |w|$ (Triangle inequality)

Proof.

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

Theorem 8 (Cauchy-Schwarz inequality). If $a_1, a_n, b_1, b_n \in \mathbb{C}$, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right| \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n |b_j|^2 \right)^{\frac{1}{2}}.$$

Interpretation: $(\vec{a}, \vec{b}) = \sum_{j=1}^n a_j \bar{b}_j$ defined on inner product on \mathbb{C}^n and $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a}, \vec{a})(\vec{b}, \vec{b})}$. (Note that $\vec{a} \cdot \vec{b} = \overline{\vec{b} \cdot \vec{a}}$)

Proof. Let $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$. We can assume 1. $B \neq 0$ because $B = 0$ is $0 \leq 0$, 2. $C \neq 0$ because $C = 0$, LHS is 0. For any $\lambda \in \mathbb{C}$, $0 \leq \sum_{j=1}^n |a_j + \lambda b_j|^2 = \sum_{j=1}^n (a_j + \lambda b_j)(\bar{a}_j + \bar{\lambda} \bar{b}_j) = \sum_{j=1}^n |a_j|^2 + \lambda \sum_{j=1}^n b_j \bar{a}_j + \bar{\lambda} \sum_{j=1}^n a_j \bar{b}_j + |\lambda|^2 \sum_{j=1}^n |b_j|^2$. Let $\lambda = tC$ for $t \in \mathbb{R}$. Then $0 \leq A + \lambda \bar{C} + \bar{\lambda} C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$. $p(t)$ is a quadratic function in terms of t and it must be non-negative. Regardless of the value of t . Therefore, the discriminant of $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$. Since $|C| \geq 0$, $|C|^2 \leq AB$. Q.E.D.

Definition 13 (Euclidean k -space). For $k \in N$, $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \dots, x_k) : x_1, x_2, \dots, x_k \in \mathbb{R}\}$ with the following properties:

Addition $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_k + y_k)$

Scalar multiplication $\lambda \vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_k)$

Inner(dot) product $(\vec{x}, \vec{y}) = \sum_{j=1}^k x_j y_j$, which is bilinear: $(\alpha \vec{x} + \beta \vec{y}) \cdot \vec{z} = \alpha \vec{x} \cdot \vec{z} + \beta \vec{y} \cdot \vec{z}$.

Norm $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^k |x_j|^2^{1/2}$

Remark. Addition and Scalar multiplication make \mathbb{R}^k into a vector space.

Theorem 9. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$. Then

- (a) $|\vec{x}| \geq 0$
- (b) $|\vec{x}| = 0 \leftrightarrow \vec{x} = \vec{0}$
- (c) $|\alpha \vec{x}| = |\alpha| |\vec{x}|$
- (d) $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| |\vec{y}|$ (special case of Cauchy-Schwarz inequality)
- (e) $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ (Triangle inequality)

Proof. $|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x}| |\vec{y}| + |\vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2$ Q.E.D.

- (f) $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$

Proof. $|\vec{x} - \vec{y}| = |(\vec{x} - \vec{z}) + (\vec{z} - \vec{y})| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$ Q.E.D.

Chapter 2

Basic Topology

Definition 14. Sets A and B have the same cardinality, if $\exists f : A \rightarrow B$ that is 1-1 and onto (i.e., bijective).

Theorem 10. Let $A \sim B$ be a relation between two sets having the same cardinality. Then \sim is an equivalence relation. That is,

- (a) $A \sim A$ (Reflexive)
- (b) $A \sim B \Rightarrow B \sim A$ (Symmetry)
- (c) $A \sim B \& B \sim C \Rightarrow A \sim C$ (Transitivity)

Definition 15. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. Let $J_n = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$.

- A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}$ (or if $A = \emptyset$).
- A set A is countably infinite if $A \sim \mathbb{N}$.
- A set A is countable if A is finite or countably infinite.

Example. \mathbb{Z} is a countably infinite. For $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$,

$$\text{Let } f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n + 1 & \text{if } n < 0 \end{cases}$$

Then f is bijective and therefore $|\mathbb{Z}| = |\mathbb{N}|$

Theorem 11. A subset of a countably infinite set is countable.

Proof. Let A be some countably infinite set and S be a infinite subset of A .

As A is a countably infinite set, we can remove duplicates and arrange A so that $A = \{a_1, a_2, a_3, \dots\}$. Let n_1 be the smallest positive integer such that $x_{n_1} \in S$. Let n_k be the smallest positive integer greater than n_{k-1} such that $x_{n_k} \in S$ for $k = 2, 3, \dots$. Let $f(k) = x_{n_k}$ for $k = 1, 2, 3, \dots$. Then this is a bijection from \mathbb{N} to S . Q.E.D.

Remark. Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

Theorem 12. Let E_1, E_2, \dots be countably infinite sets. Then $S = \bigcup_{n=1}^{\infty} E_n$ is countably infinite.

Proof. Write $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \dots\}$

$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \dots\}$

Form an array:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \dots \\ x_{21} & x_{22} & x_{23} & x_{24} & \dots \\ x_{31} & x_{32} & x_{33} & x_{34} & \dots \\ x_{41} & x_{42} & x_{43} & x_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix might have duplicates. Let T be a subset of \mathbb{N} such that $t \in T$ if and only if t is the smallest positive integer such that $x_t \in E_1 \cup E_2 \cup \dots \cup E_n$.

Then a set $\{x_t | t \in T \text{ and } \exists i \in \mathbb{N} : x_t \in E_i\}$ is S . Clearly, $|S| = |T|$, or $S \sim T$, and T is a subset of a countably infinite set, \mathbb{N} . Therefore, T is also countable, implying S is also countable. As S is infinite, S is countably infinite. Q.E.D.

Corollary. If A is countable and $n \in \mathbb{N}$, then $\{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in A\}$ is countable.

Theorem 13. Let $A = \{(b_1, b_2, b_3 \dots) | b_i \in \{0, 1\}\}$. I.e., A is a set of all infinite binary strings. Then A is uncountable.

Proof (Cantor's Diagonalization argument, 1891). Let $E \subset A$ be countably infinite. $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \mid s^{(i)} \in A\}$. It suffices to find some $s \in A \setminus E$, for this shows every countably infinite subset of A is proper construction of s . Write

$$s^{(1)} = b_1^1 b_2^1 \dots \quad (2.1)$$

$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \quad (2.2)$$

$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \quad (2.3)$$

$$\vdots$$

On diagonal, flip each bit, i.e., $0 \rightarrow 1$ and $1 \rightarrow 0$ and represent the flipped bit of b_i^i by \tilde{b}_i^i . Let $s = \tilde{b}_1^1 \tilde{b}_2^2 \tilde{b}_3^3 \dots$. Then $s \in A$ and $s \notin E$ as s differs from each $s^{(i)}$ in the i -th bit. Therefore, A is uncountable. Q.E.D.

Corollary. The set $\mathcal{P}(\mathbb{N})$ of subsets of \mathbb{N} is uncountable.

Proof. We can create $f : \mathcal{P}(\mathbb{N}) \rightarrow A$ be a bijection, where A is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \quad (2.4)$$

For example, if $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0, \dots)$. This f is a bijection, and therefore A is uncountable.

Q.E.D.

Theorem 14. \mathbb{R} is uncountable.

Proof. This is a rough sketch of the proof:

- (a) It's enough to show that $[0, 1]$ is uncountable.
- (b) Consider binary decimal representation of $x \in [0, 1]$. For example, $x = 0.101001001\dots$. Given x , choose maximal $b_1 \in \{0, 1\}$ such that $\frac{b_1}{2} \leq x$. Then choose $b_2 \in \{0, 1\}$ such that $\frac{b_1}{2} + \frac{b_2}{2} \leq x$. Continue this process to get b_1, b_2, b_3, \dots . Then $x = \sup \left\{ \sum_{i=1}^n \frac{b_i}{2^i} \right\}$. Consider any dyadic rational of the form $\frac{m}{2^n}$. Let it be $\frac{3}{2^4}$. Then this maps $\frac{3}{2^4} \rightarrow 0, 0, 1, 1, 0, 0, 0, \dots$ and never produce $0, 0, 1, 0, 1, 1, 1, \dots$, which also represents $\frac{3}{2^4}$. Let A_1 be a subset of $A = \{\text{infinite binary strings}\}$ such that A_1 does not contain any strings ending in $1, 1, 1, 1, \dots$. Then the decimal representation defines a bijection $f : [0] \rightarrow A \setminus A_1$.
- (c) A_1 is countable because $A = (A \setminus A_1) \cup A_1$, which is uncountable.

This shows that $[0, 1]$ is uncountable, and therefore \mathbb{R} is uncountable.
Q.E.D.

Definition 16 (Metric Spaces). A set X is a metric space with metric $d : X \times X \rightarrow \mathbb{R}$ if

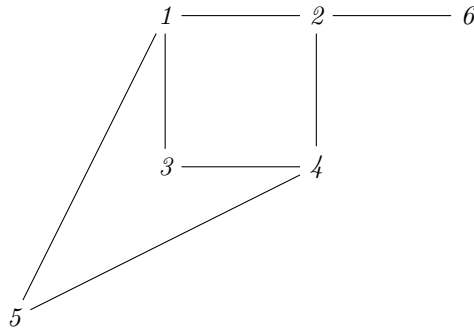
- (a) $d(p, q) > 0$ if $p \neq q$ and $d(p, q) = 0$ if $p = q$, $\forall p, q \in X$
- (b) $\forall p, q \in X : d(p, q) = d(q, p)$
- (c) $\forall p, q, r \in X : d(p, q) \leq d(p, r) + d(r, q)$ (Triangle Inequality)

Remark. A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

Example (Metric Spaces). (a) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$ are metric spaces with $d(p, q) = |p - q|$. Note the meaning of $|x|$ depends on the context.

(b) Every subset of a metric space is a metric space.

(c) $X = \{1, 2, 3, 4, 5, 6\}$



Definition 17 (Neighborhood). A neighborhood in X is a set $N_r(p) := \{q : d(q, p) < r\}$, where $p \in X, r > 0$.

Remark. If $r_1 \leq r_2$, then $N_{r_1}(p) \subset N_{r_2}(p)$.

Example.

\mathbb{R}^1 intervals, $N_r(x) = \{y \in \mathbb{R}^1 : |x - y| < r\}$

\mathbb{R}^2 disks $N_r(x) = \{y \in \mathbb{R}^2 : |x - y| < r\}$

\mathbb{R}^3 balls, $N_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}$

Given example (c), $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$, $N_2(2) = \{1, 2, 4, 6\}$, $N_3(2) = \{1, 2, 3, 4, 5, 6\} = X$.

Definition 18. Let $E \subset X$. $p \in E$ is an interior point of E if $\exists r > 0$ such that $N_r(p) \subset E$.

Example.

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$.

Definition 19. $E \subset X$ is an open set if $\forall x \in E$ is an interior point of E .

Theorem 15. Every neighborhood is an open set.

Proof. Let $g \in N_r(p)$. Then we must find $s > 0$, such that $N_s(g) \subset N_r(p)$. We know $d(p, g) < r$. Choose s such that $0 < s < r - d(p, g)$. Let $x \in N_s(g)$, then $d(g, x) < s < r - d(p, g)$. By triangle inequality, $d(p, x) \leq d(p, g) + d(g, x) < d(p, g) + r - d(p, g)$, so $x \in N_r(p)$, so $N_s(g) \subset N_r(p)$. Q.E.D.

Definition 20. Let $E \subset X$ and $p \in X$. p is a limit point of E if $\forall r > 0 \exists q \in E$ such that $q \neq p$ and $q \in N_r(p)$

Example. $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ has exactly one limit point, 0. note $0 \notin E$.

Theorem 16. If p is a limit point of $E \subset S$, then every neighborhood of p contains infinitely many points of E .

Proof. Let $N_r(p)$ be a neighborhood of p . Then $N_r(p)$ contains at least one point $q_1 \in E$ such that $q_1 \neq p$. Let $r_1 = d(p, q_1)$. Then $N_{r_1}(p)$ contains some $q_2 \in E$ such that $q_2 \neq p$. Let $r_2 = d(p, q_2)$. Then $N_{r_2}(p)$ contains some $q_3 \in E$ such that $q_3 \neq p$. Continue this process to get q_1, q_2, q_3, \dots Q.E.D.

Corollary. If $E \subset X$ is finite then E has no limit points.

Definition 21 (Closed Set). A set $E \subset X$ is closed if every limit point of E is in E .

Theorem 17. $E \subset X$ is open iff $E^c = \{x \in X : x \notin E\}$ is closed.

Proof.

- E is open $\Rightarrow E^c$ is closed.
Let p be a limit point of E^c . Then every neighborhood of p contains some $q \in E^c$ such that $q \neq p$. If $p \in E$, then because E is open, p is an interior point, i.e., there exists some neighborhood of p that is a subset of E , which does not contain any points of E^c . This implies $p \notin E$ and therefore $p \in E^c$.
- E^c is closed $\Rightarrow E$ is open.
Let $p \in E$. Then $p \notin E^c$, so p is not a limit point of E^c . Therefore, there exists some neighborhood of p that contains no points of E^c , i.e., all points of the neighborhood are in E . Thus, Every $p \in E$ is an interior point of E , and hence E is open.

Q.E.D.

Theorem 18 (De Morgan's Laws).

- (a) $(\bigcup_{\alpha} E_{\alpha})^c = \bigcap_{\alpha} E_{\alpha}^c$
- (b) $(\bigcap_{\alpha} E_{\alpha})^c = \bigcup_{\alpha} E_{\alpha}^c$

Theorem 19.

- (a) For all collection $\{G_{\alpha}\}$ of open sets : $\bigcup_{\alpha} G_{\alpha}$ is open.
- (b) For all collection $\{F_{\alpha}\}$ of closed sets : $\bigcap_{\alpha} F_{\alpha}$ is closed.
- (c) For all finite collection $\{G_1, G_2, \dots, G_n\}$ of open sets : $\bigcap_{i=1}^n G_i$ is open.
- (d) For all finite collection $\{F_1, F_2, \dots, F_n\}$ of closed sets : $\bigcup_{i=1}^n F_i$ is closed.

- Proof.** (a) Let $x \in \bigcup_{\alpha} G_{\alpha}$. Then $x \in G_{\alpha_0}$ for some α_0 . So there exists a neighborhood N of x such that $N \subset G_{\alpha_0} \subset \bigcup_{\alpha} G_{\alpha}$.
- (b) it's suffice to prove that $(\bigcap_{\alpha} F_{\alpha})^c$ is open. But $(\bigcap_{\alpha} F_{\alpha})^c = \bigcup_{\alpha} F_{\alpha}^c$ is open by (a).
- (c) Let $x \in \bigcap_{i=1}^n G_i$. Then $x \in G_i$ for $i = 1, 2, \dots, n$. So there exists a $r_i > 0$ such that $N_{r_i}(x) \subset G_i$. Let $r = \min\{r_1, r_2, \dots, r_n\}$. Then $N_r(x) \subset N_{r_i}(x) \subset G_i$ for $i = 1, 2, \dots, n$ and therefore $N_r(x) \subset \bigcap_{i=1}^n G_i$.
- (d) $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$ is open by (c).

Q.E.D.

Definition 22 (Closure). Let $E \subset X$. Let E' be a set of limit points of E in X . The set $\overline{E} = E \cup E'$ is the closure of E .

Theorem 20.

- (a) \overline{E} is closed.
- (b) $E = \overline{E} \Leftrightarrow E$ is closed.
- (c) If $F \subset X$ is closed and $E \subset F$, then $\overline{E} \subset F$. (i.e., \overline{E} is the smallest closed set containing E , and $\overline{E} = \bigcap_{F: \text{closed set with } E \subset F} F$.)

Proof. (a) Let p be a limit point of \overline{E} . It suffices to show $p \in E'$ since this implies that $p \in E' \subset E \cup E' = \overline{E}$. Let $r > 0$. $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$, i.e., $d(p, q) < \frac{r}{2}$. Since $q \in E \cup E'$, $\exists_{s \in \overline{E}}$ such that $d(q, s) < \frac{r}{2}$ (if $q \in E$, take $s = q$). But $d(p, s) \leq d(p, q) + d(q, s) < \frac{r}{2} + \frac{r}{2} = r$.

(b) (\Rightarrow) by (a)

(\Leftarrow) Suppose E is closed. Then $E' \subset E$, so $\overline{E} = E \cup E' = E$.

(c) Suppose F is closed. Then $F' \subset F$ and also $F \supset F'$. So $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

Example. Let $X = \mathbb{R}$, $d(p, q) = |p - q|$. Let $E \subset \mathbb{R}$ be nonempty and bounded above, and let $y = \sup E$. Then $y \in \overline{E}$.

Proof. Suppose for contradiction $y \notin \overline{E}$. Then y is neither a point in E nor a limit point of E , so \exists some interval $N_r(y) = (y - r, y + r)$ such that $(y - r, y + r) \cap E = \emptyset$. However, then $y - r$ is an upper bound for E since y is a least upper bound, which is a contradiction. Therefore, $y \in \overline{E}$. Q.E.D.

Definition 23 (Relative Openness). Suppose X is a metric space, so $Y \in X$ is a metric space with the same metric. Let $E \subset Y$. Then E is open relative to Y if E is an open set in the metric space Y .

Example. $X = \mathbb{R}^2 \supset \mathbb{R} = Y, E = (0, 1) \subset Y$. Then E is open relative to Y , but E is neither open nor closed in X .

Theorem 21. A set $E \subset Y \subset X$ is open relative to $Y \Leftrightarrow \exists_{\text{open set } G \subset X} : E = G \cap Y$

Proof. (\Rightarrow) Suppose $E \subset Y$ is open relative to Y . Given $p \in E$, $\exists_{r_p > 0} : N_{r_p}^Y(p) \subset E$, where $N_r^Y(p) = \{q \in Y : d(p, q) < r\}$. Then $E \subset \bigcup_{p \in E} N_{r_p}^Y(p)$ and $\bigcup_{p \in E} N_{r_p}^Y(p) \subset E$. Therefore, $E = \bigcup_{p \in E} N_{r_p}^Y(p)$.

Let $G = \bigcup_{p \in E} N_{r_p}^X(p)$. This time, we are considering p 's neighborhood in X , so each $N_{r_p}^X$ is open. Thus G is a union of open sets in X , and therefore open.

$\forall_{p \in E} : p \in N_{r_p}^X(p)$, so $E \subset G \cap Y$.

Let $p \in G \cap Y$. Then $p \in G$ and $p \in Y$. So $p \in N_{r_p}^X(p)$ for some $r_p > 0$. But $p \in Y$, so $p \in N_{r_p}^Y(p)$. Therefore, $p \in E$. This implies $G \cap Y \subset E$, and therefore $E = G \cap Y$.

(\Leftarrow) Suppose $G \subset X$ is open and $E = G \cap Y$. Then $\forall_{p \in E} : \exists_{r_p > 0} : N_{r_p}^X(p) \subset G$, so $N_{r_p}^Y(p) = N_{r_p}^X(p) \cap Y \subset G \cap Y = E$.

Q.E.D.

Note: Midterm 1 material ends here.

Definition 24 (Open Cover). An open cover of $E \subset X$ is a collection $\{G_\alpha\}$ of open subsets of X s.t $E \subset \bigcup_\alpha G_\alpha$.

Definition 25 (Compact). A set $K \subset X$ is compact if every open cover has a finite subcover; i.e., $\exists_{\alpha_1, \alpha_2, \dots, \alpha_n} : \text{s.t } K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

Example.

- If E is finite, then E is compact.
- $(0, 1) \subset \mathbb{R}$ is not compact. Bad cover: $(\frac{1}{n}, 1), n > 2$
- $[0, \infty) \subset \mathbb{R}$ is not compact. Bad cover: $(-1, n)$ for $n \in \mathbb{N}$.
- $E \subset \mathbb{R}^k$ is compact if and only if E is closed and bounded.

Theorem 22. If K is compact then K is closed.

Proof. Suppose K is compact. It suffices to prove that K^c is open. Let $p \in K^c$. We need to produce $r > 0$ s.t. $N_r(p) \subset K^c$. For $q \in K$, let $W_q = N_{r_q}(q)$, where $r_q = \frac{1}{2}d(p, q) > 0$. $\forall x \in N_{r_q}(p) : x \in W_q \Rightarrow d(x, p) + d(x, q) < 2r_q = d(p, q)$. However, X is a metric space and $p, q, x \in X$, so $d(p, q) \leq d(p, x) + d(x, q)$, leading to $d(p, q) \leq d(p, x) + d(x, q) < d(p, q)$, which is a contradiction. Hence, $\forall x \in N_{r_q}(p) : x \notin W_q$. $N_{r_q}(p) \subset W_q^c$ for $\forall q \in K$. Note that $\{W_q\}_{q \in K}$ is an open cover of K . K compact $\Rightarrow \exists$ finite number of open sets $W_{q_1}, W_{q_2}, \dots, W_{q_n}$ s.t. $K \subset \bigcup_{i=1}^n W_{q_i}$. Let $r = \min\{r_{q_1}, r_{q_2}, \dots, r_{q_n}\} > 0$.

$$\therefore N_r(p) \subset \left(\bigcap_{i \in \{1, 2, \dots, n\}} N_{r_p}(p) \right) \subset \left(\bigcap_{i \in \{1, 2, \dots, n\}} W_{q_i}^c \right) = \left(\bigcup_{i \in \{1, 2, \dots, n\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

Theorem 23. If $K \subset X$ is compact then K is bounded; i.e., $\exists M < \infty$ s.t. $\forall p, q \in K : d(p, q) \leq M$

Proof. Fix $p \in K$. An open cover of K is $\{N_n(p)\}_{n \in \mathbb{N}}$. In fact, this is an open cover of X . K compact $\Rightarrow \exists$ finite subcover $N_{n_1}(p), N_{n_2}(p), \dots, N_{n_m}(p)$. Let $R = \max\{n_1, n_2, \dots, n_m\}$. $K \subset N_R(p)$. Let $M = 2R$. $\forall q, r \in K : d(q, r) \leq d(q, p) + d(p, r) < R + R = 2R = M$. Q.E.D.

Theorem 24. If F is closed, K is compact, and $F \subset K$ then F is compact.

Proof. Suppose $F \subset K$. let $\{V_\alpha\}$ be an open cover of F . It suffice to produce a finite subcover:
Consider $\{V_\alpha\}$ together with F^c . This gives an open cover of X , hence of K , so \exists subcover of K . Drop F^c from this finite subcover. The result is a finite subcover of $\{V_\alpha\}$, which covers F Q.E.D.

Corollary. If F is closed and K is compact then $F \cap K$ is compact.

Theorem 25. Suppose $K \subset Y \subset X$. Then K is compact relative to X iff K is compact relative to Y .

Note. This is not true for open sets. For instance, let $K = Y = [0, 1] \subset X = \mathbb{R}$. Y is open and closed relative to Y , but Y is not open relative to X

Proof.

- (\Rightarrow) Suppose K is compact relative to X . Let $\{V_\alpha\}$ be an open cover of K relative to Y . For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then $\{V_\alpha\}$ is an open cover of K relative to X . Since K is compact relative to X , $\exists_{\text{finite subcover}}$.
- (\Leftarrow) Suppose K is compact relative to Y . Let $\{V_\alpha\}$ be an open cover of K relative to X . Then $\{V_\alpha \cap Y\}$ is an open cover of K relative to Y . Since K is compact relative to Y , $\exists_{\text{finite subcover}}$.

Q.E.D.

Theorem 26. Suppose $\{K_\alpha\}$ is a collection of compact sets such that $\bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$. Then $\lim_{n \rightarrow \infty} \bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$, or equivalently, $\bigcap_\alpha K_\alpha \neq \emptyset$

Example. Let $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$. Then $\{G_j\}$ is a collection of open sets, but none of them are compact. (compact sets are closed) Then $\{G_j\}$ satisfies non-empty finite intersection property but $\bigcap_{j \in \mathbb{N}} G_j = \emptyset$.

Proof. Then $K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right) = \emptyset$, so $K_{\alpha_0} \subset \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_0} (K_\alpha)^c$ and $\{(K_\alpha)^c\}_{\alpha \neq \alpha_0}$ is an open cover of K_{α_0} , so \exists a finite subcover $K_{\alpha_0} \subset \bigcup_{i=1}^n (K_{\alpha_i})^c$. But then, $K_{\alpha_0} \cap \left(\bigcap_{i=1}^n K_{\alpha_i}\right) = \emptyset$, contradiction.

Q.E.D.

Corollary. If $\{K_1, K_2, \dots\}$ are non-empty compact sets with $\forall_n : K_n \supset K_{n+1}$, then $\bigcap_{n=1}^\infty K_n \neq \emptyset$.

Proof. If $n_1 < n_N$ then $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$

Q.E.D.

Theorem 27. If K is compact and $E \subset K$ is infinite, then E has a limit point in K .

Proof. Contrapositive of the statement is : if $E \subset K$ has no limit point in K , then E is finite.

Suppose every point $q \in K$ is not a limit point of E . Then

$$\exists_{V_q = N_{r_q}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}$$

$\{V_q\}_{q \in K}$ is an open cover of K , so $\exists_{\text{finite subcover}} V_{q_1} \cup V_{q_2} \dots V_{q_n}$. Then $E = E \cap K \subset \left(\bigcup_{i=1}^n V_{q_i} \cap E\right) \subset \{q_1, q_2, \dots, q_n\}$, so E is finite.

Q.E.D.

Theorem 28. Let $I_n = [a_n, b_n] \subset \mathbb{R}$ be such that $\forall_n : I_n \supset I_{n+1}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Since $I_n \supset I_{n+1}$, $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$. Let $E = \{a_1, a_2, \dots\}$. Then $E \neq \emptyset$, every b_k is an upper bound for E , so $\exists x = \sup E$ and $a_k \leq x \leq b_k$ for all k . Therefore, $x \in I_k$ for all k , so $x \in \bigcap_{n=1}^{\infty} I_n$. Q.E.D.

Theorem 29. Let $\{I_n\}$ be a sequence of k -cells such that $I_n \supset I_{n+1}$; i.e., $I_n = \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj}, a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj} \text{ for } j = 1, 2, \dots, k\}$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof. Apply previous theorem to each component. Q.E.D.

Note. k -cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of k closed intervals on the real line.

Formally, Given real numbers a_i and b_i such that $a_i < b_i$ for every integer i from 1 to k ,

$$I = \{x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, k\}$$

Theorem 30. Let $I \subset \mathbb{R}^k$ be a k -cell. Then I is compact.

Proof. Let $I = I_0 = \{x = (x_1, x_2, \dots, x_k), a_j \leq x_j \leq b_j\}$.

Let $\Delta = \left\{ \sum_{j=1}^k (b_j - a_j)^2 \right\}^{1/2}$. Then $|\mathbf{x} - \mathbf{y}| \leq \Delta$ for $\mathbf{x}, \mathbf{y} \in I$.

Suppose for contradiction $\{G_\alpha\}$ is an open cover of I that has no finite subcover.

Let $c_j = \frac{1}{2}(a_j + b_j)$ for $j = 1, 2, \dots, k$. Using $[a_j, c_j], [c_j, b_j]$, we get 2^k k -cells Q_i with $I = \bigcup_{i=1}^{2^k} Q_i$. At least one Q_i , call it I_1 , has no finite subcover. Otherwise, every Q_i has a finite subcover, and I would have a finite subcover, namely the union of the finite subcovers of each Q_i . Repeat this step to construct $I_0 = I, I_1, I_2, \dots$. Then the sequence $\{I_n\}$ constructed by this process satisfies the following properties:

- (a) $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \dots$
- (b) $\forall_n : I_n$ has no finite subcover from $\{G_\alpha\}$
- (c) if $x, y \in I_n$ then $|x - y| \leq 2^{-n}\Delta$, where $\Delta = \text{diagonal of } I = \left(\sum_{j=1}^k (b_j - a_j)^2 \right)^{1/2}$.

By theorem 29 and (a), $\exists x^* \in \bigcap_{n=1}^{\infty} I_n$. Since $x^* \in I$, $x^* \in G_{\alpha_0}$ for some α_0 , so $\exists r > 0$ such that $N_r(x^*) \subset G_{\alpha_0}$. But by (c), $I_n \subset N_{2^{-n}\Delta}(x^*)$. As soon as n is large enough that $2^{-n}\Delta < r$, we have $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$, which contradicts (b). Q.E.D.



Note. Reverse triangle inequality

$\forall_{a,b,c \in X} : d(a,b) \geq d(a,c) - d(c,b)$ because $d(a,c) \leq d(a,b) + d(b,c)$.

Theorem 31. For $E \subset \mathbb{R}^k$, the following are equivalent:

- (a) E is closed and bounded.
- (b) E is compact.
- (c) Every infinite subset of E has a limit point in E .

Proof.

(a) \Rightarrow (b) Because E is bounded, there exists a k -cell I such that $E \subset I$; i.e., \exists_M s.t. $\forall_{x,y \in E} : |x - y| \leq M$. Therefore, E is compact.

(b) \Rightarrow (c) by theorem 28

(c) \Rightarrow (a) To see that E is bounded, suppose it were not. Then E has an infinite subset $S = \{x_1, x_2, x_3, \dots\}$ with $\forall_n : |x_n| \geq n$. S has no limit point in \mathbb{R}^k . Let $S = \{(x_1, x_2, x_3, \dots) \in E : |x_n - x_0| < \frac{1}{n}\}$. Then S is an infinite set because if S is finite, there exists a point $\mathbf{x} \in S$ such that $|\mathbf{x}| \geq |\mathbf{x}'|$ for $\mathbf{x}' \in S$. However, there exists $n \in \mathbb{N}$ such that $n > |\mathbf{x}|$ and by definition of S , there exists $x_n \in S$ such that $|x_n| \geq n > |\mathbf{x}|$, which is a contradiction. Thus, S is infinite. This S however, cannot have a limit point in E . By triangle inequality, for any $y \in \mathbb{R}^k$, $|x_n| \leq |x_n - y| + |y|$, and from archimedean property, $\exists_{m \in \mathbb{N}}$ s.t. $m > |x_n - y| + |y|$, which implies for any $y \in \mathbb{R}^k$, $r > 0$, $\exists_{m \in \mathbb{N}} : |x - y| < r < m$. However, by the definition of S , there are at most m such elements in S . Since a limit point y of E must contain an infinite number of points of E such that $d(x, y) < r$ for any $r > 0$, y cannot be a limit point, which contradicts the assumption that any infinite subset of E contains a limit point in E . Therefore, E must be bounded.

To see that E is closed, suppose it were not closed. Then $\exists_{x_0 \in E'} : x_0 \notin E$. If T has no limit point in E except $x_0 \notin E$, it contradicts (c) because T is infinite and there must be a limit point of T in E .

Therefore, we can show that E is closed by showing that T has no limit point in E except x_0 . Form an infinite sequence $(x_1, x_2, x_3, \dots), x_n \in E$ with $|x_n - x_0| < \frac{1}{n}$. Let $y \in E$, $y \neq x_0$. We'll show that y cannot be a limit point of T . $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$. Choose $n \geq \frac{2}{|y - x_0|}$, so $\frac{1}{n} \leq \frac{|y - x_0|}{2}$. Then $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$. So only finitely many x_n can lie in $N_{\frac{1}{2}|y - x_0|}(y)$. So y cannot be a limit point of S . Therefore, E is closed.

Q.E.D.

Remark. (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than \mathbb{R}^k .

Example. Failure of Heine-Borel theorem in general metric spaces.

some infinite set, discrete metric $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Then E is bounded and closed but not compact.

Theorem 32 (Weirstrass's theorem). Every bounded infinite subset $E \subset \mathbb{R}^k$ has a limit point in \mathbb{R}^k .

Proof. Choose a k -cell $I \supset E$. Since I is compact, by theorem 31, E has a limit point in I . Q.E.D.

Example. Let

$$E_0 = [0, 1] \tag{2.5}$$

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{2.6}$$

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \tag{2.7}$$

$$\vdots \tag{2.8}$$

This gives $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \dots$, where each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 26 (Cantor Set). The cantor set $P := \bigcap_n^\infty E_n$.

Proposition 1. P is compact, non-empty and contains no open intervals (a, b) and uncountable.

Proof. Compactness P is compact because $P \subset E_0 = [0, 1]$ and E_0 is compact.

Non-emptiness P is non-empty because $P \subset E_0$ and E_0 is non-empty.

No open intervals P contains no open intervals (a, b) because any (a, b) contains some $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$ and these are all removed.

Uncountability P is uncountable because P is a perfect set. Equivalently, P consists of points in $[0, 1]$ whose ternary, i.e., base 3, representation contains only 0's and 2's.

Note. ternary representation: $0.a_1a_2a_3\dots = \sum_{n=1}^\infty \frac{a_n}{3^n}$ where $a_n \in \{0, 1, 2\}$.

Q.E.D.

Example (Cantor Set). Let $E = [0, 1]$, $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, etc. Keep removing open middle third. This gives $E_0 \supset E_1 \supset E_2 \dots$. Each E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$.

Definition 27 (Separated Sets). **Separated Sets** $A, B \subset X$ are separated if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

Connected Sets $E \subset X$ is connected if there is no non-empty separated

sets $A, B \subset E$.

Example (Separated Sets). *In \mathbb{R}^1 , $[0, 1)$ and $(1, 2]$ are separated so $[0, 1) \cup (1, 2]$ is not connected. Every interval is connected (open, closed, semi-open).*

Theorem 33. $E \subset \mathbb{R}^1$ is connected if and only if E is an interval; i.e.,
 $\forall x, y \in E, x < y$ s.t. $\forall z \in (x, y) : z \in E$

Theorem 34. A metric space X is connected if and only if the only nonempty subset of X which is both open and closed is X itself.

Chapter 3

Sequence and Series

3.1 Sequences

Definition 28. In a metric space (X, d) , a sequence $\{p_n\}$ converges to p if $\forall \varepsilon > 0 \exists N$ s.t. $n \geq N \Rightarrow d(p_n, p) < \varepsilon$.
We write $\lim_{n \rightarrow \infty} p_n = p$ or $p_n \rightarrow p$.

If $\{p_n\}$ does not converge to any p then it is said to diverge.

Theorem 35. If s_n and t_n are sequences in \mathbb{C} with $s_n \rightarrow s$ and $t_n \rightarrow t$, then the following hold:

- (a) $s_n + t_n \rightarrow s + t$
- (b) $cs_n \rightarrow cs$, $c + s_n \rightarrow c + s$ for any $c \in \mathbb{C}$
- (c) $s_n t_n \rightarrow st$
- (d) $\frac{1}{s_n} \rightarrow \frac{1}{s}$ if $s \neq 0$

Lemma 2 (Squeeze Lemma). In \mathbb{R} , if $\forall n \in \mathbb{N} : 0 \leq x_n \leq s_n$ and $\lim_{n \rightarrow \infty} s_n \rightarrow 0$, then $\lim_{n \rightarrow \infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$. Then $0 \leq x_n \leq s_n < \varepsilon$ for $n \geq N$, so $x_n \rightarrow 0$. Q.E.D.

Theorem 36. (a) If $p > 0$ then $\frac{1}{n^p} \rightarrow 0$.

Proof. Let $\varepsilon > 0$. Choose N such that $\frac{1}{N^p} < \varepsilon$; i.e., $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$. Then for $n \geq N$, $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$. Q.E.D.

(b) If $p > 0$ then $\sqrt[p]{p} \rightarrow 1$.

Proof. $p = 1$ is obvious.

Suppose $p > 1$. Let $x_n = \sqrt[p]{p} - 1 > 0$. Want to show $x_n \rightarrow 0$.

Since $(x_n + 1)^n$, we have $p = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} x_n^k > \binom{n}{1} x_n = n x_n$.

Therefore, $x_n \leq \frac{p}{n}$, so $x_n \rightarrow 0$ by the Squeeze Lemma.

Suppose $p \in (0, 1)$. Let $q = \frac{1}{p} > 1$. Then $\sqrt[q]{q} \rightarrow 1$ by the previous case. By 35, $\sqrt[p]{p} = \frac{1}{\sqrt[q]{q}} \rightarrow 1$. Q.E.D.

(c) $\sqrt[n]{n} \rightarrow 1$

Proof. Let $x_n = \sqrt[n]{n} - 1 > 0$, for $n \geq 2$. $n = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} (x_n)^k > \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$. Therefore, $x_n \leq \sqrt{\frac{2}{n-1}}$. Q.E.D.

(d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$; i.e., Exponentials beat powers.

Proof. We want an upper bound on $\frac{n^\alpha}{(1+p)^n}$, so seek a lower bound on $(1+p)^n$.

$(1+p)^n = \sum_{k=0}^n \binom{n}{k} p^k > \binom{n}{k} p^k$ for $k \leq n$

$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$. Then for $k \leq \frac{n}{2}$, $\binom{n}{k} p^k >$

$(\frac{n}{2})^k \frac{p^k}{k!}$. Therefore, $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$. Let $k_0 \in \mathbb{Z}$ s.t. $k > \alpha$. Then for $n \geq 2k_0$, RHS $\rightarrow 0$ by (a).

If $|x| < 1$ then $x^n \rightarrow 0$.

Proof. $|x^n - 0| = |x|^n$, so $x^n \rightarrow 0 \Leftrightarrow |x|^n \rightarrow 0$ and $|x|^n = \frac{\frac{n_0}{|x|}}{(\frac{1}{|x|})^n} \rightarrow 0$ by (d) with $\alpha = 0$ and $1+p = \frac{1}{|x|} > 1$, so $p = \frac{1}{|x|} - 1 > 0$. Q.E.D.

Q.E.D.

Theorem 37. . Let $\{p_n\}$ be a sequence in (X, d) .

(a) $p_n \rightarrow p \Leftrightarrow \forall_{r>0} : N_r(p)$ contains all but finitely many p_n .

Proof. $\forall_{n \geq N} : p_n \in N_r(p)$ Q.E.D.

(b) If $p_n \rightarrow p$ and $p_n \rightarrow p'$ then $p = p'$.

Proof. $d(p, p') \leq d(p_n, p) + d(p_n, p')$ for all n . Fix ε . Choose N such that $d(p_n, p) < \frac{\varepsilon}{2}$ and $d(p_n, p') < \frac{\varepsilon}{2}$ for $n \geq N$. Then $d(p, p') < \varepsilon$. Then for $n \geq \max\{N, N'\}$, $d(p, p') < \varepsilon$. This is true for all $\varepsilon > 0$, so $d(p, p') = 0$. Q.E.D.

(c) If $\{p_n\}$ converges, then p_n is bounded, in a sense that $\exists_{M>0, q \in X}$ s.t. $d(p_n, q) \leq M$ for all n .

Proof. If $p_n \rightarrow p$, then $\exists N$ s.t. $d(p_n, p) < 1$ for all $n \geq N$. Thus, $\forall_{n \geq 1} : d(p_n, p) \leq \max\{1, d(p, p_1), \dots, d(p, p_{N-1})\} = M$
Q.E.D.

(d) If $E \subset X$ has a limit point p , then $\exists_{p_n \in E}$ s.t. $p_n \rightarrow p$.

Proof. We need to choose $p_n \in E$ s.t. $d(p, p_n) < \frac{1}{n}$. Let $\varepsilon > 0$. Then $d(p, p_n) < \varepsilon$ if $n > \frac{1}{\varepsilon}$
Q.E.D.

Definition 29. Given $p_n, n_1 < n_2 < n_3 < \dots$, we say $p_{n_i} = (p_{n_1}, p_{n_2}, \dots)$ is a subsequence of p_n .

Lemma 3. $p_n \rightarrow p \Leftrightarrow$ every subsequence of $\{p_n\}$ converges to p

Proof. Look at assignment 6
Q.E.D.

Theorem 38. (a) $\{p_n\}$ in X , X compact, then \exists convergent subsequence.

Proof. Let $E = \text{range of } \{p_n\}$. If E is finite, then $\exists p \in X$ and $n_1 < n_2 < \dots$ s.t. $p_n = p$ for $\forall i$. This subsequence converges to p . If E is infinite then by Theorem 28, E has a limit point $p \in X$; i.e., every neighborhood of p contains infinitely many points of E . Choose n_1 s.t. $d(p, p_{n_1}) < 1$.

Q.E.D.

(b) $\{p_n\}$ in \mathbb{R}^k , bounded, then \exists convergent subsequence.

Proof. Choose a k -cell I that contains $\{p_n\}$. I is compact. Apply (a).

Q.E.D.

Definition 30 (Cauchy Sequence). $\{p_n\}$ is a Cauchy sequence in (X, d) if $\forall \varepsilon : \exists_{N \in \mathbb{N}}$ s.t. $d(p_m, p_n) < \varepsilon \forall m, n \geq N$.

Definition 31. For $E \subset X$, $E \neq \emptyset$, we define $\text{diam } E = \sup \{d(p, q) : p, q \in E\}$. $\text{diam } E = \infty$ if the set is not bounded above.

Example. For a sequence p_n in X , let $E_n = \{p_N, p_{N+1}, \dots\}$. Then $\{p_n\}$ is a Cauchy sequence iff $\lim_{N \rightarrow \infty} \text{diam } E_N = 0$.

Theorem 39. (a) If $p_n \rightarrow p$ then $\{p_n\}$ is a Cauchy sequence.

(b) If X is a compact metric space and $\{p_n\}$ in X is a Cauchy sequence, then $\exists_{p \in X}$ s.t. $p_n \rightarrow p$.

(c) In \mathbb{R}^K every Cauchy sequence converges.

Proof. Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ is Cauchy. Let $E_N = \{p_N, p_{N+1}, \dots\}$. Then $\overline{E_N}$ is closed, hence compact. Also $\overline{E_N} \supset \overline{E_{N+1}}$ and $\lim_{N \rightarrow \infty} \text{diam } \overline{E_N} = 0$ (use Theorem 3.10(a) to see $\text{diam } \overline{E_N} = \text{diam } E_N$) By theorem 3.10(b), $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$. Claim: $p_n \rightarrow p$.

Proof of the claim: Let $\varepsilon > 0$. Choose N_0 s.t. $\text{diam } \overline{E_{N_0}} < \varepsilon$, so $d(p, q) < \varepsilon \forall q \in \overline{E_{N_0}}$, and hence $\forall q \in E_{N_0}$; i.e., $d(p, p_n) < \varepsilon$ if $n \geq N_0$.

Let $\varepsilon > 0$. Choose N s.t., $d(p_n, p) < \varepsilon/2$ if $n \geq N$. Then for $m, n \geq N$, $d(p_m, p_n) \leq d(p_n, p) + d(p, p_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Suppose $\{p_n\}$ in \mathbb{R}^k is Cauchy. Cauchy sequences are bounded in any metric space. Therefore, \exists k -cell I , which is compact, containing $\{p_n\}$. Then (b) applies Q.E.D.

Note. The converse of Theorem 39(a) does not hold in general.

Example. $X = \mathbb{Q}$ has a Cauchy sequence with no limit in \mathbb{Q} . (see assignment 6). Converse does hold if X is compact.

Theorem 40. (a) $\text{diam } \overline{E} = \text{diam } E$

(b) If $K_n \subset X$, $K_n \neq \emptyset$, K compact, $K_n \supset K_{n+1} \forall n$ and if $\lim_{n \rightarrow \infty} \text{diam } K_n = 0$, then $\bigcap_{n=1}^{\infty} K_n$ is a single point.

Proof. (a) $E \subset \overline{E} \Rightarrow \text{diam } E \leq \text{diam } \overline{E}$. For the opposite inequality, let $\varepsilon > 0$, $p, q \in \overline{E}$. Choose $p', q' \in E$ s.t. $d(p, p') < \varepsilon$, $d(q, q') < \varepsilon$. Then $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < \varepsilon + \varepsilon + \text{diam } E \leq \text{diam } E + 2\varepsilon$. Since ε is arbitrary, $\text{diam } \overline{E} \leq \text{diam } E$.

(b) Let $K = \bigcap_{n=1}^{\infty} K_n$. By Theorem 2.36, $K \neq \emptyset$. Since $K \subset K_n \forall n$, $\text{diam } K \leq \text{diam } K_n \forall n$, so $\text{diam } K = 0$. Therefore, $d(p, q) = 0 \forall p, q \in K$, so K is a simple point.

Q.E.D.

Definition 32 (Complete Metric Space). A metric space (X, d) is complete if every Cauchy sequence in X converges to a point in X .

Example. (a) X compact $\Rightarrow X$ complete.

(b) \mathbb{R}^k is complete, so is \mathbb{C} .

(c) \mathbb{Q} is not complete. (see assignment 6)

(d) See assignment 6 for reference to completion of a metric space.

We've seen that convergent sequences are bounded. $p_n = (-1)^n$ shows the converse is false. However the converse does hold for monotonic sequences.

Definition 33 (Monotone).

- A sequence $\{s_n\}$ in \mathbb{R} is monotonically increasing if $s_n \leq s_{n+1} \forall n$.
- A sequence $\{s_n\}$ in \mathbb{R} is monotonically decreasing if $s_n \geq s_{n+1} \forall n$.

Theorem 41. A monotone sequence in \mathbb{R} converges if and only if it is bounded.

Proof. \Rightarrow all convergent sequences are bounded in any metric space.

\Leftarrow **Increasing case** Let $\{s_n\}$ be monotonically increasing and $s_n \leq M \forall n$. Let $s = \sup\{s_n : n \in \mathbb{N}\}$. Then $s_n \leq s \forall n$. Let $\varepsilon > 0$. $\exists N$ s.t. $s - \varepsilon < s_N \leq s$. But then $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \leq \dots \leq s$, so $|s - s_n| < \varepsilon \forall n \geq N$, and therefore $s_n \rightarrow s$.

Q.E.D.

Definition 34 (Infinite Limits). We say

- $s_n \rightarrow \infty$ if $\forall M \in \mathbb{R} : \exists N$ s.t. $s_n \geq M \forall n \in N$.
- $s_n \rightarrow -\infty$ if $\forall M \in \mathbb{R} : \exists N$ s.t. $s_n \leq M \forall n \in N$.

Definition 35. Let $\{s_n\}$ be a sequence in \mathbb{R} . We define $\limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{\sup_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \sup_{m \geq n} \{s_m\}$.
 $\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{s_m\} = \sup_{n \geq 1} \{\inf_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{s_m\}$.

Note. Alternate definition; see ass 7 for equivalence

Remark. (a) If $a_n \leq b_n \forall n$ and $a_n \rightarrow a$ and $b_n \rightarrow b$, then $a \leq b$.

(b) $\liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n$

Example. (a) $s_n = (-1)^n(1 + \frac{1}{n^2})$ $1 \leq \sup_{m \geq n} s_m \leq 1 + \frac{1}{n^2}$, so $\limsup_{n \rightarrow \infty} s_n = 1$. Similarly, $\liminf_{n \rightarrow \infty} s_n = -1$

(b) If $\{s_n\}$ has no upper bound, then $\sup_{m \geq n} s_m = \infty$ and in this case we say $\limsup_{n \rightarrow \infty} s_n = \infty$; e.g.,

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has $\limsup_{n \rightarrow \infty} s_n = \infty$, $\liminf_{n \rightarrow \infty} s_n = -\infty$

Lemma 4. $\limsup_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = L \Leftrightarrow s_n \rightarrow L$.

Proof (L finite).

\Rightarrow This follows from $\inf_{m \geq n} s_m \leq s_n \leq \sup_{m \geq n} s_m$. $\lim_{n \rightarrow \infty} \inf_{m \geq n} s_m = \liminf_{n \rightarrow \infty} s_n$, and $\lim_{n \rightarrow \infty} \sup_{m \geq n} s_m = \limsup_{n \rightarrow \infty} s_n$. Therefore, $\lim_{n \rightarrow \infty} s_n = L$.

\Leftarrow If $s_n \rightarrow L$, then $\forall \varepsilon > 0 : \exists N$ s.t. $s_m \in [L - \varepsilon, L + \varepsilon] \forall m \geq N$. Therefore, $\forall n \geq N : L - \varepsilon \leq \inf_{m \geq n} s_m \leq \inf_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq L + \varepsilon$. Let $n \rightarrow \infty$: $L - \varepsilon \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L + \varepsilon$. Since ε is arbitrary, so $L \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L$.

Q.E.D.

3.2 Series

Definition 36 (Series). Let $\{a_n\}$ be a sequence in \mathbb{C} . Form a new sequence $\{s_n\}$, the sequence of partial sums, by $s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$. If $s_n \rightarrow s$, we say **the series** $\sum_{k=1}^{\infty} a_k$ **converges** and that $\sum_{k=1}^{\infty} a_k = s$. If $\{s_n\}$ diverges then we say $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 42. $\sum_{n \in \mathbb{N}} a_n$ converges if and only if $\exists \varepsilon > 0 : \exists N$ s.t. $|\sum_{k=m}^n a_k| < \varepsilon \forall n \geq m \geq N$.

Proof. $\sum_n a_n$ converges $\Leftrightarrow \{s_n\}$ converges $\Rightarrow \{s_n\}$ is a Cauchy sequence. Use $s_n - s_{m-1} = \sum_{k=m}^n a_k$. Q.E.D.

Corollary. If $\sum_n a_n$ converges then $a_n \rightarrow 0$.

Proof. Take $m = n$ in Theorem 3.22. $\sum_n a_n$ converges $\Rightarrow \forall \varepsilon > 0 : \exists N$ s.t. $|a_n| < \varepsilon$ if $n \geq N$. Q.E.D.

Remark. n -th term test for divergence: If $a_n \not\rightarrow 0$ then $\sum_n a_n$ diverges.

Example. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges because $\frac{n}{n+1} \rightarrow 1 \neq 0$.

Converse to Corollary is false! E.g., $\sum_n \frac{1}{n}$ diverges but $\frac{1}{n} \rightarrow 0$.

Theorem 43. If $a_n \geq 0$, then $\sum_n a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof. $\{s_n\}$ is monotone increasing, so by Theorem 41, it converges if and only if it is bounded. Q.E.D.

Theorem 44 (Comparison Test). (a) If $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges, then $\sum_n a_n$ converges.

Proof. Suppose $|a_n| \leq c_n \forall n \geq N_0$ and $\sum_n c_n$ converges. Let $\varepsilon > 0$. By theorem 3.22, $\exists N$ s.t. $\sum_{k=m}^n c_k < \varepsilon$ if $n \geq m \geq N$. Can take $N \geq N_0$. Then $|N \geq N_0|$. $|\sum_{k=m}^n a_k| \leq \sum_{k=m}^n |a_k| \leq \sum_{k=m}^n c_k < \varepsilon$ if $n \geq m \geq N$. By theorem 3.22 again, $\sum_n a_n$ converges. Q.E.D.

(b) If $a_n \geq d_n \geq 0 \forall n \geq N_0$ and if $\sum_n d_n$ diverges, then $\sum_n a_n$ diverges.

Proof. This follows from (a): if $\sum_n a_n$ converges then $\sum_n d_n$ converges. Thus it's contrapositive, (b) is true. Q.E.D.

Theorem 45 (Geometric Series). $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$, $xS_n = x + x^2 + \cdots + x^n + x^{n+1}$. Then

$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1 (\Leftrightarrow -1 < x < 1)$, then $x^{n+1} \rightarrow 0$ and $S_n \rightarrow \frac{1}{1-x}$. If $|x| \geq 1$, then x^{n+1} does not converge to 0, so $\sum_{n=0}^{\infty} x^n$ diverges. Q.E.D.

Theorem 46. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then $\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$ converges.

Proof. (\Leftarrow) We show that if $\sum_n a_n$ then $\sum_k 2^k a_{2^k}$ diverges. For this, note that $a_1 + a_2 + \cdots + a_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$ if $2^{k+1} > n$. $a_1 + a_2 + \cdots + a_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. LHS unbounded as $n \rightarrow \infty$, so RHS is also unbounded as $k \rightarrow \infty$.

(\Rightarrow) $a_1 + a_2 + a_3 + \cdots + a_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$ if $2^k \leq n$. $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$. If $\sum_n a_n$ converges, then LHS is bounded for all n so RHS is bounded for all k . Hence RHS converges since it is monotone.

Q.E.D.

Theorem 47 (p -series). $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. For $p \leq 0$, $\frac{1}{n^p} \not\rightarrow 0$, so series diverges. For $p > 0$, $\frac{1}{n^p}$ is decreasing, so $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$ converges. But $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$ converges iff $\frac{1}{2^{p-1}} < 1 (\Leftrightarrow p - 1 > 0)$, which is equivalent to $p > 1$. Q.E.D.

Theorem 48. $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. (log is to base e .)

Proof. If $p \leq 0$, then $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$, so $\sum_n \frac{1}{n(\log n)^p}$ diverges by the comparison test. If $p > 0$ then $\frac{1}{n(\log n)^p}$ decreases since $\log n$ increases. By theorem 46, $\sum_n \frac{1}{n(\log n)^p}$ converges $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k(\log 2^k)^p}$ converges $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$ converges $\Leftrightarrow p > 1$ Q.E.D.

Definition 37 (e). $e := \sum_{n=0}^{\infty} \frac{1}{n!}$.

Remark. Convergence $\frac{1}{n!} = \frac{1}{(n(n-1)(n-2)\cdots 3\cdot 2\cdot 1)} \leq \frac{1}{2\cdot 2\cdot 2\cdots 2\cdot 1} = \frac{1}{2^{n-1}}$. Therefore, $S_n = \sum_{k=0}^n \frac{1}{k!} \leq 1 + \sum_{k=1}^n \frac{1}{2^{k-1}} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1-1/2} = 3$. Hence, $e \leq 3$.

Rate of Convergence $0 < e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n+1}} \cdot \frac{1}{(n+1)!} = \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-(n+1)}} = \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+1)} = \frac{1}{(n!) \cdot n}$.

Theorem 49. $e \notin \mathbb{Q}$.

Proof. For contradiction, suppose $e = \frac{p}{q}$, $p, q \in \mathbb{N}$. Then $0 < e - S_q < \frac{1}{q \cdot q!}$, so $0 < q! \cdot e - q! \cdot S_q < \frac{1}{q}$. If we show $q! \cdot e - q! \cdot S_q \in \mathbb{Z}$, then we'll have a contradiction. But $q! \cdot e - q! \cdot S_q = q! \cdot \frac{p}{q} - \sum_{k=0}^q \frac{q!}{k!}$. Because $q! \cdot \frac{p}{q} \in \mathbb{N}$ and $\sum_{k=0}^q \frac{q!}{k!} \in \mathbb{N}$, the difference of them, $q! \cdot e - q! \cdot S_q \in \mathbb{Z}$, thus a contradiction. Q.E.D.

Theorem 50. $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$.

Proof. Let $t_n = (1 + \frac{1}{n})^n$. Then $t_n = \sum_{k=0}^n \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \cdot (\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}) \leq S_n$. So $\limsup_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_n = e$. On the other hand, for fixed m and $n \geq m$, $t_n \geq \sum_{k=0}^m \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^m \frac{1}{k!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{k-1}{n})$. Let $n \rightarrow \infty$ with m fixed. $\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!} \cdot 1 = S_m$. This is true for any m . Now let $m \rightarrow \infty$. $\liminf_{n \rightarrow \infty} t_n \geq \limsup_{m \rightarrow \infty} S_m = e$. $e \leq \liminf_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} t_n \leq e$. Therefore, $\lim_{n \rightarrow \infty} t_n$ exists and equals e . Q.E.D.

Theorem 51 (Root test). Let $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$. Then,

$$\sum a_n \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha > 1 \\ \text{inconclusive} & \text{if } \alpha = 1 \end{cases}$$

Proof (Just outline). $\alpha < 1$ Eventually $|a_n| \leq \beta^n$, thus convergence.

$\alpha > 1$ $|a_n| > 1$ for infinitely many n , thus divergence.

$\alpha = 1$ $\frac{1}{n}$ diverges, $\frac{1}{n^2}$ converges.

Q.E.D.

Theorem 52 (Ratio test). The series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ converges if $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and diverges if $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N : \left| \frac{a_{n+1}}{a_n} \right| \geq 1$. Otherwise, inconclusive.

Proof. (see textbook).

$$\text{Convergence } \sum a_n \begin{cases} \text{converges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \\ \text{diverges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \\ \text{diverges} & \text{if } \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \end{cases}$$

Note. Note that we cannot replace \liminf with \limsup in the last case.

Q.E.D.

Example. Let $s_n = \sum_{n=0}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \dots$. First, note that $a_{2k} = \frac{1}{2} \cdot \frac{1}{4^k}$, $a_{2k+1} = 2a_{2k} = \frac{1}{4^k}$ for $k \geq 0$.

Ratio test Then the ratio $\frac{a_{n+1}}{a_n}$ is the sequence $2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \dots$. Therefore, $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}$, $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$. The ratio test is inconclusive for s_n .

Root test $a_n = \begin{cases} \frac{1}{2} \cdot \frac{1}{2^n} & n \text{ even} \\ \frac{2}{2^n} & n \text{ odd} \end{cases}$, so $(\frac{1}{2})^{1/n} \cdot \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{2^{1/n}}{2}$, thus $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$. Therefore, $s_n = \sum_{n=0}^{\infty} a_n$ converges.

This is an example where the ratio test is inconclusive but the root test is conclusive.

Theorem 53. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum a_n + b_n = A + B$ and $\sum c \cdot a_n = cA$.

3.3 Power Series

Definition 38 (Power Series). For $z \in \mathbb{C}$ and a complex sequence $\{c_n\}$, $\sum_{n=0}^{\infty} c_n z^n$ is a power series.

Remark. As $z^0 = 1$ for all $z \in \mathbb{C}$, by convention $\sum_{n=0}^{\infty} c_n z^n = c_0 + \sum_{n=1}^{\infty} c_n z^n$.

Theorem 54. Let $R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}}$ such that

$$R = \begin{cases} 0 & \text{if } \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \infty \end{cases}$$

. Then $\sum c_n z^n$ $\begin{cases} \text{converges} & \text{if } |z| < R \\ \text{diverges} & \text{if } |z| > R \\ \text{inconclusive} & \text{if } |z| = R \end{cases}$

Proof. $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$. By root test, the series converges if $\frac{|z|}{R} < 1$ and diverges if $\frac{|z|}{R} > 1$.

Note. In practice, often use the ratio test to find R .

Q.E.D.

Example. (a) $\sum n! \cdot z^n$ has $R = 0$.

By ratio test $\left| \frac{(n+1)! \cdot z^{n+1}}{n! \cdot z^n} \right| = |z| \cdot (n+1) \rightarrow \infty \forall z \neq 0$. Hence, the series diverges $\forall z \neq 0$.

By root test Note $n \neq \frac{1}{2}(\frac{2}{3})^2(\frac{3}{4})^3 \dots (\frac{n-1}{n})^{n-1} n^n$ for $n \geq 2$. Then $n \neq \frac{n^n}{(1+1)^1(1+\frac{1}{2})^2(1+\frac{1}{n-1})^{n-1}}$. In the proof of Theorem 50, we saw $(1+\frac{1}{j}) \leq e$. So $n! \geq \frac{n^n}{e^{n-1}} = e \cdot (\frac{n}{e})^n$. $\sqrt[n]{n!} \geq e^{1/n} \cdot \frac{n}{e} \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, $R = \frac{1}{\infty} = 0$.

Note. Cf. Stirling's formula: $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

Definition 39 (Absolute Convergence, Conditional Convergence).

- (a) $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
- (b) $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Remark. All other convergence tests seen so far are actually tests for absolute convergence.

Example. • $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges conditionally by the alternating se-

ries test of assignment 7's practice problem 1. See also Theorem 3.43.
(MUST TAKE A LOOK!)

Given $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ and $a_n \rightarrow 0$, then $\sum (-1)^n a_n$ converges.