

# Real Variables I

November 4, 2024

# Chapter 1

## Number Systems

Natural numbers:  $\mathbb{N} = \{1, 2, 3, \dots\}$

Integers:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers:  $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

**Remark.** Note for real numbers,  $\mathbb{Q}$  has holes in it.

**Example.**  $\nexists p \in \mathbb{Q}$  s.t.  $p^2 = 2$

**Proof.** Assume  $\exists p \in \mathbb{Q}$  s.t.  $p^2 = 2$ . Then  $p = \frac{a}{b}$  where  $a, b \in \mathbb{Z}$  and  $b \neq 0$ . So,  $p^2 = \frac{a^2}{b^2} = 2 \Rightarrow a^2 = 2b^2$ . So,  $a^2$  is even  $\Rightarrow a$  is even. So,  $a = 2k$  for some  $k \in \mathbb{Z}$ . So,  $4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$ . So,  $b^2$  is even  $\Rightarrow b$  is even. So,  $b = 2l$  for some  $l \in \mathbb{Z}$ . So,  $a$  and  $b$  are both even, which contradicts the fact that  $a$  and  $b$  are coprime. So,  $\nexists p \in \mathbb{Q}$  s.t.  $p^2 = 2$ . Q.E.D.

**Definition 1 (Order).** An order on a set  $S$  is a relation  $<$  such that:

- (a) If  $a, b \in S$ , then exactly one of  $a < b$ ,  $a = b$ , or  $b < a$  is true.
- (b) If  $a, b, c \in S$  and  $a < b$  and  $b < c$ , then  $a < c$ .

**Definition 2 (Ordered Set).** An ordered set  $S$  is a set with an order  $<$ .

**Definition 3.** Let  $S$  be an ordered set. A set  $E \subset S$  is bounded above if  $\exists \beta \in S$  s.t.  $\forall x \in E : x \leq \beta$ .  
Similarly, a set  $S$  is bounded below if  $\exists \beta \in S$  s.t.  $\forall x \in E : x \geq \beta$ .

---

**Definition 4 (LUB, GLB).** Let  $S$  be an ordered set and  $E \subset S$ ,  $E \neq \emptyset$ , with  $E$  bounded above. If  $\exists \alpha$  s.t.  $\alpha$  is an upper bound for  $E$  and  $\forall \gamma < \alpha$ :  $\gamma$  is not an upper bound for  $E$ , then such  $\alpha$  is called least upper bound (LUB), or *Supremum*. Similarly, if  $\exists \alpha$  s.t.  $\alpha$  is a lower bound for  $E$  and  $\forall \gamma > \alpha$ :  $\gamma$  is not a lower bound for  $E$ . Then such  $\alpha$  is called greatest lower bound (GLB), or *Infimum*.

**Definition 5 (LUB property).** An ordered set  $S$  has the least upper bound (LUB) property if  $\forall E \subset S$  if  $E \neq \emptyset$  and  $E$  bounded above implies  $\exists \sup E \in S$ ; i.e., Every bounded subset of  $S$  has the least upper bound (LUB).

**Example.**

- $\mathbb{Z}$  has the LUB property.
- $\mathbb{Q}$  does not have the LUB property.

**Theorem 1.** Let  $S$  be an ordered set. Then  $S$  has the LUB property if and only if  $S$  has the GLB property.

**Proof.** ( $\Rightarrow$ ) Suppose  $S$  has the LUB property. Let  $B \subset S$  be non-empty and bounded below. Let  $L$  be the set of all lower bounds of  $B$ . Then  $L$  is non-empty and bounded above. Let  $\alpha = \sup L$ . We claim that  $\alpha = \inf B$ . ( $\Leftarrow$ ) Suppose  $S$  has the GLB property. Let  $E \subset S$  be non-empty and bounded above. Let  $U$  be the set of all upper bounds of  $E$ . Then  $U$  is non-empty and bounded below. Let  $\beta = \inf U$ . We claim that  $\beta = \sup E$ . Q.E.D.

**Definition 6 (Fields).** Let  $F$  be a set with two operations, addition and multiplication. Then  $F$  is a field if the following axioms are satisfied:

- (a)  $a + b = b + a$  and  $a \cdot b = b \cdot a$  for all  $a, b \in F$  (Commutative laws).
- (b)  $(a + b) + c = a + (b + c)$  and  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$  for all  $a, b, c \in F$  (Associative laws).
- (c)  $a \cdot (b + c) = a \cdot b + a \cdot c$  for all  $a, b, c \in F$  (Distributive law).
- (d)  $\exists 0 \in F$  s.t.  $a + 0 = a$  for all  $a \in F$ .
- (e)  $\exists (-a) \in F$  s.t.  $a + (-a) = 0$  for all  $a \in F$ .
- (f)  $\forall x, y \in F : xy \in F$ .
- (g)  $\forall x, y \in F : xy = yx$ .
- (h)  $\exists 1 \in F$  s.t.  $a \cdot 1 = a$  for all  $a \in F$ .
- (i) If  $a \neq 0$ , then  $\exists a^{-1} \in F$  s.t.  $a \cdot a^{-1} = 1$ .

---

(j)  $\forall x, y, z \in F : x(y + z) = xy + xz$

**Example.**

(a)  $\mathbb{Q}$  is a field, while  $\mathbb{Z}$  is not a field.

(b)  $F_p = \{0, 1, \dots, p-1\}$  with mod  $p$  arithmetic is a field.

Read Text book: 114, 115, 116, 118

**Definition 7 (Ordered Field).** An ordered field  $F$  is a field that is an ordered set such that the following properties are satisfied:

(a) If  $a, b, c \in F$  and  $a < b$ , then  $a + c < b + c$ .

(b) If  $a, b \in F$  and  $0 < a$  and  $0 < b$ , then  $0 < ab$ .

**Remark.** We say  $x$  is positive if  $x > 0$  and  $x$  is negative if  $x < 0$ .

**Example.**  $\mathbb{Q}$  is an ordered field.

**Theorem 2.**  $\exists$  an ordered field  $\mathbb{R}$  which has the LUB property and contains  $\mathbb{Q}$  as a subfield.

**Theorem 3.**

(a) Arithmetic properties of  $\mathbb{R}$ : If  $x, y \in \mathbb{R}$  and  $x > 0$  then  $\exists n \in \mathbb{N}$  such that  $nx > y$ .

(b)  $\mathbb{Q}$  is dense in  $\mathbb{R}$ : If  $x, y \in \mathbb{R}$  and  $x < y$ , then  $\exists p \in \mathbb{Q}$  such that  $x < p < y$ .

(c)  $x, y \in \mathbb{R}$  then  $\exists \alpha \in \mathbb{R} \setminus \mathbb{Q}$  such that  $x < \alpha < y$ .

**Proof.** (a) Let  $A = \{nx \mid n \in \mathbb{N}\}$ . Suppose  $\forall nx \in A : nx \leq y$ . Then  $y$  is an upper bound for  $A$ . So,  $A$  has a least upper bound  $\alpha$ . Since  $\alpha - x < \alpha$  as  $x > 0$ ,  $\alpha - x$  is not an upper bound for  $A$ . Thus,  $\exists m \in \mathbb{N} : mx > \alpha - x$ , so  $\alpha < (m+1)x \in A$ , contradicting the fact that  $\alpha$  is a supremum of  $A$ . Therefore,  $\exists n \in \mathbb{N}$  such that  $nx > y$ .

(b) Since  $y - x > 0$ , by (a),  $\exists n \in \mathbb{N}$  such that  $n(y - x) > 1$ .  $ny - nx > 1$  and therefore,  $1 + nx < ny$ . Let  $m \in \mathbb{Z}$  such that  $(m - 1) \leq nx < m$ . Such  $m$  exists by the extended version of (a). This implies there exists  $m \in \mathbb{N}$  such that  $nx < m \leq nx + 1 < ny$ . Therefore,  $x < \frac{m}{n} < y$ .

(c)  $\exists k \in \mathbb{Q}$  such that  $k^2 = 2$ ; i.e.,  $\exists \sqrt{2} \in \mathbb{R}$ .  $0 < \sqrt{2} < 2$  because if  $\sqrt{2} \geq 2$  then  $2 = \sqrt{2} \cdot \sqrt{2} \geq 2 \cdot 2 = 4$ , which is a contradiction. By (b),  $\exists p \in \mathbb{Q}$  such that  $x < p < y$  and  $\exists q \in \mathbb{Q}$  such that  $x < p < q < y$ . Let  $\alpha = p + \frac{\sqrt{2}}{2}(q - p)$ . Then  $x < p < \alpha < q < y$  and  $\alpha \notin \mathbb{Q}$  since otherwise  $\sqrt{2} = 2 \cdot \frac{\alpha - p}{q - p}$  would be rational

Q.E.D.

**Note.** (a) can be improved to:

$$\forall x, y \in \mathbb{R}, x > 0, \exists n \in \mathbb{Z} \text{ such that } (n - 1)x \leq y < nx. \quad (1.1)$$

**Proof.** Case 1:  $y \geq 0$ . Let  $A = \{m \in \mathbb{N} : y < mx\} \subset \mathbb{N}$ . By (a),  $A \neq \emptyset$ . Every non-empty subset of  $\mathbb{N}$  has a smallest element. Let  $n = \text{smallest element of } A$ . Then the inequality holds true. Case 2: Let  $y < 0$ , then there exists  $n \in \mathbb{N}$  such that  $(n - 1)x \leq -y < nx$ , which implies that (by changing sign for all terms)  $-nx < y \leq -(n - 1)x$ . Hence, the statement holds.

Q.E.D.

**Lemma 1.** Let  $a, b \in \mathbb{R}$  such that  $0 < a < b$ , then  $0 < b^n - a^n \leq nb^{n-1}(b - a)$  for some  $n \in \mathbb{N}$ .

**Proof.**

$$\begin{aligned} b^n - a^n &= (b - a) \underbrace{(b^{n-1} + ab^{n-2} + \dots + a^{n-2}b + a^{n-1})}_{n \text{ terms}} \\ &< (b - a)nb^{n-1} \end{aligned}$$

Q.E.D.

**Theorem 4.**  $\forall x \in \mathbb{R}, x > 0, n \in \mathbb{N} : \exists!(\text{unique}) y > 0 : y^n = x$  (we write  $y = x^{1/n} = \sqrt[n]{x}$ , the  $n^{\text{th}}$  root of  $x$ ).

**Proof.** Uniqueness: For any  $y_1, y_2 \in \mathbb{R}$ , if  $0 < y_1 < y_2$ , then  $0 < y_1^n < y_2^n$ , hence  $y_1^n$  and  $y_2^n$  cannot both be equal to  $x$ .

Existence: Let  $E_x = \{t \in \mathbb{R}_{>0} \mid t^n < x\}$ . If  $E \neq \emptyset$ ,  $E$  is bounded above, hence (by the least-upper-bound property) there exists a sup  $E$ . Choose  $y = \sup E$ . Consider two cases.

- (a) If  $x \leq 1$ , then  $t_0 = \frac{x}{2}$  and thereby  $t_0^n = \frac{x^n}{2^n} < x^n \leq x$  (by assumption that  $x \leq 1$ ).
- (b) If  $x > 1$ , then let  $t_0 = 1$ , leading to  $t_0^n = 1 < x$ .

In either case,  $t_0 \in E$ , and hence  $E$  is not empty. 1(a) ( $E$  is bounded above) Let  $\beta = x + 1$ . Then,  $\beta^n = (x + 1)^n > x + 1 > x$ . Then, for any  $t \in E$ , we have that  $t^n < x < \beta^n$ , hence  $t < \beta$ , making  $t$  an upper bound of  $E$ .

- (a) Assuming that  $y^n < x$ , we find  $0 < h < 1$  such that  $(y+h)^n < x$ , which leads to  $y + h \in E$ , something that contradicts with the fact that  $y = \sup E$ . This is equivalent to finding an  $0 < h < 1$  such that  $(y + h)^n - y^n < x - y^n$ . By the lemma 1, we have  $0 < (y+h)^n - y^n < n(y+1)^{n-1}h$  for any  $0 < h < 1$ . Choose  $h$  so that  $\frac{(x-y)^n}{n(y+1)^{n-1}}$ . Then  $0 < h < 1$  still holds and  $hn(y+1)^{n-1} < x - y^n$ , leading to  $(y + h)^n < x$ , and therefore  $y + h \in E$ . However, this contradicts the fact that  $y = \sup E$  as  $y + h > y$ .
- (b) Assuming that  $y^n > x$ , we find  $k > 0$  such that  $(y - k)^n > x$ , which leads to a contradiction since otherwise  $y - k$  would be an upper bound for  $E$  that's smaller than  $y$ , which is  $\sup E$ . By the lemma 1,  $y^n - (y - k)^n \leq ny^{n-1}k < y^n - x$  for any  $h < \frac{y^n - x}{ny^{n-1}}$ . Therefore,  $-(y - k)^n < -x$ , or  $x < (y - k)^n$ . Thus,  $y - k$  is also an upper bound of  $E$  and  $y - k < y = \sup E$ , which is a contradiction.

Since  $y^n < x$  and  $y^n > x$  are both contradictions,  $y^n = x$ . Q.E.D.

**Definition 8 (Cut/Dedekind Cut).** The set  $\mathbb{R}$  elements are (Dedekind) cuts, which are sets  $\alpha \subset \mathbb{Q}$  such that

- $\forall p \in \alpha, q \in \mathbb{Q} : q < p \Rightarrow q \in \alpha$
- No greatest element in  $\alpha$

**Example.**  $\alpha = \{p \in \mathbb{Q} \mid p < 0\}$ ,  $\alpha = \{p \in \mathbb{Q} \mid p \leq 0 \vee p^2 < 2\}$

**Definition 9 (Order of cuts).** For  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha < \beta := \alpha \subset \beta$

---

**Proof (test).** Let  $\gamma$  be set of cuts  $A$ , and show that  $\gamma$  is a cut and that  $\gamma = \sup A$ . Q.E.D.

**Theorem 5.** There exists an ordered field  $\mathbb{R}$  such that  $\mathbb{Q} \subset \mathbb{R}$  and  $\mathbb{R}$  has the LUB property.

**Proof.** Let  $\mathbb{R}$  be the set of all cuts with:

**order**  $a < b := a \subset b$ .

**addition**  $\alpha + \beta = \{p + q \mid p \in \alpha, q \in \beta\}$ .

**multiplication**  $\alpha \cdot \beta = \{p \cdot q \mid p \in \alpha, q \in \beta\}$ .

Q.E.D.

## Complex Numbers

**Definition 10 (Complex Field).** The underlying set is  $\mathbb{C} = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}\}$

Addition is defined as  $(a, b) + (c, d) = (a + c, b + d)$

Multiplication is defined as  $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$

Zero element is  $(0, 0)$

One element is  $(1, 0)$

**Theorem 6.**  $\mathbb{C}$  is a field.

**Proof.** Verify the 11 field axioms. For just a few axioms:

(M3):

$$x = (a, b), y = (c, d), z = (e, f). \quad x(yz) = (a, b)(ce - df, cf + de) = (a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)) = (ac - bd, ad + bc)(e, f) = (xy)z$$

(M4):

$$(a, b)(1, 0) = (a \cdot 1 - b \cdot 0, a \cdot 0 + b \cdot 1) = (a, b)$$

(M5):

$$x \neq 0 \text{ means } x = (a, b) \text{ with } a \neq 0 \text{ or } b \neq 0. \text{ That is, } a^2 + b^2 > 0. \text{ Let } \frac{1}{x} = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}). \text{ Then } x \cdot \frac{1}{x} = (a, b)(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}) = (\frac{a^2+b^2}{a^2+b^2}, \frac{-ab+ba}{a^2+b^2}) = (1, 0). \quad \text{Q.E.D.}$$

Identification of  $\mathbb{R}$  as a subfield of  $\mathbb{C}$ . Identify  $(a, 0) \in \mathbb{C}$  with  $a \in \mathbb{R}$ . Then  $(a, 0) + (b, 0) = (a + b, 0)$ ,  $(a, 0)(b, 0) = (ab, 0)$ , so we can represent them by  $a + b = a + b$ ,  $a \cdot b = a \cdot b$ . Write  $i = (0, 1)$ .  $i^2 = (0, 1)(0, 1) = (-1, 0)$ . So,  $i^2 = -1$ .  $(a, b) \leftrightarrow a + bi$ . Usually write  $z = a + bi$  for  $z \in \mathbb{C}$ .  $\text{Re}(z) = a, \text{Im}(z) = b$ .

**Definition 11.** Complex conjugate of  $z = a + bi$  is defined as  $a - bi$  and denoted by  $\bar{z}$

---

**Note.**

- (a)  $\overline{z + w} = \bar{z} + \bar{w}$
- (b)  $\overline{zw} = \bar{z} \cdot \bar{w}$
- (c)  $z + \bar{z} = 2 \cdot \operatorname{Re}(z)$
- (d)  $z - \bar{z} = 2i \cdot \operatorname{Im}(z)$
- (e)  $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2 \geq 0$ , with  $=$  if and only if  $z = 0$
- (f)  $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a-bi}{a^2+b^2}$

**Definition 12.**  $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$

In particular, if  $z = a \in \mathbb{R}$  then  $|z| = \sqrt{a^2} = |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

**Theorem 7.** For  $z, w \in \mathbb{C}$ ,

- (a)  $|z| \geq 0$  with  $=$  iff  $z = 0$
- (b)  $|z| = |\bar{z}|$
- (c)  $|zw| = |z| \cdot |w|$
- (d)  $|\operatorname{Re}(z)| \leq |z|, |\operatorname{Im}(z)| \leq |z|$

**Proof.** Let  $z = a + bi$ . Then  $|\operatorname{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|$   
Q.E.D.

- (e)  $|z + w| \leq |z| + |w|$  (Triangle inequality)

**Proof.**

$$\begin{aligned} |z + w|^2 &= (z + w)(\overline{z + w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\ &= |z|^2 + z\bar{w} + w\bar{z} + |w|^2 \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq (|z| + |w|)^2 \end{aligned}$$

Q.E.D.

**Theorem 8** (Cauchy-Schwarz inequality). If  $a_1, a_n, b_1, b_n \in \mathbb{C}$ , then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right| \leq \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^n |b_j|^2 \right)^{\frac{1}{2}}.$$



Interpretation:  $(\vec{a}, \vec{b}) = \sum_{j=1}^n a_j \bar{b}_j$  defined on inner product on  $\mathbb{C}^n$  and  $|(\vec{a}, \vec{b})| \leq \sqrt{(\vec{a}, \vec{a})(\vec{b}, \vec{b})}$ . (Note that  $\vec{a} \cdot \vec{b} = \overline{\vec{b} \cdot \vec{a}}$ )

**Proof.** Let  $A = \sum |a_j|^2$ ,  $B = \sum |b_j|^2$ ,  $C = \sum a_j \bar{b}_j$ . We can assume 1.  $B \neq 0$  because  $B = 0$  is  $0 \leq 0$ , 2.  $C \neq 0$  because  $C = 0$ , LHS is 0. For any  $\lambda \in \mathbb{C}$ ,  $0 \leq \sum_{j=1}^n |a_j + \lambda b_j|^2 = \sum_{j=1}^n (a_j + \lambda b_j)(\bar{a}_j + \bar{\lambda} \bar{b}_j) = \sum_{j=1}^n |a_j|^2 + \lambda \sum_{j=1}^n b_j \bar{a}_j + \bar{\lambda} \sum_{j=1}^n a_j \bar{b}_j + |\lambda|^2 \sum_{j=1}^n |b_j|^2$ . Let  $\lambda = tC$  for  $t \in \mathbb{R}$ . Then  $0 \leq A + \lambda \bar{C} + \bar{\lambda} C + |\lambda|^2 B = A + 2|C|^2 t + B|C|^2 t^2 = p(t)$ .  $p(t)$  is a quadratic function in terms of  $t$  and it must be non-negative. Regardless of the value of  $t$ . Therefore, the discriminant of  $p(t) = (2|C|^2)^2 - 4AB|C|^2 = 4|C|^2(|C|^2 - AB) \leq 0$ . Since  $|C| \geq 0$ ,  $|C|^2 \leq AB$ . Q.E.D.

**Definition 13 (Euclidean  $k$ -space).** For  $k \in N$ ,  $\mathbb{R}^k := \{\vec{x} = (x_1, x_2, \dots, x_k) : x_1, x_2, \dots, x_k \in \mathbb{R}\}$  with the following properties:

Addition  $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \dots, x_k + y_k)$

Scalar multiplication  $\lambda \vec{x} = (\lambda x_1, \lambda x_2, \dots, \lambda x_k)$

Inner(dot) product  $(\vec{x}, \vec{y}) = \sum_{j=1}^k x_j y_j$ , which is bilinear:  $(\alpha \vec{x} + \beta \vec{y}) \cdot \vec{z} = \alpha \vec{x} \cdot \vec{z} + \beta \vec{y} \cdot \vec{z}$ .

Norm  $|\vec{x}| = \sqrt{(\vec{x}, \vec{x})} = \sum_{j=1}^k |x_j|^2^{1/2}$

**Remark.** Addition and Scalar multiplication make  $\mathbb{R}^k$  into a vector space.

**Theorem 9.** Let  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^k$ . Then

- (a)  $|\vec{x}| \geq 0$
- (b)  $|\vec{x}| = 0 \leftrightarrow \vec{x} = \vec{0}$
- (c)  $|\alpha \vec{x}| = |\alpha| |\vec{x}|$
- (d)  $|\vec{x} \cdot \vec{y}| \leq |\vec{x}| |\vec{y}|$  (special case of Cauchy-Schwarz inequality)
- (e)  $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$  (Triangle inequality)

**Proof.**  $|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = |\vec{x}|^2 + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq |\vec{x}|^2 + 2|\vec{x}| |\vec{y}| + |\vec{y}|^2 \leq (|\vec{x}| + |\vec{y}|)^2$  Q.E.D.

- (f)  $|\vec{x} - \vec{y}| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$

**Proof.**  $|\vec{x} - \vec{y}| = |(\vec{x} - \vec{z}) + (\vec{z} - \vec{y})| \leq |\vec{x} - \vec{z}| + |\vec{z} - \vec{y}|$  Q.E.D.

## Chapter 2

# Basic Topology

**Definition 14.** Sets  $A$  and  $B$  have the same cardinality, if  $\exists f : A \rightarrow B$  that is 1-1 and onto (i.e., bijective).

**Theorem 10.** Let  $A \sim B$  be a relation between two sets having the same cardinality. Then  $\sim$  is an equivalence relation. That is,

- (a)  $A \sim A$  (Reflexive)
- (b)  $A \sim B \Rightarrow B \sim A$  (Symmetry)
- (c)  $A \sim B \& B \sim C \Rightarrow A \sim C$  (Transitivity)

**Definition 15.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Let  $J_n = \{1, 2, \dots, n\}$  for  $n \in \mathbb{N}$ .

- A set  $A$  is finite if  $A \sim J_n$  for some  $n \in \mathbb{N}$  (or if  $A = \emptyset$ ).
- A set  $A$  is countably infinite if  $A \sim \mathbb{N}$ .
- A set  $A$  is countable if  $A$  is finite or countably infinite.

**Example.**  $\mathbb{Z}$  is a countably infinite. For  $n \in \mathbb{Z} = \{0, +1, -1, +2, -2, \dots\}$ ,

$$\text{Let } f(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n + 1 & \text{if } n < 0 \end{cases}$$

Then  $f$  is bijective and therefore  $|\mathbb{Z}| = |\mathbb{N}|$

**Theorem 11.** A subset of a countably infinite set is countable.

---

**Proof.** Let  $A$  be some countably infinite set and  $S$  be a infinite subset of  $A$ .

As  $A$  is a countably infinite set, we can remove duplicates and arrange  $A$  so that  $A = \{a_1, a_2, a_3, \dots\}$ . Let  $n_1$  be the smallest positive integer such that  $x_{n_1} \in S$ . Let  $n_k$  be the smallest positive integer greater than  $n_{k-1}$  such that  $x_{n_k} \in S$  for  $k = 2, 3, \dots$ . Let  $f(k) = x_{n_k}$  for  $k = 1, 2, 3, \dots$ . Then this is a bijection from  $\mathbb{N}$  to  $S$ . Q.E.D.

**Remark.** Roughly speaking, countable sets represent the **smallest infinity**, as no uncountable set can be a subset of a countable set.

**Theorem 12.** Let  $E_1, E_2, \dots$  be countably infinite sets. Then  $S = \bigcup_{n=1}^{\infty} E_n$  is countably infinite.

**Proof.** Write  $E_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \dots\}$

$E_2 = \{x_{21}, x_{22}, x_{23}, x_{24}, \dots\}$

Form an array:

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & \dots \\ x_{21} & x_{22} & x_{23} & x_{24} & \dots \\ x_{31} & x_{32} & x_{33} & x_{34} & \dots \\ x_{41} & x_{42} & x_{43} & x_{44} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix might have duplicates. Let  $T$  be a subset of  $\mathbb{N}$  such that  $t \in T$  if and only if  $t$  is the smallest positive integer such that  $x_t \in E_1 \cup E_2 \cup \dots \cup E_n$ .

Then a set  $\{x_t | t \in T \text{ and } \exists i \in \mathbb{N} : x_t \in E_i\}$  is  $S$ . Clearly,  $|S| = |T|$ , or  $S \sim T$ , and  $T$  is a subset of a countably infinite set,  $\mathbb{N}$ . Therefore,  $T$  is also countable, implying  $S$  is also countable. As  $S$  is infinite,  $S$  is countably infinite. Q.E.D.

**Corollary.** If  $A$  is countable and  $n \in \mathbb{N}$ , then  $\{(a_1, a_2, \dots, a_n) | a_1, a_2, \dots, a_n \in A\}$  is countable.

**Theorem 13.** Let  $A = \{(b_1, b_2, b_3 \dots) | b_i \in \{0, 1\}\}$ . I.e.,  $A$  is a set of all infinite binary strings. Then  $A$  is uncountable.

---

**Proof (Cantor's Diagonalization argument, 1891).** Let  $E \subset A$  be countably infinite.  $E = \{s^{(1)}, s^{(2)}, s^{(3)}, \dots \mid s^{(i)} \in A\}$ . It suffices to find some  $s \in A \setminus E$ , for this shows every countably infinite subset of  $A$  is proper construction of  $s$ . Write

$$s^{(1)} = b_1^1 b_2^1 \dots \quad (2.1)$$

$$s^{(2)} = b_1^2 b_2^2 b_3^2 \dots \quad (2.2)$$

$$s^{(3)} = b_1^3 b_2^3 b_3^3 \dots \quad (2.3)$$

$$\vdots$$

On diagonal, flip each bit, i.e.,  $0 \rightarrow 1$  and  $1 \rightarrow 0$  and represent the flipped bit of  $b_i^i$  by  $\tilde{b}_i^i$ . Let  $s = \tilde{b}_1^1 \tilde{b}_2^2 \tilde{b}_3^3 \dots$ . Then  $s \in A$  and  $s \notin E$  as  $s$  differs from each  $s^{(i)}$  in the  $i$ -th bit. Therefore,  $A$  is uncountable. Q.E.D.

**Corollary.** The set  $\mathcal{P}(\mathbb{N})$  of subsets of  $\mathbb{N}$  is uncountable.

**Proof.** We can create  $f : \mathcal{P}(\mathbb{N}) \rightarrow A$  be a bijection, where  $A$  is the set of all infinite binary strings, by

$$f(S)_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \quad (2.4)$$

For example, if  $f(\{\text{odd natural numbers}\}) = (1, 0, 1, 0, 1, 0, 1, 0, \dots)$ . This  $f$  is a bijection, and therefore  $A$  is uncountable.

Q.E.D.

**Theorem 14.**  $\mathbb{R}$  is uncountable.

**Proof.** This is a rough sketch of the proof:

- (a) It's enough to show that  $[0, 1]$  is uncountable.
- (b) Consider binary decimal representation of  $x \in [0, 1]$ . For example,  $x = 0.101001001\dots$ . Given  $x$ , choose maximal  $b_1 \in \{0, 1\}$  such that  $\frac{b_1}{2} \leq x$ . Then choose  $b_2 \in \{0, 1\}$  such that  $\frac{b_1}{2} + \frac{b_2}{2} \leq x$ . Continue this process to get  $b_1, b_2, b_3, \dots$ . Then  $x = \sup \left\{ \sum_{i=1}^n \frac{b_i}{2^i} \right\}$ . Consider any dyadic rational of the form  $\frac{m}{2^n}$ . Let it be  $\frac{3}{2^4}$ . Then this maps  $\frac{3}{2^4} \rightarrow 0, 0, 1, 1, 0, 0, 0, \dots$  and never produce  $0, 0, 1, 0, 1, 1, 1, \dots$ , which also represents  $\frac{3}{2^4}$ . Let  $A_1$  be a subset of  $A = \{\text{infinite binary strings}\}$  such that  $A_1$  does not contain any strings ending in  $1, 1, 1, 1, \dots$ . Then the decimal representation defines a bijection  $f : [0] \rightarrow A \setminus A_1$ .
- (c)  $A_1$  is countable because  $A = (A \setminus A_1) \cup A_1$ , which is uncountable.

This shows that  $[0, 1]$  is uncountable, and therefore  $\mathbb{R}$  is uncountable.  
Q.E.D.

**Definition 16 (Metric Spaces).** A set  $X$  is a metric space with metric  $d : X \times X \rightarrow \mathbb{R}$  if

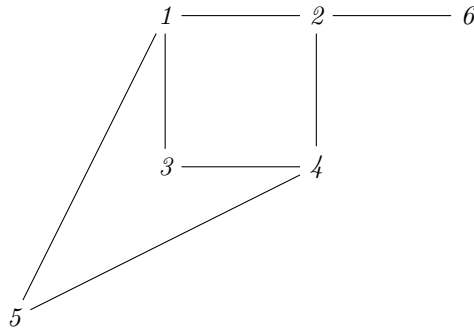
- (a)  $d(p, q) > 0$  if  $p \neq q$  and  $d(p, q) = 0$  if  $p = q$ ,  $\forall p, q \in X$
- (b)  $\forall p, q \in X : d(p, q) = d(q, p)$
- (c)  $\forall p, q, r \in X : d(p, q) \leq d(p, r) + d(r, q)$  (Triangle Inequality)

**Remark.** A metric space does not need to be an ordered set or to have addition or multiplication defined on it.

**Example (Metric Spaces).** (a)  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{R}^k$  are metric spaces with  $d(p, q) = |p - q|$ . Note the meaning of  $|x|$  depends on the context.

(b) Every subset of a metric space is a metric space.

(c)  $X = \{1, 2, 3, 4, 5, 6\}$



---

**Definition 17 (Neighborhood).** A neighborhood in  $X$  is a set  $N_r(p) := \{q : d(q, p) < r\}$ , where  $p \in X, r > 0$ .

**Remark.** If  $r_1 \leq r_2$ , then  $N_{r_1}(p) \subset N_{r_2}(p)$ .

**Example.**

$\mathbb{R}^1$  intervals,  $N_r(x) = \{y \in \mathbb{R}^1 : |x - y| < r\}$

$\mathbb{R}^2$  disks  $N_r(x) = \{y \in \mathbb{R}^2 : |x - y| < r\}$

$\mathbb{R}^3$  balls,  $N_r(x) = \{y \in \mathbb{R}^3 : |x - y| < r\}$

Given example (c),  $N_1(2) = \{2\} = N_{\frac{1}{2}}(2)$ ,  $N_2(2) = \{1, 2, 4, 6\}$ ,  $N_3(2) = \{1, 2, 3, 4, 5, 6\} = X$ .

**Definition 18.** Let  $E \subset X$ .  $p \in E$  is an interior point of  $E$  if  $\exists r > 0$  such that  $N_r(p) \subset E$ .

**Example.**

$X = \mathbb{R}^2, E = \{x \in \mathbb{R} : |x| \leq 1\}$

$X = \mathbb{N}, E \subset X$ .

**Definition 19.**  $E \subset X$  is an open set if  $\forall x \in E$  is an interior point of  $E$ .

**Theorem 15.** Every neighborhood is an open set.

**Proof.** Let  $g \in N_r(p)$ . Then we must find  $s > 0$ , such that  $N_s(g) \subset N_r(p)$ . We know  $d(p, g) < r$ . Choose  $s$  such that  $0 < s < r - d(p, g)$ . Let  $x \in N_s(g)$ , then  $d(g, x) < s < r - d(p, g)$ . By triangle inequality,  $d(p, x) \leq d(p, g) + d(g, x) < d(p, g) + r - d(p, g)$ , so  $x \in N_r(p)$ , so  $N_s(g) \subset N_r(p)$ . Q.E.D.

**Definition 20.** Let  $E \subset X$  and  $p \in X$ .  $p$  is a limit point of  $E$  if  $\forall r > 0 \exists q \in E$  such that  $q \neq p$  and  $q \in N_r(p)$

**Example.**  $E = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$  has exactly one limit point, 0. note  $0 \notin E$ .

**Theorem 16.** If  $p$  is a limit point of  $E \subset S$ , then every neighborhood of  $p$  contains infinitely many points of  $E$ .

**Proof.** Let  $N_r(p)$  be a neighborhood of  $p$ . Then  $N_r(p)$  contains at least one point  $q_1 \in E$  such that  $q_1 \neq p$ . Let  $r_1 = d(p, q_1)$ . Then  $N_{r_1}(p)$  contains some  $q_2 \in E$  such that  $q_2 \neq p$ . Let  $r_2 = d(p, q_2)$ . Then  $N_{r_2}(p)$  contains some  $q_3 \in E$  such that  $q_3 \neq p$ . Continue this process to get  $q_1, q_2, q_3, \dots$  Q.E.D.

**Corollary.** If  $E \subset X$  is finite then  $E$  has no limit points.

---

**Definition 21 (Closed Set).** A set  $E \subset X$  is closed if every limit point of  $E$  is in  $E$ .

**Theorem 17.**  $E \subset X$  is open iff  $E^c = \{x \in X : x \notin E\}$  is closed.

**Proof.**

- $E$  is open  $\Rightarrow E^c$  is closed.  
Let  $p$  be a limit point of  $E^c$ . Then every neighborhood of  $p$  contains some  $q \in E^c$  such that  $q \neq p$ . If  $p \in E$ , then because  $E$  is open,  $p$  is an interior point, i.e., there exists some neighborhood of  $p$  that is a subset of  $E$ , which does not contain any points of  $E^c$ . This implies  $p \notin E$  and therefore  $p \in E^c$ .
- $E^c$  is closed  $\Rightarrow E$  is open.  
Let  $p \in E$ . Then  $p \notin E^c$ , so  $p$  is not a limit point of  $E^c$ . Therefore, there exists some neighborhood of  $p$  that contains no points of  $E^c$ , i.e., all points of the neighborhood are in  $E$ . Thus, Every  $p \in E$  is an interior point of  $E$ , and hence  $E$  is open.

Q.E.D.

**Theorem 18 (De Morgan's Laws).**

- (a)  $(\bigcup_{\alpha} E_{\alpha})^c = \bigcap_{\alpha} E_{\alpha}^c$
- (b)  $(\bigcap_{\alpha} E_{\alpha})^c = \bigcup_{\alpha} E_{\alpha}^c$

**Theorem 19.**

- (a) For all collection  $\{G_{\alpha}\}$  of open sets :  $\bigcup_{\alpha} G_{\alpha}$  is open.
- (b) For all collection  $\{F_{\alpha}\}$  of closed sets :  $\bigcap_{\alpha} F_{\alpha}$  is closed.
- (c) For all finite collection  $\{G_1, G_2, \dots, G_n\}$  of open sets :  $\bigcap_{i=1}^n G_i$  is open.
- (d) For all finite collection  $\{F_1, F_2, \dots, F_n\}$  of closed sets :  $\bigcup_{i=1}^n F_i$  is closed.

- Proof.** (a) Let  $x \in \bigcup_{\alpha} G_{\alpha}$ . Then  $x \in G_{\alpha_0}$  for some  $\alpha_0$ . So there exists a neighborhood  $N$  of  $x$  such that  $N \subset G_{\alpha_0} \subset \bigcup_{\alpha} G_{\alpha}$ .
- (b) it's suffice to prove that  $(\bigcap_{\alpha} F_{\alpha})^c$  is open. But  $(\bigcap_{\alpha} F_{\alpha})^c = \bigcup_{\alpha} F_{\alpha}^c$  is open by (a).
- (c) Let  $x \in \bigcap_{i=1}^n G_i$ . Then  $x \in G_i$  for  $i = 1, 2, \dots, n$ . So there exists a  $r_i > 0$  such that  $N_{r_i}(x) \subset G_i$ . Let  $r = \min\{r_1, r_2, \dots, r_n\}$ . Then  $N_r(x) \subset N_{r_i}(x) \subset G_i$  for  $i = 1, 2, \dots, n$  and therefore  $N_r(x) \subset \bigcap_{i=1}^n G_i$ .
- (d)  $(\bigcup_{i=1}^n F_i)^c = \bigcap_{i=1}^n F_i^c$  is open by (c).

Q.E.D.

**Definition 22 (Closure).** Let  $E \subset X$ . Let  $E'$  be a set of limit points of  $E$  in  $X$ . The set  $\overline{E} = E \cup E'$  is the closure of  $E$ .

**Theorem 20.**

- (a)  $\overline{E}$  is closed.
- (b)  $E = \overline{E} \Leftrightarrow E$  is closed.
- (c) If  $F \subset X$  is closed and  $E \subset F$ , then  $\overline{E} \subset F$ . (i.e.,  $\overline{E}$  is the smallest closed set containing  $E$ , and  $\overline{E} = \bigcap_{F: \text{closed set with } E \subset F} F$ .)

**Proof.** (a) Let  $p$  be a limit point of  $\overline{E}$ . It suffices to show  $p \in E'$  since this implies that  $p \in E' \subset E \cup E' = \overline{E}$ . Let  $r > 0$ .  $\exists_{q \in \overline{E}, q \neq p} : q \in N_{\frac{r}{2}}(p)$ , i.e.,  $d(p, q) < \frac{r}{2}$ . Since  $q \in E \cup E'$ ,  $\exists_{s \in \overline{E}}$  such that  $d(q, s) < \frac{r}{2}$  (if  $q \in E$ , take  $s = q$ ). But  $d(p, s) \leq d(p, q) + d(q, s) < \frac{r}{2} + \frac{r}{2} = r$ .

(b)  $(\Rightarrow)$  by (a)

$(\Leftarrow)$  Suppose  $E$  is closed. Then  $E' \subset E$ , so  $\overline{E} = E \cup E' = E$ .

(c) Suppose  $F$  is closed. Then  $F' \subset F$  and also  $F \supset F'$ . So  $F = \overline{F} = F \cup F' \supset E \cup E' = \overline{E}$

Q.E.D.

**Example.** Let  $X = \mathbb{R}$ ,  $d(p, q) = |p - q|$ . Let  $E \subset \mathbb{R}$  be nonempty and bounded above, and let  $y = \sup E$ . Then  $y \in \overline{E}$ .

**Proof.** Suppose for contradiction  $y \notin \overline{E}$ . Then  $y$  is neither a point in  $E$  nor a limit point of  $E$ , so  $\exists$  some interval  $N_r(y) = (y - r, y + r)$  such that  $(y - r, y + r) \cap E = \emptyset$ . However, then  $y - r$  is an upper bound for  $E$  since  $y$  is a least upper bound, which is a contradiction. Therefore,  $y \in \overline{E}$ . Q.E.D.



---

**Definition 23 (Relative Openness).** Suppose  $X$  is a metric space, so  $Y \in X$  is a metric space with the same metric. Let  $E \subset Y$ . Then  $E$  is open relative to  $Y$  if  $E$  is an open set in the metric space  $Y$ .

**Example.**  $X = \mathbb{R}^2 \supset \mathbb{R} = Y, E = (0, 1) \subset Y$ . Then  $E$  is open relative to  $Y$ , but  $E$  is neither open nor closed in  $X$ .

**Theorem 21.** A set  $E \subset Y \subset X$  is open relative to  $Y \Leftrightarrow \exists_{\text{open set } G \subset X} : E = G \cap Y$

**Proof.**  $(\Rightarrow)$  Suppose  $E \subset Y$  is open relative to  $Y$ . Given  $p \in E$ ,  $\exists_{r_p > 0} : N_{r_p}^Y(p) \subset E$ , where  $N_r^Y(p) = \{q \in Y : d(p, q) < r\}$ . Then  $E \subset \bigcup_{p \in E} N_{r_p}^Y(p)$  and  $\bigcup_{p \in E} N_{r_p}^Y(p) \subset E$ . Therefore,  $E = \bigcup_{p \in E} N_{r_p}^Y(p)$ .

Let  $G = \bigcup_{p \in E} N_{r_p}^X(p)$ . This time, we are considering  $p$ 's neighborhood in  $X$ , so each  $N_{r_p}^X$  is open. Thus  $G$  is a union of open sets in  $X$ , and therefore open.

$\forall_{p \in E} : p \in N_{r_p}^X(p)$ , so  $E \subset G \cap Y$ .

Let  $p \in G \cap Y$ . Then  $p \in G$  and  $p \in Y$ . So  $p \in N_{r_p}^X(p)$  for some  $r_p > 0$ . But  $p \in Y$ , so  $p \in N_{r_p}^Y(p)$ . Therefore,  $p \in E$ . This implies  $G \cap Y \subset E$ , and therefore  $E = G \cap Y$ .

$(\Leftarrow)$  Suppose  $G \subset X$  is open and  $E = G \cap Y$ . Then  $\forall_{p \in E} : \exists_{r_p > 0} : N_{r_p}^X(p) \subset G$ , so  $N_{r_p}^Y(p) = N_{r_p}^X(p) \cap Y \subset G \cap Y = E$ .

Q.E.D.

**Note:** Midterm 1 material ends here.

**Definition 24 (Open Cover).** An open cover of  $E \subset X$  is a collection  $\{G_\alpha\}$  of open subsets of  $X$  s.t  $E \subset \bigcup_\alpha G_\alpha$ .

**Definition 25 (Compact).** A set  $K \subset X$  is compact if every open cover has a finite subcover; i.e.,  $\exists_{\alpha_1, \alpha_2, \dots, \alpha_n} : \text{s.t } K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \dots \cup G_{\alpha_n}$

**Example.**

- If  $E$  is finite, then  $E$  is compact.
- $(0, 1) \subset \mathbb{R}$  is not compact. Bad cover:  $(\frac{1}{n}, 1), n > 2$
- $[0, \infty) \subset \mathbb{R}$  is not compact. Bad cover:  $(-1, n)$  for  $n \in \mathbb{N}$ .
- $E \subset \mathbb{R}^k$  is compact if and only if  $E$  is closed and bounded.

**Theorem 22.** If  $K$  is compact then  $K$  is closed.

**Proof.** Suppose  $K$  is compact. It suffices to prove that  $K^c$  is open. Let  $p \in K^c$ . We need to produce  $r > 0$  s.t.  $N_r(p) \subset K^c$ . For  $q \in K$ , let  $W_q = N_{r_q}(q)$ , where  $r_q = \frac{1}{2}d(p, q) > 0$ .  $\forall x \in N_{r_q}(p) : x \in W_q \Rightarrow d(x, p) + d(x, q) < 2r_q = d(p, q)$ . However,  $X$  is a metric space and  $p, q, x \in X$ , so  $d(p, q) \leq d(p, x) + d(x, q)$ , leading to  $d(p, q) \leq d(p, x) + d(x, q) < d(p, q)$ , which is a contradiction. Hence,  $\forall x \in N_{r_q}(p) : x \notin W_q$ .  $N_{r_q}(p) \subset W_q^c$  for  $\forall q \in K$ . Note that  $\{W_q\}_{q \in K}$  is an open cover of  $K$ .  $K$  compact  $\Rightarrow \exists$  finite number of open sets  $W_{q_1}, W_{q_2}, \dots, W_{q_n}$  s.t.  $K \subset \bigcup_{i=1}^n W_{q_i}$ . Let  $r = \min\{r_{q_1}, r_{q_2}, \dots, r_{q_n}\} > 0$ .

$$\therefore N_r(p) \subset \left( \bigcap_{i \in \{1, 2, \dots, n\}} N_{r_p}(p) \right) \subset \left( \bigcap_{i \in \{1, 2, \dots, n\}} W_{q_i}^c \right) = \left( \bigcup_{i \in \{1, 2, \dots, n\}} W_{q_i} \right)^c \subset K^c$$

Q.E.D.

**Theorem 23.** If  $K \subset X$  is compact then  $K$  is bounded; i.e.,  $\exists M < \infty$  s.t.  $\forall p, q \in K : d(p, q) \leq M$

**Proof.** Fix  $p \in K$ . An open cover of  $K$  is  $\{N_n(p)\}_{n \in \mathbb{N}}$ . In fact, this is an open cover of  $X$ .  $K$  compact  $\Rightarrow \exists$  finite subcover  $N_{n_1}(p), N_{n_2}(p), \dots, N_{n_m}(p)$ . Let  $R = \max\{n_1, n_2, \dots, n_m\}$ .  $K \subset N_R(p)$ . Let  $M = 2R$ .  $\forall q, r \in K : d(q, r) \leq d(q, p) + d(p, r) < R + R = 2R = M$ . Q.E.D.

**Theorem 24.** If  $F$  is closed,  $K$  is compact, and  $F \subset K$  then  $F$  is compact.

**Proof.** Suppose  $F \subset K$ . let  $\{V_\alpha\}$  be an open cover of  $F$ . It suffice to produce a finite subcover:  
Consider  $\{V_\alpha\}$  together with  $F^c$ . This gives an open cover of  $X$ , hence of  $K$ , so  $\exists$  subcover of  $K$ . Drop  $F^c$  from this finite subcover. The result is a finite subcover of  $\{V_\alpha\}$ , which covers  $F$  Q.E.D.

**Corollary.** If  $F$  is closed and  $K$  is compact then  $F \cap K$  is compact.

**Theorem 25.** Suppose  $K \subset Y \subset X$ . Then  $K$  is compact relative to  $X$  iff  $K$  is compact relative to  $Y$ .

**Note.** This is not true for open sets. For instance, let  $K = Y = [0, 1] \subset X = \mathbb{R}$ .  $Y$  is open and closed relative to  $Y$ , but  $Y$  is not open relative to  $X$

---

**Proof.**

( $\Rightarrow$ ) Suppose  $K$  is compact relative to  $X$ . Let  $\{V_\alpha\}$  be an open cover of  $K$  relative to  $Y$ . For any subset of a metric space, there always exists an open cover relative to the whole space, as the whole metric space is trivially an open cover of itself. Then  $\{V_\alpha\}$  is an open cover of  $K$  relative to  $X$ . Since  $K$  is compact relative to  $X$ ,  $\exists_{\text{finite subcover}}$ .

( $\Leftarrow$ ) Suppose  $K$  is compact relative to  $Y$ . Let  $\{V_\alpha\}$  be an open cover of  $K$  relative to  $X$ . Then  $\{V_\alpha \cap Y\}$  is an open cover of  $K$  relative to  $Y$ . Since  $K$  is compact relative to  $Y$ ,  $\exists_{\text{finite subcover}}$ .

Q.E.D.

**Theorem 26.** Suppose  $\{K_\alpha\}$  is a collection of compact sets such that  $\bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$ . Then  $\lim_{n \rightarrow \infty} \bigcap_{i \in \{1,2,\dots,n\}} K_{\alpha_i} \neq \emptyset$ , or equivalently,  $\bigcap_\alpha K_\alpha \neq \emptyset$

**Example.** Let  $G_j = (0, \frac{1}{j}) \subset \mathbb{R}$ . Then  $\{G_j\}$  is a collection of open sets, but none of them are compact. (compact sets are closed) Then  $\{G_j\}$  satisfies non-empty finite intersection property but  $\bigcap_{j \in \mathbb{N}} G_j = \emptyset$ .

**Proof.** Then  $K_{\alpha_0} \cap \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right) = \emptyset$ , so  $K_{\alpha_0} \subset \left(\bigcap_{\alpha \neq \alpha_0} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_0} (K_\alpha)^c$  and  $\{(K_\alpha)^c\}_{\alpha \neq \alpha_0}$  is an open cover of  $K_{\alpha_0}$ , so  $\exists$  a finite subcover  $K_{\alpha_0} \subset \bigcup_{i=1}^n (K_{\alpha_i})^c$ . But then,  $K_{\alpha_0} \cap \left(\bigcap_{i=1}^n K_{\alpha_i}\right) = \emptyset$ , contradiction.  
Q.E.D.

**Corollary.** If  $\{K_1, K_2, \dots\}$  are non-empty compact sets with  $\forall_n : K_n \supset K_{n+1}$ , then  $\bigcap_{n=1}^\infty K_n \neq \emptyset$ .

**Proof.** If  $n_1 < n_N$  then  $\bigcap_{i=1}^N K_{n_i} = K_{n_N} \neq \emptyset$  Q.E.D.

**Theorem 27.** If  $K$  is compact and  $E \subset K$  is infinite, then  $E$  has a limit point in  $K$ .

**Proof.** Contrapositive of the statement is : if  $E \subset K$  has no limit point in  $K$ , then  $E$  is finite.

Suppose every point  $q \in K$  is not a limit point of  $E$ . Then

$$\exists_{V_q = N_{r_q}(q)} : V_q \cap E = \begin{cases} \emptyset & \text{if } q \notin E \\ \{q\} & \text{if } q \in E \end{cases}$$

$\{V_q\}_{q \in K}$  is an open cover of  $K$ , so  $\exists_{\text{finite subcover}} V_{q_1} \cup V_{q_2} \dots V_{q_n}$ . Then  $E = E \cap K \subset (\bigcup_{i=1}^n V_{q_i} \cap E) \subset \{q_1, q_2, \dots, q_n\}$ , so  $E$  is finite.

Q.E.D.

**Theorem 28.** Let  $I_n = [a_n, b_n] \subset \mathbb{R}$  be such that  $\forall_n : I_n \supset I_{n+1}$ . Then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Proof.** Since  $I_n \supset I_{n+1}$ ,  $\forall_{n,m} : a_n \leq a_{n+m} \leq b_{n+m} \leq b_n$ . Let  $E = \{a_1, a_2, \dots\}$ . Then  $E \neq \emptyset$ , every  $b_k$  is an upper bound for  $E$ , so  $\exists x = \sup E$  and  $a_k \leq x \leq b_k$  for all  $k$ . Therefore,  $x \in I_k$  for all  $k$ , so  $x \in \bigcap_{n=1}^{\infty} I_n$ . Q.E.D.

**Theorem 29.** Let  $\{I_n\}$  be a sequence of  $k$ -cells such that  $I_n \supset I_{n+1}$ ; i.e.,  $I_n = \{\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_{nj} \leq x_j \leq b_{nj}, a_{nj} \leq a_{n+1,j} \leq b_{n+1,j} \leq b_{nj} \text{ for } j = 1, 2, \dots, k\}$ . Then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Proof.** Apply previous theorem to each component. Q.E.D.

**Note.**  $k$ -cell is a higher dimensional analog of a rectangle or rectangular solid, which is a Cartesian product of  $k$  closed intervals on the real line.

Formally, Given real numbers  $a_i$  and  $b_i$  such that  $a_i < b_i$  for every integer  $i$  from 1 to  $k$ ,

$$I = \{x = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \dots, k\}$$

**Theorem 30.** Let  $I \subset \mathbb{R}^k$  be a  $k$ -cell. Then  $I$  is compact.

**Proof.** Let  $I = I_0 = \{x = (x_1, x_2, \dots, x_k), a_j \leq x_j \leq b_j\}$ .

Let  $\Delta = \left\{ \sum_{j=1}^k (b_j - a_j)^2 \right\}^{1/2}$ . Then  $|\mathbf{x} - \mathbf{y}| \leq \Delta$  for  $\mathbf{x}, \mathbf{y} \in I$ .

Suppose for contradiction  $\{G_\alpha\}$  is an open cover of  $I$  that has no finite subcover.

Let  $c_j = \frac{1}{2}(a_j + b_j)$  for  $j = 1, 2, \dots, k$ . Using  $[a_j, c_j], [c_j, b_j]$ , we get  $2^k$   $k$ -cells  $Q_i$  with  $I = \bigcup_{i=1}^{2^k} Q_i$ . At least one  $Q_i$ , call it  $I_1$ , has no finite subcover. Otherwise, every  $Q_i$  has a finite subcover, and  $I$  would have a finite subcover, namely the union of the finite subcovers of each  $Q_i$ . Repeat this step to construct  $I_0 = I, I_1, I_2, \dots$ . Then the sequence  $\{I_n\}$  constructed by this process satisfies the following properties:

- (a)  $I_0 = I \supset I_1 \supset I_2 \supset I_3 \supset \dots$
- (b)  $\forall_n : I_n$  has no finite subcover from  $\{G_\alpha\}$
- (c) if  $x, y \in I_n$  then  $|x - y| \leq 2^{-n}\Delta$ , where  $\Delta = \text{diagonal of } I = \left( \sum_{j=1}^k (b_j - a_j)^2 \right)^{1/2}$ .

By theorem 29 and (a),  $\exists x^* \in \bigcap_{n=1}^{\infty} I_n$ . Since  $x^* \in I$ ,  $x^* \in G_{\alpha_0}$  for some  $\alpha_0$ , so  $\exists r > 0$  such that  $N_r(x^*) \subset G_{\alpha_0}$ . But by (c),  $I_n \subset N_{2^{-n}\Delta}(x^*)$ . As soon as  $n$  is large enough that  $2^{-n}\Delta < r$ , we have  $I_n \subset N_{2^{-n}\Delta}(x^*) \subset G_{\alpha_0}$ , which contradicts (b). Q.E.D.



**Note.** Reverse triangle inequality

$\forall_{a,b,c \in X} : d(a,b) \geq d(a,c) - d(c,b)$  because  $d(a,c) \leq d(a,b) + d(b,c)$ .

**Theorem 31.** For  $E \subset \mathbb{R}^k$ , the following are equivalent:

- (a)  $E$  is closed and bounded.
- (b)  $E$  is compact.
- (c) Every infinite subset of  $E$  has a limit point in  $E$ .

**Proof.**

(a)  $\Rightarrow$  (b) Because  $E$  is bounded, there exists a  $k$ -cell  $I$  such that  $E \subset I$ ; i.e.,  $\exists_M$  s.t.  $\forall_{x,y \in E} : |x - y| \leq M$ . Therefore,  $E$  is compact.

(b)  $\Rightarrow$  (c) by theorem 28

(c)  $\Rightarrow$  (a) To see that  $E$  is bounded, suppose it were not. Then  $E$  has an infinite subset  $S = \{x_1, x_2, x_3, \dots\}$  with  $\forall_n : |x_n| \geq n$ .  $S$  has no limit point in  $\mathbb{R}^k$ . Let  $S = \{(x_1, x_2, x_3, \dots) \in E : |x_n - x_0| < \frac{1}{n}\}$ . Then  $S$  is an infinite set because if  $S$  is finite, there exists a point  $\mathbf{x} \in S$  such that  $|\mathbf{x}| \geq |\mathbf{x}'|$  for  $\mathbf{x}' \in S$ . However, there exists  $n \in \mathbb{N}$  such that  $n > |\mathbf{x}|$  and by definition of  $S$ , there exists  $x_n \in S$  such that  $|x_n| \geq n > |\mathbf{x}|$ , which is a contradiction. Thus,  $S$  is infinite. This  $S$  however, cannot have a limit point in  $E$ . By triangle inequality, for any  $y \in R^k$ ,  $|x_n| \leq |x_n - y| + |y|$ , and from archimedean property,  $\exists_{m \in \mathbb{N}}$  s.t.  $m > |x_n - y| + |y|$ , which implies for any  $y \in R^k$ ,  $r > 0$ ,  $\exists_{m \in \mathbb{N}} : |x - y| < r < m$ . However, by the definition of  $S$ , there are at most  $m$  such elements in  $S$ . Since a limit point  $y$  of  $E$  must contain an infinite number of points of  $E$  such that  $d(x, y) < r$  for any  $r > 0$ ,  $y$  cannot be a limit point, which contradicts the assumption that any infinite subset of  $E$  contains a limit point in  $E$ . Therefore,  $E$  must be bounded.

To see that  $E$  is closed, suppose it were not closed. Then  $\exists_{x_0 \in E'} : x_0 \notin E$ . If  $T$  has no limit point in  $E$  except  $x_0 \notin E$ , it contradicts (c) because  $T$  is infinite and there must be a limit point of  $T$  in  $E$ .

Therefore, we can show that  $E$  is closed by showing that  $T$  has no limit point in  $E$  except  $x_0$ . Form an infinite sequence  $(x_1, x_2, x_3, \dots), x_n \in E$  with  $|x_n - x_0| < \frac{1}{n}$ . Let  $y \in E$ ,  $y \neq x_0$ . We'll show that  $y$  cannot be a limit point of  $T$ .  $|y - x_n| \geq |y - x_0| + |x_0 - x_n| > |y - x_0| - |x_0 - x_n| > |y - x_0| - \frac{1}{n}$ . Choose  $n \geq \frac{2}{|y - x_0|}$ , so  $\frac{1}{n} \leq \frac{|y - x_0|}{2}$ . Then  $|y - x_n| \geq |y - x_0| - \frac{1}{2}|y - x_0| = \frac{1}{2}|y - x_0|$ . So only finitely many  $x_n$  can lie in  $N_{\frac{1}{2}|y - x_0|}(y)$ . So  $y$  cannot be a limit point of  $S$ . Therefore,  $E$  is closed.

Q.E.D.

**Remark.** (b) and (c) are equivalent in any metric space, but (a) does not, in general, imply (b) or (c) in a metric space other than  $\mathbb{R}^k$ .

**Example.** Failure of Heine-Borel theorem in general metric spaces.

some infinite set, discrete metric  $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ . Then  $E$  is bounded and closed but not compact.

---

**Theorem 32 (Weirstrass's theorem).** Every bounded infinite subset  $E \subset \mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .

**Proof.** Choose a  $k$ -cell  $I \supset E$ . Since  $I$  is compact, by theorem 31,  $E$  has a limit point in  $I$ . Q.E.D.

**Example.** Let

$$E_0 = [0, 1] \tag{2.5}$$

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \tag{2.6}$$

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \tag{2.7}$$

$$\vdots \tag{2.8}$$

This gives  $E_0 \supset E_1 \supset E_2 \supset E_3 \supset \dots$ , where each  $E_n$  is the union of  $2^n$  closed intervals of length  $\frac{1}{3^n}$ .

**Definition 26 (Cantor Set).** The cantor set  $P := \bigcap_n^\infty E_n$ .

**Proposition 1.**  $P$  is compact, non-empty and contains no open intervals  $(a, b)$  and uncountable.

**Proof. Compactness**  $P$  is compact because  $P \subset E_0 = [0, 1]$  and  $E_0$  is compact.

**Non-emptiness**  $P$  is non-empty because  $P \subset E_0$  and  $E_0$  is non-empty.

**No open intervals**  $P$  contains no open intervals  $(a, b)$  because any  $(a, b)$  contains some  $(\frac{3k+1}{3^n}, \frac{3k+2}{3^n})$  and these are all removed.

**Uncountability**  $P$  is uncountable because  $P$  is a perfect set. Equivalently,  $P$  consists of points in  $[0, 1]$  whose ternary, i.e., base 3, representation contains only 0's and 2's.

**Note.** ternary representation:  $0.a_1a_2a_3\dots = \sum_{n=1}^\infty \frac{a_n}{3^n}$  where  $a_n \in \{0, 1, 2\}$ .

Q.E.D.

**Example (Cantor Set).** Let  $E = [0, 1]$ ,  $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ .  $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ , etc. Keep removing open middle third. This gives  $E_0 \supset E_1 \supset E_2 \dots$ . Each  $E_n$  is the union of  $2^n$  closed intervals of length  $\frac{1}{3^n}$ .

**Definition 27 (Separated Sets).** **Separated Sets**  $A, B \subset X$  are separated if  $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ .

**Connected Sets**  $E \subset X$  is connected if there is no non-empty separated

---

sets  $A, B \subset E$ .

**Example** (Separated Sets). *In  $\mathbb{R}^1$ ,  $[0, 1)$  and  $(1, 2]$  are separated so  $[0, 1) \cup (1, 2]$  is not connected. Every interval is connected (open, closed, semi-open).*

**Theorem 33.**  $E \subset \mathbb{R}^1$  is connected if and only if  $E$  is an interval; i.e.,  
 $\forall x, y \in E, x < y$  s.t.  $\forall z \in (x, y) : z \in E$

**Theorem 34.** A metric space  $X$  is connected if and only if the only nonempty subset of  $X$  which is both open and closed is  $X$  itself.  $|\limsup_{n \rightarrow \infty} |\gamma_n|| \leq 0 + \varepsilon \alpha$ . Since  $\varepsilon$  is arbitrary, this implies  $|\limsup_{n \rightarrow \infty} |\gamma_n|| = 0$ , so  $\lim_{n \rightarrow \infty} |\gamma_n| = 0$ .



## Chapter 3

# Sequence and Series

### 3.1 Sequences

**Definition 28.** In a metric space  $(X, d)$ , a sequence  $\{p_n\}$  converges to  $p$  if  $\forall \varepsilon > 0 \exists N$  s.t.  $n \geq N \Rightarrow d(p_n, p) < \varepsilon$ .  
We write  $\lim_{n \rightarrow \infty} p_n = p$  or  $p_n \rightarrow p$ .

If  $\{p_n\}$  does not converge to any  $p$  then it is said to diverge.

**Theorem 35.** If  $s_n$  and  $t_n$  are sequences in  $\mathbb{C}$  with  $s_n \rightarrow s$  and  $t_n \rightarrow t$ , then the following hold:

- (a)  $s_n + t_n \rightarrow s + t$
- (b)  $cs_n \rightarrow cs$ ,  $c + s_n \rightarrow c + s$  for any  $c \in \mathbb{C}$
- (c)  $s_n t_n \rightarrow st$
- (d)  $\frac{1}{s_n} \rightarrow \frac{1}{s}$  if  $s \neq 0$

**Lemma 2 (Squeeze Lemma).** In  $\mathbb{R}$ , if  $\forall n \in \mathbb{N} : 0 \leq x_n \leq s_n$  and  $\lim_{n \rightarrow \infty} s_n \rightarrow 0$ , then  $\lim_{n \rightarrow \infty} x_n = 0$ .

**Proof.** Let  $\varepsilon > 0$ . Choose  $N$  such that  $n \geq N \Rightarrow 0 \leq s_n < \varepsilon$ . Then  $0 \leq x_n \leq s_n < \varepsilon$  for  $n \geq N$ , so  $x_n \rightarrow 0$ . Q.E.D.

**Theorem 36.** (a) If  $p > 0$  then  $\frac{1}{n^p} \rightarrow 0$ .

**Proof.** Let  $\varepsilon > 0$ . Choose  $N$  such that  $\frac{1}{N^p} < \varepsilon$ ; i.e.,  $N > \frac{1}{\varepsilon^{\frac{1}{p}}}$ . Then for  $n \geq N$ ,  $\frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon$ . Q.E.D.

(b) If  $p > 0$  then  $\sqrt[p]{p} \rightarrow 1$ .

**Proof.**  $p = 1$  is obvious.

Suppose  $p > 1$ . Let  $x_n = \sqrt[p]{p} - 1 > 0$ . Want to show  $x_n \rightarrow 0$ .

Since  $(x_n + 1)^n$ , we have  $p = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} x_n^k > \binom{n}{1} x_n = n x_n$ .

Therefore,  $x_n \leq \frac{p}{n}$ , so  $x_n \rightarrow 0$  by the Squeeze Lemma.

Suppose  $p \in (0, 1)$ . Let  $q = \frac{1}{p} > 1$ . Then  $\sqrt[q]{q} \rightarrow 1$  by the previous case. By 35,  $\sqrt[p]{p} = \frac{1}{\sqrt[q]{q}} \rightarrow 1$ . Q.E.D.

(c)  $\sqrt[n]{n} \rightarrow 1$

**Proof.** Let  $x_n = \sqrt[n]{n} - 1 > 0$ , for  $n \geq 2$ .  $n = (x_n + 1)^n = \sum_{k=0}^n \binom{n}{k} (x_n)^k > \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$ . Therefore,  $x_n \leq \sqrt{\frac{2}{n-1}}$ . Q.E.D.

(d) If  $p > 0$  and  $\alpha \in \mathbb{R}$ , then  $\frac{n^\alpha}{(1+p)^n} \rightarrow 0$ ; i.e., Exponentials beat powers.

**Proof.** We want an upper bound on  $\frac{n^\alpha}{(1+p)^n}$ , so seek a lower bound on  $(1+p)^n$ .

$(1+p)^n = \sum_{k=0}^n \binom{n}{k} p^k > \binom{n}{k} p^k$  for  $k \leq n$

$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\cdots(n-k+1)}{k!} p^k$ . Then for  $k \leq \frac{n}{2}$ ,  $\binom{n}{k} p^k >$

$(\frac{n}{2})^k \frac{p^k}{k!}$ . Therefore,  $\frac{n^\alpha}{(1+p)^n} < (\frac{2}{p})^k k! \frac{1}{n^{k-\alpha}}$ . Let  $k_0 \in \mathbb{Z}$  s.t.  $k > \alpha$ . Then for  $n \geq 2k_0$ , RHS  $\rightarrow 0$  by (a).

If  $|x| < 1$  then  $x^n \rightarrow 0$ .

**Proof.**  $|x^n - 0| = |x|^n$ , so  $x^n \rightarrow 0 \Leftrightarrow |x|^n \rightarrow 0$  and  $|x|^n = \frac{\frac{n_0}{|x|}}{(\frac{1}{|x|})^n} \rightarrow 0$  by (d) with  $\alpha = 0$  and  $1+p = \frac{1}{|x|} > 1$ , so  $p = \frac{1}{|x|} - 1 > 0$ . Q.E.D.

Q.E.D.

**Theorem 37.** . Let  $\{p_n\}$  be a sequence in  $(X, d)$ .

(a)  $p_n \rightarrow p \Leftrightarrow \forall_{r>0} : N_r(p)$  contains all but finitely many  $p_n$ .

**Proof.**  $\forall_{n \geq N} : p_n \in N_r(p)$  Q.E.D.

(b) If  $p_n \rightarrow p$  and  $p_n \rightarrow p'$  then  $p = p'$ .

**Proof.**  $d(p, p') \leq d(p_n, p) + d(p_n, p')$  for all  $n$ . Fix  $\varepsilon$ . Choose  $N$  such that  $d(p_n, p) < \frac{\varepsilon}{2}$  and  $d(p_n, p') < \frac{\varepsilon}{2}$  for  $n \geq N$ . Then  $d(p, p') < \varepsilon$ . Then for  $n \geq \max\{N, N'\}$ ,  $d(p, p') < \varepsilon$ . This is true for all  $\varepsilon > 0$ , so  $d(p, p') = 0$ . Q.E.D.

(c) If  $\{p_n\}$  converges, then  $p_n$  is bounded, in a sense that  $\exists_{M>0, q \in X}$  s.t.  $d(p_n, q) \leq M$  for all  $n$ .

---

**Proof.** If  $p_n \rightarrow p$ , then  $\exists N$  s.t.  $d(p_n, p) < 1$  for all  $n \geq N$ . Thus,  $\forall_{n \geq 1} : d(p_n, p) \leq \max\{1, d(p, p_1), \dots, d(p, p_{N-1})\} = M$   
Q.E.D.

(d) If  $E \subset X$  has a limit point  $p$ , then  $\exists_{p_n \in E}$  s.t.  $p_n \rightarrow p$ .

**Proof.** We need to choose  $p_n \in E$  s.t.  $d(p, p_n) < \frac{1}{n}$ . Let  $\varepsilon > 0$ . Then  $d(p, p_n) < \varepsilon$  if  $n > \frac{1}{\varepsilon}$   
Q.E.D.

**Definition 29.** Given  $p_n, n_1 < n_2 < n_3 < \dots$ , we say  $p_{n_i} = (p_{n_1}, p_{n_2}, \dots)$  is a subsequence of  $p_n$ .

**Lemma 3.**  $p_n \rightarrow p \Leftrightarrow$  every subsequence of  $\{p_n\}$  converges to  $p$

**Proof.** Look at assignment 6  
Q.E.D.

**Theorem 38.** (a)  $\{p_n\}$  in  $X$ ,  $X$  compact, then  $\exists$  convergent subsequence.

**Proof.** Let  $E = \text{range of } \{p_n\}$ . If  $E$  is finite, then  $\exists p \in X$  and  $n_1 < n_2 < \dots$  s.t.  $p_n = p$  for  $\forall i$ . This subsequence converges to  $p$ . If  $E$  is infinite then by Theorem 28,  $E$  has a limit point  $p \in X$ ; i.e., every neighborhood of  $p$  contains infinitely many points of  $E$ . Choose  $n_1$  s.t.  $d(p, p_{n_1}) < 1$ .

Q.E.D.

(b)  $\{p_n\}$  in  $\mathbb{R}^k$ , bounded, then  $\exists$  convergent subsequence.

**Proof.** Choose a  $k$ -cell  $I$  that contains  $\{p_n\}$ .  $I$  is compact. Apply (a).

Q.E.D.

**Definition 30 (Cauchy Sequence).**  $\{p_n\}$  is a Cauchy sequence in  $(X, d)$  if  $\forall \varepsilon : \exists_{N \in \mathbb{N}}$  s.t.  $d(p_m, p_n) < \varepsilon \forall m, n \geq N$ .

**Definition 31.** For  $E \subset X$ ,  $E \neq \emptyset$ , we define  $\text{diam } E = \sup \{d(p, q) : p, q \in E\}$ .  $\text{diam } E = \infty$  if the set is not bounded above.

**Example.** For a sequence  $p_n$  in  $X$ , let  $E_n = \{p_N, p_{N+1}, \dots\}$ . Then  $\{p_n\}$  is a Cauchy sequence iff  $\lim_{N \rightarrow \infty} \text{diam } E_N = 0$ .

**Theorem 39.** (a) If  $p_n \rightarrow p$  then  $\{p_n\}$  is a Cauchy sequence.

(b) If  $X$  is a compact metric space and  $\{p_n\}$  in  $X$  is a Cauchy sequence, then  $\exists_{p \in X}$  s.t.  $p_n \rightarrow p$ .

(c) In  $\mathbb{R}^K$  every Cauchy sequence converges.

**Proof.** Let  $\varepsilon > 0$ . Choose  $N$  s.t.,  $d(p_n, p) < \varepsilon/2$  if  $n \geq N$ . Then for  $m, n \geq N$ ,  $d(p_m, p_n) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ .

Suppose  $\{p_n\}$  is Cauchy. Let  $E_N = \{p_N, p_{N+1}, \dots\}$ . Then  $\overline{E_N}$  is closed, hence compact. Also  $\overline{E_N} \supset \overline{E_{N+1}}$  and  $\lim_{N \rightarrow \infty} \text{diam } \overline{E_N} = 0$  (use Theorem 3.10(a) to see  $\text{diam } \overline{E_N} = \text{diam } E_N$ ) By theorem 3.10(b),  $\exists! p \in \bigcap_{N=1}^{\infty} \overline{E_N}$ . Claim:  $p_n \rightarrow p$ .

Proof of the claim: Let  $\varepsilon > 0$ . Choose  $N_0$  s.t.  $\text{diam } \overline{E_{N_0}} < \varepsilon$ , so  $d(p, q) < \varepsilon \forall q \in \overline{E_{N_0}}$ , and hence  $\forall q \in E_{N_0}$ ; i.e.,  $d(p, p_n) < \varepsilon$  if  $n \geq N_0$ .

Let  $\varepsilon > 0$ . Choose  $N$  s.t.,  $d(p_n, p) < \varepsilon/2$  if  $n \geq N$ . Then for  $m, n \geq N$ ,  $d(p_m, p_n) \leq d(p_n, p) + d(p, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ .

Suppose  $\{p_n\}$  in  $\mathbb{R}^k$  is Cauchy. Cauchy sequences are bounded in any metric space. Therefore,  $\exists$   $k$ -cell  $I$ , which is compact, containing  $\{p_n\}$ . Then (b) applies Q.E.D.

**Note.** The converse of Theorem 39(a) does not hold in general.

**Example.**  $X = \mathbb{Q}$  has a Cauchy sequence with no limit in  $\mathbb{Q}$ . (see assignment 6). Converse does hold if  $X$  is compact.

**Theorem 40.** (a)  $\text{diam } \overline{E} = \text{diam } E$

(b) If  $K_n \subset X$ ,  $K_n \neq \emptyset$ ,  $K$  compact,  $K_n \supset K_{n+1} \forall n$  and if  $\lim_{n \rightarrow \infty} \text{diam } K_n = 0$ , then  $\bigcap_{n=1}^{\infty} K_n$  is a single point.

**Proof.** (a)  $E \subset \overline{E} \Rightarrow \text{diam } E \leq \text{diam } \overline{E}$ . For the opposite inequality, let  $\varepsilon > 0$ ,  $p, q \in \overline{E}$ . Choose  $p', q' \in E$  s.t.  $d(p, p') < \varepsilon$ ,  $d(q, q') < \varepsilon$ . Then  $d(p, q) \leq d(p, p') + d(p', q') + d(q', q) < \varepsilon + \varepsilon + \text{diam } E \leq \text{diam } E + 2\varepsilon$ . Since  $\varepsilon$  is arbitrary,  $\text{diam } \overline{E} \leq \text{diam } E$ .

(b) Let  $K = \bigcap_{n=1}^{\infty} K_n$ . By Theorem 2.36,  $K \neq \emptyset$ . Since  $K \subset K_n \forall n$ ,  $\text{diam } K \leq \text{diam } K_n \forall n$ , so  $\text{diam } K = 0$ . Therefore,  $d(p, q) = 0 \forall p, q \in K$ , so  $K$  is a simple point.

Q.E.D.

**Definition 32 (Complete Metric Space).** A metric space  $(X, d)$  is complete if every Cauchy sequence in  $X$  converges to a point in  $X$ .

**Example.** (a)  $X$  compact  $\Rightarrow X$  complete.

(b)  $\mathbb{R}^k$  is complete, so is  $\mathbb{C}$ .

(c)  $\mathbb{Q}$  is not complete. (see assignment 6)

(d) See assignment 6 for reference to completion of a metric space.

We've seen that convergent sequences are bounded.  $p_n = (-1)^n$  shows the converse is false. However the converse does hold for monotonic sequences.

**Definition 33 (Monotone).**

- A sequence  $\{s_n\}$  in  $\mathbb{R}$  is monotonically increasing if  $s_n \leq s_{n+1} \forall n$ .
- A sequence  $\{s_n\}$  in  $\mathbb{R}$  is monotonically decreasing if  $s_n \geq s_{n+1} \forall n$ .

**Theorem 41.** A monotone sequence in  $\mathbb{R}$  converges if and only if it is bounded.

**Proof.**  $\Rightarrow$  all convergent sequences are bounded in any metric space.

$\Leftarrow$  **Increasing case** Let  $\{s_n\}$  be monotonically increasing and  $s_n \leq M \forall n$ . Let  $s = \sup\{s_n : n \in \mathbb{N}\}$ . Then  $s_n \leq s \forall n$ . Let  $\varepsilon > 0$ .  $\exists N$  s.t.  $s - \varepsilon < s_N \leq s$ . But then  $s - \varepsilon < s_N \leq s_{N+1} \leq s_{N+2} \leq \dots \leq s$ , so  $|s - s_n| < \varepsilon \forall n \geq N$ , and therefore  $s_n \rightarrow s$ .

Q.E.D.

**Definition 34 (Infinite Limits).** We say

- $s_n \rightarrow \infty$  if  $\forall M \in \mathbb{R} : \exists N$  s.t.  $s_n \geq M \forall n \in N$ .
- $s_n \rightarrow -\infty$  if  $\forall M \in \mathbb{R} : \exists N$  s.t.  $s_n \leq M \forall n \in N$ .

**Definition 35.** Let  $\{s_n\}$  be a sequence in  $\mathbb{R}$ . We define  $\limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{\sup_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \sup_{m \geq n} \{s_m\}$ .  
 $\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{s_m\} = \sup_{n \geq 1} \{\inf_{m \geq n} \{s_m\}\} = \lim_{n \rightarrow \infty} \inf_{m \geq n} \{s_m\}$ .

**Note.** Alternate definition; see ass 7 for equivalence

**Remark.** (a) If  $a_n \leq b_n \forall n$  and  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a \leq b$ .

(b)  $\liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n$

**Example.** (a)  $s_n = (-1)^n(1 + \frac{1}{n^2})$   $1 \leq \sup_{m \geq n} s_m \leq 1 + \frac{1}{n^2}$ , so  $\limsup_{n \rightarrow \infty} s_n = 1$ . Similarly,  $\liminf_{n \rightarrow \infty} s_n = -1$

(b) If  $\{s_n\}$  has no upper bound, then  $\sup_{m \geq n} s_m = \infty$  and in this case we say  $\limsup_{n \rightarrow \infty} s_n = \infty$ ; e.g.,

$$s_n = \begin{cases} n & n \text{ odd} \\ -n & n \text{ even} \end{cases}$$

has  $\limsup_{n \rightarrow \infty} s_n = \infty$ ,  $\liminf_{n \rightarrow \infty} s_n = -\infty$

---

**Lemma 4.**  $\limsup_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = L \Leftrightarrow s_n \rightarrow L$ .

**Proof** ( $L$  finite).

$\Rightarrow$  This follows from  $\inf_{m \geq n} s_m \leq s_n \leq \sup_{m \geq n} s_m$ .  $\lim_{n \rightarrow \infty} \inf_{m \geq n} s_m = \liminf_{n \rightarrow \infty} s_n$ , and  $\lim_{n \rightarrow \infty} \sup_{m \geq n} s_m = \limsup_{n \rightarrow \infty} s_n$ . Therefore,  $\lim_{n \rightarrow \infty} s_n = L$ .

$\Leftarrow$  If  $s_n \rightarrow L$ , then  $\forall \varepsilon > 0 : \exists N$  s.t.  $s_m \in [L - \varepsilon, L + \varepsilon] \forall m \geq N$ . Therefore,  $\forall n \geq N : L - \varepsilon \leq \inf_{m \geq n} s_m \leq \inf_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq \sup_{m \geq n} s_m \leq L + \varepsilon$ . Let  $n \rightarrow \infty$ :  $L - \varepsilon \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L + \varepsilon$ . Since  $\varepsilon$  is arbitrary, so  $L \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq L$ .

Q.E.D.

## 3.2 Series

**Definition 36 (Series).** Let  $\{a_n\}$  be a sequence in  $\mathbb{C}$ . Form a new sequence  $\{s_n\}$ , the sequence of partial sums, by  $s_n = a_1 + a_2 + \dots + a_n = \sum_{k=1}^n a_k$ . If  $s_n \rightarrow s$ , we say **the series**  $\sum_{k=1}^{\infty} a_k$  **converges** and that  $\sum_{k=1}^{\infty} a_k = s$ . If  $\{s_n\}$  diverges then we say  $\sum_{k=1}^{\infty} a_k$  diverges.

**Theorem 42.**  $\sum_{n \in \mathbb{N}} a_n$  converges if and only if  $\exists \varepsilon > 0 : \exists N$  s.t.  $|\sum_{k=m}^n a_k| < \varepsilon \forall n \geq m \geq N$ .

**Proof.**  $\sum_n a_n$  converges  $\Leftrightarrow \{s_n\}$  converges  $\Rightarrow \{s_n\}$  is a Cauchy sequence. Use  $s_n - s_{m-1} = \sum_{k=m}^n a_k$ . Q.E.D.

**Corollary.** If  $\sum_n a_n$  converges then  $a_n \rightarrow 0$ .

**Proof.** Take  $m = n$  in Theorem 3.22.  $\sum_n a_n$  converges  $\Rightarrow \forall \varepsilon > 0 : \exists N$  s.t.  $|a_n| < \varepsilon$  if  $n \geq N$ . Q.E.D.

**Remark.**  $n$ -th term test for divergence: If  $a_n \not\rightarrow 0$  then  $\sum_n a_n$  diverges.

**Example.**  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  diverges because  $\frac{n}{n+1} \rightarrow 1 \neq 0$ .

Converse to Corollary is false! E.g.,  $\sum_n \frac{1}{n}$  diverges but  $\frac{1}{n} \rightarrow 0$ .

**Theorem 43.** If  $a_n \geq 0$ , then  $\sum_n a_n$  converges if and only if  $\{s_n\}$  is bounded.

**Proof.**  $\{s_n\}$  is monotone increasing, so by Theorem 41, it converges if and only if it is bounded. Q.E.D.

**Theorem 44 (Comparison Test).** (a) If  $|a_n| \leq c_n \forall n \geq N_0$  and  $\sum_n c_n$  converges, then  $\sum_n a_n$  converges.

**Proof.** Suppose  $|a_n| \leq c_n \forall n \geq N_0$  and  $\sum_n c_n$  converges. Let  $\varepsilon > 0$ . By theorem 3.22,  $\exists N$  s.t.  $\sum_{k=m}^n c_k < \varepsilon$  if  $n \geq m \geq N$ . Can take  $N \geq N_0$ . Then  $|N \geq N_0|$ .  $|\sum_{k=m}^n a_k| \leq \sum_{k=m}^n |a_k| \leq \sum_{k=m}^n c_k < \varepsilon$  if  $n \geq m \geq N$ . By theorem 3.22 again,  $\sum_n a_n$  converges. Q.E.D.

(b) If  $a_n \geq d_n \geq 0 \forall n \geq N_0$  and if  $\sum_n d_n$  diverges, then  $\sum_n a_n$  diverges.

**Proof.** This follows from (a): if  $\sum_n a_n$  converges then  $\sum_n d_n$  converges. Thus it's contrapositive, (b) is true. Q.E.D.

**Theorem 45 (Geometric Series).**  $\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{if } -1 < x < 1 \\ \text{diverges} & \text{otherwise} \end{cases}$

**Proof.** Let  $S_n = 1 + x + x^2 + \cdots + x^n$ ,  $xS_n = x + x^2 + \cdots + x^n + x^{n+1}$ . Then

$$S_n - xS_n = 1 - x^{n+1} \Rightarrow S_n = \frac{1 - x^{n+1}}{1 - x}$$

If  $|x| < 1 (\Leftrightarrow -1 < x < 1)$ , then  $x^{n+1} \rightarrow 0$  and  $S_n \rightarrow \frac{1}{1-x}$ . If  $|x| \geq 1$ , then  $x^{n+1}$  does not converge to 0, so  $\sum_{n=0}^{\infty} x^n$  diverges. Q.E.D.

**Theorem 46.** Suppose  $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$ . Then  $\sum_{n=1}^{\infty} a_n$  converges  $\Leftrightarrow \sum_{k=1}^{\infty} 2^k a_{2^k}$  converges.

**Proof.** ( $\Leftarrow$ ) We show that if  $\sum_n a_n$  then  $\sum_k 2^k a_{2^k}$  diverges. For this, note that  $a_1 + a_2 + \cdots + a_n \leq a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \cdots + (a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1})$  if  $2^{k+1} > n$ .  $a_1 + a_2 + \cdots + a_n \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$ . LHS unbounded as  $n \rightarrow \infty$ , so RHS is also unbounded as  $k \rightarrow \infty$ .

( $\Rightarrow$ )  $a_1 + a_2 + a_3 + \cdots + a_n \geq a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k})$  if  $2^k \leq n$ .  $a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \cdots + (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \cdots + a_{2^k}) \geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \geq \frac{1}{2}(a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k})$ . If  $\sum_n a_n$  converges, then LHS is bounded for all  $n$  so RHS is bounded for all  $k$ . Hence RHS converges since it is monotone.

Q.E.D.

**Theorem 47 ( $p$ -series).**  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

**Proof.** For  $p \leq 0$ ,  $\frac{1}{n^p} \not\rightarrow 0$ , so series diverges. For  $p > 0$ ,  $\frac{1}{n^p}$  is decreasing, so  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges iff  $\sum_k 2^k \cdot \frac{1}{(2^k)^p}$  converges. But  $\sum_k 2^k \cdot \frac{1}{(2^k)^p} = \sum_k (\frac{1}{2^{p-1}})^k$  converges iff  $\frac{1}{2^{p-1}} < 1 (\Leftrightarrow p - 1 > 0)$ , which is equivalent to  $p > 1$ . Q.E.D.

**Theorem 48.**  $\sum_{n=3}^{\infty} \frac{1}{n(\log n)^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ . (log is to base  $e$ .)

**Proof.** If  $p \leq 0$ , then  $\frac{1}{n(\log n)^p} \geq \frac{1}{n}$ , so  $\sum_n \frac{1}{n(\log n)^p}$  diverges by the comparison test. If  $p > 0$  then  $\frac{1}{n(\log n)^p}$  decreases since  $\log n$  increases. By theorem 46,  $\sum_n \frac{1}{n(\log n)^p}$  converges  $\Leftrightarrow \sum_k 2^k \cdot \frac{1}{2^k(\log 2^k)^p}$  converges  $\Leftrightarrow \sum_k \frac{1}{(\log 2)^p} \cdot \frac{1}{k^p}$  converges  $\Leftrightarrow p > 1$  Q.E.D.

**Definition 37 (e).**  $e := \sum_{n=0}^{\infty} \frac{1}{n!}$ .

**Remark. Convergence**  $\frac{1}{n!} = \frac{1}{(n(n-1)(n-2)\cdots 3\cdot 2\cdot 1)} \leq \frac{1}{2\cdot 2\cdot 2\cdots 2\cdot 1} = \frac{1}{2^{n-1}}$ . Therefore,  $S_n = \sum_{k=0}^n \frac{1}{k!} \leq 1 + \sum_{k=1}^n \frac{1}{2^{k-1}} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{1-1/2} = 3$ . Hence,  $e \leq 3$ .

**Rate of Convergence**  $0 < e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-n+1}} \cdot \frac{1}{(n+1)!} = \frac{1}{(n+1)!} \sum_{k=n+1}^{\infty} \frac{1}{(n+1)^{k-(n+1)}} = \frac{1}{(n+1)!} \cdot \frac{1}{1-1/(n+1)} = \frac{1}{(n!) \cdot n}$ .

**Theorem 49.**  $e \notin \mathbb{Q}$ .

**Proof.** For contradiction, suppose  $e = \frac{p}{q}$ ,  $p, q \in \mathbb{N}$ . Then  $0 < e - S_q < \frac{1}{q \cdot q!}$ , so  $0 < q! \cdot e - q! \cdot S_q < \frac{1}{q}$ . If we show  $q! \cdot e - q! \cdot S_q \in \mathbb{Z}$ , then we'll have a contradiction. But  $q! \cdot e - q! \cdot S_q = q! \cdot \frac{p}{q} - \sum_{k=0}^q \frac{q!}{k!}$ . Because  $q! \cdot \frac{p}{q} \in \mathbb{N}$  and  $\sum_{k=0}^q \frac{q!}{k!} \in \mathbb{N}$ , the difference of them,  $q! \cdot e - q! \cdot S_q \in \mathbb{Z}$ , thus a contradiction. Q.E.D.

**Theorem 50.**  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$ .

**Proof.** Let  $t_n = (1 + \frac{1}{n})^n$ . Then  $t_n = \sum_{k=0}^n \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^n \frac{1}{k!} \cdot (\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}) \leq S_n$ . So  $\limsup_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_n = e$ . On the other hand, for fixed  $m$  and  $n \geq m$ ,  $t_n \geq \sum_{k=0}^m \binom{n}{k} \cdot \frac{1}{n^k} = \sum_{k=0}^m \frac{1}{k!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{k-1}{n})$ . Let  $n \rightarrow \infty$  with  $m$  fixed.  $\liminf_{n \rightarrow \infty} t_n \geq \sum_{k=0}^m \frac{1}{k!} \cdot 1 = S_m$ . This is true for any  $m$ . Now let  $m \rightarrow \infty$ .  $\liminf_{n \rightarrow \infty} t_n \geq \limsup_{m \rightarrow \infty} S_m = e$ .  $e \leq \liminf_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} t_n \leq e$ . Therefore,  $\lim_{n \rightarrow \infty} t_n$  exists and equals  $e$ . Q.E.D.



---

**Theorem 51 (Root test).** Let  $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ . Then,

$$\sum a_n \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha > 1 \\ \text{inconclusive} & \text{if } \alpha = 1 \end{cases}$$

**Proof (Just outline).**  $\alpha < 1$  Eventually  $|a_n| \leq \beta^n$ , thus convergence.

$\alpha > 1$   $|a_n| > 1$  for infinitely many  $n$ , thus divergence.

$\alpha = 1$   $\frac{1}{n}$  diverges,  $\frac{1}{n^2}$  converges.

Q.E.D.

**Theorem 52 (Ratio test).** The series  $\sum_{n=1}^{\infty} a_n$  with  $a_n \neq 0$  converges if  $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$  and diverges if  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N : \left| \frac{a_{n+1}}{a_n} \right| \geq 1$ . Otherwise, inconclusive.

**Proof.** (see textbook).

$$\text{Convergence } \sum a_n \begin{cases} \text{converges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \\ \text{diverges} & \text{if } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \\ \text{diverges} & \text{if } \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \end{cases}$$

**Note.** Note that we cannot replace  $\liminf$  with  $\limsup$  in the last case.

Q.E.D.

**Example.** Let  $s_n = \sum_{n=0}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \dots$ . First, note that  $a_{2k} = \frac{1}{2} \cdot \frac{1}{4^k}$ ,  $a_{2k+1} = 2a_{2k} = \frac{1}{4^k}$  for  $k \geq 0$ .

**Ratio test** Then the ratio  $\frac{a_{n+1}}{a_n}$  is the sequence  $2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \dots$ . Therefore,  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}$ ,  $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$ . The ratio test is inconclusive for  $s_n$ .

**Root test**  $a_n = \begin{cases} \frac{1}{2} \cdot \frac{1}{2^n} & n \text{ even} \\ \frac{2}{2^n} & n \text{ odd} \end{cases}$ , so  $(\frac{1}{2})^{1/n} \cdot \frac{1}{2} \leq \sqrt[n]{a_n} \leq \frac{2^{1/n}}{2}$ , thus  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \frac{1}{2} < 1$ . Therefore,  $s_n = \sum_{n=0}^{\infty} a_n$  converges.

This is an example where the ratio test is inconclusive but the root test is conclusive.

**Theorem 53.** If  $\sum a_n = A$  and  $\sum b_n = B$ , then  $\sum a_n + b_n = A + B$  and  $\sum c \cdot a_n = cA$ .

### 3.3 Power Series

**Definition 38 (Power Series).** For  $z \in \mathbb{C}$  and a complex sequence  $\{c_n\}$ ,  $\sum_{n=0}^{\infty} c_n z^n$  is a power series.

**Remark.** As  $z^0 = 1$  for all  $z \in \mathbb{C}$ , by convention  $\sum_{n=0}^{\infty} c_n z^n = c_0 + \sum_{n=1}^{\infty} c_n z^n$ .

**Theorem 54.** Let  $R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}}$  such that

$$R = \begin{cases} 0 & \text{if } \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \infty \end{cases}$$

. Then  $\sum c_n z^n$   $\begin{cases} \text{converges} & \text{if } |z| < R \\ \text{diverges} & \text{if } |z| > R \\ \text{inconclusive} & \text{if } |z| = R \end{cases}$

**Proof.**  $\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}$ . By root test, the series converges if  $\frac{|z|}{R} < 1$  and diverges if  $\frac{|z|}{R} > 1$ .

**Note.** In practice, often use the ratio test to find  $R$ .

Q.E.D.

**Example.** (a)  $\sum n! \cdot z^n$  has  $R = 0$ .

**By ratio test**  $\left| \frac{(n+1)! \cdot z^{n+1}}{n! \cdot z^n} \right| = |z| \cdot (n+1) \rightarrow \infty \forall z \neq 0$ . Hence, the series diverges  $\forall z \neq 0$ .

**By root test** Note  $n \neq \frac{1}{2}(\frac{2}{3})^2(\frac{3}{4})^3 \dots (\frac{n-1}{n})^{n-1} n^n$  for  $n \geq 2$ . Then  $n \neq \frac{n^n}{(1+1)^1(1+\frac{1}{2})^2(1+\frac{1}{n-1})^{n-1}}$ . In the proof of Theorem 50, we saw  $(1+\frac{1}{j}) \leq e$ . So  $n! \geq \frac{n^n}{e^{n-1}} = e \cdot (\frac{n}{e})^n$ .  $\sqrt[n]{n!} \geq e^{1/n} \cdot \frac{n}{e} \rightarrow \infty$  as  $n \rightarrow \infty$ . Therefore,  $R = \frac{1}{\infty} = 0$ .

**Note.** Cf. Stirling's formula:  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$ .

**Definition 39 (Absolute Convergence, Conditional Convergence).**

- (a)  $\sum a_n$  converges absolutely if  $\sum |a_n|$  converges.
- (b)  $\sum a_n$  converges conditionally if  $\sum a_n$  converges but  $\sum |a_n|$  diverges.

**Remark.** All other convergence tests seen so far are actually tests for absolute convergence.

**Example.** •  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$  converges conditionally by the alternating se-

ries test of assignment 7's practice problem 1. See also Theorem 3.43. (MUST TAKE A LOOK!)

- Given  $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$  and  $a_n \rightarrow 0$ , then  $\sum (-1)^n a_n$  converges.
- $\sum_{n=0}^{\infty} n!2^n$  has  $R = 0$
- $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$  has  $R = \infty$  since  $R = \frac{1}{\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}} = \frac{1}{\limsup_{n \rightarrow \infty} \frac{1}{n}} = 1/0 = \infty$ , or use ratio test,  $|\frac{z^{n+1}/(n+1)^{n+1}}{z^n/n^n}| = |z| \frac{n^n}{(n+1)^{n+1}} = |z| \frac{1}{n+1} \frac{1}{(1+\frac{1}{n})^n}$ . Since  $(1 + \frac{1}{n})^n \rightarrow \frac{1}{e}$  as  $n \rightarrow \infty$ ,  $|z| \frac{1}{n+1} \frac{1}{(1+\frac{1}{n})^n} \rightarrow 0$  as  $n \rightarrow \infty \forall z \in \mathbb{C}$  so  $R = \infty$ .

**Remark.** Conditionally convergent series have a bad feature.

**Theorem 55** (Thm 3.54). Suppose  $\sum a_n$  converges conditionally. Let  $-\infty \leq \alpha \leq \beta \leq +\infty$ . Then  $\exists$  bijection  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that with  $a'_n = a_{f(n)}$  and  $S'_n = \sum_{k=1}^n a'_k$ , we have  $\liminf_{n \rightarrow \infty} s'_n = \alpha$  and  $\limsup_{n \rightarrow \infty} s'_n = \beta$

**Proof.** Take a look at the textbook Q.E.D.

**Theorem 56** (Thm 3.55). If  $\sum a_n$  converges absolutely, then every rearrangement of  $\sum a_n$  converges to the same sum.

**Proof.** Take a look at the textbook Q.E.D.

## 3.4 Products of Series

**Motivation** Consider  $z^N$  in  $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n)$ . Since  $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{n=0}^{\infty} b_n z^n) = (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) = (a_0 b_0) + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \dots$ ,  $z^N$  has coefficient  $\sum_{k=0}^N a_k b_{N-k}$ .

**Definition 40.** The product of  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  is  $\sum_{n=0}^{\infty} c_n$  where  $c_n = \sum_{k=0}^n a_k b_{n-k}$ .

**Note.** This is a discrete convolution.

**Question** If  $\sum a_n = A$  and  $\sum b_n = B$  both converge, does  $\sum c_n$  converge and if so, does it converge to  $AB$ ?

**Answer**  $\sum c_n$  converges if  $\sum a_n$  and  $\sum b_n$  converge absolutely. (ref: Thm 57). Moreover, if  $\sum c_n$  does converge, then it must converge to  $AB$  (ref: Thm ??). Maybe no otherwise (ref: Example 3.49).

---

**Theorem 57** (Thm 3.50). Suppose  $\sum a_n$  converges absolutely to  $A$  and  $\sum b_n$  converges to  $B$ . Then  $\sum c_n$  converges to  $AB$ .

**Proof.** Let  $A_n = \sum_{k=0}^n a_k$ ,  $B_n = \sum_{k=0}^n b_k$ ,  $C_n = \sum_{k=0}^n c_k$ . Then  $A_n \rightarrow A, B_n \rightarrow B$ . By definition,  $C_n = \sum_{k=0}^n \sum_{j=0}^k a_j b_{k-j} = \sum_{j=0}^n \sum_{k=j}^n a_j b_{k-j} = \sum_{j=0}^n a_j \sum_{k=j}^n b_{k-j} = \sum_{j=0}^n a_j B_{n-j} = \sum_{j=0}^n a_j B - \sum_{j=0}^n a_j (B_{n-j} - B)$ . Let  $\beta_{n-j}$ , where  $\beta_k = B_k - B$ . Then  $C_n = A_n B + \sum_{j=0}^n a_j \beta_{n-j}$ . Let  $\gamma_n = \sum_{j=0}^n a_j \beta_{n-j}$ . Note that  $A_n B \rightarrow AB, \beta_k \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $\alpha = \sum_{k=0}^{\infty} |a_k| < \infty$  ( $\because a_n$  converges absolutely by assumption). Rewrite  $\gamma_n$  as  $\gamma_n = \sum_{j=0}^n a_{n-j} \beta_j$ . We know  $\beta_j \rightarrow 0$  as  $j \rightarrow \infty$ . Let  $\varepsilon > 0$ . Choose  $N$  s.t.  $|\beta_j| < \varepsilon$  if  $j \geq N$ . Then for  $n \geq N+1$ ,  $|\gamma_n| \leq |\sum_{j=0}^N a_{n-j} \beta_j| + |\sum_{j=N+1}^n a_{n-j} \beta_j|$ . Note  $|\sum_{j=N+1}^n a_{n-j} \beta_j| \leq \varepsilon \sum_{j=N+1}^n |a_{n-j}| \leq \varepsilon \alpha$ . Let  $n \rightarrow \infty$  with  $N$  fixed. Then  $a_{n-j} \rightarrow 0$  for  $0 \leq j \leq N$  since  $|a_n| \rightarrow 0$ . Q.E.D.